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A Companion for General Physics
Study Guide for General Physics

Also known as: some physics I would have liked to know when I was a student.

Alessandro Petrolini

Always Student; Sometimes Teacher.

In memory of my friend, Prof. ***Maurizio Lo Vetere****,*
with whom I have discussed so long about many of the things written in these notes,
... although I know that not everything I wrote he would agree on ...

Always work in progress ...

Everything should be made as simple as possible, but not simpler. Albert Einstein.

First Law of Progress in Theoretical Physics: You will get nowhere by crunching equations. Stephen Weinberg.

Strive for progress, not for perfection. Richard Feynman.

You cannot get educated by this self-propagating system in which people study to pass exams, and teach others to pass exams, but nobody knows anything. You learn something by doing it yourself, by asking questions, by thinking, and by experimenting. Richard Feynman.

The first principle is that you must not fool yourself - and you are the easiest person to fool. Richard Feynman

Idiocy is when you think you know everything, without questioning; Intelligence is when you question everything you think you know.

When you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind: it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of science, whatever the matter may be. Lord Kelvin, often quoted as "If you cannot measure it, then it is not science."

> *A theorist excitedly runs into an experimentalist's office one day, exclaiming that not only has the theorist uncovered an elegant model that solves the biggest outstanding questions in particle physics, but that it makes a robust experimental prediction. The excitement is contagious, and the experimentalist applies for grants, hires postdocs, builds new laboratory equipment, and starts to take data. Many years after the initial encounter, the experimentalist's entire research group—now exclusively focused on proving this one, elegant model—is ready to unblind its data. That afternoon, the experimentalist sulks despondently into the theorist's office. "Prof. Theorist? I'm really sorry to break this to you, but the data rules out the model." The theorist looks up at the experimentalist, considers the streaks of white hair and their many years of collaboration on this project, and finally says, "What a shame! Did you know that I spent two whole weeks of my life developing that model?"*
> - Professor Tim M.P. Tait. From Just a Taste: Lectures on Flavor Physics

The history of physics is interesting but this notes will not follow the historical development. I will try to present things in a logical and coherent scheme. Many of the details will be left to the students as exercises, because this is a part of their learning process. Sometimes there will be some leaps, but I try to point the reader to the appropriate references to fill the gaps.

Much of the content of these notes is acknowledged to come from the various sources cited. **The author thinks to have fulfilled all due credits to the authors mentioned**. If you find this is not the case, please inform me, so I will take the appropriate actions.

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CHAPTER 11

Introduction

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11.01 Usage Notes

- There are plenty of beautiful books about physics (but also many others are not so good...). Therefore I feel useless to try to re-write myself what is already very well presented in the literature; of course, all according to my ideas.
- I think to have properly acknowledged all the quotations. In case of any failure to do so, I apologize in advance.
- Whenever a quote is present without any explicit citation, it is implied that it refers to the unique reference at the beginning of that particular section.
- One main emphasis is on providing reference to more advanced treatments, to stimulate curiosity and further study.

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11.02 Notations and Conventions

- Le definizioni sono indicate con il simbolo: $\equiv$.
- Gli angoli piani si intendono sempre misurati in radianti, ove non specificato altrimenti.
- Solid angles are always measured in steradians
- Whenever not specified otherwise, all physical quantities are assumed to be measured in units of the **International System** of units, **SI**; read § B.
- Si intende: $\Delta \equiv$ Variazione $\equiv$ Valore della grandezza (dopo) $-$ Valore della grandezza (prima).
- La derivazione rispetto al tempo viene talvolta indicata con un punto

$$\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}t} \quad ,$$
$$\ddot{x} \equiv \frac{d^2x}{\mathrm{d}t^2} \quad .$$

- In termodinamica le variabili intensive sono di solito indicate con un simbolo minuscolo mentre la variabili estensive con un simbolo maiuscolo. Le variabili intensive corrispondenti sono indicate con la corrispondente lettera minuscola.
- Si assume implicitamente che tutte le funzioni siano, quando necessario, differenziabili/integrabili tante volte quanto richiesto e/o sviluppabili in serie di Taylor/Fourier convergente quanto basta. That is, if any formula contains derivatives of some functions, then the functions are assumed to be differentiable as much as required; if any formula contains definite integrals, then the integrals are supposed to be convergent.
- Unless specified otherwise when dealing with tensors one always refers to three components tensors under the rotation group in euclidean three-space, for classical physics, and to four components tensors under the Lorentz group in Minkowski space, for relativistic physics.

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11.03 Nomenclature

Beware that the names below, as often in science, are not necessarily universally used in the literature: whenever reading any publication one has, unfortunately, to check always definitions and conventions at the start.

- Uniform/Homogeneous: independent from position.
- Constant: independent from time.
- Fixed: fixed quantity; it is considered to be independent from time, position and independent from anything, within the specified context.
- Stationary: everything independent from time (stationary charge, target, particle, thermodynamic state,...); but steady wave (or stationary wave) (the wave is not moving).
- Steady: something is moving, but motion is time-independent (steady current, motion, flow,...); in a steady wave (or stationary wave) there is a steady motion of something.
- Rate:= something per unit time.
- Density:= something per unit length/area/volume; that is: linear density, surface density, volume density; density alone means volume density.
- **(Volume) Flux**:= something per unit perpendicular area and per unit time; on a 3D volume.
- **Surface Flux**:= something per unit perpendicular length and per unit time; on a 2D surface.
- **Flux of a vector field** := this is the usual definition of flux of a vector field (either 2D or 3D), as from sections § 16.05.02 and § 16.06.02, whose physical dimensions are not, in general, something per unit area-time, but depend on the dimensions of the vector fields.
- **(Volume) Fluence**:= something per unit perpendicular area; that is, a flux integrated over a specified period of time, on a 3D volume.
- **Surface Fluence**:= something per unit perpendicular length; that is, a surface flux integrated over a specified period of time, on a 2D surface.
- Production Rate density: something per unit volume and per unit time.
- Sinusoidal, Cosinusoidal and Harmonic are treated as exactly synonyms.
- Vibration = Oscillation.
- Reference Frame is a concept different from Coordinate System: in any chosen Reference Frame you can define infinitely many Coordinate System.
- Numeric: one single real or complex quantity.

When dealing with the modeling of physical problems one should never forget that a model is no better than the approximations (explicit and|or implicit) made when deriving it. It is never a question of just solving the equations, forgetting what is behind them.

UNDER THE PROGRESS

11.04 Physical Constants

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CODATA RECOMMENDED VALUES OF THE FUNDAMENTAL PHYSICAL CONSTANTS: 2018

NIST SP 961 (May 2019)

An extensive list of constants is available on the NIST Physics Laboratory Web site physics.nist.gov/constants. For numerical values a number in parentheses, if present, is the one-standard-deviation uncertainty in the last two digits. For units with square brackets the full descriptions of m^{-1} and m are cycles per meter and meter per cycle, respectively. For the first radiation constant the full description of m^2 is m^{-2} (m/cycle)^4.

Quantity	Symbol	Numerical value	Unit
speed of light in vacuum	c	299 792 458 (exact)	m s^{-1}
Newtonian constant of gravitation	G	$6.674\,30(15) \times 10^{-11}$	m^3 kg^{-1} s^{-2}
Planck constant	h	$6.626\,070\,15 \times 10^{-34}$ (exact)	J Hz^{-1}
in eV s		$4.135\,667\,696\ldots \times 10^{-15}$	eV Hz^{-1}
	$\hbar$	$1.054\,571\,817\ldots \times 10^{-34}$	J s
in eV s		$6.582\,119\,569\ldots \times 10^{-16}$	eV s
elementary charge	e	$1.602\,176\,634 \times 10^{-19}$ (exact)	C
vacuum magnetic permeability $4\pi\alpha\hbar/e^2c$	μ_0	$1.256\,637\,062\,12(19) \times 10^{-6}$	N A^{-2}
$\mu_0/(4\pi \times 10^{-7})$		1.000 000 000 55(15)	N A^{-2}
vacuum electric permittivity $1/\mu_0 c^2$	ϵ_0	$8.854\,187\,8128(13) \times 10^{-12}$	F m^{-1}
Josephson constant $2e/h$	K_J	$483\,597.848\,4\ldots \times 10^9$	Hz V^{-1}
von Klitzing constant $\mu_0 c/2\alpha = 2\pi\hbar/e^2$	R_K	25 812.807 45 . . .	Ω
magnetic flux quantum $2\pi\hbar/(2e)$	Φ_0	$2.067\,833\,848\ldots \times 10^{-15}$	Wb
Bohr magneton $e\hbar/2m_e$	μ_B	$9.274\,010\,0783(28) \times 10^{-24}$	J T^{-1}
in eV T^{-1}		$5.788\,381\,8060(17) \times 10^{-5}$	eV T^{-1}
nuclear magneton $e\hbar/2m_p$	μ_N	$5.050\,783\,7461(15) \times 10^{-27}$	J T^{-1}
in eV T^{-1}		$3.152\,451\,258\,44(96) \times 10^{-8}$	eV T^{-1}
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,5693(11) \times 10^{-3}$	
inverse fine-structure constant	α^{-1}	137.035 999 084(21)	
Rydberg frequency $\alpha^2 m_e c^2/2h = E_h/2h$	cR_∞	$3.289\,841\,960\,2508(64) \times 10^{15}$	Hz
energy equivalent in eV		13.605 693 122 994(26)	eV
Rydberg constant	R_∞	10 973 731.568 160(21)	[m^{-1}]
Bohr radius $\hbar/\alpha m_e c = 4\pi\epsilon_0\hbar^2/m_e e^2$	a_0	$5.291\,772\,109\,03(80) \times 10^{-11}$	m
Hartree energy $\alpha^2 m_e c^2 = e^2/4\pi\epsilon_0 a_0 = 2(cR_\infty)h$	E_h	$4.359\,744\,722\,2071(85) \times 10^{-18}$	J
in eV		27.211 386 245 988(53)	eV
electron mass	m_e	$9.109\,383\,7015(28) \times 10^{-31}$	kg
in u		$5.485\,799\,090\,65(16) \times 10^{-4}$	u
energy equivalent in MeV	$m_e c^2$	0.510 998 950 00(15)	MeV
electron-muon mass ratio	m_e/m_μ	$4.836\,331\,69(11) \times 10^{-3}$	
electron-proton mass ratio	m_e/m_p	$5.446\,170\,214\,87(33) \times 10^{-4}$	
electron charge to mass quotient	$-e/m_e$	$-1.758\,820\,010\,76(53) \times 10^{11}$	C kg^{-1}
reduced Compton wavelength $\hbar/m_e c = \alpha a_0$	$\lambda\!\!\!^{-}_C$	$3.861\,592\,6796(12) \times 10^{-13}$	m
Compton wavelength	λ_C	$2.426\,310\,238\,67(73) \times 10^{-12}$	[m]
classical electron radius $\alpha^2 a_0$	r_e	$2.817\,940\,3262(13) \times 10^{-15}$	m
Thomson cross section $(8\pi/3)r_e^2$	σ_e	$6.652\,458\,7321(60) \times 10^{-29}$	m^2
electron magnetic moment	μ_e	$-9.284\,764\,7043(28) \times 10^{-24}$	J T^{-1}
to Bohr magneton ratio	μ_e/μ_B	−1.001 159 652 181 28(18)	
to nuclear magneton ratio	μ_e/μ_N	−1838.281 971 88(11)	
electron magnetic moment anomaly $\lvert\mu_e\rvert/\mu_B - 1$	a_e	$1.159\,652\,181\,28(18) \times 10^{-3}$	
electron g-factor $-2(1 + a_e)$	g_e	−2.002 319 304 362 56(35)	
electron-proton magnetic moment ratio	μ_e/μ_p	−658.210 687 89(20)	
muon mass in u	m_μ	0.113 428 9259(25)	u
energy equivalent in MeV	$m_\mu c^2$	105.658 3755(23)	MeV
muon-electron mass ratio	m_μ/m_e	206.768 2830(46)	
muon magnetic moment	μ_μ	$-4.490\,448\,30(10) \times 10^{-26}$	J T^{-1}
to Bohr magneton ratio	μ_μ/μ_B	$-4.841\,970\,47(11) \times 10^{-3}$	
to nuclear magneton ratio	μ_μ/μ_N	−8.890 597 03(20)	
muon magnetic moment anomaly $\lvert\mu_\mu\rvert/(e\hbar/2m_\mu) - 1$	a_μ	$1.165\,920\,89(63) \times 10^{-3}$	

Quantity	Symbol	Numerical value	Unit
muon g-factor $-2(1 + a_\mu)$	g_μ	−2.002 331 8418(13)	
muon-proton magnetic moment ratio	μ_μ/μ_p	−3.183 345 142(71)	
proton mass	m_p	$1.672\,621\,923\,69(51) \times 10^{-27}$	kg
in u		1.007 276 466 621(53)	u
energy equivalent in MeV	$m_p c^2$	938.272 088 16(29)	MeV
proton-electron mass ratio	m_p/m_e	1836.152 673 43(11)	
proton magnetic moment	μ_p	$1.410\,606\,797\,36(60) \times 10^{-26}$	J T^{-1}
to nuclear magneton ratio	μ_p/μ_N	2.792 847 344 63(82)	
proton magnetic shielding correction $1-\mu'_p/\mu_p$ (H_2O, sphere, 25 °C)	σ'_p	$2.5689(11) \times 10^{-5}$	
proton gyromagnetic ratio $2\mu_p/\hbar$	γ_p	$2.675\,221\,8744(11) \times 10^8$	s^{-1} T^{-1}
		42.577 478 518(18)	MHz T^{-1}
shielded proton gyromagnetic ratio $2\mu'_p/\hbar$ (H_2O, sphere, 25 °C)	γ'_p	$2.675\,153\,151(29) \times 10^8$	s^{-1} T^{-1}
		42.576 384 74(46)	MHz T^{-1}
neutron mass in u	m_n	1.008 664 915 95(49)	u
energy equivalent in MeV	$m_n c^2$	939.565 420 52(54)	MeV
neutron-proton mass ratio	m_n/m_p	1.001 378 419 31(49)	
neutron magnetic moment	μ_n	$-9.662\,3651(23) \times 10^{-27}$	J T^{-1}
to nuclear magneton ratio	μ_n/μ_N	−1.913 042 73(45)	
deuteron mass in u	m_d	2.013 553 212 745(40)	u
energy equivalent in MeV	$m_d c^2$	1875.612 942 57(57)	MeV
deuteron-proton mass ratio	m_d/m_p	1.999 007 501 39(11)	
deuteron magnetic moment	μ_d	$4.330\,735\,094(11) \times 10^{-27}$	J T^{-1}
to nuclear magneton ratio	μ_d/μ_N	0.857 438 2338(22)	
helion (^{3}He nucleus) mass in u	m_h	3.014 932 247 175(97)	u
energy equivalent in MeV	$m_h c^2$	2808.391 607 43(85)	MeV
shielded helion magnetic moment (gas, sphere, 25 °C)	μ'_h	$-1.074\,553\,090(13) \times 10^{-26}$	J T^{-1}
to Bohr magneton ratio	μ'_h/μ_B	$-1.158\,671\,471(14) \times 10^{-3}$	
to nuclear magneton ratio	μ'_h/μ_N	−2.127 497 719(25)	
alpha particle mass in u	m_α	4.001 506 179 127(63)	u
energy equivalent in MeV	$m_\alpha c^2$	3727.379 4066(11)	MeV
Boltzmann constant	k	$1.380\,649 \times 10^{-23}$ (exact)	J K^{-1}
Avogadro constant	N_A	$6.022\,140\,76 \times 10^{23}$ (exact)	mol^{-1}
atomic mass constant $\frac{1}{12}m(^{12}C) = 1$ u	m_u	$1.660\,539\,066\,60(50) \times 10^{-27}$	kg
energy equivalent in MeV	$m_u c^2$	931.494 102 42(28)	MeV
Faraday constant $N_A e$	F	96 485.332 12 . . .	C mol^{-1}
molar gas constant $N_A k$	R	8.314 462 618 . . .	J mol^{-1} K^{-1}
in eV K^{-1}		$8.617\,333\,262\ldots \times 10^{-5}$	eV K^{-1}
molar volume of ideal gas RT/p (T = 273.15 K, p = 101.325 kPa)	V_m	$22.413\,969\,54\ldots \times 10^{-3}$	m^3 mol^{-1}
Stefan-Boltzmann constant $\pi^2 k^4/60\hbar^3 c^2$	σ	$5.670\,374\,419\ldots \times 10^{-8}$	W m^{-2} K^{-4}
first radiation constant $2\pi hc^2$	c_1	$3.741\,771\,852\ldots \times 10^{-16}$	[W m^2]
second radiation constant hc/k	c_2	$1.438\,776\,877\ldots \times 10^{-2}$	[m K]
Wien displacement law constant $b = \lambda_{max}T = c_2/4.965\,114\,231\ldots$	b	$2.897\,771\,955\ldots \times 10^{-3}$	[m K]
Cu x unit: $\lambda(\mathrm{Cu\,K\alpha_1})/1\,537.400$	xu(Cu Kα$_1$)	$1.002\,076\,97(28) \times 10^{-13}$	m
Mo x unit: $\lambda(\mathrm{Mo\,K\alpha_1})/707.831$	xu(Mo Kα$_1$)	$1.002\,099\,52(53) \times 10^{-13}$	m

Energy equivalents

$[1\ m^{-1}]c = 299\,792\,458$ Hz	(1 Hz)$h/k = 4.799\,243\,073\ldots \times 10^{-11}$ K	(1 J) $= 6.241\,509\,074\ldots \times 10^{18}$ eV	(1 eV)$/c^2 = 1.073\,544\,102\,33(32) \times 10^{-9}$ u
$[1\ m^{-1}]hc/k = 1.438\,776\,877\ldots \times 10^{-2}$ K	(1 Hz)$h = 4.135\,667\,696\ldots \times 10^{-15}$ eV	(1 eV) $= 1.602\,176\,634 \times 10^{-19}$ J	(1 kg) $= 6.022\,140\,7621(18) \times 10^{26}$ u
$[1\ m^{-1}hc = 1.239\,841\,984\ldots \times 10^{-6}$ eV	(1 K)$k/hc = 69.503\,480\,04\ldots$ [m^{-1}]	(1 eV)$/hc = 8.065\,543\,937\ldots \times 10^5$ [m^{-1}]	(1 u) $= 1.660\,539\,066\,60(50) \times 10^{-27}$ kg
$[1\ m^{-1}]h/c = 1.331\,025\,050\,10(40) \times 10^{-15}$ u	(1 K)$k/h = 2.083\,661\,912\ldots \times 10^{10}$ Hz	(1 eV)$/h = 2.417\,989\,242\ldots \times 10^{14}$ Hz	(1 u)$c/h = 7.513\,006\,6104(23) \times 10^{14}$ [m^{-1}]
(1 Hz)$/c = 3.335\,640\,951\ldots \times 10^{-9}$ [m^{-1}]	(1 K)$k = 8.617\,333\,262\ldots \times 10^{-5}$ eV	(1 eV)$/k = 1.160\,451\,812\ldots \times 10^4$ K	(1 u)$c^2 = 9.314\,941\,0242(28) \times 10^8$ eV

NIST National Institute of Standards and Technology • U.S. Department of Commerce

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Fundamental Physical Constants — Frequently used constants

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
speed of light in vacuum	c	299 792 458	m s^{-1}	exact
Newtonian constant of gravitation	G	$6.674\,30(15) \times 10^{-11}$	m^3 kg^{-1} s^{-2}	2.2×10^{-5}
Planck constant*	h	$6.626\,070\,15 \times 10^{-34}$	J Hz^{-1}	exact
	$\hbar$	$1.054\,571\,817\ldots \times 10^{-34}$	J s	exact
elementary charge	e	$1.602\,176\,634 \times 10^{-19}$	C	exact
vacuum magnetic permeability $4\pi\alpha\hbar/e^2c$	μ_0	$1.256\,637\,062\,12(19) \times 10^{-6}$	N A^{-2}	1.5×10^{-10}
vacuum electric permittivity $1/\mu_0 c^2$	ϵ_0	$8.854\,187\,8128(13) \times 10^{-12}$	F m^{-1}	1.5×10^{-10}
Josephson constant $2e/h$	K_J	$483\,597.848\,4\ldots \times 10^9$	Hz V^{-1}	exact
von Klitzing constant $\mu_0 c/2\alpha = 2\pi\hbar/e^2$	R_K	$25\,812.807\,45\ldots$	Ω	exact
magnetic flux quantum $2\pi\hbar/(2e)$	Φ_0	$2.067\,833\,848\ldots \times 10^{-15}$	Wb	exact
conductance quantum $2e^2/2\pi\hbar$	G_0	$7.748\,091\,729\ldots \times 10^{-5}$	S	exact
electron mass	m_e	$9.109\,383\,7015(28) \times 10^{-31}$	kg	3.0×10^{-10}
proton mass	m_p	$1.672\,621\,923\,69(51) \times 10^{-27}$	kg	3.1×10^{-10}
proton-electron mass ratio	m_p/m_e	1836.152 673 43(11)		6.0×10^{-11}
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,5693(11) \times 10^{-3}$		1.5×10^{-10}
inverse fine-structure constant	α^{-1}	137.035 999 084(21)		1.5×10^{-10}
Rydberg frequency $\alpha^2 m_e c^2/2h$	cR_∞	$3.289\,841\,960\,2508(64) \times 10^{15}$	Hz	1.9×10^{-12}
Boltzmann constant	k	$1.380\,649 \times 10^{-23}$	J K^{-1}	exact
Avogadro constant	N_A	$6.022\,140\,76 \times 10^{23}$	mol^{-1}	exact
molar gas constant $N_A k$	R	8.314 462 618…	J mol^{-1} K^{-1}	exact
Faraday constant $N_A e$	F	96 485.332 12…	C mol^{-1}	exact
Stefan-Boltzmann constant $(\pi^2/60)k^4/\hbar^3c^2$	σ	$5.670\,374\,419\ldots \times 10^{-8}$	W m^{-2} K^{-4}	exact
Non-SI units accepted for use with the SI				
electron volt (e/C) J	eV	$1.602\,176\,634 \times 10^{-19}$	J	exact
(unified) atomic mass unit $\frac{1}{12}m(^{12}C)$	u	$1.660\,539\,066\,60(50) \times 10^{-27}$	kg	3.0×10^{-10}

* The energy of a photon with frequency ν expressed in unit Hz is $E = h\nu$ in J. Unitary time evolution of the state of this photon is given by $\exp(-iEt/\hbar)|\varphi\rangle$, where $|\varphi\rangle$ is the photon state at time $t = 0$ and time is expressed in unit s. The ratio $Et/\hbar$ is a phase.

GO TO BEFORE included File

GO TO AFTER included File

Fundamental Physical Constants — Extensive Listing

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
		UNIVERSAL		
speed of light in vacuum	c	299 792 458	m s^{-1}	exact
vacuum magnetic permeability $4\pi\alpha\hbar/e^2c$	μ_0	$1.256\,637\,062\,12(19) \times 10^{-6}$	N A^{-2}	1.5×10^{-10}
$\mu_0/(4\pi \times 10^{-7})$		1.000 000 000 55(15)	N A^{-2}	1.5×10^{-10}
vacuum electric permittivity $1/\mu_0c^2$	ϵ_0	$8.854\,187\,8128(13) \times 10^{-12}$	F m^{-1}	1.5×10^{-10}
characteristic impedance of vacuum $\mu_0 c$	Z_0	376.730 313 668(57)	Ω	1.5×10^{-10}
Newtonian constant of gravitation	G	$6.674\,30(15) \times 10^{-11}$	m^3 kg^{-1} s^{-2}	2.2×10^{-5}
	$G/\hbar c$	$6.708\,83(15) \times 10^{-39}$	$(\text{GeV}/c^2)^{-2}$	2.2×10^{-5}
Planck constant*	h	$6.626\,070\,15 \times 10^{-34}$	J Hz^{-1}	exact
		$4.135\,667\,696\ldots \times 10^{-15}$	eV Hz^{-1}	exact
	$\hbar$	$1.054\,571\,817\ldots \times 10^{-34}$	J s	exact
		$6.582\,119\,569\ldots \times 10^{-16}$	eV s	exact
	$\hbar c$	197.326 980 4...	MeV fm	exact
Planck mass $(\hbar c/G)^{1/2}$	m_P	$2.176\,434(24) \times 10^{-8}$	kg	1.1×10^{-5}
energy equivalent	$m_\text{P}c^2$	$1.220\,890(14) \times 10^{19}$	GeV	1.1×10^{-5}
Planck temperature $(\hbar c^5/G)^{1/2}/k$	T_P	$1.416\,784(16) \times 10^{32}$	K	1.1×10^{-5}
Planck length $\hbar/m_\text{P}c = (\hbar G/c^3)^{1/2}$	l_P	$1.616\,255(18) \times 10^{-35}$	m	1.1×10^{-5}
Planck time $l_\text{P}/c = (\hbar G/c^5)^{1/2}$	t_P	$5.391\,247(60) \times 10^{-44}$	s	1.1×10^{-5}
		ELECTROMAGNETIC		
elementary charge	e	$1.602\,176\,634 \times 10^{-19}$	C	exact
	$e/\hbar$	$1.519\,267\,447\ldots \times 10^{15}$	A J^{-1}	exact
magnetic flux quantum $2\pi\hbar/(2e)$	Φ_0	$2.067\,833\,848\ldots \times 10^{-15}$	Wb	exact
conductance quantum $2e^2/2\pi\hbar$	G_0	$7.748\,091\,729\ldots \times 10^{-5}$	S	exact
inverse of conductance quantum	G_0^{-1}	12 906.403 72...	Ω	exact
Josephson constant $2e/h$	K_J	$483\,597.848\,4\ldots \times 10^9$	Hz V^{-1}	exact
von Klitzing constant $\mu_0 c/2\alpha = 2\pi\hbar/e^2$	R_K	25 812.807 45...	Ω	exact
Bohr magneton $e\hbar/2m_\text{e}$	μ_B	$9.274\,010\,0783(28) \times 10^{-24}$	J T^{-1}	3.0×10^{-10}
		$5.788\,381\,8060(17) \times 10^{-5}$	eV T^{-1}	3.0×10^{-10}
	μ_B/h	$1.399\,624\,493\,61(42) \times 10^{10}$	Hz T^{-1}	3.0×10^{-10}
	μ_B/hc	46.686 447 783(14)	[m^{-1} T^{-1}]†	3.0×10^{-10}
	μ_B/k	0.671 713 815 63(20)	K T^{-1}	3.0×10^{-10}
nuclear magneton $e\hbar/2m_\text{p}$	μ_N	$5.050\,783\,7461(15) \times 10^{-27}$	J T^{-1}	3.1×10^{-10}
		$3.152\,451\,258\,44(96) \times 10^{-8}$	eV T^{-1}	3.1×10^{-10}
	μ_N/h	7.622 593 2291(23)	MHz T^{-1}	3.1×10^{-10}
	μ_N/hc	$2.542\,623\,413\,53(78) \times 10^{-2}$	[m^{-1} T^{-1}]†	3.1×10^{-10}
	μ_N/k	$3.658\,267\,7756(11) \times 10^{-4}$	K T^{-1}	3.1×10^{-10}
		ATOMIC AND NUCLEAR		
		General		
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,5693(11) \times 10^{-3}$		1.5×10^{-10}
inverse fine-structure constant	α^{-1}	137.035 999 084(21)		1.5×10^{-10}
Rydberg frequency $\alpha^2 m_\text{e}c^2/2h = E_\text{h}/2h$	cR_∞	$3.289\,841\,960\,2508(64) \times 10^{15}$	Hz	1.9×10^{-12}
energy equivalent	$hc\,R_\infty$	$2.179\,872\,361\,1035(42) \times 10^{-18}$	J	1.9×10^{-12}
		13.605 693 122 994(26)	eV	1.9×10^{-12}
Rydberg constant	R_∞	10 973 731.568 160(21)	[m^{-1}]†	1.9×10^{-12}
Bohr radius $\hbar/\alpha m_\text{e}c = 4\pi\epsilon_0\hbar^2/m_\text{e}e^2$	a_0	$5.291\,772\,109\,03(80) \times 10^{-11}$	m	1.5×10^{-10}
Hartree energy $\alpha^2 m_\text{e}c^2 = e^2/4\pi\epsilon_0 a_0 = 2hcR_\infty$	E_h	$4.359\,744\,722\,2071(85) \times 10^{-18}$	J	1.9×10^{-12}
		27.211 386 245 988(53)	eV	1.9×10^{-12}
quantum of circulation	$\pi\hbar/m_\text{e}$	$3.636\,947\,5516(11) \times 10^{-4}$	m^2 s^{-1}	3.0×10^{-10}

Fundamental Physical Constants — Extensive Listing

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
	$2\pi\hbar/m_e$	$7.273\,895\,1032(22) \times 10^{-4}$	$m^2\ s^{-1}$	3.0×10^{-10}
		Electroweak		
Fermi coupling constant‡	$G_F/(\hbar c)^3$	$1.166\,3787(6) \times 10^{-5}$	GeV^{-2}	5.1×10^{-7}
weak mixing angle§ θ_W (on-shell scheme)				
$\sin^2\theta_W = s_W^2 \equiv 1 - (m_W/m_Z)^2$	$\sin^2\theta_W$	0.222 90(30)		1.3×10^{-3}
		Electron, e^-		
electron mass	m_e	$9.109\,383\,7015(28) \times 10^{-31}$	kg	3.0×10^{-10}
		$5.485\,799\,090\,65(16) \times 10^{-4}$	u	2.9×10^{-11}
energy equivalent	$m_e c^2$	$8.187\,105\,7769(25) \times 10^{-14}$	J	3.0×10^{-10}
		0.510 998 950 00(15)	MeV	3.0×10^{-10}
electron-muon mass ratio	m_e/m_μ	$4.836\,331\,69(11) \times 10^{-3}$		2.2×10^{-8}
electron-tau mass ratio	m_e/m_τ	$2.875\,85(19) \times 10^{-4}$		6.8×10^{-5}
electron-proton mass ratio	m_e/m_p	$5.446\,170\,214\,87(33) \times 10^{-4}$		6.0×10^{-11}
electron-neutron mass ratio	m_e/m_n	$5.438\,673\,4424(26) \times 10^{-4}$		4.8×10^{-10}
electron-deuteron mass ratio	m_e/m_d	$2.724\,437\,107\,462(96) \times 10^{-4}$		3.5×10^{-11}
electron-triton mass ratio	m_e/m_t	$1.819\,200\,062\,251(90) \times 10^{-4}$		5.0×10^{-11}
electron-helion mass ratio	m_e/m_h	$1.819\,543\,074\,573(79) \times 10^{-4}$		4.3×10^{-11}
electron to alpha particle mass ratio	m_e/m_α	$1.370\,933\,554\,787(45) \times 10^{-4}$		3.3×10^{-11}
electron charge to mass quotient	$-e/m_e$	$-1.758\,820\,010\,76(53) \times 10^{11}$	$C\ kg^{-1}$	3.0×10^{-10}
electron molar mass $N_A m_e$	$M(e), M_e$	$5.485\,799\,0888(17) \times 10^{-7}$	$kg\ mol^{-1}$	3.0×10^{-10}
reduced Compton wavelength $\hbar/m_e c = \alpha a_0$	$\lambda\!\!\!^{-}_C$	$3.861\,592\,6796(12) \times 10^{-13}$	m	3.0×10^{-10}
Compton wavelength	λ_C	$2.426\,310\,238\,67(73) \times 10^{-12}$	[m]†	3.0×10^{-10}
classical electron radius $\alpha^2 a_0$	r_e	$2.817\,940\,3262(13) \times 10^{-15}$	m	4.5×10^{-10}
Thomson cross section $(8\pi/3)r_e^2$	σ_e	$6.652\,458\,7321(60) \times 10^{-29}$	m^2	9.1×10^{-10}
electron magnetic moment	μ_e	$-9.284\,764\,7043(28) \times 10^{-24}$	$J\ T^{-1}$	3.0×10^{-10}
to Bohr magneton ratio	μ_e/μ_B	$-1.001\,159\,652\,181\,28(18)$		1.7×10^{-13}
to nuclear magneton ratio	μ_e/μ_N	$-1838.281\,971\,88(11)$		6.0×10^{-11}
electron magnetic moment anomaly $\lvert\mu_e\rvert/\mu_B - 1$	a_e	$1.159\,652\,181\,28(18) \times 10^{-3}$		1.5×10^{-10}
electron g-factor $-2(1 + a_e)$	g_e	$-2.002\,319\,304\,362\,56(35)$		1.7×10^{-13}
electron-muon magnetic moment ratio	μ_e/μ_μ	206.766 9883(46)		2.2×10^{-8}
electron-proton magnetic moment ratio	μ_e/μ_p	$-658.210\,687\,89(20)$		3.0×10^{-10}
electron to shielded proton magnetic moment ratio (H_2O, sphere, 25 °C)	μ_e/μ'_p	$-658.227\,5971(72)$		1.1×10^{-8}
electron-neutron magnetic moment ratio	μ_e/μ_n	960.920 50(23)		2.4×10^{-7}
electron-deuteron magnetic moment ratio	μ_e/μ_d	$-2143.923\,4915(56)$		2.6×10^{-9}
electron to shielded helion magnetic moment ratio (gas, sphere, 25 °C)	μ_e/μ'_h	864.058 257(10)		1.2×10^{-8}
electron gyromagnetic ratio $2\lvert\mu_e\rvert/\hbar$	γ_e	$1.760\,859\,630\,23(53) \times 10^{11}$	$s^{-1}\ T^{-1}$	3.0×10^{-10}
		$28\,024.951\,4242(85)$	$MHz\ T^{-1}$	3.0×10^{-10}
		Muon, μ^-		
muon mass	m_μ	$1.883\,531\,627(42) \times 10^{-28}$	kg	2.2×10^{-8}
		$0.113\,428\,9259(25)$	u	2.2×10^{-8}
energy equivalent	$m_\mu c^2$	$1.692\,833\,804(38) \times 10^{-11}$	J	2.2×10^{-8}
		105.658 3755(23)	MeV	2.2×10^{-8}
muon-electron mass ratio	m_μ/m_e	206.768 2830(46)		2.2×10^{-8}
muon-tau mass ratio	m_μ/m_τ	$5.946\,35(40) \times 10^{-2}$		6.8×10^{-5}
muon-proton mass ratio	m_μ/m_p	0.112 609 5264(25)		2.2×10^{-8}

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Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
muon-neutron mass ratio	m_μ/m_n	0.112 454 5170(25)		2.2×10^{-8}
muon molar mass $N_A m_\mu$	$M(\mu), M_\mu$	$1.134\,289\,259(25) \times 10^{-4}$	kg mol^{-1}	2.2×10^{-8}
reduced muon Compton wavelength $\hbar/m_\mu c$	$\lambda\!\!\!^{-}_{C,\mu}$	$1.867\,594\,306(42) \times 10^{-15}$	m	2.2×10^{-8}
muon Compton wavelength	$\lambda_{C,\mu}$	$1.173\,444\,110(26) \times 10^{-14}$	[m]†	2.2×10^{-8}
muon magnetic moment	μ_μ	$-4.490\,448\,30(10) \times 10^{-26}$	J T^{-1}	2.2×10^{-8}
to Bohr magneton ratio	μ_μ/μ_B	$-4.841\,970\,47(11) \times 10^{-3}$		2.2×10^{-8}
to nuclear magneton ratio	μ_μ/μ_N	$-8.890\,597\,03(20)$		2.2×10^{-8}
muon magnetic moment anomaly $\lvert\mu_\mu\rvert/(e\hbar/2m_\mu) - 1$	a_μ	$1.165\,920\,89(63) \times 10^{-3}$		5.4×10^{-7}
muon g-factor $-2(1 + a_\mu)$	g_μ	$-2.002\,331\,8418(13)$		6.3×10^{-10}
muon-proton magnetic moment ratio	μ_μ/μ_p	$-3.183\,345\,142(71)$		2.2×10^{-8}
	Tau, τ^-			
tau mass[¶]	m_τ	$3.167\,54(21) \times 10^{-27}$	kg	6.8×10^{-5}
		1.907 54(13)	u	6.8×10^{-5}
energy equivalent	$m_\tau c^2$	$2.846\,84(19) \times 10^{-10}$	J	6.8×10^{-5}
		1776.86(12)	MeV	6.8×10^{-5}
tau-electron mass ratio	m_τ/m_e	3477.23(23)		6.8×10^{-5}
tau-muon mass ratio	m_τ/m_μ	16.8170(11)		6.8×10^{-5}
tau-proton mass ratio	m_τ/m_p	1.893 76(13)		6.8×10^{-5}
tau-neutron mass ratio	m_τ/m_n	1.891 15(13)		6.8×10^{-5}
tau molar mass $N_A m_\tau$	$M(\tau), M_\tau$	$1.907\,54(13) \times 10^{-3}$	kg mol^{-1}	6.8×10^{-5}
reduced tau Compton wavelength $\hbar/m_\tau c$	$\lambda\!\!\!^{-}_{C,\tau}$	$1.110\,538(75) \times 10^{-16}$	m	6.8×10^{-5}
tau Compton wavelength	$\lambda_{C,\tau}$	$6.977\,71(47) \times 10^{-16}$	[m]†	6.8×10^{-5}
	Proton, p			
proton mass	m_p	$1.672\,621\,923\,69(51) \times 10^{-27}$	kg	3.1×10^{-10}
		1.007 276 466 621(53)	u	5.3×10^{-11}
energy equivalent	$m_p c^2$	$1.503\,277\,615\,98(46) \times 10^{-10}$	J	3.1×10^{-10}
		938.272 088 16(29)	MeV	3.1×10^{-10}
proton-electron mass ratio	m_p/m_e	1836.152 673 43(11)		6.0×10^{-11}
proton-muon mass ratio	m_p/m_μ	8.880 243 37(20)		2.2×10^{-8}
proton-tau mass ratio	m_p/m_τ	0.528 051(36)		6.8×10^{-5}
proton-neutron mass ratio	m_p/m_n	0.998 623 478 12(49)		4.9×10^{-10}
proton charge to mass quotient	e/m_p	$9.578\,833\,1560(29) \times 10^7$	C kg^{-1}	3.1×10^{-10}
proton molar mass $N_A m_p$	$M(\mathrm{p}), M_p$	$1.007\,276\,466\,27(31) \times 10^{-3}$	kg mol^{-1}	3.1×10^{-10}
reduced proton Compton wavelength $\hbar/m_p c$	$\lambda\!\!\!^{-}_{C,p}$	$2.103\,089\,103\,36(64) \times 10^{-16}$	m	3.1×10^{-10}
proton Compton wavelength	$\lambda_{C,p}$	$1.321\,409\,855\,39(40) \times 10^{-15}$	[m]†	3.1×10^{-10}
proton rms charge radius	r_p	$8.414(19) \times 10^{-16}$	m	2.2×10^{-3}
proton magnetic moment	μ_p	$1.410\,606\,797\,36(60) \times 10^{-26}$	J T^{-1}	4.2×10^{-10}
to Bohr magneton ratio	μ_p/μ_B	$1.521\,032\,202\,30(46) \times 10^{-3}$		3.0×10^{-10}
to nuclear magneton ratio	μ_p/μ_N	2.792 847 344 63(82)		2.9×10^{-10}
proton g-factor $2\mu_p/\mu_N$	g_p	5.585 694 6893(16)		2.9×10^{-10}
proton-neutron magnetic moment ratio	μ_p/μ_n	$-1.459\,898\,05(34)$		2.4×10^{-7}
shielded proton magnetic moment (H_2O, sphere, 25 °C)	μ'_p	$1.410\,570\,560(15) \times 10^{-26}$	J T^{-1}	1.1×10^{-8}
to Bohr magneton ratio	μ'_p/μ_B	$1.520\,993\,128(17) \times 10^{-3}$		1.1×10^{-8}
to nuclear magneton ratio	μ'_p/μ_N	2.792 775 599(30)		1.1×10^{-8}
proton magnetic shielding correction $1 - \mu'_p/\mu_p$ (H_2O, sphere, 25 °C)	σ'_p	$2.5689(11) \times 10^{-5}$		4.2×10^{-4}

Fundamental Physical Constants — Extensive Listing

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
proton gyromagnetic ratio $2\mu_p/\hbar$	γ_p	$2.675\,221\,8744(11) \times 10^8$	$s^{-1}\ T^{-1}$	4.2×10^{-10}
		$42.577\,478\,518(18)$	MHz T^{-1}	4.2×10^{-10}
shielded proton gyromagnetic ratio $2\mu'_p/\hbar$ (H_2O, sphere, 25 °C)	γ'_p	$2.675\,153\,151(29) \times 10^8$	$s^{-1}\ T^{-1}$	1.1×10^{-8}
		$42.576\,384\,74(46)$	MHz T^{-1}	1.1×10^{-8}
Neutron, n				
neutron mass	m_n	$1.674\,927\,498\,04(95) \times 10^{-27}$	kg	5.7×10^{-10}
		$1.008\,664\,915\,95(49)$	u	4.8×10^{-10}
energy equivalent	$m_n c^2$	$1.505\,349\,762\,87(86) \times 10^{-10}$	J	5.7×10^{-10}
		$939.565\,420\,52(54)$	MeV	5.7×10^{-10}
neutron-electron mass ratio	m_n/m_e	$1838.683\,661\,73(89)$		4.8×10^{-10}
neutron-muon mass ratio	m_n/m_μ	$8.892\,484\,06(20)$		2.2×10^{-8}
neutron-tau mass ratio	m_n/m_τ	$0.528\,779(36)$		6.8×10^{-5}
neutron-proton mass ratio	m_n/m_p	$1.001\,378\,419\,31(49)$		4.9×10^{-10}
neutron-proton mass difference	$m_n - m_p$	$2.305\,574\,35(82) \times 10^{-30}$	kg	3.5×10^{-7}
		$1.388\,449\,33(49) \times 10^{-3}$	u	3.5×10^{-7}
energy equivalent	$(m_n - m_p)c^2$	$2.072\,146\,89(74) \times 10^{-13}$	J	3.5×10^{-7}
		$1.293\,332\,36(46)$	MeV	3.5×10^{-7}
neutron molar mass $N_A m_n$	$M(n), M_n$	$1.008\,664\,915\,60(57) \times 10^{-3}$	kg mol^{-1}	5.7×10^{-10}
reduced neutron Compton wavelength $\hbar/m_n c$	$\lambda\!\!\!^{-}_{C,n}$	$2.100\,194\,1552(12) \times 10^{-16}$	m	5.7×10^{-10}
neutron Compton wavelength	$\lambda_{C,n}$	$1.319\,590\,905\,81(75) \times 10^{-15}$	[m]†	5.7×10^{-10}
neutron magnetic moment	μ_n	$-9.662\,3651(23) \times 10^{-27}$	J T^{-1}	2.4×10^{-7}
to Bohr magneton ratio	μ_n/μ_B	$-1.041\,875\,63(25) \times 10^{-3}$		2.4×10^{-7}
to nuclear magneton ratio	μ_n/μ_N	$-1.913\,042\,73(45)$		2.4×10^{-7}
neutron g-factor $2\mu_n/\mu_N$	g_n	$-3.826\,085\,45(90)$		2.4×10^{-7}
neutron-electron magnetic moment ratio	μ_n/μ_e	$1.040\,668\,82(25) \times 10^{-3}$		2.4×10^{-7}
neutron-proton magnetic moment ratio	μ_n/μ_p	$-0.684\,979\,34(16)$		2.4×10^{-7}
neutron to shielded proton magnetic moment ratio (H_2O, sphere, 25 °C)	μ_n/μ'_p	$-0.684\,996\,94(16)$		2.4×10^{-7}
neutron gyromagnetic ratio $2\lvert\mu_n\rvert/\hbar$	γ_n	$1.832\,471\,71(43) \times 10^8$	$s^{-1}\ T^{-1}$	2.4×10^{-7}
		$29.164\,6931(69)$	MHz T^{-1}	2.4×10^{-7}
Deuteron, d				
deuteron mass	m_d	$3.343\,583\,7724(10) \times 10^{-27}$	kg	3.0×10^{-10}
		$2.013\,553\,212\,745(40)$	u	2.0×10^{-11}
energy equivalent	$m_d c^2$	$3.005\,063\,231\,02(91) \times 10^{-10}$	J	3.0×10^{-10}
		$1875.612\,942\,57(57)$	MeV	3.0×10^{-10}
deuteron-electron mass ratio	m_d/m_e	$3670.482\,967\,88(13)$		3.5×10^{-11}
deuteron-proton mass ratio	m_d/m_p	$1.999\,007\,501\,39(11)$		5.6×10^{-11}
deuteron molar mass $N_A m_d$	$M(d), M_d$	$2.013\,553\,212\,05(61) \times 10^{-3}$	kg mol^{-1}	3.0×10^{-10}
deuteron rms charge radius	r_d	$2.127\,99(74) \times 10^{-15}$	m	3.5×10^{-4}
deuteron magnetic moment	μ_d	$4.330\,735\,094(11) \times 10^{-27}$	J T^{-1}	2.6×10^{-9}
to Bohr magneton ratio	μ_d/μ_B	$4.669\,754\,570(12) \times 10^{-4}$		2.6×10^{-9}
to nuclear magneton ratio	μ_d/μ_N	$0.857\,438\,2338(22)$		2.6×10^{-9}
deuteron g-factor μ_d/μ_N	g_d	$0.857\,438\,2338(22)$		2.6×10^{-9}
deuteron-electron magnetic moment ratio	μ_d/μ_e	$-4.664\,345\,551(12) \times 10^{-4}$		2.6×10^{-9}
deuteron-proton magnetic moment ratio	μ_d/μ_p	$0.307\,012\,209\,39(79)$		2.6×10^{-9}
deuteron-neutron magnetic moment ratio	μ_d/μ_n	$-0.448\,206\,53(11)$		2.4×10^{-7}

Fundamental Physical Constants — Extensive Listing

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
		Triton, t		
triton mass	m_t	$5.007\,356\,7446(15) \times 10^{-27}$	kg	3.0×10^{-10}
		$3.015\,500\,716\,21(12)$	u	4.0×10^{-11}
energy equivalent	$m_t c^2$	$4.500\,387\,8060(14) \times 10^{-10}$	J	3.0×10^{-10}
		$2808.921\,132\,98(85)$	MeV	3.0×10^{-10}
triton-electron mass ratio	m_t/m_e	$5496.921\,535\,73(27)$		5.0×10^{-11}
triton-proton mass ratio	m_t/m_p	$2.993\,717\,034\,14(15)$		5.0×10^{-11}
triton molar mass $N_A m_t$	$M(t), M_t$	$3.015\,500\,715\,17(92) \times 10^{-3}$	kg mol^{-1}	3.0×10^{-10}
triton magnetic moment	μ_t	$1.504\,609\,5202(30) \times 10^{-26}$	J T^{-1}	2.0×10^{-9}
to Bohr magneton ratio	μ_t/μ_B	$1.622\,393\,6651(32) \times 10^{-3}$		2.0×10^{-9}
to nuclear magneton ratio	μ_t/μ_N	$2.978\,962\,4656(59)$		2.0×10^{-9}
triton g-factor $2\mu_t/\mu_N$	g_t	$5.957\,924\,931(12)$		2.0×10^{-9}
		Helion, h		
helion mass	m_h	$5.006\,412\,7796(15) \times 10^{-27}$	kg	3.0×10^{-10}
		$3.014\,932\,247\,175(97)$	u	3.2×10^{-11}
energy equivalent	$m_h c^2$	$4.499\,539\,4125(14) \times 10^{-10}$	J	3.0×10^{-10}
		$2808.391\,607\,43(85)$	MeV	3.0×10^{-10}
helion-electron mass ratio	m_h/m_e	$5495.885\,280\,07(24)$		4.3×10^{-11}
helion-proton mass ratio	m_h/m_p	$2.993\,152\,671\,67(13)$		4.4×10^{-11}
helion molar mass $N_A m_h$	$M(h), M_h$	$3.014\,932\,246\,13(91) \times 10^{-3}$	kg mol^{-1}	3.0×10^{-10}
helion magnetic moment	μ_h	$-1.074\,617\,532(13) \times 10^{-26}$	J T^{-1}	1.2×10^{-8}
to Bohr magneton ratio	μ_h/μ_B	$-1.158\,740\,958(14) \times 10^{-3}$		1.2×10^{-8}
to nuclear magneton ratio	μ_h/μ_N	$-2.127\,625\,307(25)$		1.2×10^{-8}
helion g-factor $2\mu_h/\mu_N$	g_h	$-4.255\,250\,615(50)$		1.2×10^{-8}
shielded helion magnetic moment (gas, sphere, 25 °C)	μ'_h	$-1.074\,553\,090(13) \times 10^{-26}$	J T^{-1}	1.2×10^{-8}
to Bohr magneton ratio	μ'_h/μ_B	$-1.158\,671\,471(14) \times 10^{-3}$		1.2×10^{-8}
to nuclear magneton ratio	μ'_h/μ_N	$-2.127\,497\,719(25)$		1.2×10^{-8}
shielded helion to proton magnetic moment ratio (gas, sphere, 25 °C)	μ'_h/μ_p	$-0.761\,766\,5618(89)$		1.2×10^{-8}
shielded helion to shielded proton magnetic moment ratio (gas/H_2O, spheres, 25 °C)	μ'_h/μ'_p	$-0.761\,786\,1313(33)$		4.3×10^{-9}
shielded helion gyromagnetic ratio $2\lvert\mu'_h\rvert/\hbar$ (gas, sphere, 25 °C)	γ'_h	$2.037\,894\,569(24) \times 10^{8}$	s^{-1} T^{-1}	1.2×10^{-8}
		$32.434\,099\,42(38)$	MHz T^{-1}	1.2×10^{-8}
		Alpha particle, α		
alpha particle mass	m_α	$6.644\,657\,3357(20) \times 10^{-27}$	kg	3.0×10^{-10}
		$4.001\,506\,179\,127(63)$	u	1.6×10^{-11}
energy equivalent	$m_\alpha c^2$	$5.971\,920\,1914(18) \times 10^{-10}$	J	3.0×10^{-10}
		$3727.379\,4066(11)$	MeV	3.0×10^{-10}
alpha particle to electron mass ratio	m_α/m_e	$7294.299\,541\,42(24)$		3.3×10^{-11}
alpha particle to proton mass ratio	m_α/m_p	$3.972\,599\,690\,09(22)$		5.5×10^{-11}
alpha particle molar mass $N_A m_\alpha$	$M(\alpha), M_\alpha$	$4.001\,506\,1777(12) \times 10^{-3}$	kg mol^{-1}	3.0×10^{-10}
		PHYSICOCHEMICAL		
Avogadro constant	N_A	$6.022\,140\,76 \times 10^{23}$	mol^{-1}	exact
Boltzmann constant	k	$1.380\,649 \times 10^{-23}$	J K^{-1}	exact
		$8.617\,333\,262\ldots \times 10^{-5}$	eV K^{-1}	exact
	k/h	$2.083\,661\,912\ldots \times 10^{10}$	Hz K^{-1}	exact

Fundamental Physical Constants — Extensive Listing

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
	k/hc	69.503 480 04 ...	$[m^{-1}\ K^{-1}]^{\dagger}$	exact
atomic mass constant$^{\|}$				
$m_u = \frac{1}{12}m(^{12}C) = 2hc\,R_\infty/\alpha^2c^2A_r(e)$	m_u	$1.660\,539\,066\,60(50) \times 10^{-27}$	kg	3.0×10^{-10}
energy equivalent	m_uc^2	$1.492\,418\,085\,60(45) \times 10^{-10}$	J	3.0×10^{-10}
		931.494 102 42(28)	MeV	3.0×10^{-10}
molar mass constant$^{\|}$	M_u	$0.999\,999\,999\,65(30) \times 10^{-3}$	kg mol^{-1}	3.0×10^{-10}
molar mass$^{\|}$ of carbon-12 $A_r(^{12}C)M_u$	$M(^{12}C)$	$11.999\,999\,9958(36) \times 10^{-3}$	kg mol^{-1}	3.0×10^{-10}
molar Planck constant	N_Ah	$3.990\,312\,712 \ldots \times 10^{-10}$	J Hz^{-1} mol^{-1}	exact
molar gas constant N_Ak	R	8.314 462 618 ...	J mol^{-1} K^{-1}	exact
Faraday constant N_Ae	F	96 485.332 12 ...	C mol^{-1}	exact
standard-state pressure		100 000	Pa	exact
standard atmosphere		101 325	Pa	exact
molar volume of ideal gas RT/p				
$T = 273.15$ K, $p = 100$ kPa	V_m	$22.710\,954\,64 \ldots \times 10^{-3}$	m^3 mol^{-1}	exact
or standard-state pressure				
Loschmidt constant N_A/V_m	n_0	$2.651\,645\,804 \ldots \times 10^{25}$	m^{-3}	exact
molar volume of ideal gas RT/p				
$T = 273.15$ K, $p = 101.325$ kPa	V_m	$22.413\,969\,54 \ldots \times 10^{-3}$	m^3 mol^{-1}	exact
or standard atmosphere				
Loschmidt constant N_A/V_m	n_0	$2.686\,780\,111 \ldots \times 10^{25}$	m^{-3}	exact
Sackur-Tetrode (absolute entropy) constant**				
$\frac{5}{2} + \ln[(m_ukT_1/2\pi\hbar^2)^{3/2}kT_1/p_0]$				
$T_1 = 1$ K, $p_0 = 100$ kPa	S_0/R	$-1.151\,707\,537\,06(45)$		3.9×10^{-10}
or standard-state pressure				
$T_1 = 1$ K, $p_0 = 101.325$ kPa		$-1.164\,870\,523\,58(45)$		3.9×10^{-10}
or standard atmosphere				
Stefan-Boltzmann constant				
$(\pi^2/60)k^4/\hbar^3c^2$	σ	$5.670\,374\,419 \ldots \times 10^{-8}$	W m^{-2} K^{-4}	exact
first radiation constant for spectral				
radiance $2hc^2\,\mathrm{sr}^{-1}$	c_{1L}	$1.191\,042\,972 \ldots \times 10^{-16}$	[W m^2 sr^{-1}]††	exact
first radiation constant $2\pi hc^2 = \pi\,\mathrm{sr}\,c_{1L}$	c_1	$3.741\,771\,852 \ldots \times 10^{-16}$	[W m^2]††	exact
second radiation constant hc/k	c_2	$1.438\,776\,877 \ldots \times 10^{-2}$	[m K]†	exact
Wien displacement law constants				
$b = \lambda_{max}T = c_2/4.965\,114\,231...$	b	$2.897\,771\,955 \ldots \times 10^{-3}$	[m K]†	exact
$b' = \nu_{max}/T = 2.821\,439\,372...\,c/c_2$	b'	$5.878\,925\,757 \ldots \times 10^{10}$	Hz K^{-1}	exact

* The energy of a photon with frequency ν expressed in unit Hz is $E = h\nu$ in J. Unitary time evolution of the state of this photon is given by $\exp(-iEt/\hbar)|\varphi\rangle$, where $|\varphi\rangle$ is the photon state at time $t = 0$ and time is expressed in unit s. The ratio $Et/\hbar$ is a phase.

† The full description of m^{-1} is cycles or periods per meter and that of m is meter per cycle (m/cycle). The scientific community is aware of the implied use of these units. It traces back to the conventions for phase and angle and the use of unit Hz versus cycles/s. No solution has been agreed upon.

‡ Value recommended by the Particle Data Group (Tanabashi, *et al.*, 2018).

§ Based on the ratio of the masses of the W and Z bosons m_W/m_Z recommended by the Particle Data Group (Tanabashi, *et al.*, 2018). The value for $\sin^2\theta_W$ they recommend, which is based on a variant of the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme, is $\sin^2\hat{\theta}_W(M_Z) = 0.231\,22(4)$.

¶ This and other constants involving m_τ are based on $m_\tau c^2$ in MeV recommended by the Particle Data Group (Tanabashi, *et al.*, 2018).

$^{\|}$ The relative atomic mass $A_r(X)$ of particle X with mass $m(X)$ is defined by $A_r(X) = m(X)/m_u$, where $m_u = m(^{12}C)/12 = 1$ u is the atomic mass constant and u is the unified atomic mass unit. Moreover, the mass of particle X is $m(X) = A_r(X)$ u and the molar mass of X is $M(X) = A_r(X)M_u$, where $M_u = N_A$ u is the molar mass constant and N_A is the Avogadro constant.

** The entropy of an ideal monoatomic gas of relative atomic mass A_r is given by $S = S_0 + \frac{3}{2}R\ln A_r - R\ln(p/p_0) + \frac{5}{2}R\ln(T/\mathrm{K})$.

†† The full description of m^2 is $\mathrm{m}^{-2} \times (\mathrm{m/cycle})^4$. See also footnote for m^{-1}.

GO TO BEFORE included File

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11.05 Quantity of Matter

L'unità di misura della quantità di materia è la mole (mol) definita come la quantità di materia che contiene tante entità elementari (atomi, molecole, particelle, ioni, etc.) quante sono contenute in 12 g dell'isotopo 12 del carbonio ($^{12}_{6}C$), pari al numero di Avogadro.

È quindi un'unità di misura di numero, per numeri molto grandi.

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11.06 Mathematical Pre-Requisites

```
*******************************************************************************
*    Prerequisiti di matematica per gli insegnamenti di Fisica Generale
*******************************************************************************

Algebra elementare
==============================================================================
* Calcolo letterale, prodotti notevoli e decomposizione in fattori.
* Proprietà delle potenze, dei radicali e dei logaritmi.
* Elevamento a potenza di un binomio.
* Progressioni aritmetiche e geometriche.
* Equazioni e disequazioni di primo grado (con discussione).
* Equazioni e disequazioni di secondo grado (con discussione).
* Equazioni speciali di grado superiore.
* Equazioni e disequazioni irrazionali.
* Sistemi di equazioni.

Nozioni di base di trigonometria piana
==============================================================================
* Definizioni e proprietà fondamentali delle funzioni trigonometriche
(seno, coseno, tangente e rispettive funzioni inverse).
* Formule di addizione, duplicazione/bisezione, prostaferesi, Werner;
funzioni trigonometriche espresse in funzione di tan(b/2);
* Soluzione di triangoli tramite la trigonometria
(teorema dei seni e teorema dei coseni).

Geometria elementare nel piano e nello spazio
==============================================================================
* Perimetri, superfici, volumi e proprietà
delle figure geometriche piane e solide più comuni.
* Proprietà dei triangoli: rettangoli, isosceli ed equilateri.
Soluzione di triangoli qualunque (in forma geometrica).
* Proprietà di quadrilateri, poligoni regolari, cerchi,
settori e segmenti circolari.
* Proprietà di parallelepipedi, piramidi, cilindri, coni, sfere,
settori e segmenti sferici.
* Soluzione di problemi con le figure piane e solide citate.

Geometria analitica
==============================================================================
* Concetto di rappresentazione cartesiana nel piano e nello spazio
  (rette, piani, circonferenze, coniche quadriche);
  rappresentazione parametrica e implicita.

Varie
==============================================================================
* Soluzioni di equazioni e disequazioni contenenti
valori assoluti, potenze, logaritmi e funzioni trigonometriche.
* Soluzione di equazioni per fattorizzazione.
```

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Exercises and Problems

11-001 ☒ EG-2023-01

1. Calcolare:

$$\mathbf{r} \equiv \{x, y, z\} \qquad \mathbf{x} \equiv \{x_1, x_2, x_3\} \tag{11.06.01}$$

$$r = |\mathbf{r}| \qquad x = |\mathbf{x}| \tag{11.06.02}$$

$$\frac{\partial r^{3/2}}{\partial y} \tag{11.06.03}$$

$$\frac{\partial x^{3/2}}{\partial x_i} \tag{11.06.04}$$

$$\frac{\partial}{\partial z} \iiint \frac{\rho[\mathbf{y}]}{|\mathbf{x} - \mathbf{y}|} \, \mathbb{d}V_{\mathbf{y}} \tag{11.06.05}$$

$$\frac{\partial}{\partial z} \iiint \frac{\rho[\mathbf{w}]}{|\mathbf{x} - \mathbf{w}|} \, \mathbb{d}V_{\mathbf{w}} \tag{11.06.06}$$

$$\frac{\partial}{\partial z} \iiint \frac{\rho[\mathbf{x}]}{|\mathbf{w} - \mathbf{x}|} \, \mathbb{d}V_{\mathbf{x}} \tag{11.06.07}$$

$$\mathbf{grad} r^{\alpha} \tag{11.06.08}$$

$$\mathbf{div}_{\mathbf{x}}(f[\mathbf{x}]\mathbf{A}[\mathbf{x}]) \tag{11.06.09}$$

$$\mathbf{div}_{\mathbf{x}}(f[\mathbf{y}]\mathbf{A}[\mathbf{x}]) \tag{11.06.10}$$

$$\nabla^2(1/r) \tag{11.06.11}$$

$$\boldsymbol{\nabla}^2(\mathbf{r}/r^2) \tag{11.06.12}$$

2. Si consideri un disco piano circolare sottile di raggio R uniformemente carico con densità superficiale ρ_S.
   a) Calcolare il potenziale elettrostatico sull'asse.
   b) Determinare l'andamento del potenziale elettrostatico sull'asse a distanza dal disco molto maggiore di R.
   c) Determinare l'andamento del potenziale elettrostatico sull'asse a distanza dal disco molto minore di R.
3. Si consideri un anello piano circolare sottile di raggio R percorso da corrente I.
   a) Calcolare il campo magnetostatico sull'asse.
   b) Determinare l'andamento del campo magnetostatico sull'asse a distanza dal disco molto maggiore di R.
   c) Determinare l'andamento del campo magnetostatico sull'asse a distanza dal disco molto minore di R.
4. Una barra cilindrica rettilinea lunga e sottile ha densità lineare di massa $\rho[z] = \alpha z$ con $\alpha = 10\ \mathrm{kg/m^2}$ e $0 \leq z \leq L$ e $L = 1$ m. La barra è ferma orizzontale, lungo l'asse z.
   a) Calcolare il peso.
   b) Calcolare la posizione del centro di massa.
   c) Calcolare il potenziale gravitazionale prodotto dalla sbarra nel punto $(0, 0, 2)$ m.
   d) Calcolare il potenziale gravitazionale prodotto dalla sbarra nel punto $(1, 1, 0)$ m.

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CHAPTER 12

Common methods in physics

12.01 On the method in Physics 1021
12.02 Tentative Anatomy of the Equations of Physics 1027
12.03 On Physics Problem Solving 1028
12.04 Common methods in Physics 1032
12.05 General Unifying Concepts in Physics 1033

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12.01 On the method in Physics

©|WEB - URL |Scientific method: Defend the integrity of physics||

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12.01.01 Some key points - unsorted

12.01.01.01 Physics Experiment Mathematics

- **In physics, one can only understands what one can measure. Beware: the fact that you can measure something, does not men you understand it**...
- **Physics is an experimental science.** The fundamental principle of science, the definition almost, is this: the sole test of the validity of any idea is experiment. R.P.Feynman.
- Any **scientific theory** must be always compared to experiments. Experimental verification is an attempt of **falsification** of the **scientific theory**. No **scientific theory** can be verified; any **scientific theory** can be falsified by a disagreement with physical reality. Any **"theory"** must be falsifiable to be a **scientific theory**.
- Any **scientific theory** must be able to make predictions (which can be falsified). A **scientific theory** with no predictive power is nothing.
- **It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment, it's wrong. In that simple statement is the key to science**. (R.P.Feynman - Cornell University Lecture, 1964).
- Whenever there is any disagreement between a theory and an experiment, that is the moment when we learn something new. On. Margaret Thatcher: "What do you do? John Ellis: "Think of things for the experiments to look for, and hope they find something different. "On. Margaret Thatcher: "Wouldn't it be better if they found what you predicted?" John Ellis: "Then we would not learn how to go further!". (John Ellis and on. Margaret Thatcher at CERN).
- **Mathematics is the language of the Universe.**
- Mathematics is the fundamental language of physics; it must be perfectly known to be able to do physics, but it is not the physics, is a prerequisite. After understanding the purely mathematical aspects, Physics hasn't even begun: one has to look at an equation and figure out what it is saying; one has to be able to translate equations into concepts and conversely.
- In physics, emphasis on physics, but the mathematical language must be perfectly known, to concentrate on physics.
- **Physics is not mathematics: usually there is no neat theorem.**
- **Lagrangian and Hamiltonian is not a starting point for physics, it is an arrival point.**

12.01.01.02 Some R.P.Feynman quotes

- Some R.P.Feynman quotes.
  - See failure as a beginning.
  - Never stop learning.
  - Assume nothing, question everything.
  - Teach others what you know.
  - Analyze objectively.
  - Practice humility.
  - Respect constructive criticism.
  - Take initiative.
  - Give credit where it's due.
  - Love what you do.
  - Don't just teach your students to read.
    - Teach them to question what they read, what they study.
    - Teach them to doubt.
    - Teach them to think.
    - Teach them to make mistakes and learn from them.
    - Teach them how to understand something.
    - Teach them how to teach others.
  - **You don't really know something until you can rebuild it yourself from basic principles!** It's a good approach - learn everything and keep that child - like curiosity, question everything... and be humble enough to admit things that you don't know (R.P.Feynman).

12.01.01.03 Scientific Authority

- Science recognizes no authority: scientific claims are not valid for having been supported by some authority: influencer, web, social network, scientific, intellectual, religious, political, television, ...
- Science is not democratic: scientific claims are not valid because they are supported|believed by the majority of people. Scientists don't vote for|against a theory|model. Nature decides. The scientific method is the only way to question Nature, understand quantitatively how it works and acquire predictive skills.
- Scientific controversies are not resolved with a cross-examination the only one scientific truth is the one deriving from experimentation and|or is consistent with experimentation.

12.01.01.04 How to

- Let us try to apply the assumption that **no interesting open physics problem hinges on getting the fine points right.** - S.Weinberg.
- **Let us try to avoid un-instructive complications and problems of detail which are predominantly in the realm of fine formal points.** - S.Weinberg.
- **First Law of Progress in Theoretical Physics: You will get nowhere by crunching equations.** - S.Weinberg.
- **Lex parsimoniae or Ockham razor**: generally recommends selecting the competing hypothesis that makes the fewest new assumptions, when the hypotheses are equal in other respects; for instance, they must both sufficiently explain available data in the first place. William of Ockham, an English monk of the 14th century, formulated the lex parsimoniae which is a fundamental tool in research: *Entia non sunt multiplicanda praeter necessitatem*. That is, the simpler and cheaper explanations of Nature are to be preferred to those more imaginative and complicated that introduce more assumptions and postulates. It is a principle at the base of physics and of the physical mindset.
- **Look for explanations that are as physical as possible, not just mathematical. Then, remember the explanation.** - R.P.Feynman
- Every topic can only be studied by successive approximations, normally infinite successive approximations...

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- Analytical approach: to do physics and understand physics there is no other way than solve problems.
- Quantitative **numerical results** always are of paramount importance in physics: exercises and problems.
- You understand (better...) a physical law when you learn what happens when it does not apply, when you go beyond it: from the ideal gas to a real gas....
- Build a model of your reality; test, predicts and try to falsify your model; keep it good until you find a better one.
- Any model is an imperfect approximation of reality. Approximation means: decide what can be ignored and what cannot.

12.01.01.05 Some key points

- Question everything, starting from what any teacher is telling you! Don't be afraid to ask stupid questions. Research is ultimately based on asking questions.
- The problem with the world is that the intelligent people are full of doubts, while the stupid ones are full of confidence. - Charles Bukowski
- "There are known knowns" is a phrase from a response United States Secretary of Defense Donald Rumsfeld gave to a question at a U.S. Department of Defense (DoD) news briefing on February 12, 2002 about the lack of evidence linking the government of Iraq with the supply of weapons of mass destruction to terrorist groups. D.Rumsfeld stated: **Reports that say that something hasn't happened are always interesting to me, because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns the ones we don't know we don't know. And if one looks throughout the history of our country and other free countries, it is the latter category that tend to be the difficult ones.**

12.01.01.06 Some schemas

- See figure 12.12 about Bloom Taxonomy.
- Workflow of physics research: see figure 12.11.

12.01.02 Falsifiability

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12.01.03 Workflow of Physics Research

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Simple schematic workflow of physics research: see figure 12.11. Others do exist, see references.

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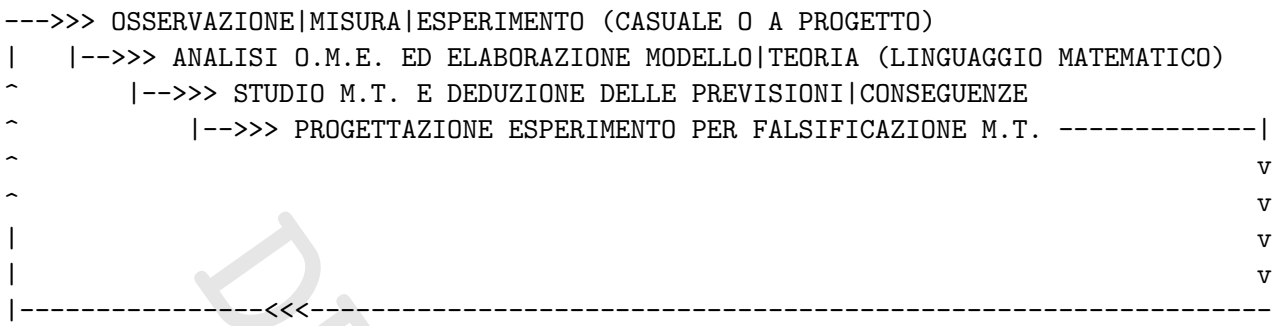

Figure 12.11: WorkFlow of physics research

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12.01.04
Physics is not meta-Physics

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I think that it is a relatively good approximation to truth - which is much too complicated to allow anything but approximations-that mathematical ideas originate in empirics, although the genealogy is sometimes long and obscure. But, once they are so conceived, the subject begins to live a peculiar life of its own and is better compared to a creative one, governed by almost entirely aesthetic motivations, than to anything else and, in particular, to an empirical science. There is, however, a further point which, I believe, needs stressing. As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from "reality" it is beset with very grave dangers. It becomes more and more purely anesthetizing, more and more purely I'art pour I'art. This need not be bad, if the field is surrounded by correlated subjects, which still have closer empirical connections, or if the discipline is under the influence of men with an exceptionally well-developed taste. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities. In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration. At the inception the style is usually classical; when it shows signs of becoming baroque, then the danger signal is up. It would be easy to give examples, to trace specific evolutions into the baroque and the very high baroque, but this, again, would be too technical.
In any event, whenever this stage is reached, the only remedy seems to me to be the rejuvenating return to the source: the re-injection of more or less directly empirical ideas. I am convinced that this was a necessary condition to conserve the freshness and the vitality of the subject and that this will remain equally true in the future.

12.01.05
Taxonomy of Knowledge

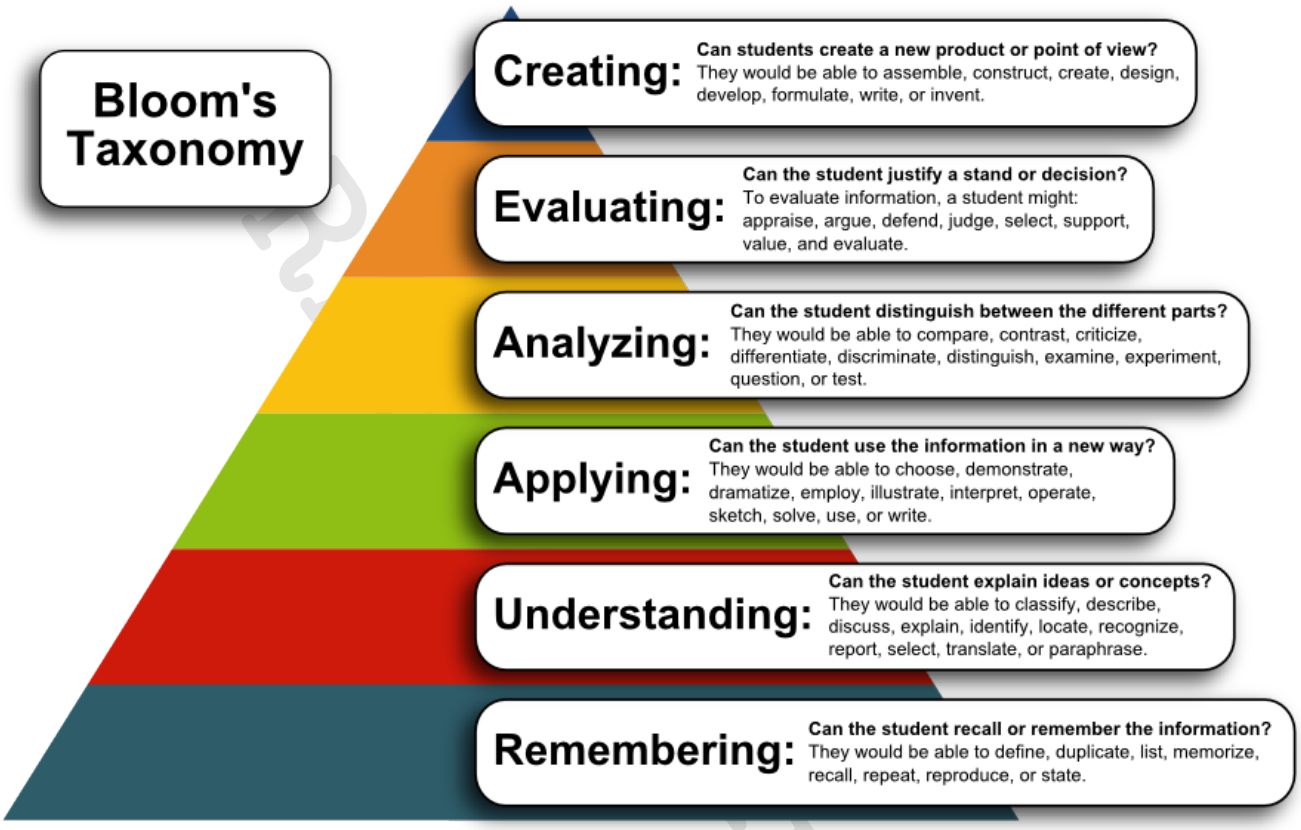

Figure 12.12: Bloom Taxonomy

12.01.06
Publication Bias

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There are two very different aspects of **publication bias**:

- Unpopular or unexpected results may not be published because either the original experimenters or some journal referees have misgivings or reservations about the results, based on the results themselves and not any independent evaluation of experimental procedures or technique.
- Expected experimental results may not be published because either the original experimenters or some journal referees do not consider them interesting enough to merit publication.

In both cases the experimental record in the literature does not fully and accurately reflect the actual experiments that have been performed. Both of these effects clearly affect the literature on experimental tests of SR. This second aspect is one reason why this list of experiments is incomplete; there have probably been many hundreds of unpublished experiments that agree with SR.
Note that this does not include papers that are rejected for other reasons, such as: inappropriate subject or style, major internal inconsistencies, or downright incompetence on the part of authors or experimenters. Such rejections are not bias, they are the proper functioning of a peer-reviewed

journal.

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12.02 Tentative Anatomy of the Equations of Physics

- Physics has not a strictly logical (mathematics-like) architecture: the **raison d'être** of physics always lies in its confrontation with Nature.
- **Operational definition**. Every physical quantity must be defined (or at least definable in principle) in a operative way: that is, a well defined measurement method, doable at least in principle, has to be agreed.
- Every physical relation must either contain only physical quantities which have been previously defined (a physical law), or it is a definition of one physical quantity in terms of already defined physical quantities (a definition).
- **Definitions**: one quantity is defined in terms of others, all of which had been defined previously; definitions are operative, that is it is necessary to clearly indicate all what to do. A definition must provide the means to get a physical quantity, either by direct measurement or indirect one.
- **Physical laws**: relations among all previously defined quantity.
- It **must be possible to falsify any physical law**, if it has to be a physical law. Any physical law allows either prediction and|or **falsifiability**.
- Physical laws can be divided into:
  - principles (postulates): few, general and important laws which summarize the experimental knowledge;
  - derived laws (theorems), or simply laws: laws which are demonstrated starting from the principles.

  The above names are far from universal use....
- **Constitutive relations | Equations of State**: relations quantifying and characterizing physical property of specific systems, thus limited in validity to the specific system.

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12.03 On Physics Problem Solving

12.03.01 Some Considerations on Solving Physics Problems

The purpose of studying Physics is to learn to deal with and solve problems.

Solving physics problems can only give maximum benefit if the student solves problems on his own. In the initial phase of the study discussion, comparison and group study can be very useful. However, the final phase of preparation must be carried out completely autonomous by each student. If only because you take the exam alone.

It is essential to address the solutions to Physics exercises and problems only after having adequately studied the theory. This involves taking classes taking notes, studying lecture notes and, necessarily, study the most important and complex parts in textbooks and, above all, the parts that were not fully understood. The arguments come presented by different sources always with a different slant, and each must find the presentation mode that best suits your preferences specifications and your own preparation.

It is essential to tackle exercises and problems gradually, starting out always from the simplest exercises, direct applications of the concepts studied, and then moving on to the problems, where more situations arise complex ones that usually require intermediate steps to reach the final solution. Knowing how to solve the exercises is not enough, it is necessary try to learn to solve problems, where the answer is not immediate application of a concept but requires intermediate steps that are not pre-defined and are not obvious a priori.

It may also be useful, in the final stage of preparation, to try to tackle more difficult problems than required in the exam. Thereby it will be easier to deal with exam problems.

Attempting to tackle complex problems without having adequately studied the theory and without having practiced with the simplest exercises and problems it's very counterproductive. In fact, there is a risk of creating preconceptions and deviations, or to base one's preparation on serious errors. All this can then be difficult to detect and correct later.

It is worth underlining that a key element in solving problems, of any kind, lies in addressing and carrying out the problem with diligence and in order. The **genius and recklessness** belongs much more to legend than reality.

In the initial study phase it is advisable to tackle exercises and problems whose solution or answer is known. This allows you to verify correctness of your solution and gain confidence in your ability to solve the problems correctly, or realize that the arguments do not have been adequately assimilated. The final stage of preparation goes however conducted by addressing problems for which the answer is not known, so as to be able to develop one's critical sense of analysis of problems and solution produced. In fact, it is necessary to prepare for the exam and the real problems, the solution of which is never known, just as we do not know whether the data is available are sufficient or perhaps redundant.

It is often difficult, especially in the initial study phase, to resolve the problems without help. However, even the unsuccessful attempts at solving the problems can be very useful, if faced with sufficient perseverance, as they develop the ability for reflection, elaboration and effort will.

We must not be discouraged by the fact that many problems cannot be solved resolve immediately. It is a recognized fact that the solution of problems, which is a type of creative work, usually proceeds according to the following scheme. The first phase is the preparatory one in which we search assiduously the solution to the problem. If the solution to the problem does not come directly found the problem is temporarily put aside and begins the second phase, that of incubation, in which you don't think about the problem and you do something else. In this phase, however, mental work proceeds unconsciously and often leads, in the long run, to the third phase, enlightenment, which allows for find the solution to the problem. It is clear however that the work is unconscious of the second phase cannot exist without intense work in the first phase preparatory. Paradoxically, therefore, when you don't know where to turn your head It's the moment when you risk learning something new. From the scheme described, it

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is clear that it is not appropriate to postpone until the last moment solving problems is useful because in this way the possibility is lost to exploit the incubation phase and therefore risk not being able to solve the most involved and interesting problems.

One warning concerns the classic end-of-chapter problems in books by text. They are obviously very useful but you cannot limit yourself to these for one complete preparation. The typical end-of-chapter exercise or problem of a textbook can be easy to solve because you know where it is located and what concepts are behind it? Every problem, if taken out of its context, it may appear or become much more difficult. In fact the problems end chapter automatically direct you towards a series of concepts, those exposed in the chapter, to solve them, help that does not exist in real problems. The simpler than end-of-chapter problems, if taken out of context, it can be much more difficult. The student, in fact, often has difficulty solve real problems because real problems are often missing from the chapters of textbooks.

During the study it may be useful to create a form that collects in a concise and summary way the fundamental facts. In fact any personal re-elaboration is useful as it allows you to analyse the more critical the problems.

You should always carefully examine and reflect on the statement of the problem. A surprisingly large number of errors arise from one reading superficial and misunderstanding of the text.

With rare exceptions, every problem must be resolved first in its entirety symbolic and numeric values inserted only in the final expression. This allows you to carry out three checks that should never be omitted once obtained a solution.

The first check consists in checking the dimensional correctness of the solution.

The second verification consists in the study of the limiting cases of the solution in which the performance of the solution is examined in the extreme cases in which the solution of problem can be obtained immediately and whose plausibility can be verified immediately.

The third check consists in evaluating the plausibility of the solution obtained by verifying that the dependencies of the result are those expected wait and that the numerical values obtained are plausible.

These three checks, while certainly not guaranteeing the correctness of the solution, they can report trivial errors. It is necessary to pay attention to the fact that in the second and third checks sometimes dependencies and unexpected trends are encountered which however are correct. In this case, the issue needs to be explored further understand what the correct result is.

In exceptional cases, when the complexity of the symbolic solutions becomes prohibitive, it may be appropriate to replace the numerical values of the problem specific at an intermediate stage of the solution and take the solution forward number of the specific problem. This can happen, for example, in presence of systems with many equations with complicated coefficients or in the presence of equations, numerical or differential, not explicitly solvable in form symbolic.

It is often possible to solve a problem or part of it in two or more completely independent ways. This, if possible, allows you to verify that alternative routes provide compatible results and provides confidence in the correctness of the result obtained. Typical examples are the solution of a problem in two different Reference Systems, the solution of a problem in two different Coordinate Systems in the same Reference System o the application of the second Cardinal Equation of mechanics with two poles different.

If the problem contains numerical data it must be solved to the end, up to numerical result. To get correct numerical results you need a good one knowledge of the dimensions and units of measurement of physical quantities is one good calculation ability. Both things can only be achieved with practice. It can be useful to convert for determining numerical results from the beginning all quantities in a single system of units of measurement (usually the International System) and express numerical values in scientific notation with a numerical coefficient of the order of unity.

It is always essential to observe the rules of approximate calculation and use the correct number of significant figures as all quantities physics are always affected by an error. As far as General Physics is concerned Precise error propagation analysis is usually required however, at least compliance with the elementary rules of the propagation of significant figures.

Always check the reasonableness of the order of magnitude of the result numeric obtained. An error in the order of magnitude is in Physics minus acceptable than an error on the significant figures with an order of magnitude Right.

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When tackling a Physics problem, it is always necessary to keep in mind that the physics builds a model of reality by introducing simplifications and approximations. On the one hand, it is therefore necessary to address real problems by avoiding them useless and inessential complications to the problem that needs to be solved. On the other hand, however, it is always necessary to verify that the simplifications and approximations made are justified.

Follow the procedures and always avoid improvising or jumping intuition: it's the best way to make mistakes.

12.03.02
General points

1. Read the text carefully several times and imagine the physical phenomenon question. Try to extract all the information from the text; often information is not expressed directly or explicitly. In one exercise all information is explicit in the text; they are necessary in a problem information or intermediate steps not explicitly indicated in the text. Note that common physical data (for example mass and radius of the Earth, mass and charge electric current of an electron or common physical constants) are often not explicitly indicated in the text.
2. Convert all numerical data to SI units immediately at the beginning (read § B), to avoid mixes of incompatible units and immediately see the relationships between them homogeneous sizes. Always convert degrees to radians to avoid errors.
3. Try to understand the physics of the problem before diving into the solution in mathematical details. Approximate answers can often be obtained by using hypotheses simplifiers; you can refine the solution later, for subsequent ones approximations.
4. Make the appropriate drawings or diagrams, which are always useful (indispensable).
5. Decide on the Reference System (if non-inertial, take note of the fictitious forces).
6. Choose the Coordinate System within the chosen reference system, the one that seems most comfortable.
7. Decide and define with extreme precision the system (or systems) to which to apply the physical laws.
8. Apply physical laws to the system, trying to translate them all into equations the information that can be deduced from the text.
9. The conserved physical quantities of the problem are always of great help in simplifying the solution of the problem. If they are present it is usually best to use them as a priority.
10. Count the unknowns and variables in written equations: if there are fewer equations and variables look for other equations, both by rereading the text for explicit information, both reflecting on the physical phenomenon for non-information explicitly proposed in the text, and by asking which quantities are conserved.
11. Don't forget that every physics problem is a model, always simplified, of a physical phenomenon. Don't pretend to take everything into account, but adapt to the text and to the hypotheses of the problem and to common sense...
12. Don't make the mistake of trying to answer questions that haven't been asked yet limit yourself to trying to obtain what is asked and why Information to obtain more information may be missing in the text.
13. Always solve equations in symbolic form to the end, replace always the numerical values to obtain the numerical result, but only in the final formula.
14. Check the correct dimensions of the written formulas.
15. Always check that the results, both the symbolic formula and the numerical results, appear to make sense.
16. If the equations give more solutions than expected, be careful to choose the right one for the problem, given the problem text. Try to interpret the other solutions, for greater safety (sometimes the interpretation of the other solutions is not at all simple...).
17. Carefully analyze all the possible limiting cases of the formulas obtained: cases with special values of the various parameters for which the solution is found immediately without solving the equations;

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verify that the solution is sensible and in accordance with the formula. Carefully analyze that parameters influence the response in the expected way.

18. Where possible and there is time, solve the same problem with different choices, such as: different reference system, different coordinate system, different choice of system, ...
19. Write everything in an orderly manner and always proceed with diligence; do all the steps in detail to avoid trivial errors.

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12.04 Common methods in Physics

12.04.01 Using approximations

12.04.01.01 Using approximate expression of a function to first order

Using approximate expression of a function to first order; sometimes also second or higher orders. For instance: Hooke law is an approximation to first order for the generic/general relation between force and elongation of an elastic medium; read § 32.04 and section § 27.01.01.

12.04.01.02 Using infinitesimal things

12.04.01.03 Large distance

12.04.01.04 Perturbative solutions

12.04.02 Effective theories

12.04.02.01 Averaging discrete things to model as a continuum

12.04.03 Check the result

- dimensions
- limiting cases
- order of magnitude makes sens.

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12.05 General Unifying Concepts in Physics

12.05.01 Covariance of physical laws

It is postulated that physical laws must be independent from the Reference Frame, in the class of Inertial Reference Frame , and/or independent from the Coordinate System. The latter comes from the arbitrariness of the choice of the Coordinate System.

For this to be true, physical laws must be expressed via tensor calculus.

Read § 20.03.06.

12.05.02 Continuity Equations

Read § 21.03.

12.05.02.01 Transport Equations

Read § 65.09.01.01.

12.05.03 Linear Systems

Read § 19.

Linear Systems are relatively easy to solve, with respect to non-linear systems. This motivates the often used linear approximation.

12.05.03.01 Sinusoidal and cosinusoidal solutions to linear equations

Use of sinusoidal and cosinusoidal solutions to linear equations: motivated by the linearity of equations; read § 19.06.

12.05.03.02 The superposition principle

The superposition principle; read § 19.01 and section § 24.03.03.01.

12.05.04 Dimensional Analysis

12.05.05 Symmetries

12.05.05.01 Symmetries and invariance

12.05.05.02 Homogeneity and Isotropy

Connected to invariance under translations and rotations in space.

Read § 21.02

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12.05.05.03 Group Theory And Applications

12.05.06
Variational Principles

Read § 21.01

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Part B

Basics of Mathematics for Physics

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13 General References 1038

14 Basic Mathematical Relations 1042

15 Elements of Vector Algebra 1052

16 Elements of Vector Calculus 1084

17 Complements of Geometry Vector Algebra Vector Calculus 1141

18 Some Elements of Fourier Analysis 1155

19 Linear Systems 1169

20 Coordinate Transformations Tensors and Physical Laws 1180

21 Some Miscellaneous Topics 1210

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CHAPTER 13

General References

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Pagina Web del Corso

Informazioni sul corso, testi di studio, problemi ed esercizi, fonti di dati, materiale vario di studio e multimediale:
WEB - URL.
Many classical texts are excellent texts, sometimes considered to be a *Bible* in their field, as they often are. However, physics progresses very very fast, so that excellent texts written many years ago remain a milestone, but may become old.

Useful General Mathematical Resources

©||S.Wolfram Mathematics Encyclopedia|*WEB - URL*|

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©||W.M.Haynes *et al.*, *CRC Handbook of Chemistry and Physics*, 2016, CRC press, 97thEd., WEB - URL.|*Source of Data*|

©||R.Blandford & K.Thorne, *Applications Of Classical Physics*, ..., ..., ...Ed., WEB - URL, WEB - URL.|*Excellent, ...*

Some Useful Physics Specific Resources/References

©||G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|*Excellent text, ...*

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©||A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|*Excellent, intermediate, modern.*|

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Some Useful Resources/References on Physics and the Real World

©||L.Bloomfield, *How Things Work*, ..., ..., ...Ed., WEB - URL.||

Other references

©|K.McDonald|WEB - URL|*A real treasure of tens of detailed and critical analyses of key physics problems, showing how ...*

The SI of Units of Measure

Read § B.

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References for this section

Some References for this section follow.

- ©|WEB - URL |Mathematical Tools for Physics, James Nearing|*Excellent!*|
- ©|S.Mauch|WEB - URL |*Excellent set of lecture notes on applied mathematics*|
- G.B.Arfken & H.J.Weber, *Mathematical Methods For Physicists*, 2005, Elsevier, 6thEd., ...
- H.Jeffreys and B.S.Jeffreys, *Mathematics for Physics*, ..., ..., ...Ed., ...

Exercises and problems for this part

- [1]
- [2]
- [3]
- [4]
- [5]
- [6]

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CHAPTER 14

Basic Mathematical Relations

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14.03 Circular and Hyperbolic Functions 1046
14.04 Miscellanea 1050

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©|S.Mauch|WEB - URL |*Excellent set of lecture notes on applied mathematics*|

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14.01 Euler Formula for the Complex Exponential

$$\boxed{\exp[\mathtt{i}\alpha] = \cos[\alpha] + \mathtt{i}\sin[\alpha]} \qquad \alpha \in \mathbb{C} \quad .$$

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14.02 Algebraic Equations

14.02.01 Algebraic Equations of Second Degree

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad .$$

Solutions satisfy Vieta [1] relations:

$$x_1 + x_2 = -\frac{b}{a} \qquad x_1 x_2 = \frac{c}{a}$$

14.02.02 Algebraic Equations of Third Degree

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Third degree (cubic) equations sometimes appear in physics problems, for instance when dealing with second-order tensors in three dimensions.

The solutions of the general algebraic equations of third degree:

$$x^3 + bx^2 + cx + d = 0 \quad ,$$

are given by:

$$x_0 = -\frac{b}{3} + \frac{\left(-2b^3 + 9cb - 27d + 3\sqrt{3}\sqrt{4db^3 - c^2b^2 - 18cdb + 4c^3 + 27d^2}\right)^{1/3}}{3\,(2)^{1/3}} + \\ - \frac{(2)^{1/3}\left(3c - b^2\right)}{3\left(-2b^3 + 9cb - 27d + 3\sqrt{3}\sqrt{4db^3 - c^2b^2 - 18cdb + 4c^3 + 27d^2}\right)^{1/3}} \quad ,$$

$$x_+ = -\frac{b}{3} - \frac{\left(1 - i\sqrt{3}\right)\left(-2b^3 + 9cb - 27d + 3\sqrt{3}\sqrt{4db^3 - c^2b^2 - 18cdb + 4c^3 + 27d^2}\right)^{1/3}}{6\,(2)^{1/3}} + \\ + \frac{\left(1 + i\sqrt{3}\right)\left(3c - b^2\right)}{3 \cdot 2^{2/3}\left(-2b^3 + 9cb - 27d + 3\sqrt{3}\sqrt{4db^3 - c^2b^2 - 18cdb + 4c^3 + 27d^2}\right)^{1/3}} \quad ,$$

$$x_- = -\frac{b}{3} - \frac{\left(1 + i\sqrt{3}\right)\left(-2b^3 + 9cb - 27d + 3\sqrt{3}\sqrt{4db^3 - c^2b^2 - 18cdb + 4c^3 + 27d^2}\right)^{1/3}}{6\,(2)^{1/3}} + \\ + \frac{\left(1 - i\sqrt{3}\right)\left(3c - b^2\right)}{3 \cdot 2^{2/3}\left(-2b^3 + 9cb - 27d + 3\sqrt{3}\sqrt{4db^3 - c^2b^2 - 18cdb + 4c^3 + 27d^2}\right)^{1/3}} \quad .$$

Solutions satisfy Vieta [2] relations:

$$x_0 + x_+ + x_- = -b \qquad x_0 x_+ + x_0 x_- + x_+ x_- = c \qquad x_0 x_+ x_- = -d \quad .$$

[1] **Vieta Formulas**: From MathWorld–A Wolfram Web Resource: WEB - URL
[2] **Vieta Formulas**: From MathWorld–A Wolfram Web Resource: WEB - URL

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14.02.03
Algebraic Equations of Fourth Degree

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Solutions satisfy Vieta [1] relations.

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[1] **Vieta Formulas**: From MathWorld–A Wolfram Web Resource: WEB - URL

14.03 Circular and Hyperbolic Functions

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See WEB - URL for the comparison of two set of functions.

Circular and Hyperbolic functions show-up in parallel for instance, in the solutions for the harmonic oscillations (second order constant coefficient linear ODE), see section § 25.04.03.

$$\sin[z] \equiv \frac{\exp[+\mathtt{i}z] - \exp[-\mathtt{i}z]}{2} \qquad \cos[z] \equiv \frac{\exp[+\mathtt{i}z] + \exp[-\mathtt{i}z]}{2} \qquad \tan[z] \equiv \frac{\sin[z]}{\cos[z]} \quad , \tag{14.03.01}$$

$$\operatorname{sih}[z] \equiv \frac{\exp[+z] - \exp[-z]}{2} \qquad \operatorname{coh}[z] \equiv \frac{\exp[+z] + \exp[-z]}{2} \qquad \operatorname{tah}[z] \equiv \frac{\operatorname{sih}[z]}{\operatorname{coh}[z]} \quad , \tag{14.03.02}$$

$$\operatorname{sih}[\mathtt{i}x] = \mathtt{i}\sin[x] \qquad \operatorname{coh}[\mathtt{i}x] = \cos[x] \qquad \text{for } x \text{ any real number} \quad . \tag{14.03.03}$$

→ 1410

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14.03.01
Circular Trigonometry

This § is referenced at pages:
[Never referenced.]

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14.03.01.01 Pythagoras Circular Theorem

$$+\cos^2[\theta]+\sin^2[\theta]=1 \qquad +1+\tan^2\theta=\sec^2\theta \qquad +1+\cot^2\theta=\csc^2\theta \quad .$$

14.03.01.02 Addition Formulas

$$\sin[\alpha\pm\beta]=\sin[\alpha]\cos[\beta]\pm\cos[\alpha]\sin[\beta] \quad , \tag{14.03.04}$$ → 1047

$$\cos[\alpha\pm\beta]=\cos[\alpha]\cos[\beta]\mp\sin[\alpha]\sin[\beta] \quad . \tag{14.03.05}$$ → 1047

From (14.03.04), (14.03.05) one has:

$$\tan[\alpha\pm\beta]=\frac{\tan[\alpha]\pm\tan[\beta]}{1\mp\tan[\alpha]\tan[\beta]} \quad . \tag{14.03.06}$$

All other common trigonometric formulas can be deduced from the addition formulas.

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14.03.02
Hyperbolic Trigonometry

This § is referenced at pages:
[2409]

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The hyperbolic functions arise in many problems of mathematics and mathematical physics in which integrals involving $\sqrt{1+x^2}$ arise (whereas the circular functions involve $\sqrt{1-x^2}$). For instance, the hyperbolic sine arises in the gravitational potential of a cylinder and the calculation of the Roche limit. The hyperbolic cosine function is the shape of a hanging cable (the so-called catenary). The hyperbolic functions arise in the calculation of rapidity of special relativity (equation (58.09.01.07)). All three appear in the Schwarzschild metric using external isotropic Kruskal coordinates in general relativity. The hyperbolic secant arises in the profile of a laminar jet. The hyperbolic cotangent arises in the Langevin function for magnetic Polarization.

14.03.02.01 Pythagoras Hyperbolic Theorem

$$+\operatorname{coh}^2[\theta]-\operatorname{sih}^2[\theta]=1 \qquad +1-\operatorname{tah}^2\theta=\operatorname{sch}^2\theta \qquad -1+\operatorname{cth}^2\theta=\operatorname{csh}^2\theta \quad .$$

14.03.02.02 Addition Formulas

$$\operatorname{sih}[\alpha\pm\beta]=\operatorname{sih}[\alpha]\operatorname{coh}[\beta]\pm\operatorname{coh}[\alpha]\operatorname{sih}[\beta] \quad , \tag{14.03.07}$$ → 1048

$$\operatorname{coh}[\alpha\pm\beta]=\operatorname{coh}[\alpha]\operatorname{coh}[\beta]\pm\operatorname{sih}[\alpha]\operatorname{sih}[\beta] \quad . \tag{14.03.08}$$ → 1048

From (14.03.07), (14.03.08) one has:

$$\tan[(\alpha\pm\beta)]=\frac{\tan[\alpha]\pm\tan[\beta]}{1\pm\tan[\alpha]\tan[\beta]} \quad . \tag{14.03.09}$$

All other common trigonometric formulas can be deduced from the addition formulas.

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14.03.03
Spherical Trigonometry

Read § 15-027, § 15-028.

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14.04 Miscellanea

14.04.01 Arithmetic/Geometric Progressions

$$\mathcal{S} \equiv \sum_{k=0}^{n} kx = \frac{kn}{2}(n+1) \quad . \tag{14.04.01}$$

$$\mathcal{S} \equiv \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \quad . \tag{14.04.02}$$

→ 2272 2277

14.04.02 Logarithm

$$a = b^{\log_b a} \qquad \Leftrightarrow \qquad \log_c a = \log_b a \log_c b \quad .$$

14.04.03 Homogeneous Functions and Euler Theorem

This § is referenced at pages:
[1616]

Any function, $U[\mathbf{x}]$, such that:

$$U[\alpha \mathbf{x}] = \alpha^n U[\mathbf{x}] \qquad \forall \alpha > 0 \quad , \tag{14.04.03}$$

is known as **homogeneous function of degree** n. Euler theorem for an homogeneous function of degree n is obtained by derivation with respect to α:

$$y_i \frac{\partial U[\mathbf{y}]}{\partial y_i} = \mathbf{y} \cdot \mathbf{grad}_{\mathbf{y}} U[\mathbf{y}] = nU[\mathbf{y}] \quad . \tag{14.04.04}$$

DRAFT - IN PROGRESS

CHAPTER 15

Elements of Vector Algebra

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ANALISI MATEMATICA

WORK-IN-PROGRESS

15.01 Introduction

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Vector algebra is part of the basic language of physics.

It is therefore necessary to master the basics of vector algebra in order to understand and use physical laws in the best way.

Necessarily and on purpose, I shall often rely on the intuition of the reader to formalize the ideas, leaving a rigorous development to a more comprehensive mathematical treatment, which is out-of-context for physics.

15.01.01 Notations Definitions and Conventions

This § is referenced at pages:
[1091]

The **convention of sum of repeated indexes** is used: whenever any index appears exactly twice in any monomial term, it is implicitly understood that the term itself is summed over all the possible values of that index, typically:

- either $1, \dots 3$ (for roman type indexes, typically in euclidean three-dimensional space)
- or $0, \dots 3$ (for Greek type indexes, typically in Minkowski pseudo-euclidean space).

The following conventions are therefore adopted:

- whenever exactly two Latin indexes appear, their summation from 1 to 3 will be implied;
- whenever exactly two Greek indexes appear, their summation from 0 to 3 will be implied;
- whenever two or more indexes appear, which must not be summed, they will be normally written inside square brackets, for the sake of clarity, whenever any ambiguity might arise.

As an example:

$$A^{[i]}\mathcal{B}^{[i]}_k g_{\alpha[i]} f_k H^{\alpha\beta}_{\cdot\cdot} \equiv \sum_{k=1}^{3}\sum_{\alpha=0}^{3} A^{[i]}\mathcal{B}^{[i]}_k g_{\alpha[i]} f_k H^{\alpha\beta}_{\cdot\cdot} \qquad \text{the index } [i] \text{ is not summed} \quad .$$

Note that the **name** given to any summed index does not matter at all: it is said to be a dumb index. That is to say, in the above formula the k might be named with any other roman letter and α might be named with any other Greek letter as the sum is implied. In fact, the dummy integration index can take any name as it does not appear in the result. This is exactly the same as for integrals: the dummy integration variable can take any name as it does not appear in the result.

Note that the **vertical position of the indexes (high/low)** might either have or have not a relevance, depending on the context: this must be seen case-by-case. In euclidean 3-space (or n-space) it is not relevant. In Minkowski space it is relevant.

Note: the limitation to **two and only two indexes** is because the summation must be between one contravariant and one covariant indexes, a topic which can only be understood within the general framework described in section § 20.

DRAFT - IN PROGRESS

15.01.02
Scalars Vectors and tensors

This § is referenced at pages:
[1205]

15.01.02.01 Scalars and Vectors: Physics Versus Mathematics

The discussion is limited to Cartesian Coordinate Coordinate System: for a full discussion read § 20.

One should note that the words **scalar** and **vector** may have a different meaning in Calculus (Mathematics) and Physics.

In Calculus a scalar function is just a real-valued function giving real values and a vector function is a real-valued function giving an n-tuple (an ordered list) or real values.

In physics, one deals with physical quantities and the term scalar/vector has a well defined meaning with respect to changes of systems of coordinates (for a full discussion read § 20). Therefore, in physics, not all real numbers are scalars and not all n-tuples of real numbers (lists of real numbers) are vectors.

15.01.02.02 Scalars and Vectors: the case of three-dimensional euclidean space

Let us consider a orthonormal Cartesian Coordinates Coordinate System.

- a scalar is a physical quantity which does not change its value under a rotation of the orthonormal Cartesian Coordinates Coordinate System (examples: the distance between two points);
- it takes more than an oriented direction ad a magnitude to make a vector; a vector is a physical quantity which:
  - is a member of a vector space;
  - its components transform, under rotation of the orthonormal Cartesian Coordinates Coordinate System, as the position vector $\mathbf{r} \equiv \{x, y, z\}$.

Counter-examples:

- a number which is not a scalar: any single component of a vector;
- a triplet of numbers which is not a vector:
  - a triplet of three distances (the three components do not change as the components of the position vector under rotation, as distances do not change under rotation);
  - the oriented arrows defining finite rotations (magnitude: angle of rotation, direction: the axis of rotation, orientation: given by the right-hand rule) do not add as elements of a vector space, as the addition of finite rotations is not commutative;
  - E.M.Purcell squrl operator is not a vector [a].
  - the wavelength, the fundamental concept for MPW, is a length which cannot be treated as a **wavelength vector** λ , but it is, on the other hand, the reciprocal of the module of the **wave-vector**.

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15.02 Coordinate Systems

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15.02.01
Cartesian Coordinates Coordinate System

Consider an orthonormal Cartesian Coordinates Coordinate System in three dimensions. This type of Coordinate System will always be implicitly used, unless specified otherwise.

Let either $\mathbf{r} \equiv \{x, y, z\}$ or $\mathbf{x} \equiv \{x_1, x_2, x_3\}$ be the three Cartesian coordinates of any point.

The three unit vectors of the three orthonormal Cartesian axes are either $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

The components of a generic vector $\mathbf{v}$ along the three orthonormal Cartesian axes are either $\{v_x, v_y, v_z\}$ or $\{v_1, v_2, v_3\}$.

Le relazioni tra i vettori base, $\mathbf{e}_1$, $\mathbf{e}_2$ ed $\mathbf{e}_3$, di un Coordinate System Cartesiane ortonormali sono:

$$\boxed{\mathbf{e}_i \cdot \mathbf{e}_k \equiv \delta_{ik}^{\cdot\cdot} \qquad i,k = 1,2,3 \qquad \text{orthonormality}} \quad , \tag{15.02.01}$$

→ 1055

$$\boxed{\left\{ \mathbf{e}_i \wedge \mathbf{e}_k \equiv \epsilon_{ikj}^{\cdot\cdot\cdot} \mathbf{e}_j \right\} \quad \Leftrightarrow \quad \left\{ \mathbf{e}_j \cdot (\mathbf{e}_i \wedge \mathbf{e}_k) \equiv \epsilon_{ikj}^{\cdot\cdot\cdot} \right\} \qquad i,j,k = 1,2,3 \qquad \text{right-handedness}} \quad . \tag{15.02.02}$$

→ 1055 1073

Il simbolo $\delta_{ik}^{\cdot\cdot}$ (simbolo delta di **Kronecker symbol**), read § 17.04.

Il simbolo $\delta_{ik}^{\cdot\cdot}$ (simbolo delta di **Kronecker symbol**) è definito qui dalla equazione (15.02.01), come l'oggetto simmetrico nei suoi due indici:

$$\boxed{\delta_{ik}^{\cdot\cdot} = \begin{cases} 1 & \text{se } i = k \\ 0 & \text{se } i \neq k \end{cases} \quad ,} \quad .$$

Il simbolo $\epsilon_{ikj}^{\cdot\cdot\cdot}$ è **Levi-Civita symbol** , read § 17.04.

Il simbolo $\epsilon_{ikj}^{\cdot\cdot\cdot}$ in tre dimensioni (**Levi-Civita symbol** in tre dimensioni) è definito qui dalla equazione (15.02.02), come l'oggetto totalmente antisimmetrico in una qualunque coppia dei suoi tre indici:

$$\boxed{\epsilon_{ijk}^{\cdot\cdot\cdot} = \begin{cases} +1 & \text{se } i,j,k \text{ è una permutazione pari } [1] \text{ di } 1,2,3 \quad , \\ -1 & \text{se } i,j,k \text{ è una permutazione dispari } [2] \text{ di } 1,2,3 \quad , \\ 0 & \text{se almeno due indici uguali} \quad , \end{cases}} \quad .$$

→ 1072

The derivation of equations (15.02.01) can be carried on by explicit calculation from the basic properties of the unit vectors of an orthonormal basis.

In principle, given an orthonormal basis, equations (15.02.01) can be considered to be the definition of the delta and epsilon symbols.

Read § 15-009 for some properties of the delta and epsilon symbols.

Any generic vector is written as:

$$\boxed{\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 \equiv v_k \mathbf{e}_k} \quad .$$

DRAFT - UNDER PEER REVIEW

15.02.02
Polar Coordinates in Plane

Le coordinate polari nel piano sono definite dalle relazioni

$$\begin{cases} x = r\cos[\theta] \\ y = r\sin[\theta] \end{cases},$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \begin{cases} \theta = \theta_0 = \arccos(x/r) & \text{se } y \geq 0; \text{ con } 0 \leq \theta_0 \leq \pi \\ \theta = 2\pi - \theta_0 & \text{se } y < 0 \end{cases} \end{cases}, \quad \begin{cases} \theta = \arctan y/x & \text{se } x > 0 \\ \theta = +\pi/2 & \text{se } x = 0 \text{ e } y > 0 \\ \theta = -\pi/2 & \text{se } x = 0 \text{ e } y < 0 \\ \theta = \arctan y/x + \pi & \text{se } x < 0 \end{cases},$$

15.02.03
Cylindrical Coordinates in Space

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See figure **??**.

15.02.04
Spherical Coordinates in Space

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See figure **??**.

15.02.05
Change Between Different Systems of Coordinates

15.03 Vector Algebra - a Short Summary of the Main Results

15.03.01
Prodotto Scalare E Vettoriale in Componenti Cartesiane Ortonormali

Given any two vectors one can define two types of multiplication between them: either a scalar product, giving a scalar, or a vector product, giving a vector, while remaining inside the same vector space.

Il prodotto scalare e vettoriale dei due vettori $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ e $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$ si può esprimere in termini delle componenti Cartesiane ortonormali.

More generally, given two vectors, $\mathbf{a}$ and $\mathbf{b}$, one can multiply them in three ways:

- one can get a scalar, $\mathbf{a}\cdot\mathbf{b}$;
- one can get a vector, $\mathbf{a}\wedge\mathbf{b}$;
- one can get a symmetrical second order tensor, $(a_r b_s + a_s b_r)$.

Read § 20.06.03.

DRAFT - LW - PROGRESS

15.03.01.01 Prodotto Scalare

Il prodotto scalare si esprime in termini delle componenti cartesiane ortogonali come:

$$\boxed{\mathbf{a}\cdot\mathbf{b} = \delta_{kj}^{\cdot\cdot} a_k b_j = a_k b_k} \quad . \tag{15.03.01}$$

15.03.01.02 Prodotto Vettoriale

Le componenti del prodotto vettoriale si esprimono in termini delle componenti Cartesiane ortonormali come:

$$\boxed{(\mathbf{a}\wedge\mathbf{b})_k = \epsilon_{kij}^{\cdot\cdot\cdot} a_i b_j} \quad . \tag{15.03.02}$$

The vector product is useful, from the geometrical point of view, to create either a vector perpendicular to a given vector or to a pair of vector.

Prodotto Vettoriale Espresso Attraverso Determinante

Il prodotto vettoriale di due vettori, $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ e $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$, si può esprimere, formalmente, in termini delle componenti Cartesiane ortonormali dei vettori, come il determinante:

$$\boxed{\mathbf{a}\wedge\mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}} \quad . \tag{15.03.03}$$

15.03.02
Geometrical Properties Connected to the Multiplication Between Two Vectors

15.03.02.01 Prodotto Vettoriale E Area Di Un Triangolo E Di Un Parallelogramma

Se due vettori **a** e **b** sono applicati nello stesso punto, il loro prodotto vettoriale è un vettore il cui modulo è uguale all'area del parallelogramma, S_P, definito dai due vettori.

Quindi il loro prodotto vettoriale è anche uguale al doppio dell'area, S_T, del triangolo delimitato dai due vettori e avente come terzo lato il segmento che unisce i vertici dei due vettori:

$$\boxed{S_P = |\mathbf{a}\wedge\mathbf{b}| \qquad S_T = \frac{1}{2}|\mathbf{a}\wedge\mathbf{b}|} \quad . \tag{15.03.04}$$

→ 1057 1078 1239

15.03.02.02 Description of Parallelogram-shaped area by Means of a Vector

This § is referenced at pages:
[1074]

The equation (15.03.04) is at the basis of the description of a parallelogram by means of a vector. In fact, starting from equation (15.03.04), one can describe a parallelogram, in terms of its two sides, **a** and **b**, by means of one of the two vectors:

$$\boxed{\mathbf{S} = \pm(\mathbf{a}\wedge\mathbf{b})} \quad , \tag{15.03.05}$$

→ 1078

whose module gives the area of the parallelogram and whose direction is perpendicular to the parallelogram itself. The choice of the sign, that is orientation, of the vector is usually dictated by other considerations. When, for instance, the parallelogram is just a piece of a closed surface the convention is to orient the vector exiting from the volume enclosed by the surface.

It is obvious that the vector **S** does not completely describe the parallelogram, but only its orientation and area.

DRAFT IN PROGRESS

15.03.02.03 Area Di Un Triangolo

L'area di un triangolo i cui vertici sono individuati dai tre vettori $\mathbf{a}$, $\mathbf{b}$ e $\mathbf{c}$, aventi lo stesso punto di applicazione, vale:

$$S = \frac{1}{2}|\,\mathbf{a}\wedge\mathbf{b} + \mathbf{b}\wedge\mathbf{c} + \mathbf{c}\wedge\mathbf{a}| \quad . \tag{15.03.06}$$

15.03.02.04 Teorema Dei Coseni

Dati tre vettori $\mathbf{a}$, $\mathbf{b}$ e $\mathbf{c}$ tali che $\mathbf{a} = \mathbf{b} - \mathbf{c}$, e detti rispettivamente α, β e γ gli angoli opposti ai lati $\mathbf{a}$, $\mathbf{b}$ e $\mathbf{c}$ del triangolo, si ha il teorema dei coseni:

$$a^2 = b^2 + c^2 - 2bc\cos[\alpha] \quad . \tag{15.03.07}$$

15.03.02.05 Teorema Dei Seni

Dati tre vettori $\mathbf{a}$, $\mathbf{b}$ e $\mathbf{c}$ tali che $\mathbf{a} = \mathbf{b} - \mathbf{c}$, e detti rispettivamente α, β e γ gli angoli opposti ai lati $\mathbf{a}$, $\mathbf{b}$ e $\mathbf{c}$ del triangolo, si ha il teorema dei seni:

$$\frac{\sin[\alpha]}{a} = \frac{\sin[\beta]}{b} = \frac{\sin[\gamma]}{c} \quad . \tag{15.03.08}$$

WORKED SOLUTION

$$\mathbf{a} = \mathbf{b} - \mathbf{c} \implies |\,\mathbf{b}\wedge\mathbf{a}| = |-\mathbf{b}\wedge\mathbf{c}|$$

15.03.03 Triple Vector Products

It is possible to define two different products involving three vectors and multiplications among them: a scalar product, giving a scalar, a vector product, giving a vector.

15.03.03.01 Triplo Prodotto Scalare E Volume Di Un Parallelepipedo

This § is referenced at pages:
[1074]

Dati tre vettori $\mathbf{a}$, $\mathbf{b}$ e $\mathbf{c}$ si può formare un triplo prodotto scalare il cui valore è invariante per permutazione dei tre vettori:

$$\mathbf{a}\cdot(\,\mathbf{b}\wedge\mathbf{c}) = \mathbf{b}\cdot(\,\mathbf{c}\wedge\mathbf{a}) = \mathbf{c}\cdot(\,\mathbf{a}\wedge\mathbf{b}) = \epsilon_{ijk}a_ib_jc_k = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \quad . \tag{15.03.09}$$

→ 1074 1075

Il triplo prodotto scalare $\mathbf{a}\cdot(\,\mathbf{b}\wedge\mathbf{c})$ di tre vettori non nulli si annulla se e solo se i tre vettori sono complanari.

DRAFT - IN PROGRESS

15.03.03.02 Triplo Prodotto Vettoriale

Dati tre vettori $\mathbf{a}$, $\mathbf{b}$ e $\mathbf{c}$ si può formare un triplo prodotto vettoriale. Si hanno le identità, tra loro equivalenti,

$$\boxed{\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})} \quad , \qquad \boxed{(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c} = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{a}(\mathbf{c}\cdot\mathbf{b})} \quad .$$

→ 1059 1066 1074 1075

15.03.04 Geometrical Properties of Multiplication Between Three Vectors

15.03.04.01 Volume Di Un Parallelepipedo

Dati tre vettori $\mathbf{a}$, $\mathbf{b}$ e $\mathbf{c}$ aventi lo stesso punto di applicazione il volume V del parallelepipedo generato dai tre vettori è uguale al modulo del triplo prodotto scalare dei tre vettori:

$$\boxed{V = |\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})|} \quad .$$

15.03.04.02 Scomposizione Di Un Vettore in Componenti Rispetto Ad Un Unit Vector

This § is referenced at pages:
[1142]

Un vettore $\mathbf{x}$ può essere scomposto nelle sue componenti parallela, $\mathbf{x}_{\parallel}$, ed ortogonale, $\mathbf{x}_{\perp}$, ad un unit vector $\hat{\mathbf{n}}$ tramite la relazione:

$$\boxed{\mathbf{x} = (\mathbf{x}\cdot\mathbf{n})\,\hat{\mathbf{n}} + \mathbf{n}\wedge(\mathbf{x}\wedge\mathbf{n}) \equiv \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}} \quad , \tag{15.03.10}$$

→ 1066 1130

che deriva direttamente dalla (15.03.03.02).

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15.04 Systems of Vectors and Vector Fields

This § is referenced at pages:
[1651, 1651]

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- **A free vector is any vector with no specified line of action or point of application**.
- **A sliding vector is any vector that has a specified line of action or axis, but has no specified point of application**.
- An **applied vector** is any vector that has both a specified line of action and a specified point of application.

15.04.01
Applied Vectors and Applied Vector Fields - Vector Spaces

Dato un vettore **f** applicato in un punto P si definisce retta d'azione del vettore la retta parallela al vettore e passante per il punto P.

Dato un punto arbitrario dello spazio, il polo O, si definisce braccio del vettore rispetto al polo O la distanza del punto dalla retta d'azione del vettore.

A vector field is a map such that at any point in some domain is associated a vector (applied at the point).

Important examples of applied vector fields in mechanics are:

- force (momentum of the force);
- momentum of the force (whose momentum is not useful);
- momentum of a mass point (momentum of the momentum aka orbital angular momentum);
- orbital angular momentum of a mass point (whose momentum is not useful);
- gravitational fields.

Important examples of applied vector fields outside mechanics include electric/magnetic fields.

15.04.02
Momentum of an Applied Vector

Both the polar and axial moments of a vector are uniquely defined with respect to a pole (the former) or an oriented straight line, that is an axis, (the latter). Either the pole or the axis must always be specified when dealing with the momentum of a vector.

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15.04.02.01 Momentum of an Applied Vector With Respect to an Axis (Axial Momentum)

Sia $\mathbf{f}$ un vettore applicato in P. Il momento assiale rispetto ad una retta orientata arbitraria è la proiezione lungo la retta orientata del momento polare del vettore rispetto ad un qualunque polo giacente sulla retta. Se $\hat{\mathbf{u}}$ è un unit vector che individua la retta orientata ed R un qualunque punto della retta:

$$\boxed{\gamma_{\hat{\mathbf{u}}} \equiv \hat{\mathbf{u}} \cdot (\mathbf{x}_{PR} \wedge \mathbf{f})} \quad . \tag{15.04.01}$$

→ 1066

Dalla relazione (15.04.03) segue che il momento assiale così definito è indipendente dalla scelta del polo lungo la retta.

15.04.02.02 Momento of an Applied Vector With Respect to a Point (Polar Momentum)

Sia $\mathbf{f}$ un vettore applicato in P. Il momento (polare) del vettore $\mathbf{f}$ rispetto ad un polo O, γ_O, è definito come

$$\boxed{\gamma_O \equiv \mathbf{x}_{PO} \wedge \mathbf{f} \qquad \text{applied in } P; \qquad \text{with} \qquad \mathbf{x}_{PO} \equiv (\mathbf{x}_P - \mathbf{x}_O)} \quad . \tag{15.04.02}$$

→ 1061

Il vettore $\mathbf{x}_{PO} \equiv (\mathbf{x}_P - \mathbf{x}_O)$ è il vettore che individua la posizione di P rispetto ad O, cioè la posizione di P relativa ad O.

Il momento di un vettore applicato $\mathbf{f}$ è un vettore che va considerato come un **vettore applicato nel punto di applicazione** del vettore $\mathbf{f}$. In tal modo si definisce per un qualunque polo, a partire da un campo vettoriale $\mathbf{f}$, un altro campo vettoriale: il campo vettoriale dei momenti rispetto al polo O, fisso.

Therefore for a given vector field and a pole one can build a second vector field, the field of momentum of the vector with respect to the pole, such that at any point is space two vectors are applied: the vector itself and its momentum with respect to the pole.

Il modulo del momento del vettore è uguale al prodotto tra il modulo del vettore e il braccio del vettore rispetto al polo dato. Equivalentemente il modulo del momento del vettore è uguale alla distanza del punto di applicazione del vettore dal polo moltiplicata per il modulo della proiezione del vettore nella direzione perpendicolare al vettore $\mathbf{x}_{PO}$ nel piano contenente la retta d'azione del vettore e il polo.

Pole-Dependent Momentum - Change of Pole

Il momento di un vettore $\mathbf{f}$ varia al variare del polo secondo la relazione

$$\boxed{\gamma_B = \gamma_A + \mathbf{x}_{AB} \wedge \mathbf{f} = \gamma_A + (\mathbf{x}_A - \mathbf{x}_B) \wedge \mathbf{f}} \quad , \tag{15.04.03}$$

→ 1061 1066 1536

dove $\mathbf{x}_{AB} \equiv \mathbf{x}_A - \mathbf{x}_B$ è il vettore posizione del polo A rispetto al polo B.

The relation shows that the momentum with respect to a certain pole is given by the momentum around any other pole plus the momentum, with respect to the pole, of the vector as if applied at the other pole.

Note that the total momentum includes a contribution from a intrinsic momentum, which is independent from the pole and therefore does not appear in (15.04.03).

When, for a pole-dependent momentum, the pole is not written it is assumed to be the origin of the Coordinate System.

Pole-Independent Momentum (Intrinsic Momentum)

Often in physics, **momenta independent from any pole appear** (**spin** and **torque**, for instance) and have a major role.

Pole-independent momenta will be denoted as:

$$\boxed{\tilde{\gamma}[\mathbf{x}] \qquad \text{with} \qquad \tilde{\gamma}_A[\mathbf{x}] = \tilde{\gamma}_B[\mathbf{x}] \qquad \forall A, B} \quad . \tag{15.04.04}$$

Pole-independent momenta must be thought as real momenta not arising by a vector $\mathbf{f}$, that is not arising from the (15.04.02) relation, applied at a point.

Pole-independent momenta do not affect sums of the applied vectors but do affect total momenta of applied vectors: they must be included in the sum to find the total momentum: see equations (15.04.05) and (15.04.06).

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15.04.02.02.1 Examples

Force

Let us consider a system of applied forces and call torque a momentum of a system of forces which is independent form the pole. This is a primitive concept, which can be also seen as a limiting case of systems of vectors with zero sum and non-zero total momentum, for instance any set of n identical vectors applied at the vertexes of a regular polygon with n-sides making the same angles with respect the line joining the vertex with the center of the polygon.

Read § 36.05.02.02 and § 36.05.04.

- Force: the force applied vector field.
- Polar momentum of the force.
- Torque: the pole-independent momentum applied vector field; applied vector at the point where the torque acts.
- Total Momentum of the force: torque plus polar momentum of the force.

Linear Momentum

- Linear Momentum: the Linear Momentum applied vector field.
- Polar Linear Momentum: orbital angular momentum.
- Spin: the pole-independent angular momentum applied vector field; applied vector at the point where the spin acts.
- Total Angular Momentum: orbital plus spin angular momentum.

15.04.02.03 Sum and Sum of Momenta of a System of Applied Vectors

Discrete case

Sia dato un sistema di vettori $\mathbf{f}_k$ e di torques $\tilde{\gamma}_k$ applicati nei punti P_k. Si definiscono la risultante, $\mathbf{F}$, e il momento risultante rispetto ad un polo O arbitrario, $\mathbf{\Gamma}_0$, come:

$$\boxed{\mathbf{F} \equiv \sum_k \mathbf{f}_k \qquad \text{a free vector}} \quad ,$$

$$\boxed{\mathbf{\Gamma}_0 \equiv \sum_k \left(\mathbf{x}_{P_k 0} \wedge \mathbf{f}_k + \tilde{\gamma}_k \right) \qquad \text{a vector applied in the pole O;}} \quad . \tag{15.04.05}$$

→ 1061

Continuous case

Let the vector and torque fields, $\mathbf{f}[\mathbf{x}]$, $\tilde{\gamma}[\mathbf{x}]$.

$$\boxed{\mathbf{F} \equiv \iiint \mathrm{d}\mathbf{f}[\mathbf{x}] \qquad \text{a free vector}} \quad ,$$

$$\boxed{\mathbf{\Gamma}_0 \equiv \iiint \left((\mathbf{x} - \mathbf{x}_0) \wedge \mathrm{d}\mathbf{f}[\mathbf{x}] + \mathrm{d}\tilde{\gamma}[\mathbf{x}] \right) \qquad \text{a vector applied in the pole O;}} \quad . \tag{15.04.06}$$

→ 1061

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Properties

La risultante di un sistema di vettori applicati è un vettore libero, non dipendendo la sua definizione da alcun punto privilegiato, mentre **il momento risultante può essere considerato un vettore applicato nel polo**. Il momento risultante può essere quindi considerato un campo vettoriale al variare del polo.

La relazione tra i momenti del sistema rispetto a due poli A e B è data da

$$\boxed{\boldsymbol{\Gamma}_B = \boldsymbol{\Gamma}_A + \mathbf{x}_{AB}\wedge\mathbf{F} = \boldsymbol{\Gamma}_A + (\mathbf{x}_A - \mathbf{x}_B)\wedge\mathbf{F}} \tag{15.04.07}$$

→ 1063 1064 1442

dove $\mathbf{x}_{AB} \equiv \mathbf{x}_A - \mathbf{x}_B$ è il vettore posizione del polo A rispetto al polo B.

In particolare se la risultante è nulla il momento è indipendente dal polo.

Dalla (15.04.07) si deduce che

$$\boxed{I \equiv \mathbf{F}\cdot\boldsymbol{\Gamma}_O = \text{indipendente da } O} \quad .$$

La grandezza $I \equiv \mathbf{F}\cdot\boldsymbol{\Gamma}_O$, indipendente dal polo, è detta invariante scalare del sistema di vettori applicati.

15.04.02.04 Central Axis

If $\mathbf{F} = 0$ the resultant momentum does not depend on the pole.

If $\mathbf{F} \neq 0$ the resultant momentum do depend on the pole but its projection on $\mathbf{F}$ is an invariant for changes of pole.

Let us assume $\mathbf{F} \neq 0$. Consider the projections of $\boldsymbol{\Gamma}_P$ along and perpendicular to $\mathbf{F}$. The component along $\mathbf{F}$ does not depend on the pole while the component perpendicular to $\mathbf{F}$ does depend on it.

Starting from

$$\boldsymbol{\Gamma}_P = \frac{I\mathbf{F}}{F^2} + \frac{\mathbf{F}\wedge(\boldsymbol{\Gamma}_P\wedge\mathbf{F})}{F^2} \quad ,$$

one can write:

$$\boldsymbol{\Gamma}_P = \frac{I\mathbf{F}}{F^2} + \mathbf{F}\wedge\left(\frac{\boldsymbol{\Gamma}_O\wedge\mathbf{F}}{F^2} + (\mathbf{x}_P - \mathbf{x}_O)\right) \quad .$$

The central axis is defined as the locus of the points (poles) in space such that the momentum reduces to the part parallel to $\mathbf{F}$ only. Therefore one can choose an arbitrary point and solve for the unknown point P such that:

$$\mathbf{F}\wedge\left(\frac{\boldsymbol{\Gamma}_O\wedge\mathbf{F}}{F^2} + (\mathbf{x}_P - \mathbf{x}_O)\right) = \mathbf{0} \quad .$$

The vector equation, according to § 17.02, is solved by:

$$\frac{\boldsymbol{\Gamma}_O\wedge\mathbf{F}}{F^2} + (\mathbf{x}_P - \mathbf{x}_O) = \lambda[P]\mathbf{F} \quad ,$$

for any real $\lambda[P]$, that is:

$$\boxed{(\mathbf{x}_P - \mathbf{x}_O) = \lambda[P]\mathbf{F} - \frac{\boldsymbol{\Gamma}_O\wedge\mathbf{F}}{F^2}} \quad , \tag{15.04.08}$$

→ 1063 1064

showing that the result is a straight line parallel to $\mathbf{F}$.

The particular solution of equation (15.04.08)

$$\boxed{(\mathbf{x}_P - \mathbf{x}_O) = \frac{\mathbf{F}\wedge\boldsymbol{\Gamma}_O}{F^2}} \quad ,$$

determines in a unique way the point of the central axis in the plane perpendicular to the vector $\mathbf{F}$ and passing by the pole O.

If any point C belongs to the central axis the resultant momentum can be written as:

$$\boldsymbol{\Gamma}_C = \frac{I\mathbf{F}}{F^2} \quad , \tag{15.04.09}$$

and the transformation law becomes

$$\boldsymbol{\Gamma}_P = \boldsymbol{\Gamma}_C + (\mathbf{x}_C - \mathbf{x}_P)\wedge\mathbf{F} = \frac{I\mathbf{F}}{F^2} + (\mathbf{x}_C - \mathbf{x}_P)\wedge\mathbf{F} \quad . \tag{15.04.10}$$

→ 1064

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Note that a central axis can be constructed only for systems with non zero resultant, as it is clear from relation (15.04.08).

15.04.02.05 Equivalent Systems of Applied Vectors

Due sistemi di vettori applicati si dicono **equivalenti se hanno lo stesso risultante e lo stesso momento rispetto ad ogni polo**.

Se due sistemi di vettori applicati **hanno la stessa risultante e lo stesso momento rispetto ad un polo qualunque allora sono equivalenti**, grazie alla (15.04.07).

Esistono tre categorie elementari di sistemi di vettori applicati.

1. Il sistema nullo, formato dal vettore nullo, per cui si ha $\mathbf{F} = 0$, $\mathbf{\Gamma}_0 = 0$ ed $I = 0$.
2. Il sistema costituito da un singolo vettore $\mathbf{f}$ applicato in un punto P. Questo sistema ha sempre invariante scalare nullo, $I = 0$. Infatti si ha $\mathbf{F} = \mathbf{f}$ e, in generale, $\mathbf{\Gamma}_0 \neq 0$. Però per ogni punto C della retta d'azione del vettore si ha $\mathbf{\Gamma}_C = 0$. L'invariante scalare è quindi nullo per il polo C e quindi nullo per ogni polo. Alternativamente si può osservare che il momento rispetto ad un polo qualunque è sempre ortogonale al vettore risultante.
3. Una coppia, cioè il sistema costituito da due vettori $\mathbf{f}_1$ e $\mathbf{f}_2$, uguali ed opposti, applicati in due punti P_1 e P_2. La distanza tra le rette d'azione dei due vettori è detto braccio della coppia. Si ha $\mathbf{F} = 0$, e, in generale, $\mathbf{\Gamma}_0 \neq 0$ (eccetto il caso di coppia a braccio nullo, quando cioè le rette d'azione dei due vettori coincidono). Si ha quindi $I = 0$. Il momento non dipende dal polo ed è detto semplicemente momento della coppia.

I tre sistemi sono in generale tra loro non equivalenti, salvo casi particolari di vettori nulli oppure coppia a braccio nullo. Inoltre tutti e tre hanno invariante scalare nullo.

Given any system of vectors with null scalar invariant two cases are possible, as follows.

- Suppose $\mathbf{F} = 0$. In this case the momentum is independent on the pole and the system is equivalent to a torque of momentum Γ_O;
- Suppose $\mathbf{F} \neq 0$. In this case the system admits a central axis. Fixed an arbitrary point, C, on the central axis, as $I = 0$, one has, from equation (15.04.10):

$$\mathbf{\Gamma}_P = (\mathbf{x}_C - \mathbf{x}_P) \wedge \mathbf{F} \quad .$$

The system is thus equivalent to a single force, equal to the resultant, applied at any point on the central axis.

Sistemi Di Vettori Applicati Riducibili Ed Irriducibili

Un sistema di vettori applicati è detto riducibile se è equivalente ad uno dei tre sistemi elementari di vettori.

La condizione di invariante scalare nullo è necessaria e sufficiente affinché un sistema sia riducibile.

Si ha il fondamentale risultato che **un qualunque sistema irriducibile di vettori applicati, cioè ad invariante scalare diverso da zero, può sempre essere ridotto ad un sistema equivalente composto da un vettore applicato e da una coppia**, che è il più semplice sistema equivalente ad un sistema irriducibile.

Let us determine the simplest system to which an irreducible system can be reduced, by equivalence. As $I \neq 0$ one has $\mathbf{F} \neq 0$ and therefore an irreducible system admits a central axis.

- Equation (15.04.10) implies that the system has the same total force and total torque as a system made of a vector, $\mathbf{V}$, applied on the central axis, plus a torque having momentum $\mathbf{\Gamma}$, with:

$$\boxed{\mathbf{V} = \mathbf{F} \qquad \text{applied on the central axis;} \qquad \mathbf{\Gamma} = I\mathbf{F}/F^3} \quad .$$

Note that in this case $\mathbf{V}$ and $\mathbf{\Gamma}$ are collinear.
- One second alternative equivalent set is given by a vector, $\mathbf{V}$, with line of action passing by O, plus a torque having momentum $\mathbf{\Gamma}$, with:

$$\boxed{\mathbf{V} = \mathbf{F} \qquad \text{with line of action passing by } O; \qquad \mathbf{\Gamma} = \mathbf{\Gamma}_0} \quad .$$

- As a particular case of the previous case, and one often used in practice, one can choose:

$$\boxed{\mathbf{V} = \mathbf{F} \qquad \text{applied in } O; \qquad \mathbf{\Gamma} = \mathbf{\Gamma}_0} \quad .$$

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15.04.02.06 Effects of a System of Forces on a Mechanical System

The effect of a system of force on a system is the following.

The sum of a system of forces (that is the total force) affects the linear motion of the Center-Of-Mass.

The total torque, that is the sum of the momenta of forces (that is the total momentum of the forces) with respect to any pole, plus the sum of the intrinsic, independent from the pole, torques (that is the total intrinsic torque), affect the angular momentum of the system with respect to that pole.

Any system of Force|Torque with zero total force and zero total torque with respect to any pole has no effect on the total momentum nor on the total angular momentum of the system around the pole. Nevertheless, it may have important effects on the motion, causing deformations of the system.

15.04.03
Free Vectors and Free Vector Fields - Affine Spaces

Heuristic introduction.

In physical words, an affine space is a vector field where all vectors with the same magnitude, direction and sense, are identified regardless of the application point.

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15.05 Examples and Physical Applications

15.05.01
✓ Change of Pole in the Momentum
Demonstrate equation (15.04.03).

15.05.02
✓ Arm of an Applied Vector With Respect to a Pole
Let the vector $\mathbf{f}$ be applied in the pole P. Write a vector relation giving the vector originating from the pole P and directed to the point of closest approach of the line of action of the vector to the pole, whose module is the arm.

15.05.03
✓ Axial Moment
Show that the axial moment defined in (15.04.01) is independent from the choice of the point R.

15.05.04
✓ Independence From the Origin of the Center-Of-Mass
Show that the position of the Center-Of-Mass of a set of point particles is independent from the choice of the origin of the Coordinate System.

15.05.05
✓ Centrifugal Force
Consider a non-inertial Rest Frame such that $\mathbf{\Omega}$ has a constant fixed direction. Let the origin of the Coordinate System located on the rotation axis. Let $\mathbf{r}$ be the position vector of a point with respect to the non-Inertial Reference Frame . Consider the expression of the centrifugal force,

$$\mathbf{f}_c = -m\,\mathbf{\Omega}\wedge(\mathbf{\Omega}\wedge\mathbf{r}) = +m\,\mathbf{\Omega}\wedge(\mathbf{r}\wedge\mathbf{\Omega}) = +m\Omega^2\,\mathbf{n}\wedge(\mathbf{r}\wedge\mathbf{n}) \quad ,$$

in terms of the unit vector in the direction of the angular velocity Ω, $\widehat{\mathbf{n}}$, and write it in a suitable alternative form.

What happens if the Coordinate System has its origin outside the axis of rotation? Is the above formula still exact?

WORKED SOLUTION

Use (15.03.03.02):

$$\mathbf{f}_c = m\left(\Omega^2\mathbf{r} - \mathbf{\Omega}\,(\mathbf{r}\cdot\mathbf{\Omega})\right) \quad .$$

This shows that the force only depends on the distance from the rotation axis and is always radially directed away from it. It follows from equation (15.03.10) that the centrifugal force is proportional to the component of $\mathbf{r}$ perpendicular to the direction of $\mathbf{\Omega}$.
Equation (15.03.10) shows that the centrifugal force is proportional to the component of the position vector $\mathbf{r}$ perpendicular to the direction of $\mathbf{\Omega}$.

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15.05.06
✓ Legge Di Snell Della Riflessione

This § is referenced at pages:
[1142, 1252]

Read § 48.02.
Show that:

$$\boxed{\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_0 - 2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\alpha}_0\right)\widehat{\mathbf{n}}} \quad . \tag{15.05.01}$$

15.05.07
✓ Legge Di Snell Della Rifrazione

This § is referenced at pages:
[1142, 1252, 1854, 1897]

Read § 48.02.
Si consideri un piano ed un unit vector $\mathbf{u}$ normale al piano e applicato nel punto O del piano. Si consideri un vettore $\boldsymbol{\alpha_0}$ applicato in O che forma un angolo θ_0 con $\mathbf{u}$ ed un secondo vettore $\boldsymbol{\alpha_1}$ applicato in O che forma un angolo θ_1 con $\mathbf{u}$. Se la relazione tra i due angoli è $\sin[\theta_1] = k\sin[\theta_0]$ (legge di Snell della rifrazione) dimostrare che l'espressione esplicita di $\boldsymbol{\alpha_1}$ in funzione di $\boldsymbol{\alpha_0}$ e $\mathbf{u}$ è:

$$\boxed{\boldsymbol{\alpha}_1 = -\mathbf{u}\cos[\theta_1] + k\left(\boldsymbol{\alpha}_0 - \mathbf{u}\left(\boldsymbol{\alpha_0}\cdot\mathbf{u}\right)\right)/\alpha_0} \quad . \tag{15.05.02}$$

15.05.08
✓ Kinematic Constraints - A Wheel Rolling on a Plane

This § is referenced at pages:
[Never referenced.]

Consider a wheel of radius R moving on a horizontal plane with velocity of its center $\mathbf{v}_C$ and angular velocity $\boldsymbol{\omega}$.

1. Determine the rolling without slipping condition.
2. In case of rolling without slipping, determine the axis of instantaneous rotation.
3. In case of rolling without slipping, determine the acceleration of all the points of the edge.

Read also, for more complex cases, section § 26-029, § 26-030.

15.05.09
☒ (Non-)Relativistic Charged Particle in a Uniform and Constant Magnetic Field

This § is referenced at pages:
[1070, 1070, 2425, 2475]

Consider a charged point particle inside a constant and uniform magnetic field $\mathbf{B}$. The particle has, at time t_0, the velocity $\mathbf{v}_0$. Determine the trajectory of the particle:

1. for a low-speed motion;
2. for a relativistic motion.

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WORKED SOLUTION

The non-relativistic motion is just special case of the generic relativistic one:

1. for a low-speed motion assume, in the calculations below, $\gamma[v] = 1$;
2. for a relativistic motion assume, in the calculations below, $\gamma[v] = 1/\sqrt{1-v^2}$.

Let $\mathbf{B} \equiv B\widehat{\mathbf{e}}_z$ and decompose the velocity and linear momentum in its components parallel and perpendicular to the magnetic field:

$$\begin{aligned} \mathbf{v} &\equiv \mathbf{v}_\parallel + \mathbf{v}_\perp \quad , \\ \gamma = \gamma[v] &\neq \gamma[\mathbf{v}_\parallel] \quad , \\ \gamma = \gamma[v] &\neq \gamma[\mathbf{v}_\perp] \quad , \\ \mathbf{p} \equiv m\gamma\mathbf{v} \equiv m\gamma\left(\mathbf{v}_\parallel + \mathbf{v}_\perp\right) &\equiv m\gamma\mathbf{v}_\parallel + m\gamma\mathbf{v}_\perp \equiv \mathbf{p}_\parallel + \mathbf{p}_\perp \quad . \end{aligned}$$

The equation of motion in presence of magnetic field only is:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\,\mathbf{v}\wedge\mathbf{B} = q\left(\mathbf{v}_\perp + \mathbf{v}_\parallel\right)\wedge\mathbf{B} = q\,\mathbf{v}_\perp\wedge\mathbf{B} = \frac{\mathrm{d}\mathbf{p}_\parallel}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{p}_\perp}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{p}_\perp}{\mathrm{d}t} \quad , \tag{15.05.03}$$

→ 1068

where the last passage derives from the fact that the Lorentz force lies in the plane perpendicular to $\mathbf{B}$ and therefore

$$\frac{\mathrm{d}\mathbf{p}_\parallel}{\mathrm{d}t} = 0 \quad ,$$

because

$$\frac{\mathrm{d}\mathbf{p}_\parallel}{\mathrm{d}t} \parallel \mathbf{B} \quad ,$$

due to the fact that

$$\mathbf{v}_\parallel \equiv v_\parallel\widehat{\mathbf{e}}_z \implies \frac{\mathrm{d}\mathbf{v}_\parallel}{\mathrm{d}t} = \frac{\mathrm{d}v_\parallel}{\mathrm{d}t}\widehat{\mathbf{e}}_z \parallel \widehat{\mathbf{e}}_z \implies \frac{\mathrm{d}\mathbf{p}_\parallel}{\mathrm{d}t} = m\frac{\mathrm{d}\gamma[v]\mathbf{v}_\parallel}{\mathrm{d}t} = m\gamma[v]\frac{\mathrm{d}\mathbf{v}_\parallel}{\mathrm{d}t} \equiv m\gamma[v]\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}\widehat{\mathbf{e}}_z \parallel \mathbf{B} \quad ,$$

because energy conservation implies:

$$\frac{\mathrm{d}\gamma[v]}{\mathrm{d}t} = 0 \quad .$$

In other words the equation of motion (15.05.03) implies:

$$\widehat{\mathbf{e}}_z\cdot(\,\mathbf{v}_\perp\wedge\mathbf{B}) = 0 \implies 0 = \frac{\mathrm{d}\mathbf{p}_\parallel}{\mathrm{d}t} = m\gamma\frac{\mathrm{d}\mathbf{v}_\parallel}{\mathrm{d}t} = m\gamma\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}\widehat{\mathbf{e}}_z = 0 \implies \boxed{v_\parallel = \text{constant}} \quad .$$

Alternatively, in a simpler way, one can write:

$$0 = \mathbf{v}\cdot\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\gamma\,\mathbf{v}\cdot\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{p}\cdot\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\,\mathbf{p}\cdot\mathbf{p}) \implies |\mathbf{p}| = \text{constant} \implies |\mathbf{v}| = \text{constant} \quad ,$$

to find again that that the module of the velocity is constant, as well as the γ factor.
The same result could be obtained from the energy equation for a point particle:

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \pi = \mathbf{f}\cdot\mathbf{v} \quad .$$

The equation of motion (15.05.03) is reduced to a plane equation of motion in the plane perpendicular to the magnetic field plus a constant velocity motion along the direction of the magnetic field:

$$\frac{\mathrm{d}\mathbf{p}_\perp}{\mathrm{d}t} = q\,\mathbf{v}_\perp\wedge\mathbf{B} \quad ,$$

$$\frac{\mathrm{d}\mathbf{p}_\parallel}{\mathrm{d}t} = 0 \implies \mathbf{p}_\parallel = \text{constant} \implies \mathbf{v}_\parallel = \text{constant} \qquad \text{because } \gamma \text{ is constant} \quad .$$

Therefore not only the module of the velocity is constant but also the module of the component of the velocity parallel to the magnetic field is constant and thus also the module of the component of the velocity perpendicular to the magnetic field is constant as well:

$$|\mathbf{p}_\perp| = \text{constant} \quad .$$

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We can therefore limit ourselves to only study motion in the plane perpendicular to $\mathbf{B}$. In this plane:

$$\frac{\mathrm{d}\mathbf{v}_\perp}{\mathrm{d}t} = \widehat{\mathbf{T}}\frac{\mathrm{d}v_\perp}{\mathrm{d}t} + \widehat{\mathbf{N}}\frac{v_\perp^2}{\rho} = 0 + \widehat{\mathbf{N}}\frac{v_\perp^2}{\rho} = \frac{q}{m\gamma}(\mathbf{v}_\perp \wedge \mathbf{B}) = \frac{qv_\perp B}{m\gamma}\left(\widehat{\mathbf{T}}\wedge\widehat{\mathbf{e}}_z\right) \quad .$$

Taking the modules of the resulting equation,

$$\boxed{\widehat{\mathbf{N}}\frac{v_\perp^2}{\rho} = \frac{q}{m\gamma}(\mathbf{v}_\perp \wedge \mathbf{B})} \quad ,$$

where all quantities are already known to be constant except for the radius of curvature, ρ, and accounting for the fact that $\mathbf{v}_\perp$ and $\mathbf{B}$ are perpendicular, one finds:

$$\boxed{q\rho B = m\gamma[v]v_\perp \equiv p_\perp} \quad , \tag{15.05.04}$$

→ 1069

showing that the radius of curvature is also constant, that is the trajectory is a circumference. Note that in this relation (15.05.04): $\gamma = \gamma[v]$ and $\gamma \neq \gamma[v_\perp]$.
The **Larmor Frequency** follows from equation (15.05.04):

$$\boxed{\Omega_\mathrm{L} \equiv \frac{2\pi}{T} = \frac{v_\perp}{\rho} = \frac{qB}{m\gamma}} \quad . \tag{15.05.05}$$

→ 1069

The **Larmor Frequency** equation (15.05.05) determines, among many other things, the way circular particle accelerators work, showing, in particular, that the revolution frequency is only constant for non-relativistic particles (cyclotron) while it changes for relativistic particles (synchrotron, sometimes known as a synchro-cyclotron).
Equation (15.05.04) shows that in a circular particle accelerator the final energy for any given particle will only depend on the product ρB: this is one of reasons why one is lead to build larger and larger circular particle accelerators to get higher energies; another reason is that increasing ρ the energy irradiated by the particles tends to decrease.
As in a cyclotron the magnetic field does not depend on time the radius of the orbit will increase, as it is shown by equation (15.05.05).
When the particle becomes relativistic the **Larmor Frequency** changes and the orbital period changes: the synchro-cyclotron is built to change the frequency of the acceleration fields to be synchronous with the **Larmor Frequency** .
As in a synchrotron the radius of the orbit is kept constant the magnetic field has to increase.
See[a] for more details.
The equation in vector form also determines the handedness of the curvature of the trajectory in the plane, showing that its sign depends on the charge of the particle.
Another way to obtain the same result is the one proposed in problem § 15.05.10.
If an electric field is also present the motion due to the electric field can be superimposed to the motion due to the magnetic field. However, in this case:

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \pi = q\,\mathbf{v}\cdot\mathbf{E} \quad .$$

[a] .....

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15.05.10
☒ Spin Precession in a Constant and uniform magnetic field

This § is referenced at pages:
[1067, 2425, 2475]

Determine the solution to the (classical) spin-precession equation:

$$\boxed{\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = g\,\mathbf{s}\wedge\mathbf{B}} \quad , \tag{15.05.06}$$

→ 1070

for a fixed spin vector, s, where g is a constant and $\mathbf{B}$ is a constant and uniform magnetic field.

WORKED SOLUTION

One can take the advantage of using complex vectors as follows.

First Solution

$$\dot{\mathbf{s}} = g\,\mathbf{s}\wedge\mathbf{B} \quad ,$$
$$\mathbf{s} = \mathbf{s}_0 \exp[\mathrm{i}\omega t] \qquad \text{with } \mathbf{s} \text{ and } \mathbf{s}_0 \text{ complex vectors;} \quad ,$$
$$\mathbf{s}_0 = \frac{g}{\mathrm{i}\omega}\mathbf{s}_0\wedge\mathbf{B} \quad ,$$
$$\mathbf{s} \equiv \mathbf{s}_\mathrm{R} + \mathrm{i}\mathbf{s}_\mathrm{I} \quad ,$$
$$\mathbf{s}_\mathrm{R} = +\frac{g}{\omega}\mathbf{s}_\mathrm{I}\wedge\mathbf{B} \tag{15.05.07}$$

→ 1070

$$\mathbf{s}_\mathrm{I} = -\frac{g}{\omega}\mathbf{s}_\mathrm{R}\wedge\mathbf{B} \tag{15.05.08}$$

→ 1070

$$\Longrightarrow \mathbf{s}_\mathrm{R}\cdot\mathbf{s}_\mathrm{I} = 0 \quad ,$$
$$\Longrightarrow \mathbf{s}_\mathrm{R}\cdot\mathbf{B} = 0 \quad ,$$
$$\Longrightarrow \mathbf{s}_\mathrm{I}\cdot\mathbf{B} = 0 \quad .$$

From (15.05.07), (15.05.08) (take one of either $\mathbf{s}_\mathrm{R}$ or $\mathbf{s}_\mathrm{I}$ and replace in the other equation) one finds that, in order to have a non zero s, it is required that $\omega = \gamma B$.
Then take $\mathbf{B} = B\widehat{\mathbf{e}}_3$ and

$$\mathbf{s}_\mathrm{R} = |\mathbf{s}_0|\{+\cos[\alpha], +\sin[\alpha], 0\} \quad ,$$
$$\mathbf{s}_\mathrm{I} = |\mathbf{s}_0|\{-\sin[\alpha], +\cos[\alpha], 0\} \quad .$$

that is a solution. Finally:

$$\mathbf{s}_0 = \exp[\mathrm{i}\omega t](\mathbf{s}_\mathrm{R} + \mathrm{i}\mathbf{s}_\mathrm{I}) \Longrightarrow \begin{cases} \mathbf{s}_0\big|_x = |\mathbf{s}_0|\exp[\mathrm{i}\omega t]\exp[\mathrm{i}\alpha] & , \\ \mathbf{s}_0\big|_y = |\mathbf{s}_0|\exp[\mathrm{i}\omega t]\exp[\mathrm{i}(\alpha - \pi/2)] \end{cases} .$$

Second Solution

By following exactly the same steps and usign the same conventions as in problem § 15.05.09, with the simplification given by $\gamma \equiv 1$ in the curretn case, one passes from equation (15.05.06) to equation:

$$\frac{\mathrm{d}\mathbf{s}_\perp}{\mathrm{d}t} = g\,\mathbf{s}_\perp\wedge\mathbf{B} \qquad \frac{\mathrm{d}\mathbf{s}_\parallel}{\mathrm{d}t} = 0 \qquad \text{with} \qquad \mathbf{s} = s_\parallel\widehat{\mathbf{e}}_\parallel + s_\perp\widehat{\mathbf{e}}_\perp \quad . \tag{15.05.09}$$

It follows:

$$g\,\mathbf{s}_\perp\wedge\mathbf{B} = \frac{\mathrm{d}\mathbf{s}_\perp}{\mathrm{d}t} \quad ,$$

and

$$0 = \mathbf{s}_\perp \cdot \frac{\mathrm{d}\mathbf{s}_\perp}{\mathrm{d}t} \qquad \Longrightarrow \qquad |\mathbf{s}_\perp| = \text{constant} \quad .$$

Now, in the plane perpendicular to $\mathbf{B}$, one can write:

$$\mathbf{s}_\perp = s_\perp \widehat{\mathbf{e}}_r \quad ,$$

$$\frac{\mathrm{d}\mathbf{s}_\perp}{\mathrm{d}t} = s_\perp \dot{\theta} \widehat{\mathbf{e}}_\theta = g\, \mathbf{s}_\perp \wedge \mathbf{B} = -g s_\perp B \widehat{\mathbf{e}}_\theta \quad .$$

Note the minus sign deriving from the vector product:

$$\widehat{\mathbf{e}}_r \wedge \mathbf{B} \sim -\widehat{\mathbf{e}}_\theta \quad .$$

The **Larmor Frequency** is thus:

$$\boxed{\dot{\theta} = -gB} \quad .$$

This is coherent with the results of problem § 15.05.09 as

$$g \Leftrightarrow \frac{q}{m\gamma} \quad .$$

15.05.11

✓ Center of Gravity

Consider an extended mass body with mass density $\rho[\mathbf{x}]$. Weight acts on every infinitesimal mass as $\mathrm{d}\mathbf{f} = \mathbf{g}\ \mathrm{d}m$.

Show that the total force and total momentum with respect to a pole O are:

$$\mathbf{F} = M\mathbf{g} \qquad \boldsymbol{\Gamma}_0 = M\,(\mathbf{X}_{\mathrm{CM}} - \mathbf{x}_0) \wedge \mathbf{g} \quad . \tag{15.05.10}$$

15.05.12

☒ Distributed Load

Su un segmento orizzontale AB di lunghezza L (asse x orizzontale) agisce un sistema continuo di vettori applicati, distribuito con densità $\mathbf{f}[x] = -p\,(x/L)\,\widehat{\mathbf{e}}_2$ dove la costante p ha le dimensioni di una forza ed x è la distanza da A del generico punto P di AB. Determinare risultante e momento risultante rispetto ad A del sistema di forze e determinare un sistema di vettori applicati equivalente formato da un solo vettore.

15.05.13

Forces and Torques Produced by a Cylindrically Symmetrical Mass Distribution

Read § 29.05.01.02.

15.05.14

✓ General Elliptic Polarization for a transverse wave

Read § 46.03.01.03.

WORK IN PROGRESS

15.05.15

✓ Hubble's law

E.Hubble discovered that distant galaxies are receding from the Earth with a radial velocity proportional to their distance from the Earth:

$$\mathbf{v}_i = H_0 \left(\mathbf{r}_i - \mathbf{r}_E \right) \qquad \text{Hubble's law} \quad , \tag{15.05.11}$$

for any distant galaxy i, where $\mathbf{R}_E$ is the Earth position.

Demonstrate that Hubble's law does not imply that the Earth is at the center of the universe showing that taking any the galaxy as the observer|origin Hubble's law is still obeyed.

15.05.16

The Levi-Civita symbol in 2D

This § is referenced at pages:
[1073]

In two dimensions, the **Levi-Civita symbol** is defined by:

$$\epsilon_{ij} = \begin{cases} +1 & \text{if } (i,j) = (1,2) \\ -1 & \text{if } (i,j) = (2,1) \\ 0 & \text{if } i = j \end{cases} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \quad . \tag{15.05.12}$$

Read § 17.04.

15.05.17

The Levi-Civita symbol in 3D

This § is referenced at pages:
[1074]

See equation (15.02.01).
Read § 17.04.

T.R.A.F.T.W. - IN PROGRESS

15.06 Exercises Problems and Physical Applications

©|M.R.Spiegel *et al.*, *Vector Analysis*, 2009, McGraw-Hill, 2ndEd., ...||*A lot of problems on Vector Algebra and Calculus ...*

15-005 ✓ Diagonali Interne Di Un Cubo

Determinare l'angolo θ formato da due diagonali interne di un cubo.

WORKED SOLUTION

$$\mathbf{a} = +\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \quad , \mathbf{b} = +\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3 \implies \cos[\theta] = \frac{\mathbf{a}\cdot\mathbf{b}}{ab} = \frac{1}{3}$$

15-006 ☒ Definition of the Levi-Civita symbol in 3D Cartesian Coordinates

Demonstrate the equivalence of the two expressions in equation (15.02.02).

15-007 A closed expression of the Levi-Civita symbol in 3D

This § is referenced at pages:
[Never referenced.]

Show that the formula for **Levi-Civita symbol** in 3D Cartesian Coordinates is:

$$\epsilon^{\dots}_{ijk} \equiv \frac{(i-j)\,(j-k)\,(k-i)}{2} \quad .$$

15-008 ☒ Properties of the Levi-Civita symbol in 2D

This § is referenced at pages:
[Never referenced.]

Read § 15.05.16.
Show, possibly by explicit calculation, the following properties:

$$\epsilon_{ij}\epsilon^{mn} = \delta_i{}^m\delta_j{}^n - \delta_i{}^n\delta_j{}^m \quad , \tag{15.06.01}$$

$$\epsilon_{ij}\epsilon^{in} = \delta_j{}^n \quad , \tag{15.06.02}$$

$$\epsilon_{ij}\epsilon^{ij} = 2. \quad . \tag{15.06.03}$$

DRAFT IN PROGRESS

15-009 ☒ Properties of the Levi-Civita symbol in 3D

This § is referenced at pages:
[1055, 1133]

Read § 15.05.17.
Show, possibly by explicit calculation, the following properties:

$$\delta^{\cdot\cdot}_{ij}\epsilon^{\cdot\cdot\cdot}_{ijk} = 0 \quad ,$$
$$S^{\cdot\cdot}_{ij}\epsilon^{\cdot\cdot\cdot}_{ijk} = 0 \qquad \text{for any symmetrical object, such that: } S^{\cdot\cdot}_{ij} = S^{\cdot\cdot}_{ji} \quad ,$$
$$\epsilon^{\cdot\cdot\cdot}_{ijk}\epsilon^{\cdot\cdot\cdot}_{ijk} = 6 \quad ,$$
$$\boxed{\epsilon^{\cdot\cdot\cdot}_{ijk}\epsilon^{\cdot\cdot\cdot}_{pqk} = \delta^{\cdot\cdot}_{ip}\delta^{\cdot\cdot}_{jq} - \delta^{\cdot\cdot}_{iq}\delta^{\cdot\cdot}_{jp}} \quad .$$

15-010 Properties of the Levi-Civita symbol for hyper-volume calculations

This § is referenced at pages:
[Never referenced.]

1. Show how the 2D **Levi-Civita symbol** can be used to calculate the hyper-volume in 2 dimensions, that is area, by using it to express the area spanned by two vectors. Read § 15.03.02.02
2. Show how the 3D **Levi-Civita symbol** can be used to calculate the hyper-volume in 3 dimensions, that is volume, by using it to express the volume spanned by three vectors. Read § 15.03.03.01.

15-011 ☒ Perpendicularity of Vectors

Show that two vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if:

$$|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}| \quad .$$

15-012 ✓ Some Algebraic Vector Identities

Let the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{p}$ and the unit vector $\widehat{\mathbf{n}}$.
Show that:

$$|\,\mathbf{n}\wedge\mathbf{p}| = |\,\mathbf{n}\wedge(\,\mathbf{p}\wedge\mathbf{n})| \quad .$$

WORKED SOLUTION

Show that:

$$|\,\mathbf{a}\wedge\mathbf{b}|^2 = a^2b^2 - (\,\mathbf{a}\cdot\mathbf{b})^2 \quad .$$

WORKED SOLUTION

Show that:

$$(\,\mathbf{a}\wedge\mathbf{b})\cdot(\,\mathbf{a}\wedge\mathbf{c}) = a^2\,(\,\mathbf{b}\cdot\mathbf{c}) - (\,\mathbf{a}\cdot\mathbf{c})\,(\,\mathbf{a}\cdot\mathbf{b}) \quad . \qquad (15.06.04)$$

→ 1080

WORKED SOLUTION

Deriva dalla (15.03.09) e dalla (15.03.03.02).

Show that:

$$(\,\mathbf{a}\wedge\mathbf{b})\wedge(\,\mathbf{a}\wedge\mathbf{c}) = \mathbf{a}\,(\,\mathbf{c}\cdot(\,\mathbf{a}\wedge\mathbf{b})) \quad . \qquad (15.06.05)$$

→ 1080

BRIEF NEW-PROGRESS

WORKED SOLUTION

Deriva dalla (15.03.09) e dalla (15.03.03.02).

Show that:

$$(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \quad . \tag{15.06.06}$$

WORKED SOLUTION

Deriva dalla (15.03.09) e dalla (15.03.03.02).

Show that:

$$(\mathbf{a}\wedge\mathbf{b})\wedge(\mathbf{c}\wedge\mathbf{d}) = \mathbf{b}(\mathbf{a}\cdot(\mathbf{c}\wedge\mathbf{d})) - \mathbf{a}(\mathbf{b}\cdot(\mathbf{c}\wedge\mathbf{d})) = \mathbf{c}(\mathbf{d}\cdot(\mathbf{a}\wedge\mathbf{b})) - \mathbf{d}(\mathbf{c}\cdot(\mathbf{a}\wedge\mathbf{b})) \quad . \tag{15.06.07}$$

→ 1077

WORKED SOLUTION

Deriva dalla (15.03.09) e dalla (15.03.03.02).

Show that:

$$(\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}))\wedge\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\wedge\mathbf{c}) \quad . \tag{15.06.08}$$

WORKED SOLUTION

Show that:

$$(\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}))\cdot(\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})) = (\mathbf{a}\cdot\mathbf{c})^2 b^2 + (\mathbf{a}\cdot\mathbf{b})^2 b^2 c^2 - 2(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{a}) \quad . \tag{15.06.09}$$

WORKED SOLUTION

Show that:

$$\mathbf{a}\wedge(\mathbf{b}\wedge(\mathbf{c}\wedge\mathbf{d})) = \mathbf{b}(\mathbf{a}\cdot(\mathbf{c}\wedge\mathbf{d})) - \mathbf{a}\cdot\mathbf{b}\,\mathbf{c}\wedge\mathbf{d} \quad . \tag{15.06.10}$$

→ 2252

WORKED SOLUTION

15-013 ✓ Distance of a Point From a Line

This § is referenced at pages:
[1142]

Consider a line passing through the point O, whose position vector is $\mathbf{r}_0$, and is parallel to the unit vector $\widehat{\mathbf{n}}$. If the point P has position vector $\mathbf{r}$ demonstrate that the distance of P from the line is:

$$\boxed{d = |(\mathbf{r}-\mathbf{r}_0)\wedge\widehat{\mathbf{n}}|} \quad .$$

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15-014 ✓ Distance of a Point From a Plane

This § is referenced at pages:
[Never referenced.]

Write a formula for the distance of a point from a plane as a function of the three vectors, $\widehat{\mathbf{n}}$, $\mathbf{x}_0$, $\mathbf{r}_0$:

$$0 = \widehat{\mathbf{n}}\cdot(\mathbf{x}-\mathbf{x}_0) \qquad \mathbf{r}_0 \quad , \tag{15.06.11}$$

defining the point and the plane.

15-015 ✓ Distance Between Two Lines

This § is referenced at pages:
[1142, 1601]

Consider a line that passes through the point (1), and is parallel to the vector $\widehat{\mathbf{n}}_1$, and a second line passes through (2), and is parallel to the vector $\widehat{\mathbf{n}}_2$.

If $\widehat{\mathbf{n}}_1$ and $\widehat{\mathbf{n}}_2$ and are not parallel show that the distance between the lines is:

$$\boxed{d = \frac{|\,(\mathbf{r}_1-\mathbf{r}_2)\cdot(\,\widehat{\mathbf{n}}_1\wedge\widehat{\mathbf{n}}_2)|}{|(\,\widehat{\mathbf{n}}_1\wedge\widehat{\mathbf{n}}_2)|}} \quad .$$

If $\widehat{\mathbf{n}}_1$ and $\widehat{\mathbf{n}}_2$ and are parallel ($\widehat{\mathbf{n}}_1 = \widehat{\mathbf{n}}_2 \equiv \widehat{\mathbf{n}}$) show that the distance between the lines is:

$$\boxed{d_{\parallel} = \frac{|\,(\mathbf{r}_1-\mathbf{r}_2)\cdot\widehat{\mathbf{n}}|}{|\widehat{\mathbf{n}}|}} \quad .$$

15-016 ✓ Coplanarity of Two Straight Lines

This § is referenced at pages:
[1142]

Let

$$\mathbf{r}_1[t] = \mathbf{r}_{1|0} + \mathbf{v}_1 t \quad ,$$
$$\mathbf{r}_2[t] = \mathbf{r}_{2|0} + \mathbf{v}_2 t \quad .$$

Show that they are co-planar if and only if:

$$\Delta\mathbf{r}\cdot(\,\mathbf{v}_2\wedge\mathbf{v}_1) = 0 \quad ,$$

for any $\Delta\mathbf{r} \equiv \mathbf{r}_2[t_2] - \mathbf{r}_1[t_1]$.

WORKED SOLUTION

Co-planarity condition for 3 vectors: $\mathbf{a}\cdot(\,\mathbf{b}\wedge\mathbf{c}) = 0$.

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CC BY NC ND

15-017 Angle between Two planes

Show how to use equation (15.06.07) to write an expression for the angle between two planes.

15-018 ✓ Direction Cosines

This § is referenced at pages:
[Never referenced.]

Given any fixed vector in a Cartesian Coordinate System:

1. show that the sum of the squares of the cosines of the angles between the vector and the three unit vectors (axes direction cosines) is: 1;
2. show that the sum of the squares of the cosines of the angles between the vector and the three coordinate planes (plane direction cosines) is: 2.

15-019 Division of a Segment

This § is referenced at pages:
[1142]

©|M.R.Spiegel *et al.*, *Vector Analysis*, 2009, McGraw-Hill, 2ndEd., ...|1.54 1.55||

Consider the two points P and Q, defined by the position vectors $\mathbf{r}_P$ and $\mathbf{r}_Q$. Show that the point R that divides the segment PQ into two parts whose lengths are in the ratio $\overline{PR} \div \overline{QR} = \alpha \div \beta$ is given by:

$$\mathbf{r}_R = \frac{\alpha \mathbf{r}_P + \beta \mathbf{r}_Q}{\alpha + \beta} \quad .$$

Show the position vector $\mathbf{r}_R$ is independent from the choice of the origin of the Coordinate System.

15-020 Identità Trigonometrica

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|3.29||

Sia N un numero intero, $N > 1$. Dimostrare che

$$\sum_{k=0}^{k=N-1} \cos\left[\frac{2\pi k}{N}\right] = 0 \quad ,$$

$$\sum_{k=0}^{k=N-1} \sin\left[\frac{2\pi k}{N}\right] = 0 \quad .$$

WORKED SOLUTION

Si consideri la somma di N vettori applicati nell'origine e aventi lo stesso modulo ciascuno dei quali forma un angolo $2\pi/N$ con il precedente.

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15-021 ✓ A Triangle

This § is referenced at pages:
[Never referenced.]

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|2.11||
©|M.R.Spiegel *et al.*, *Vector Analysis*, 2009, McGraw-Hill, 2ndEd., ...|2.34||

A triangle, that is a plane figure with three vertexes and three sides is the simplest plane figure, that is the one with the smaller number of sides and vertexes. Denote by $\boldsymbol{\ell}_k$, with $k = 1, \dots 3$, the three vectors describing the three sides.

1. Show that one has:
$$\sum_{k=1}^{3} \boldsymbol{\ell}_k = 0 \quad . \tag{15.06.12}$$
This result can be generalised to generic closed polygon and, in the limiting case, to any closed curve.
2. Consider, in the plane, the three vectors perpendicular to each side and module proportional to the length of the side. Show that they sum to zero

15-022 ✓ A Tetrahedron

This § is referenced at pages:
[Never referenced.]

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|2.11||
©|M.R.Spiegel *et al.*, *Vector Analysis*, 2009, McGraw-Hill, 2ndEd., ...|2.34||

A tetrahedron, that is a solid figure with four vertexes and four triangular faces is the simpler solid figure, that is the one with the smaller number of sides, vertexes and faces. Denote by $\mathbf{a}_k$, with $k = 1, \dots 4$, the four vectors describing the four faces, according to a straightforward generalisation of equation (15.03.05): the vector is normal to the triangular face and its modulus is equal to the are of the face.

Show that one has:
$$\sum_{k=1}^{4} \mathbf{a}_k = 0 \quad . \tag{15.06.13}$$
This result can be generalised to generic closed polyhedra and, in the limiting case, to any closed surface; read § 16-005.

WORKED SOLUTION

The tetrahedron can be generated by any of the three sides with a common vertex, by the three position vectors originating at the common vertex and ending on the opposite vertex of each side. Express $\mathbf{a}_k$ in terms of the six sides via (15.03.04), (15.03.05) and sum.
Consider any closed regular surface approximated by triangulation. Consider any triangular face, an additional point on the surface and the corresponding tetrahedron. Thanks to the relation demonstrated for the tetrahedron, the original face has a surface vector equal to the sum of the three new faces.
By induction, the property is true for four faces and if true for n faces is also true for $n + 1$.

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15-023 ✓ A corner reflector

This § is referenced at pages:
[Never referenced.]

Read § 48.02.

Three mutually perpendicular plane reflecting surfaces are hit by a ray of light striking all three surfaces and reflected back. Show that it is reflected back along a direction parallel to the direction of incidence.

15-024 Random Walk

This § is referenced at pages:
[2753]

©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|problem 2.19||

A point follows a path in space that consists of N equal steps, each of length s. The direction in space of each step is entirely random, with no relation or correlation between any two steps.

Show that the mean square displacement between the initial and the final positions is: $\langle S^2 \rangle = Ns^2$.

WORKED SOLUTION

The assumption that the direction of every step is independent of the direction of every other step means that $\langle \mathbf{s}_k \cdot \mathbf{s}_j \rangle = s^2 \delta_{kj}^{\cdot\cdot}$.

15-025 Interaction Energy Between Two Dipoles

This § is referenced at pages:
[1761]

Read § 36.20.12.

The interaction energy between two dipoles is given in equation (36.20.12).

Show that it can be written:

$$\mathcal{E}[\mathbf{r} \equiv (\mathbf{r}_2 - \mathbf{r}_1), \mathbf{d}_1, \mathbf{d}_2] \propto \frac{|\mathbf{d}_1||\mathbf{d}_2|}{r^3}(2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\phi) \quad , \tag{15.06.14}$$

where θ_1 and θ_2 are the polar angles of $\mathbf{d}_1$ and $\mathbf{d}_2$ relative to $\mathbf{r}$, while ϕ is the azimuth of the azimuth of $\mathbf{d}_2$ relative to the $\mathbf{d}_1$-+$\mathbf{r}$ plane or the azimuth of $\mathbf{d}_1$ relative to the $\mathbf{d}_2$-−$\mathbf{r}$ plane.

WORKED SOLUTION

The addition theorem for spherical harmonics can be useful.

DRAFT - ROUGH

15-026 Magnetic Force between two non relativistic point charges and Newton third law

This § is referenced at pages:
[2394]

15-027 Sine Theorem in Spherical Trigonometry

This § is referenced at pages:
[1049]

©|E.W.Weisstein - **Spherical Trigonometry**. From MathWorld – A Wolfram Web Resource|WEB - URL ||

©|M.R.Spiegel *et al.*, *Vector Analysis*, 2009, McGraw-Hill, 2ndEd., ...|2.51||

See WEB - URL.

Prove the Sine Theorem in spherical trigonometry using equations (15.06.04), (15.06.05).

15-028 Cosine Theorem in Spherical Trigonometry

This § is referenced at pages:
[1049]

©|E.W.Weisstein - **Spherical Trigonometry**. From MathWorld – A Wolfram Web Resource|WEB - URL ||

©|M.R.Spiegel *et al.*, *Vector Analysis*, 2009, McGraw-Hill, 2ndEd., ...|2.101||

See WEB - URL.

Prove the Cosine Theorem in spherical trigonometry using equations (15.06.04), (15.06.05).

15-029 Metodo Del Raddoppio Dell'angolo Di Prua

©|H.C.Ohanian, , ..., ..., ...Ed., ...|3.12||

Un possibile metodo per misurare la distanza di una nave da un faro è il seguente. Si supponga che la nave si muova su una traiettoria rettilinea e si misuri l'angolo α tra la traiettoria della nave e la direzione in cui è visto il faro. Si misuri lo spazio percorso dalla nave dall'istante in cui si misura l'angolo α fino a quando lo stesso angolo, tra la traiettoria della nave e la direzione in cui è visto il faro vale, 2α. Dimostrare che la distanza dal faro in questo istante vale quanto lo spazio percorso dalla nave tra le due misure.

UNDER PEER REVIEW - IN PROGRESS

15-030 Property of the Ellipse

Consider an ellipses with foci F_1 and F_2. Let P be any point of the ellipses.

1. Show that the vectors $\overrightarrow{\mathbf{PF_1}}$ and $\overrightarrow{\mathbf{PF_2}}$ form equal angles with the tangent to the ellipse.
2. Interpret the result in relation with Snell law of reflection by describing what happens if a point light source is placed at one of the foci of an elliptical mirror.

15-031 Reciprocal Vectors

©|M.R.Spiegel *et al.*, *Vector Analysis*, 2009, McGraw-Hill, 2ndEd., ...|1.11 2.53 2.54 2.103 2.104||
©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|||

Reciprocal vectors enter in the study of crystalline solids.

Consider three non-coplanar vectors, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Express any vector $\mathbf{v}$ as a linear combination of the vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$:

$$\mathbf{v} = a\mathbf{A} + b\mathbf{B} + c\mathbf{C} \qquad \mathbf{A}\cdot(\mathbf{B}\wedge\mathbf{C}) = 0 \quad .$$

1. Consider the reciprocal vectors $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$ defined as:

$$\mathbf{A}' \equiv \frac{\mathbf{B}\wedge\mathbf{C}}{\mathbf{A}\cdot(\mathbf{B}\wedge\mathbf{C})} \quad ,$$

$$\mathbf{B}' \equiv \frac{\mathbf{C}\wedge\mathbf{A}}{\mathbf{A}\cdot(\mathbf{B}\wedge\mathbf{C})} \quad ,$$

$$\mathbf{C}' \equiv \frac{\mathbf{A}\wedge\mathbf{B}}{\mathbf{A}\cdot(\mathbf{B}\wedge\mathbf{C})} \quad .$$

2. Show that:

$$\mathbf{A}\cdot\mathbf{A}' = \mathbf{B}\cdot\mathbf{B}' = \mathbf{C}\cdot\mathbf{C}' = 1 \quad ,$$
$$\mathbf{B}\cdot\mathbf{A}' = \mathbf{C}\cdot\mathbf{A}' = \mathbf{C}\cdot\mathbf{B}' = \mathbf{A}\cdot\mathbf{B}' = \mathbf{A}\cdot\mathbf{C}' = \mathbf{B}\cdot\mathbf{C}' = 0 \quad ,$$

.

3. Show that the volume of the parallelepiped spanned by the three vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ is the reciprocal of the volume of the parallelepiped spanned by the three reciprocal vectors.
4. Show that:

$$a = \mathbf{v}\cdot\mathbf{A}' \quad ,$$
$$b = \mathbf{v}\cdot\mathbf{B}' \quad ,$$
$$c = \mathbf{v}\cdot\mathbf{C}' \quad .$$

5. Show that the set of vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are the reciprocal vectors of the set $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$.
6. Show that the only right handed set of vectors which are self-reciprocal are the unit vectors $\widehat{\mathbf{e}}_1$, $\widehat{\mathbf{e}}_2$ and $\widehat{\mathbf{e}}_3$.

15-032 System of Forces

Find examples of a system of forces having zero total force and zero total momentum but capable to determine a strain and|or induce a deformation on a system.

WORKED SOLUTION

Some examples follow.

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- – $\mathbf{f}_1 = -2a\widehat{\mathbf{e}}_3$, applied in $\{0, 0, 0\}$;
  – $\mathbf{f}_2 = a\widehat{\mathbf{e}}_3$, applied in $\{+1, 0, 0\}$;
  – $\mathbf{f}_3 = a\widehat{\mathbf{e}}_3$, applied in $\{-1, 0, 0\}$.
- Consider a spherical mass of plasma like a star. Any shell spherical shell suffers a pressure form the external layers. The total force and total torque with respect to the center are zero but the external layers tend to compress the spherical shell of plasma.
- Consider a spring which is acted by two opposite and identical forces at its extremes.
- Consider the surface tension of an inflated ball.

DRAFT-IN-PROGRESS

CHAPTER 16

Elements of Vector Calculus

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16.13 Exercises Problems and Physical Applications 1133

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Best single reference:

©|P.J.Olver|Vector Calculus in 3 dimensions - WEB - URL |Excellent: all and only what is needed ...

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|||

©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|||

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|||

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1; 2||

©|T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...||*Excellent*|

©|P.J.Olver|Vector Calculus in 3 dimensions - WEB - URL |*Excellent: all and only what is needed here.*|

©|R.B.McQuistan|Scalar and Vector Fields: A Physical Interpretation|*Eccellente introduzione avanzata*|

©|WEB - URL |Mathematical Tools for Physics, James Nearing|*Excellent!*|

©|G.B.Arfken & H.J.Weber, *Mathematical Methods For Physicists*, 2005, Elsevier, 6thEd., ...|||

©|WEB - URL |Schaum Outline of Advanced Calculus, 3rd Ed., By R.C.Wrede, M.Spiegel|*Exercises/problems*|

©|WEB - URL |Vector analysis and an introduction to tensor analysis, By S.Lipschutz, M.R.Spiegel, D.Spellma- ...

©|S.Mauch|WEB - URL |*Excellent set of lecture notes on applied mathematics*|

©|S.J.Cowley|WEB - URL |*Excellent*|

©|J.D.Romano and R.H.Price - *Why no shear in "Div, grad, rotor, and all that"?* |WEB - URL |*A MUST*|

©|WEB - URL |ElectroMagnetic Field and Energy - H.A.Haus & J.R.Melcher|*A must...*|

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ALGEBRA E GEOMETRIA

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ANALISI MATEMATICA

16.01 Introduction

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|2||

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 2,3||

Vector calculus is part of the basic language of physics; vector calculus is in fact a subset of **tensor calculus, the basic language of physics**.

It is therefore necessary to master the basics of vector calculus in order to understand and use physical laws in the best way.

Necessarily and on purpose, I shall often rely on the intuition of the reader to formalize the ideas, leaving a rigorous development to a more comprehensive mathematical treatment, which is out-of-context for physics.

In general, **assume all functions to be sufficiently regular**, such as C^∞, unless specified otherwise.

Heuristically, carry on derivation/integration of vectors as follows. Use an **orthonormal Cartesian Coordinates Coordinate System** and write any vector in components; then take the derivative|integral as if every components were a scalar, for all the components, while keeping the Cartesian unit vectors fixed. By definition, this is the derivative|integral of a vector, in a **orthonormal Cartesian Coordinates Coordinate System**.

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16.02 Calculus in Many Variables - Differentiation - Cartesian Coordinates

©|R.B.McQuistan|Scalar and Vector Fields: A Physical Interpretation|*Eccellente introduzione avanzata*|

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- Partial derivative, easy.
- Differentiation (more difficult than derivation).
- Differentiation and derivation are not the same: the former implies the latter, not vice-versa.
- Vector gradient, shear, rotor, divergence: read § 16.06.01.01.
- Second derivatives: quadrupole moments.
- Taylor formula in many variables, scalar:

$$\boxed{\phi[\mathbf{x}] \simeq \phi[\mathbf{x}_0] + (\mathbf{x}-\mathbf{x}_0)\cdot\boldsymbol{\nabla}\phi[\mathbf{x}_0]} \quad .$$

  Example: potential energy, stability of small oscillations.
- Taylor formula in many variables, vector:

$$\boxed{V_j[\mathbf{x}] \simeq V_j[\mathbf{x}_0] + (\mathbf{x}-\mathbf{x}_0)\cdot\boldsymbol{\nabla}V_j[\mathbf{x}_0] \implies \mathbf{V}[\mathbf{x}] \simeq \mathbf{V}[\mathbf{x}_0] + (\mathbf{x}-\mathbf{x}_0)\cdot\boldsymbol{\nabla}\mathbf{V}[\mathbf{x}_0]} \quad .$$

  Example: force|torque on a **Electric Dipole** | **Magnetic Dipole** in external field. Read § 36.05.06.01 and § 36.05.06.02

By definition:

$$\boxed{(\mathbf{x}-\mathbf{x}_0)\cdot\boldsymbol{\nabla} \equiv (\mathbf{x}-\mathbf{x}_0)_j\,(\boldsymbol{\nabla})_j \equiv \sum_j (\mathbf{x}-\mathbf{x}_0)_j\,(\boldsymbol{\nabla})_j} \quad . \tag{16.02.01}$$

Examples in physics:

-

16.03 Calculus in Many Variables - Integration - Cartesian Coordinates

©|R.B.McQuistan|Scalar and Vector Fields: A Physical Interpretation|*Eccellente introduzione avanzata*|

It is exceedingly important to understand the concept definitions of integrals in this section, because it is the way integration is written down in physics.

16.03.01
Integration on a 1D 2D 3D Domain

- Integral in n dimensions of a scalar:

$$\int f[x]\,\mathrm{d}x \equiv \lim_{\max\{\Delta L(x)\}\to 0}\sum_k \Delta L(x)_k f[x_k] \qquad \Delta L(x)_k \equiv (\Delta x)_k \quad ,$$

$$\iint f[x,y]\,\mathrm{d}x\,\mathrm{d}y \equiv \lim_{\max\{\Delta S(x,y)\}\to 0}\sum_k \Delta S(x,y)_k f[x_k,y_k] \qquad \Delta S(x,y)_k \equiv (\Delta x\Delta y)_k \quad ,$$

$$\iiint f[x,y,z]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \equiv \lim_{\max\{\Delta V(x,y,z)\}\to 0}\sum_k \Delta V(x,y,z)_k f[\mathbf{x}_k] \qquad \Delta V(x,y,z)_k \equiv (\Delta x\Delta y\Delta z)_k \quad .$$

  Examples:
  - line, surface, volume: charge and|or mass;
  - quadrupole tensors, equation (36.14.04), (36.14.05).

- Integral in n dimensions of a vector:

$$\int \mathbf{V}[x]\,\mathrm{d}x \equiv \lim_{\max\{\Delta L(x)\}\to 0}\sum_k \Delta L(x)_k \mathbf{V}[x_k] \qquad \Delta L(x)_k \equiv (\Delta x)_k \quad ,$$

$$\iint \mathbf{V}[x,y]\,\mathrm{d}x\,\mathrm{d}y \equiv \lim_{\max\{\Delta S(x,y)\}\to 0}\sum_k \Delta S(x,y)_k \mathbf{V}[x_k,y_k] \qquad \Delta S(x,y)_k \equiv (\Delta x\Delta y)_k \quad ,$$

$$\iiint \mathbf{V}[x,y,z]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \equiv \lim_{\max\{\Delta V(x,y,z)\}\to 0}\sum_k \Delta V(x,y,z)_k \mathbf{V}[\mathbf{x}_k] \qquad \Delta V(x,y,z)_k \equiv (\Delta x\Delta y\Delta z)_k \quad .$$

  Examples:
  - average Electric|Magnetic field in matter;
  - position of the Center-Of-Mass;
  - force on a volume current;
  - **Electric Dipole** | **Magnetic Dipole** , equation (36.14.06), (36.14.09).

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16.03.02
Integration on a 1D Curve

Infinitesimal vector line element, $\mathbb{d}\mathbf{L}$, as a function of the curve parameterization, $\mathbf{x}[w]$:

$$\mathbb{d}\mathbf{L} = \pm\mathcal{L}\widehat{\mathbf{n}}\,\mathbb{d}w \qquad \widehat{\mathbf{n}} \equiv \frac{\partial_w \mathbf{x}[w]}{|\partial_w \mathbf{x}[w]|} = \qquad \mathcal{L} \equiv |\partial_w \mathbf{x}[w]| \quad . \tag{16.03.01}$$

16.03.02.01 Integration on a 1D Curve in Space

This § is referenced at pages:
[1090]

- Line integrals, from integration on a straight line to integration on a generic curve:

$$\int_C f[\mathbf{x}]\;\mathbb{d}L \equiv \lim_{\max\{\Delta L(\mathbf{x})\}\to 0} \sum_k f[\mathbf{x}_k]\Delta L(\mathbf{x}_k) \quad .$$

  Similar for a vector:

$$\int_C \mathbf{V}[\mathbf{x}]\;\mathbb{d}L \equiv \lim_{\max\{\Delta L(\mathbf{x})\}\to 0} \sum_k \mathbf{V}[\mathbf{x}_k]\Delta L(\mathbf{x}_k) \quad .$$

  Examples:
  - length of a curve;
  - line charge and|or mass;
  - Center-Of-Mass;

- Line integrals, from integration on a straight line to integration on a generic curve, with vector line elements:

$$\int_C f[\mathbf{x}]\;\mathbb{d}\mathbf{L} \equiv \lim_{\max\{\Delta\mathbf{L}(\mathbf{x})\}\to 0} \sum_k f[\mathbf{x}_k]\Delta\mathbf{L}(\mathbf{x}_k) \quad .$$

  Examples:
  - ...
  - ...

- Line integrals, from integration on a straight line to integration on a generic curve, with vector line elements:

$$\int_C \mathbf{V}[\mathbf{x}]\cdot\,\mathbb{d}\mathbf{L} \equiv \lim_{\max\{\Delta\mathbf{L}(\mathbf{x})\}\to 0} \sum_k \mathbf{V}[\mathbf{x}_k]\cdot\Delta\mathbf{L}(\mathbf{x}_k) \quad .$$

  Examples:
  - circulation of a vector field;
  - work of a force.

- Line integrals, from integration on a straight line to integration on a generic curve, with vector line elements:

$$\int_C \mathbf{V}[\mathbf{x}]\wedge\,\mathbb{d}\mathbf{L} \equiv \lim_{\max\{\Delta\mathbf{L}(\mathbf{x})\}\to 0} \sum_k \mathbf{V}[\mathbf{x}_k]\wedge\Delta\mathbf{L}(\mathbf{x}_k) \quad .$$

  Examples:
  - Biot-Savart law for a wire (36.06);
  - force on a line current.

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16.03.03
Integration on a 2D Surface

Infinitesimal vector surface element, $\mathbb{d}\mathbf{S}$, as a function of the surface parameterization, $\mathbf{x}[p,q]$:

$$\mathbb{d}\mathbf{S} = \pm \mathcal{S}\widehat{\mathbf{n}}\,\mathbb{d}p\,\mathbb{d}q = \qquad \widehat{\mathbf{n}} \equiv \frac{\partial_p\mathbf{x}[p,q]\wedge\partial_q\mathbf{x}[p,q]}{\left|\partial_p\mathbf{x}[p,q]\wedge\partial_q\mathbf{x}[p,q]\right|} \qquad \mathcal{S} \equiv \left|\partial_p\mathbf{x}[p,q]\wedge\partial_q\mathbf{x}[p,q]\right| \quad . \tag{16.03.02}$$

16.03.03.01 Integration on a 2D Surface in Space

This § is referenced at pages:
[Never referenced.]

- Surface integrals, from integration on a plane to integration on a generic surface:
$$\iint_\Sigma f[\mathbf{x}]\;\mathbb{d}S \equiv \lim_{\max\{\Delta S(\mathbf{x})\}\to 0}\sum_k f[\mathbf{x}_k]\Delta S(\mathbf{x}_k) \quad .$$
Similar for a vector:
$$\iint_\Sigma \mathbf{V}[\mathbf{x}]\;\mathbb{d}S \equiv \lim_{\max\{\Delta S(\mathbf{x})\}\to 0}\sum_k \mathbf{V}[\mathbf{x}_k]\Delta S(\mathbf{x}_k) \quad .$$
Examples:
  - area of a surface;
  - surface charge and|or mass;
  - position of the Center-Of-Mass;
- Surface integrals, from integration on a plane to integration on a generic surface, with vector surface elements:
$$\iint_\Sigma f[\mathbf{x}]\;\mathbb{d}\mathbf{S} \equiv \lim_{\max\{\Delta\mathbf{S}(\mathbf{x})\}\to 0}\sum_k f[\mathbf{x}_k]\Delta\mathbf{S}(\mathbf{x}_k) \quad .$$
Examples:
  - force from pressure.
  - ..
- Surface integrals, from integration on a plane to integration on a generic surface, with vector surface elements:
$$\iint_\Sigma \mathbf{V}[\mathbf{x}]\cdot\mathbb{d}\mathbf{S} \equiv \lim_{\max\{\Delta\mathbf{S}(\mathbf{x})\}\to 0}\sum_k \mathbf{V}[\mathbf{x}_k]\cdot\Delta\mathbf{S}(\mathbf{x}_k) \quad .$$
Examples:
  - flux of a vector field;
  - current density, equation (36.03.08).
- Surface integrals, from integration on a plane to integration on a generic surface, with vector surface elements:
$$\iint_\Sigma \mathbf{V}[\mathbf{x}]\wedge\mathbb{d}\mathbf{S} \equiv \lim_{\max\{\Delta\mathbf{S}(\mathbf{x})\}\to 0}\sum_k \mathbf{V}[\mathbf{x}_k]\wedge\Delta\mathbf{S}(\mathbf{x}_k) \quad .$$
Examples:
  - Biot-Savart law for a surface current (36.06);
  - force on a surface current.

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16.03.04
Integration on a 1D Curve on a 2D Surface

This § is referenced at pages:
[Never referenced.]

All is exactly as in section § 16.03.02.01 with the curve, C, belonging to the surface, Σ:

$$C \in \Sigma \quad . \tag{16.03.03}$$

Examples:

- Surface current density, equation (36.03.10).

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16.04 Curvilinear Coordinates Systems

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©|G.B.Arfken & H.J.Weber, *Mathematical Methods For Physicists*, 2005, Elsevier, 6thEd., ...|||

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- **General contravariant coordinates: defined as a distance along curves.**
- **General covariant coordinates: defined as a distance from surfaces.**

Read § 58.11.03.04.

16.04.01 General Orthonormal Coordinate System

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...||*Appendix A*|

No sum over repeated indexes in this section.

In this section the convention of sum of repeated indexes defined in section § 15.01.01 is temporarily suspended.

Let the three coordinates of a generic orthonormal Coordinate System in three-dimensional space be u^i, with $i = 1, 2, 3$.

Every point in space, $\mathbf{r}$, shall be described by the bi-univocal (at least locally) transformation:

$$\mathbf{r} = \mathbf{r}[u^1, u^2, u^3] \equiv \mathbf{r}[u^k] \quad ,$$

which functionally relates the coordinates of the generic orthonormal coordinate system to a orthonormal Cartesian Coordinates Coordinate System system.

The three unit vectors are defined in terms of the scale factors, h_i, as:

$$\mathbf{e}_i \equiv \frac{1}{h_i}\frac{\partial \mathbf{r}}{\partial u^i} \qquad \text{with} \qquad h_i \equiv \left|\frac{\partial \mathbf{r}}{\partial u^i}\right| \quad .$$

For an orthonormal Coordinate System the three unit vectors satisfy:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}^{\cdot\cdot} \qquad \mathbf{e}_i \wedge \mathbf{e}_j = \epsilon_{ijk}^{\cdot\cdot\cdot}\mathbf{e}_k \quad .$$

The infinitesimal displacement, $\mathbb{d}\mathbf{r}$, is:

$$\mathbb{d}\mathbf{r} = \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial u^i}\,\mathbb{d}u^i = \sum_{i=1}^{3} \mathbf{e}_i h_i\,\mathbb{d}u^i \quad .$$

The squared infinitesimal line element for the orthonormal Coordinate System becomes:

$$\mathbb{d}s^2 \equiv \mathbb{d}\mathbf{r}\cdot\,\mathbb{d}\mathbf{r} \equiv \sum_{i,j=1}^{3} \delta_{ij}^{\cdot\cdot} h_i h_j\,\mathbb{d}u^i\,\mathbb{d}u^j = \sum_{i=1}^{3} h_i^2\left(\mathbb{d}u^i\right)^2 \quad .$$

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The surface elements on the surfaces $u^s = \text{constant}$ are given by:

$$\mathbb{d}\boldsymbol{\Sigma}_1 = \left(\frac{\partial \mathbf{r}}{\partial u^2} \wedge \frac{\partial \mathbf{r}}{\partial u^3}\right) \mathbb{d}u^2\, \mathbb{d}u^3 = \mathbf{e}_1 h_2 h_3\, \mathbb{d}u^2\, \mathbb{d}u^3 \qquad u^1 = \text{constant} \quad ,$$

$$\mathbb{d}\boldsymbol{\Sigma}_2 = \left(\frac{\partial \mathbf{r}}{\partial u^3} \wedge \frac{\partial \mathbf{r}}{\partial u^1}\right) \mathbb{d}u^3\, \mathbb{d}u^1 = \mathbf{e}_2 h_3 h_1\, \mathbb{d}u^3\, \mathbb{d}u^1 \qquad u^2 = \text{constant} \quad ,$$

$$\mathbb{d}\boldsymbol{\Sigma}_3 = \left(\frac{\partial \mathbf{r}}{\partial u^1} \wedge \frac{\partial \mathbf{r}}{\partial u^2}\right) \mathbb{d}u^1\, \mathbb{d}u^2 = \mathbf{e}_3 h_1 h_2\, \mathbb{d}u^1\, \mathbb{d}u^2 \qquad u^3 = \text{constant} \quad .$$

In case two coordinates parameterize a surface (for instance u^s is constant) the $\mathbb{d}\boldsymbol{\Sigma}_s$ gives the area of the infinitesimal element of surface.

One can also write the above oriented projection on the plane perpendicular to the Cartesian axis k of the area spanned by an infinitesimal change of coordinates $\mathbb{d}u^p$ and $\mathbb{d}u^q$ as:

$$\boxed{\mathbb{d}\mathbf{S}_k = \sum_{r,s=1}^{3} \epsilon_{krs}^{\cdots} \frac{\partial r_r}{\partial u^p} \frac{\partial r_s}{\partial u^q}\, \mathbb{d}u^p\, \mathbb{d}u^q}$$

The infinitesimal volume element for the orthonormal Coordinate System becomes:

$$\boxed{\mathbb{d}V = h_1 h_2 h_3\, \mathbb{d}u^1\, \mathbb{d}u^2\, \mathbb{d}u^3} \quad .$$

16.04.01.01 Cartesian Coordinates Coordinate System

16.04.01.02 Cylindrical Coordinate System

$$\{r, \theta, z\} \implies h_r = 1 \qquad h_\theta = r \qquad h_z = 1 \quad .$$

See figure **??**.

16.04.01.03 Spherical Coordinate System

$$\{r, \theta, \phi\} \implies h_r = 1 \qquad h_\theta = r \qquad h_\phi = r \sin[\theta] \quad .$$

See figure **??**.

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16.05 Integral Operators for Vector Fields

This § is referenced at pages:
[Never referenced.]

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16.05.01 Circulation

This § is referenced at pages:
[Never referenced.]

Consider a vector field $\mathbf{V}[\mathbf{x}]$.
Its circulation around any line, C, which is partitioned by segments $\Delta\mathbf{L}_k$, is defined as:

$$\Gamma_C \equiv \int_C \mathbf{V}[\mathbf{x}]\cdot \mathbb{d}\mathbf{L} \equiv \lim_{\max\{|\Delta\mathbf{L}_k|\}\to 0} \sum_k \mathbf{V}_k[\mathbf{x}]\cdot\Delta\mathbf{L}_k \quad .$$

The circulation around a closed line, which is the boundary of a surface Σ ($C = \partial\Sigma$), is:

$$\Gamma_{C=\partial\Sigma} \equiv \oint_{C=\partial\Sigma} \mathbf{V}[\mathbf{x}]\cdot \mathbb{d}\mathbf{L} \quad .$$

Orientation of the curve and surface must be compatible with the usual **right-hand rule**.
If the surface is partitioned in two parts,

$$\Sigma_1 \quad \Sigma_2 \quad : \quad \Sigma = \Sigma_1 \cup \Sigma_2 \quad \& \quad \Sigma_1 \cap \Sigma_2 = \emptyset \quad ,$$

the circulation over C is the sum of the circulations over the boundaries of C_1 and C_2, as the common line gives two equal and opposite contributions to the two circulations:

$$\Gamma_{C=\partial\Sigma} = \Gamma_{C_1=\partial\Sigma_1} + \Gamma_{C_2=\partial\Sigma_2} \quad ,$$

provided consistent orientations for the lines are chosen.
This can be generalized to partitioning into any number of surfaces/circulations.

16.05.02 Flux

This § is referenced at pages:
[1004]

Consider a vector field $\mathbf{V}[\mathbf{x}]$.
Its flux across any surface, Σ, which is partitioned by triangles $\Delta\mathbf{S}_k$, is defined as:

$$\Phi_\Sigma \equiv \iint_\Sigma \mathbf{V}[\mathbf{x}]\cdot \mathbb{d}\mathbf{S} \equiv \lim_{\max\{|\Delta\mathbf{S}_k|\}\to 0} \sum_k \mathbf{V}_k[\mathbf{x}]\cdot\Delta\mathbf{S}_k \quad .$$

The flux across a closed surface, which is the boundary of a volume Ω ($\Sigma = \partial\Omega$), is:

$$\Phi_{\Sigma=\partial\Omega} \equiv \oiint_{\Sigma=\partial\Omega} \mathbf{V}[\mathbf{x}]\cdot \mathbb{d}\mathbf{S} \quad .$$

Orientation of the surface and volume must be compatible with the usual **right-hand rule**.
If the volume is partitioned in two parts,

$$\Omega_1 \quad \Omega_2 \quad : \quad \Omega = \Omega_1 \cup \Omega_2 \quad \& \quad \Omega_1 \cap \Omega_2 = \emptyset \quad ,$$

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the flux over Σ is the sum of the fluxes over the boundaries of Ω_1 and Ω_2, as the common surface gives two equal and opposite contributions to the two fluxes:

$$\Phi_{\Sigma=\partial\Omega} = \Phi_{\Sigma_1=\partial\Omega_1} + \Phi_{\Sigma_2=\partial\Omega_2} \quad ,$$

provided consistent orientations for the surfaces are chosen.

This can be generalized to partitioning into any number of volumes/fluxes.

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16.06 Linear First-Order Differential Operators

This § is referenced at pages:
[1118]

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©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 2, 2,3||
©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Appendix A||

Linear first-order differential operators can be thought as deriving from the attempt to derive point properties from the well-known integral properties:

- of scalar fields (the difference of the values at two points);
- of vector fields (flux and circulation).

In the case of vector fields in orthonormal Cartesian Coordinates Coordinate System one might consider the partial derivatives to get the Jacobian matrix. However, it has been found that some suitable combinations of the pure partial derivatives are much more useful. These gives rise to divergence, rotor and strain. The latter, while very important in general, is not relevant in ElectroMagnetism.

Moreover, it is necessary to generalize to the case of arbitrary Coordinate Systems.

16.06.01 General Definition of Linear First-Order Differential Operators in 3D

©|Invariant Definitions for Vector Calculus|WEB - URL ||

One can build three different linear first-order differential operators linking scalar and|or (at least one) vector functions:

- the gradient: scalar $\longrightarrow$ vector;
- the divergence: vector $\longrightarrow$ scalar, it is the flux volume density (in 3 dimensions) as flux is additive and therefore there exists a flux density;
- the rotor: vector $\longrightarrow$ vector, it is the circulation surface density (in 3 dimensions) as circulation is additive and therefore there exists a circulation density.

These operators are often introduced by giving their expression in Cartesian coordinates. However, in physics it is better to use a coordinate-independent description: a coordinate-independent definition of the differential operators is more useful. Moreover, it gives a direct physical interpretation.

Once one has introduced the differential operators in a coordinate-free way the three fundamental integral theorems of calculus (see section § 16.07) provide the expressions of the operators in the chosen Coordinate System.

In short: while always using a small (infinitesimal) sphere around the point; then:

- **the flux across a spherical surface per unit volume gives the divergence at the point**;
- **the circulation around a great circle per unit area gives the rotor component in the direction perpendicular to the circle at the point**.

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16.06.01.01 General First-Order Vector Derivatives

This § is referenced at pages:
[1086, 1101]

©|J.D.Romano and R.H.Price - *Why no shear in "Div, grad, rotor, and all that"?* |WEB - URL |*A MUST*|

Gradient:

$$\boxed{\frac{\partial V_j}{\partial x_i}} \quad . \tag{16.06.01}$$

Shear, traceless symmetrical second-order tensor:

$$\boxed{\frac{1}{2}\left(\frac{\partial V_r}{\partial x_s}+\frac{\partial V_s}{\partial x_r}\right)-\frac{1}{3}\delta_{rs}\mathbf{div}\,\mathbf{V}} \quad . \tag{16.06.02}$$

© - QUOTE

... remind the reader that div and rotor are not the entire story when it comes to vector derivatives ...
... After learning about div, grad, and rotor ..., one might end up with the impression that these differential operators contain all there is to know about the differential calculus of scalar and vector fields. But this is not the case. For example, the gradient of a vector field $\mathbf{V}$, or equivalently the derivative of $\mathbf{V}$ in the (arbitrary) direction of $\mathbf{n}$ at a point $\mathbf{x}$, is defined by:

$$\boxed{\boldsymbol{\nabla}_{\mathbf{n}}\mathbf{V}\equiv(\,\mathbf{n}\cdot\boldsymbol{\nabla}\,)\,\mathbf{V}\equiv\lim_{\epsilon\to 0}\left(\frac{\mathbf{V}[\mathbf{x}+\epsilon\mathbf{n}]-\mathbf{V}[\mathbf{x}]}{\epsilon}\right)} \quad . \tag{16.06.03}$$

This operation is not simply contained in the divergence and rotor operations. ... The reader should note that our statement about the necessity of knowing the gradient of a vector field does not conflict with Helmholtz theorem..... ... To see that the divergence and rotor of a vector field do not suffice to specify it... This leaves five components unaccounted for, and these correspond to a symmetric, trace-free derivative operation that has the physical interpretation of shear. ...

Read § 16.12.17.

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16.06.01.02 General Definition of the Gradient

This § is referenced at pages:
[1100]

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Consider a scalar field $U[\mathbf{x}]$. The gradient is defined by the relation:

$$\boxed{\mathbf{grad}U\cdot \mathbb{d}\mathbf{L} \equiv (\widehat{\mathbf{n}}\cdot\mathbf{grad}U)\ \mathbb{d}L \equiv\ \mathbb{d}U[\mathbf{x}] \qquad \text{a vector field}} \quad . \tag{16.06.04}$$

The projection of the gradient of a scalar field along any infinitesimal displacement, $\mathbb{d}\mathbf{L}$, at any given point, thus provides the infinitesimal change of the value of the scalar field along the infinitesimal displacement $\mathbb{d}\mathbf{L}$.

In detail, the projection of the grad of a scalar field along the direction $\widehat{\mathbf{n}}$ is defined as the limit, for infinitesimally small path starting from the given point along the direction $\widehat{\mathbf{n}}$, of the ratio

$$\boxed{\widehat{\mathbf{n}}\cdot\mathbf{grad}U[\mathbf{x}] = \lim_{\Delta L\longrightarrow 0}\frac{U[\mathbf{x}+\widehat{\mathbf{n}}\Delta L]-U[\mathbf{x}]}{\Delta L} \qquad \text{a vector field}} \quad , \tag{16.06.05}$$

for any path ΔL starting from the point P and directed along $\widehat{\mathbf{n}}$.

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16.06.01.03 General Definition of the Divergence

This § is referenced at pages:
[1107]

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©|T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...|Vol. 2, 2nd edition, § 12.20|*Excellent*|

©|WEB - URL |ElectroMagnetic Field and Energy - H.A.Haus & J.R.Melcher||

©|WEB - URL |ElectroMagnetic Field and Energy - H.A.Haus & J.R.Melcher|*Appendix 2*|

©|D.Styer|Geometrical definition of divergence and curl|*WEB - URL*|

Consider a vector field $\mathbf{V}[\mathbf{x}]$ and its flux across any closed surface, Σ:

$$\Phi_\Sigma \equiv \oiint_\Sigma \mathbf{V}\cdot d\mathbf{S} \quad .$$

In order to pick-up a property of the field in the neighborhood of some point, $\mathbf{x}$, consider the flux across the boundary (frontier) of smaller and smaller volumes enclosing the given point, that is, containing the given point.

As the flux is additive with respect to partitioning of the volume one may wonder what about the limit for infinitesimally small volume.

The divergence of a vector field is defined as the limit, for infinitesimally small volume containing the given point, of the ratio

$$\boxed{\mathbf{div}\,\mathbf{V}[\mathbf{x}] \equiv \lim_{\Delta V_\mathbf{x}\longrightarrow 0} \frac{1}{\Delta V_\mathbf{x}} \oiint_{\partial\Omega_\mathbf{x}} \mathbf{V}\cdot d\mathbf{S} \qquad \text{a scalar field}} \quad , \tag{16.06.06}$$

→ 1098 1107

for any volume $\Omega_\mathbf{x}$ surrounding the point P.

It can be shown that, if the limit exists, it does not depend neither on the shape of the volume chosen to perform the limit, nor on the limiting procedure.

The divergence of a vector field, at any given point, is thus the flux of the vector field per unit volume around the point.

It can be shown that the divergence is indeed a scalar [1], that is it transforms as a scalar under coordinate transformations § 20.

Defining the divergence via definition (16.06.06) makes it possible to give an immediate physical interpretation to the concept of divergence. Moreover, that definition does not depend on any specific Coordinate System. If one chooses a orthonormal Cartesian Coordinates Coordinate System and a cubic volume with edges parallel to the xyz axes it is easy to deduce the expression of the divergence in Cartesian coordinates. In a similar way the expression of the divergence in any other Coordinate System can be determined.

Following the terminology of hydrodynamics it is often said that **points where divergence is different from zero give the Sources|Sinks of the field**.

The divergence can be also viewed as a weighted derivative along the direction of the field, or along the field hose [2]

[1] WEB - URL

[2] ©|WEB - URL |ElectroMagnetic Field and Energy - H.A.Haus & J.R.Melcher|*A must...*|

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16.06.01.04 General Definition of the Rotor

This § is referenced at pages:
[Never referenced.]

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©|T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...|Vol. 2, 2nd edition, § 12.20|*Excellent*|
©|WEB - URL |ElectroMagnetic Field and Energy - H.A.Haus & J.R.Melcher||
©|WEB - URL |ElectroMagnetic Field and Energy - H.A.Haus & J.R.Melcher|*Appendix 2*|
©|D.Styer|Geometrical definition of divergence and curl|*WEB - URL*|

Consider a vector field $\mathbf{V}[\mathbf{x}]$ and its circulation around any closed line, C:

$$\Gamma_C \equiv \oint_C \mathbf{V}\cdot \mathbb{d}\mathbf{L} \quad .$$

In order to pick-up a property of the field in the neighborhood of some point, $\mathbf{x}$, consider the circulation around the boundary (frontier) of smaller and smaller surfaces enclosing the given point, that is, containing the given point. However, since the loops can have different orientations one is forced to consider surfaces in a plane passing for the given point and with fixed orientation in space, perpendicular to the unit vector $\widehat{\mathbf{n}}$.

As the circulation is additive with respect to partitioning of the area one may wonder what about the limit for infinitesimally small area.

The projection of the rotor of a vector field along the direction $\widehat{\mathbf{n}}$ is defined as the limit, for infinitesimally small area containing the given point, of the ratio

$$\boxed{\widehat{\mathbf{n}}\cdot\mathbf{rot}\,\mathbf{V}[\mathbf{x}] \equiv \lim_{|\Delta\mathbf{S}_\mathbf{x}|\longrightarrow 0} \frac{1}{|\Delta\mathbf{S}_\mathbf{x}|}\oint_{\partial\Sigma_\mathbf{x}} \mathbf{V}\cdot \mathbb{d}\mathbf{L} \qquad \text{a vector field}} \quad , \tag{16.06.07}$$

→ 1099

for any surface $\Sigma_\mathbf{x}$ lying in the plane perpendicular to $\widehat{\mathbf{n}}$ and surrounding the point P.

It can be shown that, if the limit exists, it does not depend neither on the shape of the surface chosen to perform the limit, nor on the limiting procedure.

The projection of the rotor of a vector field along any unit vector, at any given point, is thus the circulation of the vector field per unit area in the plane passing by the point and perpendicular to the unit vector.

The direction of the rotor vector is perpendicular to the plane such that the limit in equation (16.06.07) is maximum.

It can be shown that the rotor is indeed a vector [1], that is it transforms as a vector under coordinate transformations § 20.

Defining the rotor via definition (16.06.07) makes it possible to give an immediate physical interpretation to the concept of rotor. Moreover, that definition does not depend on any specific Coordinate System. If one chooses a orthonormal Cartesian Coordinates Coordinate System and a square surface with edges parallel to the xyz axes it is easy to deduce the expression of the rotor in Cartesian coordinates. In a similar way the expression of the rotor in any other Coordinate System can be determined.

Following the terminology of hydrodynamics it is often said that **points where rotor is different from zero give the vortexes of the field**. Nothing enters/exits vertexes.

The rotor can be viewed in terms of a field hose: the transverse components of the rotor can be regarded as derivatives with respect to transverse directions of the vector field weighted by incremental line elements [2]

[1] WEB - URL
[2] ©|WEB - URL |ElectroMagnetic Field and Energy - H.A.Haus & J.R.Melcher|*A must...*|

16.06.01.05 Alternative Definitions in Terms of Volume Integrals Only

©|T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...|||
©|M.R.Spiegel *et al.*, *Vector Analysis*, 2009, McGraw-Hill, 2ndEd., ...|||
©|Vector and Tensor Analysis with Applications - Borisenko, A.I. and Tarapov, I.E. |||

Alternative definitions can be given in terms of volume integral only, for instance, by exploiting the vector operators theorem to get equivalent expressions:

$$\mathbf{div}\,\mathbf{V}[\mathbf{x}] = \lim_{\Delta V_{\mathbf{x}} \longrightarrow 0} \frac{1}{\Delta V_{\mathbf{x}}} \oiint_{\partial\Omega_{\mathbf{x}}} \mathbb{d}\mathbf{S}\cdot\mathbf{V} \quad \text{weighted radial flux on a surface surrounding } P \quad ,$$

$$\mathbf{rot}\,\mathbf{V}[\mathbf{x}] = \lim_{\Delta V_{\mathbf{x}} \longrightarrow 0} \frac{1}{\Delta V_{\mathbf{x}}} \oiint_{\partial\Omega_{\mathbf{x}}} \mathbb{d}\mathbf{S}\wedge\mathbf{V} \quad \text{weighted tangential circulation on a surface surrounding } P \quad ,$$

$$\mathbf{grad}\,U[\mathbf{x}] = \lim_{\Delta V_{\mathbf{x}} \longrightarrow 0} \frac{1}{\Delta V_{\mathbf{x}}} \oiint_{\partial\Omega_{\mathbf{x}}} U\ \mathbb{d}\mathbf{S} \quad \text{weighted mean on a surface surrounding } P \quad ,$$

for any volume $\Omega_{\mathbf{x}}$ surrounding the point P.

16.06.02 General Definition of Linear First-Order Differential Operators in 2D

This § is referenced at pages:
[1004, 1723]

16.06.02.01 2D Gradient

The gradient is defined as the two-dimensional scalar product, just as in section § 16.06.01.02.

16.06.02.02 2D Flux and Divergence

Flux and Divergence, in 2D, are the only concepts requiring some modification with respect to 3D.

Flux and Divergence, **on a surface, measure the flow of the vector field across a curve, on the surface**. Here, flow is used on purpose as a qualitative term, waiting for a quantitative definition below.

Flux in two dimensions is defined as follows. In a plane, consider a vector field $\mathbf{V}[\mathbf{x}]$ and its flux around any line, C, 2D flux is defined as:

$$\widehat{\mathbf{N}} \equiv \frac{\mathbb{d}\mathbf{L}\wedge\widehat{\mathbf{n}}}{\mathbb{d}L} \qquad \Phi_C^{(2)} \equiv \int_C \left(\mathbf{V}\cdot\widehat{\mathbf{N}}\right)\mathbb{d}L \qquad \Phi_C^{(2)} = \int_C (\widehat{\mathbf{n}}\wedge\mathbf{V})\cdot\mathbb{d}\mathbf{L} \quad , \tag{16.06.08}$$

→ 1100

with $\widehat{\mathbf{n}}$ a function which gives an outward-facing normal unit vector on the surface at each point on the curve, and the implicit usual limiting definition of the integral.

Orientation of the curve and surface must be compatible with the usual **right-hand rule**.

Equation (16.06.08) shows how the flux can be calculated via a circulation, the circulation of $\widehat{\mathbf{n}}\wedge\mathbf{V}$.

It is used in the concept of surface current density § 36.03.02.02.

The divergence is defined as the two-dimensional flux per unit area as follows. On a surface consider a small area $\Sigma_{\mathbf{x}}$ around the point $\mathbf{x}$ and let the area tend to zero to find the limit

$$\mathbf{div}_{\mathbf{S}}\mathbf{F}[\mathbf{x}] = \lim_{\Delta S_{\mathbf{x}} \longrightarrow 0} \frac{1}{\Delta S_{\mathbf{x}}} \oint_{\partial\Sigma_{\mathbf{x}}} \left(\mathbf{V}\cdot\widehat{\mathbf{N}}\right)\mathbb{d}L \qquad \text{a scalar field} \quad , \tag{16.06.09}$$

for any area $\Sigma_{\mathbf{x}}$ surrounding the point $\mathbf{x}$ and where $\widehat{\mathbf{N}}$ is a unit-vector tangent to the surface and perpendicular to the curve at any point.

It is a scalar field, in a pure two-dimensional space.

DRAFT In Progress

16.06.02.03 2D Circulation and Rotor

Circulation in two dimensions is obviously defined as in 3D.

The rotor is defined as the two-dimensional circulation per unit area as follows, exactly as the projection on a plane of the three-dimensional rotor.

$$\mathbf{rot}_S\mathbf{F}[\mathbf{x}] = \lim_{\Delta S_\mathbf{x}\longrightarrow 0} \frac{1}{\Delta S_\mathbf{x}} \oint_{\partial\Sigma_\mathbf{x}} \mathbf{V}\cdot \mathbb{d}\mathbf{L} \qquad \text{a scalar field} \quad , \tag{16.06.10}$$

for any surface $\Sigma_\mathbf{x}$ surrounding the point P.

It is a scalar field, in a pure two-dimensional space.

16.06.03 Linear First-Order Differential Operators in Different Coordinate Systems

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|Appendix||

16.06.03.01 Linear First-Order Differential Operators in Cartesian Coordinates Coordinate System

This § is referenced at pages:
[1118]

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...- ...

Let x_i $(i = 1, 2, 3)$, a system of orthonormal Cartesian Coordinates Coordinate System. It can be shown that the following relations apply. Read also § 16.06.04 for a derivation of these results.

$$\mathbf{grad}\phi[\mathbf{x}] = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i \equiv \mathbf{e}_i\partial_i\phi \quad . \tag{16.06.11}$$

$$\mathbf{div}\,\mathbf{V}[\mathbf{x}] = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3} = \frac{\partial V_i}{\partial x_i} \equiv \partial_i V_i \quad . \tag{16.06.12}$$

$$\mathbf{rot}\,\mathbf{V}[\mathbf{x}] = \left(\frac{\partial V_2}{\partial x_3} - \frac{\partial V_3}{\partial x_2}\right)\mathbf{e}_1 + \left(\frac{\partial V_3}{\partial x_1} - \frac{\partial V_1}{\partial x_3}\right)\mathbf{e}_2 + \left(\frac{\partial V_1}{\partial x_2} - \frac{\partial V_2}{\partial x_1}\right)\mathbf{e}_3 = \epsilon^{\dots}_{ijk}\frac{\partial V_k}{\partial x_j}\mathbf{e}_i \equiv \mathbf{e}_i\epsilon^{\dots}_{ijk}\partial_j V_k \quad . \tag{16.06.13}$$

Second-Order Tensor of Derivatives in Cartesian Coordinates Coordinate System

The second-order Cartesian tensor

$$\frac{\partial V_i}{\partial x_j} \quad ,$$

can be decomposed into its:

- trace, a scalar: the divergence;
- anti-symmetrical part, a second-order anti-symmetrical tensor: the rotor;
- symmetrical part, a second-order trace-less symmetrical tensor: the shear tensor.

Read § 16.06.01.01 and section § 16.12.17.

DRAFT-WIP-PRESS

Nabla Vector Operator and Its Pitfalls

This § is referenced at pages:
[Never referenced.]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.8||

In orthonormal Cartesian Coordinates Coordinate System the linear first-order differential operators can be expressed via the nabla vector operator:

$$\boldsymbol{\nabla} \equiv \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} \quad . \tag{16.06.14}$$

Then one has:

$$\mathbf{grad}\phi \equiv \boldsymbol{\nabla}\phi \qquad \mathbf{div}\,\mathbf{A} \equiv \boldsymbol{\nabla}\cdot\mathbf{A} \qquad \mathbf{rot}\,\mathbf{A} \equiv \boldsymbol{\nabla}\wedge\mathbf{A} \quad .$$

Note that the straightforward algebraic use of the nabla vector operator gives sometimes wrong results [1].

A simple example is: $\mathbf{grad}a \wedge \mathbf{grad}b$. In general this is not zero while the blind use of nabla would give zero as the direction would seem to be carried by nabla so that the vector product of two vectors with the same direction is zero.

Read § 16-015.

16.06.03.02 Linear First-Order Differential Operators in Cylindrical Coordinates

Read § 16.11.

16.06.03.03 Linear First-Order Differential Operators in Spherical Coordinates

Read § 16.11.

16.06.03.04 Linear First-Order Differential Operators in General Orthonormal Curvilinear Coordinates

The gradient of a function $\phi[\mathbf{r}]$ is given in orthonormal curvilinear coordinates by:

$$\mathbf{grad}\phi[\mathbf{r}] = \frac{1}{h_1}\frac{\partial\phi}{\partial u^1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial u^2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial u^3}\mathbf{e}_3 \quad .$$

The divergence of a vector field $\mathbf{V}[\mathbf{r}]$ is given in orthonormal curvilinear coordinates by:

$$\mathbf{div}\,\mathbf{V}[\mathbf{r}] = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial\left(h^2 h^3 V_1\right)}{\partial u^1} + \frac{\partial\left(h^3 h^1 V_2\right)}{\partial u^2} + \frac{\partial\left(h^1 h^2 V_3\right)}{\partial u^3}\right) \quad . \quad \rightarrow 1103$$

The rotor of a vector field $\mathbf{V}[\mathbf{r}]$ is given in orthonormal curvilinear coordinates by:

$$\mathbf{rot}\,\mathbf{V}[\mathbf{r}] = \frac{\mathbf{e}_1}{h_2 h_3}\left(\frac{\partial\left(h^3 V_3\right)}{\partial u^2} - \frac{\partial\left(h^2 V_2\right)}{\partial u^3}\right) + \frac{\mathbf{e}_2}{h_3 h_1}\left(\frac{\partial\left(h^2 V_2\right)}{\partial u^3} - \frac{\partial\left(h^3 V_3\right)}{\partial u^1}\right) + \frac{\mathbf{e}_3}{h_1 h_2}\left(\frac{\partial\left(h^1 V_1\right)}{\partial u^1} - \frac{\partial\left(h^1 V_1\right)}{\partial u^2}\right) \quad . \quad \rightarrow 1103$$

The Laplacian of a scalar field $\phi[\mathbf{r}]$ is given in orthonormal curvilinear coordinates by:

$$\nabla^2\phi[\mathbf{r}] = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u^1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\phi}{\partial u^2}\right) + \frac{\partial}{\partial u^3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\phi}{\partial u^3}\right)\right) \quad .$$

[1] R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ... 2.2.8

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16.06.04
Meaning of Linear First-Order Differential Operators

This § is referenced at pages:
[1101]

The interpretation of the gradient of a scalar field is well known from Calculus: the gradient gives, at any point, the direction along which the scalar field is increasing most rapidly.

The expressions in general orthonormal coordinates of the divergence and rotor (equations (16.06.03.04), (16.06.03.04)) clarify in the general case the interpretation of the meaning of the two operators which can be suspected from the analysis of the special cases of orthonormal Cartesian Coordinates Coordinate System, as well as in Cylindrical and Spherical coordinates. In fact it is clear that the divergence measures the rate of change of each component of the field along the direction defined by the component itself. On the other hand the rotor measures the rate of change of each component of the field along the two directions orthogonal to the one defined by the component itself.

Following the terminology of hydrodynamics it is often said that

- points where the divergence is different from zero give the **Sources|Sinks** of the field;
- points where the rotor is different from zero give the **Vortexes** of the field.

Note that also saddle points are significant points for a function of more variables, that is critical points (zero gradient) which are not local extrema.

16.06.05
Expressions of Divergence and Rotor in Cartesian Coordinates

This § is referenced at pages:
[1139]

Consider the flow of a fluid in Hydrodynamics as it provides an intuitive example.

The mass flux is expressed in terms of the mass density, ρ, and the velocity field of the fluid, $\mathbf{v}[\mathbf{x},t]$, a vector field giving, at any point and time, the velocity of the fluid particles passing by that point at time t.

The mass flux is given by the current density of mass

$$\boxed{\mathbf{j}_{\mathtt{M}}[\mathbf{x},t] \equiv \rho[\mathbf{x},t]\mathbf{v}[\mathbf{x},t]} \quad , \tag{16.06.15}$$

which can be expressed, in the case of all identical particles, also in terms of the number density, $\rho_{\mathtt{N}}[\mathbf{x},t]$, and mass of the single particle, m, as:

$$\mathbf{j}_{\mathtt{M}}[\mathbf{x},t] \equiv \rho_{\mathtt{M}}[\mathbf{x},t]\mathbf{v}[\mathbf{x},t] \longrightarrow m\rho_{\mathtt{N}}[\mathbf{x},t]\mathbf{v}[\mathbf{x},t] \quad ,$$

This is to be compared with the definition of current density of particles (section § 30.06) and the current density of charge, the usual current density of ElectroMagnetism (section § 36.03.04.01). In the present case one has a current density of mass, instead of a current density charge of particles or a current density of charges.

In a similar way one can consider the particle flux, that is the number of particles per unit area per unit time.

The particle flux is given by the current density of particles

$$\boxed{\mathbf{j}_{\mathtt{N}}[\mathbf{x},t] \equiv \rho_{\mathtt{N}}[\mathbf{x},t]\mathbf{v}[\mathbf{x},t]} \quad . \tag{16.06.16}$$

→ 2735

DRAFT - IN PROGRESS

16.06.05.01 Divergence in Cartesian Coordinates Coordinate System: Hydrodynamics

The flux of $\mathbf{j}_{\mathtt{M}}$ across any closed surface gives the **mass out-flow per unit time**: if it is positive there is an overall out-flow of mass, if it is negative there is an overall in-flow of mass.

In order to appreciate the meaning of the divergence let us consider its application to particle fluxes.

Consider a small rectangular parallelepiped with sides along the x, y and z axis and length Δx, Δy and Δz.

It is understood that a limit will be taken such that: $\Delta x \to 0$, $\Delta y \to 0$ and $\Delta z \to 0$.

From the definition of current density of particles the number of particles crossing a surface described by $\hat{\mathbf{n}}\ \mathbb{d}\mathbf{S}$ in time $\mathbb{d}t$,

$$\boxed{\mathbb{d}N = (\mathbf{j}\cdot\mathbf{n})\ \mathbb{d}S\ \mathbb{d}t} \quad ,$$

one finds, by applying the theorem of the mean of integral calculus:

$$\mathbb{d}N^x_{\mathtt{OUT}} \simeq (j_x[x+\Delta x, \widetilde{y}_+, \widetilde{z}_+] - j_x[x, \widetilde{y}_-, \widetilde{z}_-])\,\Delta y\Delta z\ \mathbb{d}t \to \frac{\partial j_x}{\partial x}[x,y,z]\Delta x\Delta y\Delta z\ \mathbb{d}t + \mathcal{O}\left(\Delta x\Delta y\Delta z\right)^2 \quad ,$$

$$\mathbb{d}N^y_{\mathtt{OUT}} \simeq (j_y[\widetilde{x}_+, y+\Delta y, \widetilde{z}_+] - j_x[\widetilde{x}_-, y, \widetilde{z}_-])\,\Delta z\Delta x\ \mathbb{d}t \to \frac{\partial j_y}{\partial y}[x,y,z]\Delta x\Delta y\Delta z\ \mathbb{d}t + \mathcal{O}\left(\Delta x\Delta y\Delta z\right)^2 \quad ,$$

$$\mathbb{d}N^z_{\mathtt{OUT}} \simeq (j_z[\widetilde{x}_+, \widetilde{y}_+, z+\Delta z] - j_z[\widetilde{x}_-, \widetilde{y}_-, z])\,\Delta x\Delta y\ \mathbb{d}t \to \frac{\partial j_z}{\partial z}[x,y,z]\Delta x\Delta y\Delta z\ \mathbb{d}t + \mathcal{O}\left(\Delta x\Delta y\Delta z\right)^2 \quad .$$

In general:

$$\begin{aligned} \widetilde{x}_+ \neq \widetilde{x}_- \qquad & 0 \leq \widetilde{x}_+ | \widetilde{x}_- \leq x + \Delta x \quad , \\ \widetilde{y}_+ \neq \widetilde{y}_- \qquad & 0 \leq \widetilde{y}_+ | \widetilde{y}_- \leq y + \Delta y \quad , \\ \widetilde{z}_+ \neq \widetilde{z}_- \qquad & 0 \leq \widetilde{z}_+ | \widetilde{z}_- \leq z + \Delta z \quad . \end{aligned}$$

The total particle out-flow is (definition of divergence):

$$\boxed{\mathbb{d}N_{\mathtt{OUT}} \equiv\ \mathbb{d}t \oiint_{\Omega} \mathbf{j}\cdot\ \mathbb{d}\mathbf{S} \simeq \Delta x\Delta y\Delta z\ \mathbb{d}t \left(\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}\right) = (\mathbf{div}\mathbf{j})\,\Delta x\Delta y\Delta z\ \mathbb{d}t} \quad .$$

Therefore the infinitesimal particle flux per unit volume is given by the divergence.

One can then write:

$$\oiint_{\Omega} \mathbf{j}\cdot\mathbf{n}\ \mathbb{d}\mathbf{S} \simeq \Delta x\Delta y\Delta z \left(\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}\right) = \Delta x\Delta y\Delta z \mathbf{div}\mathbf{j} \quad .$$

Note that the total out-flow from any closed surface might be, for instance, positive, but some of its components might be, for instance, negative.

16.06.05.02 Rotor in Cartesian Coordinates Coordinate System: Hydrodynamics

Consider a 2D flow of fluid, in the xy plane. The circulation of $\mathbf{j}_{\mathtt{M}}$ around any point gives the **vorticity**: if it is positive there is an overall fluid vortex circulation in the clockwise direction; if it is negative there is an overall fluid vortex circulation in the counterclockwise direction;

Consider a small ship floating on the fluid. The tendency to rotate as a consequence of the different velocity of the fluid at different points can be approximately calculated as follows. Approximate the ship as a small rectangle, with sides, of length Δx and Δy, respectively in the directions parallel to the x and y axes.

It is understood that a limit will be taken such that: $\Delta x \to 0$ and $\Delta y \to 0$.

The contributions to the circulation of the velocity field can be approximately estimated as follows, taking only first-order expressions, thanks to Lagrange mean value theorem.

The two contributions along x at the two sides parallel to the x axis are, thanks to Lagrange mean value theorem:

$$\Delta x v_x[\widetilde{x}_-, y] \qquad \Delta x \left(v_x[\widetilde{x}_-, y] + \Delta y \frac{\partial v_x}{\partial y}[\widetilde{x}_+, y+\Delta y]\right) \qquad \text{for some } x \leq \widetilde{x}_- | \widetilde{x}_+ \leq x + \Delta x \quad ,$$

tending to rotate in opposite directions, with an overall result that only depends on the derivative term.

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The two contributions along y at the two sides parallel to the x axis are, thanks to Lagrange mean value theorem:

$$\Delta y v_y[x, \widetilde{y}_+] \qquad \Delta y \left(v_y[x, \widetilde{y}_+] + \Delta x \frac{\partial v_y}{\partial x}[x + \Delta x, \widetilde{y}_-] \right) \qquad \text{for some } y \leq \widetilde{y}_- | \widetilde{y}_+ \leq y + \Delta y \quad ,$$

tending to rotate in opposite directions, with an overall result that only depends on the derivative term.

The overall effect is thus given by the combined effect of the two derivative terms, whose effect is to tend to rotate in opposite directions. Therefore the overall effect is given by the difference of the two derivative terms, that is just the z component of the rotor, that is the circulation per unit area:

$$\oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{L} \approx \Delta x \Delta y \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \equiv \Delta x \Delta y \, \mathbf{rot}\mathbf{v}\Big|_z \quad .$$

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16.07 Fundamental Integral Theorems of Vector Calculus

This § is referenced at pages:
[1095]

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|2||
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 2, 2,3||

The three fundamental theorems of Vector Calculus relate integrals on some region (line, surface and volume) to expressions over the boundary of the region itself.

16.07.01
Gradient Theorem

The line integral of the gradient of some function can be evaluated from from the knowledge of the function on the boundaries of the line alone via the **gradient theorem** .

$$\phi[2]-\phi[1]=\int_1^2 \mathbf{grad}\phi\cdot \mathbb{d}\mathbf{L} \quad . \tag{16.07.01}$$

→ 1113

© - QUOTE

D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...
Geometrical Interpretation: Suppose you wanted to determine the height of the Eiffel Tower. You could climb the stairs, using a ruler to measure the rise at each step, and adding them all up (that's the left side of Eq. 1.55), or you could place altimeters at the top and the bottom, and subtract the two readings (that's the right side); you should get the same answer either way (that's the fundamental theorem).

WORK IN PROGRESS

16.07.02
Divergence Theorem

The volume integral of the divergence of some function can be evaluated from from the knowledge of the function on the boundaries of the volume alone.

$$\iiint_{\Omega} \mathbf{div}\,\mathbf{A}\ \mathrm{d}V = \iiint_{\Omega} \boldsymbol{\nabla}\cdot\mathbf{A}\ \mathrm{d}V = \oiint_{\partial\Omega} \mathbf{A}\cdot\mathrm{d}\mathbf{S} \quad . \tag{16.07.02}$$

→ 1107

16.07.02.01 Two Corollaries to the Divergence Theorem

©|G.B.Arfken & H.J.Weber, *Mathematical Methods For Physicists*, 2005, Elsevier, 6thEd., ...|||

Show that:

$$\iiint_{\Omega} \boldsymbol{\nabla}\phi\ \mathrm{d}V = \oiint_{\partial\Omega} \phi\hat{\mathbf{n}}\ \mathrm{d}S = \oiint_{\partial\Omega} \phi\ \mathrm{d}\mathbf{S} \quad . \tag{16.07.03}$$

→ 1107 1131 1847

Show that:

$$\iiint_{\Omega} \boldsymbol{\nabla}\wedge\mathbf{A}\ \mathrm{d}V = \oiint_{\partial\Omega} (\hat{\mathbf{n}}\wedge\mathbf{A})\ \mathrm{d}S = \oiint_{\partial\Omega} \mathrm{d}\mathbf{S}\wedge\mathbf{A} \quad . \tag{16.07.04}$$

→ 1107 1131 1889

Note that the two theorems (16.07.03), (16.07.04) might be used as alternative definitions for gradient and rotor, as it is done in (16.06.06) for defining the divergence.

Note that the two theorems (16.07.03), (16.07.04) as well as the **divergence theorem** (16.07.02) are all three of the form:

$$\iiint_{\Omega} \boldsymbol{\nabla}\otimes(\ldots)\ \mathrm{d}V = \oiint_{\partial\Omega} \hat{\mathbf{n}}\otimes(\ldots)\ \mathrm{d}S = \oiint_{\partial\Omega} \mathrm{d}\mathbf{S}\otimes(\ldots) \quad , \tag{16.07.05}$$

where the $\otimes$ operator is the kind of multiplication defined by the nabla vector operator on the left-side.

16.07.02.02 Examples and Applications

When using $\phi = 1$ in equation (16.07.03) one finds the simple result

$$\oiint_{\partial\Omega} \mathrm{d}\mathbf{S} = 0 \quad . \tag{16.07.06}$$

The two corollaries (16.07.03), (16.07.04) provide an alternative coordinate-independent definition of the gradient and rotor, based on a volume integral, that is similar to the original coordinate-independent definition of the divergence in section § 16.06.01.03.

The corollary (16.07.03) can be used to demonstrate the Archimedes principle of hydrostatics.

DRAFT WORK IN PROGRESS

16.07.03
Rotor Theorem

The surface integral of the rotor of some function can be evaluated from from the knowledge of the function on the boundaries of the surface alone.

$$\iint_{\Sigma} \mathbf{rot}\,\mathbf{A}\cdot \mathbb{d}\mathbf{S} = \iint_{\Sigma} \boldsymbol{\nabla}\wedge\mathbf{A}\cdot \mathbb{d}\mathbf{S} = \oint_{\partial\Sigma} \mathbf{A}\cdot \mathbb{d}\mathbf{L} \quad . \tag{16.07.07}$$

→ 1108 1131 1711

The orientations of the surface Σ and its boundary $\partial\Sigma$ must be compatible.

16.07.03.01 Two Corollaries to the Rotor Theorem

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Show that:

$$\iint_{\Sigma} \mathbb{d}\mathbf{S}\wedge\boldsymbol{\nabla}\phi = \iint_{\Sigma} \widehat{\mathbf{n}}\wedge\boldsymbol{\nabla}\phi\ \mathbb{d}S = \oint_{\partial\Sigma} \phi\ \mathbb{d}\mathbf{L} \quad . \tag{16.07.08}$$

→ 1108 1711 1739

Show that:

$$\iint_{\Sigma} (\ \mathbb{d}\mathbf{S}\wedge\boldsymbol{\nabla})\wedge\mathbf{A} = \iint_{\Sigma} (\widehat{\mathbf{n}}\wedge\boldsymbol{\nabla})\wedge\mathbf{A}\ \mathbb{d}S = \oint_{\partial\Sigma} \mathbb{d}\mathbf{L}\wedge\mathbf{A} \quad . \tag{16.07.09}$$

→ 1108 1781

Note that the two theorems (16.07.08), (16.07.09) as well as the **rotor theorem** (16.07.07) are all three of the form:

$$\iint_{\Sigma} (\ \mathbb{d}\mathbf{S}\wedge\boldsymbol{\nabla})\otimes(\ldots) = \iint_{\Sigma} (\widehat{\mathbf{n}}\wedge\boldsymbol{\nabla})\otimes(\ldots)\ \mathbb{d}S = \oint_{\partial\Sigma} \mathbb{d}\mathbf{L}\otimes(\ldots) \quad , \tag{16.07.10}$$

→ 1781

where the $\otimes$ operator is the kind of multiplication defined by the nabla vector operator on the left-side.

16.07.03.02 Examples and Applications

When using $\phi = 1$ in equation (16.07.08) one finds the simple result:

$$\oint_{\partial\Sigma} \mathbb{d}\mathbf{L} = 0 \quad . \tag{16.07.11}$$

The three theorems (16.07.08), (16.07.09) as well as the **rotor theorem** (16.07.07) all show that the left-hand side integrals are independent of the specific surface of integration chosen, once the boundary is fixed to $\partial\Sigma$.

DRAFT IN PROGRESS

16.07.04
General Theorems

©|F.Wegner, *Classical Electrodynamics*, 2003, ..., ...Ed., WEB - URL.|||

16.07.05
Helmholtz Theorem

This § is referenced at pages:
[1662, 1852, 1894]

Helmholtz theorem, under the appropriate hypotheses, provides the explicit expressions for a vector field when the field divergence and rotor are given, showing that, under the appropriate hypotheses, any vector field can be reconstructed from its divergence and rotor.

Suppose a vector field is defined by the set of equations:

$$\mathbf{div}\,\mathbf{V} = \rho \quad ,$$
$$\mathbf{rot}\,\mathbf{V} = \mathbf{J} \qquad \text{this implies, for consistency,} \qquad \mathbf{div}\,\mathbf{J} = 0 \quad ,$$

with

$$\mathbf{div}\,\mathbf{V}\Big|_{\infty} = 0 \qquad \mathbf{rot}\,\mathbf{V}\Big|_{\infty} = 0 \quad ,$$

that is the divergence and rotor vanish at infinity.

The vector field may be written as the sum of an irrotational part plus a solenoidal part:

$$\boxed{\mathbf{V} = -\mathbf{grad}\phi + \mathbf{rot}\,\mathbf{A}} \quad ,$$

where for a Vector Field, $\mathbf{V}$, one has

$$\boxed{\phi[\mathbf{x}] = \frac{1}{4\pi}\iiint\limits_{\Omega} \frac{\rho[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}V_{\mathbf{y}}} \quad ,$$

$$\boxed{\mathbf{A}[\mathbf{x}] = \frac{1}{4\pi}\iiint\limits_{\Omega} \frac{\mathbf{J}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}V_{\mathbf{y}}} \quad ,$$

with the integral extended to any region Ω including the set of points where $\mathbf{div}\,\mathbf{V}$ and $\mathbf{rot}\,\mathbf{V}$ are different from zero.

In other words Helmholtz theorem states that **any sufficiently smooth, rapidly decaying vector field can be resolved into an irrotational (rotor-free) component plus a solenoidal (divergence-free) component**. It is also known as the fundamental theorem of vector calculus.

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16.08 Second-Order Differential Operators

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|2||
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 2, 2,3||

One can build five second-order differential operators:

$$\textbf{divgrad}\,\phi \equiv \nabla^2\phi \tag{16.08.01}$$ → 1110

$$\textbf{rotgrad}\,\phi = 0 \tag{16.08.02}$$

$$\textbf{graddiv}\,\mathbf{A} \tag{16.08.03}$$ → 1110

$$\textbf{divrot}\,\mathbf{A} = 0 \tag{16.08.04}$$

$$\textbf{rotrot}\,\mathbf{A} \equiv \textbf{graddiv}\,\mathbf{A} - \boldsymbol{\nabla}^2\mathbf{A} \qquad \text{a definition for } \boldsymbol{\nabla}^2\mathbf{A} \quad . \tag{16.08.05}$$ → 1110 1133

Equation (16.08.01) is actually the definition of the scalar Laplacian operator:

$$\boxed{\textbf{divgrad}\,\phi = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\phi \equiv \nabla^2\phi} \quad . \tag{16.08.06}$$

Equation (16.08.05) is actually the definition of the vector Laplacian operator:

$$\boxed{\boldsymbol{\nabla}^2\mathbf{A} \equiv \textbf{graddiv}\,\mathbf{A} - \textbf{rotrot}\,\mathbf{A}} \quad . \tag{16.08.07}$$ → 1110

In fact, in orthonormal Cartesian Coordinates Coordinate System, equation (16.08.05) can be straightforwardly derived for each of the orthonormal Cartesian Coordinates Coordinate System of the vector field by developing the **rotrot**(...):

$$\{\textbf{rotrot}\,\mathbf{A}\}_k = \frac{\partial}{\partial x_k}(\textbf{div}\,\mathbf{A}) - \nabla^2 A_k \quad . \tag{16.08.08}$$ → 1110

Equation (16.08.07) is then taken as the definition of the vector Laplacian in any generic Coordinate System. In a general Coordinate System the expression of the vector Laplacian has nothing to do with the Cartesian Coordinate expression (16.08.08) but it has to be worked-out explicitly from equation (16.08.05). In Cartesian Coordinates one finds:

$$\boxed{\left\{\boldsymbol{\nabla}^2\mathbf{A}\right\}_k = \nabla^2 A_k \qquad \text{in orthonormal Cartesian Coordinates Coordinate System only}} \quad . \tag{16.08.09}$$ → 2087

Note that equations (16.08.03), (16.08.05) both include the same operator: **graddiv**$\mathbf{A}$.

The operators which are identically zero are those two which are built with one and only one rotor operator. In fact the rotor definition includes the **Levi-Civita symbol** and the second-order operator has two ∂. It follows that this kind of operator is identically zero because it is a product made of an anti-symmetrical and a symmetrical part.

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16.09 Some Elements of Potential Theory

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©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 2, 2,3||
©|G.De Marco, **Analisi 1, Analisi 2**, Decibel-Zanchelli|||
©|J.Cantarella *et al.*|Vector Calculus and the Topology of Domains in 3-Space|*§ 6, 16, 17; DOI: 10.2307/2695643* - ...

Given a vector field defined on a bounded domain in 3D-space one may ask whether the vector field is the gradient of some function or the rotor of another vector field. The answer to these and similar questions requires to understand the relationship between the calculus of vector fields and the topology of their domains of definition and, therefore, it is not a trivial matter at all. Unfortunately in most books, and not only physics books, the topic is treated in a simplified and partly wrong way.

16.09.01 Domain and Connectedness

©|T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...||*Fair discussion.*|

In the following definitions the word **regular** is intended in a non-technical sense, as the detailed discussion of the properties required to the domains is far beyond what can be presented here. For a fair discussion and more details, see the references.

- A subset $\Omega \subseteq \mathbb{R}^3$ is said to be 0-connected (or path-wise connected) if there exist a curve $\mathcal{C} \subseteq \Omega$ connecting every two points belonging to Ω such that the two points are the boundary of the curve.
- A subset $\Omega \subseteq \mathbb{R}^3$ is said to be 1-connected if every **regular** closed curve $\mathcal{C} \subseteq \Omega$ is the boundary of an oriented **regular** surface $\mathcal{S} \subseteq \Omega$ such that $\mathcal{C} = \partial\mathcal{S}$.
- A subset $\Omega \subseteq \mathbb{R}^3$ is said to be 2-connected if every **regular** closed surface $\mathcal{S} \subseteq \Omega$ is the boundary of a **regular** domain $\mathcal{D} \subseteq \Omega$ such that $\mathcal{S} = \partial\mathcal{D}$.

16.09.02 Conservative Fields

Note that usage of names may be not universal.

16.09.02.01 Conservative - Irrotational Fields

This § is referenced at pages:
[Never referenced.]

A field, $\mathbf{V}$, is said to be **conservative for the circulation** if the circulation around any closed line is zero.
In this case: $\mathbf{rot}\mathbf{V} = 0$, that is **the field is rotor-less**, commonly named irrotational. The circulation is conserved in the sense that, given any two points, it does not depend on the path having the two points as the boundary.
If the field can be written as the gradient of a scalar potential, it is just known as, in short terms, **a conservative field**, the name deriving from mechanical energy conservation in point mechanics. Read § 16.09.03 form more details.

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16.09.02.02 Conservative - Divergenceless Fields

This § is referenced at pages:
[1132]

A field, $\mathbf{V}$, is said to be **conservative for the flux** if the flux across any closed surface is zero. In this case: $\mathbf{div}\mathbf{V} = 0$, that is **the field is divergence-less**.

The flux is conserved in the sense that, given any closed line, it does not depend on the surface having the closed line as the boundary.

If the field can be written as the rotor of a **vector potential** , it is just known as, in short terms, **a solenoidal field** or an incompressible (fluid) field. Read § 16.09.03 form more details.

16.09.03 Conditions for a Vector Field to Admit Potentials

This § is referenced at pages:
[1111, 1112, 1730]

©|T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...|Vol. 2 - § 10 and 12||

©|P.J.Olver|Vector Calculus in 3 dimensions - WEB - URL |*Excellent: all and only what is needed here.*|

©|W.Kaplan|Advanced Calculus, Addison Wesley, 5 edition (July 25, 2002)|*§ 5.13*|

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|1.6.2||

16.09.03.01 Gradient-less fields

This § is referenced at pages:
[Never referenced.]

Theorem

A scalar field defined on all of a 0-connected set, has vanishing gradient if and only if it has a fixed value.

16.09.03.02 Scalar Potential and Reconstruction of a Scalar Field From Its Gradient

This § is referenced at pages:
[1114]

Let us determine under which conditions a vector field is the gradient of a scalar field.

A vector field, $\mathbf{V}$, defined on a subset $\mathcal{D}$ of $\mathbb{R}^3$ with simple linear connection (1-connected), admits a (non-unique) scalar potential if and only if it is irrotational:

$$\mathcal{D} \text{ is 1-connected : } \quad \{\mathbf{rot}\,\mathbf{V} = 0 \quad \Leftrightarrow \quad \mathbf{V} = -\mathbf{grad}\,\phi\} \quad .$$

The scalar potential is defined to within an arbitrary constant.

In any convex domain, an explicit expression to deduce the scalar potential from the field, is given by:

$$\phi[x,y,z] - \phi[0] = -\int_0^{\{x,y,z\}} \mathbf{F}\cdot \mathbb{d}\mathbf{L} \quad . \tag{16.09.01}$$

This expression can be applied in any convex sub-domain but, for a generic domain, one might be forced to use different vector potentials in different convex sub-domains.

Note that the minus sign in the definition is just conventional in physics. In mathematics the definition is often given without the minus sign.

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Theorem

A vector field defined on all of a 1-connected set, has vanishing rotor if and only if a scalar field exists (the scalar potential) whose gradient is the vector field itself; the scalar potential is defined to within a fixed term.

Counterexample

©|T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...|Vol. 2 - § 10.16||

© - QUOTE

C. Amrouche, C. Bernardi, M. Dauge, and V. Girault,
Vector potentials in three dimensional non-smooth domains,
Math. Methods Appl. Sci., 21, No. 9, 823-864 (1998).
WEB - URL
However, in the three-dimensional case, the situation is much more complicated: a potential can also be used for handling the divergence-free property, However, the conditions that must be enforced in order to ensure its existence and uniqueness are not so simple. Three main difficulties appear:
(i) In contrast to the two-dimensional case, the potential is no longer a scalar function, but a vector function with three components, so it is known as vector potential. As the rotor of a gradient is always zero, a global condition must be added to eliminate such gradients; it is known as the "**gauge fixing condition** " and it usually concerns the divergence of the potential. More precisely, the condition that we propose imposes that the vector potential is itself divergence-free. But such a condition is generally not sufficient for uniqueness: for instance, the gradient of any harmonic scalar function is rotor-free and divergence-free.
(ii) Adequate boundary conditions must be enforced on the vector potential: they concern either its normal or its tangential component (but not both). Consequently, the functional spaces in which the vector potential is sought for, are no longer standard Sobolev spaces and their embedding in these Sobolev spaces depend on the boundary conditions and on the regularity of the domain.
(iii) The geometry of the domain is more complex than in the two-dimensional case, even when it is bounded, since the simple-connexity is no longer linked to the number of connected components of the boundary. So, two different parameters are necessary to characterize the geometry of the domain, related to the number of components of the boundary and to the homotopy group of the domain. They are closely related to the dimensions of the kernels of the problems that we consider, which are spaces of rotor- and divergence-free vector fields with null boundary conditions.

16.09.03.03 Vector Potential and Reconstruction of a Vector Field From Its Rotor

This § is referenced at pages:
[1114, 1743, 1807]

©|T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...|Vol. 2 - § 12.16|*Explicit discussion*|

Let us determine under which conditions a vector field is the rotor of a vector field.

A vector field, $\mathbf{F}$, defined on a subset $\mathcal{D}$ of $\mathbb{R}^3$ with simple surface connection (2-connected), admits a (non-unique) vector potential if and only if it is divergence-less:

$$\mathcal{D} \text{ is 2-connected : } \quad \{\mathbf{div}\,\mathbf{V} = 0 \quad \Leftrightarrow \quad \mathbf{F} = \mathbf{rot}\,\mathbf{A}\} \quad .$$

The vector potential is defined to within the gradient of an arbitrary scalar function.

An explicit expression to deduce the vector potential from the field, the analogous of equation (16.07.01), can be found in [1]. In fact, in any convex domain, an explicit expression to deduce the vector potential

[1] T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...

from the field, is given by:

$$\mathbf{A}[x,y,z] = \int_0^1 t\,\mathbf{F}[tx,ty,tz]\wedge(x\widehat{\mathbf{e}}_1 + y\widehat{\mathbf{e}}_2 + z\widehat{\mathbf{e}}_3)\,\mathrm{d}t \quad . \tag{16.09.02}$$

→ 1807

This expression can be applied in any convex sub-domain but, for a generic domain, one might be forced to use different vector potentials in different convex sub-domains.

Theorem

A vector field defined on all of a 2-connected set, has vanishing divergence if and only if a vector field exists (the vector potential) whose rotor is the vector field itself; the vector potential is defined to within the gradient of an arbitrary function.

Counterexample

©|T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...|Vol. 2 - § 12.16||

16.09.03.04 Independence of Line|Surface Integrals From the Path/surface

According to sections § 16.09.03.02, § 16.09.03.03 the following propositions apply:

- A line integral on a 1-connected domain is independent of the integration line, and only depends on the boundaries of the integration line, if and only if the vector fields is irrotational. In this case the circulation along all curves having the same extremes is the same, and one can just speak of circulation between two points.
- A surface integral on a 2-connected domain is independent of the integration surface, and only depends on the boundaries of the integration surface, if and only if the vector fields is solenoidal. In this case the flux across all surfaces having the same boundary is the same, and one can just speak of flux across a line or flux concatenated to a line.

16.09.03.05 Connections Between the Three Fundamental Theorems

In order for the **rotor theorem** to be well defined it is necessary that the surface-integral of the rotor must be independent of the surface having as a boundary the chosen closed line. That is the flux of a rotor across a closed surface must be zero. Applying the **divergence theorem** to the rotor gives zero as the divergence of a rotor is zero. But it is necessary that the domain is 2-connected.

In order for the fundamental theorem to be be well defined it is necessary that the line-integral of the gradient must be independent of the line having as a boundary the two chosen points. That is the circulation of a gradient across a closed line must be zero. Applying the **rotor theorem** to the gradient gives zero as the rotor of a gradient is zero. But it is necessary that the domain is 1-connected.

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16.10 Vector Field Derivatives

16.10.01 Directional Derivative

$$\boxed{\mathbf{n}\cdot\boldsymbol{\nabla}} \quad . \tag{16.10.01}$$

16.10.02 Convective/Substantial Derivative

This § is referenced at pages:
[Never referenced.]

©|S.Bobbio & E.Gatti, *Elettromagnetismo E Ottica*, 1991, Bollati-Boringhieri, ...Ed., ...|||

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|6.1||

©|E.J.Rothwell and M. J.Cloud|ElectroMagnetics|*A.2*|

©|T.Frankel|The Geometry of Physics, 3rd-Ed|*problem 4.3, pag. 143*|

Let $\mathbf{A}[\mathbf{x},t]$ and $\mathbf{B}[\mathbf{x},t]$ be time-dependent vector fields and let $\rho[\mathbf{x},t]$ be a time-dependent scalar field.
The following classical expressions for the time derivatives of line, surface and volume integrals over moving domains apply:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{C[t]}\mathbf{A}\cdot\mathrm{d}\mathbf{L}=\int\limits_{C[t]}\left(\partial_t\mathbf{A}+\mathbf{grad}(\mathbf{v}\cdot\mathbf{A})-\mathbf{v}\wedge\mathbf{rot}\,\mathbf{A}\right)\cdot\mathrm{d}\mathbf{L}} \quad , \quad ,$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Sigma[t]}\mathbf{B}\cdot\mathrm{d}\mathbf{S}=\iint\limits_{\Sigma[t]}\left(\partial_t\mathbf{B}+(\mathbf{div}\mathbf{B})\,\mathbf{v}-\mathbf{rot}(\mathbf{v}\wedge\mathbf{B})\right)\cdot\mathrm{d}\mathbf{S}} \quad , \quad ,$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{\Omega[t]}\rho\,\mathrm{d}V=\iiint\limits_{\Omega[t]}\left(\partial_t\rho+\mathbf{div}(\rho\mathbf{v})\right)\,\mathrm{d}V} \quad .$$

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16.11 Summary of Formulae

This § is referenced at pages:
[1102, 1102]

The page at figure 16.11 is taken unmodified from Physics Formulary by Johan Wevers.
Web link at: WEB - URL.

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The ∇-operator

In cartesian coordinates (x, y, z) holds:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z \ , \ \text{grad} f = \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z$$

$$\text{div}\ \vec{a} = \vec{\nabla}\cdot\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \ , \ \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\text{rot}\ \vec{a} = \vec{\nabla}\times\vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{e}_z$$

In cylinder coordinates (r, φ, z) holds:

$$\vec{\nabla} = \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial \varphi}\vec{e}_\varphi + \frac{\partial}{\partial z}\vec{e}_z \ , \ \text{grad} f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi + \frac{\partial f}{\partial z}\vec{e}_z$$

$$\text{div}\ \vec{a} = \frac{\partial a_r}{\partial r} + \frac{a_r}{r} + \frac{1}{r}\frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z} \ , \ \nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\text{rot}\ \vec{a} = \left(\frac{1}{r}\frac{\partial a_z}{\partial \varphi} - \frac{\partial a_\varphi}{\partial z}\right)\vec{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\vec{e}_\varphi + \left(\frac{\partial a_\varphi}{\partial r} + \frac{a_\varphi}{r} - \frac{1}{r}\frac{\partial a_r}{\partial \varphi}\right)\vec{e}_z$$

In spherical coordinates (r, θ, φ) holds:

$$\begin{aligned}
\vec{\nabla} &= \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\vec{e}_\varphi \\
\text{grad} f &= \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi \\
\text{div}\ \vec{a} &= \frac{\partial a_r}{\partial r} + \frac{2a_r}{r} + \frac{1}{r}\frac{\partial a_\theta}{\partial \theta} + \frac{a_\theta}{r\tan\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial \varphi} \\
\text{rot}\ \vec{a} &= \left(\frac{1}{r}\frac{\partial a_\varphi}{\partial \theta} + \frac{a_\theta}{r\tan\theta} - \frac{1}{r\sin\theta}\frac{\partial a_\theta}{\partial \varphi}\right)\vec{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial \varphi} - \frac{\partial a_\varphi}{\partial r} - \frac{a_\varphi}{r}\right)\vec{e}_\theta + \\
&\quad \left(\frac{\partial a_\theta}{\partial r} + \frac{a_\theta}{r} - \frac{1}{r}\frac{\partial a_r}{\partial \theta}\right)\vec{e}_\varphi \\
\nabla^2 f &= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\tan\theta}\frac{\partial f}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}
\end{aligned}$$

General orthonormal curvelinear coordinates (u, v, w) can be obtained from cartesian coordinates by the transformation $\vec{x} = \vec{x}(u, v, w)$. The unit vectors are then given by:

$$\vec{e}_u = \frac{1}{h_1}\frac{\partial \vec{x}}{\partial u} \ , \ \vec{e}_v = \frac{1}{h_2}\frac{\partial \vec{x}}{\partial v} \ , \ \vec{e}_w = \frac{1}{h_3}\frac{\partial \vec{x}}{\partial w}$$

where the factors h_i set the norm to 1. Then holds:

$$\begin{aligned}
\text{grad} f &= \frac{1}{h_1}\frac{\partial f}{\partial u}\vec{e}_u + \frac{1}{h_2}\frac{\partial f}{\partial v}\vec{e}_v + \frac{1}{h_3}\frac{\partial f}{\partial w}\vec{e}_w \\
\text{div}\ \vec{a} &= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u}(h_2 h_3 a_u) + \frac{\partial}{\partial v}(h_3 h_1 a_v) + \frac{\partial}{\partial w}(h_1 h_2 a_w)\right) \\
\text{rot}\ \vec{a} &= \frac{1}{h_2 h_3}\left(\frac{\partial(h_3 a_w)}{\partial v} - \frac{\partial(h_2 a_v)}{\partial w}\right)\vec{e}_u + \frac{1}{h_3 h_1}\left(\frac{\partial(h_1 a_u)}{\partial w} - \frac{\partial(h_3 a_w)}{\partial u}\right)\vec{e}_v + \\
&\quad \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 a_v)}{\partial u} - \frac{\partial(h_1 a_u)}{\partial v}\right)\vec{e}_w \\
\nabla^2 f &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial w}\right)\right]
\end{aligned}$$

Figure 16.11: FIGURE

16.12 Examples and Physical Applications

16.12.01
Unit Vectors in the Plane in Polar Coordinates

Given the time derivative of one of the two unit vectors for polar coordinates in the plane, $\widehat{\mathbf{e}}_r$ and $\widehat{\mathbf{e}}_\theta$, deduce the second one from the orthonormality condition.

WORKED SOLUTION

$$0 = \widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{e}}_\theta \quad ,$$

take the time derivative

In 3D the trick generalizes as in section § 23.06.02.02 and equation (23.06.03).

16.12.02
☒ Gradient in Spherical Coordinates

Esprimere il gradiente in coordinate sferiche e in coordinate curvilinee generiche.

WORKED SOLUTION

Dato un campo scalare $U[\mathbf{r}]$ si ha, per definizione di gradiente:

$$\mathrm{d}U[\mathbf{r}] \equiv \mathbf{grad}U \cdot \mathrm{d}\mathbf{r} \quad ,$$

che è invariante, cioé indipendente dal Coordinate System.
Esprimendo $\mathrm{d}\mathbf{r}$ in coordinate polari si ha:

$$\mathrm{d}\mathbf{r} = \mathrm{d}r\widehat{\mathbf{e}}_r + r\,\mathrm{d}\theta\widehat{\mathbf{e}}_\theta + r\sin[\theta]\,\mathrm{d}\phi\widehat{\mathbf{e}}_\phi \quad ,$$

da cui seguono le componenti di $\mathbf{grad}U$ in coordinate sferiche:

$$\mathbf{grad}U = \begin{cases} \mathbf{grad}U\big|_r = \dfrac{\partial U}{\partial r} \quad , \\ \mathbf{grad}U\big|_\theta = \dfrac{1}{r}\dfrac{\partial U}{\partial \theta} \quad , \\ \mathbf{grad}U\big|_\phi = \dfrac{1}{r\sin[\theta]}\dfrac{\partial U}{\partial \phi} \quad . \end{cases}$$

Per un generale sistema di coordinate ortonormali si ha:

$$\mathrm{d}\mathbf{r} = \sum_{k=1}^{3} h_k\,\mathrm{d}\ell_k \mathbf{e}_k \quad ,$$

da cui

$$\mathrm{d}U \equiv \mathbf{grad}U \cdot \mathrm{d}\mathbf{r} = \sum_{k=1}^{3} h_k\,\mathbf{grad}U\Big|_k\,\mathrm{d}\ell_k = \sum_{k=1}^{3}\frac{\partial U}{\partial \ell_k}\,\mathrm{d}\ell_k \quad ,$$

da cui segue

$$\mathbf{grad}U\Big|_k = \frac{1}{h_k}\frac{\partial U}{\partial \ell_k} \quad .$$

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16.12.03
☒ Visualization of 3D ElectroMagnetic Fields

©|WEB - URL||*Excellent for visualization of many examples of ElectroStatics 3D fields*|

©|WEB - URL||*Excellent for visualization of many examples of MagnetoStatics 3D fields*|

©|WEB - URL||*Excellent for visualization of many examples of vector 3D fields*|

Visualization of many examples of 3D fields

16.12.04
✓ Graphical Examples of Vector Fields in 2D

In this section a number of graphical representations of two-dimensional vector fields are shown. The limitation to two-dimensional fields is not essential and it is used only to simplify and make clearer the graphical representations. In fact the general definitions in section § 16.06 can be straightforwardly restricted to two-dimensional fields. Alternatively one can consider the fields as three-dimensional fields with no z-dependence, $\frac{\partial}{\partial z}\{\ldots\} = 0$, and zero z-component, $V_z = 0$. In this case the expressions in section § 16.06.03.01 show that the gradient has a zero z-component, the divergence is limited to the partial derivatives with respect to x and y while the rotor only has a non-zero z-component.

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16.12.04.01 Example: Radial Linear Field

This § is referenced at pages:
[1125]

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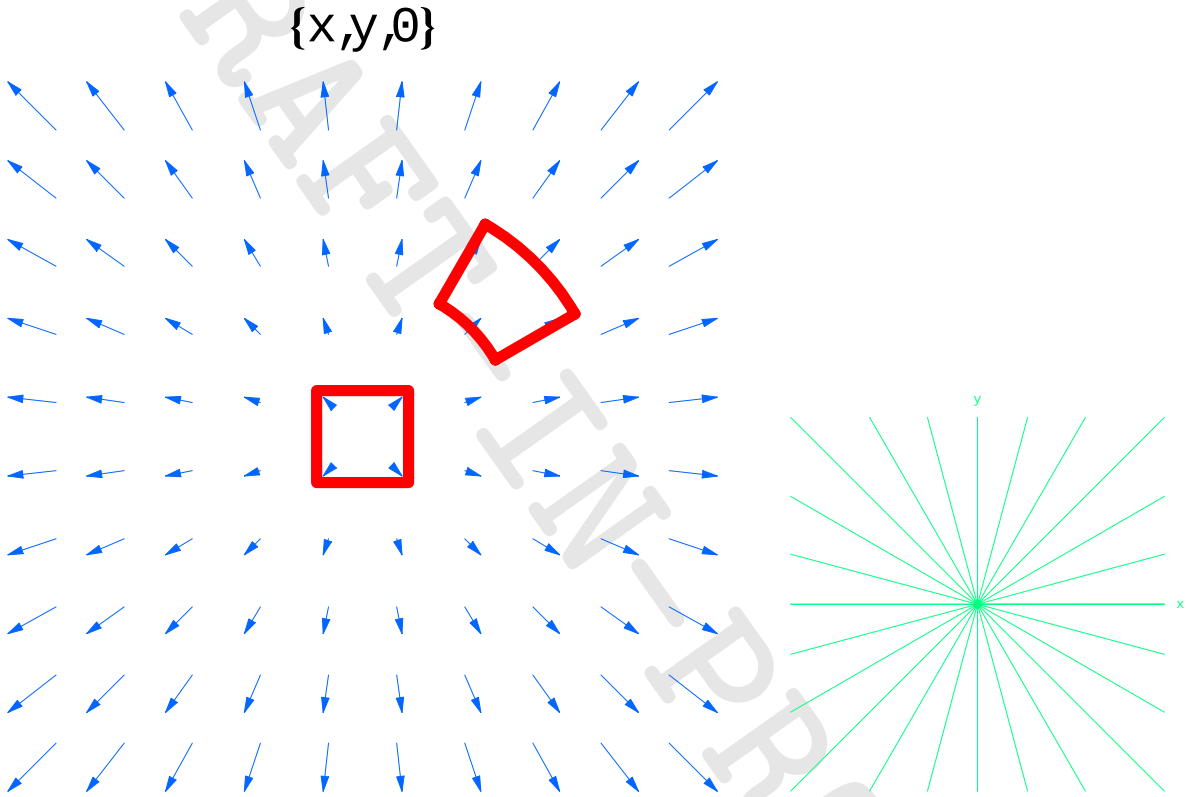


(a) The Vector Field (b) The Field Lines

Figure 16.12: ...

It is a central field, that is a radial and conservative field.
In 3D:

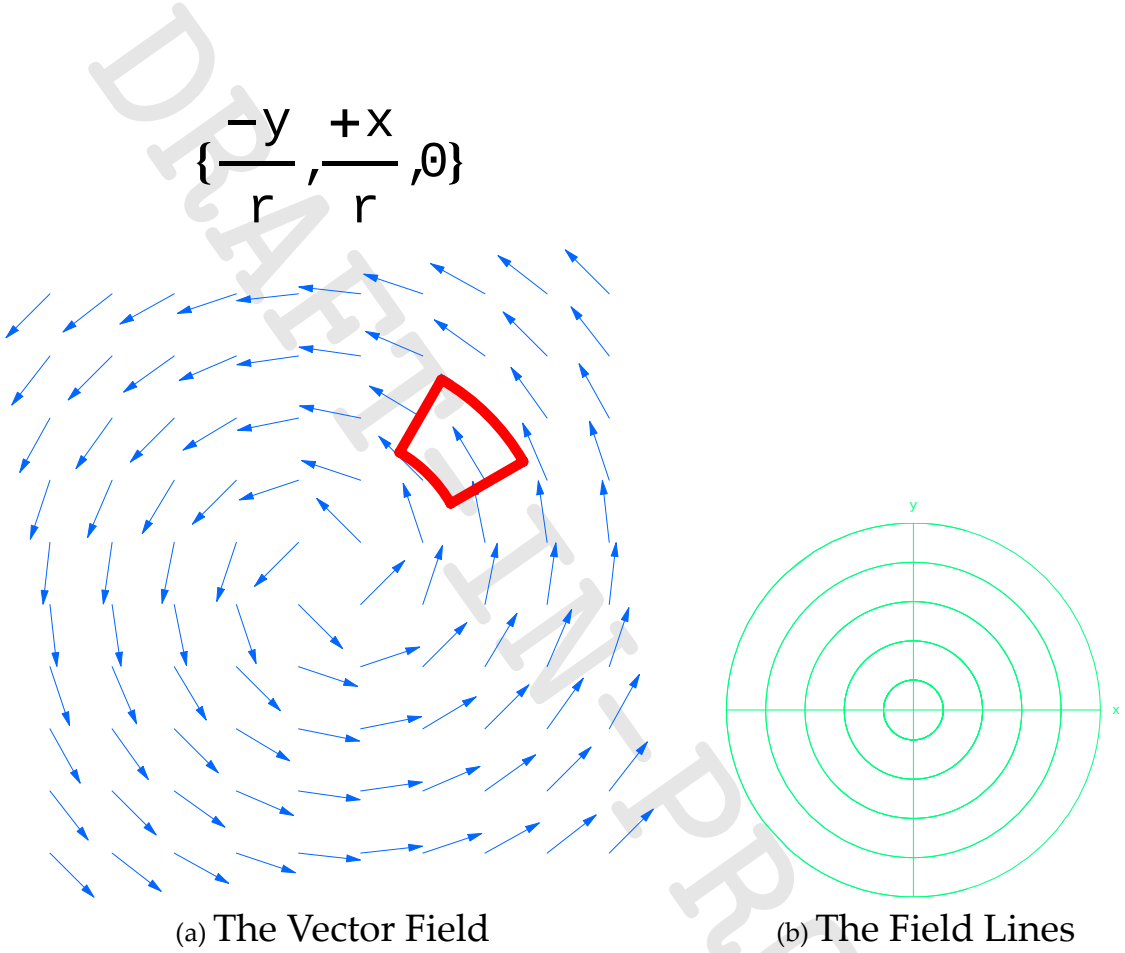
$$\mathbf{V}[\mathbf{r}] = \alpha\mathbf{r} \propto \mathbf{r} \quad ,$$
$$\alpha > 0 \qquad \text{in the figure} \quad ,$$
$$\mathbf{div}\,\mathbf{V} = 3\alpha \qquad (|\mathbf{V}| \text{ increases with } |\mathbf{r}|; \text{ then } \Longrightarrow \mathbf{div}\,\mathbf{V} > 0) \quad ,$$
$$\mathbf{rot}\,\mathbf{V} = 0 \qquad \text{(it is a conservative field)}$$

Examples:

- a uniformly charged sphere (inside the sphere);
- for $\alpha < 0$: elastic force.

16.12.04.02 Example: Tangential Constant Module Field

This § is referenced at pages:
[1125]



(a) The Vector Field (b) The Field Lines

Figure 16.13: ...

It is a purely tangential field. Its magnitude is independent from r and θ.
In 3D:

$$\mathbf{V}[\mathbf{r}] = \alpha \widehat{\mathbf{e}}_\psi = \alpha \left(\frac{-y}{r} \widehat{\mathbf{e}}_1 + \frac{+x}{r} \widehat{\mathbf{e}}_2 \right) \quad ,$$

$$\alpha > 0 \qquad \text{in the figure} \quad ,$$

$$\mathbf{div}\,\mathbf{V} = 0 \qquad \text{(So much enters, so much exits.)} \quad ,$$

$$\mathbf{rot}\,\mathbf{V} = \widehat{\mathbf{e}}_3 \frac{\alpha}{r} \qquad \text{(The magnitude is independent from } r\text{.)}$$

Examples:

- the magnetic field inside an infinite circular cylinder wire with $\mathbf{j} \propto \widehat{\mathbf{e}}_3 / r$.

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16.12.04.03 Example: Tangential Inverse Distance Module Field

This § is referenced at pages:
[Never referenced.]

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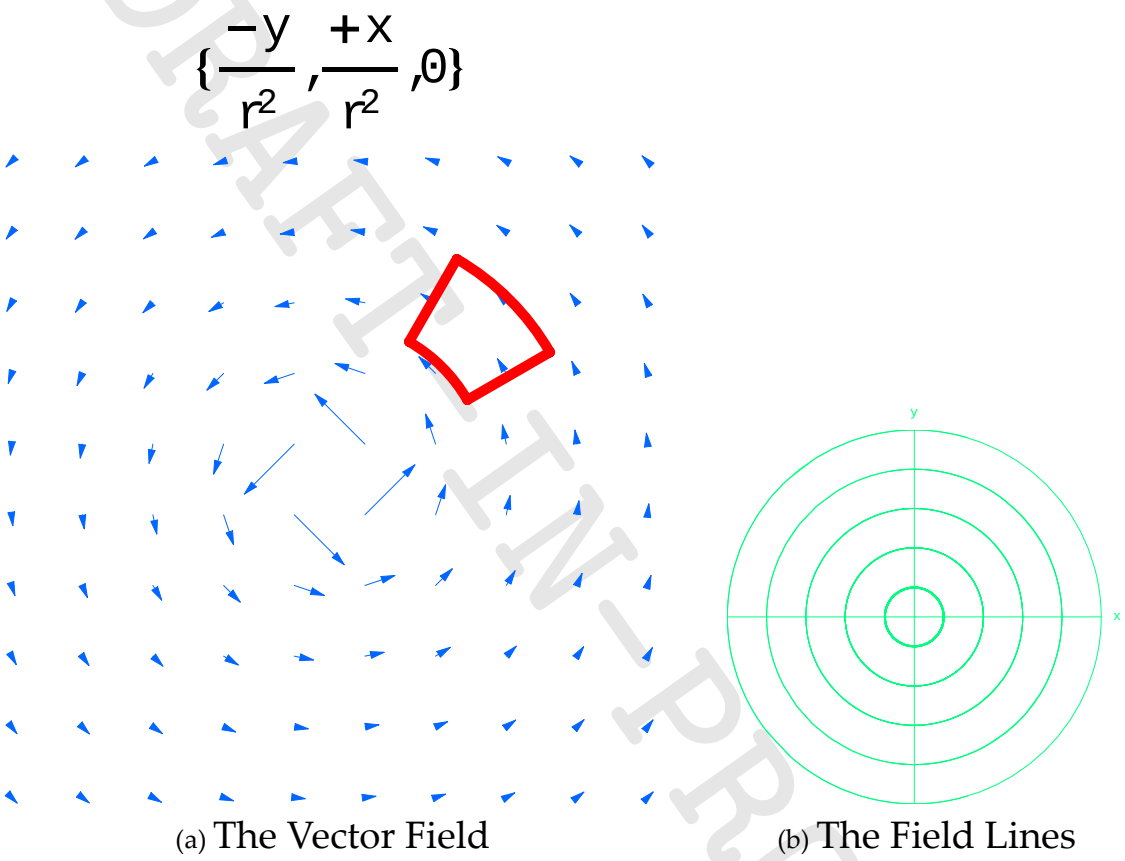


(a) The Vector Field (b) The Field Lines

Figure 16.14: ...

It is a purely tangential field. Its magnitude decreases with increasing r.
In 3D:

$$\mathbf{V}[\mathbf{r}] = \frac{\beta}{r}\widehat{\mathbf{e}}_\psi = \alpha\left(\frac{-y}{r^2}\widehat{\mathbf{e}}_1 + \frac{+x}{r^2}\widehat{\mathbf{e}}_2\right) \quad ,$$

$$\beta > 0 \qquad \text{in the figure} \quad ,$$

$$\mathbf{div}\,\mathbf{V} = 0 \qquad \text{(So much enters, so much exits.)} \quad ,$$

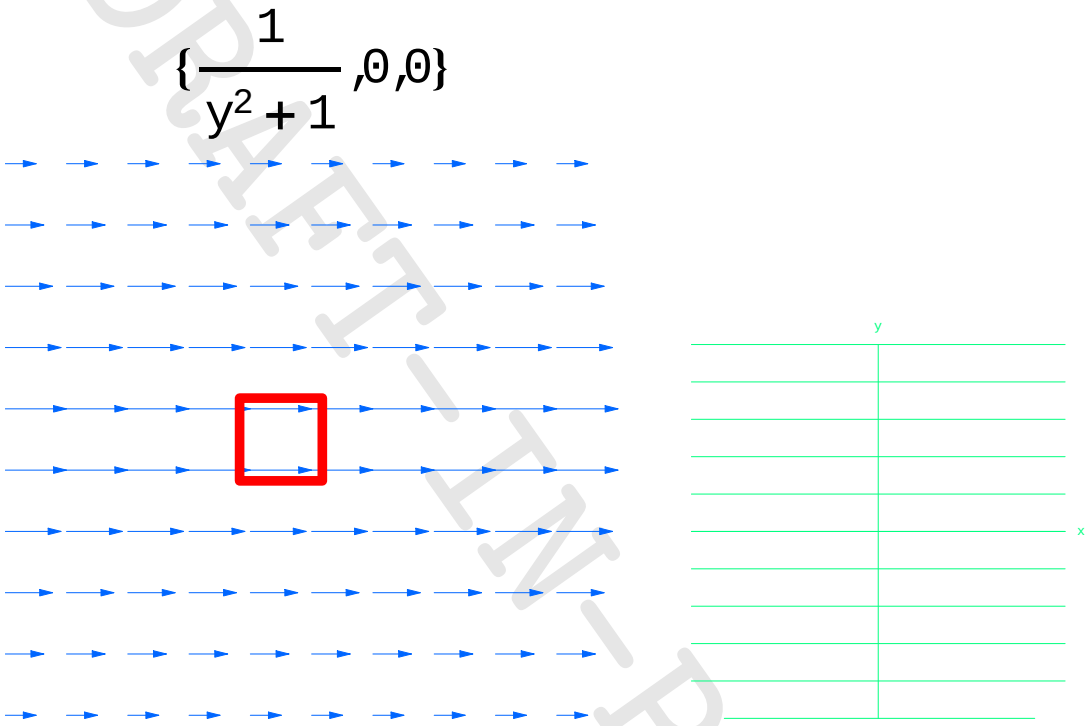
$$\mathbf{rot}\,\mathbf{V} = 0 \qquad \text{(The module decreases with } r\text{: circulation might be zero.)}$$

Examples:

- the magnetic field outside an infinite circular cylinder wire (it is consistent with **Maxwell Equations**);

16.12.04.04 Example: X Field Y-Dependent

This § is referenced at pages:
[Never referenced.]



(a) The Vector Field (b) The Field Lines

Figure 16.15: ...

In 3D:

$$\mathbf{V}[\mathbf{r}] = f[y]\hat{\mathbf{e}}_1 \qquad f[y] \text{ an arbitrary function.} \quad ,$$

$$\mathbf{div}\,\mathbf{V} = \frac{\partial f[y]}{\partial x} = 0 \qquad \text{(So much enters, so much exits.)} \quad ,$$

$$\mathbf{rot}\,\mathbf{V} \neq 0 \qquad \text{(the integral around the path is not zero)} \quad ,$$

$$\left(\frac{\partial V_x}{\partial y} = \frac{\partial f[y]}{\partial y} \neq 0\right) \quad .$$

Examples:

- water flow in a river;

Note:

- the field is divergence-less but field lines are not closed: compare with the magnetic field; read § 16.12.19.
- the field has a rotor different from zero, but there is no curling/rotoring of the field lines.

DRAFT-IN-PROGRESS

16.12.04.05 Example: X Field X-Dependent

This § is referenced at pages:
[Never referenced.]

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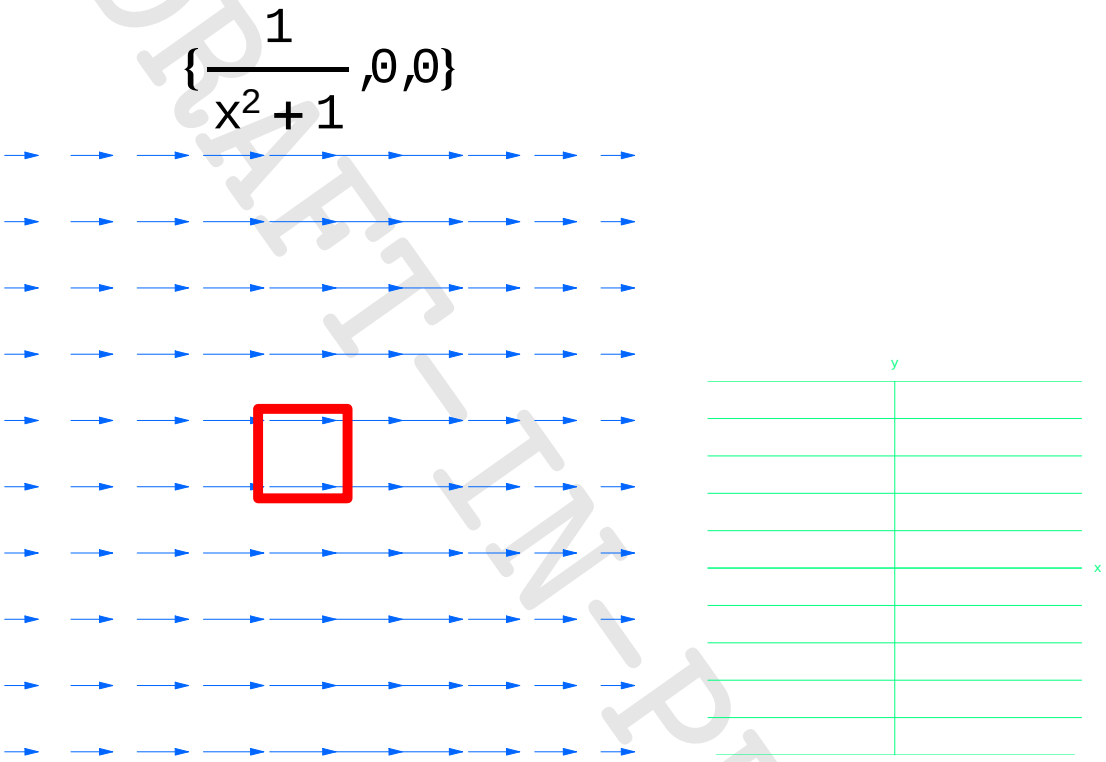

(a) The Vector Field (b) The Field Lines

Figure 16.16: ...

In 3D:

$$\mathbf{V}[\mathbf{r}] = f[x]\widehat{\mathbf{e}}_1 \qquad f[x] \text{ an arbitrary function.} \quad ,$$

$$\mathbf{div}\,\mathbf{V} = \frac{\partial f[x]}{\partial x} \neq 0 \qquad \text{(In general what enters is different from what exits.)} \quad ,$$

$$\mathbf{rot}\,\mathbf{V} = 0 \qquad \text{(the integral around the path is zero)} \quad ,$$

$$(\frac{\partial V_x}{\partial y} = \frac{\partial V_x}{\partial z} = 0.)$$

Examples:

- ...

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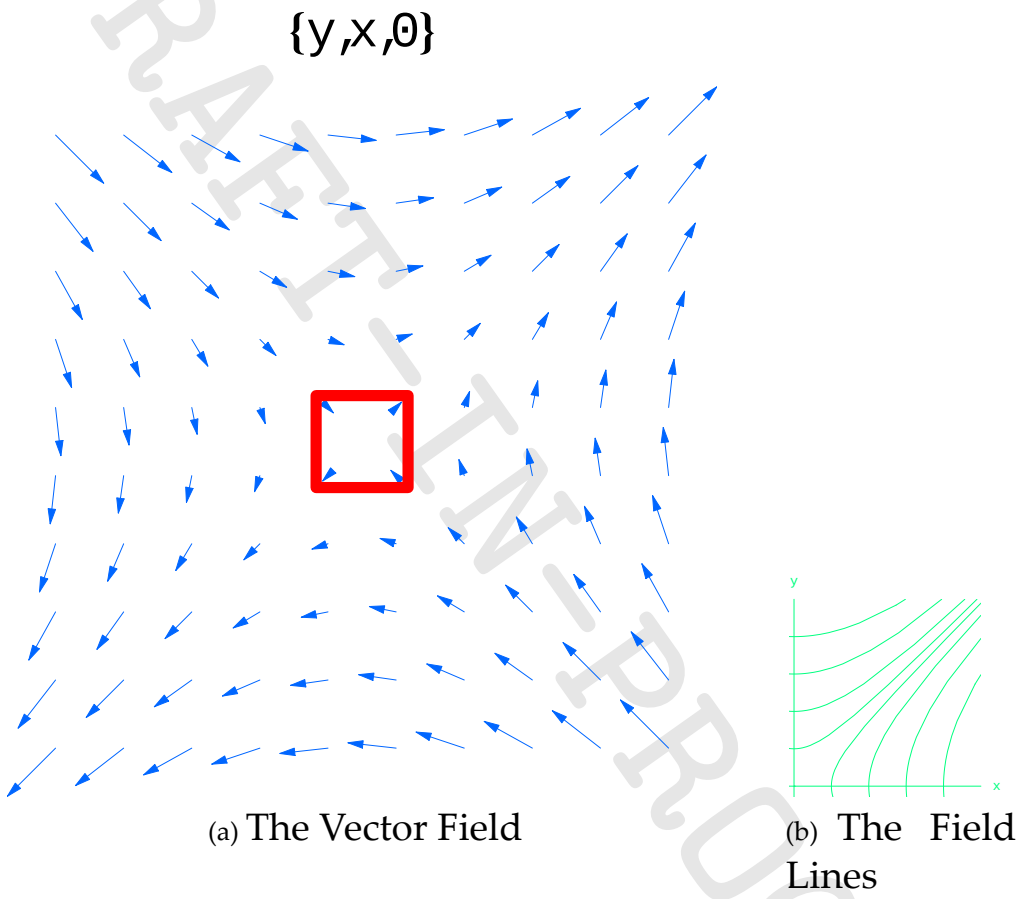
16.12.04.06 Example: Quadrupole Field

This § is referenced at pages:
[1795]



(a) The Vector Field (b) The Field Lines

Figure 16.17: ...

In 3D:

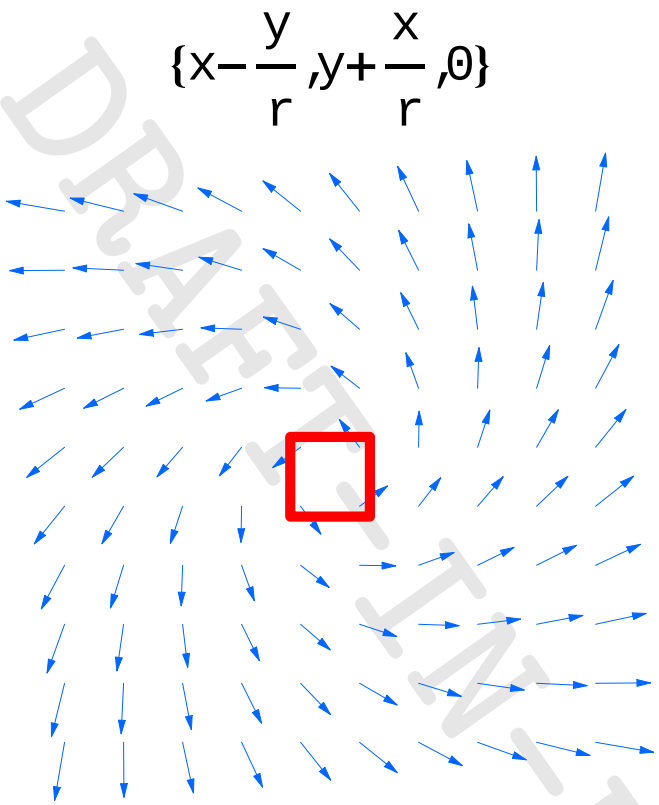
$$\mathbf{V}[\mathbf{r}] = k\,(y\widehat{\mathbf{e}}_1 + x\widehat{\mathbf{e}}_2) \quad \dots \quad ,$$
$$\mathbf{div}\,\mathbf{V} = 0 \quad \text{(The divergence might be zero.)} \quad ,$$
$$\mathbf{rot}\,\mathbf{V} = 0 \quad \text{(The rotor might be zero.)}$$

Example (consistent with **Maxwell Equations** outside sources):

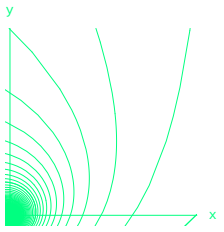
- electric field of four point equal charges at the four corners (ElectroStatic **Quadrupole**): positive in the upper-left and lower-right corners; negative in the lower-left and upper-right corners;
- magnetic field of four infinite rectilinear equal current wires at the center of the four sides, perpendicular to the plane (magnetic **Quadrupole**): entering in the left and right sides; exiting in the lower and upper sides;

DRAFT-IN-PROGRESS

16.12.04.07 Sum of a Radial and a Tangential Field



(a) The Vector Field



(b) The Field Lines

Figure 16.18: ...

The coefficient $\alpha = 1$ is introduced in order to fix the physical dimensions.
Sum of the vector fields of sections § 16.12.04.02 and § 16.12.04.01: linearity of the operators.
In 3D:

$$\mathbf{V}[\mathbf{r}] = \left(x - \alpha\frac{y}{r}\right)\widehat{\mathbf{e}}_1 + \left(y + \alpha\frac{+x}{r}\right)\widehat{\mathbf{e}}_2 \qquad \text{...} \quad ,$$

$$\mathbf{div}\,\mathbf{V} \neq 0 \qquad \text{(The divergence is clearly not zero.)} \quad ,$$

$$\mathbf{rot}\,\mathbf{V} \neq 0 \qquad \text{(The rotor is clearly not zero.)}$$

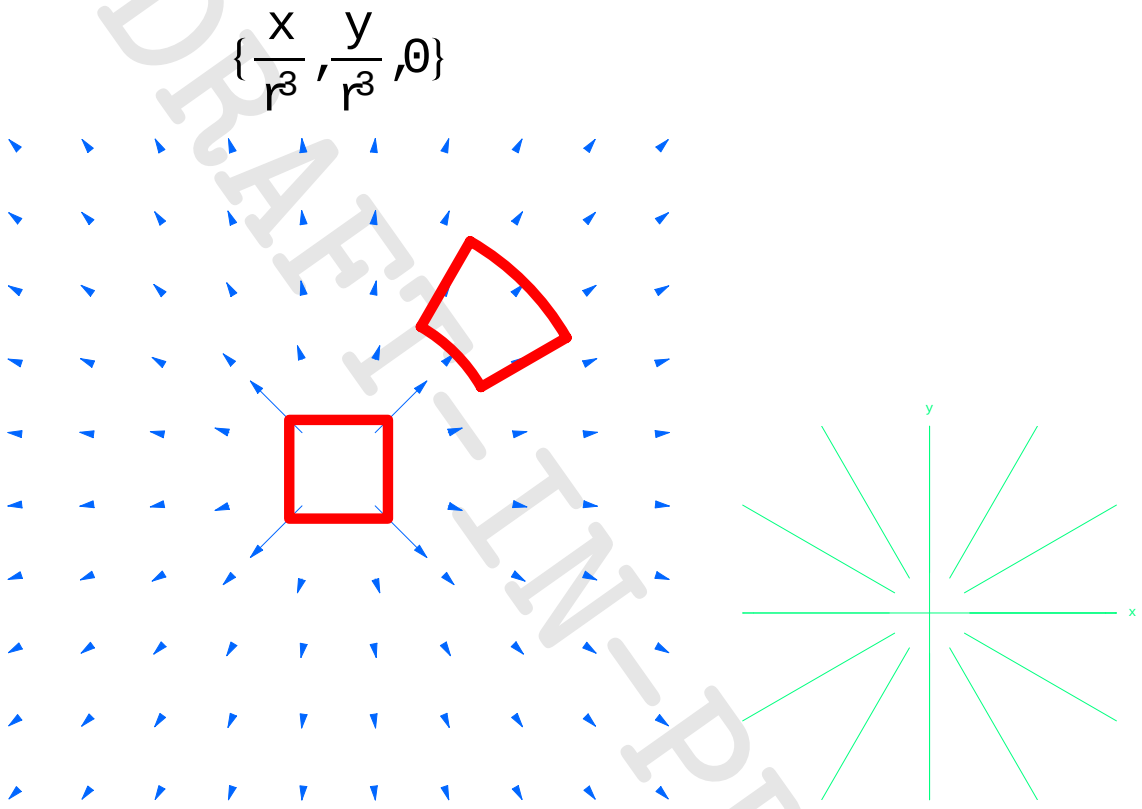
Examples:

- ...

DRAFT-IN-PROGRESS

16.12.04.08 Divergence-Less Functions

This example shows an important warning: while the r^{-2} function is divergence-less in three dimensions, it is not divergence-less in two-dimension! The r^{-1} function is divergence-less in two dimensions.



(a) The Vector Field (b) The Field Lines

Figure 16.19: ...

DRAFT-IN-PROGRESS

16.12.05
Spherical Mathematics

This § is referenced at pages:
[Never referenced.]

©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|§ 4.5, 4.6, 4.7||

16.12.06
Large-Distance Multipole Development

This § is referenced at pages:
[1569]

©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|§ 4.4|*Excellent*|

Read § 29.05, § 36.14, § 36.14, § 36.14.06, § 36.14.07, § 29.05.04.

$$Q^{\cdot\cdot}_{rs} = \frac{1}{2}(c_r p_s + c_s p_r)$$

16.12.07
Properties of a Distribution of Currents with compact support

This § is referenced at pages:
[1739]

Consider an arbitrary current distribution, with limited extension in space, that is:

$$\text{a sphere of radius } R \text{ exists such that} \qquad \mathbf{j}[\mathbf{r}] = 0 \qquad \text{for} \qquad |\mathbf{r}| > R \quad ,$$

that is **a function with compact support**, that is **a function which vanishes at infinity**.

1. Assume a steady current distribution, that is:

$$\mathbf{j}[\mathbf{r}, t] = \mathbf{j}[\mathbf{r}] \quad .$$

The following identity can be derived integrating over the sphere of radius R (volume Ω), at the frontier of which $\mathbf{j}[\mathbf{r}] = 0$:

$$0 = \oiint_{\partial\Omega} g[\mathbf{r}]\mathbf{j}[\mathbf{r}] \cdot \mathbb{d}\mathbf{S} = \iiint_{\Omega} \mathbf{div}(g[\mathbf{r}]\mathbf{j}[\mathbf{r}])\, \mathbb{d}V = \iiint_{\Omega} \mathbf{j}[\mathbf{r}] \cdot \mathbf{grad} g[\mathbf{r}]\, \mathbb{d}V + \iiint_{\Omega} g[\mathbf{r}] \mathbf{div}\mathbf{j}[\mathbf{r}]\, \mathbb{d}V \quad .$$

From the steady condition and the continuity equation one has:

$$\mathbf{div}\mathbf{j}[\mathbf{r}] = 0 \quad .$$

It follows:

$$\boxed{\iiint_{\Omega} \mathbf{j}[\mathbf{r}] \cdot \mathbf{grad} g[\mathbf{r}]\, \mathbb{d}V = 0 \qquad \text{for any function } g[\mathbf{r}]} \quad . \qquad (16.12.01) \quad \rightarrow 1736$$

A particular case:

$$g[\mathbf{r}] = \mathbf{n}\cdot\mathbf{r} \implies \boxed{\mathbf{n} \cdot \iiint_{\Omega} \mathbf{j}\, \mathbb{d}V = 0 \qquad \text{for any function } \widehat{\mathbf{n}} \implies 0 = \iiint_{\Omega} \mathbf{j}\, \mathbb{d}V} \quad . \qquad (16.12.02) \quad \rightarrow 1782$$

2. With no further assumptions than compact support:

$$\boxed{\iiint_{\Omega} \mathbf{div}\mathbf{j}[\mathbf{r}, t]\, \mathbb{d}V = 0} \quad , \qquad (16.12.03) \quad \rightarrow 1740$$

as it is easily shown with the above derivation and $g[\mathbf{r}] = 1$.

DRAFT IN PROGRESS

16.12.08
Applications to Equations of Physics

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.12||

16.12.09
☒ Coulomb Law

Deduce the Coulomb law from the expression of the electric potential in ElectroStatics:

$$\mathbf{E} = -\mathbf{grad}\Phi \qquad \Phi[\mathbf{x}] = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}V_{\mathrm{y}} \quad .$$

Use equation (16.13.01).

16.12.10
☒ Biot-Savart Law

Deduce the Biot-Savart law from the expression of the **vector magnetic potential** in MagnetoStatics:

$$\mathbf{B} = \mathbf{rot}\,\mathbf{A} \qquad \mathbf{A}[\mathbf{x}] = \frac{\mu_0}{4\pi}\iiint \frac{\mathbf{j}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}V_{\mathrm{y}} \quad .$$

Use equation (16.13.01).

16.12.11
A Rigid Motion

Consider the velocity field, $\mathbf{v}[\mathbf{x}]$, of the points at rest with a Rest Frame in arbitrary motion with respect to another fixed Rest Frame:

$$\mathbf{v}[\mathbf{x}] = \mathbf{v}_0 + \boldsymbol{\omega}\wedge(\mathbf{x}-\mathbf{x}_0) \quad ,$$

with $\mathbf{x}_0$, $\mathbf{v}_0$ and $\boldsymbol{\omega}$ (angular velocity) fixed vectors. Note that this is, in fact, the distrobution of the velocities of a rigid body, see equation (26.03.02).

Show that the rotor of the vector field is twice the angular velocity:

$$\mathbf{rot}\,\mathbf{v}[\mathbf{x}] = 2\boldsymbol{\omega} \quad . \tag{16.12.04}$$

This problem shows that the rotor of a vector field has to do with *rotational* properties of the field.

WORKED SOLUTION

By direct computation:

$$\mathbf{rot}\,\mathbf{v}[\mathbf{x}] = \mathbf{rot}(\boldsymbol{\omega}\wedge\mathbf{x}) = 2\boldsymbol{\omega} \quad .$$

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16.12.12

☒ Electric Dipole | Magnetic Dipole - From Potentials to Fields

This § is referenced at pages:
[1714, 1714, 2506]

1. Determine the electric field, equation (36.07.01), produced by an **Electric Dipole** $\mathbf{p}$ whose electric potential is:

$$\Phi[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \quad . \tag{16.12.05}$$ → 1759

2. Determine the magnetic field, equation (36.07.02), produced by a **Magnetic Dipole** $\mathbf{m}$ whose **vector magnetic potential** is:

$$\mathbf{A}[\mathbf{r}] = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\wedge\mathbf{r}}{r^3} \quad . \tag{16.12.06}$$

16.12.13

☒ Dipole Field Lines

Determinare l'equazione delle linee di forza del campo prodotto da un dipolo ideale.

WORKED SOLUTION

Consideriamo, per fissare le idee, il campo elettrico di dipolo elettrico ideale:

$$\mathbf{E}[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\,(\mathbf{p}\cdot\mathbf{r})\,\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}\right) \quad .$$

è la somma di un termine radiale non isotropo e di un termine a direzione e verso fissati, entrambi con dipendenza dall'inverso del cubo della distanza dal dipolo.
Si consideri un Coordinate System sferiche centrato sul dipolo e con asse z parallelo al momento di dipolo $\mathbf{p}$. Proiezione sul unit vector radiale del campo elettrico:

$$E_r\widehat{\mathbf{e}}_r \equiv (\,\mathbf{E}\cdot\widehat{\mathbf{e}}_r\,)\,\widehat{\mathbf{e}}_r = \frac{1}{4\pi\varepsilon_0}\left(\frac{2\,(\,\mathbf{p}\cdot\mathbf{r}\,)}{r^4}\right)\widehat{\mathbf{e}}_r = \frac{1}{4\pi\varepsilon_0}\left(\frac{2p\cos[\theta]}{r^3}\right)\widehat{\mathbf{e}}_r \quad . \tag{16.12.07}$$

Proiezione sul unit vector polare del campo elettrico:

$$E_\theta\widehat{\mathbf{e}}_\theta \equiv (\,\mathbf{E}\cdot\widehat{\mathbf{e}}_\theta\,)\,\widehat{\mathbf{e}}_\theta = \mathbf{E} - E_r\widehat{\mathbf{e}}_r = \frac{1}{4\pi\varepsilon_0}\left(\frac{(\,\mathbf{p}\cdot\mathbf{r}\,)\,\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}\right) \quad . \tag{16.12.08}$$ → 1130

The radial component of the field is thus:

$$E_r = \frac{1}{4\pi\varepsilon_0}\left(\frac{2p\cos[\theta]}{r^3}\right) \quad . \tag{16.12.09}$$

In order to find an explicit expression for the component E_θ one can proceed in three ways, at least. The result is:

$$E_\theta = \frac{1}{4\pi\varepsilon_0}\left(\frac{p\sin[\theta]}{r^3}\right) \quad . \tag{16.12.10}$$ → 1130

A first method to find an explicit expression for the component E_θ is the following.
Data la scelta del Coordinate System si ha:

$$\widehat{\mathbf{e}}_3 \parallel \mathbf{p} \quad .$$

Allora si può scrivere:

$$\widehat{\mathbf{e}}_\psi \parallel \mathbf{r}\wedge\widehat{\mathbf{e}}_3 \quad ,$$
$$\widehat{\mathbf{e}}_\theta \parallel \widehat{\mathbf{e}}_r\wedge\widehat{\mathbf{e}}_\psi$$

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ovvero

$$\widehat{\mathbf{e}}_\theta \parallel \mathbf{r}\wedge(\mathbf{p}\wedge\mathbf{r}) = r^2\mathbf{p} - \mathbf{r}\,(\mathbf{p}\cdot\mathbf{r}) \quad .$$

Normalizzando e definendo il segno nel modo corretto ($\widehat{\mathbf{e}}_\theta$ nel verso di θ crescente) si ha:

$$\widehat{\mathbf{e}}_\theta = \frac{\mathbf{r}\,(\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{pr^2\sin[\theta]} \qquad \text{per } \theta \neq (0,\pi) \quad . \tag{16.12.11}$$

→ 1130

Dal confronto tra (16.12.11), (16.12.08) si deduce il risultato (16.12.10). One should keep in mind that $\sin[\theta]$ is always positive in this problem. The sign to assign to E_θ has to be checked carefully.
A second method to find the explicit expression (16.12.10) for the component E_θ is the following. It consists in calculating $\mathbf{E}_\theta$ directly from the module of expression (16.12.08) by taking the scalar product of expression (16.12.08) with itself.
A third method to find an explicit expression (16.12.10) for the component E_θ is the following. It consists in calculating calculating $\mathbf{E}_\theta$ as the components of vector $\mathbf{E}$ perpendicular to $\mathbf{r}$, as in expression (15.03.10).
Le linee di forza del campo elettrico, $\mathbf{r}[s]$, sono determinate dall'equazione:

$$\boxed{\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \mathbf{E}} \quad . \tag{16.12.12}$$

→ 1796

One has:

$$\mathrm{d}\mathbf{r} = \mathrm{d}r\widehat{\mathbf{e}}_r + r\,\mathrm{d}\theta\widehat{\mathbf{e}}_\theta \quad .$$

Da quanto ricavato sopra si scrivono dunque le componenti del vettore campo elettrico in coordinate polari e si eguagliano alle componenti del campo di linee di forza (taking into account that the factor $\frac{1}{4\pi\varepsilon_0}$ is irrelevant):

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \frac{2p\cos[\theta]}{r^3} \quad ,$$
$$r\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{p\sin[\theta]}{r^3} \quad .$$

Per risolvere il sistema di equazioni si divide membro a membro e si separano le variabili:

$$r = r[\theta[s]] \quad \Longrightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}s} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s} \quad \Longrightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}s}\frac{1}{r} = 2\frac{\mathrm{d}\theta}{\mathrm{d}s}\cot\theta \quad ,$$

da cui la soluzione:

$$r = c\,(\sin[\theta])^2 \quad ,$$

dove la costante di integrazione, c, corrisponde alla distanza della linea di forza dal dipolo sul piano equatoriale dello stesso ($\theta = \pi/2$). Nota: $\int\cot\theta = \log[|\sin[\theta]|]$.
In orthonormal Cartesian Coordinates Coordinate System the equation becomes:

$$\left(x^2+y^2\right)^3 = c^2y^4 \quad .$$

16.12.14

☒ Dipole Field in Spherical Coordinates

Show that a **Dipole** field (either electric or magnetic), $\mathbf{F}$,

$$\mathbf{F}[\mathbf{r}] \propto (2\cos[\theta]\,\widehat{\mathbf{e}}_r + \sin[\theta]\,\widehat{\mathbf{e}}_\theta) \quad ,$$

where the **Dipole** moment, $\mathbf{p}$, located at the origin of the Coordinate System, is either the **Electric Dipole** moment, $\mathbf{p}$, or the **Magnetic Dipole** moment, $\mathbf{m}$, can be written, in spherical coordinates, as:

$$\mathbf{F}[\mathbf{r}] \propto \frac{|\mathbf{p}|}{r^3}\left(3\frac{(\mathbf{p}\cdot\mathbf{r})}{r^5}\mathbf{r} - \frac{\mathbf{p}}{r^3}\right) \quad .$$

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16.12.15
☒ Archimedes Principle

This § is referenced at pages:
[1514, 1516, 1662]

Demonstrate the Archimedes' principle using equation (16.07.03):
1. calculate the total force on an object immersed in a fluid, given the pressure field;
2. calculate the total torque on an object immersed in a fluid, given the pressure field.

WORKED SOLUTION

1. Apply equation (16.07.03) using as $\phi[\mathbf{x}]$ the pressure field.
2. Use as a pole the origin of the Coordinate System.

$$\boldsymbol{\Gamma} = \oiint \mathbf{r} \wedge \mathbb{d}\mathbf{f}[\mathbf{r}] = \quad ,$$

$$\oiint \mathbf{r} \wedge (-p[\mathbf{r}]\, \mathbb{d}\mathbf{S}) = \quad ,$$

$$\iiint \mathbf{rot}(-p[\mathbf{r}]\mathbf{r})\, \mathbb{d}V = \quad ,$$

$$\mathbf{e}_i \iiint -\epsilon_{ijk}^{\cdot\cdot\cdot} \frac{\partial}{\partial x^j}(p[\mathbf{r}] x_k)\, \mathbb{d}V = \quad ,$$

$$\mathbf{e}_i \iiint -\epsilon_{ijk}^{\cdot\cdot\cdot} \left(x_k \frac{\partial}{\partial x_j} p + p \delta_{kj}^{\cdot\cdot} \right) \mathbb{d}V = \quad ,$$

$$\mathbf{e}_i \iiint -\epsilon_{ijk}^{\cdot\cdot\cdot} x_k \frac{\partial}{\partial x_j} p\, \mathbb{d}V \quad ,$$

so that:

$$\boxed{\boldsymbol{\Gamma} = \iiint \mathbf{r} \wedge (-\mathbf{grad} p[\mathbf{r}])\, \mathbb{d}V} \quad . \tag{16.12.13}$$

Compare the result with the one obtained via (16.07.04) and (16.13.08). It follows that the force per unit volume $-\mathbf{grad} p[\mathbf{r}]$ can be used to calculate the total torque of the pressure forces, in addition to the total force.

16.12.16
☒ Green formula in 2D

Rotor theorem (16.07.07) in 2D becomes Green formula:

$$\mathbf{A} = A_x[y]\hat{\mathbf{e}}_1 + A_y[x]\hat{\mathbf{e}}_2 \quad ,$$

$$\mathbf{rot}\,\mathbf{A} = \hat{\mathbf{e}}_3 \left(\frac{\partial A_y[x]}{\partial x} - \frac{\partial A_x[y]}{\partial y} \right) \quad ,$$

$$\boxed{\iint \left(\frac{\partial A_y[x]}{\partial x} - \frac{\partial A_x[y]}{\partial y} \right) \mathbb{d}S = \oint (A_x\, \mathbb{d}x + A_y\, \mathbb{d}y) \qquad \text{Green formula in 2D}} \quad . \tag{16.12.14}$$

→ 1135

It is also called Gauss theorem or the **divergence theorem** in the plane, others call it Stokes theorem in the plane.

16.12.17
Example of shear

This § is referenced at pages:
[1096, 1101, 1490]

©|J.D.Romano and R.H.Price - *Why no shear in "Div, grad, rotor, and all that"?* |WEB - URL |*A MUST*|

For a cube of moving fluid, the shear represents the rate at which each side is deviating from a square, and the nature of that deviation. A simple example is the two-dimensional flow given by the velocity field:

$$\mathbf{v} = (x + y)\,\widehat{\mathbf{e}}_1 + (x - y)\,\widehat{\mathbf{e}}_2 + 0\widehat{\mathbf{e}}_3 \quad . \tag{16.12.15}$$

It is easy to verify that both the divergence and the rotor of this vector field are zero, but the shearing of the fluid represents a real physical effect.

16.12.18
Shear and Gradient in cylindrical and spherical coordinates

This § is referenced at pages:
[1490]

©|J.D.Romano and R.H.Price - *Why no shear in "Div, grad, rotor, and all that"?* |WEB - URL |*A MUST*|

16.12.19
Solenoidal fields and field lines

This § is referenced at pages:
[1122, 1716, 1791]

©|J.Slepian|Lines of Force in Electric and Magnetic Fields|*Am. J. Phys. 1 February 1951; 19 (2): 87-90.* WEB - URL|

©|K.L.McDonald|Topology of Steady Current Magnetic Fields|*Am. J. Phys. 1 December 1954; 22 (9): 586-596.* ...

©|M.Lieberherr|The magnetic field lines of a helical coil are not simple loops|*Am. J. Phys. 1 November 2010; 78 (11):* ...

Read § 16.09.02.02.
If any field is such that its flux across any closed surface is zero, it is called a solenoidal field.
However, this does not mean that its field lines are necessarily closed curves.
See, for instance:

- a uniform and constant magnetic field (with sources at infinity);
- or the magnetic field on the axis of a real solenoid with circular cross-section (whose sources are not at infinity);
- or the field lines around a toroidal current which can, under suitable conditions, loop around the toroid without closing (sources not at infinity and field lines not going to infinity).

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16.13 Exercises Problems and Physical Applications

©|M.R.Spiegel *et al.*, *Vector Analysis*, 2009, McGraw-Hill, 2ndEd., ...||*A lot of problems on Vector Algebra and Calculus ...*

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16-001 ☒ Second-Order Differential Operators - Rotor of Rotor

Demonstrate equation (16.08.05) in orthonormal Cartesian Coordinates Coordinate System.

WORKED SOLUTION

$$\{\mathbf{rotrot}\,\mathbf{A}\}_p = \epsilon_{pji}\epsilon_{irs}\partial_j\partial_r A_s = \partial_p\partial_j A_j - \partial_j\partial_j A_p \quad .$$

16-002 ✓ Irrotational Fields

Consider an irrotational field:

$$\epsilon_{pji}\frac{\partial V_i}{\partial x_j} = 0 \quad ,$$

and demonstrate, using the relations in section § 15-009, that it can be written as:

$$\frac{\partial V_i}{\partial x_j} = \frac{\partial V_j}{\partial x_i} \quad .$$

16-003 ✓ Some Useful Identities

Demonstrate the following relations (where $\widehat{\mathbf{n}} \equiv \mathbf{r}/r$ is the radial unit vector):

$$\frac{\partial r}{\partial x^k} = \frac{x^k}{r} \quad ,$$
$$\mathbf{grad}\, r = \mathbf{r}/r \quad ,$$
$$\mathbf{grad}\, r^2 = 2\mathbf{r} \quad ,$$
$$\mathbf{div}\,\mathbf{r} = 3 \quad ,$$
$$\mathbf{rot}\,\mathbf{r} = 0 \quad ,$$
$$\mathbf{div}\,\mathbf{n} = 2/r \quad ,$$
$$\mathbf{rot}\,\mathbf{n} = 0 \quad ,$$
$$\mathbf{grad}\, r^\alpha = \alpha r^{\alpha-2}\mathbf{r} \quad .$$

16-004 ☒ Identity for ElectroMagnetism

Demonstrate the following relation:

$$\mathbf{grad}_{\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{a}|} = -\frac{(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|^3} = -\mathbf{grad}_{\mathbf{a}}\frac{1}{|\mathbf{x}-\mathbf{a}|} \qquad \text{for any} \qquad \mathbf{x} \neq \mathbf{a} \quad . \tag{16.13.01}$$

→ 1128 1847 1889

Be careful with the signs above: the values of the function $|\mathbf{x}-\mathbf{a}|^{-1}$ is symmetrical in $\mathbf{x}$ and $\mathbf{a}$ and the symmetry of the result is broken by the variable of the gradient.

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16-005 ✓ Some Identities From the Divergence Theorem

This § is referenced at pages:
[1078]

Let Ω be an arbitrary volume and Σ a closed surface; let Σ_1 and Σ_2 be two arbitrary open surfaces with the same closed curve as boundary: $\partial\Sigma_1 = \partial\Sigma_2 = \Gamma$.

Demonstrate the following relations:

$$\Omega = \iiint_{\Omega} \mathrm{d}V = \frac{1}{3} \oiint_{\partial\Omega} \mathbf{r} \cdot \mathrm{d}\mathbf{S} \quad ,$$

$$\oiint_{\Sigma} \mathrm{d}\mathbf{S} = 0 \qquad \text{apply to a fixed vector field} \quad ,$$

$$\iint_{\Sigma_1} \mathrm{d}\mathbf{S} = \iint_{\Sigma_2} \mathrm{d}\mathbf{S} \quad .$$

Read § 16-006 for a geometrical application.

16-006 ✓ Volume of a Generic Pyramid or Cone

This § is referenced at pages:
[1134]

Calculate the volume of a generic pyramid or cone, with a generic base of surface A and height H, by means of the corollary to the divergence theorem:

$$\Omega = \iiint_{\Omega} \mathrm{d}V = \frac{1}{3} \oiint_{\partial\Omega} \mathbf{r} \cdot \mathrm{d}\mathbf{S} \quad .$$

WORKED SOLUTION

Take the origin of the Coordinate System at the vertex of the pyramid/cone and the base of the pyramid/cone parallel to the xy plane. In the surface integral

$$\frac{1}{3} \oiint_{\partial\Omega} \mathbf{r} \cdot \mathrm{d}\mathbf{S} \quad ,$$

the lateral surface give no contribution at all. The contribution of the base of the pyramid/cone can be written as:

$$\frac{1}{3} \oiint_{\partial\Omega} z \, \mathrm{d}x \, \mathrm{d}y = \frac{H}{3} \oiint_{\partial\Omega} \mathrm{d}x \, \mathrm{d}y = \frac{AH}{3} \quad .$$

PRELIMINARY PROOFS

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16-007 ✓ Some Identities From the Rotor Theorem

Demonstrate that:

$$\oint_C \mathbb{d}\mathbf{L} = 0 \quad . \tag{16.13.02}$$

Demonstrate that given two different curves C_1 and C_2 joining two points:

$$\int_{C_1} \mathbb{d}\mathbf{L} = \int_{C_2} \mathbb{d}\mathbf{L} \quad . \tag{16.13.03}$$

16-008 ✓ Area of a plane surface - Ellipse and Hyperbola

1. Consider a plane surface in the xy plane and use (16.12.14) in the xy plane to express the area of the surface in terms of a line integral over the contour as follows:
$$\iint \left(\frac{\partial A_y[x]}{\partial x} - \frac{\partial A_x[y]}{\partial y} \right) \mathbb{d}S = \frac{1}{2} \oint (y\, \mathbb{d}x - x\, \mathbb{d}y) \quad . \tag{16.13.04}$$
Compare with equation (16.13.05).
2. An ellipse is described by:
$$\mathbf{r}[t] = a_x \widehat{\mathbf{e}}_1 \cos t + a_y \widehat{\mathbf{e}}_2 \sin t \qquad a_x > 0 \qquad a_y > 0 \quad .$$
Show that the area of the ellipse is $\pi a_x a_y$.
3. Set the origin at one of the focii of the ellipse and use the formula to discuss the second Kepler law, equation (29.02.03).
4. An hyperbola is described by:
$$\mathbf{r}[t] = a_x \widehat{\mathbf{e}}_1 \coh t + a_y \widehat{\mathbf{e}}_2 \sih t \qquad a_x > 0 \qquad a_y > 0 \quad .$$
Calculate the area of a sector of hyperbola (a region of the Cartesian plane bounded by a hyperbola and two segments from the origin to it) symmetrical with respect to the x axis, as a function of the x coordinate of the point on the hyperbola.
5. Set the origin at one of the focii of the hyperbola and use the formula to discuss the second Kepler law, equation (29.02.03).

16-009 ☒ Identities for Magnetic dipole moments

This § is referenced at pages:
[1780, 1781, 1781]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 1.62||

Use the **rotor theorem** with vector field $\mathbf{A}[\mathbf{r}] = {}^1\!/_2\,(\mathbf{r}\wedge\mathbf{n})$, with $\widehat{\mathbf{n}}$ a constant and uniform field, to show that:

$$\mathbf{S} \equiv \frac{1}{2} \oint_{\partial\Sigma} \mathbf{r} \wedge \mathbb{d}\mathbf{r} = \iint_{\Sigma} \mathbb{d}\mathbf{S} \qquad \text{with } \partial\Sigma = \Gamma \quad . \tag{16.13.05}$$

→ 1135 1781

Show that, for any fixed vector $\mathbf{c}$, one has:

$$\oint_{\partial\Sigma} (\mathbf{c}\cdot\mathbf{r})\ \mathbb{d}\mathbf{r} = \mathbf{c}\wedge\mathbf{S} \qquad \mathbf{S} \equiv \frac{1}{2} \oint_{\partial\Sigma} \mathbf{r} \wedge \mathbb{d}\mathbf{r} = \iint_{\Sigma} \mathbb{d}\mathbf{S} \quad . \tag{16.13.06}$$

→ 1739

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16-010 ✓ Gradient of a product

Demonstrate that:

$$\boxed{\mathbf{grad}(\phi_1\phi_2) = \phi_1\mathbf{grad}\phi_2 + \phi_2\mathbf{grad}\phi_1} \quad .$$

16-011 ✓ Divergence of a product

Dimostrare che:

$$\boxed{\mathbf{div}(f[\mathbf{x}]\mathbf{A}[\mathbf{x}]) = f[\mathbf{x}]\mathbf{div}\,\mathbf{A}[\mathbf{x}] + \mathbf{grad}\,f[\mathbf{x}]\cdot\mathbf{A}[\mathbf{x}]} \quad . \tag{16.13.07}$$

→ 1847

16-012 ✓ Rotor of a product

Dimostrare che:

$$\boxed{\mathbf{rot}(f[\mathbf{x}]\mathbf{A}[\mathbf{x}]) = f[\mathbf{x}]\mathbf{rot}\,\mathbf{A}[\mathbf{x}] + \mathbf{grad}\,f[\mathbf{x}]\wedge\mathbf{A}[\mathbf{x}]} \quad . \tag{16.13.08}$$

→ 1131 1889

16-013 ☒ Radial and Conservative Force Fields

This § is referenced at pages:
[1137, 1550]

Consider the radial force field:

$$\mathbf{f} = f_r[r,\theta,\phi]\widehat{\mathbf{e}}_r \quad ,$$

where $f_r[r,\theta,\phi]$ is an arbitrary function, and show that, in order to be conservative, in a 1-connected domain, it is necessary and sufficient that:

$$\frac{\partial f_r}{\partial\theta} = \frac{\partial f_r}{\partial\phi} = 0 \quad .$$

Such a field is known as a Central Force Field, read § 29.

Explain the physical meaning of this result showing that, if the above condition is not satisfied, it is easy to find a closed path along which the work done by the force is not zero.

Read also § 16-017.

WORKED SOLUTION

Use the expression of the rotor in spherical coordinates.
For a central field any trajectory at fixed distance from the force center implies no work. If the module of the force depends on the orientation one could find a closed trajectory going along a radial path, followed by a constant r path up to a point where the radial path to the starting distance gives a non null value, to close the path at fixed r. In this way a closed path with non zero work is found, demonstrating that the field is non-conservative.

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16-014 ☒ Vector Identity for use with the Poynting vector

Demonstrate that:

$$\boxed{\mathbf{div}(\mathbf{A}\wedge\mathbf{B}) = \mathbf{B}\cdot\mathbf{rot}\,\mathbf{A} - \mathbf{A}\cdot\mathbf{rot}\,\mathbf{B}} \tag{16.13.09}$$

16-015 ☒ Care With Nabla

This § is referenced at pages:
[1102]

Demonstrate that the naive application of the nabla vector operator brings to the wrong result.

1.
$$\mathbf{grad}(\mathbf{A}\cdot\mathbf{B}) \neq \mathbf{B}\mathbf{div}\,\mathbf{A} + \mathbf{A}\mathbf{div}\,\mathbf{B} \quad .$$

2.
$$\mathbf{grad}\phi_1\wedge\mathbf{grad}\phi_2 \neq 0 \quad ,$$
as one would naively say that the vector product of two **parallel** ∇ is zero.

16-016 Use of the Divergence Theorem

Verify the **divergence theorem** for the vector field

$$\mathbf{f}[\mathbf{x}] = |\mathbf{x}|^{\alpha}\widehat{\mathbf{e}}_r \qquad \alpha > -2 \quad ,$$

on a sphere of radius R centered on the origin.

WORKED SOLUTION

Use $\alpha > -2$ to avoid problems of convergence at the origin.
One finds: $4\pi R^{2+\alpha}$.

16-017 ☒ Divergence and Rotor of a central power Law 3D field

This § is referenced at pages:
[1136, 1138, 1550]

Dimostrare che un campo del tipo:

$$\mathbf{A}[\mathbf{r}] \propto \mathbf{r}r^{\alpha} \quad ,$$

è irrotazionale e calcolarne la divergenza.

Quindi se il campo rappresenta una forza è una forza conservativa, essendo la forza definita definito su un insieme 1-connesso (tutto lo spazio per $\alpha \geq 0$; tutto lo spazio meno l'origine per $\alpha \leq 0$).

Read also § 16-013.

Note that this problem does not show that a conservative field does not depend on the angles, but only that if it is a power-law in the distance then it is a conservative field.

WORKED SOLUTION

Si consideri, per $i \neq j$, la derivata:

$$\frac{\partial}{\partial x_j}A_i = \frac{\partial}{\partial x_j}(x_i r^{\alpha}) = \alpha x_i x_j r^{\alpha-1} \quad .$$

Essendo il risultato simmetrico in (i,j) il rotore fa zero, a causa della definizione di rotore in coordinate cartesiane:

$$\mathbf{rot}\,\mathbf{A}[\mathbf{x}]\Big|_k = \epsilon^{\cdots}_{kij}\frac{\partial A_j}{\partial x_i} \quad .$$

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La divergenza risulta:

$$\mathbf{div}\,\mathbf{A} = (\alpha + 3)\, r^{\alpha} \quad .$$

La divergenza è nulla per $\alpha = -3$, che corrisponde al campo gravitazionale ed elettrico.

16-018 ☒ Divergence and Rotor of a central power Law 2D field

This § is referenced at pages:
[Never referenced.]

Si risolva il problema § 16-017 nel caso di due dimensioni. Cambia qualcosa rispetto al caso di tre dimensioni?

WORKED SOLUTION

Il campo è irrotazionale.
La divergenza risuta:

$$\mathbf{div}\,\mathbf{A} = (\alpha + 2)\, r^{\alpha} \quad .$$

La divergenza è nulla per $\alpha = -2$.

16-019 ✓ Field Lines and Equipotential Surfaces

Show that the equation defining the field lines of a vector field can be written as:

$$\mathbf{u}\cdot\mathbf{rot}\,\mathbf{u} = 0 \quad .$$

16-020 Examples of vector fields in 2D

This § is referenced at pages:
[Never referenced.]

In the xy plane, consider the vector field $\mathbf{r}$.

1. Calculate the 2D-flux and 2D-circulation around the circle of radius one and centered at the origin.
2. Calculate the 2D-flux and 2D-circulation around the circle of radius one and centered $\{1, 1\}$.
3. Calculate the 2D-flux and 2D-circulation around the square of side one and centered at the origin.
4. Calculate the 2D-flux and 2D-circulation around the square of side one and centered $\{1, 1\}$.

16-021 Plane Angle

This § is referenced at pages:
[1152]

In a plane.

1. Show that the plane angle subtended by any curve with respect to a point O is the 2D flux of the vector field $\mathbf{r}/r^2$ over the curve.
2. Show that the result is independent of the curve chosen, provided one considers curves having the same boundary.

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16-022 Solid Angle

This § is referenced at pages:
[1152]

In space.

1. Show that the solid angle subtended by any surface with respect to a point O is the 3D flux of the vector field $\mathbf{r}/r^3$ over the surface.
2. Show that the result is independent of the surface chooses, provided one considers surfaces with the same boundary.

16-023 Vector Operators in Cylindrical and Spherical Coordinate System

This § is referenced at pages:
[Never referenced.]

©|WEB - URL |ElectroMagnetic Field and Energy - H.A.Haus & J.R.Melcher|*§ 2 - problem 2.1.4, 2.1.6, 2.4.3, 2.4.4.*|

Reformulate the arguments as in section § 16.06.05, using infinitesimal volume elements in Cylindrical and Spherical Coordinate System, to derive the expressions of divergence and rotor in Cylindrical and Spherical Coordinate System, as listed in figure 16.11.

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CHAPTER 17

Complements of Geometry Vector Algebra Vector Calculus

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17.01 Analytic Geometry

17.01.01
Equazione Della Retta

L'equazione parametrica di una retta nello spazio si esprime in termini di un parametro t, di un vettore $\mathbf{u}$ la cui direzione coincide con quella della retta e di un punto $\mathbf{x}_0$ per cui passa la retta:

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{u} \quad .$$

La corrispondente equazione cartesiana è

$$(\mathbf{x} - \mathbf{x}_0) \wedge \mathbf{u} = 0 \quad .$$

L'equazione parametrica della retta passante per due punti assegnati, $\mathbf{x}_A$ e $\mathbf{x}_B$, è data, in forma parametrica, da ciascuna delle due relazioni:

$$\mathbf{x} - \mathbf{x}_A = \lambda\,(\mathbf{x}_B - \mathbf{x}_A) \qquad || \qquad \mathbf{x} - \mathbf{x}_B = \lambda\,(\mathbf{x}_A - \mathbf{x}_B) \quad ,$$

con λ che varia nell'intervallo $(-\infty, +\infty)$.

La corrispondente equazione cartesiana è ciascuna delle due relazioni:

$$(\mathbf{x} - \mathbf{x}_A) \wedge (\mathbf{x}_B - \mathbf{x}_A) = 0 \qquad || \qquad (\mathbf{x} - \mathbf{x}_B) \wedge (\mathbf{x}_A - \mathbf{x}_B) = 0 \quad .$$

17.01.02
Equazione Del Piano

L'equazione parametrica di un piano, in termini dei due parametri t_1 e t_2, si esprime attraverso due vettori paralleli al piano e tra loro non paralleli, $\mathbf{u}_1$ e $\mathbf{u}_2$:

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{u}_1 + t_2\mathbf{u}_2 \qquad \text{con } \mathbf{u}_1 \wedge \mathbf{u}_2 \neq 0 \quad .$$

L'equazione di un piano ortogonale ad un vettore $\mathbf{n}$ è:

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0 \qquad \Leftrightarrow \qquad \mathbf{n} \cdot \mathbf{x} + b = 0 \qquad \text{con } b = -\,\mathbf{n} \cdot \mathbf{x}_0 \quad .$$

Valori diversi di b individuano piani diversi tra loro paralleli.

L'equazione del piano passante per tre punti non allineati, $\mathbf{x}_A$, $\mathbf{x}_B$ e $\mathbf{x}_C$, si esprime in termini del triplo prodotto scalare come

$$(\mathbf{x} - \mathbf{x}_A) \cdot (\,(\mathbf{x}_B - \mathbf{x}_A) \wedge (\mathbf{x}_C - \mathbf{x}_A)) = 0 \qquad \text{and/or similar permutations of the three points} \quad .$$

17.01.03
Lines and Planes intersections

$$\mathbf{x}[t] = \mathbf{x}_0 + t\mathbf{u} \tag{17.01.01}$$

$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{a}_0) = 0 \tag{17.01.02}$$

$$\boxed{\mathbf{x}^\star = \frac{\mathbf{u}}{\mathbf{a} \cdot \mathbf{u}} \mathbf{a} \cdot (\mathbf{a}_0 - \mathbf{x}_0) + \mathbf{x}_0} \quad . \tag{17.01.03}$$

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17.01.04
Quadratic curves - Conics
Read § 29.01.

17.01.05
Quadratic surfaces - Quadrics
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17.01.06
Applications of Vector Algebra to Analytic Geometry of Lines and Planes
Read § 15-013.
Read § 15-015.
Read § 15-016.
Read § 15.05.06.
Read § 15.05.07.
Read § 15-019.
Read § 15.03.04.02
Read § 17.03.01.

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17.02 Some Vector Equations and Their Solution

This § is referenced at pages:
[1063, 1440]

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17.02.01 First-Order Vector Equations

17.02.01.01 Solution of First-Order Vector Equation: Scalar Product

Dimostrare che l'equazione vettoriale

$$\mathbf{A}\cdot\mathbf{X} = p \tag{17.02.01}$$

→ 1143 1144

può venire risolta univocamente rispetto a $\mathbf{X}$ se e solo se

$$\mathbf{A} \neq 0 \quad . \tag{17.02.02}$$

Dimostrare che in tal caso la soluzione generale è

$$\mathbf{X} = \frac{p\mathbf{A}}{A^2} + \mathbf{V}\wedge\mathbf{A} \qquad \text{con } \mathbf{V} \text{ vettore arbitrario} \quad . \tag{17.02.03}$$

→ 1143 1144

WORKED SOLUTION

Discussione dell'equazione (17.02.01).

- Se $\mathbf{A} = 0$ e $p \neq 0$ l'equazione è impossibile.
- Se $\mathbf{A} = 0$ e $p = 0$ l'equazione è indeterminata, qualunque vettore $\mathbf{X}$ la soddisfa.
- Se $\mathbf{A} \neq 0$ l'equazione è determinata e la soluzione generale è la (17.02.03), qualunque sia p.

In fact the homogeneous equation $\mathbf{A}\cdot\mathbf{X} = 0$ admits as a solution $\mathbf{X}_\mathrm{H}$ any $\mathbf{X}$ perpendicular to $\mathbf{A}$, that is $\mathbf{X}_\mathrm{H} = \mathbf{V}\wedge\mathbf{A}$ with $\mathbf{V}$ any arbitrary vector. Note that actually only the component of $\mathbf{V}$ perpendicular to $\mathbf{A}$ is relevant in the expression of $\mathbf{X}_\mathrm{H}$.
The expression (17.02.03) is obviously a solution of the equation.
To demonstrate that expression (17.02.03) includes all the solutions consider any two solutions of the non-homogeneous equation (17.02.01): their difference satisfies the homogeneous equation

$$\mathbf{A}\cdot\mathbf{X} = 0 \quad . \tag{17.02.04}$$

→ 1143

The general solution to the homogeneous equation (17.02.04), $\mathbf{X}_\mathrm{H}$, can be easily found via geometrical methods: it is given by all and only all the non null vectors perpendicular to $\mathbf{A}$. This can be written as:

$$\mathbf{X}_\mathrm{H} = \mathbf{V}\wedge\mathbf{A} \qquad \text{with } \mathbf{V} \text{ any arbitrary vector} \quad .$$

In fact the above solution gives only vectors $\mathbf{X}_\mathrm{H}$ perpendicular to $\mathbf{A}$ and any vector perpendicular to $\mathbf{A}$, let us say $\mathbf{Y}$, can be written in the above form, for a suitable choice of the vector $\mathbf{V}$, such as:

$$\mathbf{V} = -\frac{1}{A^2}\mathbf{Y}\wedge\mathbf{A} \qquad \mathbf{Y}\cdot\mathbf{A} = 0 \quad .$$

A particular solution of the non-homogeneous equation is: $\mathbf{X}_\mathrm{P} = \frac{p\mathbf{A}}{A^2}$.
Any other solution differs from $\mathbf{X}_\mathrm{P}$ by $\mathbf{X}_\mathrm{H}$.

Equation (17.02.03) represents thus the full set of solutions.
Note that only the component of $\mathbf{V}$ perpendicular to $\mathbf{A}$ is relevant and therefore there are only two degrees of freedom involved in the choice of $\mathbf{V}$. This is consistent with the fact that equation (17.02.01) is one linear equation with three unknowns.

Geometrical Interpretation
The geometrical interpretation is very useful.
One is looking for a vector, $\mathbf{X}$, whose projection along $\mathbf{A}$ is p.

There are clearly infinite solutions. The solution has a component along $\mathbf{A}$, $\frac{p\mathbf{A}}{A^2}$, and an arbitrary component perpendicular to $\mathbf{A}$, $\mathbf{V}\wedge\mathbf{A}$.

17.02.01.02 Solution of First-Order Vector Equation: Vector Product

Dimostrare che l'equazione vettoriale

$$\boxed{\mathbf{A}\wedge\mathbf{X}=\mathbf{B}} \tag{17.02.05}$$

→ 1144 1145

può venire risolta univocamente rispetto a $\mathbf{X}$ se e solo se

$$\boxed{\mathbf{A}\cdot\mathbf{B}=0 \qquad \text{e} \qquad \mathbf{A}\neq 0} \quad . \tag{17.02.06}$$

Dimostrare che in tal caso la soluzione generale è

$$\boxed{\mathbf{X}=\lambda\mathbf{A}+\frac{\mathbf{B}\wedge\mathbf{A}}{A^2} \qquad \lambda \text{ scalare arbitrario}} \quad . \tag{17.02.07}$$

→ 1144 1145

Notare che il risultato precedente dimostra l'impossibilità di definire in modo univoco il quoziente di due vettori. Infatti il risultato precedente dimostra che una generica equazione vettoriale come la (17.02.05) non può mai ammettere una ed una sola soluzione: o non ha soluzione o ne ha infinite.
Il risultato mostra anche che, come ci si aspetta, $\mathbf{B}\cdot\mathbf{X}=0$ in quanto si ha $\mathbf{A}\cdot\mathbf{B}=0$.

WORKED SOLUTION

Discussione dell'equazione (17.02.05).

- Se $\mathbf{A}=0$ e $\mathbf{B}\neq 0$ l'equazione è impossibile.
- Se $\mathbf{A}\cdot\mathbf{B}\neq 0$ l'equazione è impossibile, perchè il triplo prodotto che si ottiene moltiplicando entrambi i membri scalarmente per $\mathbf{A}$ si annulla. Questo è ovvio se si considera che $\mathbf{A}\wedge\mathbf{X}$ è perpendicolare ad $\mathbf{A}$ e quindi solo se $\mathbf{C}$ è perpendicolare ad $\mathbf{A}$ (cioè $\mathbf{A}\cdot\mathbf{B}=0$) può esistere una soluzione.
- Se $\mathbf{A}=0$ e $\mathbf{B}=0$ l'equazione è indeterminata, qualunque vettore $\mathbf{X}$ la soddisfa.
- Se $\mathbf{A}\neq 0$ e $\mathbf{A}\cdot\mathbf{B}=0$ l'equazione è determinata e la soluzione generale è la (17.02.07), qualunque sia $\mathbf{B}$.

In fact the homogeneous equation $\mathbf{A}\wedge\mathbf{X}=0$ admits as a solution $\mathbf{X}_H$ any $\mathbf{X}$ parallel to $\mathbf{A}$, that is $\mathbf{X}_H=\lambda\mathbf{A}$ with λ any arbitrary real number.
The expression (17.02.07) is obviously a solution of the equation.
To demonstrate that expression (17.02.07) includes all the solutions consider any two solutions of the non-homogeneous equation (17.02.05): their difference satisfies the homogeneous equation

$$\boxed{\mathbf{A}\wedge\mathbf{X}=0} \quad . \tag{17.02.08}$$

→ 1144

The general solution to the homogeneous equation (17.02.08), $\mathbf{X}_H$, can be easily found via geometrical methods: it is given by all and only all the non null vectors parallel to $\mathbf{A}$. This can be written as:

$$\mathbf{X}_H=\lambda\mathbf{A} \qquad \text{with } \lambda \text{ any arbitrary scalar} \quad .$$

To determine the particular solution one should consider that one needs a vector function of the two vectors $\mathbf{A}$ and $\mathbf{B}$. The simplest possibility is just a vector proportional to $\mathbf{B}\wedge\mathbf{A}$. A particular solution of the non-homogeneous equation is then: $\mathbf{X}_P=\frac{\mathbf{B}\wedge\mathbf{A}}{A^2}$, thanks to the condition $\mathbf{A}\cdot\mathbf{B}=0$.

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Any other solution differs from $\mathbf{X}_{\mathrm{P}}$ by $\mathbf{X}_{\mathrm{H}}$.
Equation (17.02.07) represents thus the full set of solutions.
Note that there is only one degree of freedom involved in the general solution of the homogeneous equation.
This is consistent with the fact that equation (17.02.05) is a set of three linear equations with three unknowns and the rank of the incomplete matrix of the coefficients is two, as it can be easily shown by noting that the determinant of the incomplete matrix of the coefficients is zero but it has minors of rank two.

Geometrical Interpretation
The geometrical interpretation is very useful.
One is looking for a vector, $\mathbf{X}$, which is perpendicular to $\mathbf{B}$, as $\mathbf{B}$ comes out as the result af the vector product between $\mathbf{A}$ and $\mathbf{X}$. Moreover, $\mathbf{B}$ must be perpendicular to $\mathbf{A}$ for the same reason.
We need a vector perpendicular to both $\mathbf{A}$ and $\mathbf{C}$ in order to re-obtain the direction of $\mathbf{C}$ after vector product with $\mathbf{A}$. Any component of $\mathbf{X}$ along $\mathbf{A}$ can be added to the solution so that there are clearly infinite solutions.

17.02.02
Miscellaneous Linear Vector Equations and Systems of Equations

This § is referenced at pages:
[Never referenced.]

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The following results can be demonstrated. Note the precise statements on the conditions imposed on the coefficients to make the results valid.

For the demonstration one can assume that the solution must be expressed as a linear combination of all the vectors which can be formed with $\mathbf{a}$ and $\mathbf{b}$, that is:

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\,\mathbf{a}\wedge\mathbf{b} \quad .$$

Equations are special cases of the general linear first order system of equations.

17.02.02.01 System of Linear vector equations (1)

Under suitable conditions on the coefficients one has:

$$\{\mathbf{a} \neq 0 \qquad \mathbf{b} \neq 0 \qquad \mathbf{b}\cdot\mathbf{c} = 0\} \qquad \Longrightarrow \begin{cases} \mathbf{x}\cdot\mathbf{a} = p \quad , \\ \mathbf{x}\wedge\mathbf{b} = \mathbf{c} \end{cases} \qquad \Longrightarrow \mathbf{x} = \frac{\mathbf{a}\wedge\mathbf{c} + p\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}} \quad .$$

17.02.02.02 System of Linear vector equations (2)

Under suitable conditions on the coefficients one has:

$$\{\mathbf{a} \neq 0 \qquad \mathbf{b} \neq 0 \qquad \mathbf{c} \neq 0\} \qquad \begin{cases} \mathbf{x}\cdot\mathbf{a} = \alpha \quad , \\ \mathbf{x}\cdot\mathbf{b} = \beta \quad , \\ \mathbf{x}\cdot\mathbf{c} = \gamma \quad , \end{cases} \qquad \Longrightarrow \mathbf{x} = \frac{\alpha\,\mathbf{b}\wedge\mathbf{c} + \beta\,\mathbf{c}\wedge\mathbf{a} + \gamma\,\mathbf{a}\wedge\mathbf{b}}{\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})} \quad .$$

PROGRESS

17.02.02.03 Linear vector equation (1)

Under suitable conditions on the coefficients one has:

$$\mathbf{x} + \mathbf{b}\wedge\mathbf{x} = \mathbf{a} \implies \mathbf{x} = \frac{\mathbf{a} + \mathbf{a}\wedge\mathbf{b} + (\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}}{1 + b^2} \quad .$$

17.02.02.04 Linear vector equation (2)

Under suitable conditions on the coefficients one has:

$$\mathbf{a}\wedge\mathbf{x} + \mathbf{b}\,(\mathbf{c}\cdot\mathbf{x}) = \mathbf{d} \implies \mathbf{x} = \frac{\mathbf{d}\wedge\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}} + \mathbf{a}\left(\frac{\mathbf{a}\cdot\mathbf{d} + \mathbf{c}\cdot(\mathbf{b}\wedge\mathbf{d})}{(\mathbf{a}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{b})}\right) \quad .$$

17.02.02.05 Coefficients for linear combination of vectors

Under suitable conditions on the coefficients one has:

$$\mathbf{p} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \implies \begin{cases} \alpha = \dfrac{\mathbf{p}\cdot(\mathbf{b}\wedge\mathbf{c})}{\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})} & , \\ \beta = \dfrac{\mathbf{p}\cdot(\mathbf{c}\wedge\mathbf{a})}{\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})} & , \\ \gamma = \dfrac{\mathbf{p}\cdot(\mathbf{a}\wedge\mathbf{b})}{\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})} & \end{cases} \quad .$$

17.02.02.06 ☒ Coefficients for linear combination of scalar products

Under suitable conditions on the coefficients one has:

$$\mathbf{p} = \alpha\,\mathbf{b}\wedge\mathbf{c} + \beta\,\mathbf{c}\wedge\mathbf{a} + \gamma\,\mathbf{a}\wedge\mathbf{b} \implies \begin{cases} \alpha = \dfrac{\mathbf{p}\cdot\mathbf{a}}{\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})} & , \\ \beta = \dfrac{\mathbf{p}\cdot\mathbf{b}}{\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})} & , \\ \gamma = \dfrac{\mathbf{p}\cdot\mathbf{c}}{\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})} & \end{cases} \quad .$$

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(10–12)

Solutions of Some Vector Equations

$$\left.\begin{aligned}(\mathbf{x}\cdot\mathbf{c}) &= p\\ (\mathbf{x}\times\mathbf{a}) &= \mathbf{b}\end{aligned}\right\}: \qquad \mathbf{x} = \frac{\mathbf{c}\times\mathbf{b} + p\mathbf{a}}{\mathbf{a}\cdot\mathbf{c}} \tag{10–13}$$

(Here $\mathbf{a}\cdot\mathbf{b} = 0$ if the original equations are possible.)

$$\left.\begin{aligned}\mathbf{x}\cdot\mathbf{a} &= \alpha\\ \mathbf{x}\cdot\mathbf{b} &= \beta\\ \mathbf{x}\cdot\mathbf{c} &= \gamma\end{aligned}\right\}: \qquad \mathbf{x} = \frac{\alpha\mathbf{b}\times\mathbf{c} + \beta\mathbf{c}\times\mathbf{a} + \gamma\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}} \tag{10–14}$$

$$\mathbf{x} + \mathbf{b}\times\mathbf{x} = \mathbf{a}: \qquad \mathbf{x} = \frac{\mathbf{a} + \mathbf{a}\times\mathbf{b} + (\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{1 + b^2} \tag{10–15}$$

$$\mathbf{a}\times\mathbf{x} + \mathbf{b}(\mathbf{c}\cdot\mathbf{x}) = \mathbf{d}: \qquad \mathbf{x} = \frac{\mathbf{d}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}} + \mathbf{a}\left[\frac{\mathbf{a}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{b}\times\mathbf{d}}{(\mathbf{a}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b})}\right] \tag{10–16}$$

$$\mathbf{p} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}: \begin{cases} x = \dfrac{\mathbf{p}\cdot\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}} \\ y = \dfrac{\mathbf{p}\cdot\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}} \\ z = \dfrac{\mathbf{p}\cdot\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}} \end{cases} \tag{10–17}$$

$$\mathbf{p} = x(\mathbf{b}\times\mathbf{c}) + y(\mathbf{c}\times\mathbf{a}) + z(\mathbf{a}\times\mathbf{b}):$$

$$x = \frac{\mathbf{p}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \quad y = \frac{\mathbf{p}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \quad z = \frac{\mathbf{p}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}} \tag{10–18}$$

Rotations

If all of space is rotated about an axis along $\boldsymbol{\delta}$ and by an angle of δ radians (clockwise, looking out along $\boldsymbol{\delta}$ and not necessarily less than 2π), then $\mathbf{r}$ goes into $\mathbf{r}'$ where

$$\mathbf{r}' = \mathbf{r} + \boldsymbol{\delta}\times\mathbf{r}\,\frac{\sin\delta}{\delta} + \boldsymbol{\delta}\times(\boldsymbol{\delta}\times\mathbf{r})\,\frac{1-\cos\delta}{\delta^2} \tag{10–19}$$

The inverse, of course, is given by changing the sign of $\boldsymbol{\delta}$. (Proof: resolve $\mathbf{r}$ into a sum of two vectors, one parallel to and one perpendicular to $\boldsymbol{\delta}$.)

(10–20)

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17.03 Rotations of a Vector Around an Axis - Geometrical Approach

This § is referenced at pages:
[1192, 1196, 1241, 1439]

©|H.Goldstein *et al.*, *Classical Mechanics*, 2014, Pearson Education, 3rdEd., ...|4.7|*Demonstration*|

Read also sections § 20.05, § 23.06, § 26.03. This section presents the geometrical/vector-algebra approach.

Let us use here the so-called **active approach**: the Coordinate System is kept fixed while the vectors are **rotated**.

17.03.01 Rotation of a Vector With Origin on the Rotation Axis

This § is referenced at pages:
[1142]

If one vector, $\mathbf{v}$, having origin on an axis defined by the unit vector $\widehat{\mathbf{n}}$, is rotated by an angle $\Delta\phi$ around the axis $\widehat{\mathbf{n}}$, with sign convention defined by the right-hand rule, then $\mathbf{v}$ goes into $\mathbf{v}'$ where:

$$\boxed{\mathbf{v}' = \mathbf{v} + \Delta\mathbf{v} = \mathbf{v} + \sin[\Delta\phi]\,(\mathbf{n}\wedge\mathbf{v}) + (1-\cos[\Delta\phi])\,(\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{v}))} \quad , \tag{17.03.01}$$

→ 1149 1150

sometimes known as the Rodrigues rotation formula, which can be derived by geometry and a suitable geometrical construction [1] by observing that the components of the original and rotated vectors on the rotation axis are the same and the components perpendicular to the rotation axis can be deduced from geometrical considerations in the plane perpendicular to the rotation axis.

The expression of an infinitesimal rotation can be deduced from equation (17.03.01). To first-order in $\delta\phi$ one obtains:

$$\boxed{\mathbf{v}' = \mathbf{v} + \delta\mathbf{v} = \mathbf{v} + \delta\boldsymbol{\phi}\wedge\mathbf{v} + \mathcal{O}(\delta\phi)^2 \quad \Leftrightarrow \quad \delta\mathbf{v} = \delta\boldsymbol{\phi}\wedge\mathbf{v} + \mathcal{O}(\delta\phi)^2} \quad . \tag{17.03.02}$$

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The above expression for the infinitesimal rotation, equation (17.03.02), can be used to show easily that infinitesimal (first-order) rotations commute. It cannot be obviously used to show that finite rotations do not commute, as the expression is limited to infinitesimal (first-order) rotations.

To first order, rotations (that is infinitesimal rotations) do commute; in fact:

$$\mathbf{v}' \equiv \mathbf{v} + \delta\mathbf{v} \simeq \mathbf{v} + \delta\boldsymbol{\phi}'\wedge\mathbf{v} \quad ,$$
$$\mathbf{v}'' \equiv \mathbf{v}' + \delta\mathbf{v}' \simeq \mathbf{v}' + \delta\boldsymbol{\phi}''\wedge\mathbf{v}' \quad ,$$
$$\mathbf{v}'' \simeq \mathbf{v} + \delta\boldsymbol{\phi}'\wedge\mathbf{v} + \delta\boldsymbol{\phi}''\wedge(\mathbf{v} + \delta\boldsymbol{\phi}'\wedge\mathbf{v}) = \mathbf{v} + \delta\boldsymbol{\phi}'\wedge\mathbf{v} + \delta\boldsymbol{\phi}''\wedge\mathbf{v} + \delta\boldsymbol{\phi}''\wedge(\delta\boldsymbol{\phi}'\wedge\mathbf{v}) \simeq \mathbf{v} + (\delta\boldsymbol{\phi}' + \delta\boldsymbol{\phi}'')\wedge\mathbf{v} \quad ,$$
$$\mathbf{v}'' \simeq \mathbf{v} + (\delta\boldsymbol{\phi}' + \delta\boldsymbol{\phi}'')\wedge\mathbf{v} \quad .$$

In order to show that finite rotation do not commute it is enough to think of two successive rotations by $\pi/2$ around two perpendicular axes to see that the final result is different depending on which rotation has been performed first.

From the algebraic point of view one knows that rotations are described by a non-commutative group, SO(3), read § 20.05.

Non-commutativity can also be shown by using the expression to second-order in $\delta\phi$ of equation (17.03.02)

$$\boxed{\mathbf{v}' \equiv \mathbf{v} + \delta\mathbf{v} = \mathbf{v} + \delta\boldsymbol{\phi}\wedge\mathbf{v} + \frac{\delta\boldsymbol{\phi}\wedge(\delta\boldsymbol{\phi}\wedge\mathbf{v})}{2} + \mathcal{O}(\delta\phi)^3} \quad . \tag{17.03.03}$$

[1] See: ...

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It can be easily shown that, when performing two successive rotations, the second-order expression gives a result depending on the order of the two rotations. Clearly, it might be even possible that successive orders restore the commutativity, but this is not the case as SO(3) is a non-commutative group.

Note finally that, $\delta\phi$ is indeed a vector, while there is no such vector as ϕ, as finite rotations do not belong to a vector space, being non commutative.

17.03.02
Relation of Angle and Oriented Direction With the Rotation Matrix

Using equation (17.03.02) in components and converting vector products into anti-symmetric matrices one can write the rotation equations in matrix form. Let

$$\mathbb{K} \equiv \begin{pmatrix} 0 & -n_3 & +n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & +n_1 & 0 \end{pmatrix} \propto \epsilon^{\cdot\cdot\cdot}_{ikj} n^j_{\cdot} \quad ,$$

then the rotation matrix is:

$$\mathbb{R} = \mathbb{I} + \sin[\Delta\phi]\,\mathbb{K} + (1 - \cos[\Delta\phi])\,\mathbb{K}^2$$

Even if the derivation usually involves the position vector, the formula (17.03.01) can be applied for the rotation of any vector.

17.03.03
Rotation of a Vector With Origin Outside the Rotation Axis

It is sufficient to note that, give any point, O, on the rotation axis, the vector with origin at A and end at B can be written as $\mathbf{v}_B - \mathbf{v}_A = (\mathbf{v}_B - \mathbf{v}_O) - (\mathbf{v}_A - \mathbf{v}_O)$.

As the formula for infinitesimal rotations is linear in the angle, the use of the linearity results in the rotation formula being unchanged.

For finite rotations, due to the non-linearity of the general rotation formula, the above simple result does not apply.

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17.04 General Kronecker symbol and Levi-Civita symbols

This § is referenced at pages:
[1055, 1055, 1072, 1072]

The extension of the **Kronecker symbol** and **Levi-Civita symbol** to a generic number of dimensions is straightforward.

The general **Kronecker symbol** in n dimensions is defined as:

$$\delta^{\cdot\cdot}_{ij} = \begin{cases} 1 & \text{if and only if } i = j \\ 0 & \text{if and only if } i \neq j \end{cases} \quad , \quad \text{for } i,j = 1,n \quad ,$$

The general **Levi-Civita symbol** in n dimensions is defined as:

$$\epsilon^{\cdot\cdot\cdot\cdot}_{i_1 i_2 \ldots i_n} = \begin{cases} +1 & \text{if and only if } i_1, i_2, \ldots i_n \text{ is an even permutation of } 1, 2, \ldots n \\ -1 & \text{if and only if } i_1, i_2, \ldots i_n \text{ is an odd permutation of } 1, 2, \ldots n \\ 0 & \text{if any two indexes are equal} \end{cases} \quad ,$$

.

The latter enters the determinant:

$$\det[\mathbb{A}] = \frac{1}{n!} \epsilon^{\cdot\cdot\cdot\cdot}_{r_1 r_2 \ldots r_n} \epsilon^{\cdot\cdot\cdot\cdot}_{s_1 s_2 \ldots s_n} A^{\cdot\cdot}_{r_1 s_1} A^{\cdot\cdot}_{r_2 s_2} \ldots A^{\cdot\cdot}_{r_n s_n} \quad . \tag{17.04.01}$$

PRELIMINARY-PROGRESS

17.05 Examples and Physical Applications

17.05.01
Plane Angles

Read § 16-021.

17.05.02
Solid Angles

Read § 16-022.

17.05.03
✓ Another Expression for the Rotation of a Vector

Show that an equivalent expression for the space rotation of one vector, $\mathbf{v} \to \mathbf{v}'$, whose origin is on the axis $\widehat{\mathbf{n}}$ is given by:

$$\boxed{\mathbf{v}' - \mathbf{v} = \mathbf{w} \wedge (\mathbf{v}' + \mathbf{v}) \qquad \mathbf{w} \equiv \widehat{\mathbf{n}} \tan[\phi/2] \qquad \text{for any } \phi \neq \pi} \quad . \tag{17.05.01}$$

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17.06 Exercises Problems and Physical Applications

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CHAPTER 18

Some Elements of Fourier Analysis

18.01 Fourier Series for Periodic Functions 1156
18.02 Functions Periodic in Time/Space 1157
18.03 Fourier Integral (Fourier Transform) for Non-Periodic Functions 1160
18.04 Propagation of a Pulse in a Dispersive Medium 1163
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This § is referenced at pages:
[1997, 2015, 2015, 2143, 2143]

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Fourier analysis is a crucial tool in many branches of physics.

One case where Fourier analysis intervene is wave propagation. In this case one should be careful to distinguish between:

- at fixed time: the wave function must be considered as a function of the position, the spatial **Fourier Transform** must be considered;
- at a fixed position: the wave function must be considered as a function of time, the time **Fourier Transform** must be considered.

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MATEMATICA

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18.01 Fourier Series for Periodic Functions

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18.02 Functions Periodic in Time/Space

This § is referenced at pages:
[1406]

Any **periodic function** of period $\mathcal{P}$ can be expressed as a **series of harmonic functions**.
The periodicity is most often:

- either a periodicity in time (in this case the period will be a time: $\mathcal{P} = T$);
- or in space (in this case the period will be a length: $\mathcal{P} = L$).

In case of a periodic function in time, ($\mathcal{P} = T$), one often introduces the **angular frequency**:

$$\omega \equiv \frac{2\pi}{T} \quad .$$

In case of a periodic function in space, ($\mathcal{P} = L$), one often introduces the **wave-number**:

$$k \equiv \frac{2\pi}{L} \quad .$$

18.02.01 Real Fourier Series for Periodic Functions

The **Fourier Series** of a periodic function with period $\mathcal{P}$ is given in terms of a **quantised frequency spectrum**:

$$f[w] = f_0 + \sum_{n=1}^{\infty} \left(a_n \sin\left[\frac{2\pi n w}{\mathcal{P}}\right] + b_n \cos\left[\frac{2\pi n w}{\mathcal{P}}\right] \right) \quad . \tag{18.02.01}$$

→ 1157

Note that the difference between two neighboring space/time frequencies are:

$$\Delta k \equiv k_{n+1} - k_n = \frac{2\pi}{\mathcal{P}} \equiv k \qquad \Delta\omega \equiv \omega_{n+1} - \omega_n = \frac{2\pi}{\mathcal{P}} \equiv \omega \quad ,$$

and this tends to zero when the period goes to infinity.

Assuming that the series is convergent the coefficients can be calculated (u arbitrary) from:

$$a_m \equiv \frac{2}{\mathcal{P}} \int_u^{u+\mathcal{P}} f[w] \sin\left[\frac{2\pi m w}{\mathcal{P}}\right] \, \mathrm{d}w \quad , \tag{18.02.02}$$

→ 1157

$$b_m \equiv \frac{2}{\mathcal{P}} \int_u^{u+\mathcal{P}} f[w] \cos\left[\frac{2\pi m w}{\mathcal{P}}\right] \, \mathrm{d}w \quad , \tag{18.02.03}$$

→ 1157

$$f_0 \equiv \frac{1}{\mathcal{P}} \int_u^{u+\mathcal{P}} f[w] \, \mathrm{d}w \qquad \text{mean value of the function} \quad . \tag{18.02.04}$$

→ 1157

The integration must be carried on one period but the starting point is not relevant thanks to the periodicity. In fact integrating in the interval $u \le w \le u + \mathcal{P}$ the **Fourier Series** (18.02.01), the **Fourier Series** (18.02.01) multiplied by $\sin[2\pi m w/\mathcal{P}]$ and the **Fourier Series** (18.02.01) multiplied by $\cos[2\pi m w/\mathcal{P}]$, one can find, respectively, the expression of the coefficients in equations (18.02.04), (18.02.02), (18.02.03).

While the coefficients a_n, b_n and f_0 can be formally defined for any function for which the integrals make sense, whether the series so defined actually converges to $f[w]$ depends on the properties of $f[w]$.

Sufficient conditions for the expandibility in a convergent Fourier Series of a periodic function in the interval $w_0 \le w \le w_0 + \mathcal{P}$ are provided by the Dirichlet conditions: the interval can be subdivided

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into a finite number of intervals such that in any interval the function is continuous and monotone. Such a function can be expanded in a **Fourier Series** which converges to the function at the points where the function is continuous and it converges to the mean of the positive and negative limits at the points of discontinuity of the function.

Note that any realistic and physically reasonable function can be expanded in **Fourier Series** convergent at any point.

18.02.01.01 Asymptotic Behavior of the Fourier Coefficients

It can be shown that for a piece-wise continuous limited function the large n behavior of the coefficients is:

$$a_n \sim \mathcal{O}\left(1/n\right) \qquad b_n \sim \mathcal{O}\left(1/n\right) \quad .$$

In case the first derivative of the function is a piece-wise continuous limited function the large n behavior of the coefficients is:

$$a_n \sim \mathcal{O}\left(1/n^2\right) \qquad b_n \sim \mathcal{O}\left(1/n^2\right) \quad ,$$

and so on for higher derivatives.

18.02.01.02 Asymptotic Behavior of the Fourier Coefficients

If a periodic function and its derivatives up to k-th order are continuous then for $n \to \infty$ both expressions $a_n n^{k+1}$ and $b_n n^{k+1}$ tend to zero.

18.02.02
Complex Fourier Series for Periodic Functions

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The **Fourier Series** can be expressed as a **complex Fourier series** as:

$$f[w] = f_0 + \sum_{n=1}^{\infty}\left(a_n \sin\left[\frac{2\pi n w}{\mathcal{P}}\right] + b_n \cos\left[\frac{2\pi n w}{\mathcal{P}}\right]\right) = \sum_{m=-\infty}^{m=+\infty} c_m \exp\left[\frac{2\mathrm{i}\pi m w}{\mathcal{P}}\right] \quad . \tag{18.02.05}$$

The coefficients are given by:

$$c_m \equiv \frac{1}{\mathcal{P}}\int_{u}^{u+\mathcal{P}} f[w] \exp\left[\frac{-2\mathrm{i}\pi m w}{\mathcal{P}}\right] \mathrm{d}w \quad . \tag{18.02.06}$$

$$c_0 = f_0 \qquad c_m = \frac{b_{+m} - \mathrm{i}a_{+m}}{2} \quad \text{for } m \geq +1 \qquad c_m = \frac{b_{-m} + \mathrm{i}a_{-m}}{2} \quad \text{for } m \leq -1 \quad . \tag{18.02.07}$$

18.02.03
Frequency and Phase Spectrum for Periodic Functions

The **amplitude of the Fourier coefficients**, that is the modulus, gives the so-called **frequency spectrum**, a discrete spectrum in case of a periodic function. Note, however, that the phase of the coefficients is necessary as well to reconstruct the function. The **phase of the Fourier coefficients** gives the so-called **phase spectrum**, a discrete spectrum in case of a periodic function.

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18.02.04
Energy for Periodic Functions

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1-50.5||

The energy of a signal might have both terms proportional to its amplitude squared and terms proportional to the time derivative of the amplitude squared. It is of interest to express them in terms of the Fourier coefficients (Parseval theorem).

One has:

$$\int_u^{u+\mathcal{P}} f^2[w]\,\mathrm{d}w = \int_u^{u+\mathcal{P}} \left(f_0 + \sum_{n=1}^{\infty} \left(a_n \sin\left[\frac{2\pi n w}{\mathcal{P}}\right] + b_n \cos\left[\frac{2\pi n w}{\mathcal{P}}\right] \right) \right)^2 \mathrm{d}w = \mathcal{P} f_0^2 + \frac{\mathcal{P}}{2} \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) \quad , \tag{18.02.08}$$

→ 1167

because all the cross terms vanish when integrated.

One also has:

$$\int_u^{u+\mathcal{P}} \dot{f}^2[w]\,\mathrm{d}w = \int_u^{u+\mathcal{P}} \left(\sum_{n=1}^{\infty} \left(a_n \frac{2n\pi}{\mathcal{P}} \cos\left[\frac{2\pi n w}{\mathcal{P}}\right] - b_n \frac{2n\pi}{\mathcal{P}} \sin\left[\frac{2\pi n w}{\mathcal{P}}\right] \right) \right)^2 \mathrm{d}w = \frac{\mathcal{P}}{2} \left(\frac{2\pi}{\mathcal{P}}\right)^2 \sum_{n=1}^{\infty} n^2 \left(a_n^2 + b_n^2\right) \quad . \tag{18.02.09}$$

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18.03 Fourier Integral (Fourier Transform) for Non-Periodic Functions

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A **non periodic function** can be considered as a **periodic function with very large, actually infinite, period**.

Consider then a periodic function, develop it in **Fourier Series** and then let the period go to infinity.

The discussion about **Fourier Series** of periodic functions leads to the expectation that the spacing between the different frequencies reduces to zero, at the limit of infinite period, and that the frequency spectrum becomes an essentially continuous spectrum, as the spacing decreases. In fact the spacing between time/space frequencies is $2\pi/\mathcal{P}$.

Consider a piece-wise continuous function such that $\int_{-\infty}^{+\infty} |f[w]| \, \mathrm{d}w$ exist.

The following relation is valid (**Fourier integral** or **Fourier transform** theorem):

$$f[w] = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f[s] \exp\left[\mp \mathrm{i} p s\right] \, \mathrm{d}s \right) \exp\left[\pm \mathrm{i} p w\right] \, \mathrm{d}p \quad .$$

Different conventions exist for both the choice of the factors and the sign in the exponential. One often used in physics is the following (symmetrical) one (ω for the case of a time function, k for the case of a space-function):

$$\widetilde{f}[\omega] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f[s] \exp\left[-\mathrm{i}\omega s\right] \, \mathrm{d}s \qquad \widetilde{f}[k] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f[s] \exp\left[-\mathrm{i} k s\right] \, \mathrm{d}s \quad ,$$

corresponding to:

$$f[w] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widetilde{f}[\omega] \exp\left[+\mathrm{i}\omega w\right] \, \mathrm{d}\omega \qquad f[w] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widetilde{f}[k] \exp\left[+\mathrm{i} k w\right] \, \mathrm{d}k \quad .$$

18.03.01 Frequency and Phase Spectrum for Non-Periodic Functions

The **module of the Fourier transform**, gives the so-called **frequency spectrum**, a continuous spectrum in case of a non-periodic function. Note, however, that the phase of the coefficients is necessary as well to reconstruct the function. The **phase of the Fourier transform** gives the so-called **phase spectrum**, a continuous spectrum in case of a non-periodic function.

The frequency spectrum may be either a time-frequency spectrum (in case of the transform of a time function), or a space-frequency spectrum (in case of the transform of a space function).

18.03.02 Indeterminacy Relation

One has the indeterminacy relations, in terms of the mean squared deviation, $\Delta\ldots$, for a generic function $f[s]$ with **Fourier Transform** $\widetilde{f}[p]$:

$$(\Delta s)^2 \equiv \frac{\int_{-\infty}^{+\infty} (s-\overline{s})^2 \, |f[s]|^2 \, \mathrm{d}s}{\int_{-\infty}^{+\infty} |f[s]|^2 \, \mathrm{d}s} \qquad (\Delta p)^2 \equiv \frac{\int_{-\infty}^{+\infty} (p-\overline{p})^2 \left|\widetilde{f}[p]\right|^2 \, \mathrm{d}p}{\int_{-\infty}^{+\infty} \left|\widetilde{f}[p]\right|^2 \, \mathrm{d}p} \quad , \qquad (18.03.01)$$

and (the indeterminacy relations)

$$\Delta s \Delta p \geq 1/2 \implies \{ \quad \text{space function } \Delta z \Delta k \geq 1/2 \qquad \text{time function } \Delta t \Delta\omega \geq 1/2 \quad \} \quad .$$

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Sinusoidal Signal of Finite Duration

This § is referenced at pages:
[Never referenced.]

©|J.P.Pérez *et al.*, *Optique*, ..., DUNOD, ...Ed., WEB - URL.|18||

Consider a Sinusoidal|Cosinusoidal signal of finite duration and frequency Ω (truncated sinusoid):

$$f[t] = \begin{cases} f_0 \cos[\Omega t] & |t| \leq \dfrac{T_0}{2} \\ 0 & |t| > \dfrac{T_0}{2} \end{cases} , \qquad \Omega > 0 \quad .$$

The **Fourier Transform** is:

$$\widetilde{f}[\omega] = \frac{f_0}{\sqrt{2\pi}} \left(\frac{\sin\left[\dfrac{T_0(\Omega-\omega)}{2}\right]}{\Omega-\omega} + \frac{\sin\left[\dfrac{T_0(\Omega+\omega)}{2}\right]}{\Omega+\omega} \right) \quad .$$

The **Fourier Transform** is an even function, as it should be, because the function is an even function. Thus the function has two parts, peaked at $\omega = \pm\Omega$, as $\cos[+\Omega t] = \cos[-\Omega t]$. In fact the original function is not changed when replacing $\Omega \to -\Omega$.

The result shows that most energy is contained in the region between the first two zeros of the frequency distribution, around $\omega = \pm\Omega$, defined by:

$$\omega = +\Omega \pm \frac{2\pi}{T_0} \qquad \omega = -\Omega \pm \frac{2\pi}{T_0} \quad .$$

The width of the frequency spectrum can thus be estimated as: $\Delta\omega \approx \dfrac{2\pi}{T_0}$. It decreases when T_0 increases. The time-length on the pulse is $\Delta t = T_0$. Thus one has: $\Delta\omega\Delta t \approx 2\pi$.

A parallel discussion can be carried on for a finite Sinusoidal|Cosinusoidal signal in space.

Square Signal of Finite Duration

Consider a square signal of finite duration and duration T_0:

$$f[t] = \begin{cases} f_0 & |t| \leq \dfrac{T_0}{2} \\ 0 & |t| > \dfrac{T_0}{2} \end{cases} , \qquad .$$

The **Fourier Transform** is:

$$\widetilde{f}[\omega] = f_0 \sqrt{\frac{2}{\pi}} \frac{\sin\left[\dfrac{\omega T_0}{2}\right]}{\omega} \quad .$$

The **Fourier Transform** is an even function, as it should be, because the function is an even function.

The result shows that most energy is contained in the region between the first two zeros of the frequency distribution, defined by:

$$\omega = \pm\frac{2\pi}{T_0} \quad .$$

The width of the frequency spectrum can thus be estimated as: $\Delta\omega \approx \dfrac{2\pi}{T_0}$. It decreases when T_0 increases The time-length on the pulse is $\Delta t = T_0$. Thus one has: $\Delta\omega\Delta t \approx 2\pi$.

A parallel discussion can be carried on for a finite square signal in space.

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A Gaussian Packet

$$f[z] = \exp\left[-a^2 z^2\right] \implies \widetilde{f}[k] = \frac{1}{|a|\sqrt{2}} \exp -\frac{k^2}{4a^2} \quad .$$

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18.04 Propagation of a Pulse in a Dispersive Medium

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||

©|J.Franklin, *Classical ElectroMagnetism*, 2017, 2ndEd, ...Ed., WEB - URL.|§ 11.6.3|*Excellent!*|

©|WEB - URL |||

18.04.01
Superposition of MPW solution to D'Alembert wave-equation

A general solution to the **D'Alembert wave-equation** can be built from the superposition of MPW:

$$\xi[z,t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A[k] \exp\left[\mathtt{i}\,(kz - \omega[k]t)\right] \,\mathrm{d}k \quad .$$

Moreover, the uncertainty principle states:

$$\Delta z \Delta k \geq 1/2 \quad .$$

There is a tendency for the original shape to be lost and for the pulse to become distorted in shape.

One may verify that, for reasonable pulses or wave-packets that do not cut off too violently, the values of the root-mean squares lie near the lower limiting value, so that the equality sign can be often used.

18.04.02
Group-Velocity and pulse propagation

This § is referenced at pages:
[1165, 2010, 2010, 2149, 2164, 2164]

©|J.Franklin, *Classical ElectroMagnetism*, 2017, 2ndEd, ...Ed., WEB - URL.|§ 11.6|*Detailed demo.*|

Read § 18.05.05.
Read § 46.17.12.
Read § 44.03.11 and equation (44.03.08).

The general cases of a highly dispersive medium and of a very sharp pulse with a great spread of wave-numbers are very difficult to treat.

On the contrary, the propagation of a pulse which is not too broad in its wave-number spectrum in a medium for which the frequency depends weakly on the wave-number, can be easily handled in an approximate way:

$$\omega[k] = \omega_0 + \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k=k_0} (k - k_0) + \mathcal{O}\,(k - k_0)^2 \quad .$$

The integral gives:

$$\xi[z,t] \simeq \frac{1}{\sqrt{2\pi}} \exp\left[\mathtt{i}\left(k_0 t \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k=k_0} - \omega_0 t\right)\right] \int_{-\infty}^{+\infty} A[k] \exp\left[\mathtt{i}\left(kz - kt \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k=k_0}\right)\right] \,\mathrm{d}k \quad ,$$

which is a rigidly moving profile with velocity given by the **group-velocity** :

$$v_g[k=k_0] \equiv \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k=k_0} = \frac{\mathrm{d}(kv_p)}{\mathrm{d}k}[k=k_0] = v_p[k=k_0] + k_0 \frac{\mathrm{d}v_p}{\mathrm{d}k}[k=k_0] \quad .$$

WORK-IN-PROGRESS

If an energy is associated with the amplitude of the wave (typically to its absolute square), in this approximation the transport of energy then occurs with the **group-velocity** , since that is the rate at which the pulse travels along.

The **group-velocity** may differ greatly from the **phase-velocity** , often becoming larger, or even negative.

group-velocity is generally not a useful concept in regions of strong dispersion as the approximations made are no longer valid.

Read § 44.03.11.

18.04.03
Spread of a Wave-Packet

This § is referenced at pages:
[2015]

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For a Gaussian wave-packet, to second-order in k, with $\omega = \omega[k]$, it can be shown that the packet propagates at the **group-velocity** while its spread (standard deviation) increased with time as:

$$\sigma[t] = \sigma\sqrt{1+\left(\frac{\omega_0'' t}{\sigma^2}\right)^2} \qquad \omega'' \equiv \frac{\mathrm{d}v_g}{\mathrm{d}k} = \frac{d^2\omega}{\mathrm{d}k^2} \quad . \tag{18.04.01}$$

18.04.04
No Signal Can Travel Faster Than Light

This § is referenced at pages:
[2231]

©|J.Franklin, *Classical ElectroMagnetism*, 2017, 2ndEd, ...Ed., WEB - URL.|§ 11.6|*Detailed demo.*|

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18.05 Examples and Physical Applications

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18.05.01 Damped Sinusoidal|Cosinusoidal Signal

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18.05.02 Functions of Finite Duration (Pulses)

This applies to both functions in time and functions in space. A function of finite duration (a pulse) is defined as a function such that for both large enough and small enough values of w the function is zero:

$$f[w] = 0 \qquad \text{if, for some } W,\ |w| \geq W/2 \quad .$$

From the physical point of view functions of finite duration are the rule, as it is hardly conceivable any real physical function which has an infinite duration either in space or in time.

One can describe such a kind of functions as a periodic function with period, for instance, $\mathcal{P} = W$.

18.05.03 Coherence Time of Light

Read § 52.02.

18.05.04 Modulation

18.05.05 Expressions for the group-velocity

This § is referenced at pages:
[1163, 2149]

Read § 18.04.02.
Read § 46.17.12.

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Read § 44.03.11 and equation (44.03.08).

$$c = 1 \tag{18.05.01}$$

$$v_p \equiv v_p[k] \qquad \tilde{v}_p \equiv \tilde{v}_p[\lambda] \qquad \boxed{\tilde{v}_p[\lambda] \equiv v_p[k]} \tag{18.05.02}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}k} v_p[k]} = \frac{\mathrm{d}}{\mathrm{d}k}\tilde{v}_p[1/k] = \tilde{v}'_p[1/k]\frac{\mathrm{d}\lambda}{\mathrm{d}k} = \tilde{v}'_p[\lambda]\frac{\mathrm{d}\lambda}{\mathrm{d}k}\boxed{= -\lambda^2\tilde{v}'_p[\lambda]} \tag{18.05.03}$$

$$v_g \equiv v_g[k] \qquad \tilde{v}_g \equiv \tilde{v}_g[\lambda] \qquad \tilde{v}_g[\lambda] \equiv v_g[k] \tag{18.05.04}$$

$$v_p[k] \equiv \frac{\omega[k]}{k} \qquad v_g[k] \equiv \frac{\mathrm{d}\omega[k]}{\mathrm{d}k} \tag{18.05.05}$$

$$\boxed{v_g[k] =} \frac{\mathrm{d}\omega[k]}{\mathrm{d}k} = \frac{\mathrm{d}}{\mathrm{d}k}(\omega[k]) = \frac{\mathrm{d}}{\mathrm{d}k}(kv_p[k]) = v_p[k] + k\frac{\mathrm{d}v_p[k]}{\mathrm{d}k} \equiv v_p[k] + kv'_p[k] = \boxed{\tilde{v}_p[\lambda] - \lambda\tilde{v}'_p[\lambda]} \tag{18.05.06}$$

$$v_g[k] = \frac{\mathrm{d}\omega[k]}{\mathrm{d}k} = \frac{\mathrm{d}}{\mathrm{d}k}(\omega[k]) = \frac{\mathrm{d}}{\mathrm{d}k}(kv_p[k]) = \frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{k}{n_p[k]}\right) = \frac{1}{n_p[k]} - k\frac{n'_p[k]}{n_p^2[k]} = \frac{n_p[k] - kn'_p[k]}{n_p^2[k]} \quad . \tag{18.05.07}$$

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18.06 Exercises Problems and Physical Applications

18-001 Odd and Even Functions

Show that for an even(odd) periodic function only the cosinus(sinus) term exist in the **Fourier Series** .

18-002 Energy of a Periodic Function

Demonstrate the equalities (18.02.08), (18.02.09) and explain their meaning in terms of energy of a function.

18-003 Spreading of a Wave Packet

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|6.3, 293||

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CHAPTER 19

Linear Systems

19.01 Superposition Principle 1170
19.02 Addition of Sinusoidal|Cosinusoidal Functions of the Same Frequency 1171
19.03 Algebra of Complex Scalars 1173
19.04 Algebra of Complex Vectors 1174
19.05 Algebra With Time-Harmonic Quantities and Their Complex Representation 1176
19.06 Linear and Time-Invariant Systems (LTI) 1178

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This § is referenced at pages:
[1033]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.22; 1.23; 1.24; 1.25||

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19.01 Superposition Principle

This § is referenced at pages:
[1033, 1408, 1411]

As any source can be developed in **Fourier Series** | **Fourier Integral** the linearity thus implies that one can only consider the solution to fixed frequency sources and later sum/integrate all the solutions with the appropriate coefficients to get the final solution.

The previous discussions form the basis of the standard method of solution based on Fourier decomposition.

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19.02 Addition of Sinusoidal|Cosinusoidal Functions of the Same Frequency

This § is referenced at pages:
[1406, 1999, 2008, 2256, 2265]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|Appendix||

Read also § 44.04.01.

19.02.01
Addition of Two Real Sinusoidal|Cosinusoidal Functions of the Same Frequency

Consider the cosinusoidal functions:

$$f_1[s] = A_1 \cos[\alpha s - \phi_1] \qquad f_2[s] = A_2 \cos[\alpha s - \phi_2] \qquad \text{with } A_1 > 0 \text{ and } A_2 > 0 \quad . \tag{19.02.01}$$

→ 1171

The positiveness condition on A_1 and A_2 is not limiting at all because one can always make them positive via a suitable change of the **phase constants** ϕ_1 and ϕ_2.

$$f[s] = f_1[s] + f_2[s] = A \cos[\alpha s - \phi] \quad ,$$

$$A = \sqrt{A_1^2 + A_1^2 + 2A_1 A_2 \cos[\phi_2 - \phi_1]} \geq 0 \tag{19.02.02}$$

→ 1172

$$\cos[\phi] = \frac{A_1 \cos[\phi_1] + A_2 \cos[\phi_2]}{A} \qquad \sin[\phi] = \frac{A_1 \sin[\phi_1] + A_2 \sin[\phi_2]}{A} \quad .$$

→ 1172 2022

19.02.02
Addition of Two Complex Sinusoidal|Cosinusoidal Functions of the Same Frequency

The complex representation of the cosinusoidal functions (19.02.01), in terms of the so-called **phasors**, is:

$$\overline{f}_1 = A_1 \exp[+\mathtt{i}(\alpha s - \phi_1)] = A_1 \exp[-\mathtt{i}\phi_1] \exp[\mathtt{i}\alpha s] \equiv \overline{A}_1 \exp[+\mathtt{i}\alpha s] \quad ,$$

$$\overline{f}_2 = A_2 \exp[+\mathtt{i}(\alpha s - \phi_2)] = A_2 \exp[-\mathtt{i}\phi_2] \exp[\mathtt{i}\alpha s] \equiv \overline{A}_2 \exp[+\mathtt{i}\alpha s] \quad .$$

It is implicit in the complex representation that the phase information is kept in the complex constant:

$$\overline{A} \equiv A \exp[-\mathtt{i}\phi] \quad .$$

The sum in terms of the complex representation of cosinusoidal functions reads:

$$\overline{f} = \overline{f}_1 + \overline{f}_2 = (A_1 \exp[-\mathtt{i}\phi_1] + A_2 \exp[-\mathtt{i}\phi_2]) \exp[\mathtt{i}\alpha s] \equiv \overline{A} \exp[\mathtt{i}\alpha s] \quad ,$$

and:

$$\overline{A} \equiv A_1 \exp[-\mathtt{i}\phi_1] + A_2 \exp[-\mathtt{i}\phi_2] \quad , \tag{19.02.03}$$

→ 1172

and then:

$$f[s] \equiv f_1[s] + f_2[s] \equiv \mathfrak{Re}[\overline{f}] = \frac{1}{2}\left(\overline{A} \exp[+\mathtt{i}\alpha s] + \overline{A}^\star \exp[-\mathtt{i}\alpha s]\right) \equiv A \cos[\alpha s - \phi] \quad .$$

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19.02.03
Addition of an Arbitrary Number of Sinusoidal|Cosinusoidal Functions of the Same Frequency

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The result will be obtained by iteration of the results (19.02.02), (19.02.01), (19.02.03).
The real representation gives:

$$f[s] = \sum_{k=1}^{n} f_k[s] = \sum_{k=1}^{n} A_k \cos[\alpha s - \phi_k] = A \cos[\alpha s - \phi] \quad ,$$

$$A^2 = \sum_{r=1}^{n} \sum_{s=1}^{n} A_r A_s \cos[\phi_r - \phi_s] \geq 0 \tag{19.02.04}$$

$$\cos[\phi] = \frac{1}{A} \sum_{q=1}^{n} A_q \cos[\phi_q] \qquad \sin[\phi] = \frac{1}{A} \sum_{q=1}^{n} A_q \sin[\phi_q] \quad .$$

The complex representation gives:

$$\overline{A} \equiv \sum_{q=1}^{n} A_q \exp[-\mathtt{i}\phi_q] \quad . \tag{19.02.05}$$

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19.03 Algebra of Complex Scalars

This § is referenced at pages:
[2018, 2061, 2067, 2086, 2086]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.22||

It is just similar to, and simpler of, the case of complex vectors, as in section § 19.04.

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19.04 Algebra of Complex Vectors

This § is referenced at pages:
[1173, 2018, 2061, 2067, 2076]

Complex three-vectors are defined either as triplets of complex numbers,

$$\overline{\mathbf{V}} \equiv \{\overline{V}_1, \overline{V}_2, \overline{V}_3\} \quad ,$$

or by building a complex vector from two real vectors,

$$\overline{\mathbf{V}} \equiv \mathbf{V}_{\mathrm{R}} + \mathrm{i}\mathbf{V}_{\mathrm{I}} \quad ,$$

the two definitions being clearly equivalent.

Moreover:

$$\mathbf{V}_{\mathrm{R}} = \frac{1}{2}\left(\overline{\mathbf{V}} + \overline{\mathbf{V}}^{\star}\right) \equiv \mathfrak{Re}\left[\overline{\mathbf{V}}\right] \qquad \mathbf{V}_{\mathrm{I}} = \frac{1}{2}\left(\overline{\mathbf{V}} - \overline{\mathbf{V}}^{\star}\right) \equiv \mathfrak{Im}\left[\overline{\mathbf{V}}\right] \quad .$$

Sum is defined as the usual sum of complex numbers.

The product of a vector with a complex number is calculated as the usual product of complex numbers.

The scalar product of two vectors:

$$\overline{\mathbf{A}} \equiv \{\mathbf{A}_{\mathrm{R}} + \mathrm{i}\mathbf{A}_{\mathrm{I}}\} \qquad \overline{\mathbf{B}} \equiv \{\mathbf{B}_{\mathrm{R}} + \mathrm{i}\mathbf{B}_{\mathrm{I}}\} \quad ,$$

is defined, exploiting the distributive property of the scalar product, as the complex number:

$$\overline{\mathbf{A}}\cdot\overline{\mathbf{B}} \equiv \overline{\mathbf{B}}\cdot\overline{\mathbf{A}} \equiv (\mathbf{A}_{\mathrm{R}} + \mathrm{i}\mathbf{A}_{\mathrm{I}})\cdot(\mathbf{B}_{\mathrm{R}} + \mathrm{i}\mathbf{B}_{\mathrm{I}}) = (\mathbf{A}_{\mathrm{R}}\cdot\mathbf{B}_{\mathrm{R}} - \mathbf{A}_{\mathrm{I}}\cdot\mathbf{B}_{\mathrm{I}}) + \mathrm{i}\,(\mathbf{A}_{\mathrm{R}}\cdot\mathbf{B}_{\mathrm{I}} + \mathbf{A}_{\mathrm{I}}\cdot\mathbf{B}_{\mathrm{R}}) \quad .$$

Note that this is **different from the Hermitian scalar product defined for a complex vector space**:

$$\cancel{\langle\overline{\mathbf{A}\mathbf{B}}\rangle \equiv \overline{\mathbf{A}}^{\star}\cdot\overline{\mathbf{B}} \neq \overline{\mathbf{B}}^{\star}\cdot\overline{\mathbf{A}} \equiv \langle\overline{\mathbf{B}\mathbf{A}}\rangle} \quad .$$

In the latter case, in fact, the scalar product of the vector with itself must be real and positive to define the norm.

The vector product of two vectors:

$$\overline{\mathbf{A}} \equiv \{\mathbf{A}_{\mathrm{R}} + \mathrm{i}\mathbf{A}_{\mathrm{I}}\} \qquad \overline{\mathbf{B}} \equiv \{\mathbf{B}_{\mathrm{R}} + \mathrm{i}\mathbf{B}_{\mathrm{I}}\} \quad ,$$

is defined, in a similar way to the scalar product, exploiting the distributive property of the vector product as the complex vector:

$$\overline{\mathbf{A}}\wedge\overline{\mathbf{B}} \equiv (\mathbf{A}_{\mathrm{R}} + \mathrm{i}\mathbf{A}_{\mathrm{I}})\wedge(\mathbf{B}_{\mathrm{R}} + \mathrm{i}\mathbf{B}_{\mathrm{I}}) = (\mathbf{A}_{\mathrm{R}}\wedge\mathbf{B}_{\mathrm{R}} - \mathbf{A}_{\mathrm{I}}\wedge\mathbf{B}_{\mathrm{I}}) + \mathrm{i}\,(\mathbf{A}_{\mathrm{R}}\wedge\mathbf{B}_{\mathrm{I}} + \mathbf{A}_{\mathrm{I}}\wedge\mathbf{B}_{\mathrm{R}}) \quad .$$

The scalar product of a complex vector with its complex conjugate is calculated with standard algebra, and it provides the definition of the positive squared module of the vector:

$$\left|\overline{\mathbf{A}}\right|^2 \equiv \overline{\mathbf{A}}\cdot\overline{\mathbf{A}}^{\star} = \overline{\mathbf{A}}^{\star}\cdot\overline{\mathbf{A}} = (\mathbf{A}_{\mathrm{R}} + \mathrm{i}\mathbf{A}_{\mathrm{I}})\cdot(\mathbf{A}_{\mathrm{R}} - \mathrm{i}\mathbf{A}_{\mathrm{I}}) = \mathbf{A}_{\mathrm{R}}^2 + \mathbf{A}_{\mathrm{I}}^2 \geq 0 \quad .$$

Note that for a complex vector $\overline{\mathbf{A}}$ the two conditions $\overline{\mathbf{A}}\cdot\overline{\mathbf{A}} = 1$ and $\left|\overline{\mathbf{A}}\right|^2 = 1$ are not equivalent. In fact:

$$\overline{\mathbf{A}}^2 \equiv \overline{\mathbf{A}}\cdot\overline{\mathbf{A}} = (\mathbf{A}_{\mathrm{R}} + \mathrm{i}\mathbf{A}_{\mathrm{I}})\cdot(\mathbf{A}_{\mathrm{R}} + \mathrm{i}\mathbf{A}_{\mathrm{I}}) = \mathbf{A}_{\mathrm{R}}^2 - \mathbf{A}_{\mathrm{I}}^2 + 2\mathrm{i}\,\mathbf{A}_{\mathrm{R}}\cdot\mathbf{A}_{\mathrm{I}} \in \mathbb{C} \quad ,$$

$$\left|\overline{\mathbf{A}}\right|^2 = \mathbf{A}_{\mathrm{R}}^2 + \mathbf{A}_{\mathrm{I}}^2 \geq 0 \quad , \tag{19.04.01}$$

which are different, unless $\mathbf{A}_{\mathrm{I}} = 0$. All in all:

$$\overline{\mathbf{A}}^2 = \mathbf{A}_{\mathrm{R}}^2 - \mathbf{A}_{\mathrm{I}}^2 + 2\mathrm{i}\,\mathbf{A}_{\mathrm{R}}\cdot\mathbf{A}_{\mathrm{I}} \in \mathbb{C} \quad ,$$

$$\left|\overline{\mathbf{A}}\right|^2 = \mathbf{A}_{\mathrm{R}}^2 + \mathbf{A}_{\mathrm{I}}^2 \geq 0 \quad . \tag{19.04.02}$$

Note, in particular, that if the scalar product of a complex vector with itself is zero this does not imply that the complex vector is zero but only that:

$$\overline{\mathbf{A}}\cdot\overline{\mathbf{A}} = 0 \implies \{ \quad \mathbf{A}_{\mathrm{R}}\cdot\mathbf{A}_{\mathrm{I}} = 0 \quad \text{and} \quad |\mathbf{A}_{\mathrm{R}}| = |\mathbf{A}_{\mathrm{I}}| \quad \} \quad ,$$

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19.04.01
Complex Direction Angles

This § is referenced at pages:
[2087]

Direction angles are better defined via $k \equiv \sqrt{\mathbf{k}\cdot\mathbf{k}}$ and not via the module of the complex vector, $|\overline{\mathbf{k}}|$:

$$\boxed{k \equiv \sqrt{\mathbf{k}\cdot\mathbf{k}} \qquad k \in \mathbb{C}} \quad ,$$

$$\boxed{k_x \equiv k\cos[\alpha_x] \equiv k\sin[\theta]\cos[\phi]} \quad ,$$

$$\boxed{k_y \equiv k\cos[\alpha_y] \equiv k\sin[\theta]\sin[\phi]} \quad ,$$

$$\boxed{k_z \equiv k\cos[\alpha_z] \equiv k\cos[\theta]} \quad .$$

Similarly, the complex direction cosines are defined as:

$$\boxed{\cos[\alpha_k] \equiv \frac{\overline{V}_k}{\sqrt{\overline{\mathbf{V}}\cdot\overline{\mathbf{V}}}}} \quad .$$

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19.05 Algebra With Time-Harmonic Quantities and Their Complex Representation

This § is referenced at pages:
[Never referenced.]

Physical quantities varying in a harmonic way are often well represented by complex numbers. Read § 46. The usual convention is that the physical quantity is represented by the real part of the complex number.

A Sinusoidal|Cosinusoidal varying quantity can be expressed as a complex number, denoted conventionally with $\overline{A}$, which keeps the information on both the amplitude, ρ, and phase, θ. The convention is used that the real quantity is given by the real part of the complex number:

$$A\cos[\omega t+\phi] \quad \Leftrightarrow \quad A\exp[\mathtt{i}(\omega t+\phi)] \equiv \overline{A}\exp[\mathtt{i}\omega t] \quad \text{with} \quad \overline{A} \equiv A\exp[\mathtt{i}\phi] \quad .$$

The convention to use $\exp[-\mathtt{i}\omega t]$, instead of $\exp[+\mathtt{i}\omega t]$, is the one normally used in Relativistic Quantum Physics, and most often in Optics. In Electric Engineering the other convention is most often used.

Moreover, one has:

$$\overline{A} \equiv A\exp[\mathtt{i}\phi] \equiv A(\cos[\phi]+\mathtt{i}\sin[\phi]) = A\cos[\phi]+\mathtt{i}A\sin[\phi] \equiv A_R+\mathtt{i}A_I \quad ,$$

as well as:

$$A = \mathfrak{Re}\left[\overline{A}\exp[\mathtt{i}\omega t]\right] = \frac{1}{2}\left(\overline{A}\exp[+\mathtt{i}\omega t]+\overline{A}^{\star}\exp[-\mathtt{i}\omega t]\right) \quad .$$

Whenever the product of two physical time-harmonic quantities is expressed with the complex representation, one has to be careful to select the correct expression for the result.

The time-averaged value of the product of two time-harmonic quantities having the same frequency, averaged over an integer number of periods, is found as follows:

$$A\cos[\omega t+\phi_A]\,B\cos[\omega t+\phi_B] = \frac{AB}{2}(\cos[\phi_A-\phi_B]+\cos[2\omega t+\phi_A+\phi_B]) \quad \longrightarrow \quad \frac{AB}{2}\cos[\phi_A-\phi_B] \quad . \tag{19.05.01}$$

→ 1176

It can be easily shown, by direct calculation with the polar representation of complex numbers, that the above result (19.05.01) can be expressed, via complex numbers, as:

$$\langle A\cos[\omega t+\phi_A]\,B\cos[\omega t+\phi_B]\rangle = \frac{AB}{2}\cos[\phi_A-\phi_B] = \frac{1}{2}\mathfrak{Re}\left[\overline{AB}^{\star}\right] = \frac{1}{2}\mathfrak{Re}\left[\overline{A}^{\star}\overline{B}\right] = \frac{\overline{AB}^{\star}+\overline{A}^{\star}\overline{B}}{4} \quad . \tag{19.05.02}$$

→ 1176

The same result applies for the time averaged value of scalar and vector products of vector fields.

The same important result in equation (19.05.02) cam be obtained as follows.

Suppose to have two time-harmonic quantities:

$$\overline{A}[t] = \overline{A}_0\exp[-\mathtt{i}\omega t] \qquad \overline{B}[t] = \overline{B}_0\exp[-\mathtt{i}\omega t] \quad .$$

The quantities are assumed to be scalar, but the result will be easily generalized to the scalar and vector product of two vectors. Typical application are for the power per unit volume transferred by the ElectroMagnetic fields on matter ($\mathbf{j}\cdot\mathbf{E}$) and the Poynting vector (energy flow via ElectroMagnetic fields).

By definition the product of the two real quantities is:

$$A[t]B[t] \equiv \mathfrak{Re}\left[\overline{A}[t]\right]\mathfrak{Re}\left[\overline{B}[t]\right] \equiv \left(\frac{\overline{A}[t]+\overline{A}^{\star}[t]}{2}\right)\left(\frac{\overline{B}[t]+\overline{B}^{\star}[t]}{2}\right) = \frac{\overline{A}[t]\overline{B}[t]+\overline{A}^{\star}[t]\overline{B}^{\star}[t]+\overline{A}[t]\overline{B}^{\star}[t]+\overline{A}^{\star}[t]\overline{B}[t]}{4} \quad .$$

The time-averaged expression is now easily found to be:

$$\langle A[t]B[t]\rangle = \langle\frac{\overline{A}[t]\overline{B}[t]+\overline{A}^{\star}[t]\overline{B}^{\star}[t]+\overline{A}[t]\overline{B}^{\star}[t]+\overline{A}^{\star}[t]\overline{B}[t]}{4}\rangle = \frac{\overline{A}_0\overline{B}_0{}^{\star}+\overline{A}_0{}^{\star}\overline{B}_0}{4} = \frac{1}{2}\mathfrak{Re}\left[\overline{A}_0\overline{B}_0{}^{\star}\right] = \frac{1}{2}\mathfrak{Re}\left[\overline{A}_0{}^{\star}\overline{B}_0\right] \quad .$$

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Therefore:

$$\boxed{\langle A[t]B[t]\rangle = \frac{1}{2}\mathfrak{Re}\left[\overline{A}_0\overline{B}_0^{\star}\right] = \frac{1}{2}\mathfrak{Re}\left[\overline{A}_0{}^{\star}\overline{B}_0\right] = \frac{\overline{A}_0\overline{B}_0{}^{\star} + \overline{A}_0{}^{\star}\overline{B}_0}{4}} \quad .$$

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19.06 Linear and Time-Invariant Systems (LTI)

This § is referenced at pages:
[1033, 1960]

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The eigenfunctions of Linear and Time Invariant systems (LTI) are Sinusoids|Cosinusoids (or, in complex form, exponentials).

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An important property of sinusoids at a particular frequency is that they are closed with respect to addition. In other words, if you take a sinusoid, make many copies of it, scale them all by different gains, delay them all by different time intervals, and add them up, you always get a sinusoid at the same original frequency. This is a non-trivial property. It obviously holds for any constant signal (which we may regard as a sinusoid at zero frequency), but it is not obvious for non constant functions.
Since every linear and time-invariant (LTI) system operates by copying, scaling, delaying, and summing its input signals to create its output signals, it follows that when a sinusoid at a particular frequency is input to an LTI system, a sinusoid at that same frequency always appears at the output. Only the amplitude and phase can be changed by the system. Therefore sinusoids are eigenfunctions of LTI systems. Conversely, if the system is non-linear or time-varying, new frequencies are created at the system output.

CHAPTER 20

Coordinate Transformations Tensors and Physical Laws

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This § is referenced at pages:
[1053, 1054, 1054, 1098, 1099, 2411, 2418, 2418, 2418, 2419, 2420, 2430]

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©|WEB - URL ||*Excellent*|
©|I.S.Sokolnikoff|Tensor Analysis, theory and applications, John Wiley and Sons, second edition, 1964||
©|R.Blandford & K.Thorne, *Applications Of Classical Physics*, ..., ..., ...Ed., WEB - URL, WEB - URL.|§ 1|*Excellent*|

WORK-IN-PROGRESS

20.01 Introduction

Note that in mathematics the word scalar is typically used for a generic one-component quantity, and the word vector for a generic n-uple of scalar quantities.
In physics the terms scalars and vectors (as well as tensors) are used in a more restrictive way referring to quantities which transform in a well defined way under coordinate transformations, as described in this section.

In fact not any physical quantity defined by a single number is a scalar. A typical counterexample is any componente of a vector, which is not a scalar. Scalars must be invariant under the set of prescribed transformation of coordinates.

There are also physical quantities which require a magnitude and an oriented direction for their complete specification. Not any physical quantity defined by a magnitude and an oriented direction (or, equivalently, by an n-uple of numbers) is a vector. Vectors must transform as prescribed by the set of prescribed transformation of coordinates. Moreover, vectors must obey all the properties of mathematical vectors in linear spaces. A typical counterexample is any finite angular displacement, which may be characterized by magnitude and oriented direction but is not a vector, for the addition of two or more angular displacements is not, in general, commutative.

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20.02 General Definition of Tensors

Tensors are of paramount importance in connection with transformations of coordinates. They serve to isolate intrinsic geometric and physical properties from those that merely depend on the choice of the coordinates, which always has a degree of arbitrariness.

Physical laws must be expressed in terms that do not depend at all on the arbitrary choice of the Coordinate System. Note that Coordinate System must be interpreted here in the most general sense of space-time Coordinate System, because in most important situations (relativity) the time is a coordinate on the same footing as space coordinates.

It must be clear, however, that in many cases it can be useful to introduce explicitly a Coordinate System, but valuable results must be finally expressed in terms of tensors law, independent of any preferred Coordinate System.

There are (at least) two ways of approaching the definition of tensors.

- Physicists usually define tensors in terms of objects whose components transform according to certain rules, introducing the ideas of contravariant and covariant transformations.
- Mathematicians usually define tensors in a way which involves defining certain vector spaces without fixing any Coordinate Systems until suitable bases are introduced when needed. contravariant vectors are just the elements of the vector space while covariant vectors (or just co-vectors) are the one-forms, the elements of the dual space to the contravariant vectors.

Physicists and engineers are among the first to recognise that vectors and tensors have a physical significance as entities, which goes beyond the (often arbitrary) co-ordinate system in which their components are enumerated. Similarly, mathematicians find there are some tensor relations which are more conveniently derived in a co-ordinate notation.

Since a sensible physical law must not depend upon an arbitrary choice of coordinates system, physical laws must be expressed in terms of tensors, that is objects which transform in a suitable way under coordinate transformations.

One should first note that by coordinate transformations we usually mean transformations of both the space and time coordinates.

20.02.01 Changes of Coordinates

This § is referenced at pages:
[Never referenced.]

Consider the set of n coordinates,

$$x^a \qquad \text{with } a = 1, \dots n \quad ,$$

and a general coordinate transformation

$$x^a \longrightarrow \overline{x}^a = \overline{x}^a[x^b] \qquad a, b = 1, \dots n \quad , \tag{20.02.01}$$

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which is locally invertible at P:

$$\frac{\partial \overline{x}^a}{\partial x^b}[P] \equiv M^a_{\cdot b} \neq 0 \quad , \tag{20.02.02}$$

where the partial derivatives matrix (Jacobian) is defined in such a way that its first (row) index labels the new coordinate, while its second (column) index labels the old coordinate.

In parallel we consider the inverse coordinate transformation:

$$\overline{x}^a \longrightarrow x^a = x^a[\overline{x}^b] \qquad a, b = 1, \dots n \quad , \tag{20.02.03}$$

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Consequently the partial derivatives matrix (Jacobian) of the inverse coordinate transformation is by the above definition,

$$\boxed{\frac{\partial x^a}{\partial \overline{x}^b}[P] \equiv \overline{M}^{a\cdot}_{\cdot b} \neq 0} \quad . \tag{20.02.04}$$

20.02.02 Some Conventions

We will introduce the recipe that up and low indexes are different and that summation can be done only on a up and a low index. The full understanding of this recipe will gradually show up in the following, with the understanding that up/low indexes represent different ways of transformation.

We will use the recipe to label the first index of a matrix as the row index (irrespective of its up/low level) and the second index of a matrix as the column index (irrespective of its up/low level).

20.02.03 Two Types of Transformations

This § is referenced at pages:
[2411]

20.02.03.01 Transformation of Contravariant Indexes

Differentials, locally at P, transforms like:

$$\boxed{\mathrm{d}x^a \longrightarrow \mathrm{d}\overline{x}^a = \frac{\partial \overline{x}^a}{\partial x^b}[P]\, \mathrm{d}x^b = \frac{\partial \overline{x}^a}{\partial x^b}\, \mathrm{d}x^b \equiv M^{a\cdot}_{\cdot b}\, \mathrm{d}x^b} \quad ,$$

where the Jacobian matrix, $\dfrac{\partial \overline{x}^a}{\partial x^b}[P]$, is evaluated at the point P. This is known as a contravariant transformation. Since now on the explicit mention of point P will be avoided and implicit.

20.02.03.02 Transformation of Covariant Indexes

Partial derivatives, locally at P, transforms like:

$$\boxed{\frac{\partial}{\partial x^a} \equiv \partial_a \longrightarrow \frac{\partial}{\partial \overline{x}^a} \equiv \overline{\partial}_a = \frac{\partial x^b}{\partial \overline{x}^a}[P]\frac{\partial}{\partial x^b} = \frac{\partial x^b}{\partial \overline{x}^a}\frac{\partial}{\partial x^b} \equiv \overline{M}^{b\cdot}_{\cdot a}\frac{\partial}{\partial x^b} = \overline{M}^{b\cdot}_{\cdot a}\partial_b} \quad ,$$

where the Jacobian matrix, $\dfrac{\partial x^b}{\partial \overline{x}^a}[P]$, is evaluated at the point P. This is known as a covariant transformation. Since now on the explicit mention of point P will be avoided and implicit.

20.02.03.03 Relations Between Contravariant and Covariant Transformations

One has:

$$\boxed{\frac{\partial \overline{x}^a}{\partial x^r}\frac{\partial x^r}{\partial \overline{x}^b} = M^{a\cdot}_{\cdot r}\overline{M}^{r\cdot}_{\cdot b} = \delta^{a\cdot}_{\cdot b}} \quad ,$$

and

$$\boxed{\frac{\partial x^a}{\partial \overline{x}^r}\frac{\partial \overline{x}^r}{\partial x^b} = \overline{M}^{a\cdot}_{\cdot r}M^{r\cdot}_{\cdot b} = \delta^{a\cdot}_{\cdot b}} \quad .$$

The two equations imply that $\overline{\mathbb{M}}$ is the inverse of $\mathbb{M}$:

$$\boxed{\left(\overline{\mathbb{M}}\right)^{a\cdot}_{\cdot b} = \left(\mathbb{M}^{-1}\right)^{a\cdot}_{\cdot b}} \quad .$$

and that in general one needs both relations to demonstrate the $\overline{\mathbb{M}}$ is both the inverse right and inverse left matrix of $\mathbb{M}$.

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Therefore, for a given coordinate transformation (20.02.01), contravariant indexes transform like the Jacobian matrix of the transformation, $\mathbb{M}$, while covariant indexes transform like the inverse of the Jacobian matrix of the transformation.

Note the conventional location of indexes: upper contravariant indexes and lower covariant indexes.

Note that:

- in the case of contravariant indexes the matrix is summed on the second index, that is the normal multiplication between the $\mathbb{M}$ matrix on the left and the column vector of the coordinates on the right;
- in the case of covariant indexes the matrix is summed on the first index, that is the normal multiplication between the row vector of the coordinates on the left and the the $\overline{\mathbb{M}}$ matrix on the right.

Given the above comments if one wants to express both transformations in the same way, for instance as the normal multiplication between a matrix on the left and the column vector of the coordinates on the right, one needs to introduce an additional transposition of indexes:

$$\{.\}^a_{\cdot} = M^a_{\cdot b}\{.\}^b_{\cdot} = \left(\overline{\mathbb{M}}^{-1}\right)^{a\cdot}_{\cdot b}\{.\}^b_{\cdot} = \left(\overline{\mathbb{M}}^{-1^{\mathsf{T}}}\right)^{\cdot a}_{b\cdot}\{.\}^b_{\cdot} \quad ,$$

$$\{.\}^{\cdot}_a = \overline{M}^{b\cdot}_{\cdot a}\{.\}^{\cdot}_b = \left(\mathbb{M}^{-1}\right)^{b\cdot}_{\cdot a}\{.\}^{\cdot}_b = \left(\mathbb{M}^{-1^{\mathsf{T}}}\right)^{\cdot b}_{a\cdot}\{.\}^{\cdot}_b \quad .$$

The reason why one uses two different types of indexes, upper contravariant indexes and lower covariant indexes, is just the fact that we have discovered that there are two different ways of transforming under a change of coordinates.

When handling matrices and indexes one should carefully remember the following rules/conventions:

- the first index of the matrix is the row index, while the second index is the column index: that is the horizontal location of the index refers to the matrix;
- the first index of both matrices, $\mathbb{M}$ and $\overline{\mathbb{M}}$, is taken from the numerator of the Jacobian matrix of the coordinate transformation, while the second one is taken from the denominator;
- the vertical location of the index, that is high or low, has nothing to do with the matrix itself but it only reminds the way that index transforms;

In many significant cases the coordinate transformation is a linear one and the transformation matrix, therefore, is a matrix which does not depend on the point.

20.02.03.04 Scalars (Rank Zero Tensors)

A scalar is a quantity which does not transform under a change of coordinates $x^p \longrightarrow \overline{x}^p = \overline{x}^p[x^q]$:

$$\overline{\phi} = \phi \quad .$$

20.02.03.05 Vectors (Rank One Tensors)

A contravariant Vector is a set of n quantities, V^a, which transforms, under a change of coordinates $x^p \longrightarrow \overline{x}^p = \overline{x}^p[x^q]$, as the differentials

$$V^a \longrightarrow \overline{V}^a = \frac{\partial \overline{x}^a}{\partial x^b} V^b \quad . \tag{20.02.05}$$

A covariant Vector is a set of n quantities, V_a, which transforms, under a change of coordinates $x^p \longrightarrow \overline{x}^p = \overline{x}^p[x^q]$, as the derivatives

$$V_a \longrightarrow \overline{V}_a = \frac{\partial x^b}{\partial \overline{x}^a} V_b \quad . \tag{20.02.06}$$

If one considers the vectors as **column vectors** and the operation on the right-hand side of equations (20.02.05), (20.02.06) as the standard row times column multiplication of matrices it is evident that the matrix defining the transformation of a covariant vector is just the inverse and transposed of the matrix defining the transformation of a contravariant vector.

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20.02.03.06 Tensors of Rank Two and Higher Rank (Arbitrary Rank)

A tensor of rank $r \equiv r_u + r_l$, with r_u contravariant (upper) indexes and r_l covariant (lower) indexes, (called a tensor of type $\{r_u, r_l\}$), is a set of n^m quantities, $T^{a\cdot cd\ldots\,\cdot}_{\cdot b\cdot\cdot\cdot\ldots}$, which transforms, under a change of coordinates $x^p \longrightarrow \overline{x}^p = \overline{x}^p[x^q]$, as m_u contravariant (upper) indexes and m_l covariant (lower) indexes:

$$\{x^p \longrightarrow \overline{x}^p = \overline{x}^p[x^q] \qquad p,q = 1,\ldots n\} \Longrightarrow \left\{T^{a\cdot cd\ldots\,\cdot}_{\cdot b\cdot\cdot\cdot\ldots} \longrightarrow \overline{T}^{a\cdot cd\ldots\,\cdot}_{\cdot b\cdot\cdot\cdot\ldots} = M^{a}_{\cdot A}\overline{M}^{B}_{\cdot b}M^{c}_{\cdot C}M^{d}_{\cdot D}T^{A\cdot CD\ldots\,\cdot}_{\cdot B\cdot\cdot\cdot\ldots}\right\} \quad . \tag{20.02.07}$$

→ 1186

20.02.03.07 Tensors Fields of Arbitrary Rank

A tensor field (of any rank $r \equiv r_u + r_l$, and type $\{r_u, r_l\}$) is defined as a function of the coordinates transforming as a tensor whenever the coordinate argument changes in a way to always describe the same point:

$$\begin{aligned} & x^p \longrightarrow \overline{x}^p \qquad p,q = 1,\ldots n = \overline{x}^p[x^q] \quad , \\ \Longrightarrow \quad & T^{a\cdot cd\ldots\,\cdot}_{\cdot b\cdot\cdot\cdot\ldots}[x^p] \longrightarrow \overline{T}^{a\cdot cd\ldots\,\cdot}_{\cdot b\cdot\cdot\cdot\ldots}[\overline{x}^p] = M^{a}_{\cdot A}\overline{M}^{B}_{\cdot b}M^{c}_{\cdot C}M^{d}_{\cdot D}T^{A\cdot CD\ldots\,\cdot}_{\cdot B\cdot\cdot\cdot\ldots}[x^p] \quad , \\ \Leftrightarrow \quad & T^{a\cdot cd\ldots\,\cdot}_{\cdot b\cdot\cdot\cdot\ldots}[x^p] \longrightarrow \overline{T}^{a\cdot cd\ldots\,\cdot}_{\cdot b\cdot\cdot\cdot\ldots}[\overline{x}^p] = \frac{\partial \overline{x}^a}{\partial x^A}\frac{\partial x^B}{\partial \overline{x}^b}\frac{\partial \overline{x}^c}{\partial x^C}\frac{\partial \overline{x}^c}{\partial x^D}T^{A\cdot CD\ldots\,\cdot}_{\cdot B\cdot\cdot\cdot\ldots}[x^p] \quad . \end{aligned}$$

As a mnemonics it can be emphasised that the indexes on the left, $\{a, b, c, d, \ldots\}$, appear attached to the new coordinate set $\overline{x}$ in the Jacobian derivative matrix on the right-hand side.

20.02.03.08 Tensors Fields: A More General Definition

This § is referenced at pages:
[1185]

©|S.Weinberg, *The Quantum Theory of Fields, Vol. 1,2,3*, 1995, CUP, ...Ed., WEB - URL.|2.12||

Consider a locally invertible coordinate transformation

$$x^p \longrightarrow \overline{x}^p = \overline{x}^p[x^q] \qquad p,q = 1,\ldots n \quad ,$$

in a space of dimension n. and an arbitrary set of m functions of the coordinates

$$f_i[x^p], \qquad i = 1,\ldots m \quad .$$

Suppose that there exist a mapping of the set of functions $f_i[x^p]$ that is isomorphic to the coordinate transformation. The m functions then represent the m components of a tensor.

In particular, if the coordinate transformations form a group, the group representations can be used as the $f_i[x^p]$ functions.

20.02.03.09 Tensors Under Linear Transformations of Coordinates

In many significant cases the relevant coordinate transformation is a set of linear transformations forming a group. In this case what introduced in section § 20.02.03.08 is straightforwardly applied via any representation of the group.

It is in fact possible to start to define a tensor from linear transformations only, as it is often done. In this way, however, one loses the possibility to use non linear transformations of the coordinates, quite often important (non Cartesian set of coordinates, for instance).

Note that any quantity transforming like a tensor for general transformation of coordinates will transform as a tensor for linear transformations, obviously. However, there might be quantities, (namely the coordinates themselves) which are vectors for linear transformations but are not vectors for general non-linear transformations.

Also in Euclidean space one can find non-linear coordinates transformations, for instance when changing from orthonormal Cartesian Coordinates Coordinate System to Spherical Coordinates.

DRAFT - IN PROGRESS

20.03 Tensor Algebra and Calculus

20.03.01 Linear Combination of Tensors

Any linear combination with constant coefficients of tensors with the same kind of indexes is still a tensor of the same kind. It follows straightforwardly from (20.02.07).

20.03.02 Tensor (Direct) Product of Tensors

Given any two tensors, of rank m and n, the set of quantities produced by the tensor (direct) product of the two sets of numbers is a tensor of rank $m+n$ with contravariant/covariant indexes as in the original tensors. It follows straightforwardly from (20.02.07).

20.03.03 Contraction

This § is referenced at pages:
[1191]

The contraction between any contravariant index and any other covariant index is defined by summing the pair of indexes. This gives a tensor of rank two units less than the original tensor:

$$T^{a\cdot cs\ldots\cdot}_{\cdot s\cdot\cdot\cdot\ldots} \longrightarrow \overline{T}^{a\cdot cs\ldots\cdot}_{\cdot s\cdot\cdot\cdot\ldots} = M^{a\cdot}_{\cdot A}\overline{M}^{B\cdot}_{\cdot s}M^{c\cdot}_{\cdot C}M^{s\cdot}_{\cdot D}T^{A\cdot CD\ldots\cdot}_{\cdot B\cdot\cdot\cdot\ldots} = M^{a\cdot}_{\cdot A}M^{c\cdot}_{\cdot C}\delta^{B\cdot}_{\cdot D}T^{A\cdot CD\ldots\cdot}_{\cdot B\cdot\cdot\cdot\ldots} = M^{a\cdot}_{\cdot A}M^{c\cdot}_{\cdot C}T^{A\cdot CS\ldots\cdot}_{\cdot S\cdot\cdot\cdot\ldots} \quad .$$

The conclusion obviously comes from the fact that the contraction introduces a product of the transformation matrix times its inverse, giving a **Kronecker tensor** .

This shows that, in order to get a tensor, one contravariant index and one covariant index must be contracted.

20.03.04 Zero Tensor

Any zero tensor is invariant, that is it is zero after any coordinate transformation. It follows straightforwardly from (20.02.07).

20.03.05 Kronecker tensor

The **Kronecker tensor** is an invariant tensor:

$$\delta^{a\cdot}_{\cdot b} \longrightarrow \overline{\delta}^{a\cdot}_{\cdot b} = M^{a\cdot}_{\cdot A}\overline{M}^{B\cdot}_{\cdot b}\delta^{A\cdot}_{\cdot B} = M^{a\cdot}_{\cdot S}\overline{M}^{S\cdot}_{\cdot b} = \delta^{a\cdot}_{\cdot b} \quad .$$

DRAFT – IN PROGRESS

20.03.06
Principle of Covariance

This § is referenced at pages:
[1033, 1523, 1740, 2417, 2422]

If a particular law, written in tensor form, is valid in a particular Coordinate System, then it is valid in any general Coordinate System.

Note that, depending on the group of transformation considered, Coordinate System must be taken in its most general form, that is, not necessarily space-coordinates only:

- when one considers classical tensors, that is based on the group of rotation in space, SO(3), a change of Coordinate System is a rotation of a Cartesian Coordinates Coordinate System, or, more generally, a rotation plus translation;
- when one considers special-relativistic tensors, that is based on the Lorentz group, a change of Coordinate System is a **Lorentz Transformation** between Inertial Reference Frame , or, more generally, a **Poincaré Transformation** ;
- when one considers general-relativistic tensors, a general change of space-time Coordinate System is considered.

Read § 58.11.09.

WORKED SOLUTION

First Demonstration

Assume that the following equation is valid, with $T^{ab}_{\cdot\cdot}$ and $S^{ab}_{\cdot\cdot}$ tensors,

$$T^{ab}_{\cdot\cdot} = S^{ab}_{\cdot\cdot} \quad .$$

Then we can multiply both members by $M^{p\cdot}_{\cdot a} M^{q\cdot}_{\cdot b}$

$$M^{p\cdot}_{\cdot a} M^{q\cdot}_{\cdot b} T^{ab}_{\cdot\cdot} = M^{p\cdot}_{\cdot a} M^{q\cdot}_{\cdot b} S^{ab}_{\cdot\cdot} \implies \overline{T}^{pq}_{\cdot\cdot} = \overline{S}^{pq}_{\cdot\cdot} \quad .$$

Second Demonstration

Another alternative derivation follows.

Assume that the tensor quantity assumes a certain expression $\mathcal{F}^{a\cdot\cdot}_{\cdot bc}$ in a certain special Coordinate System. Then the expression in an arbitrary Coordinate System is jut obtained by tensor transformation:

$$x^a \longrightarrow \overline{x}^a = \overline{x}^a[x^b] \implies \mathcal{F}^{a\cdot\cdot}_{\cdot bc} \longrightarrow \overline{\mathcal{F}}^{p\cdot\cdot}_{\cdot qr} = M^{p\cdot}_{\cdot a} \overline{M}^{b\cdot}_{\cdot q} \overline{M}^{c\cdot}_{\cdot r} \mathcal{F}^{a\cdot\cdot}_{\cdot bc} \quad .$$

The demonstration goes by assuming, ab-absurdo, that, in an arbitrary Coordinate System, there might exist two different expressions, $\overline{\mathcal{F}}^{a\cdot\cdot}_{\cdot bc}$ and $\overline{\mathcal{F}}^{a\cdot\cdot}_{\cdot bc}$, reducing to $\mathcal{F}^{a\cdot\cdot}_{\cdot bc}$ in the special Rest Frame. Consider then the transformation of the difference $\overline{\mathcal{F}_1}^{a\cdot\cdot}_{\cdot bc} - \overline{\mathcal{F}_2}^{a\cdot\cdot}_{\cdot bc}$ back to the special Rest Frame: this gives zero, by hypothesis, because both $\overline{\mathcal{F}_1}^{a\cdot\cdot}_{\cdot bc}$ and $\overline{\mathcal{F}_2}^{a\cdot\cdot}_{\cdot bc}$ have the same expression in the special Rest Frame. It follows, form the invariance of the zero tensor, that the difference between the two expressions is the same in any Rest Frame, that is $\overline{\mathcal{F}_1}^{a\cdot\cdot}_{\cdot bc} = \overline{\mathcal{F}_1}^{a\cdot\cdot}_{\cdot bc}$.

This second derivation can be seen as an application of the first one when one of the two members of the tensor equality is the zero tensor.

It follows that:

- **if a tensor expression reduces to the correct particular expression in a certain Coordinate System, then that is the only correct covariant extension of that particular expression**;
- if all the components of a tensor vanish in one frame, they vanish in every frame;
- **two tensors equal in one Coordinate System are equal in every Coordinate System**.

Read § 20.10.02 for counter-examples.

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20.03.07
Quotient Law

This § is referenced at pages:
[1444, 1652, 2417, 2438, 2489]

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Consider any tensor relation among tensors such as, for instance:

$$K^a_{\cdot\cdot} A^{\cdot\cdot}_{aj} = B^{\cdot}_{j} \qquad \text{or} \qquad K^{a\,\cdot}_{\cdot\,b} T^{\cdot\cdot}_{aj} = S^{\cdot\cdot}_{bj} \qquad \dots \quad ,$$

where $A^{\cdot\cdot}_{aj}$ and $B^{\cdot}_{j}$, or $T^{\cdot\cdot}_{aj}$ and $S^{\cdot\cdot}_{bj}$, are known to be tensors. One can then infer that K is a tensor provided that all the components of A_k or $T^{\cdot\cdot}_{km}$, that is the tensor multiplying the quantity with unknown properties, K, can be varied independently.

Note, in particular, that the tensors A_k or $T^{\cdot\cdot}_{km}$ must be absolutely arbitrary, no symmetry or anti-symmetry, for instance, because one must be able to set any of their components to 1 and, at the same time, all the others to zero in order to demonstrate the theorem.

WORKED SOLUTION

Let us show it for a particular case, which will show that the same demonstration can be applied to any case. Assume we know that the relation

$$T^{\cdot\cdot}_{kj} C^{j}_{\cdot} = A^{\cdot}_{k} \quad ,$$

is always valid, whatever the Coordinate System, and that $A^{\cdot}_{k}$ and $C^{j}_{\cdot}$ are tensors. Let us demonstrate that $T^{\cdot\cdot}_{kj}$ is also a tensor. We know, by hypothesis, that the transformed relation is satisfied,

$$\overline{T}^{\cdot\cdot}_{kj} \overline{C}^{j}_{\cdot} = \overline{A}^{\cdot}_{k} \quad ,$$

with $A^{\cdot}_{k}$ and $C^{j}_{\cdot}$ transforming as tensors, and we want to determine the relation between $T^{\cdot\cdot}_{kj}$ and $\overline{T}^{\cdot\cdot}_{kj}$, showing that $\mathbb{T}$ is actually a tensor.
We can proceed as follows:

$$\begin{aligned}
\overline{T}^{\cdot\cdot}_{kj} \overline{C}^{j}_{\cdot} &= \overline{A}^{\cdot}_{k} \quad , \\
\overline{T}^{\cdot\cdot}_{kj} M^{j\,\cdot}_{\cdot\,b} C^{b}_{\cdot} &= \overline{M}^{a\,\cdot}_{\cdot\,k} A^{\cdot}_{a} \quad , \\
M^{k\,\cdot}_{\cdot\,s} \overline{T}^{\cdot\cdot}_{kj} M^{j\,\cdot}_{\cdot\,b} C^{b}_{\cdot} &= \overline{M}^{a\,\cdot}_{\cdot\,k} A^{\cdot}_{a} M^{k\,\cdot}_{\cdot\,s} \qquad \text{multiplying both members by } M^{k\,\cdot}_{\cdot\,s} \quad , \\
M^{k\,\cdot}_{\cdot\,s} \overline{T}^{\cdot\cdot}_{kj} M^{j\,\cdot}_{\cdot\,b} C^{b}_{\cdot} &= A^{\cdot}_{a} \delta^{a\,\cdot}_{\cdot\,s} \quad , \\
M^{k\,\cdot}_{\cdot\,s} \overline{T}^{\cdot\cdot}_{kj} M^{j\,\cdot}_{\cdot\,b} C^{b}_{\cdot} &= A^{\cdot}_{s} \quad , \\
M^{k\,\cdot}_{\cdot\,s} \overline{T}^{\cdot\cdot}_{kj} M^{j\,\cdot}_{\cdot\,b} C^{b}_{\cdot} &= T^{\cdot\cdot}_{sb} C^{b}_{\cdot} \qquad \text{using } T^{\cdot\cdot}_{kj} C^{j}_{\cdot} = A^{\cdot}_{k} \quad , \\
\left(\overline{T}^{\cdot\cdot}_{kj} M^{k\,\cdot}_{\cdot\,s} M^{j\,\cdot}_{\cdot\,b} - T^{\cdot\cdot}_{sb} \right) C^{b}_{\cdot} &= 0 \quad , \\
\Longrightarrow \; T^{\cdot\cdot}_{sb} &= \overline{T}^{\cdot\cdot}_{kj} M^{k\,\cdot}_{\cdot\,s} M^{j\,\cdot}_{\cdot\,b} \qquad \text{as it is valid for any } C^{j}_{\cdot} \quad , \\
\Longrightarrow \; \overline{T}^{\cdot\cdot}_{pq} &= T^{\cdot\cdot}_{sb} \overline{M}^{s\,\cdot}_{\cdot\,p} \overline{M}^{b\,\cdot}_{\cdot\,q} \qquad \text{multiplying by } \overline{M}^{s\,\cdot}_{\cdot\,p} \overline{M}^{b\,\cdot}_{\cdot\,q} \quad .
\end{aligned}$$

It is stressed again that **it must be possible to vary the components of the tensor C, of arbitrary rank in general, in an arbitrary manner in order to justify the last passage, while no similar condition is necessary on the tensor A.**

20.03.08
Derivative of a Tensor

In general it is not a tensor, because the necessity to derive the Jacobian matrix introduces additional terms. However, it is a tensor in the case of linear transformations.

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20.04 General Curvilinear Coordinates and Metrics

We take as an example a space described by three coordinates because the generalisation to an arbitrary number of coordinates is straightforward.

Consider the problem of representing a generic vector $\mathbf{v}$ in terms of three non orthogonal and non normalized vectors: $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. This bases is naturally induced when one consider a curvilinear system of coordinates: $\{x^1, x^2, x^3\}$. In this case the base vectors are defined by:

$$\boxed{\mathbf{e}_j \equiv \frac{\partial P[x^a]}{\partial x^j}} \quad .$$

One can write:

$$\mathbf{v} = v^k \mathbf{e}_k \quad .$$

But, for a non-orthonormal basis, we do not expect: $v^k = \mathbf{v}\cdot\mathbf{e}_k$. In fact, for general non orthonormal bases,

$$\mathbf{v}\cdot\mathbf{e}_k = v^j\, \mathbf{e}_j\cdot\mathbf{e}_k \neq v^k \quad .$$

However, if we manage to find a set of vectors, $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$, such that:

$$\mathbf{e}_j\cdot\mathbf{e}^k = \delta^{k\,\cdot}_{\cdot\,j} = \delta^{\cdot\,k}_{j\,\cdot} \quad , \tag{20.04.01}$$

→ 1206

we can recover:

$$\mathbf{v}\cdot\mathbf{e}^k = v^k \quad .$$

Note, however, that $\{\mathbf{e}_k\} \neq \{\mathbf{e}^k\}$.

Therefore

$$\boxed{\mathbf{v} = v^k\mathbf{e}_k \qquad v^k = \mathbf{v}\cdot\mathbf{e}^k \qquad \mathbf{e}_j\cdot\mathbf{e}^k = \delta^{k\,\cdot}_{\cdot\,j} = \delta^{\cdot\,k}_{j\,\cdot}} \quad .$$

Similarly:

$$\boxed{\mathbf{v} = v_k\mathbf{e}^k \qquad v_k = \mathbf{v}\cdot\mathbf{e}_k \qquad \mathbf{e}_j\cdot\mathbf{e}^k = \delta^{k\,\cdot}_{\cdot\,j} = \delta^{\cdot\,k}_{j\,\cdot}} \quad .$$

The bases $\{\mathbf{e}^k\}$ is known as the reciprocal base of the base $\{\mathbf{e}_k\}$.

20.04.01 A Metric

So far the space to which the coordinates x^a refer to has been without any structure. One can impose a structure by defining the distance between all pairs of neighboring points by means of a metric, not necessarily positive-definite:

$$\boxed{\mathbb{d}s^2 = \mathbb{d}\mathbf{x}\cdot\,\mathbb{d}\mathbf{x} = \mathbb{d}x^a\,\mathbb{d}x^b\,\mathbf{e}_a\cdot\mathbf{e}_b \equiv g^{\cdot\,\cdot}_{ab}[P]\,\mathbb{d}x^a\,\mathbb{d}x^b \qquad \mathbb{d}s^2 = \mathbb{d}x\cdot\,\mathbb{d}x = \mathbb{d}x_a\,\mathbb{d}x_b\,\mathbf{e}^a\cdot\mathbf{e}^b \equiv g^{ab}_{\cdot\,\cdot}[P]\,\mathbb{d}x_a\,\mathbb{d}x_b} \quad .$$

(20.04.02)

→ 1191

where the $g^{\cdot\,\cdot}_{ab}[P]$ and $g^{ab}_{\cdot\,\cdot}[P]$, the metric tensor, are in general, functions of the position:

$$\boxed{g^{\cdot\,\cdot}_{ab}[P] \equiv \mathbf{e}_a\cdot\mathbf{e}_b \qquad g^{ab}_{\cdot\,\cdot}[P] \equiv \mathbf{e}^a\cdot\mathbf{e}^b} \quad .$$

Moreover, we have the prescription for raising and lowering the indexes via the metric tensor:

$$v_k = \mathbf{v}\cdot\mathbf{e}_k \equiv v^j\,\mathbf{e}_j\cdot\mathbf{e}_k = v^j g^{\cdot\,\cdot}_{jk} \qquad v^k = \mathbf{v}\cdot\mathbf{e}^k \equiv v_j\,\mathbf{e}^j\cdot\mathbf{e}^k = v_j g^{jk}_{\cdot\,\cdot} \quad .$$

Finally note that

$$\mathbf{e}^j \equiv e^s[\mathbf{e}^j]\mathbf{e}_s = \delta^{j\,\cdot}_{\cdot\,s}\mathbf{e}^s \quad ,$$

$$e^s[\mathbf{e}^j] = \mathbf{e}^j\cdot\mathbf{e}^s \quad ,$$

$$\delta^{j\,\cdot}_{\cdot\,k} = \mathbf{e}^j\cdot\mathbf{e}_k = e^s[\mathbf{e}^j]\,\mathbf{e}_s\cdot\mathbf{e}_k = e^s[\mathbf{e}^j]g^{\cdot\,\cdot}_{sk} = \left(\mathbf{e}^j\cdot\mathbf{e}^s\right) g^{\cdot\,\cdot}_{sk} = g^{js}_{\cdot\,\cdot}g^{\cdot\,\cdot}_{sk} \quad ,$$

showing that the two contravariant and covariant metric tensor matrices are inverse of one another and that the mixed metric tensor matrices is just the **Kronecker tensor** .

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Note that the difference between contravariant and covariant indexes become noteworthy when the metric tensor is a function of the position.

The $g_{ab}^{\cdot\cdot}[P]$ is in fact a tensor as it is easly shown from the invariance of the distance with respect to the choice of the coordinates implying that $g_{ab}^{\cdot\cdot}[P]$ is a second-rank covariant tensor. We can assume

$$g_{ab}^{\cdot\cdot}[P] = g_{ba}^{\cdot\cdot}[P] \quad .$$

without any loss of generality.

It is straightforward to check the request of invariance of $\mathbb{d}s^2$ implies that $g_{ab}^{\cdot\cdot}$ is a rank-two covariant tensor.

20.04.01.01 Alternative Derivation of the Raising and Lowering Indexes

A less formal but shortened derivation of the raising and lowering indexes via the metric tensor goes as follows.

We know (section § 20.03.03) that in order to get an invariant we can contract one contravariant index with one covariant index from a rank two tensor. It looks then natural, from inspection of equation (20.04.02),

$$\mathbb{d}s^2 = g_{ab}^{\cdot\cdot}[P]\, \mathbb{d}x^a\, \mathbb{d}x^b \sim \mathbb{d}x^a\, \mathbb{d}x_a \sim \mathbb{d}x_a\, \mathbb{d}x^a \quad .$$

We can then define the general operation of raising and lowering the indexes as:

$$T^{a\cdot c\ldots\cdot}_{\cdot b\cdot\cdot\ldots} \equiv g_{bB}^{\cdot\cdot} T^{aBc\ldots\cdot}_{\cdot\cdot\cdot\cdot\ldots} \qquad T^{abc\ldots\cdot}_{\cdot\cdot\cdot\cdot\ldots} \equiv g^{bB}_{\cdot\cdot} T^{a\cdot c\ldots\cdot}_{\cdot B\cdot\cdot\ldots} \quad .$$

20.04.01.02 Transformations Defined via Invariant Quantities

Proper Rotations in Euclidean Three-Dimensional Space

In Euclidean three-dimensional space, the proper rotation group SO(3) (or, similarly, O(3)), can be defined as the set of linear transformations leaving invariant both the handness, $\det[\mathbb{R}] = 1$, and the distance $\mathbb{d}s^2$ with the Euclidean metric $g_{ab}^{\cdot\cdot} = \delta_{ab}^{\cdot\cdot}$:

$$\boxed{\mathbb{d}s^2 = \mathbb{d}x^a\, \mathbb{d}x_a = \mathbb{d}x_1{}^2 + \mathbb{d}x_2{}^2 + \mathbb{d}x_3{}^2} \quad .$$

It follows that there is no distinction at all between covariant and contravariant indexes under transformations of SO(3).

Proper Lorentz Transformation in Minkowski Space

In Minkowski space the proper **Lorentz Transformation** group (or, similarly, the full **Lorentz Transformation** group), can be defined as the set of linear transformations leaving invariant the time direction, $\mathbb{L}^{0\cdot}_{\cdot 0} \geq 1$, the handness of the space axes, $\det[\mathbb{L}] = 1$, and the Minkowski distance $\mathbb{d}s^2$ with the Minkowski metric $g_{ab}^{\cdot\cdot} = \eta_{ab}^{\cdot\cdot}$:

$$\boxed{\mathbb{d}\tau^2 = \eta_{ab}^{\cdot\cdot}\, \mathbb{d}x^a\, \mathbb{d}x^b = c^2\, \mathbb{d}t^2 - \mathbb{d}x_1{}^2 - \mathbb{d}x_2{}^2 - \mathbb{d}x_3{}^2} \quad .$$

It follows that to change between covariant and contravariant indexes under proper **Lorentz Transformation** one needs to change the sign of the space components of the four-vector.

Note that the metric in Minkowski space is not positive-definite.

The invariant $\mathbb{d}\tau$ is the proper time that is the time measured in the proper Rest Frame, because that gives the time interval when $\mathbb{d}\mathbf{x} = 0$, that is the time measured by an observer at rest with respect to the system.

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20.05 Group of Rotations in Physics

This § is referenced at pages:
[1149, 1149, 1241, 1439]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.18; 1.20|*Introduction*|
©|H.Goldstein *et al.*, Classical Mechanics (3rd Edition), Pearson Education International, 2013|Chapters 1, 4|*WEB - URL* |
©|Gerrit Coddens. Spinors for everyone. 2017. cea-01572342|WEB - URL |*Full critical discussion.*|

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see H. Goldstein *et al.*, Classical Mechanics (3rd Edition), Pearson Education International, 2013; chapters: 1, 4.

Read also sections § 17.03, § 23.06, § 26.03. This section presents the algebraic approach.
Let us consider only orthonormal Cartesian Coordinates Coordinate Systems.

20.05.01
Basics: Active and Passive Points of View

Two approaches are possible.

Active Rotations

The **passive** approach: the vector is a fixed fixed and the Coordinate System (that is the basis in the vector space) is rotated. The new components of the same vector in the new basis are given by:

$$\boxed{\mathbf{v} \to \mathbf{v}} \quad \text{unchanged: a vector is defined without reference to any basis} \quad , \quad ,$$

$$\boxed{\{\mathbf{e}_k\} \to \{\overline{\mathbf{e}}_k\}} \quad \text{just a change of basis in the vector space} \quad , \quad ,$$

$$\boxed{\mathbf{v} = v_k \mathbf{e}_k = \overline{v}_k \overline{\mathbf{e}}_k} \quad \text{the same vector represented in two different bases} \quad , \quad ,$$

$$\overline{v}_j = v_k \left(\overline{\mathbf{e}}_j \cdot \mathbf{e}_k\right) \Longrightarrow \boxed{\overline{v}_j \equiv R^{\cdot\cdot}_{jk} v_k} \Longrightarrow \overline{\mathbf{e}}_k = \left(\overline{\mathbf{e}}_k \cdot \mathbf{e}_j\right) \mathbf{e}_j = R^{\cdot\cdot}_{kj} \mathbf{e}_j \quad , \quad ,$$

$$\boxed{R^{\cdot\cdot}_{jk} \equiv \overline{\mathbf{e}}_j \cdot \mathbf{e}_k} \quad ,$$

that is: the same vector is represented in different bases; the new components are the components of the same vector in a different basis.

The matrix in the passive approach will be called $\mathbb{R}$, as this is the most elementary meaning of rotation: a rotation of the Coordinate System.

Passive Rotations

The **active** approach: the Coordinate System (that is the basis in the vector space) is fixed and the vector is rotated. The operator $\mathcal{G}$ **rotates** the vector. The components of the new vector in the old basis are given by:

$$\mathbf{v} \to \overline{\mathbf{v}} \equiv \mathcal{G}[\mathbf{v}] \qquad \text{the rotated vector} \quad , \quad ,$$
$$\{\mathbf{e}_k\} \to \{\overline{\mathbf{e}}_k\} \equiv \mathcal{G}[\{\mathbf{e}_k\}] \qquad \text{the basis vectors rotated as any other vector} \quad , \quad ,$$
$$v_k \equiv \mathbf{v}\cdot\mathbf{e}_k = \overline{\mathbf{v}}\cdot\overline{\mathbf{e}}_k = \mathcal{G}[\mathbf{v}]\cdot\mathcal{G}[\mathbf{e}_k] \implies \mathcal{G}[\mathbf{e}_k] \equiv \overline{\mathbf{e}}_k = \left(\overline{\mathbf{e}}_k\cdot\mathbf{e}_j\right)\mathbf{e}_j \equiv G^{\cdot\cdot}_{jk}\mathbf{e}_j \quad , \quad ,$$
$$\overline{v}_k = \overline{\mathbf{v}}\cdot\mathbf{e}_k = \mathcal{G}[\mathbf{v}]\cdot\mathbf{e}_k = \mathcal{G}\left[\left(\mathbf{v}\cdot\mathbf{e}_j\right)\mathbf{e}_j\right]\cdot\mathbf{e}_k = v_j\left(\mathcal{G}[\mathbf{e}_j]\cdot\mathbf{e}_k\right) \qquad \overline{v}_k = G^{\cdot\cdot}_{kj}v_j \quad , \quad ,$$
$$G^{\cdot\cdot}_{kj} \equiv \overline{\mathbf{e}}_j\cdot\mathbf{e}_k$$

The vector is actively **rotated**, the basis is not changed; the basis vectors are rotated as any other vector.

The rotated vector has on the rotated basis the same components as the original vector in the original basis.

Active Versus Passive Rotations

Clearly:

$$\boxed{\mathbb{R}^{-1} = \mathbb{R}^{\mathsf{T}} \equiv \mathbb{G} \qquad \mathbb{G}^{-1} = \mathbb{G}^{\mathsf{T}} \equiv \mathbb{R}} \quad .$$

Note that, in order to compare the two approaches, it should be emphasized that, geometrically, the correspondence between the basis vectors in the two cases are:

$$\{\{\mathbf{e}_k\} \quad \text{passive}\} \quad \Leftrightarrow \quad \{\{\overline{\mathbf{e}}_k\} \quad \text{active}\} \quad ,$$
$$\{\{\mathbf{e}_k\} \quad \text{active}\} \quad \Leftrightarrow \quad \{\{\overline{\mathbf{e}}_k\} \quad \text{passive}\} \quad .$$

20.05.02

Group of Rotations Algebra

The present discussion is carried on by using the position vector as example; however, by definition of vector, it can be applied to any vector.

Let us use exclusively the passive approach, unless specified otherwise: the same vector as seen by two different rotated Coordinate Systems.

Si considerino due Coordinate System cartesiani ortogonali i cui unit vectors degli assi (due terne ortonormali) siano rispettivamente $\{\mathbf{e}_k\}$, con $k = 1, 2, 3$, e $\{\overline{\mathbf{e}}_k\}$, con $k = 1, 2, 3$.

When the Coordinate System $\{\mathbf{e}_k\}$ is rotated one obtains the Coordinate System $\{\overline{\mathbf{e}}_k\}$.

Si definisca la matrice di trasformazione tra i due Coordinate System come:

$$\mathbf{e}_j \longrightarrow \overline{\mathbf{e}}_j \qquad \implies \qquad \boxed{\mathbb{R}^{\cdot\cdot}_{kj} \equiv \overline{\mathbf{e}}_k\cdot\mathbf{e}_j} \qquad (20.05.01) \quad \to 1193$$

$$\overline{\mathbf{e}}_k \longrightarrow \mathbf{e}_k \qquad \implies \qquad \boxed{\mathbb{R}^{-1\,\cdot\cdot}_{\quad jk} \equiv \mathbf{e}_j\cdot\overline{\mathbf{e}}_k = \mathbb{R}^{\cdot\cdot}_{kj} = \mathbb{R}^{\mathsf{T}\,\cdot\cdot}_{\quad jk}} \qquad (20.05.02)$$

$$\boxed{\mathbb{R}^{-1} = \mathbb{R}^{\mathsf{T}} \equiv \mathbb{G}} \quad .$$

Note that from the very definition (20.05.01) it is clear that the matrix for the inverse transformation is given by the transpose matrix.

La matrice di trasformazione è una matrice ortogonale a determinante unitario (gruppo SO(3)), cioé la sua inversa coincide con la sua trasposta, per cui si ha:

$$\boxed{\mathbb{R}^{\mathsf{T}}\mathbb{R} = \mathbb{R}\mathbb{R}^{\mathsf{T}} = \mathbb{I} \qquad \Leftrightarrow \qquad R^{\cdot\cdot}_{ik}R^{\cdot\cdot}_{jk} = R^{\cdot\cdot}_{ki}R^{\cdot\cdot}_{kj} = \delta^{\cdot\cdot}_{ij} \qquad \implies \qquad \det[\mathbb{R}] = \pm 1} \quad .$$

(20.05.03) → 1198

In forma matriciale si ha:

$$\mathbb{R}^{\mathsf{T}}\mathbb{R} = \mathbb{R}\mathbb{R}^{\mathsf{T}} = \mathbb{I} \qquad \implies \qquad \boxed{\mathbb{R}^{\mathsf{T}} = \mathbb{R}^{-1} \equiv \mathbb{G}} \quad . \qquad (20.05.04) \quad \to 1195$$

Dalla relazione di ortogonalità segue:

$$\mathbb{I} = \mathbb{R}\mathbb{R}^{\mathsf{T}} \implies 0 = \dot{\mathbb{R}}\mathbb{R}^{\mathsf{T}} + \mathbb{R}\dot{\mathbb{R}}^{\mathsf{T}} \implies \mathbb{R}\dot{\mathbb{R}}^{\mathsf{T}} = -\dot{\mathbb{R}}\mathbb{R}^{\mathsf{T}} \implies \dot{\mathbb{R}}\mathbb{R}^{\mathsf{T}} \qquad \text{è antisimmetrica} \quad .$$

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And also:

$$\mathbb{I} = \mathbb{G}\mathbb{G}^{\mathsf{T}} \implies 0 = \dot{\mathbb{G}}\mathbb{G}^{\mathsf{T}} + \mathbb{G}\dot{\mathbb{G}}^{\mathsf{T}} \implies \mathbb{G}\dot{\mathbb{G}}^{\mathsf{T}} = -\dot{\mathbb{G}}\mathbb{G}^{\mathsf{T}} \implies \dot{\mathbb{G}}\mathbb{G}^{\mathsf{T}} \quad \text{è antisimmetrica} \quad .$$

Passive Approach

The direct and inverse transformations are therefore given, in the passive approach, by:

$$\mathbf{x}_{\mathrm{P}} = \mathbb{R}\mathbf{x} \qquad \mathbf{x}_{\mathrm{P}}\text{: new components of the same vector} \quad , \quad ,$$
$$\implies \mathbf{x} = \mathbb{G}\mathbf{x}_{\mathrm{P}} \quad .$$

Therefore, for any vector fixed in the new frame, assumed to be moving with time (the moving frame):

$$\{\mathbf{x} = \mathbb{G}\mathbf{x}_{\mathrm{P}} \qquad \dot{\mathbf{x}}_{\mathrm{P}} = 0\} \implies \dot{\mathbf{x}} = \dot{\mathbb{G}}\mathbf{x}_{\mathrm{P}} = \dot{\mathbb{G}}\mathbb{R}\mathbf{x} = \dot{\mathbb{R}}^{\mathsf{T}}\mathbb{R}\mathbf{x} \equiv \mathbb{A}\mathbf{x} \qquad \mathbb{A} \equiv \dot{\mathbb{R}}^{\mathsf{T}}\mathbb{R} \quad , \tag{20.05.05}$$

so that:

$$\boxed{\dot{\mathbb{R}}^{\mathsf{T}}\mathbb{R}\mathbf{x} = -\dot{\mathbb{R}}^{\mathsf{T}}\dot{\mathbb{R}}\mathbf{x} \equiv \mathbb{A}_{\mathbb{R}}\mathbf{x}} \quad . \tag{20.05.06}$$

→ 1195

The action of any 3×3 anti-symmetrical matrix can be represented as a vector product, read § 20.05.04, and therefore the Poisson relations have been recovered.

Active Approach

The direct and inverse transformations are therefore given, in the active approach, by:

$$\mathbf{x}_{\mathrm{A}} = \mathbb{G}\mathbf{x} \quad ,$$
$$\mathbf{x} = \mathbb{R}\mathbf{x}_{\mathrm{A}} \quad .$$

Therefore, for any moving vector in the absolute frame:

$$\{\mathbf{x}_{\mathrm{A}} = \mathbb{G}\mathbf{x} \qquad \dot{\mathbf{x}} = 0\} \implies \dot{\mathbf{x}}_{\mathrm{A}} = \dot{\mathbb{G}}\mathbf{x} = \dot{\mathbb{G}}\mathbb{R}\mathbf{x}_{\mathrm{A}} = \dot{\mathbb{G}}\mathbb{G}^{\mathsf{T}}\mathbf{x}_{\mathrm{A}} \equiv \mathbb{A}_{\mathbb{G}}\mathbf{x}_{\mathrm{A}} \tag{20.05.07}$$

The action of any 3×3 anti-symmetrical matrix can be represented as a vector product, read § 20.05.04, and therefore the Poisson relations have been recovered.

20.05.03

Angular Velocity

As the matrix $\mathbb{A}$ is anti-symmetric, one can set, without loss of generality:

$$\boxed{A_{kj}^{\cdot\cdot} \equiv \epsilon_{jks}^{\cdot\cdot\cdot}\omega_s = \begin{vmatrix} 0 & -\omega_3 & +\omega_2 & , \\ +\omega_3 & 0 & -\omega_1 & , \\ -\omega_2 & +\omega_1 & 0 & , \end{vmatrix}} \quad .$$

→ 1205

It is then necessary to invert the equation to determine $\boldsymbol{\omega}$:

$$\boxed{\omega_s = \frac{1}{2}\epsilon_{sjk}^{\cdot\cdot\cdot}A_{kj}^{\cdot\cdot}} \quad . \tag{20.05.08}$$

Read § 20.05.04.

20.05.04

Matrici Antisimmetriche Del Secondo Ordine E Prodotto Vettoriale

This § is referenced at pages:
[1194, 1194, 1194, 1198, 1199, 2490]

Let $\mathbb{A}$ be an antisymmetric second-order tensor: $A_{kj}^{\cdot\cdot} = -A_{jk}^{\cdot\cdot}$.

As a 3×3 real antisymmetric matrix has three parameters, one may wonder whether the information carried by the matrix can be expressed in some way via a vector.

Let $\mathbf{x}$ be any vector; the vector $\mathbf{y} = \mathbb{A}\mathbf{x}$ defined as $y_i = A_{ir}^{\cdot\cdot}x_r$ can be expressed as a vector product, in terms of a suitable vector $\boldsymbol{\omega}$, as follows:

$$\boxed{y_i = A_{ir}^{\cdot\cdot}x_r \stackrel{?}{=} \epsilon_{isr}^{\cdot\cdot\cdot}\omega_s x_r \qquad \Leftrightarrow \qquad \mathbf{y} \stackrel{?}{=} \boldsymbol{\omega}\wedge\mathbf{x}} \quad .$$

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In other words we are looking for a vector $\boldsymbol{\omega}$ satisfying:

$$\boxed{A^{\cdot\cdot}_{ir} = \epsilon^{\cdot\cdot\cdot}_{ris}\omega_s \implies -A^{\cdot\cdot}_{ir} = \epsilon^{\cdot\cdot\cdot}_{irs}\omega_s} \quad .$$

In fact, using equation (20.06.05), one finds:

$$-\epsilon^{\cdot\cdot\cdot}_{ipq}A^{\cdot\cdot}_{ir} = \epsilon^{\cdot\cdot\cdot}_{ipq}\epsilon^{\cdot\cdot\cdot}_{irs}\omega_s \quad ,$$
$$-\epsilon^{\cdot\cdot\cdot}_{ipq}A^{\cdot\cdot}_{ir} = \omega_s\left(\delta^{\cdot\cdot}_{pr}\delta^{\cdot\cdot}_{qs} - \delta^{\cdot\cdot}_{ps}\delta^{\cdot\cdot}_{qr}\right) \quad ,$$
$$-\epsilon^{\cdot\cdot\cdot}_{ipq}A^{\cdot\cdot}_{ir} = \omega_q\delta^{\cdot\cdot}_{pr} - \omega_p\delta^{\cdot\cdot}_{qr} \quad .$$

As in the latest equation r, p and q are arbitrary free indexes we can freely chose $p = r$ to find:

$$-\epsilon^{\cdot\cdot\cdot}_{irq}A^{\cdot\cdot}_{ir} = \omega_q\delta^{\cdot\cdot}_{rr} - \omega_r\delta^{\cdot\cdot}_{qr} = 3\omega_q - \omega_q = 2\omega_q \quad ,$$

that is:

$$\boxed{\omega_q = \frac{1}{2}\epsilon^{\cdot\cdot\cdot}_{qri}A^{\cdot\cdot}_{ir}} \tag{20.05.09}$$

Therefore:

$$\mathbf{y} = \mathbb{A}\mathbf{x} \implies y_i = A^{\cdot\cdot}_{ir}x_r = \epsilon^{\cdot\cdot\cdot}_{isr}\omega_s x_r \implies \mathbf{y} = \boldsymbol{\omega}\wedge\mathbf{x}$$

20.05.05
Infinitesimal Transformations and Rotation Matrices

Write the rotation matrix, $\mathbb{U}$ (either $\mathbb{R}$ or $\mathbb{G}$), for an infinitesimal transformation as:

$$\boxed{\mathbb{U} = \mathbb{I} + \mathbb{B} + \mathcal{O}\left(\mathbb{B}\right)^2} \quad ,$$

where $\mathbb{B}$ is matrix depending on small (infinitesimal) parameters, three in the most general case.

The orthogonality relation (20.05.04) reads:

$$\mathbb{I} = \mathbb{U}\mathbb{U}^{\mathsf{T}} = \left(\mathbb{I} + \mathbb{B}\right)\left(\mathbb{I} + \mathbb{B}\right)^{\mathsf{T}} = \mathbb{I} + \left(\mathbb{B} + \mathbb{B}^{\mathsf{T}}\right) + \mathcal{O}\left(\mathbb{B}\right)^2 \qquad \implies \qquad \boxed{\mathbb{B} + \mathbb{B}^{\mathsf{T}} = 0} \quad .$$

The matrix $\mathbb{B}$ is thus antisymmetric.

It follows that for an infinitesimal rotation:

$$\boxed{\delta\mathbf{x} = \mathbb{B}\mathbf{x} \qquad \Delta x_k = \mathbb{B}^{\cdot\cdot}_{kj}x_j} \quad . \tag{20.05.10}$$

→ 1195

Note that the above equation (20.05.10) cannot be used to evaluate $\dot{x}$, because it is a first-order development already. To this end matrix $\mathbb{A}$ defined in equation (20.05.06) must be used. In other words, $\mathbb{A}$ describes exactly any rotation, while $\mathbb{B}$ describes an infinitesimal rotation. One has:

$$\mathbb{A} \equiv \dot{\mathbb{U}}^{\mathsf{T}}\mathbb{U} \simeq \dot{\mathbb{B}}^{\mathsf{T}}\left(\mathbb{I} + \mathbb{B}\right) \simeq \dot{\mathbb{B}}^{\mathsf{T}} \implies \left\{\mathbb{A}^{\mathsf{T}} \simeq \dot{\mathbb{B}} \qquad \mathbb{A} \simeq \dot{\mathbb{B}}^{\mathsf{T}}\right\}$$

20.05.06
Relations Between Rotation Matrix and Rotation Angle-Axis - Euler Theorem

This § is referenced at pages:
[1440]

The following Euler theorem can be demonstrated.

For every proper rotation around a certain point O and described by the matrix $\mathbb{R}$, there exist one axis, passing by O and with unit vector $\hat{\mathbf{n}}$, such that $\mathbb{R}$ results in a rotation about $\{O;\hat{\mathbf{n}}\}$:

- by an angle θ given by
$$\cos\left[\theta\right] = \left(\mathrm{Tr}\, R - 1\right)/2 \quad ,$$
- the three eigenvalues of the proper rotation matrix, $\mathbb{R}$, are:
$$1 \qquad \exp\left[\pm\mathrm{i}\theta\right] \qquad \text{with} \qquad \cos\left[\theta\right] = \left(\mathrm{Tr}\, R - 1\right)/2 \quad .$$

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20.05.07
Rotations of a Vector Around an Axis: Geometric Approach

Read § 17.03 for the geometrical approach.

20.05.08
Isotropic Tensors Under Rotation

This § is referenced at pages:
[1208, 1658]

Isotropic tensors under rotation are tensors whose components are invariant under any rotation [1].

All scalars are isotropic rank-zero tensors.

It can be shown that no first rank isotropic tensor exist. In fact, a first rank tensor is just a vector which is endowed with a direction.

It can be shown that there is only one second rank isotropic tensor: the scalar multiples of

$$\boxed{\delta^{\cdot\cdot}_{jk}} \quad .$$

It can be shown that there is only one third rank isotropic tensor: the scalar multiples of

$$\boxed{\epsilon^{\cdots}_{ijk}} \quad .$$

It can be shown that there are three independent fourth rank isotropic tensors: the scalar multiples of

$$\begin{cases} \boxed{\delta^{\cdot\cdot}_{jk}\delta^{\cdot\cdot}_{rs}} \quad , \\ \boxed{\delta^{\cdot\cdot}_{jr}\delta^{\cdot\cdot}_{ks} + \delta^{\cdot\cdot}_{js}\delta^{\cdot\cdot}_{kr}} \quad , \\ \boxed{\delta^{\cdot\cdot}_{jr}\delta^{\cdot\cdot}_{ks} - \delta^{\cdot\cdot}_{js}\delta^{\cdot\cdot}_{kr} = \epsilon^{\cdots}_{jki}\epsilon^{\cdots}_{rsi}} \end{cases}$$

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[1] WEB - URL

20.06 Cartesian Tensors Under the Group of Rotations

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Cartesian tensors are often used in various branches of classical physics, such as fluid mechanics and elasticity. In classical continuum mechanics, the space of interest is usually the three-dimensional Euclidean space. If one restricts the local coordinates to be orthonormal Cartesian Coordinates Coordinate System with the same scale centered at the point of interest, the metric tensor is the **Kronecker tensor** . Therefore there is no need to distinguish covariant and contravariant components. Usually all the indexes of Cartesian tensors are written as subscripts. Cartesian tensors achieve considerable computational simplification at the cost of generality and of some theoretical insight.

In generale una grandezza tensoriale è definita in termini delle sue proprietà di trasformazione rispetto ad un gruppo di trasformazioni. In fisica classica il gruppo di trasformazioni è il gruppo di Galilei. In fisica moderna il gruppo di trasformazioni è il gruppo di Poincarè.

Nella fisica classica le grandezze fisiche scalari, vettoriali e tensoriali sono definite, in generale, in termini delle loro proprietà di trasformazione per trasformazioni di Galilei. Nella fisica relativistica si generalizza il concetto di tensore considerando le proprietà di trasformazioni tra Reference Frame Inerziali arbitrari secondo le trasformazioni di Lorentz.

However, only the rotational, SO(3), part of the Galilei group is interesting, as far as tensor transformation properties are concerned, as the boosts and translations are commutative. Similarly only the homogeneous Lorentz group, out of the full Poincarè group, is interesting, as far as tensor transformation properties are concerned, as the translations are commutative.

Le leggi fisiche devono essere espresse tramite relazioni tensoriali in quanto l'espressione funzionale della legge non può dipendere dalla scelta del Coordinate System, o del Reference Frame inerziale, che è arbitraria.

In this section we will describe with some more details a particularly important class of tensors, Cartesian Tensors under SO(3). We restrict our attention to orthonormal Coordinate System in Euclidean space with three dimensions.

Data una generica grandezza fisica sia G la sua espressione nel primo Coordinate System e sia G' la sua espressione nel secondo Coordinate System. La grandezza eventualmente può essere un campo che dipende dalla posizione: $G[\mathbf{r}]$.

20.06.01 Scalari Cartesiani (Tensori Cartesiani Di Ordine Zero)

Una grandezza scalare è una grandezza il cui valore non cambia (è invariante) al cambiare del Coordinate System (Cartesiane ortonormali) dal Coordinate System $\mathbf{r}$ al Coordinate System $\mathbf{r}'$:

$$\phi'[\mathbf{r}'] = \phi[\mathbf{r}] \quad . \tag{20.06.01}$$

→ 1197

Esempio tipico di grandezza scalare è il modulo di un vettore o il prodotto scalare di due vettori. Esempio tipico di grandezza che non è una grandezza scalare è la componente di un vettore rispetto ad un asse qualunque.

Note that it is sometimes said that a scalar is a physical quantity represented by one single number. This is wrong. A scalar is not only represented by a single number, but it has also to satisfy to the definition (20.06.01).

20.06.02
Vettori Cartesiani (Tensori Cartesian Del Primo Ordine)

Al variare dal Coordinate System $\mathbf{r}$ al Coordinate System $\mathbf{r}'$ le componenti di un generico vettore $\mathbf{v}$ rispetto al primo Coordinate System (Cartesiane ortonormali) sono legate alle componenti rispetto al secondo Coordinate System (Cartesiane ortonormali) come segue:

$$v'_k[\mathbf{r}'] = R^{\cdot\cdot}_{kj} v_j[\mathbf{r}] \quad , \tag{20.06.02}$$

→ 1198

dove $R^{\cdot\cdot}_{ij}$ è la matrice di trasformazione tra i due Coordinate System cartesiani ortogonali. Infatti si ha:

$$\boxed{v'_k \equiv \mathbf{v}\cdot\mathbf{e}'_k = \mathbf{v}\cdot\left(R^{\cdot\cdot}_{kj}\mathbf{e}_j\right) = R^{\cdot\cdot}_{kj}\left(\mathbf{v}\cdot\mathbf{e}_j\right) \equiv R^{\cdot\cdot}_{kj} v_j} \quad .$$

La relazione (20.06.02) definisce un insieme di grandezze, v_k, con $k = 1, 2, 3$, (le componenti di un vettore) che, per definizione, costituiscono un vettore (ovvero un tensore del primo ordine).

Dalla (20.05.03) si ricava la relazione inversa:

$$\boxed{v_k = v'_j R^{\cdot\cdot}_{jk}} \quad . \tag{20.06.03}$$

20.06.03
Tensori Cartesiani Del Secondo Ordine

This § is referenced at pages:
[1056, 1204]

Un tensore Cartesiano del secondo ordine è un insieme di grandezze $T^{\cdot\cdot}_{ij}$ con $i, j = 1, 2, 3$ che per un cambiamento del Coordinate System (Cartesiane ortonormali), dal Coordinate System $\mathbf{r}$ al Coordinate System $\mathbf{r}'$, si trasforma come

$$T'^{\cdot\cdot}_{ij}[\mathbf{r}'] = R^{\cdot\cdot}_{ip} R^{\cdot\cdot}_{jq} T^{\cdot\cdot}_{pq}[\mathbf{r}] \quad , \tag{20.06.04}$$

dove $R^{\cdot\cdot}_{ij}$ è la matrice di trasformazione tra i due Coordinate System cartesiani ortogonali. Esempi di comuni tensori del secondo ordine sono: il tensore di inerzia, il tensore di **Electric-Permittivity** , il tensore di **Magnetic-Permeability** il tensore di **conductivity** , il tensore momento di quadrupolo gravitazionale, il tensore momento di quadrupolo elettrico, il tensore momento di quadrupolo magnetico, il tensore di deformazione e il tensore degli sforzi.

Any time there is a linear relation between two vectors one has to expect the relation to pass via a second-order tensor.

Un caso particolare è rappresentato dai tensori simmetrici del secondo ordine ($T^{\cdot\cdot}_{ij} = T^{\cdot\cdot}_{ji}$) che sono costituiti da sei grandezze indipendenti.

Un altro caso particolare è costituito dai tensori antisimmetrici del secondo ordine ($T^{\cdot\cdot}_{ij} = -T^{\cdot\cdot}_{ji}$) che sono costituiti da tre grandezze indipendenti. Le tre componenti indipendenti di un tensore antisimmetrico del secondo ordine si trasformano come le componenti di un opportuno vettore e questo permette quindi di assimilare il tensore antisimmetrico come un vettore opportuno, as shown in section § 20.05.04.

Ogni tensore del secondo ordine può essere espresso come somma di una componente simmetrica ed una antisimmetrica:

$$T^{\cdot\cdot}_{ij} = \frac{T^{\cdot\cdot}_{ij} + T^{\cdot\cdot}_{ji}}{2} + \frac{T^{\cdot\cdot}_{ij} - T^{\cdot\cdot}_{ji}}{2} \equiv \mathbb{T}^{\mathtt{S}}_{\mathtt{ij}} + \mathbb{T}^{\mathtt{A}}_{\mathtt{ij}} \quad .$$

La parte simmetrica, a sua volta, può essere decomposta in una parte a traccia nulla più una parte proporzionale al tensore identità:

$$\boxed{\mathbb{T}^{\mathtt{S}}_{\mathtt{ij}} = \left(\mathbb{T}^{\mathtt{S}}_{\mathtt{ij}} - \frac{1}{3}\mathbb{T}^{\mathtt{S}}_{\mathtt{kk}}\delta^{\cdot\cdot}_{ij}\right) + \frac{1}{3}\mathbb{T}^{\mathtt{S}}_{\mathtt{kk}}\delta^{\cdot\cdot}_{ij}} \quad .$$

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20.06.03.01 Invarianti Di Un Tensore Simmetrico Del Secondo Ordine

Si consideri un tensore simmetrico del secondo ordine. Questo ha tre invarianti scalari che possono essere formati dalle sue componenti:

- la traccia,
$$I_1 \equiv T_{11} + T_{22} + T_{33} = \delta_{ij}^{\cdot\cdot} T_{ij}^{\cdot\cdot} = T_{ii}^{\cdot\cdot} \quad ,$$
lineare nelle componenti del tensore,
- la somma dei suoi minori diagonali,
$$I_2 \equiv \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{33} & T_{31} \\ T_{13} & T_{11} \end{vmatrix} = \frac{1}{2}\left(T_{ii}^{\cdot\cdot} T_{jj}^{\cdot\cdot} - T_{ij}^{\cdot\cdot} T_{ij}^{\cdot\cdot}\right) = \frac{1}{2}\epsilon_{ij_1k_1}\epsilon_{ij_2k_2} T_{j_1j_2} T_{k_1k_2} \quad ,$$
quadratica nelle componenti del tensore,
- il determinante,
$$I_3 \equiv \det\left\{T_{ij}^{\cdot\cdot}\right\} = \epsilon_{ijk}^{\cdot\cdot\cdot} T_{1i} T_{2j} T_{3k} \quad ,$$
cubica nelle componenti del tensore.

I tre invarianti scalari sono i coefficienti dell'equazione cubica le cui soluzioni costituiscono gli autovalori del tensore simmetrico del secondo ordine:
$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \quad .$$

20.06.03.02 Tensori Antisimmetrici Del Secondo Ordine E Prodotto Vettoriale

Read § 20.05.04

20.06.04
Tensori (Cartesiani) Di Ordine Superiore

Analogamente a quanto mostrato si possono definire tensori di ordine superiore. Intervengono, ad esempio, nella meccanica dei corpi continui.

Il numero di indici di un tensore cartesiano si definisce il rango (rank) del tensore.

20.06.05
Contravariant and Covariant Indices for Cartesian Tensors

While the distinction between covariant and contravariant indices must be made for general tensors, the two are equivalent for tensors in n-dimensional Euclidean space equipped with orthonormal Cartesian Coordinates Coordinate System, and such tensors are known as Cartesian tensors.

In fact the metric tensors for an Euclidean space equipped with Cartesian coordinates is just the **Kronecker tensor** .

20.06.06
Levi-Civita tensor in Euclidean Space With n Dimensions

The **Levi-Civita tensor** in an Euclidean space with n dimensions equipped with an orthonormal basis is defined as (compare with § **??**):

$$\epsilon_{i_1\ldots i_n} = \begin{cases} +1 \text{ if } \{i_1 \ldots i_n\} \text{ is an even permutation of } 1, \ldots n \quad , \\ -1 \text{ if } \{i_1 \ldots i_n\} \text{ is an odd permutation of } 1, \ldots n \quad , \\ 0 \text{ if any two indexes are equal} \quad . \end{cases} \quad .$$

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20.06.06.01 Properties of the Levi-Civita tensor in 3 Dimensions

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The product of two **Levi-Civita tensor** can be expressed as a function of the **Kronecker tensor** :

$$\epsilon^{\dot{i}\dot{j}\dot{k}}\epsilon^{\dot{l}\dot{m}\dot{n}} =$$

$$+\delta^{\dot{i}\dot{l}}\delta^{\dot{j}\dot{m}}\delta^{\dot{k}\dot{n}} + \delta^{\dot{i}\dot{m}}\delta^{\dot{j}\dot{n}}\delta^{\dot{k}\dot{l}} + \delta^{\dot{i}\dot{n}}\delta^{\dot{j}\dot{l}}\delta^{\dot{k}\dot{m}} - \delta^{\dot{i}\dot{m}}\delta^{\dot{j}\dot{l}}\delta^{\dot{k}\dot{n}} - \delta^{\dot{i}\dot{l}}\delta^{\dot{j}\dot{n}}\delta^{\dot{k}\dot{m}} - \delta^{\dot{i}\dot{n}}\delta^{\dot{j}\dot{m}}\delta^{\dot{k}\dot{l}} \quad .$$

It can be, in fact, written as a 3×3 determinant of **Kronecker tensor** .

Product of two **Levi-Civita tensor** summed on one index:

$$\epsilon^{\dot{i}\dot{p}\dot{q}}\epsilon^{\dot{i}\dot{r}\dot{s}} = \delta^{\dot{p}\dot{r}}\delta^{\dot{q}\dot{s}} - \delta^{\dot{p}\dot{s}}\delta^{\dot{q}\dot{r}} \quad . \tag{20.06.05}$$

→ 1195

Product of two **Levi-Civita tensor** summed on two indexes:

$$\epsilon^{\dot{i}\dot{j}\dot{r}}\epsilon^{\dot{i}\dot{j}\dot{s}} = 2\delta^{\dot{r}\dot{s}} \quad . \tag{20.06.06}$$

Product of two **Levi-Civita tensor** summed on all three indexes:

$$\epsilon^{\dot{i}\dot{j}\dot{k}}\epsilon^{\dot{i}\dot{j}\dot{k}} = 3! \quad . \tag{20.06.07}$$

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20.07 Lorentz Group in Physics

20.08 Tensors Under the Lorentz Group

20.09 Tensors and Physical Laws

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.31||

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20.10 Examples and Physical Applications

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20.10.01 Reciprocal Basis in Euclidean Three-Dimensional Space

If $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are contravariant basis vectors of the euclidean three-dimensional space (not necessarily orthogonal nor of unit norm) then the covariant basis vectors of their reciprocal system are:

$$\mathbf{e}_1 = \frac{\mathbf{e}^2 \wedge \mathbf{e}^3}{\mathbf{e}^1 \cdot (\mathbf{e}^2 \wedge \mathbf{e}^3)} \quad ,$$

$$\mathbf{e}_2 = \frac{\mathbf{e}^3 \wedge \mathbf{e}^1}{\mathbf{e}^1 \cdot (\mathbf{e}^2 \wedge \mathbf{e}^3)} \quad ,$$

$$\mathbf{e}_3 = \frac{\mathbf{e}^1 \wedge \mathbf{e}^2}{\mathbf{e}^1 \cdot (\mathbf{e}^2 \wedge \mathbf{e}^3)} \quad .$$

Read § 20-006.

Then the contravariant coordinates of any vector can be obtained by the scalar product of the vector with the contravariant basis vectors:

$$q^1 = v \cdot \mathbf{e}^1 \qquad q^2 = v \cdot \mathbf{e}^2 \qquad q^3 = v \cdot \mathbf{e}^3 \quad .$$

Likewise, the covariant components of the vector can be obtained from the dot product with covariant basis vectors:

$$q_1 = \mathbf{v} \cdot \mathbf{e}_1 \qquad q_2 = \mathbf{v} \cdot \mathbf{e}_2 \qquad q_3 = \mathbf{v} \cdot \mathbf{e}_3 \quad .$$

Then the vector can be expressed in two (reciprocal) ways:

$$\mathbf{v} = q_i \mathbf{e}^i = q_1 \mathbf{e}^1 + q_2 \mathbf{e}^2 + q_3 \mathbf{e}^3 \quad ,$$

and

$$\mathbf{v} = q^i \mathbf{e}_i = q^1 \mathbf{e}_1 + q^2 \mathbf{e}_2 + q^3 \mathbf{e}_3 \quad .$$

The indices of covariant coordinates, vectors, and tensors are subscripts (but see above, note on notation convention). If the contravariant basis vectors are orthonormal then they are equivalent to the covariant basis vectors, so there is no need to distinguish between the covariant and contravariant coordinates, and all indices are subscripts.

20.10.02 Counter-Example to the Principle of Covariance

This § is referenced at pages:
[1187]

Consider the momentum of a force with respect to the origin, $\boldsymbol{\gamma} = \mathbf{r} \wedge \mathbf{f}$. Its z component is: $\gamma_z = r_x f_y - r_y f_x$ Other combinations, such as $\gamma_z = r_x f_z - f_y r_z$, would not transform m as the components of a vector, $\boldsymbol{\gamma}$.

Imagine to study the free-fall of a mass-point in the uniform gravity field, with the y axis vertical towards the sky, and trajectory of the falling mass-point: $x[t] = d > 0$, $y[t]$, $z = 0$. Suppose to find that it satisfies, for the angular momentum, a second-cardinal-equation-like, non vector equation $m d \dot{v}_y = -mgd$ and translate it in components as: $\dot{\ell}_z = r_x f_y$. Unless we write our experimental result in a tensor form, we are in trouble because, while $\dot{\ell}_z$ is a component of a vector, $r_x f_y$ is not! In fact, if we decided instead to use a rotated Coordinate System, with $y = y'$, $z = x'$, $x = -z'$, within our freedom for an arbitrary definition of the experimental Coordinate System, we would find a different physical law, and if trying to apply our original law, we would find $\dot{\ell}_{z'} = r_{x'} f_{y'}$, which is, $0 = 0$, at most, useless.

A significant example is the introduction of the viscosity, read § 28.05. The naive definition, equation (28.05.01), must be improved to the real definition, equation (28.05.03), in order to obtain, a relation independent on the choice of the Coordinate System.

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20.10.03
Inertia Matrix|Tensor

Explicit demonstration that it is a second rank tensor, from the definition of second rank tensor. -

20.10.04
Electric-Permittivity Tensor

In loss-less linear media (real **Electric-Permittivity**) the **Electric-Permittivity** tensor is symmetric.

20.10.05
Magnetic-Permeability Tensor

In loss-less linear media (real **Magnetic-Permeability**) the **Magnetic-Permeability** tensor is symmetric.

20.10.06
Conductivity Tensor

20.10.07
Electric Quadrupole Tensor

Read § 36.14.04.

20.10.08
Magnetic Quadrupole Tensor

Read § 36.14.05.

20.10.09
Gravitational Quadrupole Tensor

20.10.10
Example of Application of the Principle of Covariance

Read § 28.05.02.

20.10.11
Examples of Contravariant and Covariant Vector Components

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If no metric exists at all then contravariant Vectors (that is Vectors) and covariant Vectors (that is coVectors) are different objects. If a metric exists then there is a canonical isomorphism between them. So one introduces vectors and, after fixing a Coordinate System, one speaks about their contravariant and covariant components. The distinction is necessary when dealing with non-orthonormal Cartesian Coordinates Coordinate Systems.

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20.10.12
Product of Vectors

Let $\mathbf{A} = \{A_1, A_2, A_3\}$ and $\mathbf{B} = \{B_1, B_2, B_3\}$. To form the product of the two vectors one must use the nine components of the second-order tensor

$$A_j B_k = \begin{pmatrix} A_1B_1 & A_1B_2 & A_1B_3 & , \\ A_2B_1 & A_2B_2 & A_2B_3 & , \\ A_3B_1 & A_3B_2 & A_3B_3 & \end{pmatrix} \quad .$$

The scalar and vector product are a particular combination of those nine quantities. A second-order tensor can be decomposed, as described in section § 20.06.03, into a symmetric second rank tensor (that is a spherical tensor of rank two), plus an anti-symmetric second rank tensor (that is equivalent to a vector, that is a spherical tensor of rank one), plus a scalar, (that is a spherical tensor of rank zero).

The anti-symmetric second rank tensor is equivalent to taking the vector product $\mathbf{A} \wedge \mathbf{B}$ and the scalar is equivalent to taking the scalar product $\mathbf{A} \cdot \mathbf{B}$.

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20.11 Exercises Problems and Physical Applications

20-001 Not a Scalar/Vector Physical Quantity

Read § 15.01.02.

1. Find a examples of physical quantities represented by one single number that are not a scalar physical quantities.
2. Find a examples of physical quantities represented by one single number that are not a vector physical quantities.

20-002 Translation

Show that under a coordinate translation the transformation matrix is just the identity matrix, than the definition implies that there is no change in the components.

20-003 When the Difference Is Not Relevant

Consider the case of linear orthogonal transformations in a linear vector space, that is linear transformations such that the transformation matrix $\mathbb{O}$ satisfies:

$$\mathbb{O}\mathbb{O}^{\mathsf{T}} = \mathbb{O}^{\mathsf{T}}\mathbb{O} = \mathbb{I} \quad .$$

Show that contravariant and covariant indexes transform in the same way, so that there is no difference, and setting indexes high/low makes no difference.

20-004 Rotation About a Fixed Axis - Active Approach

Consider the fixed $\mathcal{I}$ and the moving $\mathcal{I}'$ rotating around the fixed $\mathbf{e}_3 = \mathbf{e}'_3$, that is the common $z = z'$ axes.

$$\mathbb{G} = \mathbb{R}^{\mathsf{T}} = \begin{pmatrix} +\cos[\theta] & -\sin[\theta] & 0 \\ +\sin[\theta] & +\cos[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

To first-order:

$$\mathbb{B} \equiv \mathbb{R} - \mathbb{I} = \begin{pmatrix} 0 & +\theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Longrightarrow \dot{\mathbb{B}} = \begin{pmatrix} 0 & +\dot{\theta} & 0 \\ -\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, = \mathbb{A}^{\mathsf{T}} \Longrightarrow \mathbb{A} = \begin{pmatrix} 0 & -\dot{\theta} & 0 \\ +\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Comparison with equation (20.05.03) shows that $\omega_3 = \dot{\theta}$ as it should by hypothesis.

20-005 Reciprocal Bases

1. Demonstrate that the reciprocal base of a reciprocal base is the original base.
2. Demonstrate that the reciprocal base of a right-handed reciprocal base is still a right-handed reciprocal base.

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20-006 Reciprocal Basis in Euclidean Three-Dimensional Space

This § is referenced at pages:
[1202]

Determine the reciprocal basis $\{\mathbf{e}^k\}$ of the basis $\{\mathbf{e}_k\}$, defined by (20.04.01).

WORKED SOLUTION

Observe that the requirement (20.04.01) implies that $\mathbf{e}^1$ must be orthogonal to both $\mathbf{e}_2$ and $\mathbf{e}_3$, $\mathbf{e}^2$ must be orthogonal to both $\mathbf{e}_3$ and $\mathbf{e}_2$, $\mathbf{e}^3$ must be orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_1$.
Therefore we can set

$$\mathbf{e}^k \propto \epsilon^{kij}_{\cdots}\, \mathbf{e}_i \wedge \mathbf{e}_j \quad .$$

The normalisation is determined to be

$$\boxed{\mathbf{e}^k = \frac{\epsilon^{kij}_{\cdots}\, \mathbf{e}_i \wedge \mathbf{e}_j}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \wedge \mathbf{e}_3)}} \quad .$$

20-007 Non-Orthogonal Cartesian Coordinates Coordinate Systems in a Plane

If the two axes form an angle α the metric is given by:

$$\begin{pmatrix} 1 & \cos[\alpha] \\ \cos[\alpha] & 1 \end{pmatrix}$$

From this is clear that contravariant and covariant components are identical if an only if $\alpha = \pi/2$.

20-008 Polar Coordinates in the Plane and Spherical Coordinates in the Space

1. Determine the metric tensor for polar coordinates in the plane.

   WORKED SOLUTION

   $$\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

2. Determine the metric tensor for spherical coordinates in the space.

20-009 Angular Velocity

1. Show it can be defined as:

   $$\boxed{\boldsymbol{\omega} \equiv \frac{1}{2}\mathbf{e}'_{\mathbf{k}} \wedge \dot{\mathbf{e}}'_{\mathbf{k}}} \quad . \tag{20.11.01}$$

2. Show that il vettore

   $$\boxed{\omega_q = \frac{1}{2}\epsilon^{\cdots}_{ijq} R^{\cdot\cdot}_{js} \dot{R}^{\cdot\cdot}_{is} = \frac{1}{2}\epsilon^{\cdots}_{ijq} R^{\cdot\cdot}_{js} \dot{G}^{\cdot\cdot}_{si} = \frac{1}{2}\epsilon^{\cdots}_{ijq} \left(\mathbb{R}\dot{\mathbb{G}}\right)^{\cdot\cdot}_{ji}} \tag{20.11.02}$$

   è la velocità angolare.

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20-010 Angle Preservation Under Rotation

Show that the angle between any two vectors is preserved in a rotation of Euclidean three-dimensional space.

WORKED SOLUTION

The angle between two vectors can be written in terms of lengths only as:

$$2\,\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}+\mathbf{y}|^2 - |\mathbf{x}|^2 - |\mathbf{y}|^2 \quad ,$$

and lengths are preserved in rotations.

20-011 Invarianza per Rotazioni Del Prodotto Scalare Di Due Vettori

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Dimostrare che il prodotto scalare di due vettori, $\mathbf{a}$ e $\mathbf{b}$, è invariante per rotazioni del Coordinate System. Si supponga, per semplicità, di lavorare nel piano xy anziché nello spazio three-dimensionale. Si considerino le componenti dei due vettori $\mathbf{a}$ e $\mathbf{b}$ rispetto ad un sistema di assi $x'y'$ ruotato di una angolo θ rispetto al sistema originale xy. Si calcoli il prodotto scalare usando le componenti dei due vettori rispetto al Coordinate System $x'y'$. Si verifichi che il risultato ottenuto coincide con quello che si ottiene usando le componenti rispetto al sistema xy. Il prodotto dei due vettori è cioè indipendente dall'angolo θ che misura la rotazione del Coordinate System $x'y'$ rispetto al Coordinate System xy.

WORKED SOLUTION

Si ha:

$$a'_i = \mathbb{R}_{ij} a_j \qquad b'_i = \mathbb{R}_{ij} b_j$$

con $\mathbb{R}$ matrice ortogonale speciale. Quindi:

$$\mathbb{R}\mathbb{R}^{\mathsf{T}} = \mathbb{R}^{\mathsf{T}}\mathbb{R} = \mathbb{I} \quad .$$

Allora:

$$\mathbf{a'}\cdot\mathbf{b'} = a'_i b'_i = \mathbb{R}_{is}\mathbb{R}_{ir} a_s b_r = \mathbb{R}_{is}\mathbb{R}^{\mathsf{T}}_{ri} a_s b_r = a_i b_i \quad .$$

20-012 Invariance of the Trace

Demonstrate that the trace of a second-order tensor is an invariant.

20-013 Invariance of the Symmetrical and Anti-Symmetrical Part of a Tensor

Show that a symmetrical tensor is always transformed into a symmetrical tensor, and an anti-symmetrical tensor is always transformed into an anti-symmetrical tensor.

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20-014 Levi-Civita tensor and the Determinant

Show how the determinant of any $n \times n$ matrix, $\mathbb{A}^{\cdot\cdot}_{rs}$, can be written in terms of the **Levi-Civita tensor** in n dimensions.

WORKED SOLUTION

$$\det[\mathbb{A}] = \epsilon_{i_1 \ldots i_n} \mathbb{A}_{1i_1} \mathbb{A}_{2i_2} \mathbb{A}_{ni_n} \tag{20.11.03}$$

→ 1208

The **Levi-Civita tensor** is an isotropic, completely anti-symmetric, rank n tensor.

20-015 Levi-Civita tensor Values in Different Coordinate System

Show that the **Levi-Civita tensor** assumes the same value in any Coordinate System

WORKED SOLUTION

Use (20.11.03) and the fact that the determinant is invariant under SO(3) to show that the value of any single element of the **Levi-Civita tensor** transforms to the same value.

20-016 Isotropic Tensors

Read § 20.05.08.

1. Show that **Kronecker tensor** is the only isotropic second-rank tensor by performing explicitly a few suitable rotations and impose that the components do not change.
2. Show that**Levi-Civita tensor** is the only isotropic third-rank tensor by performing explicitly a few suitable rotations and impose that the components do not change.

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CHAPTER 21

Some Miscellaneous Topics

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21.01 Variational Principles

This § is referenced at pages:
[1034]

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21.02 Homogeneity and Isotropy

This § is referenced at pages:
[1033, 1853, 1896]

The two properties are a priori independent.

Homogeneity is linked to invariance under space translation (Poincarè group). It implies that properties are described by quantities (tensors) that do not depend on the position.

Isotropy is linked to invariance under space rotations (SO(3)). It implies that properties are described by isotropic tensors.

Some examples follow.

- Homogeneous but non isotropic.
  - An homogeneous current conducting material with a non isotropic crystalline structure.
- Isotropic but not homogeneous.
  - A non homogeneous fluid has isotropic properties, locally at least. For instance the physical properties of the atmosphere are changing with height but locally isotropic. For a fluid, for instance, the stress tensor is isotropic.

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21.03 Global and Local Balance of Extensive Physical Quantities

This § is referenced at pages:
[1033, 1922, 2461, 2461, 2481, 2493, 2578]

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21.03.01
Extensive Physical Quantities

Intensive quantities are quantities which do not depend at all on the size of the system considered. Extensive quantities are quantities which do depend on the size of the system considered. Let X be an extensive physical quantity: such a quantity scales proportionally to the size of the system.

For extensive quantities one can define a volume density:

$$x_v \equiv \frac{\mathrm{d}X}{\mathrm{d}V} \Longrightarrow X = \iiint_\Omega x_v \, \mathrm{d}V \quad . \tag{21.03.01}$$

→ 1212

One can equally well define a mass density:

$$x_m \equiv \frac{\mathrm{d}X}{\mathrm{d}M} \Longrightarrow X = \iiint_\Omega \rho x_m \, \mathrm{d}V \quad . \tag{21.03.02}$$

→ 1212

In general the following relation between mass and volume density apply:

$$x_v = \rho x_m \quad .$$

Note that equation (21.03.01) is somewhat more satisfactory than equation (21.03.02), as it refers to a volume density, while mass is an extensive (and conserved) quantity itself.

Some examples follow.

- The number of particles is an extensive quantity:
$$N = \iiint_\Omega n_v \, \mathrm{d}V \quad ,$$
showing that the concentration, n_v, is the volume density of the number of particles.
- The linear momentum of a system is an extensive quantity:
$$\mathbf{P} = \iiint_\Omega \rho \mathbf{v} \, \mathrm{d}V \quad ,$$
showing that the velocity is the mass density of the linear momentum.

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21.03.02
Intensive Physical Quantities

Intensive physical quantities can be written in the form of a ratio of the extensive quantities.

They are defined locally, that is at any point, in contrast to extensive quantities which are defined globally for a system.

21.03.03
Kinematics of Transport

Read § 65.09.01.01.

21.03.03.01 Volume Current Density

The transport of an extensive physical quantity X is locally described, at any point and any time, by a vector, $\mathbf{J}_X$, the volume current density, defined in such a way that

$$\boxed{\delta X \equiv (\mathbf{J}_X \cdot \widehat{\mathbf{n}})\ \mathrm{d}S\ \mathrm{d}t = J_X \cos[\theta]\ \mathrm{d}S\ \mathrm{d}t} \quad ,$$

denotes the amount $\mathrm{d}X$ of the quantity X crossing a fixed, small and flat element of surface $\mathrm{d}\mathbf{S} \equiv \widehat{\mathbf{n}}\,\mathrm{d}S$ in the oriented direction of the normal, $\widehat{\mathbf{n}}$, during time $\mathrm{d}t$.

The amount δX_{R} of the quantity X received/entering during time $\mathrm{d}t$ by a system with a fixed closed boundary S is:

$$\boxed{\delta X_{\mathrm{R}} = -\,\mathrm{d}t \oiint (\mathbf{J}_X \cdot \widehat{\mathbf{n}})\ \mathrm{d}S} \quad ,$$

using the usual convention to orient the normal unit vector $\widehat{\mathbf{n}}$ exiting the surface.

The quantity crossing any surface per unit time, along the oriented normal $\widehat{\mathbf{n}}$, is given by the flux of the vector $\mathbf{J}_X$ over the surface:

$$\boxed{\delta X = \mathrm{d}t \iint_S (\mathbf{J}_X \cdot \widehat{\mathbf{n}})\ \mathrm{d}S} \quad .$$

21.03.03.02 Current Density for Omni-Directional Motion

Example: random motion of particles inside fluids.

Example: black-body radiation.

Read § 65.09.01.01.

21.03.03.03 Convective and Non Convective Currents

This § is referenced at pages:
[1894, 2484]

The transport of an extensive physical quantity can be either convective or not convective.

Convective motion exists whenever there is a global motion of matter. This can be seen as the motion of the Center-Of-Mass of any infinitesimal part of the matter. The transport of matter implies a convective transport and a convective contribution to the total current having the form:

$$\boxed{\delta X = x_v\,(\mathbf{v} \cdot \widehat{\mathbf{n}})\ \mathrm{d}S\ \mathrm{d}t} \quad \Longrightarrow \quad \boxed{\mathbf{J}_{X|\mathrm{C}} = x_v \mathbf{v}} \quad .$$

The non convective volume current vector is thus:

$$\boxed{\mathbf{J}_{X|\mathrm{NC}} \equiv \mathbf{J}_X - x_v \mathbf{v}} \quad .$$

The transport of an extensive physical quantity is non convective in a certain Rest Frame, when matter is locally at rest relative to this Rest Frame:

$$\boxed{\langle \mathbf{v} \rangle = 0} \quad \Longrightarrow \quad \boxed{\mathbf{J}_X = \mathbf{J}_{X|\mathrm{NC}}} \quad .$$

Read also § 61.02.02.04.

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21.03.04
Global Balance of Extensive Physical Quantities

The global balance of extensive physical quantities, that is the continuity equation, is a general relation which is encountered in many different topics.

Consider a generic system bounded by a closed surface S. The variation ΔX of the amount of an extensive quantity X of a system between two instants has two contributions. The first one comes from the exchanges with the external world, considered as positive if the quantity is given to the system. The second one comes from the production of the quantity inside the system, considered as positive is the quantity is created inside the system.

One has:

$$\boxed{\Delta X = \Delta X_{\mathtt{Received}} + \Delta X_{\mathtt{Created}} \equiv \Delta X_{\mathtt{R}} + \Delta X_{\mathtt{C}}} \quad .$$

The balance is often written in differential form considering two infinitesimally close instants:

$$\boxed{\mathrm{d}X = \delta X_{\mathtt{R}} + \delta X_{\mathtt{C}}} \quad .$$

For instance consider a town bounded by a closed surface and its population. Then $X_{\mathtt{R}}$ is due to the emigrated versus immigrant balance while $X_{\mathtt{C}}$ is due to the balance between newborn and deaths.

An extensive quantity is conservative if it is conserved when the system is isolated, that is:

$$\boxed{X \quad \text{is conservative} \quad \Leftrightarrow \quad \{\{\text{Isolated System} \quad \Leftrightarrow \quad \Delta X_{\mathtt{R}} = 0\} \implies \Delta X_{\mathtt{C}} = 0\}} \quad .$$

One should notice that occasionally a physical quantity might be conserved without being conservative. This is the case, for instance, of the mechanical energy when the internal non conservative forces do no work.

21.03.05
Local Balance of Extensive Physical Quantities

21.03.05.01 Local Sources of an Extensive Physical Quantity

The local creation or destruction of a an extensive physical quantity is described by its creation or destruction rate per unit volume, σ_X. It follows:

$$\boxed{\delta X_{\mathtt{C}} = \mathrm{d}t \iiint_{\Omega} \sigma_X \, \mathrm{d}V} \quad .$$

21.03.05.02 Continuity Equation

Consider the local balance of an extensive physical quantity, X, in a system delimited by a fixed closed surface S. One has:

$$\delta X_{\mathtt{R}} = -\,\mathrm{d}t \oiint (\mathbf{J}_{\mathtt{X}} \cdot \widehat{\mathbf{n}}) \, \mathrm{d}S = -\,\mathrm{d}t \iiint_{\Omega} \mathbf{div}\mathbf{J}_{\mathtt{X}} \, \mathrm{d}V \quad ,$$

$$\delta X_{\mathtt{C}} = \mathrm{d}t \iiint_{\Omega} \sigma_X \, \mathrm{d}V \quad ,$$

$$\mathrm{d}X = X[t + \mathrm{d}t] - X[t] = \iiint_{\Omega} (x_v[M, t + \mathrm{d}t] - x_v[M, t]) \, \mathrm{d}V = \mathrm{d}t \iiint_{\Omega} \partial_t x_v \, \mathrm{d}V \quad .$$

As this is valid for any fixed region of space one can deduce the continuity equation for the extensive physical quantity X:

$$\boxed{\partial_t x_v + \mathbf{div}\mathbf{J}_{\mathtt{X}} = \sigma_{\mathtt{X}}} \quad . \qquad (21.03.03) \quad \rightarrow 2738$$

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21.03.06
Local Conservation Laws

R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...- 27.1

Consider energy, the same considerations apply to linear momentum and angular momentum.

Energy is conserved, but the ElectroMagnetic field can carry energy, as any other field. Energy is conserved, and the detailed way energy is conserved is based on local conservation. Energy can be transferred only by energy flow. Energy inside the volume enclosed by a closed surface can change if and only if there is energy flow across the boundaries of the region.

This is much stronger than the statement that energy is conserved.

Consider a world-wide conservation law: energy disappears from one closed region and it simultaneously appears inside another closed region, far away from the first one. This world-wide conservation law would be incompatible with relativity as the concept of simultaneity is relative and another inertial observer would not see energy conserved at any time.

The only known way to make conservation law covariant is by making it local.

As a corollary local conservation implies that if anything moves from one place to the other there must be something happening in between.

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21.04 Time Dispersion and Spatial Non-Locality

This § is referenced at pages:
[1853, 1896, 1961, 1961, 2090]

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Let:

- $R[t]$: response
- $S[t]$: stimulus

Response cannot be instantaneous: it takes time. If the stimulus is too rapidly changing with respect to the characteristic response times of the system there is a delay in the response. Therefore the response will depend, a priori, on the stimulus at all the times before, that is from the history of events.

$$R[t] = \int_{w=-\infty}^{t} K[t,w]S[w]\,\mathrm{d}w \quad ,$$

$$K[t,w] \to K[t-w] \qquad \text{if the system is time-invariant, as it often happens} \quad ,$$

$$R[t] = \int_{w=-\infty}^{t} K[t-w]S[w]\,\mathrm{d}w \quad ,$$

$$R[t] = \int_{w=-\infty}^{w=+\infty} K[t-w]S[w]\,\mathrm{d}w \qquad K=0 \qquad \text{for} \qquad t<w \quad ,$$

$$S[w] = S_0 \exp[\mathtt{i}\omega t] \implies R[\omega] = K[\omega]S[\omega] \quad .$$

It sounds reasonable that for harmonic time dependence the answer is simple as an harmonic oscillation has always existed and therefore the transient part of the response has died away.

- Time dispersion: response depends on the stimulus at all the times before t (simply known as dispersion).
- Space dispersion: response depends on the stimulus at all the points in space (non-locality). It is typically less important and less common than time dispersion.

An important examples is in ElectroMagnetism in presence of matter, read § 41.04.05. In fact, the relation between $\mathbf{D}|\mathbf{H}$ and $\mathbf{E}|\mathbf{B}$ may be non-local:

$$\boxed{Y[\mathbf{x},t] = \iiint \mathrm{d}\mathbf{z} \int \mathrm{d}w\,(K[\mathbf{z},w]X[\mathbf{x}-\mathbf{z},t-w])} \quad ,$$

where $K[\mathbf{z},w]$ may be localized around $\mathbf{z} \simeq 0$ and|or $w \simeq 0$ but it is non vanishing for same range away from the origin. In terms of the **Fourier Transforms** it becomes:

$$\boxed{\widetilde{Y}[\mathbf{k},\omega] = \widetilde{K}[\mathbf{k},\omega]\widetilde{X}[\mathbf{k},\omega]} \quad .$$

It is often possible to neglect the non-locality in space.

21.05 External Documents

21.06 Perturbation Methods

21.06.01 Method of Variations of the Parameters

©|H.Pollard, *Mathematical Introduction To Celestial Mechanics*, 1966, Prentice-Hall, ...Ed., ...|4||

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21.06.02
Method of Successive Approximations

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Part C

Basics of Mechanics

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22 General References 1222

23 Kinematics of Points and Reference Frames 1226

24 Galilei-Newton Mechanics of General Systems 1285

25 Oscillations of a few mass points 1402

26 Introduction to Mechanics of Rigid-Bodies 1436

27 Introduction to Mechanics of Elastic Solids 1486

28 Introduction to Mechanics of Fluids 1514

29 Introduction to Central Force Fields and Gravitation 1545

30 Introduction to Many-Particles Systems 1599

31 Introduction to Celestial Mechanics and Astrodynamics 1621

32 Introduction to the Mechanics of Continuous Media 1643

33 Some Additional Topics in Mechanics 1675

34 Other Topics 1683

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CHAPTER 22

General References

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Pagina Web del Corso

Informazioni sul corso, testi di studio, problemi ed esercizi, fonti di dati, materiale vario di studio e multimediale:
WEB - URL.
Many classical texts are excellent texts, sometimes considered to be a *Bible* in their field, as they often are. However, physics progresses very very fast, so that excellent texts written many years ago remain a milestone, but may become old.

Useful General Mathematical Resources

©||S.Wolfram Mathematics Encyclopedia|*WEB - URL*|

©||J.Wevers Mathematics Formulary|*WEB - URL*|

©||WEB - URL|*Nice summary website.*|

Some Useful General Physics Resources/References

©||D.Halliday & R.Resnick & K.S.Krane, *Physics*, 2001, J.Wiley & Sons, 5thEd., ...|*Excellent, basic, modern; but do not ...*

©||Berkeley Physics Course, in 5 Vol., , ..., McGraw-Hill, ...Ed., ...|*Excellent text, but do not forget it was written many ...*

©||D.V.Sivuchin, , ..., ..., ...Ed., ...|*Excellent text, but do not forget it was written many years ago, and physics progresse- ...*

©||J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|*Excellent, basic, modern.*|

©||R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|*Excellent text, but do not ...*

©||L.D.Landau *et al.*, , ..., ..., ...Ed., ...|*Excellent text, but do not forget it was written many years ago, and physics ...*

©||W.M.Haynes *et al.*, *CRC Handbook of Chemistry and Physics*, 2016, CRC press, 97thEd., WEB - URL.|*Source of Data*|

©||R.Blandford & K.Thorne, *Applications Of Classical Physics*, ..., ..., ...Ed., WEB - URL, WEB - URL.|*Excellent, ...*

Some Useful Physics Specific Resources/References

©||G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|*Excellent text, ...*

©||Barger & Olsson, , ..., ..., ...Ed., ...||

©||M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|*Excellent, advanced text; but do not forget it was ...*

©||H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|*Excellent, advanced text; but do not forget it was ...*

©|M.dePodesta, *Understanding The Properties Of Matter*, 2002, CRC Press, 2ndEd., ...|Now freely available: ...

©||H.Goldstein *et al.*, *Classical Mechanics*, 2014, Pearson Education, 3rdEd., ...|*Excellent, advanced text; but do not ...*

©||D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|*Excellent, basic, modern.*|

©||E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|*Excellent, basic, modern.*|

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©||W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|*Excellent, ...*

©||F.Reif, *Fundamentals Of Statistical And Thermal Physics*, 2009, Waveland Press, ...Ed., ...|*Excellent, basic text; but ...*

©||A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|*Excellent, intermediate, modern.*|

©||M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|*Excellent, basic text; but ...*

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Some Useful Resources/References on Physics and the Real World

©||L.Bloomfield, *How Things Work*, ..., ..., ...Ed., WEB - URL.||

Other references

©|K.McDonald|WEB - URL|*A real treasure of tens of detailed and critical analyses of key physics problems, showing how ...*

The SI of Units of Measure

Read § B.

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References for this section

Some References for this section follow.

- ©|WEB - URL |Newtonian Dynamics|*Richard Fitzpatrick*|
- ©|WEB - URL |Classical Mechanics|*Nicholas Wheeler Physics Lectures*|

Exercises and problems for this part

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CHAPTER 23

Kinematics of Points and Reference Frames

23.01 Reference Frames and Coordinate Systems 1226
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23.01 Reference Frames and Coordinate Systems

23.01.01 Reference Frames

A Reference Frame is a set of fixed - rigid - axes to be used as reference for defining the position of any point of space plus a clock defining the time, which is an absolute parameter in Galilei-Newton Mechanics . Space, in Galilei-Newton Mechanics , is the euclidean $\mathbb{R}^3$.

23.01.02 Coordinate Systems

A Coordinate System is any specific set of coordinates which can be defined for any given Reference Frame, fixed in time allowing to unambiguously define each point in space in a univocal way. These are not necessarily Cartesian coordinates. For any given Reference Frame, infinite possibility exist to define a coordinate set.

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23.02 Basic Kinematics

23.02.01 Posizione Velocità Accelerazione

Sia x[t] la posizione di un punto in funzione del tempo. Velocità ed accelerazione sono definite da

$$\mathbf{x}[t] \implies \mathbf{v}[t] \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \implies \mathbf{a}[t] \equiv \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{d^2\mathbf{x}}{\mathrm{d}t^2} \quad .$$

Le relazioni inverse sono

$$\mathbf{x}[t] = \mathbf{x}[t_0] + \int_{t_0}^{t} \mathbf{v}[t]\ \mathrm{d}t \quad ,$$

$$\mathbf{v}[t] = \mathbf{v}[t_0] + \int_{t_0}^{t} \mathbf{a}[t]\ \mathrm{d}t \quad .$$

23.02.02 Posizione Velocità Accelerazione Relative

Date due particelle, A e B, la posizione relativa di A rispetto a B, $\mathbf{x}_{\mathtt{AB}}[t]$ la velocità relativa di A rispetto a B, $\mathbf{v}_{AB}[t]$, e l'accelerazione relativa di A rispetto a B, $\mathbf{a}_{AB}[t]$, sono definite da

$$\mathbf{x}_{\mathtt{AB}}[t] \equiv \mathbf{x}_{\mathtt{A}} - \mathbf{x}_{\mathtt{B}} \implies \mathbf{v}_{AB}[t] \equiv \mathbf{v}_{\mathtt{A}} - \mathbf{v}_{\mathtt{B}} \implies \mathbf{a}_{AB}[t] \equiv \mathbf{a}_{\mathtt{A}} - \mathbf{a}_{\mathtt{B}} \quad .$$

23.02.03 Jerk or Jolt

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In physics, jerk or jolt is the rate at which an object's acceleration changes with respect to time:

$$\boxed{\mathbf{j}[t] \equiv \dot{\mathbf{a}}} \quad . \tag{23.02.01}$$

It has considerable importance in many fields of engineering.

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23.03 Point Kinematics and Differential Geometry of Curves

©|WEB - URL |A First Course in Curves and Surfaces|*T. Shifrin*|
©|R.N. Jazar|Advanced Dynamics|*Unusually extensive treatment*|

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23.03.01
Parametrization of Curves

A point of a curve is parametrized by:

$$\mathbf{x}[t] = \begin{cases} x = x[t] & , \\ y = y[t] & , \\ z = z[t] & , \end{cases} \qquad \text{with} \qquad \mathbf{v} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \neq 0 \qquad \text{and suitable limits on } t\text{, if any} \quad .$$

It is implicitly assumed that the functions involved are differentiable just enough.
A curve is said to be regular when $v \neq 0$ at every point.
A point on the curve such that $v = 0$ is called a singular point on the curve.

23.03.02
Ascissa Curvilinea

L'ascissa curvilinea misura la lunghezza della curva $\mathbf{x}[t]$ ed è definita da

$$s[t] = s[t_0] + \int_{t_0}^{t} v[t]\ \mathrm{d}t = s[t_0] + \int_{t_0}^{t} \sqrt{\mathbf{v}\cdot\mathbf{v}}\ \mathrm{d}t \quad .$$

Si ha dunque

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v \quad .$$

23.03.03
Terna Di Unit Vectors Intrinseci Di Una Curva

Per ogni punto di una curva x[t] si definisce la terna intrinseca di unit vectors ortogonali $\widehat{\mathbf{T}}$, $\widehat{\mathbf{N}}$ e $\widehat{\mathbf{B}}$.

$$\widehat{\mathbf{T}} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} \equiv \dot{\mathbf{x}}[s] = \frac{1}{v}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathbf{v}}{v} \quad , \tag{23.03.01}$$

→ 1248 1271

$$\widehat{\mathbf{N}} \equiv \rho\frac{\mathrm{d}\widehat{\mathbf{T}}}{\mathrm{d}s} \equiv \frac{\ddot{\mathbf{x}}[s]}{|\ddot{\mathbf{x}}[s]|} = \frac{\rho}{v}\frac{\mathrm{d}\widehat{\mathbf{T}}}{\mathrm{d}t} \quad , \tag{23.03.02}$$

→ 1228 1248 1271

$$\widehat{\mathbf{B}} \equiv \mathbf{T}\wedge\mathbf{N} = \frac{\dot{\mathbf{x}}[s]\wedge\ddot{\mathbf{x}}[s]}{|\ddot{\mathbf{x}}[s]|} \quad . \tag{23.03.03}$$

→ 1248 1271

Note that, by definition, **the intrinsic unit vectors are dimensionless quantities** as the natural parametrization is given by the curvilinear abscissa. However, in the context of kinematics, the natural parameter is time, and therefore a factor v must be accounted for, whenever the kinematic character matters.

Il unit vector $\widehat{\mathbf{T}}$ è il unit vector tangente alla curva.

La relazione (23.03.02) definisce sia il unit vector $\widehat{\mathbf{N}}$ che il raggio di curvatura ρ (paragrafo § 23.03.04) della curva nel punto dato; ρ è il raggio del cerchio osculatore alla curva nel punto dato. Il unit vector $\widehat{\mathbf{N}}$ è diretto verso il centro di curvatura della curva nel punto dato.

I unit vectors $\widehat{\mathbf{N}}$ e $\widehat{\mathbf{T}}$, ortogonali tra di loro, definiscono il piano osculatore alla curva nel punto dato.

Il unit vector $\widehat{\mathbf{B}}$ è perpendicolare al piano osculatore alla curva nel punto dato.

The following definitions are used.

- Rectifying plane is the plane defined by $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{B}}$.
- Normal plane is the plane defined by $\widehat{\mathbf{N}}$ and $\widehat{\mathbf{B}}$, orthogonal to $\widehat{\mathbf{T}}$.
- Principal normal is line which contains $\widehat{\mathbf{N}}$ and passes through $\mathbf{x}[s]$.
- Binormal is the line which contains $\widehat{\mathbf{B}}$ and passes through $\mathbf{x}[s]$.

23.03.04
Curvatura E Torsione

This § is referenced at pages:
[1228]

La curvatura κ, il raggio di curvatura ρ e la torsione τ, il raggio di torsione σ sono definiti da:

$$\kappa \equiv \frac{1}{\rho} \equiv -\frac{\mathrm{d}\widehat{\mathbf{N}}}{\mathrm{d}s}\cdot\widehat{\mathbf{T}} = +\widehat{\mathbf{N}}\cdot\frac{\mathrm{d}\widehat{\mathbf{T}}}{\mathrm{d}s} \geq 0 \quad ,$$

$$\tau \equiv \frac{1}{\sigma} \equiv +\frac{\mathrm{d}\widehat{\mathbf{N}}}{\mathrm{d}s}\cdot\widehat{\mathbf{B}} = -\widehat{\mathbf{N}}\cdot\frac{\mathrm{d}\widehat{\mathbf{B}}}{\mathrm{d}s} \gtreqless 0 \quad .$$

Il raggio di curvatura è un numero positivo; la torsione è un numero positivo per una curva destrorsa ed è un numero negativo per una curva sinistrorsa.

Il raggio di curvatura e di torsione si esprimono in termini di $\mathbf{x}[t]$ tramite le relazioni:

$$\kappa = \frac{1}{\rho} = \frac{|\,\mathbf{x}'\wedge\mathbf{x}''|}{|\mathbf{x}'|^3} = \frac{|\,\mathbf{v}\wedge\mathbf{a}|}{v^3} \qquad (23.03.04)$$

→ 1270

$$\tau = \frac{1}{\sigma} = \frac{\mathbf{x}'\cdot(\,\mathbf{x}''\wedge\mathbf{x}''')}{|\,\mathbf{x}'\wedge\mathbf{x}''|^2} = \frac{\mathbf{v}\cdot(\,\mathbf{a}\wedge\mathbf{x}''')}{|\,\mathbf{v}\wedge\mathbf{a}|^2} \qquad (23.03.05)$$

Si hanno inoltre le proprietà seguenti:

$$\kappa = 0 \quad \Leftrightarrow \quad \text{retta} \quad ,$$
$$\tau = 0 \quad \Leftrightarrow \quad \text{curva piana (se } \kappa \neq 0) \quad .$$

La conoscenza di κ e τ caratterizza completamente la curva a meno di uno spostamento rigido. Cioè due curve che hanno gli stessi κ e τ sono sovrapponibili. Si ha quindi il teorema seguente. If two single-valued continuous functions $\kappa[s]$ (curvature) and $\tau[s]$ (torsion) are given for $s > 0$, then there exists exactly one space curve, determined except for orientation and position in space (i.e., up to a Euclidean motion), where s is the arc length, κ is the curvature, and τ is the torsion.

23.03.05
Frenet Formulas

In termini dell'ascissa curvilinea si hanno le formule di Frenet:

$$\frac{\mathrm{d}\widehat{\mathbf{T}}}{\mathrm{d}s} = +\kappa\widehat{\mathbf{N}} \quad ,$$

→ 1230

$$\frac{\mathrm{d}\widehat{\mathbf{N}}}{\mathrm{d}s} = -\kappa\widehat{\mathbf{T}} + \tau\widehat{\mathbf{B}} \quad ,$$

$$\frac{\mathrm{d}\widehat{\mathbf{B}}}{\mathrm{d}s} = -\tau\widehat{\mathbf{N}} \quad .$$

→ 1230

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In forma matriciale

$$\begin{bmatrix} \frac{\mathrm{d}\widehat{\mathbf{T}}}{\mathrm{d}t} \\ \frac{\mathrm{d}\widehat{\mathbf{N}}}{\mathrm{d}t} \\ \frac{\mathrm{d}\widehat{\mathbf{B}}}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} 0 & +\kappa & 0 \\ -\kappa & 0 & +\tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{T}} \\ \widehat{\mathbf{N}} \\ \widehat{\mathbf{B}} \end{bmatrix} \quad . \tag{23.03.06}$$

Inoltre dalle formule di Frenet si hanno le seguenti relazioni:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \widehat{\mathbf{T}} \quad ,$$
$$\frac{d^2\mathbf{x}}{\mathrm{d}s^2} = \kappa\widehat{\mathbf{N}} \quad ,$$
$$\frac{d^3\mathbf{x}}{\mathrm{d}s^3} = \frac{\mathrm{d}\kappa}{\mathrm{d}s}\widehat{\mathbf{N}} + \kappa\tau\widehat{\mathbf{B}} - \kappa^2\widehat{\mathbf{T}} \quad ,$$
$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\cdot\left(\frac{d^2\mathbf{x}}{\mathrm{d}s^2}\wedge\frac{d^3\mathbf{x}}{\mathrm{d}s^3}\right) = \kappa^2\tau \quad .$$

23.03.06
Darboux Vector

This § is referenced at pages:
[Never referenced.]

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Le formule di Frenet-Serret si possono scrivere in forma più compatta in termini del vettore di Darboux, che rappresenta la velocità angolare della terna intrinseca:

$$\boldsymbol{\Omega} = \tau\widehat{\mathbf{T}} + \kappa\widehat{\mathbf{B}} = \frac{\boldsymbol{\omega}}{v} \qquad \boldsymbol{\omega} = v\left(\tau\widehat{\mathbf{T}} + \kappa\widehat{\mathbf{B}}\right) = v\boldsymbol{\Omega} \quad , \tag{23.03.07}$$

→ 1248

where $\boldsymbol{\omega}$ is the real kinematics angular velocity. In fact, κ and τ are just purely geometrical parameters of the curve and the angular velocity also depend on the speed along the curve.

In terms of Darboux vector, the angular velocity of the intrinsic Reference Frame is given by two components: one along $\widehat{\mathbf{B}}$ and the other along $\widehat{\mathbf{T}}$ with angular speed respectively:

$$\omega_B = v\kappa \quad ,$$
$$\omega_T = v\tau \quad .$$

Therefore curvature describes rotation of the intrinsic Coordinate System around the bi-normal vector and torsion describes rotation of the intrinsic Coordinate System around the tangent vector.

The fact that $\omega_N = 0$ is understood from equations (23.03.05), (23.03.05), showing that the changes of $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{B}}$ are along $\widehat{\mathbf{N}}$ and therefore $\widehat{\mathbf{N}}$ is orthogonal to the angular velocity.

Finally:

$$\frac{\mathrm{d}\widehat{\mathbf{T}}}{\mathrm{d}s} = \boldsymbol{\Omega}\wedge\mathbf{T} \qquad \frac{\mathrm{d}\widehat{\mathbf{N}}}{\mathrm{d}s} = \boldsymbol{\Omega}\wedge\mathbf{N} \qquad \frac{\mathrm{d}\widehat{\mathbf{B}}}{\mathrm{d}s} = \boldsymbol{\Omega}\wedge\mathbf{B} \quad .$$

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Darboux vector

From Wikipedia, the free encyclopedia

In differential geometry, especially the theory of space curves, the **Darboux vector** is the angular velocity vector of the Frenet frame of a space curve.[1] It is named after Gaston Darboux who discovered it.[2] It is also called **angular momentum vector**, because it is directly proportional to angular momentum.

In terms of the Frenet-Serret apparatus, the Darboux vector **ω** can be expressed as[3]

$$\boldsymbol{\omega} = \tau \mathbf{T} + \kappa \mathbf{B} \qquad (1)$$

and it has the following symmetrical properties:[2]

$$\boldsymbol{\omega} \times \mathbf{T} = \mathbf{T}',$$
$$\boldsymbol{\omega} \times \mathbf{N} = \mathbf{N}',$$
$$\boldsymbol{\omega} \times \mathbf{B} = \mathbf{B}',$$

which can be derived from Equation (1) by means of the Frenet-Serret theorem (or vice versa).

Let a rigid object move along a regular curve described parametrically by **β**(*t*). This object has its own intrinsic coordinate system. As the object moves along the curve, let its intrinsic coordinate system keep itself aligned with the curve's Frenet frame. As it does so, the object's motion will be described by two vectors: a translation vector, and a rotation vector **ω**, which is an areal velocity vector: the Darboux vector.

Note that this rotation is kinematic, rather than physical, because usually when a rigid object moves freely in space its rotation is independent of its translation. The exception would be if the object's rotation is physically constrained to align itself with the object's translation, as is the case with the cart of a roller coaster.

Consider the rigid object moving smoothly along the regular curve. Once the translation is "factored out", the object is seen to rotate the same way as its Frenet frame. The total rotation of the Frenet frame is the combination of the rotations of each of the three Frenet vectors:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathbf{T}} + \boldsymbol{\omega}_{\mathbf{N}} + \boldsymbol{\omega}_{\mathbf{B}}.$$

Each Frenet vector moves about an "origin" which is the centre of the rigid object (pick some point within the object and call it its centre). The areal velocity of the tangent vector is:

$$\begin{aligned}\boldsymbol{\omega}_{\mathbf{T}} &= \lim_{\Delta t \to 0} \frac{\mathbf{T}(t) \times \mathbf{T}(t + \Delta t)}{2\,\Delta t} \\ &= \frac{\mathbf{T}(t) \times \mathbf{T}'(t)}{2}.\end{aligned}$$

Likewise,

$$\boldsymbol{\omega}_{\mathbf{N}} = \frac{1}{2}\,\mathbf{N}(t) \times \mathbf{N}'(t),$$
$$\boldsymbol{\omega}_{\mathbf{B}} = \frac{1}{2}\,\mathbf{B}(t) \times \mathbf{B}'(t).$$

Now apply the Frenet-Serret theorem to find the areal velocity components:

$$\boldsymbol{\omega}_{\mathbf{T}} = \frac{1}{2}\mathbf{T} \times \mathbf{T}' = \frac{1}{2}\kappa \mathbf{T} \times \mathbf{N} = \frac{1}{2}\kappa \mathbf{B}$$

$$\boldsymbol{\omega}_{\mathbf{N}} = \frac{1}{2}\mathbf{N}\times\mathbf{N}' = \frac{1}{2}(-\kappa\mathbf{N}\times\mathbf{T} + \tau\mathbf{N}\times\mathbf{B}) = \frac{1}{2}(\kappa\mathbf{B} + \tau\mathbf{T})$$

$$\boldsymbol{\omega}_{\mathbf{B}} = \frac{1}{2}\mathbf{B}\times\mathbf{B}' = -\frac{1}{2}\tau\mathbf{B}\times\mathbf{N} = \frac{1}{2}\tau\mathbf{T}$$

so that

$$\boldsymbol{\omega} = \frac{1}{2}\kappa\mathbf{B} + \frac{1}{2}(\kappa\mathbf{B} + \tau\mathbf{T}) + \frac{1}{2}\tau\mathbf{T} = \kappa\mathbf{B} + \tau\mathbf{T},$$

as claimed.

The Darboux vector provides a concise way of interpreting curvature κ and torsion τ geometrically: curvature is the measure of the rotation of the Frenet frame about the binormal unit vector, whereas torsion is the measure of the rotation of the Frenet frame about the tangent unit vector.[2]

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2. Farouki, Rida T. (2008), *Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable*, Geometry and Computing **1**, Springer, p. 181, ISBN 9783540733980.
3. Oprea, John (2007), *Differential Geometry and Its Applications*, Mathematical Association of America Textbooks, MAA, p. 21, ISBN 9780883857489.

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Categories: Differential geometry | Vectors

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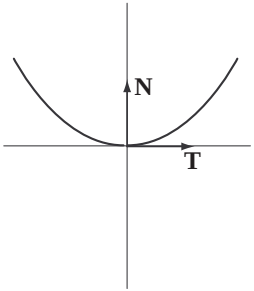
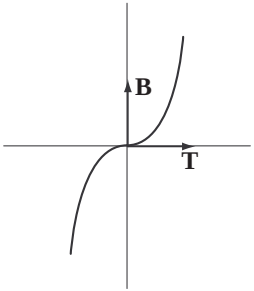
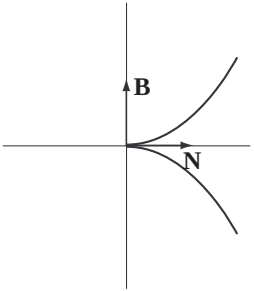


osculating plane



rectifying plane



normal plane

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23.03.07
Espressione Della Velocità in Coordinate Intrinseche

In termini della terna di vettori intrinseci alla traiettoria la velocità si esprime come:

$$\boxed{\mathbf{v} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = v\widehat{\mathbf{T}}} \quad .$$

23.03.08
Espressione Dell'accelerazione in Coordinate Intrinseche

In termini della terna di vettori intrinseci alla traiettoria l'accelerazione si esprime come:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(v\widehat{\mathbf{T}}\right) = \frac{\mathrm{d}v}{\mathrm{d}t}\widehat{\mathbf{T}} + v\frac{\mathrm{d}\widehat{\mathbf{T}}}{\mathrm{d}t} \quad . \tag{23.03.08}$$

$$\boxed{\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}t}\widehat{\mathbf{T}} + \frac{v^2}{\rho}\widehat{\mathbf{N}} \equiv a_{\mathrm{T}}\widehat{\mathbf{T}} + a_{\mathrm{R}}\widehat{\mathbf{N}}} \quad . \tag{23.03.09}$$

→ 1270 1591

Il vettore accelerazione giace quindi nel piano definito dai unit vectors $\widehat{\mathbf{T}}$ e $\widehat{\mathbf{N}}$, il piano osculatore.

Dalla relazione sopra si deduce che l'accelerazione tangenziale è legata solo al cambiamento del modulo della velocità mentre quella radiale dipende solo dal cambiamento di direzione. Infatti $\mathbf{a}_{\mathrm{T}}$ esiste se e solo se v non è costante nel tempo, mentre $\mathbf{a}_{\mathrm{N}}$ esiste se e solo se $\kappa \neq 0$ cioè se e solo se cambia la direzione di $\mathbf{v}$.

23.03.09
Rectilinear motion

23.03.09.01 Moto Rettilineo Uniforme

Se $\mathbf{v}$ è costante, motion along the direction of $\mathbf{v}$:

$$\boxed{\mathbf{x}[t] = \mathbf{x}[t_0] + \mathbf{v}\,(t - t_0)} \quad . \tag{23.03.10}$$

23.03.09.02 Moto Rettilineo Uniformemente Accelerato

Se $\mathbf{a}$ è costante:

$$\boxed{\mathbf{x}[t] = \mathbf{x}[t_0] + \mathbf{v}\,(t - t_0) + {}^1\!/_2\mathbf{a}\,(t - t_0)^2} \quad ,$$

$$\boxed{\mathbf{v}[t] = \mathbf{v}[t_0] + \mathbf{a}\,(t - t_0)} \quad ,$$

$$\boxed{v^2 = v_0^2 + 2\,\mathbf{a}\cdot(\mathbf{x} - \mathbf{x}_0)} \quad .$$

→ 1347 1362 1461

La velocità media è

$$\boxed{\langle\mathbf{v}\rangle = \frac{\Delta\mathbf{x}}{\Delta t} = \mathbf{v}_0 + \frac{1}{2}\mathbf{a}\Delta t} \quad . \tag{23.03.11}$$

DRAFT - IN PROGRESS

23.03.09.03 Moto Armonico

This § is referenced at pages:
[1239]

A one-dimensional harmonic motion along the z axis is such that:

$$\boxed{z[t] = z_0 + A\cos\left[\omega\left(t - t_0\right)\right] = z[t] = z_0 + A\cos\left[\omega t - \phi_0\right]} \quad ,$$

dove A è l'ampiezza del moto e $\omega = 2\pi\nu$ la pulsazione, legata al periodo T dalla relazione $\omega T = 2\pi$.

23.03.09.04 Moto Rettilineo Generico

This § is referenced at pages:
[1238, 1240]

UNDER-IT-IN-PROGRESS

23.04 Point Kinematics and Differential Geometry of Surfaces

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DRAFT IN PROGRESS

23.04.01 Parametrization of Surfaces

A point of a surface is parametrized by:

$$\mathbf{x}[u,v] = \begin{cases} x = x[u,v] \\ y = y[u,v] \\ z = z[u,v] \end{cases} , \qquad \text{with} \quad \widehat{\mathbf{N}} \equiv \frac{\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v}}{\left| \frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \right|} \neq 0 \qquad \text{and suitable limits on } u \text{ and } v, \text{ if any} \quad ,$$

then the three vectors:

$$\left\{ \frac{\partial \mathbf{x}}{\partial u}; \frac{\partial \mathbf{x}}{\partial v}; \widehat{\mathbf{N}} \neq 0 \right\} \quad ,$$

make, at any point on the surface, a basis, in general not orthogonal and made by non normalized vectors. The third vector is always orthogonal to the surface at the point.

Si assume implicitamente che le funzioni che intervengono siano differenziabili quanto basta.

Una superficie è detta regolare quando $\widehat{\mathbf{N}} \neq 0$ in ogni punto.

Un punto della superficie tale che $\widehat{\mathbf{N}} = 0$ è detto punto singolare della superficie.

23.04.02 Curvature

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At each point P of a differentiable surface in 3-dimensional Euclidean space one may choose a unit normal vector. A normal plane at P is one that contains the normal vector, and will therefore also contain a unique direction tangent to the surface and cut the surface in a plane curve, called normal section. This curve will in general have different curvatures for different normal planes at P. The principal curvatures at P, denoted k_1 and k_2, are the maximum and minimum values of this curvature. Here the curvature of a curve is by definition the reciprocal of the radius of the osculating circle. The curvature is taken to be positive if the curve turns in the same direction as the surface chosen normal, and otherwise negative. The directions in the normal plane where the curvature takes its maximum and minimum values are always perpendicular, if k_1 does not equal k_2, a result of Euler (1760), and are called principal directions. From a modern perspective, this theorem follows from the spectral theorem because these directions are as the principal axes of a symmetric tensor - the second fundamental form.

The product $k_1 k_2$ of the two principal curvatures is the Gaussian curvature, K, and the average $(k_1 + k_2)/2$ is the mean curvature, H. If at least one of the principal curvatures is zero at every point, then the Gaussian curvature will be 0 and the surface is a developable surface. For a minimal surface, the mean curvature is zero at every point.

23.04.03
Plane Motion in Cartesian Coordinates

This § is referenced at pages:
[Never referenced.]

It is just the superposition of two generic uni-dimensional motions along the x and y directions, as in section § 23.03.09.04.

23.04.04
Plane Motion in Polar Coordinates

This § is referenced at pages:
[Never referenced.]

Convention: angles in a plane are always measured starting from a/the fixed direction on the plane and going towards the mobile direction.

23.04.04.01 Moto Piano Generale in Coordinate Polari

Il vettore posizione in termini dei unit vectors delle coordinate polari $\widehat{\mathbf{e}}_r$ ed $\widehat{\mathbf{e}}_\theta$, è

$$\boxed{\mathbf{r} = r\widehat{\mathbf{e}}_r}$$

(definizione del unit vector radiale $\widehat{\mathbf{e}}_r$). Il unit vector tangenziale $\widehat{\mathbf{e}}_\theta$ è ortogonale ad $\widehat{\mathbf{e}}_r$ e il verso è quello di θ crescente.

In termini dei unit vectors $\widehat{\mathbf{e}}_1$ ed $\widehat{\mathbf{e}}_2$ delle coordinate cartesiane nel ortogonali nel piano si ha

$$\boxed{\widehat{\mathbf{e}}_r = +\widehat{\mathbf{e}}_1 \cos[\theta] + \widehat{\mathbf{e}}_2 \sin[\theta]} \quad ,$$

$$\boxed{\widehat{\mathbf{e}}_\theta = -\widehat{\mathbf{e}}_1 \sin[\theta] + \widehat{\mathbf{e}}_2 \cos[\theta]} \quad .$$

La derivata temporale dei unit vectors posizione è dunque

$$\boxed{\frac{\mathrm{d}\widehat{\mathbf{e}}_r}{\mathrm{d}t} = +\dot{\theta}\widehat{\mathbf{e}}_\theta} \quad ,$$

$$\boxed{\frac{\mathrm{d}\widehat{\mathbf{e}}_\theta}{\mathrm{d}t} = -\dot{\theta}\widehat{\mathbf{e}}_r} \quad .$$

L'espressione della velocità in coordinate polari è:

$$\boxed{\mathbf{v} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \dot{r}\widehat{\mathbf{e}}_r + r\dot{\theta}\widehat{\mathbf{e}}_\theta} \quad . \qquad (23.04.01) \quad \rightarrow 1239$$

L'espressione dell'accelerazione in coordinate polari è:

$$\boxed{\mathbf{a} \equiv \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\mathbf{e}}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\widehat{\mathbf{e}}_\theta} \quad . \qquad (23.04.02) \quad \rightarrow 1239$$

DRAFT WORK IN PROGRESS

23.04.04.02 Velocità Areolare

This § is referenced at pages:
[1550]

Sia $\Delta\mathbf{S}[t]$ l'area spazzata dal vettore posizione $\mathbf{x}[t]$ in funzione del tempo. Dalla (15.03.04) si ha

$$\Delta S[t] \simeq \frac{1}{2}|\,\mathbf{x}\wedge\Delta\mathbf{x}|\quad .$$

Ne segue per la velocità areolare

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{1}{2}|\,\mathbf{x}\wedge\mathbf{v}|}\quad . \tag{23.04.03}$$

→ 1272 1551 1563

In a more rigorous fashion:

$$\Delta\mathbf{S}[t]\equiv\int\!\!\int \mathrm{d}x\;\mathrm{d}y=\int_{\theta_1}^{\theta_2}\int_{0}^{\rho[\theta]}\rho[\theta]\;\mathrm{d}\rho\;\mathrm{d}\theta=\frac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2[\theta]\;\mathrm{d}\theta\quad ,$$

from which

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t}=\dot\theta\frac{\mathrm{d}A}{\mathrm{d}\theta}=\frac{1}{2}\rho^2[\theta]\dot\theta=\frac{1}{2}|\,\mathbf{x}\wedge\mathbf{v}|}\quad .$$

23.04.04.03 Moto Circolare

Nel moto circolare la traiettoria è una circonferenza. Se si sceglie come origine di un sistema di coordinate polari il centro della circonferenza si ha allora $\dot r=0$. Risulta allora dalle espressioni (23.04.01), (23.04.02)

$$\boxed{\mathbf{v}=r\dot\theta\widehat{\mathbf{e}}_\theta\qquad \mathbf{a}=-r\dot\theta^2\widehat{\mathbf{e}}_r+r\ddot\theta\widehat{\mathbf{e}}_\theta}\quad .$$

Se la velocità angolare $\dot\theta$ è costante il moto è detto circolare uniforme.

23.04.04.04 Moto Armonico Bi-Dimensionale

It is just the superposition of two harmonic uni-dimensional motions along the x and y directions, as in section § 23.03.09.03:

$$x[t]=x_0+r_x\cos\left[\omega_x t-\phi_x\right]\quad ,$$
$$y[t]=y_0+r_y\cos\left[\omega_y t-\phi_y\right]\quad .$$

- Same frequencies: $\omega_x=\omega_y$. In general, it gives an ellipse, $r_x\neq r_y$; it is a circle if and only if $r_x=r_y$.
- Different frequencies: $\omega_x\neq\omega_y$. In general, non-closed curves; they are known as Lissajous figures.

Read also § 46.03.01.03.

DRAFT-IN-PROGRESS

23.05 Point Kinematics in Three Dimensions

23.05.01 Space Motion in Cartesian Coordinates

This § is referenced at pages:
[Never referenced.]

It is just the superposition of three generic uni-dimensional motions along the x y z directions, as in section § 23.03.09.04.

23.05.02 Space Motion in Cylindrical Coordinates

Write the components of velocity and acceleration.

23.05.03 Space Motion in Spherical Coordinates

Write the components of velocity and acceleration.

DRAFT-IN-PROGRESS

23.06 Kinematics of the Relative Motion Between Two Reference Frame

This § is referenced at pages:
[1149, 1192, 1439]

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See H. Goldstein *et al.*, Classical Mechanics (3rd Edition), Pearson Education International, 2013; chapters: 1, 4.

Read also sections § 17.03, § 20.05, § 26.03. This section presents the implementation to the relative kinematics of motion among different Reference Frame.

23.06.01 Describing Rotations

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Describing rotations via geometrical variables can be obviously accomplished in many ways.

23.06.01.01 Euler Angles

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The traditional set of variables used to geometrically describe rotations are the Euler angles. Euler angles representing rotations about $\widehat{\mathbf{z}}$, $\widehat{\mathbf{N}}$, and $\widehat{\mathbf{Z}}$ axes are shown in figure 23.11. The xyz (original) system is shown in blue, the XYZ (rotated) system is shown in red.

The angle α in figure is often called ϕ and named **angle of precession**.

The angle β in figure is often called θ and named **angle of nutation**.

The angle γ in figure is often called ψ and named **angle of proper rotation**.

The line of nodes (N) is the line given by the intersection between the xy and XY planes and shown in green in figure 23.11.

Euler angles are particularly intuitive when the body follows a rotational motion.

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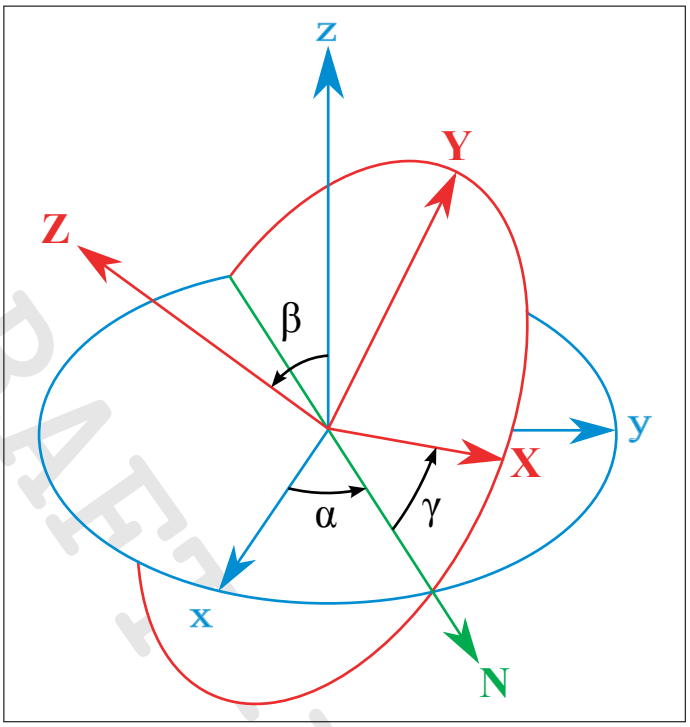

Figure 23.11: FIGURE

23.06.01.02 Yaw Pitch Roll

Another common parametrization, often used in engineering applications and more intuitive is given by the variables yaw, pitch and roll, as shown in figure 23.12.

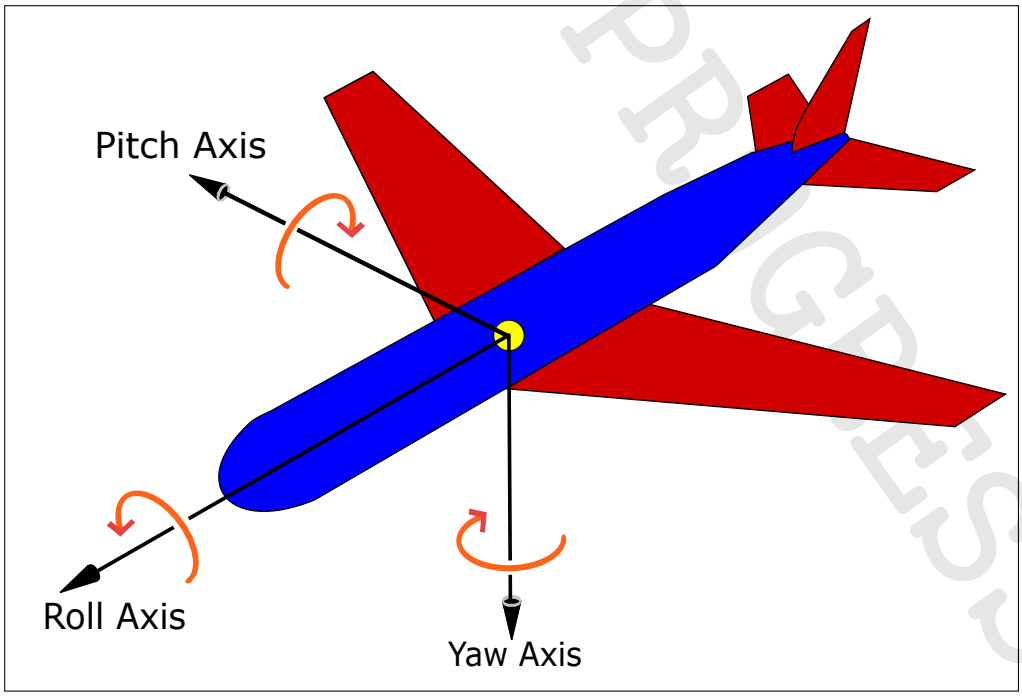

Figure 23.12: FIGURE

These are the three angles of rotation about the xyz axes of the Coordinate System: the roll axis is the longitudinal axis; the roll and pitch axes define the vehicle plane; the yaw axis is the vertical axis.

Yaw, pitch and roll are particularly intuitive when the body follows a translational/rotational motion.

DRAFT - IN PROGRESS

23.06.01.03 Direction Cosines

A generic set of parameters, closely related to the algebraic approach, is given by the direction cosines, that is the elements of the rotation matrix:

$$\boxed{\mathbb{R}^{\cdot\cdot}_{pq} \equiv \mathbf{e}'_p \cdot \mathbf{e}_q} \quad .$$

23.06.01.04 Angle Variables

Note that, at variance with the position vector, it not possible, in general, to define the **angular position** as something as:

$$\cancel{\boldsymbol{\theta} \equiv \alpha\widehat{\mathbf{e}}_\alpha + \beta\widehat{\mathbf{e}}_\beta + \gamma\widehat{\mathbf{e}}_\gamma} \quad ,$$

in terms of three angular positions with respect to three suitable axes, for the simple reason that the sum is not well defined, as finite angles do not form a vector space because addition is not commutative. Clearly this can be possible in special cases, for instance for rotations around one fixed axis.

23.06.02
Time-Derivation Angular Velocity and Poisson Formulas

Consider an Inertial Reference Frame , $\mathcal{I}$, and a generic (in general non-inertial) Reference Frame, $\mathcal{I}'$, such that the origin of the second Reference Frame, $\mathcal{O}'$, moves with respect to the origin, $\mathcal{O}$, of the first one with velocity $\mathbf{u}$ and the angular velocity (to be defined shortly) of the second Reference Frame with respect to the first one is $\boldsymbol{\omega}$. Let $\{\mathbf{e}_k\}$ be an orthonormal basis in $\mathcal{I}$ and $\{\mathbf{e}'_k\}$ be an orthonormal basis in $\mathcal{I}'$. Let us call, conventionally and for clarity purposes only, the Inertial Reference Frame $\mathcal{I}$ **absolute** (A) (or **fixed** or **space**) frame and the moving Reference Frame $\mathcal{I}'$, **relative** (R) (or **moving** or **body**) frame. These are just conventional names, with no physical nor meta-physical significance.

In short, symbolically:

$$\mathcal{I} \equiv \{\mathcal{O}; \mathbf{e}_k\} \qquad \mathcal{I}' \equiv \{\mathcal{O}'; \mathbf{e}'_k\} \quad .$$

Note that, as long as one is limited to the field of pure kinematics (not dynamics) it is not strictly necessary that the first Reference Frame be an Inertial one.

Let us consider how time-changing quantities change from one Reference Frame to another. Time is an absolute parameter. It should be clear that changing of time derivation refers to the fact that different observers, fixed in different Reference Frames, see their own orthonormal base vectors are fixed in time for each of the observers. Therefore, time derivation, for each observer, means to take the time derivative of the components only; however, every other observer can describe the vector in terms of another basis (not its own) such that the orthonormal base vectors are time dependent.

PRELIMINARY-IN-PROGRESS

23.06.02.01 Time-Derivative of Scalar/Numerical Quantities

When one considers time derivation, there is no difference in the absolute and relative time-derivative of a scalar/numerical quantity, because it is independent from any chosen basis:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{A}} s = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{R}} s \qquad s[\mathbf{x}] = s'[\mathbf{x}'] \qquad \text{for any scalar/numerical quantity} \quad . \tag{23.06.01}$$

In fact no moving base of vectors is involved.

23.06.02.02 Time-Derivative of Vectors and Poisson Formulas

This § is referenced at pages:
[1117]

When one considers time derivation, there is a difference in the absolute and relative time-derivative of a vector, as vectors are referred to the two different bases of the two Reference Frame, which are in relative motion:

$$\mathbf{v}[t] = v_k \mathbf{e}_k = v'_k \mathbf{e}'_k \to v_k[t]\mathbf{e}_k = v'_k \mathbf{e}'_k[t] \qquad \text{fixed vector in the moving frame} \quad ,$$
$$\mathbf{v}[t] = v_k \mathbf{e}_k = v'_k \mathbf{e}'_k \to v_k[t]\mathbf{e}_k = v'_k[t] \mathbf{e}'_k[t] \qquad \text{arbitrary vector in the moving frame} \quad .$$

In fact, every vector, can be described in terms of the fixed orthonormal basis of axes, $\{\mathbf{e}_k\}$, which is time-independent, or in terms of the moving orthonormal basis of axes, $\{\mathbf{e}'_k\}$, fixed with respect to the moving Reference Frame, $\{\mathbf{e}_k\}$, which is time-dependent, whereas the components, with respect to the moving Reference Frame, are constant, by hypothesis.

Vector Fixed in the Moving Frame

For any vector fixed in the moving frame one has:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{A}} \mathbf{v} = \boldsymbol{\omega} \wedge \mathbf{v} \qquad \text{vector fixed in the moving frame} \quad . \tag{23.06.02}$$

→ 1244

Consider, in fact, two unit vectors, $\mathbf{a}$ and $\mathbf{b}$ fixed in the moving frame:

$$\mathbf{a}\cdot\mathbf{a} = 1 \qquad \mathbf{b}\cdot\mathbf{b} = 1 \qquad \mathbf{a}\cdot\mathbf{b} = \text{constant}$$

Therefore:

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\cdot\mathbf{a} = 0 \qquad \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}t}\cdot\mathbf{b} = 0 \qquad \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\cdot\mathbf{b} + \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}t}\cdot\mathbf{a} = 0$$

Now, if $\dot{\mathbf{v}}\cdot\mathbf{v} = 0$ one can always find a vector $\boldsymbol{\omega}[t]$ such that

$$\dot{\mathbf{v}}\cdot\mathbf{v} = 0 \implies \{\dot{\mathbf{v}} = \boldsymbol{\omega}\wedge\mathbf{v} \qquad \text{with} \qquad \boldsymbol{\omega}[t] \equiv \mathbf{v}\wedge\dot{\mathbf{v}}\}$$

A priori, $\boldsymbol{\omega}[t]$ depends on time. However, $\boldsymbol{\omega}[t]$ is the same for any vector fixed in the moving frame. In fact:

$$\left\{\dot{\mathbf{a}} = \boldsymbol{\omega}_{\mathbf{a}}\wedge\mathbf{a} \qquad \dot{\mathbf{b}} = \boldsymbol{\omega}_{\mathbf{b}}\wedge\mathbf{b}\right\} \implies 0 = \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\cdot\mathbf{b} + \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}t}\cdot\mathbf{a} = (\boldsymbol{\omega}_{\mathbf{a}} - \boldsymbol{\omega}_{\mathbf{b}})\cdot(\mathbf{a}\wedge\mathbf{b}) = 0 \quad ,$$

which implies $\boldsymbol{\omega}_{\mathbf{a}} = \boldsymbol{\omega}_{\mathbf{b}}$ as $\mathbf{a}$ and $\mathbf{b}$ are arbitrary vectors fixed in the moving frame. One can take, for instance, the three unit vectors of a base fixed in the moving frame, in pairs, and find that the three corresponding $\boldsymbol{\omega}$ are all equal: $\boldsymbol{\omega} \equiv \boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \boldsymbol{\omega}_3$. By decomposing an arbitrary vector on the basis and applying equation (23.06.02), which is a linear one it is found that the same $\boldsymbol{\omega}$ applies to any fixed vector.

The result of the above derivation can be applied to the three unit vectors of the moving base.

Therefore, for any rigid motion, there exist a unique vector, $\boldsymbol{\omega}$, the angular velocity vector, which describes rotation via equation (23.06.02).

For each of the three unit vectors of the moving base, $\mathbf{e}'$, one has:

$$\dot{\mathbf{e}}' = \boldsymbol{\omega}\wedge\mathbf{e}' \implies \mathbf{e}'\wedge\dot{\mathbf{e}}' = \mathbf{e}'\wedge(\boldsymbol{\omega}\wedge\mathbf{e}') = \boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathbf{e}')\,\mathbf{e}' \quad . \tag{23.06.03}$$

→ 1117

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Summing over the three orthonormal unit vectors of the moving base, one finds:

$$\boldsymbol{\omega} = \frac{1}{2}\mathbf{e'}_k \wedge \dot{\mathbf{e}}'_k \quad . \tag{23.06.04}$$

→ 1248

Arbitrary Moving Vector in the Moving Frame - Poisson Formula

Consider now any arbitrary moving vector in the moving frame.

To avoid any misunderstanding let us make clear that:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{A}} \mathbf{v} \equiv \dot{v}_k \mathbf{e}_k \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{R}} \mathbf{v} \equiv \dot{v}'_k \mathbf{e}'_k \quad . \tag{23.06.05}$$

→ 1245

Therefore:

$$\mathbf{v} \equiv v_k \mathbf{e}_k = v'_k \mathbf{e}'_k \implies \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{A}} \left(v'_k \mathbf{e}'_k\right) = \dot{v}'_k \mathbf{e}'_k + v'_k \dot{\mathbf{e}}'_k = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{R}} \mathbf{v} + v'_k \dot{\mathbf{e}}'_k \quad .$$

Poisson relation (23.06.06), accounts for the fact that, in general, both the moving unit vectors of the basis $\{\mathbf{e}'\}$ and the components of any vector with respect to the moving base, v'_k, are time-varying:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{A}} \mathbf{v} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{R}} \mathbf{v} + \boldsymbol{\omega} \wedge \mathbf{v} \qquad \text{arbitrary vector in the moving frame - Poisson formula} \quad . \tag{23.06.06}$$

→ 1245 1246 1247

23.06.02.03 Angular Velocity

Angular velocity is a free vector, only depending on the relative motion between two Reference Frame, and not on the position.

Clearly,

$$\boldsymbol{\omega}_{2|1} = -\boldsymbol{\omega}_{1|2} \quad . \tag{23.06.07}$$

→ 1245

In fact, as the choice of the two Reference Frame is arbitrary, Poisson relation (23.06.06) and the definitions in equation (23.06.05), imply that the angular velocity of Reference Frame A with respect to Reference Frame R is $-\boldsymbol{\omega}$, as one can show by moving the $\boldsymbol{\omega} \wedge \mathbf{v}$ on the other side of the equation and expressing $\mathbf{v}$ in terms of the basis vector of Reference Frame A.

As any vector, it can be represented both in terms of the fixed and of the moving basis:

$$\boldsymbol{\omega} = \omega_k \mathbf{e}_k = \omega'_k \mathbf{e}'_k$$

Moreover, it has the very special properties that its time derivative is the same in both Reference Frame, from (23.06.06).

Note that the angular velocity is not the derivative of any vector, as no **angle** vector exists.

23.06.02.04 Angular Acceleration

The time derivative of the angular velocity is the same in both Reference Frame, from (23.06.06); this defines the angular acceleration:

$$\dot{\boldsymbol{\omega}} \equiv \boldsymbol{\alpha} \equiv \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{A}} \boldsymbol{\omega} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{R}} \boldsymbol{\omega} \quad .$$

Clearly, from equation (23.06.07):

$$\boldsymbol{\alpha}_{2|1} = -\boldsymbol{\alpha}_{1|2} \quad . \tag{23.06.08}$$

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23.06.02.05 Composition of Angular Velocities

This § is referenced at pages:
[1329]

It can be shown that the angular velocity resulting from the composition of a number of angular velocities is the vector sum of the individual angular velocities.

If the Reference Frame $\mathcal{I}_2$ rotates with angular velocity $\boldsymbol{\omega}_{2|1}$ with respect to the Reference Frame $\mathcal{I}_1$ and the Reference Frame $\mathcal{I}_3$ rotates with angular velocity $\boldsymbol{\omega}_{3|2}$ with respect to the Reference Frame $\mathcal{I}_2$ then the angular velocity of the Reference Frame $\mathcal{I}_3$ with respect to the Reference Frame $\mathcal{I}_1$, $\boldsymbol{\omega}_{3|1}$, is:

$$\boxed{\boldsymbol{\omega}_{3|1} = \boldsymbol{\omega}_{3|2} + \boldsymbol{\omega}_{2|1}} \quad . \tag{23.06.09}$$

→ 1246

The above relation can be extended to the addition of the angular velocities of an arbitrary number of Reference Frame.

It is worth emphasizing here that the above addition formula, equation (23.06.09), if interpreted as the sum component by component, is only valid whenever all the vectors are referred to the same, thought arbitrary, Reference Frame. This is of course always true for the sum of vectors. However, in relative kinematics, due to the constant use of at least two Reference Frame, it might be easier to use it erroneously.

23.06.02.06 Composition of Angular Accelerations

This § is referenced at pages:
[1329]

It can be shown that the angular acceleration resulting from the composition of a number of angular acceleration is not the vector sum of the individual angular acceleration.

If the Reference Frame $\mathcal{I}_2$ rotates with angular acceleration $\dot{\boldsymbol{\omega}}_{2|1}$ with respect to the Reference Frame $\mathcal{I}_1$ and the Reference Frame $\mathcal{I}_3$ rotates with angular acceleration $\dot{\boldsymbol{\omega}}_{3|2}$ with respect to the Reference Frame $\mathcal{I}_2$ then the angular acceleration of the Reference Frame $\mathcal{I}_3$ with respect to the Reference Frame $\mathcal{I}_1$, $\dot{\boldsymbol{\omega}}_{3|1}$, is:

$$\boxed{\dot{\boldsymbol{\omega}}_{3|1} = \dot{\boldsymbol{\omega}}_{3|2} + \dot{\boldsymbol{\omega}}_{2|1} + \boldsymbol{\omega}_{2|1} \wedge \boldsymbol{\omega}_{3|2}} \quad . \tag{23.06.10}$$

In fact:

$$\boldsymbol{\alpha}_{3|1} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_1 \boldsymbol{\omega}_{3|1} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_1 (\boldsymbol{\omega}_{2|1} + \boldsymbol{\omega}_{3|2}) = \boldsymbol{\alpha}_{2|1} + \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_2 \boldsymbol{\omega}_{3|2} + \boldsymbol{\omega}_{2|1} \wedge \boldsymbol{\omega}_{3|2} = \boldsymbol{\alpha}_{2|1} + \boldsymbol{\alpha}_{3|2} + \boldsymbol{\omega}_{2|1} \wedge \boldsymbol{\omega}_{3|2}$$

Therefore, the angular velocity addition theorem does not extend to angular acceleration.

23.06.02.07 Second-Order Poisson Formula

From equation (23.06.06) it is possible to derive the general relation for the second order time derivative:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\mathrm{A}} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\mathrm{A}} \mathbf{v} = \dots} \quad . \tag{23.06.11}$$

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23.07 Relative Kinematics

23.07.01
Position Velocity Acceleration

Consider two Reference Frame in relative motion: $\mathcal{I}$ and $\mathcal{I}'$.

Siano $\mathbf{X}$, $\mathbf{V}$ and $\mathbf{A}$, rispettivamente, la posizione, velocità e accelerazione dell'origine di $\mathcal{I}'$ rispetto ad $\mathcal{I}$ e $\boldsymbol{\omega}$ la velocità angolare di $\mathcal{I}'$ rispetto ad $\mathcal{I}$.

La relazione tra l'operatore di derivazione temporale in due sistemi di riferimento in moto relativo con velocità angolare $\boldsymbol{\omega}$ è data dalla formula di Poisson (23.06.06).

Siano $\mathbf{x}$, $\mathbf{v}$ and $\mathbf{a}$ la posizione, velocità ed accelerazione di un punto rispetto al Reference Frame $\mathcal{I}$ ed $\mathbf{x}'$, $\mathbf{v}'$ and $\mathbf{a}'$ la posizione, velocità ed accelerazione di un punto rispetto al Reference Frame $\mathcal{I}'$.

La relazione che lega $\mathbf{x}$ e $\mathbf{x}'$, al vettore posizione dell'origine di $\mathcal{I}'$ rispetto ad $\mathcal{I}$, $\mathbf{X}$, è

$$\boxed{\mathbf{x} = \mathbf{x}' + \mathbf{X}} \tag{23.07.01}$$

La velocità rispetto al Reference Frame $\mathcal{I}$ è pari alla somma della velocità rispetto al Reference Frame $\mathcal{I}'$ e della velocità di trascinamento $\mathbf{v}_T$ di $\mathcal{I}'$ rispetto ad $\mathcal{I}$:

$$\boxed{\mathbf{v} = \mathbf{v}' + \mathbf{v}_T \quad \Leftrightarrow \quad \mathbf{v} = \mathbf{v}' + \left(\mathbf{V} + \boldsymbol{\omega}\wedge\mathbf{x}'\right)} \quad . \tag{23.07.02}$$

→ 1439

L'accelerazione rispetto al Reference Frame $\mathcal{I}$ è pari alla somma dell'accelerazione rispetto al Reference Frame $\mathcal{I}'$, dell'accelerazione di trascinamento $\mathbf{a}_T$ di $\mathcal{I}'$ rispetto ad $\mathcal{I}$ e dell'accelerazione di Coriolis $\mathbf{a}_C$:

$$\boxed{\mathbf{a} = \mathbf{a}' + \mathbf{a}_T + \mathbf{a}_C \quad \Leftrightarrow \quad \mathbf{a} = \mathbf{a}' + \left(\mathbf{A} + \dot{\boldsymbol{\omega}}\wedge\mathbf{x}' + \boldsymbol{\omega}\wedge\left(\boldsymbol{\omega}\wedge\mathbf{x}'\right)\right) + 2\,\boldsymbol{\omega}\wedge\mathbf{v}'} \quad . \tag{23.07.03}$$

→ 1329

Note that **the angular velocity is only dependent of the relative motion** of one Reference Frame with respect to the other; that is it does not depend on the reference point used as a CRV nor on anything else.

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23.08 Examples and Physical Applications

23.09 Exercises Problems and Physical Applications

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23-001 Angular Velocity Is a Vector

Demonstrate that the angular velocity vector components transform as a vector, equation (23.06.04).

23-002 Perpendicularity Among the Three Unit Vectors Tangent, Normal and Binormal

Show that the three unit vectors, tangent, normal and binormal, form an orthonormal basis.

WORKED SOLUTION

From the definitions (23.03.01), (23.03.02), (23.03.03) it is obvious that only the perpendicularity between $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$ has to be demonstrated. The demonstrations follows:

$$2\,\frac{\mathrm{d}\widehat{\mathbf{T}}}{\mathrm{d}s}\cdot\widehat{\mathbf{T}}=\frac{\mathrm{d}}{\mathrm{d}s}(\,\mathbf{T}\cdot\mathbf{T})=\frac{\mathrm{d}}{\mathrm{d}s}(1)=0\implies\mathbf{T}\cdot\mathbf{N}=0\quad.$$

23-003 Darboux Vector

Using Frenet formulas to demonstrate equation (23.03.07).
Show that the contributions to the angular velocity of the three intrinsic unit vectors are given by:

$$\boldsymbol{\omega}=\boldsymbol{\omega}_{\mathbf{T}}+\boldsymbol{\omega}_{\mathbf{N}}+\boldsymbol{\omega}_{\mathbf{B}}\quad,$$

$$\boldsymbol{\omega}_{\mathbf{T}}\equiv\frac{1}{2}\widehat{\mathbf{T}}\wedge\frac{\mathrm{d}\widehat{\mathbf{T}}}{\mathrm{d}t}=\frac{1}{2}\kappa\widehat{\mathbf{B}}\quad,$$

$$\boldsymbol{\omega}_{\mathbf{N}}\equiv\frac{1}{2}\widehat{\mathbf{N}}\wedge\frac{\mathrm{d}\widehat{\mathbf{N}}}{\mathrm{d}t}=\frac{1}{2}\left(\kappa\widehat{\mathbf{B}}+\tau\widehat{\mathbf{T}}\right)\quad,$$

$$\boldsymbol{\omega}_{\mathbf{B}}\equiv\frac{1}{2}\widehat{\mathbf{B}}\wedge\frac{\mathrm{d}\widehat{\mathbf{B}}}{\mathrm{d}t}=\frac{1}{2}\tau\widehat{\mathbf{T}}\quad.$$

Write the above expression as the Poisson relations, with Darboux vector as the angular velocity.

23-004 Radius of Curvature

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1. Show that the radius of curvature of a parametric curve,

$$x=x[s]\qquad y=y[s]\qquad z=z[s]\quad,$$

parametrized by the curvilinear abscissa s, is given by:

$$\rho=\sqrt{\left(\frac{d^2x}{\mathrm{d}s^2}\right)^2+\left(\frac{d^2y}{\mathrm{d}s^2}\right)^2+\left(\frac{d^2z}{\mathrm{d}s^2}\right)^2}\quad.$$

2. Show that the radius of curvature of a parametric curve (trajectory) is given by:

$$\boxed{\rho=\frac{v^3}{|\,\mathbf{v}\wedge\mathbf{a}|}}\quad.\tag{23.09.01}$$

23-005 Helix

Consider:

$$x[t] = R\cos[\omega t] \quad ,$$
$$y[t] = R\sin[\omega t] \quad ,$$
$$z[t] = ct \quad .$$

Find the curvature and torsion.

WORKED SOLUTION

$$\kappa = \frac{R}{R^2 + c^2/\omega^2} \quad ,$$
$$\tau = \frac{c/\omega}{R^2 + c^2/\omega^2} \quad .$$

23-006 Il Jet in Volo Radente

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23-007 L'auto Da Corsa

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|2.D.4||

Un'automobile da corsa, durante una batteria di qualificazione di due giri, percorre il primo giro con una velocità scalare media di $90\ \mathrm{km/h}$. Il conducente vuole percorrere il secondo giro in modo che la velocità scalare media complessiva sui due giri salga a $180\ \mathrm{km/h}$, per poter ottenere la qualificazione. Dimostrare che ciò è impossibile.

WORKED SOLUTION

Sia L la lunghezza della pista, Δt_1 il tempo impiegato a percorrere il primo giro e Δt_2 il tempo (incognito) con cui deve essere percorso il secondo giro per ottenere la velocità scalare media desiderata. Si ha

$$\overline{v}_1 = \frac{L}{\Delta t_1} \quad ,$$
$$\overline{v} = \frac{2L}{\Delta t_1 + \Delta t_2} = \frac{2}{\dfrac{1}{v_1} + \dfrac{1}{v_2}} \qquad (23.09.02)$$

→ 1249

Ne segue che, qualunque sia $\Delta t_2 > 0$, si ha

$$\overline{v} = \frac{2}{\dfrac{1}{v_1} + \dfrac{1}{v_2}} < \frac{2}{\dfrac{1}{v_1}} = 2v_1 = 180\ \mathrm{km/h}$$

Per ottenere il risultato voluto il conducente dovrebbe percorrere il secondo giro in 0 s.
Nel caso di n giri la (23.09.02) può essere scritta (e generalizzata) come

$$\overline{v} = \frac{nL}{\Delta t_1 + \Delta t_2 + \ldots + \Delta t_n} = \frac{n}{\dfrac{1}{v_1} + \dfrac{1}{v_2} + \ldots + \dfrac{1}{v_n}}$$

23-008 Moto in Una Dimensione

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|2.12||

23-009 Lo Scontro Fra I Treni

Due treni viaggiano in verso opposto su un binario rettilineo. Se il primo treno si muove a velocità costante $v_1 = 120$ km/h, il secondo a velocità costante $v_2 = 30$ ms e al tempo $t = 0$ si trovano a distanza $a = 80$ km, calcolare dopo quanto tempo avviene lo scontro e la posizione dei treni a quell'istante rispetto alla posizione iniziale del primo treno. Si risolva il problema sia dal punto di vista del capostazione a terra che da quello dei macchinisti. Si risolva inoltre il problema anche per via grafica oltre che analitica.

WORKED SOLUTION

Si scelga un Coordinate System avente l'origine nella posizione del primo treno all'istante iniziale e con l'asse x diretto lungo il binario verso il secondo treno. Per il capostazione a terra le equazioni del moto dei due treni sono:

$$x_1(t) = v_1 t \quad ,$$
$$x_2(t) = -v_2 t + a \quad .$$

I treni si incontrano al tempo t_0 tale che $x_1(t_0) = x_2(t_0)$, cioè:

$$t_0 = \frac{a}{v_1 + v_2} = 1.26{\cdot}10^3 \text{ s} \quad ,$$

$$x_1(t_0) = x_2(t_0) = v_1 t_0 = \frac{a v_1}{v_1 + v_2} = 42.1{\cdot}10^3 \text{ m} \quad .$$

dove si sono usati i fattori di conversione

$$1 \text{ km/h} = 0.2778 \text{ ms} \qquad 1 \text{ ms} = 3.6 \text{ km/h} \tag{23.09.03}$$

Le soluzioni trovate per $x_1(t_0)$, $x_2(t_0)$ e t_0 si riducono ai valori corretti nei casi in cui $v_1 \gg v_2$ e $v_2 \gg v_1$. Inoltre essendo $v_1 \simeq v_2$ i due treni si incontrano quasi a metà strada.
Per analizzare la situazione dal punto di vista del macchinista sul primo treno si scelga un Coordinate System solidale con il primo treno, asse x' parallelo all'asse x e origine coincidente con il primo treno. Dal punto di vista del macchinista del primo treno il secondo treno si trova a distanza a al tempo $t = 0$ e viaggia a velocità, relativa al primo treno, $\mathbf{v}_r = \mathbf{v}_2 - \mathbf{v}_1$ che passando alle componenti lungo l'asse x' diventa $v_r = -(v_1 + v_2)$. In modo del tutto equivalente la velocità relativa del secondo treno rispetto al Reference Frame solidale con il primo treno può essere ricavata dalla legge di composizione delle velocità (23.09.18). Dal punto di vista del macchinista del primo treno le equazioni del moto sono quindi:

$$x''1(t) = 0 \qquad \text{il macchinista sta fermo nell'origine del sistema di coordinate} \quad ,$$
$$x''2(t) = -(v_1 + v_2)t + a \quad .$$

da cui seguono le soluzioni

$$t_0 = \frac{a}{v_1 + v_2} \qquad \text{(la stessa ottenuta dal capostazione)} \quad ,$$
$$x'_1(t_0) = x'_2(t_0) = 0 \qquad \text{(lo scontro avviene nell'origine per il macchinista)}$$

Per cui ne segue lo stesso risultato di sopra (come deve essere) per t_0 e un risultato consistente col precedente per $x'_1(t_0)$ e $x'_2(t_0)$.

WORKED SOLUTION

$t_0 = 1.26{\cdot}10^3$ s; $x_1(t_0) = 42.1{\cdot}10^3$ m.

23-010 ✓ Achille E La Tartaruga

Il filosofo greco Zenone propose nell'antichità il seguente paradosso (di Achille e la tartaruga). Supponiamo che al tempo $t = 0$ Achille si metta a correre per raggiungere una tartaruga che dista da lui a e che si mette a scappare. Sia V la velocità di Achille e v la velocità della tartaruga con $V \gg v$. Il veloce Achille non potrà mai raggiungere la lenta tartaruga perché giunto alla posizione a, occupata dalla tartaruga al tempo $t = 0$, questa si sarà spostata (pur se di poco) in un altra posizione; quando Achille raggiungerà questa seconda posizione la tartaruga si sarà ancora spostata (pur se di poco) in un'altra posizione e così via all'infinito. Achille non raggiungerà dunque mai la tartaruga in quanto per fare ciò sarebbero necessari un numero infinito di passi. Spiegare l'apparente paradosso.

WORKED SOLUTION

Si ricordi la formula della somma della serie geometrica: $S = \sum_{k=0}^{+\infty} x^k = 1/(1-x)$ per $|x| < 1$.

WORKED SOLUTION

Si scelga un sistema di coordinate con origine nella posizione occupata da Achille al tempo $t = 0$ e asse x diretto verso la tartaruga. Sia $X(t)$ la posizione di Achille e $x(t)$ quella della tartaruga:

$$X(t=0) = 0 \quad ,$$
$$x(t=0) = a \quad .$$

L'intervallo di tempo Δt_1 in cui Achille raggiunge la posizione a occupata dalla tartaruga a $t = 0$ e lo spostamento della tartaruga in questo intervallo di tempo valgono:

$$\Delta t_1 = \frac{a}{V} \quad ,$$
$$\Delta x_1 = v\Delta t_1 = \frac{av}{V} \quad .$$

Nei passi seguenti si ha:

$$\Delta t_2 = \frac{\Delta x_1}{V} = \frac{av}{V^2} \quad ,$$
$$\Delta x_2 = v\Delta t_2 = \frac{av^2}{V^2} \quad .$$

In generale, al passo n-esimo si ha:

$$\Delta t_n = \frac{a}{V}\left(\frac{v}{V}\right)^{(n-1)} \quad ,$$
$$\Delta x_n = \frac{av}{V}\left(\frac{v}{V}\right)^{(n-1)} \quad .$$

In generale per passare da Δt_i a Δx_i si moltiplica per v. Per passare da Δx_i a Δt_{i+1} si divide per V. Il tempo totale per questa serie di eventi si può calcolare usando il risultato della somma di una serie geometrica

$$S = \sum_{k=0}^{+\infty} x^k = \frac{1}{1-x} \qquad \text{per } |x| < 1. \tag{23.09.04}$$

si ricava che l'intervallo di tempo dopo cui Achille raggiunge la tartaruga vale

$$\Delta t = \sum_{n=1}^{+\infty} \Delta t_n = \frac{a}{V}\sum_{k=0}^{+\infty}\left(\frac{v}{V}\right)^k = \frac{a}{V}\left(\frac{1}{1-v/V}\right) = \frac{a}{V-v} \qquad \text{per } v < V.$$

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Analogamente lo spazio percorso dalla tartaruga prima di essere raggiunta vale

$$\Delta x = \sum_{n=1}^{+\infty} \Delta x_n = \frac{av}{V} \sum_{k=0}^{+\infty} \left(\frac{v}{V}\right)^k = \frac{av}{V-v} \qquad \text{per } v < V.$$

e lo spazio percorso da Achille vale

$$\Delta X = \Delta x + a = \frac{aV}{V-v} \qquad \text{per } v < V.$$

I risultati sono in accordo con quanto si sarebbe ottenuto risolvendo le equazioni del moto:

$$x(t) = vt + a \quad ,$$
$$X(t) = Vt \quad .$$

L'apparente paradosso nasce dal fatto che la somma di un numero infinito di intervalli di tempo (di lunghezza decrescente) può avere somma finita per cui di fatto Achille raggiunge la tartaruga in un tempo finito.

23-011 ✓ Il Problema Del Rimbalzo E Il Principio Del Tempo Minimo

This § is referenced at pages:
[Never referenced.]

Due amici si trovano alle distanze h_1 e h_2 da un muro verticale rettilineo. Uno dei due vuole lanciare orizzontalmente una palla all'altro, facendola rimbalzare sul muro, in modo che la palla raggiunga l'amico nel minor tempo possibile.

Determinare le caratteristiche della traiettoria che deve seguire il la palla per raggiungere l'amico nel minore tempo possibile. Si assuma che i tratti di traiettoria siano rettilinei e orizzontali, e il rimbalzo ideale.

Tale traiettoria è quella che seguirebbe la luce. Il principio analogo in ottica è detto principio di Fermat. La traiettoria risultante soddisfa alla legge della riflessione (di Snell) dell'ottica geometrica, equazione (55.03.01).

Read § 15.05.06.

23-012 ✓ Il Problema Del Bagnino E Il Principio Del Tempo Minimo

This § is referenced at pages:
[2360]

Un bagnino avvista una bagnante che chiede aiuto. La velocità con cui il bagnino può correre sulla spiaggia vale V, la velocità con cui può nuotare vale v, $v < V$, la distanza tra il bagnino e la riva vale D, quella della bagnante dalla riva vale d e la distanza tra il bagnino e la bagnante, misurata lungo la riva supposta rettilinea, vale h.

Determinare le caratteristiche della traiettoria che deve seguire il bagnino per raggiungere la bagnante nel minore tempo possibile.

Tale traiettoria è quella che seguirebbe la luce. Il principio analogo in ottica è detto principio di Fermat. La traiettoria risultante soddisfa alla legge della rifrazione (di Snell) dell'ottica geometrica, equazione (55.03.02).

Read § 15.05.07.

WORKED SOLUTION

Essendo la velocità del bagnino costante in spiaggia e in acqua la traiettoria che consente il tempo minimo di percorrenza sarà una spezzata composta da due tratti rettilinei in spiaggia e in mare. Infatti nel caso di moto con velocità costante il tempo di percorrenza più breve tra due punti corrisponde alla distanza più breve tra i due punti cioè un segmento di retta. Il bagnino percorrerà

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allora una traiettoria costituita da una spezzata cambiando direzione al momento del tuffo in mare. Sia x la distanza lungo la riva a cui il bagnino deve tuffarsi in acqua ($0 \le x \le h$). Il tempo, T, che impiega il bagnino a raggiungere la bagnante vale

$$T = \frac{\sqrt{x^2 + D^2}}{V} + \frac{\sqrt{d^2 + (h-x)^2}}{v}$$

Per trovare i punti di stazionarietà si calcola la derivata prima rispetto ad x e la si pone uguale a zero:

$$\frac{dT}{dx} = \frac{x}{V\sqrt{x^2 + D^2}} - \frac{h-x}{v\sqrt{d^2 + (h-x)^2}} = 0$$

Osserviamo che

$$\left.\frac{dT}{dx}\right|_{x=0} < 0 \qquad \left.\frac{dT}{dx}\right|_{x=h} > 0$$

La condizione di stazionarietà può essere riscritta introducendo gli angoli θ_I e θ_R che la traiettoria rettilinea del bagnino forma con la normale alla riva rispettivamente dal lato spiaggia e dal lato mare. Si trova:

$$\boxed{\frac{\sin\theta_I}{V} = \frac{\sin\theta_R}{v}} \tag{23.09.05}$$

→ 1253

Per verificare che il punto di stazionarietà trovato è effettivamente un minimo si può calcolare la derivata seconda di T rispetto ad x:

$$\frac{d^2T}{dx^2} = \frac{D^2}{V(x^2 + D^2)^{3/2}} + \frac{d^2}{v(d^2 + (h-x)^2)^{3/2}} > 0$$

che è sempre positiva per cui il punto di stazionarietà è un punto di minimo e la funzione $T(x)$ è convessa. Il fatto che la derivata seconda sia sempre positiva significa che la derivata prima è crescente ed essendo la derivata prima negativa per $x = 0$ e positiva per $x = h$ ne deduciamo che esiste uno ed un solo valore di x per cui la derivata si annulla, cioè uno ed un solo punto di minimo. L'equazione che pone la derivata prima uguale a zero determina dunque il valore di x che rende minimo il tempo di percorrenza e tale valore è unico. La soluzione esplicita dell'equazione per x non è banale.

Il problema risolto (equazione (23.09.05)) ha un analogo nel Principio di Fermat dell'ottica geometrica che afferma che nell'andare tra due punti un raggio luminoso segue il cammino per cui impiega un tempo stazionario (e non minimo come richiesto nel problema del bagnino). Nel caso dell'ottica geometrica, introducendo l'indice di rifrazione di un mezzo materiale, $n \equiv c/v$, con c la velocità della luce nel vuoto, l'equazione (23.09.05) si riscrive nella forma della legge di Snell della rifrazione:

$$\boxed{n_I \sin\theta_I = n_R \sin\theta_R} \tag{23.09.06}$$

Il problema mostra che la traiettoria cui corrisponde il tempo minimo di percorrenza in un mezzo in cui la velocità è variabile non è rettilineo.

23-013 L'uccello Che Vola Tra I Due Treni

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Due treni viaggiano lungo un binario rettilineo in versi opposti alla velocità di $34\ \mathrm{km/h}$. Un uccello che vola alla velocità costante di $58\ \mathrm{km/h}$ parte da uno dei treni quando questi si trovano alla distanza di $102\ \mathrm{km}$ e vola verso l'altro treno lungo il binario. Raggiunto il secondo treno inverte il suo moto e si dirige verso il primo treno raggiunto il quale inverte ancora il suo moto e così via. 1) Quanti voli fa l'uccello prima che i due treni si scontrino? 2) Quale è lo spazio totale percorso dall'uccello?

WORKED SOLUTION

1) Infiniti. 2) 87 km

23-014 Il Cane E Il Plotone Di Soldati

Un plotone di soldati si muove in linea retta a velocità costante v_P. Un cane, che cammina a velocità costante v_C, parte dalla coda del plotone diretto verso la testa e quando la raggiunge inverte il suo moto dirigendosi verso la coda. Il cane raggiunge la coda del plotone nell'istante in cui questa si trova nella posizione in cui si trovava la testa del plotone all'istante in cui il cane è partito. Determinare il rapporto tra la velocità del cane e quella del plotone.

WORKED SOLUTION

Sia L la lunghezza del plotone di soldati, T_1 il tempo che il cane impiega ad andare dalla coda alla testa del plotone e T_2 il tempo che il cane impiega per tornare dalla testa alla coda del plotone. Si possono allora scrivere le tre equazioni

$$\begin{aligned} L &= v_P\,(T_1 + T_2) \quad , \\ v_C T_1 &= L + v_F T_1 \quad , \\ v_C T_2 &= L - v_F T_2 \quad . \end{aligned}$$

nelle quattro incognite v_C, v_P, T_1 e T_2. Risolvendo il sistema rispetto al rapporto delle due velocità si trova

$$\frac{v_C}{v_P} = 1 \pm \sqrt{2}$$

di cui solo la soluzione positiva è accettabile per il problema in quanto v_C e v_P sono moduli di velocità cioè sono positivi.

WORKED SOLUTION

$v_C/v_P = 1 + \sqrt{2}$

23-015 Moto in Una Dimensione

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23-016 Moto Unidimensionale Generico

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Un oggetto si muove lungo l'asse x secondo la legge oraria

$$x(t) = at^2 - bt^3 + k \sin \alpha t$$

dove x è espresso in metri e t in secondi. 1) Determinare le dimensioni delle costanti a, b, k e α. Si supponga che, in unità del Sistema Internazionale di unità di misura, le costanti valgano $a = 3$, $b = 1$, $k = 1$ e $\alpha = \pi/2$. 2) Determinare la posizione della particella dopo 4 s. 3) Determinare la velocità della particella al tempo $t = 0$ e dopo 4 s. 4) Determinare la velocità media nell'intervallo di tempo tra $t = 0$ s e $t = 4$ s. 5) Se $k = 0$ determinare dopo quanto tempo l'oggetto raggiunge il massimo valore di x.

WORKED SOLUTION

1) $[a] = \mathcal{L}\mathcal{T}^{-2}$, $[b] = \mathcal{L}\mathcal{T}^{-3}$, $[k] = \mathcal{L}$, $[\alpha] = \mathcal{T}^{-1}$; 2) $x(t = 4) = -16$ m;

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23-017 Accelerazione Di Un Elettrone

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23-018 Controllo Automatico Di Velocità

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23-019 Velocità Di Caduta Delle Gocce Di Pioggia

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23-020 Accelerazione Di Gravità Su Un Altro Pianeta

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23-021 Misura Del Tempo Di Reazione

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23-022 La Caduta Da Un Edificio

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23-023 Proprietà Del Moto Di Un Grave

This § is referenced at pages:
[1258]

Un grave è lanciato verso l'alto con velocità iniziale $v_0 = 5\,\mathrm{ms}$ a partire dalla quota $x_0 = 1.8\ \mathrm{m}$. Determinare, trascurando la resistenza dell'aria, 1) dopo quanto tempo il grave raggiunge il punto di quota massima; 2) quanto vale la quota massima; 3) quanto tempo impiega a ridiscendere alla quota da cui è partito; 4) quale è la sua velocità nell'istante in cui si ritrova alla quota da cui è partito; 5) dopo quanto tempo cade a terra. Dimostrare che il moto è simmetrico rispetto all'istante in cui il grave si trova alla quota massima.

WORKED SOLUTION

L'equazione del moto rettilineo uniformemente accelerato nel campo di gravità, avendo scelto un sistema di assi cartesiani con asse x rivolto verso l'alto, e origine al suolo è:

$$x(t) = x_0 + v_0 t - \frac{1}{2} g t^2 \quad ,$$
$$v(t) = v_0 - g t \quad .$$

La quota massima si ha al tempo t_0 a cui $v(t_0) = 0$. Il tempo t_0 e la quota massima sono dunque

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dati dalle equazioni

$$t_0 = \frac{v_0}{g} \quad ,$$
$$x(t_0) = x_0 + \frac{v_0^2}{2g} \geq x_0 \quad .$$

Si osservi che il segno di t_0 dipende dal segno della componente x, v_{0x}, della velocità iniziale $\mathbf{v}_0$. Si osservi ora che

$$x(2t_0) = x_0 \quad ,$$
$$v(2t_0) = -v_0 \quad .$$

per cui il grave, nel caso in cui al tempo $t = 0$ fosse in fase ascendente, cioè $\mathbf{v}_0$ diretta verso l'alto e quindi $v_{0x} > 0$, è ricaduto, al tempo $2t_0$, alla stessa quota che aveva al tempo $t = 0$ impiegando nella fase di ricaduta dal punto di massimo, lo stesso tempo che ha impiegato per raggiungere il massimo e ritrovandosi con velocità avente lo stesso modulo e verso opposto rispetto al tempo $t = 0$. Analogo discorso si può fare nel caso in cui il grave al tempo $t = 0$ si trovi in fase discendente, cioè $\mathbf{v}_0$ diretta verso il basso e quindi $v_{0x} < 0$, da cui seguirebbe $t_0 < 0$. Con i dati del problema si trova dunque

$$t_0 = 0.510 \text{ s} \quad ,$$
$$x(t_0) - x_0 = 1.27 \text{ m} \quad ,$$
$$\text{tempo per ricadere alla quota da cui è partito} = t_0 \quad ,$$
$$\text{velocità alla quota da cui è partito} = -v_0 \qquad (23.09.07)$$

→ 1256

L'equazione è simmetrica rispetto a $t = t_0$, infatti detto $t' = t - t_0$, si può riscrivere la (23-023) come

$$x'(t') \equiv x(t' + t_0) = x_0 + \frac{v_0^2}{2g} - \frac{1}{2}gt'^2 \qquad (23.09.08)$$

→ 1256

che è simmetrica per $t' \to -t'$ e dalla quale risulta evidente il valore della quota massima raggiunta dal grave.
Il tempo a cui il grave arriva al suolo si trova imponendo $x = 0$ nell'equazione (23.09.08) (o, alternativamente, nell'equazione (23-023)).

$$t = t_0 + t' = \frac{v_0}{g} \pm \sqrt{\frac{2}{g}\left(x_0 + \frac{v_0^2}{2g}\right)} = \begin{cases} t_1 = -0.282 \text{ s} \\ t_2 = +1.301 \text{ s} \end{cases} \quad ,$$

Delle due soluzioni quella positiva rappresenta l'istante di tempo a cui il grave arriva al suolo. Quella negativa rappresenta l'istante a cui un grave, che si trova al tempo $t = 0$ nelle condizioni iniziali del problema, sarebbe passato per la quota $x = 0$, durante la fase di salita, se il suo moto fosse definito per tempi negativi. Si veda anche il problema § 23-024.
1) $t_0 = 0.510$ s; 2) $x(t_0) - x_0 = 1.27$ m; 3) t_0; 4) $-v_0$; 5) 1.301 s.

23-024 La Palla Lanciata Verso Il Suolo

This § is referenced at pages:
[1255]

Una palla viene lanciata verso il suolo con velocità iniziale $v_0 = 1$ ms diretta verso il basso da una quota $H = 100$ m. 1) Determinare dopo quanto tempo raggiunge il suolo. 2) Determinare da quale quota avrebbe dovuto partire da fermo il grave per trovarsi nelle condizioni iniziali date. 3) Determinare la velocità media del grave nella caduta.

WORKED SOLUTION

Scegliamo un sistema di coordinate con l'asse x orientato verso il basso e origine al suolo. L'equazione del moto e la velocità della palla risultano dunque:

$$x(t) = x_0 + v_0 t + \frac{1}{2} g t^2 = -H + v_0 t + \frac{1}{2} g t^2 \quad ,$$
$$v(t) = v_0 + g t \quad .$$

L'arrivo la suolo è definito dalla condizione $x(t) = 0$ che ha due soluzioni:

$$t_1 = \frac{-v_0 - \sqrt{v_0^2 + 2Hg}}{g} = -4.62 \text{ s} < 0 \quad ,$$
$$t_2 = \frac{-v_0 + \sqrt{v_0^2 + 2Hg}}{g} = +4.41 \text{ s} > 0 \quad .$$

simmetriche rispetto a $t_0 = -v_0/g = -0.10$ s, l'istante in cui la velocità del grave vale zero e la quota è massima. L'equazione presenta due soluzioni; quella cercata è t_2 che è positiva. L'equazione impostata risolve il problema di trovare a quale tempo passa per la quota $x = 0$ un grave che al tempo $t = 0$ si trova a quota H dal suolo ($x = -H$ con il sistema di coordinate scelto) e con velocità v_0 rivolta verso il basso. In altri termini l'equazione così impostata non contiene l'informazione che il grave parte al tempo $t = 0$. Tale informazione va posta in più all'equazione specificando che tale equazione è definita solo per $t > 0$. In altri termini l'insieme di definizione della funzione fisica del tempo $x(t)$ è $t \geq 0$, più ristretto rispetto all'insieme di esistenza della funzione matematica $x(t)$ che è $-\infty < t < +\infty$. In tal modo la soluzione t_1 viene automaticamente ad essere inaccettabile per il problema dato. Il significato della soluzione t_1 è che, date queste condizioni iniziali al tempo $t = 0$, un grave in moto già a $t < 0$ sarebbe transitato per la quota $x = 0$ al tempo $t_1 < 0$ nella fase di salita antecedente la ricaduta.

Sia $h + H$ la quota massima a cui si troverebbe il grave al tempo t_0, $x(t_0) = -(h + H)$. Ricaviamo h per determinare a che quota si trova il grave al tempo t_0. Sostituendo l'espressione per t_0, che sappiamo essere l'istante in cui la quota è massima, nell'equazione del moto si trova

$$-(h + H) = -H - \frac{v_0^2}{g} + \frac{v_0^2}{2g}$$

da cui segue

$$h = \frac{v_0^2}{2g} = 5.10 \cdot 10^{-2} \text{ m}$$

L'ipotetico tempo di caduta da $x(t_0) = -(h + H)$ al suolo sarebbe in tal caso:

$$\Delta T = \sqrt{\frac{2(H + h)}{g}} = \frac{\sqrt{v_0^2 + 2Hg}}{g} = t_2 - t_0$$

La prima espressione per ΔT nell'equazione sopra è quella canonica per un grave che cade da altezza $H + h$ con velocità iniziale nulla. La seconda si ottiene introducendo nella prima l'espressione di h ricavata sopra e permette di verificare che il risultato è proprio uguale a $t_2 - t_0$, come deve.

Calcoliamo la velocità media della caduta:

$$\overline{v} \equiv \frac{H}{t_2} = 22.7 \text{ ms}$$

La velocità media in un moto rettilineo uniformemente accelerato vale, in base alla definizione generale di velocità media,

$$\boxed{\overline{v} = \frac{\Delta x}{\Delta t} = v_0 + \frac{1}{2} a \Delta t}$$

la quale fornisce, con $a = g$, $\overline{v} = 22.7$ ms, come il calcolo precedente. Nel caso di moto rettilineo uniformemente accelerato (e solo in questo caso) la velocità media si può calcolare come il valore

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medio della velocità iniziale e finale, poiché la velocità cresce linearmente col tempo:

$$\overline{v} = \frac{v + v_0}{2} = v_0 + \frac{1}{2}a\Delta t$$

WORKED SOLUTION

1) 4.41 s; 2) 100.051 m; 3) 22.7 ms.

23-025 Metodo per Misurare L'accelerazione Di Gravità

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Il seguente metodo è stato impiegato per misurare il valore dell'accelerazione di gravità g. Si lancia una palla verticalmente verso l'alto entro un tubo a vuoto e la si osserva mentre, salendo prima e ricadendo poi, passa davanti a due traguardi U e L posti a distanza nota H. Se Δt_L rappresenta l'intervallo di tempo tra i due passaggi al livello L e Δt_U rappresenta l'intervallo di tempo tra i due passaggi al livello U, dimostrare che l'accelerazione di gravità g vale:

$$g = \frac{8H}{\Delta t_L^2 - \Delta t_U^2}$$

WORKED SOLUTION

Siano v_U e v_L i moduli della velocità della palla quando passa davanti la traguardo U ed L rispettivamente. Come discusso nel problema § 23-023, in assenza della resistenza dell'aria il modulo della velocità della palla mentre passa davanti al traguardo è lo stesso sia nella fase di salita che in quella di discesa. Inoltre il tempo che la palla impiega ad arrivare al punto di massima altezza (quando $v = 0$) partendo dal traguardo è lo stesso che impiega a ridiscendere e vale quindi metà dell'intervallo totale (Δt_L o Δt_U). Scegliendo un sistema di coordinate con asse x orientato verso l'alto si hanno le tre equazioni:

$$\begin{aligned} v_U^2 - v_L^2 &= -2gH \quad , \\ 0 &= v_U - g\frac{\Delta t_U}{2} \quad , \\ 0 &= v_L - g\frac{\Delta t_L}{2} \quad . \end{aligned}$$

che risolte forniscono l'equazione per g richiesta.

23-026 La Palla Che Cade Davanti Alla Finestra

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Una palla d'acciaio viene lasciata cadere dal tetto di un edificio con velocità iniziale nulla. Un osservatore ad una finestra alta $d = 120$ cm nota che la palla impiega $\Delta t = 0.125$ s ad attraversare in lunghezza la finestra. La palla continua a cadere, urta il suolo elasticamente (si supponga istantaneo l'urto) e riappare al bordo inferiore della finestra $\Delta T = 2$ s dopo essere passata per lo stesso punto durante la caduta. Quanto è alto l'edificio?

WORKED SOLUTION

Sia H l'altezza dal suolo del davanzale della finestra e sia h la distanza dal bordo superiore della finestra al tetto dell'edificio. L'altezza dell'edificio è dunque $H+d+h$. Il tempo che la palla impiega a cadere dal davanzale al suolo è uguale a quello che impiega a risalire e vale dunque $\Delta T/2$. Siano v_1 e v_2 rispettivamente le velocità della palla al passaggio al bordo superiore della finestra e al davanzale. Si usi un sistema di coordinate con asse x diretto verticalmente verso il basso e origine al suolo. Sia il tempo $t=0$ l'istante in cui la palla passa al davanzale della finestra. Si ha:

$$v_2 - v_1 = g\Delta t \quad ,$$
$$v_2^2 - v_1^2 = 2gd \quad ,$$
$$0 = -H + v_2\frac{\Delta T}{2} + \frac{g}{2}\left(\frac{\Delta T}{2}\right)^2 \quad ,$$
$$0 = -(H+d) + v_1\left(\frac{\Delta T}{2}+\Delta t\right) + \frac{g}{2}\left(\frac{\Delta T}{2}+\Delta t\right)^2 \quad ,$$
$$v_1^2 = 2gh \quad ,$$
$$v_2^2 = 2g(h+d) \quad .$$

Dalle prime due equazioni si ricavano i valori di v_1 e v_2:

$$v_1 = \frac{1}{2}\left[\frac{2d}{\Delta t} - g\Delta t\right] = 8.99\,\text{ms} \quad ,$$
$$v_2 = \frac{1}{2}\left[\frac{2d}{\Delta t} + g\Delta t\right] = 10.2\,\text{ms} \quad .$$

Dalla terza e quinta equazione si ricavano poi i valori di H e h.

$$H = +v_2\frac{\Delta T}{2} + \frac{g}{2}\left(\frac{\Delta T}{2}\right)^2 = 15.1\ \text{m} \quad ,$$
$$h = \frac{v_1^2}{2g} = 4.12\ \text{m} \quad .$$

Tali risultati sono cosistenti con la quarta e sesta equazione scritta sopra. L'altezza globale del palazzo risulta dunque

$$H+d+h = (15.1+1.2+4.12)\ \text{m} = 20.4\ \text{m}$$

WORKED SOLUTION

20.4 m.

23-027 Oggetto Visto Da Una Finestra

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23-028 Moto in Fluido Viscoso

Si supponga che un punto materiale sia in caduta non libera nel campo di gravità terrestre con equazione del moto

$$x(t) = \frac{m}{k}\left(\frac{mg}{k} - v_0\right)\left[\exp -\frac{kt}{m} - 1\right] + \frac{mgt}{k} + x_0$$

dove m è la massa del corpo, g l'accelerazione di gravità e k una costante assegnata. 1) Si determini le dimensioni della costante k, 2) il significato fisico delle due costanti x_0 e v_0, 3) il valore limite cui tendono la velocità e l'accelerazione in funzione del tempo.

WORKED SOLUTION

Le dimensioni della costante k si possono determinare in vari modi dalla equazione del moto data. Una possibilità è considerare l'argomento dell'esponenziale che deve essere adimensionale. Ne segue

$$[k] = \mathcal{MT}^{-1}$$

Il significato fisico della costante x_0 è manifesto se si calcola

$$x(t=0) = x_0$$

Il significato fisico della costante v_0 può essere determinato calcolando l'andamento della velocità

$$v(t) \equiv \frac{dx(t)}{dt} = -\left(\frac{mg}{k} - v_0\right)\exp -\frac{kt}{m} + \frac{mg}{k}$$

Ne segue

$$v(t=0) = v_0$$

La velocità tende, per $t \to \infty$ al valore limite

$$\lim_{t\to\infty} v(t) = \frac{mg}{k}$$

L'accelerazione vale

$$a(t) \equiv \frac{dv(t)}{dt} = \frac{k}{m}\left(\frac{mg}{k} - v_0\right)\exp -\frac{kt}{m}$$

e tende a zero al crescere del tempo. Il moto può essere descritto qualitativamente come segue. Il corpo accelera con accelerazione sempre decrescente e che tende a zero, mentre la velocità aumenta finché non raggiunge, asintoticamente, una velocità limite e accelerazione nulla.
Questo moto è il moto di un corpo in un fluido viscoso descritto in section § 24-040

WORKED SOLUTION

1) $[k] = \mathcal{MT}^{-1}$; 2) x_0 e v_0 sono rispettivamente posizione e velocità al tempo $t = 0$; 3) $\lim_{t\to\infty} v(t) = mg/k$ e $\lim_{t\to\infty} a(t) = 0$.

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23-029 Oggetto Che Cade E Rimbalza

Un grave è lasciato cadere con velocità iniziale nulla, rimbalza elasticamente al suolo, invertendo istantaneamente la propria velocità, e risale. 1) Determinare l'espressione del tempo T che impiega dal momento del lancio, alla quota x, fino al momento in cui risale ad una quota h assegnata 2) Determinare da quale quota va lasciato cadere affinché tale tempi sia minimo.

WORKED SOLUTION

1) $T = \sqrt{2/g}\left[2\sqrt{x} - \sqrt{x-h}\right]$; 2) Il tempo minimo si ha per $x = 4h/3$.

23-030 La Profondità Del Pozzo

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|3.63||

23-031 Il Problema Dei Semafori Sincronizzati

©|Symon, , ..., ..., ...Ed., ...|x.yy||

©|Wal, , ..., ..., ...Ed., ...|x.yy||

Su una strada sono disposti semafori a distanza regolare d. Il periodo del ciclo del semaforo vale T. Passando per un verde a che velocità costante si deve proseguire per continuare a prendere sempre il verde?

23-032 Il Cacciatore Che Spara All'uccello Su Un Ramo

23-033 Il Centometrista

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23-034 Il Tubo a Raggi Catodici

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23-035 Il Tiro Delle Freccette

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23-036 Il Moto Del Proiettile

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23-037 Il Bombardiere E La Contraerea - Il Cacciatore E La Lepre

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Si supponga di avvistare un bombardiere che, volando a quota $h = 1600$ m e con velocità $v = 500$ km/h, si dirige verso una postazione di contraerea. Si vuole colpire l'aereo quando si trova sopra un punto a distanza $a = 2$ km dalla postazione, cioè appena entrato entro la portata della batteria di contraerea ma prima che si avvicini troppo. Si trascura inizialmente l'effetto della forza di gravità sul moto del proiettile e si punta perciò il cannone nella direzione del bombardiere con alzo pari a $\theta = \arctan(h/a)$. 1) Sapendo che la velocità con cui vengono sparati i proiettili dalla contraerea vale $v_0 = 600$ ms a che distanza b (misurata al suolo) deve essere l'aereo al momento dello sparo? 2) Si discuta l'errore commesso con questa approssimazione. 3) Se immediatamente prima di essere colpito l'aereo sgancia una bomba dove cadrà la bomba rispetto alla postazione? Si applichi il risultato trovato al problema del cacciatore che, avvistata una lepre in corsa con $v = 15$ ms, punta il fucile perpendicolarmente alla traiettoria rettilinea della lepre e si calcoli l'angolo di anticipo con cui deve sparare per colpirla nel momento un cui questa passa davanti al fucile.

WORKED SOLUTION

Si scelga un sistema di assi cartesiani con l'asse x avente origine nella postazione e diretto verso il bombardiere e asse y verticale. Sia $t = 0$ l'istante in cui il proiettile colpisce il bombardiere e $t_0 < 0$ l'istante in cui si deve sparare. Le equazioni del moto del bombardiere e del proiettile sono:

$$\begin{aligned} x_a(t) &= -vt + a \quad , \\ y_a(t) &= h \quad , \\ x_p(t) &= (t - t_0) v_0 \cos\theta \quad , \\ y_p(t) &= (t - t_0) v_0 \sin\theta \quad . \end{aligned}$$

con $\tan\theta = h/a$ e $\theta = 38.7°$. Il bombardiere viene colpito quando $x_a(t) = x_p(t)$ e $y_a(t) = y_p(t)$. La sola incognita risulta essere t_0, essendo già noto che l'istante in cui il proiettile colpisce il bombardiere è $t = 0$ per come è stata scelta l'origine dei tempi. La seconda incognita cinematica sarebbe l'angolo di puntamento θ che è già stato determinato come $\tan\theta = h/a$. Si trova

$$t_0 = \frac{-h}{v_0 \sin\theta} = -\frac{\sqrt{a^2 + h^2}}{v_0} = -4.27 \text{ s} \qquad \text{non dipende da } v \text{ per come è posto il problema}$$

Il valore t_0 risulta, come deve, uguale allo spazio percorso dal proiettile diviso la sua velocità. Valutando la posizione dell'aereo al tempo $t = t_0$ si trova:

$$\begin{aligned} x_a(t_0) &= \frac{v\sqrt{a^2 + h^2}}{v_0} + a \equiv b = 2.59 \text{ km} \quad , \\ y_a(t_0) &= h = 1.6 \text{ km} \quad . \end{aligned}$$

Per soddisfare le richieste si deve quindi sparare con un anticipo di $|t_0|$ rispetto all'istante in cui il bombardiere raggiunge la posizione stabilita per l'abbattimento mentre questo si trova alla posizione b. L'angolo di vista del bombardiere al momento dello sparo (α) vale dunque:

$$\alpha = \arctan\left[\frac{h}{\frac{v\sqrt{a^2 + h^2}}{v_0} + a}\right] = 31.7°$$

Nel limite $v_0 \gg v$ si ha $\tan\alpha \to \tan\theta$. Nel limite $v \gg v_0$ si ha $\tan\alpha \to 0$.

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Per valutare l'errore commesso nel trascurare l'effetto della gravità sul proiettile si consideri che il proiettile vola per un tempo $\Delta t = \sqrt{a^2+h^2}/v_0$ e che in tale intervallo di tempo cade sulla verticale di una distanza

$$\Delta y_{\texttt{corr}} = -\frac{g}{2}(\Delta t)^2 = -\frac{g}{2}\left(\frac{a^2+h^2}{v_0^2}\right) = -89.4 \text{ m} < 0$$

Questo risultato significa che sparando con le condizioni calcolate il proiettile si ritrova, all'istante $t = 0$, all'ascissa x prevista, essendo le equazioni del moto orizzontale indipendenti da g, ma con una ordinata y che differisce di $\Delta y_{\texttt{corr}}$ rispetto a quella desiderata. L'approssimazione fatta è dunque non accettabile essendo l'errore commesso paragonabile con le dimensioni del bombardiere. Il calcolo andrebbe rifatto correttamente considerando l'effetto della forza peso sul moto del proiettile. Come prima approssimazione si può pensare di correggere il tiro modificando la puntata del cannone in modo da mirare ad una posizione $-\Delta y_{\texttt{corr}}$ rispetto a quella calcolata ignorando l'effetto dell'accelerazione di gravità, cioè puntando $|\Delta y_{\texttt{corr}}|$ più in alto. L'angolo approssimato di puntamento del cannone si può valutare dunque come

$$\tan\theta' \simeq \frac{h-\Delta y_{\texttt{corr}}}{a} = \frac{h}{a} + \frac{g(h^2+a^2)}{2av_0^2} = 40.2^\circ \neq 38.7^\circ$$

Ma questa è solo una soluzione approssimata.
La bomba lasciata cadere dal bombardiere immediatamente prima di essere colpito ha equazioni del moto:

$$x_b(t) = -vt + a \qquad \text{la stessa del bombardiere} \quad ,$$

$$y_b(t) = h - \frac{1}{2}gt^2 \qquad \text{supponendo } v_y(t=0) = 0 \quad .$$

La bomba arriva a terra dopo un tempo $\Delta t_1 = \sqrt{2h/g} = 18.1$ s e si trova a $x_b = -v\sqrt{2h/g} + a = -0.508$ km, oltre la postazione della contraerea che quindi non è al sicuro nelle condizioni date potendo essere colpita da una bomba lanciata dal bombardiere prima che questo sia abbattuto.
Il calcolo corretto della traiettoria, senza trascurare l'effetto della forza di gravità, procede come segue. Le equazioni del moto diventano:

$$\begin{aligned} x_a(t) &= -vt + a \quad , \\ y_a(t) &= h \quad , \\ x_p(t) &= (t-t_0)v_0\cos\theta \quad , \\ y_p(t) &= (t-t_0)v_0\sin\theta - \frac{1}{2}g(t-t_0)^2 \quad . \end{aligned}$$

Si noti che solo la quarta equazione si è modificata. Le incognite sono t_0, il tempo a cui si deve sparare, e θ, l'alzo del cannone per colpire, al tempo $t = 0$ il bombardiere nel punto di coordinate $x = a$ e $y = h$. Si ottengono quindi le equazioni

$$x_p(t=0) = a = (-t_0)v_0\cos\theta \quad ,$$

$$y_p(t=0) = h = (-t_0)v_0\sin\theta - \frac{1}{2}g(-t_0)^2 \quad .$$

che risolte rispetto a t_0 e $\tan\theta$ forniscono

$$t_0 = -\frac{a}{v_0\cos\theta} \qquad \text{come prima ma ora } \theta \text{ è incognito} \quad ,$$

$$\frac{ga^2}{2v_0^2}\tan^2\theta - a\tan\theta + h + \frac{ga^2}{2v_0^2} = 0 \quad .$$

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Le due soluzioni per $\tan\theta$ sono:

$$\tan\theta = \frac{v_0^2\left(1 \pm \sqrt{1 - \frac{g}{v_0^2}\left(2h + \frac{ga^2}{v_0^2}\right)}\right)}{ga} = \begin{cases} \theta_1 = 40.3^\circ \implies t_{0\,1} = -4.37\ \mathrm{s} \\ \theta_2 = 88.4^\circ \implies t_{0\,2} = -120.\ \mathrm{s} \end{cases},$$

→ 1264

La soluzione che interessa sarà quella con il minore valore di $\tan\theta$ perché corrisponde a colpire l'aereo direttamente e quindi in minore tempo, anziché nella fase di ricaduta del proiettile. Questo è confermato dalla soluzione per t_0 che per il minore valore di $\tan\theta$ fornisce il minore valore del tempo di volo $|t_0|$. Si ha, come deve, che nel limite $g \to 0$, $\tan\theta_1 \to h/a$ e $t_0 \to -\sqrt{a^2+h^2}/v_0$, che è la soluzione trovata nel caso in cui si è trascurata la gravità. Lo sviluppo al primo ordine in g della soluzione per $\tan\theta_1$ usando lo sviluppo del binomio

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \mathcal{O}(x^3) \qquad -1 < x \leq +1 \tag{23.09.09}$$

→ 1264

fornisce

$$\tan\theta_1 \simeq \frac{h}{a}\left[1 + \frac{g}{2hv_0^2}\left(a^2+h^2\right)\right] = 40.2^\circ \simeq \tan\theta$$

ed è, come ovvio, maggiore di h/a per compensare la caduta del proiettile sotto l'effetto della forza di gravità. L'angolo θ_1 così calcolato coincide con l'angolo θ' calcolato precedentemente. Nello sviluppo del binomio occorre utilizzare anche il termine di secondo grado nella formula (23.09.09) in quanto il g a denominatore della (23-037) semplifica il termine di primo ordine in g al numeratore. Si noti che il termine da sviluppare nella (23-037) include sia g che g^2 per cui il termine di ordine x^2 nella (23.09.09) fornisce anche un contributo di ordine g^2.
Nel caso del cacciatore e la lepre si ha $a = 0$ e l'applicazione della formula per α fornisce il risultato

$$\text{angolo di anticipo} \equiv 90^\circ - \alpha = 1.43^\circ$$

WORKED SOLUTION

1) $b = 2.59$ km; 2) Errore sulla quota del proiettile $\Delta y_{\mathtt{corr}} = -89.4$ m; 3) 0.508 km oltre la postazione.

23-038 La Pietra Lanciata Verso L'altura

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23-039 Gittata Massima Del Cannone E Gittata Lungo Un Piano Inclinato

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This § is referenced at pages:
[1266, 1267]

Sia dato un cannone che spara proiettili con velocità iniziale $v_0 = 500$ ms ad un angolo θ dal suolo. 1) Determinare l'equazione della traiettoria del proiettile e l'angolo a cui deve essere puntato il cannone per ottenere la gittata massima trascurando la resistenza dell'aria. 2) Nel caso il cannone si trovi alla base di un pendio rettilineo che forma un angolo α con l'orizzontale determinare la gittata lungo il pendio. 3) Nel caso di sparo in pianura determinare l'angolo di lancio per cui la massima altezza raggiunta dal proiettile è uguale alla gittata.

WORKED SOLUTION

Scelto un sistema di coordinate con origine nel cannone, asse x orizzontale e asse y verticale, le equazioni del moto del proiettile sono

$$x(t) = v_0 t \cos\theta \quad ,$$
$$y(t) = v_0 t \sin\theta - \frac{1}{2} g t^2 \quad .$$

L'equazione della traiettoria si ottiene eliminando la variabile tempo dalle equazioni del moto:

$$\boxed{y = x\tan\theta - \frac{g}{2v_0^2}\left(1+\tan^2\theta\right)x^2} \qquad (23.09.10)$$

→ 1266

Il punto di caduta del proiettile, cioè la gittata G del cannone, sarà dunque

$$\boxed{G = \frac{v_0^2 \sin 2\theta}{g}}$$

Da cui segue per la massima gittata:

$$\theta_{\max} = \pi/4 \quad ,$$
$$G_{\max} = \frac{v_0^2}{g} = 25.5 \text{ km} \quad .$$

Nel caso il cannone si trovi alla base di un pendio che forma un angolo α con l'orizzontale il punto di caduta si determina dal sistema

$$y = x\tan\theta - \frac{g}{2v_0^2}\left(1+\tan^2\theta\right)x^2 \quad ,$$
$$y = x\tan\alpha \quad .$$

che fornisce

$$x = \frac{2v_0^2\cos^2\theta}{g}(\tan\theta - \tan\alpha)$$

da cui segue la gittata lungo il pendio G

$$\boxed{G = \frac{2v_0^2\cos^2\theta}{g\cos\alpha}(\tan\theta - \tan\alpha)}$$

WORKED SOLUTION

2) $G = 2v_0^2\cos^2\theta\,(\tan\theta - \tan\alpha)\,/\,(g\cos\alpha)$.

23-040 Parabola Di Sicurezza

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Dato un cannone che spara colpi con una velocità iniziale di v_0 1) determinare il luogo dei punti che non sono raggiungibili dai proiettili sparati dal cannone; 2) determinare l'altezza massima raggiunta dal proiettile.

WORKED SOLUTION

Si scelga un sistema di coordinate con origine nel cannone, asse x parallelo al suolo e asse y verticale.

L'equazione della traiettoria del proiettile è (equazione (23.09.10)):

$$y = x\tan\theta - \frac{g}{2v_0^2}\left(1+\tan^2\theta\right)x^2$$

dove θ è l'alzo del cannone. Siano x' e y' le coordinate di un generico punto bersaglio. Determiniamo l'alzo che deve avere il cannone per colpire il bersaglio posto in (x', y'). Ciò si ottiene risolvendo l'equazione

$$y' = x'\tan\theta - \frac{g}{2v_0^2}\left(1+\tan^2\theta\right)x'^2$$

nell'incognita θ. Si trova

$$\tan\theta = \frac{x' \pm \sqrt{x'^2 - 4kx'^2\left(y'+kx'^2\right)}}{2kx'^2} \qquad \text{con } k \equiv \frac{g}{2v_0^2}$$

Tale equazione ha due soluzioni reali per $\tan\theta$ (eventualmente coincidenti) se e solo se

$$x'^2 - 4kx'^2\left(y'+kx'^2\right) \geq 0 \qquad \Leftrightarrow \qquad \boxed{y' \leq \frac{1}{4k} - kx'^2}$$

che rappresenta una parabola (detta di sicurezza). I punti sulla parabola possono essere colpiti dal cannone scegliendo opportunamente l'alzo. I punti all'interno della parabola possono essere colpiti con due alzi diversi del cannone, uno colpisce il bersaglio nella fase di salita, l'altro nella fase di discesa. Il vertice della parabola si trova alla quota $y'_{\max} = 1/(4k) = v_0^2/2g$, la massima quota raggiungibile dal cannone con $\theta = \pi/2$. La gittata massima si ha per $|x'_{\max}| = 1/(2k) = v_0^2/g$

23-041 Salto in Lungo

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Nelle Olimpiadi del 1968 a Città del Messico Bob Beamon stabilì il record mondiale con un salto in lungo di $l = 8.90$ m. Supponendo che la velocità iniziale al momento del salto sia stata pari a $v_0 = 9.50$ ms, pari circa a quella di un corridore dei 100 metri, e sapendo che l'accelerazione di gravità a Città del Messico vale $g = 9.78\ \mathrm{m/s^2}$ quale è la differenza tra il record e la massima gittata ottenibile in assenza di aria? Se Beamon avesse compiuto lo stesso salto a Berlino ($g = 9.81\ \mathrm{m/s^2}$), mantenendo gli stessi valori di θ e v_0, quanto avrebbe saltato?

WORKED SOLUTION

Si usino i risultati del problema § 23-039 e sia G la gittata del cannone.

$$G = \frac{v_0^2\sin 2\theta}{g} \quad ,$$
$$\theta_{\max} = \pi/4 \quad ,$$
$$G_{\max} = \frac{v_0^2}{g} \quad .$$

Nel caso di Beamon si ottiene $G_{\max} = 9.23$ m. Se si fosse incluso l'effetto della resistenza dell'aria tale valore sarebbe risultato ancora più basso. La dipendenza da g, a parità di v_0 e θ, si determina come segue:

$$\Delta G = \Delta\left(\frac{v_0^2\sin 2\theta}{g}\right) \quad ,$$
$$\Longrightarrow \frac{\Delta G}{G} = -\frac{\Delta g}{g} \quad .$$

da cui si ricava $\Delta G = -2.7$ cm

23-042 Il Cannone Sulla Collina

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23-043 Aggiustamento Del Tiro Del Cannone

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In artiglieria è pratica abituale sparare una successione di colpi di prova prima di iniziare a sparare per ottenere l'effetto voluto. L'artigliere prima spara prima un colpo al di qua del bersaglio, poi un colpo al di là e apporta le necessarie correzioni al tiro in base ai risultati dei due colpi in modo da colpire il bersaglio. Si supponga che il primo colpo, sparato con alzo $7°20''$, atterri $180\ \mathrm{m}$ al di qua del bersaglio mentre il secondo, sparato con angolo di tiro $7°35''$, atterri $120\ \mathrm{m}$ oltre il bersaglio. Quanto vale l'angolo di proiezione corretto per colpire il bersaglio?

WORKED SOLUTION

Nel problema dato v_0, la velocità con cui i proiettili sono sparati dal cannone, è fissata ed ignota. Dal problema § 23-039 si ha

$$\Delta G = \left.\frac{dG}{d\alpha}\right|_{\alpha=\alpha_0} \Delta\alpha = \frac{2v_0^2 \cos 2\alpha}{g}\Delta\alpha$$

e quindi

$$\Delta G_1 = \left.\frac{dG}{d\alpha}\right|_{\alpha=\alpha_0} \Delta\alpha_1 = -180\ \mathrm{m} \quad ,$$

$$\Delta G_2 = \left.\frac{dG}{d\alpha}\right|_{\alpha=\alpha_0} \Delta\alpha_2 = +120\ \mathrm{m} \quad .$$

da cui segue

$$\frac{\Delta G_1}{\Delta G_2} = \frac{\Delta\alpha_1}{\Delta\alpha_2} = \frac{\alpha-\alpha_1}{\alpha-\alpha_2} \equiv k \qquad \alpha \text{ incognita, } k \text{ nota}$$

Si ricava

$$\alpha = \frac{\alpha_1 - k\alpha_2}{1-k}$$

Tale relazione ha i limiti corretti nei casi $\Delta G_1 = 0$ ($\alpha = \alpha_1$), $\Delta G_2 = 0$ ($\alpha = \alpha_2$), $\Delta G_1 = -\Delta G_2$ ($\alpha = (\alpha_1 + \alpha_2)/2$). Usando i dati del problema si ottiene che l'angolo corretto di puntamento vale $\alpha = 7°29''$.

23-044 La Ruota Panoramica

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23-045 Stelle Di Neutroni

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23-046 Distanza Minima Tra Due Aereoplani

Le equazioni del moto di due aereoplani, in un Coordinate System in cui l'origine coincide con l'aereoporto, gli assi x e y sono paralleli al suolo e l'asse z è verticale e orientato verso l'alto, sono

$$\mathbf{x}_1(t) = a_1 t \mathbf{e}_x + b_1 t \mathbf{e}_y + c_1 t^2 \mathbf{e}_z \quad ,$$
$$\mathbf{x}_2(t) = (a_2 t + a_{20})\,\mathbf{e}_x + (b_2 t + b_{20})\,\mathbf{e}_y + \left(c_{20} - c_2 t^2\right)\mathbf{e}_z \quad .$$

Il primo aereoplano è in fase di decollo mentre il secondo è in fase di atterraggio. Determinare a quale istante i due aereoplani si trovano alla distanza minima e il valore della distanza minima tra i due aereoplani supponendo che le varie costanti, tutte espresse in unità del Sistema Internazionale, valgano $a_1 = 1$, $b_1 = 2$, $c_1 = +3$, $a_2 = 0$, $b_2 = 2$, $c_2 = -3$, $a_{20} = 1$, $b_{20} = 1$, $c_{20} = 10$.

WORKED SOLUTION

La distanza è stazionaria rispetto al tempo se e solo se

$$(\mathbf{x}_1 - \mathbf{x}_2)\cdot(\mathbf{v}_1 - \mathbf{v}_2)$$

WORKED SOLUTION

$t_0 = 1$ s, $d = 101$.

23-047 Distanza Minima Tra Due Oggetti in Moto Arbitrario

Un punto materiale si muove nello spazio con equazione del moto generica

$$\mathbf{x}(t) = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2 + x_3(t)\mathbf{e}_3 \tag{23.09.11}$$

1) Dimostrare che condizione sufficiente affinché la distanza del punto dall'origine abbia un minimo relativo è

$$\mathbf{x}\cdot\mathbf{v} = 0 \qquad v^2 + \mathbf{x}\cdot\mathbf{a} > 0 \tag{23.09.12}$$

→ 1268

2) Si generalizzi il risultato per determinare la condizione sufficiente affinché la distanza tra due punti materiali, in moto con equazione del moto $\mathbf{x}_1(t)$ e $\mathbf{x}_2(t)$, un minimo relativo.

WORKED SOLUTION

Si studi la condizione di minimo del quadrato della distanza del punto dall'origine in funzione del tempo: $d^2 = \mathbf{x}\cdot\mathbf{x}$.

WORKED SOLUTION

2) La condizione (23.09.12) viene espressa in termini della posizione e velocità relative $\mathbf{x}_R \equiv \mathbf{x}_1 - \mathbf{x}_2$ e $\mathbf{v}_R \equiv \mathbf{v}_1 - \mathbf{v}_2$.

DRAFT - Proof

23-048 Oggetti Super-Luminali

Gli astronomi hanno scoperto oggetti che emettono getti di materia ad altissima velocità. Se la distanza dell'oggetto dalla Terra è nota (sia d) la misura della velocità angolare alla quale si vede il getto muoversi sulla sfera celeste permette di determinare la velocità con cui il getto di materia è emesso dal nucleo centrale. In alcuni casi sono stati osservati getti in moto con velocità apparente superiore alla velocità della luce, fatto in apparente contraddizione con la teoria della relatività speciale. Mostrare come sia possibile ottenere velocità apparenti superiori a quella della luce e come ciò non contrasti con la relatività speciale considerando l'emissione del getto ad un angolo θ rispetto alla linea di vista dell'osservatore terrestre. 1) Determinare la relazione tra velocità apparente e quella reale del getto di materia. 2) Determinare la velocità apparente di un sistema in cui la direzione di emissione del getto forma un angolo $\theta = 9.2°$ con la direzione di vista supponendo la velocità di emissione pari a $v = 0.987c$ (questa si può determinare dallo studio dello spostamento **Doppler** delle linee spettrali). 3) Determinare il valore dell'angolo θ per cui la velocità apparente del getto sulla sfera celeste è massima e il valore di tale velocità massima.

DRAFT

WORKED SOLUTION

Sia t_0 l'istante di emissione del getto di materia in direzione θ rispetto alla linea di vista. L'emissione del getto viene osservata da Terra al tempo t_1:

$$t_1 = t_0 + d/c$$

Il getto viene osservato da Terra, dopo che ha percorso la distanza $v\Delta t$, all'istante

$$t_2 = t_0 + \Delta t + \frac{d - v\Delta t\cos\theta}{c}$$

L'intervallo di tempo Δt tra le due osservazioni da Terra è

$$t_2 - t_1 = \Delta t(1 - \frac{v\cos\theta}{c}) < \Delta t \qquad (\text{se } \cos\theta > 0)$$

L'astronomo conosce la distanza d dell'oggetto e misura uno spostamento angolare $\Delta\theta$ sulla sfera celeste da cui deduce uno spostamento apparente Δx del getto. Lo spostamento apparente Δx sulla sfera celeste vale

$$\Delta x \equiv \Delta\theta d = \frac{v\Delta t\sin\theta}{d}d = v\Delta t\sin\theta$$

è da notare il fatto che la relazione precedente è esatta essendo $d \gg v\Delta t$ per cui non è necessario distinguere fra la distanza dell'oggetto che emette il getto, d, e la distanza del getto al tempo t_2 che vale $\simeq d - v\Delta t\cos\theta$. La velocità apparente osservata risulta dunque:

$$v_{\text{app}}(\theta) = \frac{\Delta x}{t_2 - t_1} = \frac{d\Delta\theta}{t_2 - t_1} = \frac{v\sin\theta}{1 - (v/c)\cos\theta}$$

che può essere maggiore di c.
Il massimo della velocità al variare dell'angolo di emissione θ si trova derivando rispetto a θ e ponendo la derivata uguale a zero:

$$\frac{dv_{\text{app}}(\theta)}{d\theta} = \frac{v\cos\theta - v^2/c}{(1 - v/c\cos\theta)^2} = 0$$

da cui segue che l'angolo a cui si ottiene la velocità apparente massima e la velocità apparente massima valgono

$$\cos\theta_{\text{max}} = v/c \qquad v_{\text{app}}\Big|_{\text{max}} = \frac{v}{\sqrt{1 - v^2/c^2}}$$

Con i dati del problema si ottiene $v_{\text{app}} = 6.14c$.

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WORKED SOLUTION

1) $v_{\text{app}}(\theta) = v\sin\theta/\left(1-(v/c)\cos\theta\right)$; 2) $v_{\text{app}} = 6.14c$; 3) $\cos\theta_{\text{max}} = v/c$.

23-049 Raggio Di Curvatura Della Traiettoria Di Un Grave

Si consideri il moto di un proiettile sparato da un cannone con velocità iniziale $v_0 = 450\,\text{ms}$ con un alzo pari a $\theta = 31°$. Si determini in quale punto della traiettoria l'accelerazione normale è massima, quanto vale, il raggio di curvatura e il valore dell'accelerazione tangenziale in quel punto. Si determini inoltre l'andamento del raggio di curvatura della traiettoria in funzione del tempo.

WORKED SOLUTION

Si utilizzi la rappresentazione intrinseca dell'accelerazione

WORKED SOLUTION

Scegliendo un sistema di assi cartesiani con origine nel cannone, asse x diretto parallelamente al suolo e asse y verticale orientato verso l'alto l'equazione del moto del proiettile è:

$$\begin{aligned} x(t) &= x_0 + v_0 t\cos\theta \quad , \\ y(t) &= y_0 + v_0 t\sin\theta - \frac{1}{2}gt^2 \quad . \end{aligned}$$

l'andamento della velocità e dell'accelerazione sono

$$\begin{aligned} \dot{x}(t) &= v_0\cos\theta \quad , \\ \dot{y}(t) &= v_0\sin\theta - gt \quad . \\ \dot{v}_x(t) &= 0 \quad , \\ \dot{v}_y(t) &= -g \quad . \end{aligned}$$

L'accelerazione durante il moto può essere scomposta nelle due componenti normale e tangenziale la cui somma da sempre g. All'istante in cui il proiettile raggiunge la quota massima la velocità è diretta orizzontalmente e quindi l'accelerazione normale è diretta verticalmente verso il basso. Il verso si deduce dal fatto che l'accelerazione normale punta sempre verso il centro del cerchio osculatore. In tale posizione l'accelerazione è dunque tutta normale e il suo modulo vale g. Dalla (23.03.09) si ha allora

$$\rho = \frac{v^2}{g} = \frac{v_0^2\cos^2\theta}{g} = 15.2\ \text{km}$$

dove si è tenuto conto che nel punto di massima altezza della traiettoria si ha $v_y = 0$ e $v = v_x = v_0\cos\theta$.
L'accelerazione normale nel punto di massima altezza della traiettoria vale dunque g e quella tangenziale vale zero (infatti il modulo della velocità ha un minimo in queste condizioni).
L'andamento del raggio di curvatura in funzione del tempo si può dedurre dalla relazione (23.03.04) usando le equazioni del moto date sopra. Si trova

$$\frac{1}{\rho} = \frac{v_0 g|\cos\theta|}{\left|v_0^2 + g^2t^2 - 2gv_0 t\sin\theta\right|^{3/2}}$$

che calcolato al punto di massimo della traiettoria ($t = v_0\sin\theta/g$) si riduce al valore calcolato sopra. Il calcolo può essere effettuato in modo più generale. Il modulo della velocità può essere calcolato dalle relazioni ricavate sopra

$$v = \sqrt{v_0^2 + g^2t^2 - 2gv_0 t\sin\theta}$$

Se ne deduce

$$a_T = \frac{dv}{dt} = \frac{g}{v}(gt - v_0\sin\theta)$$

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L'accelerazione tangenziale vale zero per $t = v_0 \sin\theta/g$, coerentemente con quanto ricavato sopra, in quanto questo è l'istante al quale il grave si trova alla quota massima. L'accelerazione del grave è sempre g per cui si può dedurre l'accelerazione normale in ogni punto delle traiettoria

$$a_N = \sqrt{g^2 - a_T^2} = \frac{g}{v} v_0 |\cos\theta| = \frac{v^2}{\rho}$$

ed è massima nel punto in cui v è minima che è il punto in cui la quota è massima essendo v_x costante e $|v_y|$ zero in quel punto. In tali circostanze vale (essendo $v = v_0|\cos\theta|$) $a_N\big|_{\max} = g$ come deve. Tale metodo permette di calcolare l'accelerazione normale e tangenziale (e di conseguenza le componenti tangenziale e normale della forza) senza calcolare il raggio di curvatura della traiettoria che ora può invece essere dedotto dall'espressione di a_N

$$\frac{1}{\rho} = \frac{g v_0 |cos\theta|}{v^3}$$

in accordo con quanto ricavato sopra.

23-050 Accelerazione Tangenziale Ed Accelerazione Normale

This § is referenced at pages:
[Never referenced.]

Dato un punto materiale in moto con equazione del moto $\mathbf{x}(t)$ determinare le componenti tangenziale e normale dell'accelerazione.

WORKED SOLUTION

Dalle formule (23.03.01), (23.03.02), (23.03.03).

$$\frac{1}{\rho} = \left|\frac{d\widehat{\mathbf{T}}}{dt}\right| \frac{1}{v}$$

si ricava

$$\mathbf{v} \equiv \frac{d\mathbf{x}}{dt} \equiv v\widehat{\mathbf{T}}$$

Il vettore $d\widehat{\mathbf{T}}/ds$ è ortogonale al versore $\widehat{\mathbf{T}}$ perché

$$0 = \frac{d}{ds}1 = \frac{d}{ds}\mathbf{T}\cdot\mathbf{T} = 2\frac{d\widehat{\mathbf{T}}}{ds}\cdot\widehat{\mathbf{T}}$$

Derivando

$$\frac{d\mathbf{v}}{dt} = v\widehat{\mathbf{T}} = \frac{dv}{dt}\widehat{\mathbf{T}} + v\frac{d\widehat{\mathbf{T}}}{dt}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\widehat{\mathbf{T}} + \frac{v^2}{\rho}\widehat{\mathbf{N}}$$

Il vettore accelerazione giace dunque nel piano definito dai versori $\widehat{\mathbf{T}}$ e $\widehat{\mathbf{N}}$ Dalla relazione sopra si deduce che l'accelerazione tangenziale è legata al cambiamento del modulo della velocità mentre quella radiale non influisce sul modulo ma solo sulla direzione. Infatti $\mathbf{a}_T$ esiste se e solo se $dv/dt \neq 0$; mentre $\mathbf{a}_N$ esiste se e solo se $1/\rho \neq 0$ cioè se e solo se $d\widehat{\mathbf{T}}/dt \neq 0$ cioè se e solo se cambia la direzione di $\mathbf{v}$. Si ha

$$\frac{1}{\rho} = \frac{|\mathbf{x}'\wedge\mathbf{x}''|}{|\mathbf{x}'|^3} = \frac{|\mathbf{v}\wedge\mathbf{a}|}{v^3} \qquad (23.09.13)$$

$$\frac{1}{\tau} = \frac{|\mathbf{x}\wedge\mathbf{x}''|}{\mathbf{x}'\cdot(\mathbf{x}''\wedge\mathbf{x}''')} \qquad (23.09.14)$$

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Si hanno inoltre le proprietà seguenti:

$$\frac{1}{\rho} = 0 \Leftrightarrow \qquad \text{retta} \quad ,$$
$$\frac{1}{\tau} = 0 \Leftrightarrow \qquad \text{curva piana} \quad .$$

La conoscenza di ρ e τ caratterizza completamente la curva a meno di uno spostamento rigido. Cioè due curve che hanno gli stessi ρ e τ sono sovrapponibili.
In termini dell'ascissa curvilinea si hanno le formule di Frenet:

$$\frac{d\widehat{\mathbf{T}}}{ds} = \frac{1}{\rho}\widehat{\mathbf{N}} \quad ,$$
$$\frac{d\widehat{\mathbf{N}}}{ds} = -\frac{1}{\rho}\widehat{\mathbf{T}} - \frac{1}{\tau}\widehat{\mathbf{B}} \quad ,$$
$$\frac{d\widehat{\mathbf{B}}}{ds} = \frac{1}{\tau}\widehat{\mathbf{N}} \quad .$$

23-051 Velocità E Accelerazione Nel Moto Armonico Bidimensionale

23-052 Velocità E Accelerazione in Coordinate Polari

23-053 Velocità Areolare

Demonstrate equation (23.04.03).

23-054 Il Cannone Sulla Collina E Quello in Basso

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|3.77||

23-055 Il Missile Che Insegue L'aereo

©|BBM, , ..., ..., ...Ed., ...|2.1.15||

23-056 Determinazione Della Traiettoria Di Una EAS

Si osserva lo sciame prodotto nell'atmosfera da un raggio cosmico primario (EAS). Sapendo la velocità con cui si muove lo sciame e osservandolo con un rivelatore avente una risoluzione temporale di 10 ns determinare la traiettoria.

23-057 L'auto Di Passaggio

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EN-PROGRESS

23-058 La Battaglia Navale

Una nave da battaglia è sul lato ovest di una piccola isola montuosa, la cui vetta è a $h = 1800\ \mathrm{m}$. Una nave nemica può avvicinarsi fino a $D = 2500\ \mathrm{m}$ dal lato est dell'isola e può sparare proiettili con una velocità iniziale di $v_0 = 250\ \mathrm{m/s}$. Se la costa orientale è, in orizzontale, a $d = 300\ \mathrm{m}$ dall vetta, quali sono le distanze dalla riva orientale in cui la nave può essere al sicuro da bombardamento della nave nemica?

23-059 Una Misura Di Accelerazione Di Gravità

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Al National Physical Laboratory in Gran Bretagna hanno eseguito una misurazione di g lanciando verticalmente in alto una sfera di vetro in un tubo a vuoto e lasciandola ricadere.

Sia Δt_I l'intervallo di tempo fra i due passaggi della sfera a un livello inferiore, Δt_S l'intervallo di tempo fra i due passaggi a un livello superiore. Posta H la distanza tra i due livelli, si dimostri che:

$$g = \frac{8H}{\Delta t_I^2 - \Delta t_S^2} \quad .$$

WORKED SOLUTION

$$\Delta t = \frac{4v_0}{g} \qquad v_S^2 - v_I^2 = -2gH$$

23-060 Conversione Di Velocità Angolari

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23-061 Moto Del Sole Nella Galassia

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23-062 La Freccia Lanciata Verso Una Ruota Che Gira

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23-063 La Ruota Della Diligenza

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DRAFT-IN-PROGRESS

23-064 Giorno Solare E Giorno Siderale

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23-065 La Pulsar

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23-066 Periodo Sinodico Dei Pianeti

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This § is referenced at pages:
[Never referenced.]

Si supponga che il moto della Terra, di Marte e di Venere attorno al Sole sia un moto circolare uniforme con centro nel Sole, che le orbite siano tutte coplanari e che il periodo di rivoluzione (periodo siderale) dei tre pianeti, sia rispettivamente $T_T = 365.26$ giorni, $T_M = 687$ giorni e $T_V = 224.7$ giorni. Determinare il periodo sinodico di Marte e Venere, cioè il periodo di tempo tra due successive congiunzioni o due successive opposizioni del pianeta. Per un pianeta interno, quale Venere, la congiunzione è definita come la circostanza in cui il pianeta, la Terra e il Sole sono allineati. Si distingue tra congiunzione inferiore, quando il pianeta è tra il Sole e la Terra, e congiunzione superiore quando il pianeta e la Terra si trovano da parti opposte rispetto al Sole. Anche per un pianeta esterno, quale Marte, sono possibili due circostanze in cui il pianeta, la Terra e il Sole sono allineati. Quando il pianeta e la Terra si trovano da parti opposte rispetto al Sole si parla di congiunzione mentre se la Terra si trova tra il Sole e il pianeta si parla di opposizione.

WORKED SOLUTION

Fissiamoci sul caso di opposizione del pianeta (gli altri casi procedono in modo del tutto analogo). Al tempo $t = 0$ il pianeta sia in opposizione. Allora detti θ_1 e θ_2 gli angoli sotto cui la Terra e il pianeta sono visti dal Sole, si ha

$$\theta_1(t = 0) = \theta_2(t = 0) = 0$$

Le equazioni del moto sono

$$\theta_1(t) = \omega_1 t \quad ,$$
$$\theta_2(t) = \omega_2 t \quad .$$

In generale si ha opposizione all'istante t tale che

$$\theta_1(t) = \theta_2(t) + 2\pi k \qquad k \text{ intero qualunque}$$

la prima opposizione si ha allora per

$$\theta_1(t) - \theta_2(t) = 2\pi \qquad (\text{supponendo } \omega_1 > \omega_2)$$

Cioè, detto t_0 il tempo al quale avviene la opposizione, per

$$(\omega_1 - \omega_2)t_0 = 2\pi \implies t_0 = \frac{2\pi}{\omega_1 - \omega_2}$$

Ne segue

$$\boxed{\frac{1}{t_0} = \frac{1}{T_1} - \frac{1}{T_2} \qquad (\text{per } T_1 < T_2)}$$

La relazione ha i limiti corretti nel caso $T_2 \gg T_1$ ($t_0 = T_1$) e nel caso $T_2 \simeq T_1$ ($t_0 \to \infty$). Con i dati del problema si trovano, per Marte e Venere, i periodi sinodici $T'_M = 2.135$ anni e $T'_V = 584$ giorni.

WORKED SOLUTION

$T'_V = 584$ giorni; $T'_M = 2.135$ anni.

23-067 Moto Circolare Non Uniforme

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23-068 La Cicloide

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23-069 Velocità Alla Superficie Terrestre

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23-070 Misura Della Velocità Della Luce

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23-071 La Cinghia Di Trasmissione

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23-072 Moto Con Accelerazione Angolare Costante

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23-073 Cinematica Di Un CD/DVD

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23-074 Misura Della Velocità Della Luce Con Il Metodo Di Fizeau

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Uno dei primi metodi per la misura della velocità della luce è il cosiddetto interferometro di Fizeau. Un fascio di luce attraversa una fenditura del bordo esterno di una ruota dentata, viaggia verso uno uno specchio e, riflesso da questo, torna sulla ruota dentata per passare attraverso una fenditura successiva della ruota dentata.

Si supponga che la ruota abbia raggio di $R = 5.0$ cm e che sul suo bordo siano stati ricavati $n = 500$ denti. Misurando la velocità della luce con lo specchio posto a $L = 500$ m dalla ruota si è ottenuto $c = 3.0 \cdot 10^8$ m/s.

- Si calcoli la velocità, angolare, supposta costante, della ruota dentata.

DRAFT - IN PROGRESS

WORKED SOLUTION

$$\omega = \frac{\pi c}{nL} = 3770 \text{ rad/s}$$

23-075 L'auto Sotto La Pioggia

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Si supponga che la pioggia cada verticalmente con velocità $v = 7.8\,\text{ms}$ rispetto al suolo. Un'auto si muove in linea retta con velocità $V = 55$ km/h. 1) Quale angolo formano con la verticale le tracce delle gocce di pioggia sul finestrino dell'auto? 2) Quale è la velocità delle gocce di pioggia osservata dall'auto.

WORKED SOLUTION

Detta $\mathbf{v'}$ la velocità delle gocce di pioggia nel Reference Frame dell'auto si ha:

$$\mathbf{v'} = \mathbf{v} - \mathbf{V}$$

Detto θ l'angolo con la verticale, considerato positivo per una traiettoria orientata all'indietro rispetto alla velocità dell'auto, ne segue:

$$\tan\theta = \frac{V}{v}$$

Se il lunotto posteriore di un auto ha un angolo di inclinazione θ_0 con la verticale questo non si bagnerà se e solo se la velocità dell'auto è sufficientemente grande che:

$$V > v \tan\theta_0$$

Si noti che la velocità della pioggia che si potrebbe calcolare usando questo problema risulterebbe inferiore della velocità vera. Infatti l'auto trascina parzialmente attorno a sé l'aria circostante.

23-076 L'aereo E Il Vento

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Un pilota vola verso est da A a B e quindi torna verso ovest in A. Sia $\mathbf{v}$ la velocità dell'aereo rispetto all'aria, $\mathbf{u}$ quella dell'aria rispetto la suolo e l la distanza tra A e B. Si supponga che l'aereo si muova a velocità costante. Se $u = 0$ mostrare che la durata del viaggio è $t_0 = 2l/v$. Supponendo che l'aria si muova verso est (o ovest) mostrare che la durata del viaggio è

$$t_E = \frac{t_0}{1 - u^2/v^2}$$

Supponendo che l'aria si muova verso nord (o sud) mostrare che la durata del viaggio è

$$t_N = \frac{t_0}{\sqrt{1 - u^2/v^2}}$$

WORKED SOLUTION

Nel caso il vento soffi verso est il modulo della velocità dell'aereo relativa al suolo vale $v'_1 = v + u$ all'andata e $v'_2 = v - u$ al ritorno. Ne segue

$$t_E = \frac{l}{v+u} + \frac{l}{v-u} = \frac{t_0}{1 - u^2/v^2} > t_0 \qquad \text{per } u < v$$

Si noti che deve essere $u < v$ perché altrimenti l'aereo non riesce a ritornare al punto A.
Nel caso il vento soffi verso nord l'aereo, per volare lungo la congiungente $A - B$, deve dirigersi, rispetto all'aria, ad una direzione che forma un angolo α verso sud rispetto alla direzione originaria

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$A-B$, per compensare la traslazione verso nord dell'aria rispetto alla quale la sua velocità è $\mathbf{v}$. L'angolo α è determinato dalla relazione

$$v_\perp + u = -v\sin\alpha + u = 0$$

In tal modo l'aereo si muove lungo la direzione $A-B$ puntando sempre la prua ad un angolo α verso sud. In altri termini il vettore velocità dell'aereo è diretto lungo la congiungente $A-B$ mentre la direzione dell'aereo (della sua prua) punta costantemente ad un angolo α rispetto a tale congiungente. La componente della velocità lungo la direzione della congiungente $A-B$ vale

$$v_\parallel = v\cos\alpha$$

Ne segue:

$$t_N = \frac{2l}{v\cos\alpha} = \frac{2l}{v\sqrt{1-u^2/v^2}} = \frac{t_0}{\sqrt{1-u^2/v^2}} > t_0 \qquad \text{per } u < v$$

Si noti che deve essere $u < v$ perché altrimenti l'aereo non riesce a raggiungere il punto B dal punto A. Infatti non esiste un angolo α che soddisfi la condizione $-v\sin\alpha + u = 0$ se $u > v$ e per $u = v$ si ha $v_\parallel = v\cos\alpha = 0$.

23-077 Moto Di Un Grave Dentro Un Ascensore

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Un ascensore sale con un'accelerazione $a = 1.22\ \mathrm{m/s^2}$ diretta verso l'alto. Nell'istante in cui la sua velocità vale $v_0 = 2.44\,\mathrm{ms}$ un bullone mal fissato cade dal soffitto posto ad un'altezza $h = 2.74\ \mathrm{m}$ dal suolo dell'ascensore. Calcolare il tempo che impiega il bullone ad arrivare al suolo dell'ascensore e la distanza percorsa dal bullone rispetto alla tromba dell'ascensore. Calcolare la velocità del bullone quando tocca il pavimento dell'ascensore sia rispetto al Reference Frame dell'ascensore che a quello fisso a terra. Si risolva il problema sia nel Reference Frame dell'ascensore che in quello fisso a terra e si confrontino i risultati.

WORKED SOLUTION

Si scelga l'asse x orientato verso l'alto e si scelga come origine la posizione del pavimento dell'ascensore al tempo $t = 0$ in cui il bullone si stacca. Le equazioni del moto del pavimento dell'ascensore e del bullone sono allora:

$$x_{\mathtt{asc}}(t) = v_0 t + \frac{1}{2}at^2 \tag{23.09.15}$$

→ 1278

$$x_B(t) = h + v_0 t - \frac{1}{2}gt^2 \tag{23.09.16}$$

→ 1278

Il bullone arriva la suolo all'istante t_0 in cui $x_{\mathtt{asc}}(t_0) = x_B(t_0)$:

$$h = \frac{1}{2}(g+a)t_0^2 \implies t_0 = \pm\sqrt{\frac{2h}{a+g}} = \pm 0.705\ \mathrm{s}$$

La soluzione corretta per questo problema è quella positiva. Si noti che nelle formule ricavate sopra per h e t_0 tutto va come se l'accelerazione effettiva del bullone fosse $a+g$. Il cambiamento di posizione rispetto al Reference Frame a terra vale:

$$\Delta x_B = x_B(t_0) - x_B(0) = v_0 t_0 - \frac{1}{2}gt_0^2 = -0.717\ \mathrm{m} < 0$$

Nel Reference Frame solidale con l'ascensore per l'accelerazione del bullone si ha, essendo $\mathbf{a}_T = \mathbf{a}$ e $\mathbf{a}_A = \mathbf{g}$,

$$\mathbf{a}_A = \mathbf{a}_R + \mathbf{a}_T + \mathbf{a}_C \implies \mathbf{g} = \mathbf{a}_R + \mathbf{a} \implies \mathbf{a}_R = \mathbf{g} - \mathbf{a}$$

Passando alle componenti sull'asse x si ha

$$a_R = -(g+a)$$

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L'osservatore nell'ascensore allora, essendo nulla la velocità iniziale del bullone nel Reference Frame solidale con l'ascensore, attribuirà al bullone l'equazione del moto

$$x'_B(t) = h + \frac{1}{2}a_R t^2 = h - \frac{1}{2}(g+a)t^2 \tag{23.09.17}$$

→ 1278

e calcolerà il tempo di caduta come

$$t_0 = \sqrt{\frac{-2h}{a_R}} = \sqrt{\frac{2h}{a+g}}$$

in accordo con l'osservatore a terra. Il segno meno davanti a $2h$ nella formula precedente deriva dalla scelta dell'orientazione verso l'alto dell'asse x che rende $a_R < 0$. Per il cambiamento di posizione si ha dalle equazioni del moto (23.09.15), (23.09.16), (23.09.17):

$$\Delta x'_B = x'_B(t_0) - x'_B(0) = -h \quad ,$$

$$\Delta x_{\mathsf{asc}} = v_0 t_0 + \frac{1}{2}a t_0^2 \quad ,$$

$$\Delta x_B = v_0 t_0 - \frac{1}{2}g t_0^2 \quad .$$

che soddisfano come deve alla

$$\Delta x_B = \Delta x'_B + \Delta x_{\mathsf{asc}}$$

Il calcolo della velocità si può effettuare usando la relazione

$$\boxed{v^2 = v_0^2 + 2a(x - x_0)}$$

Applicando la relazione sopra all'osservatore solidale con l'ascensore si ottiene:

$$v'^2 = 0 - 2(a+g)(\Delta x'_B) = 2h(a+g) = \left(\frac{2h}{t_0}\right)^2 \implies v' = -\frac{2h}{t_0} = -7.77\,\mathrm{ms}$$

Mentre applicandola al Reference Frame solidale con il suolo si ha:

$$v^2 = v_0^2 - 2(g)(\Delta x_B) = (v_0 - g t_0)^2 \implies v = v_0 - g t_0 = -4.47\,\mathrm{ms}$$

In entrambi i casi si è scelto il segno corretto estraendo la radice quadrata. La legge di composizione delle velocità

$$\boxed{\mathbf{v} = \mathbf{v}' + \mathbf{v}_T} \tag{23.09.18}$$

→ 1250

se si usa la relazione

$$v_T = v_0 + a t_0 = 3.30\,\mathrm{ms}$$

diventa

$$v = v' + v_T = v' + v_0 + a t_0$$

che è correttamente verificata usando i valori calcolati di v, v_T e v'.

23-078 La Barca Che Attraversa Il Fiume

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|4.79||

Una barca può muoversi alla velocità di $6.44\ \mathrm{km/h}$, rispetto all'acqua, e la corrente del fiume ha velocità di $3.22\ \mathrm{km/h}$. 1) In quale direzione deve vogare se vuole raggiungere la riva opposta nel punto di fronte a quello di partenza? 2) Se il fiume è largo $6.44\ \mathrm{km}$ quanto tempo impiega ad attraversarlo? 3) Quanto tempo impiega a remare $3.22\ \mathrm{km}$ più a valle e quindi nuovamente verso il punto di partenza? 4) Quanto tempo impiega a remare $3.22\ \mathrm{km}$ più a monte e quindi nuovamente verso il punto di partenza? 5) In quel direzione deve vogare per raggiungere la riva opposta nel minor tempo possibile e quanto vale questo tempo?

23-079 Aberrazione Non Relativistica Della Luce Stellare (Di Bradley)

La direzione di provenienza della luce stellare è soggetta al fenomeno dell'aberrazione (scoperta da Bradley nel 1727) a seguito del quale la direzione apparente della stella cambia durante l'anno a causa del moto della Terra attorno al Sole. Supponendo che la velocità della luce rispetto al Reference Frame in cui il Sole è in quiete sia $\mathbf{c}$ e che la Terra ruoti attorno al Sole con velocità $\mathbf{v}_T$ determinare l'angolo di aberrazione, cioè l'angolo tra la direzione vera della stella e quella in cui viene osservata per effetto del moto della Terra. Sapendo che il valore massimo dell'angolo di aberrazione misurato dagli astronomi vale $\Delta\alpha = 20.5''$ determinare la velocità media della Terra nella sua orbita e il raggio dell'orbita sapendo che il periodo siderale della Terra vale $T = 365.26$ giorni

WORKED SOLUTION

Applicando la legge di composizione delle velocità alla luce della stella si ottiene

$$\mathbf{c} = \mathbf{v}_T + \mathbf{c}' \tag{23.09.19}$$

→ 1279

Sia α l'angolo tra $\mathbf{c}$ e $-\mathbf{v}_T$ e sia α' l'angolo tra $\mathbf{c}'$ e $-\mathbf{v}_T$. Moltiplicando scalarmente e vettorialmente entrambi i membri dell'equazione (23.09.19) per il vettore $-\mathbf{v}_T$ e tenendo conto delle definizioni degli angoli α e α' si ottengono le relazioni

$$\begin{aligned} c\cos\alpha &= c'\cos\alpha' - v_T \quad , \\ c\sin\alpha &= c'\sin\alpha' \end{aligned}$$

da cui si deduce:

$$\boxed{\tan\alpha' = \frac{c\sin\alpha}{c\cos\alpha + v_T}}$$

Nel caso $v_T \to 0$ la formula si riduce correttamente a $\alpha' = \alpha$ (nessuna aberrazione). In pratica la condizione è soddisfatta per $v_T \ll c|\cos\alpha|$. Ma $\cos\alpha$ varia durante l'anno per cui la condizione risulta $v_T \ll c\min|\cos\alpha|$

Sia $\Delta\alpha \equiv \alpha - \alpha'$. Dal teorema di Carnot si ha:

$$c'^2 = c^2 + v_T^2 + 2cv_T\cos\alpha \qquad \text{e} \qquad \left|\frac{v_T}{c}\right| \ll 1 \implies c' \simeq c$$

Dal teorema dei seni si ha:

$$\frac{\sin\Delta\alpha}{v_T} = \frac{\sin\alpha'}{c} = \frac{\sin\alpha}{c'} \implies \sin\Delta\alpha = \frac{v_T}{c'}\sin\alpha \simeq \frac{v_T}{c}\sin\alpha$$

Da cui il valore massimo dell'aberrazione $\Delta\alpha$ misurato dagli astronomi (per $\alpha \simeq 90°$) vale

$$\sin\Delta\alpha\Big|_{\max} = \frac{v_T}{c}$$

Ne segue $v_T = 29.8$ km/s per la Terra. Per il raggio dell'orbita terrestre si ha

$$R = \frac{v_T T}{2\pi} = 1.496{\cdot}10^{11}\ \text{m}$$

23-080 Moto Libero Osservato Da Un Reference Frame Ruotante

Un punto materiale si muove con velocità costante $\mathbf{v_0}$ rispetto al Reference Frame del Laboratorio xyz. Il punto materiale viene osservato da un Reference Frame costituito da un disco orizzontale di raggio $R = 30$ cm che ruota con velocità angolare $\omega_0 = 3$ giri/s attorno all'asse z del Reference Frame del Laboratorio. All'istante iniziale il punto si trova a distanza $r_0 = 7$ cm dall'asse ed ha velocità $v_0 = 15$ ms diretta perpendicolarmente al vettore $\mathbf{r_0}$. Determinare il modulo della velocità del punto rispetto al Reference Frame solidale col disco ruotante quando questo raggiunge il bordo del disco. Determinare l'andamento in funzione del tempo delle componenti radiali della velocità del punto materiale rispetto al Reference Frame del Laboratorio.

WORKED SOLUTION

L'equazione del moto del punto rispetto al Reference Frame del Laboratorio, con una scelta opportuna dell'orientamento degli assi xy, è:

$$x(t) = r_0 \quad ,$$
$$y(t) = v_0 t \quad .$$

Dalla relazione generale che lega le velocità di due Reference Frame xyz e $x'y'z'$, detta O l'origine del Reference Frame $x'y'z'$, si ha

$$\mathbf{v} = \mathbf{v'} + \mathbf{v}_O + \boldsymbol{\omega} \wedge \mathbf{x'}$$

si trova

$$\mathbf{v'} = \mathbf{v_0} - \boldsymbol{\omega} \wedge \mathbf{x'}$$

Detto ϕ l'angolo tra l'asse x e la posizione del punto materiale quando questo raggiunge il bordo si ha $\cos\phi = r_0/R$. Le componenti radiale e tangenziale di $\mathbf{v'}$ valgono

$$\mathbf{v}'_R = v_0 \sin\phi \quad ,$$
$$\mathbf{v}'_T = v_0 \cos\phi - \omega R \quad .$$

Ne segue il modulo della velocità $\mathbf{v'}$

$$v'^2 = v_0^2 + \omega^2 R^2 - 2\omega v_0 r_0$$

Sostituendo i valori dati, dopo aver osservato che $\omega = 2\pi\omega_0$, si trova $v' = 14.7\,\mathrm{ms}$.
Studiamo il moto del punto materiale nel Reference Frame del Laboratorio in coordinate polari. Dalle equazioni del moto in coordinate cartesiane si ricavano quelle in coordinate polari:

$$r(t) \equiv \sqrt{x^2(t) + y^2(t)} = \sqrt{r_0^2 + v_0^2 t^2} \quad ,$$

$$\theta(t) \equiv \arctan\frac{y}{x} = \arctan\frac{v_0 t}{r_0}$$

Ne seguono le espressioni

$$\dot{r}(t) = \frac{v_0^2 t}{r} \quad ,$$

$$\dot{\theta}(t) = \frac{v_0 r_0}{r^2}$$

Il modulo della velocità, calcolato usando le componenti in coordinate polari, vale

$$v^2 = \dot{r}^2(t) + r^2\dot{\theta}(t)$$

ed è uguale a v_0^2 come deve.

23-081 Il Cannoncino Sulla Giostra

Su una giostra di raggio R che ruota attorno al suo asse con velocità angolare costante Ω è sistemato un cannoncino, la cui bocca è situata esattamente sull'asse della giostra, che spara un proiettile con velocità v_0 parallela al suolo. Descrivere il moto del proiettile, dallo sparo al suo arrivo al bordo della giostra, dal punto di vista dell'osservatore solidale con la giostra e mostrare che i risultati che si ottengono sono equivalenti a quelli dell'osservatore solidale con il suolo.

WORKED SOLUTION

Si consideri il Reference Frame $\mathcal{A}$ solidale con il suolo (convenzionalmente il Reference Frame assoluto) e il Reference Frame $\mathcal{R}$ solidale con la giostra (convenzionalmente il Reference Frame relativo). Si fissino per i due Sistemi di Riferimento $\mathcal{A}$ e $\mathcal{R}$ i sistemi di coordinate xyz e $x'y'z'$, rispettivamente solidali con il suolo e con la giostra, e sia O l'origine del sistema di coordinate $x'y'z'$. I due sistemi di coordinate abbiano origine nel centro della giostra, gli assi z e z' coincidenti tra loro e con l'asse della giostra e gli assi x e x' coincidenti tra loro e con la direzione e verso del cannoncino all'istante dello sparo $t = 0$. Sia Ω la velocità angolare del Reference Frame $\mathcal{R}$ rispetto al Reference

Frame $\mathcal{A}$. Per l'osservatore solidale con il suolo il moto del proiettile è rettilineo e uniforme lungo l'asse x e il proiettile arriva dopo un tempo $\Delta t_B = R/v_0$ al bordo della giostra ($x_B = R$ e $y_B = 0$). Dal punto di vista dell'osservatore solidale con la giostra la velocità e l'accelerazione si esprimono come

$$\boxed{\mathbf{v}_A = \mathbf{v}_R + \mathbf{v}_T \implies \mathbf{v} = \{\mathbf{v}'\} + \{\mathbf{v}_O + \boldsymbol{\Omega}\wedge\mathbf{r}'\}}$$

$$\boxed{\mathbf{a}_A = \mathbf{a}_R + \mathbf{a}_T + \mathbf{a}_C \implies \mathbf{a} = \{\mathbf{a}'\} + \left\{\mathbf{a}_O + \dot{\boldsymbol{\Omega}}\wedge\mathbf{x}' + \boldsymbol{\Omega}\wedge\left(\boldsymbol{\Omega}\wedge\mathbf{x}'\right)\right\} + \{2\,\boldsymbol{\Omega}\wedge\mathbf{v}'\}} \qquad (23.09.20)$$

da cui segue

$$\begin{aligned} \mathbf{v}' &= \mathbf{v}_0 - \boldsymbol{\Omega}\wedge\mathbf{r}' \qquad \mathbf{v} = \mathbf{v}_0 \quad , \\ \mathbf{a}' &= -\,\boldsymbol{\Omega}\wedge\left(\boldsymbol{\Omega}\wedge\mathbf{r}'\right) - 2\,\boldsymbol{\Omega}\wedge\mathbf{v}' \qquad \mathbf{a} = \mathbf{0} \end{aligned}$$

La velocità angolare della giostra si esprime come

$$\boldsymbol{\Omega} = \Omega\mathbf{e}_3 = \Omega\mathbf{e}_3'$$

L'accelerazione di Coriolis diventa dunque

$$\mathbf{a}_C = 2\,\boldsymbol{\Omega}\wedge\mathbf{v}' = 2\Omega\left[\dot{r}\widehat{\mathbf{e}}_\theta - r\dot{\theta}\widehat{\mathbf{e}}_r\right]$$

Si noti che la componente radiale dell'accelerazione di Coriolis deriva da v_θ' mentre quella tangenziale deriva da v_r'
mentre l'accelerazione di trascinamento (che ha solo la componente dell'accelerazione centripeta) diventa

$$\mathbf{a}_T = \boldsymbol{\Omega}\wedge\left(\boldsymbol{\Omega}\wedge\mathbf{r}'\right) = \boldsymbol{\Omega}\left(\boldsymbol{\Omega}\cdot\mathbf{r}'\right) - \mathbf{r}'\Omega^2 = -r\Omega^2\widehat{\mathbf{e}}_r$$

Ne seguono le componenti radiale e tangenziale dell'accelerazione:

$$\mathbf{a}_r' = \left(r\Omega^2 + 2\Omega r\dot{\theta}\right)\widehat{\mathbf{e}}_r = \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\mathbf{e}}_r$$

$$\mathbf{a}_\theta' = \left(-2\Omega\dot{r}\right)\widehat{\mathbf{e}}_\theta = \left(2\dot{\theta}\dot{r} + r\ddot{\theta}\right)\widehat{\mathbf{e}}_\theta$$

L'equazione per le componenti tangenziali può essere integrata, dopo aver moltiplicato per r, ottenendo

$$r^2\dot{\theta} + r^2\Omega = cost \equiv 0 \qquad \text{vale zero per le condizioni iniziali a } t = 0$$

Ne segue, essendo r non identicamente nullo,

$$\dot{\theta} = -\Omega$$

Cioè la velocità angolare del proiettile nel Reference Frame solidale con la giostra è uguale e opposta alla velocità angolare della giostra. Lo spostamento angolare nell'intervallo di tempo Δt si trova allora dalla

$$\Delta\theta = -\Omega\Delta t$$

Usando il risultato trovato si può ricavare l'accelerazione radiale:

$$\mathbf{a}_r' = -r\Omega^2\widehat{\mathbf{e}}_r = \left(\ddot{r} - r\Omega^2\right)\widehat{\mathbf{e}}_r$$

da cui si deduce l'equazione del moto radiale $\ddot{r} = 0$. Le condizioni iniziali sono $r(t=0) = 0$ e $\dot{r}(t=0) = v_0$ (segue dalla espressione ricavata $\mathbf{v}' = \mathbf{v}_0 - \boldsymbol{\Omega}\wedge\mathbf{r}'$ quando $\mathbf{r}' = 0$). La legge del moto nel Reference Frame solidale con la giostra è dunque

$$r(t) = v_0 t$$

Anche per l'osservatore sulla giostra il proiettile raggiunge dunque il bordo della giostra dopo un intervallo di tempo $\Delta t_B = R/v_0$. Il Δt_B è lo stesso per i due Reference Frame in quanto nella Meccanica Classica gli intervalli di tempo sono assoluti e uguali per tutti gli osservatori. Lo spostamento angolare del proiettile mentre raggiunge il bordo vale dunque $\Delta\theta = -\Omega\Delta t_B = -\Omega R/v_0 < 0$ che è uguale ed opposto allo spostamento angolare della giostra nello stesso intervallo di tempo. A questo corrisponde uno spostamento ΔL misurato lungo il bordo della giostra pari a $\Delta L = R\Delta\theta = -v_b\Delta t_B < 0$ dove $v_b = \Omega R$ indica la velocità del bordo della giostra. Tale risultato

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è consistente con la descrizione dell'osservatore solidale con il suolo che vede il bordo della giostra ruotare in modo tale che, durante il tragitto del proiettile, il bordo ruota di una quantità pari a ΔL. L'analisi ha mostrato che la velocità radiale del proiettile misurata dai due osservatori è la stessa. Ciò è ovvio perché le coordinate radiali dei due Reference Frame coincidono sempre. Il valore assoluto della velocità vale costantemente v_0 per l'osservatore al suolo mentre aumenta nel tempo per l'osservatore solidale con la giostra

$$V = \sqrt{v_0^2 + r^2\Omega^2}$$

Infatti dall'equazione per $\mathbf{a}_\theta$ si ha che il proiettile è soggetto ad un'accelerazione tangenziale costante e con $\ddot{\theta} = 0$. Questo implica che $\theta(t)$ cresce linearmente col tempo. Poiché però anche $r(t)$ aumenta col tempo la componente tangenziale della velocità ($v'_\theta = r\dot{\theta}$) cresce. Si noti che $\mathbf{a}'_\theta$ deriva da v'_r ed è quindi costante, mentre $\mathbf{a}'_r$ deriva dall'accelerazione di trascinamento $\mathbf{a}_T$ e da v'_θ.

23-082 Barca Che Attraversa Il Fiume Puntando Il Timone Al Punto Di Arrivo

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|1.144||

23-083 Il Problema Della Barca

Un uomo deve attraversare un fiume di larghezza L la cui corrente si muove a velocità costante $\mathbf{u}$, con un barca che si può muovere con velocità $\mathbf{V}$ rispetto all'acqua. L'uomo può correre sulla riva con velocità $\mathbf{v}$. Quale traiettoria conviene seguire per arrivare nel più breve tempo possibile in un punto dell'altra sponda esattamente di fronte a quallo di partenza ?

23-084 Il Volo Intercontinentale

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|4.E41||

Un volo intercontinentale di $D = 4300$ km richiede nel verso da est a ovest 50 min di più che nel verso opposto. La velocità dell'aereo rispetto all'aria è di $v = 960$ km/h. Che cosa si può dire riguardo i venti in quota che esso incontra?

23-085 Pistone E Biella

23-086 ☒ La Scala Che Scivola Giù Dal Muro

Una scala lunga L è appoggiata ad un muro verticale con il piede di base a distanza D dal muro verticale.

1. Se la scala inizia a scivolare verso terra determinare l'equazione della traiettoria del punto centrale della scala.
2. Determinare l'equazione della traiettoria di un generico punto a distanza x, con $0 \leq x \leq$, dal piede di base della scala.

WORK-IN-PROGRESS

23.10 External Documents

DRAFT-IN-PROGRESS

CHAPTER 24

Galilei-Newton Mechanics of General Systems

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©|D.Morin, *Introduction to Classical Mechanics With Problems and Solutions*, ..., ..., ...Ed., TO ADD:.||*Many problems* ...
©|L.Eisenbud - On the Classical Laws of Motion|Am.J.Phys, , ..., ..., ...Ed., ... 26, 144 (1958); doi: 10.1119/1.1934608||
©|R.Weinstock - Laws of Classical Motion: ...|Am.J.Phys, , ..., ..., ...Ed., ... 29, 698 (1961); doi: 10.1119/1.193755- ...
©|J.B.Keller - Newton second law|Am.J.Phys, , ..., ..., ...Ed., ... 55, 1145 (1987); doi: 10.1119/1.15268||
©|WEB - URL |On revisiting the axiomatic foundations of Newtonian mechanics|*Nice!*|
©|WEB - URL |Jean-Marie Berthelot Mécanique des Solides Rigides |*Francese!*|

DRAFT-IN-PROGRESS

24.01 Introduction

© - QUOTE

Robert Weinstock: Am.J.Phys, , ..., ..., ...Ed., ... 29, 698 (1961); doi: 10.1119/1.1937555
We assume means for measuring time and length, and thus for measuring velocity and acceleration, as known; in brief, the kinematics of a particle relative to any given coordinate system stands available for use.
We accept the notion of a particle as a body whose linear dimensions are negligible compared with lengths we ordinarily measure; **as such it will be called point-particle**.
Our space is assumed to be euclidean, our relativity Galilean.

We assume tools and methods for measuring time and lengths, and thus for measuring velocity and acceleration, as known: the kinematics of a point-particle relative to any given Coordinate System in any given Reference Frame is known.

Space is euclidean.

Time is a universal parameter.

Galilean relativity is at work,

A **point-particle is a body whose linear dimensions are negligible compared with relevant dimensions** of our phenomena/models. we ordinarily measure; **as such it will be called point-particle**.

Except when stated otherwise, the systems will be assumed to be closed, that is contain a fixed set of particles.

- I use the term **Classical Mechanics** for the non-relativistic and non-quantum mechanics; also, the terms **Newtonian Mechanics** and|or **Galilean Mechanics** will be used as synonyms or as **Galilei-Newton Mechanics** .
- Relativistic mechanics will be normally mentioned as: **Relativistic Mechanics**.
- Quantum mechanics will be normally mentioned as: **Quantum Mechanics**.
- The term **mass** will normally mean **inertial mass**; when referring to gravitational mass the term **gravitational mass** will be normally used.
- Whenever the momentum of any vector is written without putting any subscript to indicate the pole, it is assumed that the momentum is independent on the pole.
- Tentatively, the following nomenclature will be used. Assume mass-points.

WIP - Work In Progress

- Force|Torque.
  - Force: $\mathbf{f}$, an applied vector in P.
  - Torque: $\tilde{\gamma}$, a momentum of forces independent from the pole; it is not necessarily built from a set of forces, but it may be elementary, as for ideal **Electric Dipoles** | **Magnetic Dipoles** .
  - Orbital momentum of forces with respect to the pole O: $\gamma_0 = (P-O)\wedge\mathbf{f}$.
  - Total momentum of the forces (or Total Torque) with respect to the pole O: $\gamma_0 = (P-O)\wedge\mathbf{f} + \tilde{\gamma}$.
- Linear momentum and angular momentum.
  - Linear momentum: $\mathbf{p}$, an applied vector in P.
  - Intrinsic Angular momentum or spin: $\mathbf{s}$, an intrinsic angular momentum independent from the pole; it is not necessarily built from a set of linear momenta, but it may be elementary.
  - Orbital angular momentum with respect to the pole O: $\ell_0 = (P-O)\wedge\mathbf{p}$.
  - Total angular momentum with respect to the pole O: $\ell_P = (P-O)\wedge\mathbf{p} + \mathbf{s}$.
- Force, torque and total torque as well as linear momentum, spin, and total angular momentum, form a **screw/torseur**, respectively in English/French, the object of screw theory [a].
- For a vector field:
  - sources|sinks are points where the divergence is different from zero;
  - vortexes are points where the rotor is different from zero.
- For ElectroMagnetic fields:
  - positive|negative charges are sources|sinks for electric fields;
  - currents are vortexes, as well as points where $\partial_t\mathbf{E} \neq 0$ or $\partial_t\mathbf{B} \neq 0$ are vortexes as well.
- To avoid a sloppy language, **the term *generator* will be tentatively used for sources|sinks|vortexes**.
- Beware, all this is not a universal nomenclature, as many different naming schemes and tastes do exist.

[a] See, for instance, WEB - URL

- .....

DRAFT - IN PROGRESS

24.02

Space and Time in Classical Physics

©|V.Moretti|Fondamenti di FISICA MATEMATICA|*WEB - URL* |

DRAFT-IN-PROGRESS

24.03 Modeling of mechanical systems and their evolution

- Define the **system**, read § 24.03.01.
- Determine the expressions for the **kinematical/dynamical quantities** of the system, read § 24.03.02.
- Determine the expressions for the **interactions of the system**, both internal and with the universe, read § 24.03.03.
- Determine how the system exchanges dynamical quantities with the universe: the **law of physics**.

Read § 24.03.04 for more details.

24.03.01 Systems

This § is referenced at pages:
[1289]

The definition of the **system** in physics is arbitrary, but a crucial one.

A system is well-defined whenever, for any entity, it can be unambiguously stated whether it belongs to the system or not. The complement of the system is generically called **universe**. Read also § 64.06.

It is very easy, in physics, to get things wrong due to a insufficiently precise definition/characterization of **the system**.

24.03.02 Modeling Material Systems

This § is referenced at pages:
[1289]

©|H.C. Ohanian|What is spin?|*Am.J.Phys, , ..., ..., ...Ed., ... 54 (6), June 1986*|

©|J.M.Lévy-Leblond - Nonrelativistic particles and wave equations|WEB - URL |*Commun. Math. Phys. 6 (1967) 286*|

©|M.Rivas|kinematic theory of spinning particles: classical and quantum formalism|*Extensive*|

©|D.Sen|Of Mass, Charge and Spin, the basic attributes of matter|*https://doi.org/10.13140/2.1.3419.7445*|

©|S.Forte - Spin in Quantum Field theory|WEB - URL |*Definitive*|

©|E.Castellani - Galilean Particles|WEB - URL |*Excellent detailed complete*|

©|A.Heslot - Classical mechanics and the electron spin|WEB - URL |*Am.J.Phys, , ..., ..., ...Ed., ... 51, 1096 (1983).*|

©|L.H. Thomas B.A.| I. The kinematics of an electron with an axis - 1927|*DOI:10.1080/14786440108564170*|

© - QUOTE

©|A.Heslot - Classical mechanics and the electron spin|WEB - URL |*Am.J.Phys, , ..., ..., ...Ed., ... 51, 1096 (1983).*|

... since Dirac's natural introduction of the spin and derivation of the correct value for the **gyromagnetic ratio** in relativistic quantum mechanics, it is often believed that relativity is necessarily involved in these issue. In fact, spin has specifically nothing to do with relativity or quantum mechanics, but arises from the representations of the rotation group, which already plays a role in **Galilei-Newton Mechanics** . Furthermore, the expression for the **gyromagnetic ratio** , namely, the electron charge divided by its mass, does not contain the speed of light nor Planck's constant,

DRAFT - IN - PROGRESS

so it probably has a non-relativistic, non-quantum origin.

A mass point is a geometrical point endowed with mass. It is the basic building block of all mechanics and physics. As such the mechanical concepts related to mass points are introduced at first.

The mass point, in the modern view of particle physics, corresponds to a spin-less particle.

However, for more generality, it is necessary to extend the concept.

© - QUOTE

©|H.C.Corben and P.Stehle|Classical Mechanics|*§ 78*|

...
In the definition of a particle given in Section 2, the only part of the description of a particle which could depend upon the time was taken to be its position. We now want to relax this restriction and consider a particle with an "intrinsic" angular momentum or spin, independent of the orbital motion of the particle but which may be subjected to torques and forces. In particular, we are interested in a spin of constant magnitude with which is associated a magnetic moment ...
... This particular system is interesting because it is a reasonable classical analogue to the quantum-mechanical electron of the Dirac theory. There the electron has a spin $\hbar/2$ which vanishes in the classical limit so that the analogy is not exact in the sense of the correspondence principle, but nonetheless certain features of interest emerge from a classical model.
...

24.03.02.01 Mass Points

This § is referenced at pages:
[2396]

The minimal and simplest system in **Galilei-Newton Mechanics** is the mass point, a geometrical point with an associated intrinsic property, the inertial mass, $m > 0$. In fact, in **Galilei-Newton Mechanics** massless particles do not exist, or, more precisely, even if they exist, they do not interact in any way with the rest of the universe.

Mass points are defined to only interact with the rest of the universe through forces: no torques, no other multiple interactions...

Mass points are endowed with another intrinsic property, the gravitational mass, describing the gravitational forces among particles.

It is a fundamental axiom that inertial mass and gravitational mass are coincident.

Mass points can be endowed with other intrinsic properties, generally called charges. A charged mass point (aka a point electric charge) is a mass point endowed with an additional intrinsic property, electric charge, describing electric forces. other charges, in the language of modern particle physics do exist, such as color charges, flavour charges, ... Note that the gravitational mass is one charge.

The Center-Of-Mass of any system is exactly a mass point.

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24.03.02.02 Point Particles

This § is referenced at pages:
[Never referenced.]

©|M.Rivas|kinematic theory of spinning particles: ...|*Excellent full detailed explanation*|

© - QUOTE

L.D.Landau *et al.*, *Mechanics*, ..., ..., ...Ed., WEB - URL - ISBN: 978-0-7506-2896-9.
One of the fundamental concepts of mechanics is that of a particle. By this we mean a body whose dimensions may be neglected in describing its motion. The possibility of so doing depends, of course, on the conditions of the problem concerned. For example, the planets may be regarded as particles in considering their motion about the Sun, but not in considering their rotation about their axes.

© - QUOTE

©|S.Forte - Spin in Quantum Field theory|WEB - URL ||

Both in non-relativistic and relativistic theories, spin is one of the quantum numbers that classify elementary excitations.

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©|E.Castellani - Galilean Particles|WEB - URL ||

Both in the non-quantum and quantum case, the characteristic invariant quantities one obtain by constructing the irreducible (projective) representations of the Galilei group are the mass m (as can be seen also from the relations representing the Lie brackets for the Galilei group generators), the internal energy (that is, the energy in the rest system) and a quantity which can be interpreted as an internal angular momentum or spin. It is known that the internal energy of an isolated particle is an arbitrary parameter.

Un punto materiale in **Galilei-Newton Mechanics** è un oggetto grande abbastanza da non essere un oggetto quantistico, ma piccolo abbastanza da poter essere trattato geometricamente come puntiforme, nell'ambito della sensibilità dell'esperimento. Da questo punto di vista può avere, come per un corpo rigido piccolo, una struttura interna: momento angolare intrinseco, dipolo, quadrupolo ...

In the modern view of particle physics, this amounts to generalize Newton spinless particle to particles with any spin.

It can be shown that, in all generality, particles can be classified according to their mass, internal energy and spin, according to the properties of the irreducible representations of the Galilei group.

Tale modello di un sistema è utile se e solo se le dimensioni del sistema sono trascurabili rispetto alle altre dimensioni in gioco e se gli eventuali gradi di libertà interni del sistema, ad eccezione dello spin, non intervengono in modo rilevante.

Si assume quindi che ogni particella sia caratterizzata da una massa e da un momento angolare di spin, il momento angolare della particella invariante per scelta del polo.

Tale definizione generalizza quella di punto materiale ed è utile, per esempio, per descrivere la fisica di un dipolo elettrico o magnetico ideali nonché dal fatto che le particelle elementari (pur non descrivibili, a rigore, attraverso la fisica classica) sono caratterizzate di un momento angolare intrinseco oltre che di una massa.

In ogni caso specifico occorre poi determinare:

1. il legame tra lo spin e le eventuali proprietà della particella, quali momento di dipolo elettrico o magnetico, o momenti di ordine superiore;
2. il legame cinematico (vincolo costitutivo) tra velocità e impulso;
3. il legame cinematico (vincolo costitutivo) tra velocità angolare e momento angolare, se ha senso, altrimenti si studia semplicemente la dinamica del vettore di spin;
4. la legge della Force|Torque che definiscono, per lo specifico sistema, la Force|Torque che agiscono sulla particella.

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©|M.Rivas|kinematic theory of spinning particles: classical and quantum formalism||

In **Galilei-Newton Mechanics** the simplest geometrical object is a point of mass m. Starting with massive points we can construct arbitrary systems of any mass and shape, and thus any distribution of matter. The massive point can be considered as the elementary particle of **Galilei-Newton Mechanics** . In the modern view of particle physics it corresponds to a spin-less particle. We know that there exist **spinning objects** like electrons, muons, photons, neutrinos, quarks and perhaps many others, that can be considered as elementary particles in the sense that they cannot be considered as compound systems of other objects. Even more, we do not find in Nature any spin-less elementary particles (except for the Higgs boson, as of today). It is clear that the **Galilei-Newton Mechanics** point mass does not give account of the spin structure of particles and the existence of spin is a fundamental intrinsic attribute of an elementary particle, which is lacking in **Galilei-Newton Mechanics** , but it has to be accounted for.
In quantum mechanics, Wigner work on the representations of the non-homogeneous Lorentz group provides a very precise mathematical definition of the concept of elementary particle. An elementary particle is a quantum mechanical system whose Hilbert space of pure states is the representation space of a projective unitary irreducible representation of the Poincaré group. Irreducible representations of the Poincaré group are characterized by two invariant parameters m and $\mathfrak{s}$, the mass and the spin of the system, respectively. By finding the different irreducible representations, we can obtain the quantum description of massless and massive particles of any spin.

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©|J.M.Lévy-Leblond|Non-Relativistic particles and wave equations|*WEB - URL*; *Comm. Math. Phys. 6 (1967)*, ...

This paper is devoted to a detailed study of non-relativistic particles and their properties, as described by Galilei invariant wave equations, in order to obtain a precise distinction between the specifically relativistic properties of elementary quantum mechanical systems and those which are also shared by non-relativistic systems. After having emphasized that spin, for instance, is not such a specifically relativistic effect, we construct wave equations for non-relativistic particles with any spin.

Spin is an intrinsic property of any particle. It was discovered by the Stern-Gerlach experiment. It appeared like a spinning electric charge, but nothing is actually spinning.
Also, spin is a conserved quantity, it is an intrinsic property.

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©|S.Forte - Spin in Quantum Field Theory|WEB - URL ||

We see that spin is one of the three numbers which classify non-relativistic elementary excitations, along with mass and internal energy.
...
Summarizing, both in non-relativistic and relativistic theories, spin is one of the quantum numbers that classify elementary excitations. In quantum mechanics, the state vectors of physical systems are expanded on a basis of irreducible representations of the rotation group (in the non-relativistic case) or the Lorentz group (in the relativistic case). In quantum field theory, one-particle states are, respectively, Galilei or Poincaré irreducible representations. In the relativistic case, rotations are implicitly defined by the Pauli-Lubanski vector as the subgroup of the Lorentz group which leaves the Four-Momentum invariant.

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©|R.J.Gould|The intrinsic magnetic moment of elementary particles|*WEB - URL; Am.J.Phys, , ..., ..., ...Ed., ...*

In fact, spin has specifically nothing to do with relativity or quantum mechanics, but arises from the representations of the rotation group, which already play a role in **Galilei-Newton Mechanics** . Furthermore, the expression for the **gyromagnetic ratio** , namely, the electron charge divided by its mass, does not contain the speed of light nor Planck constant, so it probably has a non relativistic, non quantum origin.

Starting with mass points one can construct arbitrary systems, of any mass, shape, properties, that is any distribution of matter. Other kinds of systems, defined as geometrical points with mass, may have additional properties, such as: electric charge, electric or magnetic **Dipole** moment, spin, electric or magnetic **Quadrupole** moments, ... (all in all generically called **particles**).

In general, a system is treated as a geometrical point whenever there is no experimental indication of a size larger than zero, within the present experimental capabilities (for instance, as of today, the electron). Moreover, whenever the system is known to be an extended system but its size is very much smaller than all other lengths relevant to the problem, it can be modeled as a point system. In general, it is possible whenever the internal structure and details are not relevant.

Many different models for physical systems do exist, some of the most common are listed below.

- A mass point, which is implicitly assumed to have no size (that is it is a geometrical point) and to be isotropic (it has zero spin). This is the mass point of **Galilei-Newton Mechanics** .
- A point charge, that is a mass point carrying a property (the electric charge) such that it is subject to the ElectroMagnetic force; it is implicitly assumed to have no size and to be isotropic (zero spin).
- A **spinning** mass point, which is a mass point with intrinsic angular momentum, that is it has no size but it is not intrinsically isotropic (spin greater than zero). No real **spinning** is implied by the definition!
- A **Dipole** : a mass point endowed with some intrinsic property defined by a vector, such as an electric or **Magnetic Dipole** moment, which is implicitly assumed to have no size and to be not isotropic; eventually it might have an electric charge and|or other charges.
- An elementary particle: a mass point with an intrinsic angular momentum whose module cannot change (as in quantum physics), which is implicitly assumed to have no size and to be not isotropic; eventually it might have an electric charge and|or other charges.
- A classical particle: a mass point endowed with an intrinsic angular momentum whose module can change, which is implicitly assumed to have no size and to be not isotropic.
- A particle: in its most general and generic sense this name will be used to refer to a point-like

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massive object, possibly with some other properties to be specified; it might possibly have an internal structure which is not taken into account, as discussed in section § 65.05.04. Particles are the recognizable units composing a system at any time. Therefore a particle can be either a point-like object or an extended object studied from distances much larger than its dimensions and whose internal structure is either ignored or modeled on top of the point-like object.

- ...

24.03.02.03 What Is Spin?

Spin is an internal degree of freedom of any particle.

24.03.02.04 Other Model Systems

- **A discrete system made of points**: mass points, **Dipoles** or particles, or others.
- A continuous system made of of mass points, **Dipoles** or particles, or others.
- A continuous fluid system made of mass points, **Dipoles** or particles, or others.
- A continuous solid system made of of mass points, **Dipoles** or particles, or others.
- A continuous elastic solid system made of of mass points, **Dipoles** or particles, or others.
- The so-called micro-polar media, solids or fluids
- A rigid-body.
- ...

When studying any system, a suitable model must be chosen. Some examples follow.

- An atom, as long as its internal state does not change, can be considered as a mass point endowed with (a possibly null) spin, that is as a particle.
- A ion can be modeled as an atom plus an electric charge.
- The Earth, in its motion around the Sun, can be considered, to a first approximation, as a rigid-body of negligible size, and modeled as a particle, that is a mass point endowed with an intrinsic angular momentum which may eventually change. When a more refined analysis is carried on one realizes that the Earth shape is deformed by tides such that the gravitational interaction between Earth and Sun cannot be simply described as the interaction of two point masses.

The mass point is a model which perfectly fits the Center-Of-Mass concept.

24.03.03 Modeling Interactions and Dynamical Quantities

This § is referenced at pages:
[1289]

Many complex systems can be modeled as made of a large number of point systems, regardless the specific model of point system one uses (such as mass points, point **Dipoles** , point particles or any others): the elementary constituents.

The laws of mechanics for the complex systems may then be often deduced from the laws of the elementary constituents. However, as a more general approach, one can assume the validity of the cardinal equations of mechanics as postulates, and use them to describe the mechanics of any system. The equations for the elementary constituents can be deduced, in any case, as particular cases of the general systems.

The two essential and preliminary steps, in order to study any system, are as follows.

- The definition of the dynamical quantities relevant to the description of the system: kinetic energy, linear momentum, angular momentum.
- The modeling of the interactions which can affect the physical system and originated by the universe, that is the definition of the mathematical form of the interaction with the universe, such as force and torque. Many systems can be described interacting via a force and|or torque only; However, other systems might require more complex interactions, such as a **Quadrupole** moment interaction. Work/power and potential energy are thus also defined.

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Note that, in general, fields also are typically present, which need to be accounted for in the modeling of the systems (interactions and dynamical quantities).

24.03.03.01 Superposition Principle of Interactions

This § is referenced at pages:
[1033]

The superposition principle of the interactions is assumed to hold: whenever any system is subject to any external Force|Torque|Interaction, the resulting interaction can be found by summing (typically a vector sum) the different Force|Torque|Interaction.

That is, Force|Torque (and other interactions) belong to a vector space, that is they add linearly according to a vector sum.

24.03.04 Synopsis - the Logical Structure of Mechanics

This § is referenced at pages:
[1289]

- Definition of the system: define which objects belongs to the systems and which of the objects do not; clearly and unambiguously define the boundary surface separating the system from the rest of the universe.
- Decide the model for the system: simple enough and detailed enough for the problems under study. Simple enough to be tractable; complex enough to be sensitive to what is of interest. Do understand which effects need to be included and which can be excluded.
  - Define the model.
    - Any point particle has strictly positive mass and greater or equal zero spin.
    - Discrete/continuous systems: always built with point particles.
  - Define the mechanical quantities for the model system: mass, linear momentum, angular momentum (including spin, if any), kinetic energy, work and potential energy. Use general properties:
    - additivity over subsystems;
    - the Center-Of-Mass;
    - general properties such as **Koënig theorem** ;
    - ....
  - Define the interactions for the model system: forces, torques, others, ... Which interactions are relevant for the problem? Which interactions can be neglected, given the task?
  - Choose a suitable Reference Frame and, in the Reference Frame, a suitable Coordinate System. Time is an absolute parameter, the same for all observers. Each observer lives in a euclidean 3D space.
  - Use general properties: the **superposition principle** , **Galilei-Invarian*** , ...
- Implement the axioms: the law of inertia and the three **Cardinal Equations** .
  - Put into the laws the mechanical quantities and the interactions and do predictions to falsify the theory.
  - Once a result is found, that is a prediction can be made, make sure it is not manifestly wrong: check dimensions, limiting cases of the parameters, plausibility of numerical results.
  - Experiments are the only check for physics.

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24.04 Dynamical Quantities

24.04.01 Properties of Dynamical Quantities

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L.D.Landau *et al.*, *Mechanics*, ..., ..., ...Ed., WEB - URL - ISBN: 978-0-7506-2896-9.
Not all integrals of the motion, however, are of equal importance in mechanics. There are some whose constancy is of profound significance, deriving from the fundamental homogeneity and isotropy of space and time. The quantities represented by such integrals of the motion are said to be conserved, and have an important common property of being additive: their values for a system composed of several parts whose interaction is negligible are equal to the sums of their values for the individual parts. - It is to this additivity that the quantities concerned owe their especial importance in mechanics. Let us suppose, for example, that two bodies interact during a certain interval of time. Since each of the additive integrals of the whole system is, both before and after the interaction, equal to the sum of its values for the two bodies separately, the conservation laws for these quantities immediately make possible various conclusions regarding the state of the bodies after the interaction, if their states before the interaction are known.

24.04.01.01 Mass

©|https://arxiv.org/abs/1501.07489|mass additivity||

Mass is an intrinsic property in **Galilei-Newton Mechanics** .
Mass of any system is, by definition, a quantity additive over sub-systems. That is, additivity of mass is the good way to define mass such that it is a useful quantity in the formulation of **Cardinal Equations** . This is an experimental fact.
In **Galilei-Newton Mechanics** mass for and isolated system is a conserved quantity if and only if the total linear momentum for an isolated system is conserved; this follows from Galilei invariance, read § 24.10.04.
The principle of mass additivity states that the mass of a composite object is the sum of the masses of its elementary components.
Mass additivity is true in **Galilei-Newton Mechanics** , but false in relativistic physics.
That mass is additive is an empirical fact, in **Galilei-Newton Mechanics** , which we use without question in constructing our physical and mathematical models of the universe.

24.04.01.02 Linear Momentum

linear momentum of any system, is, by definition, a quantity additive over sub-systems. That is, additivity of linear momentum is the good way to define linear momentum such that it is a useful quantity in the formulation of **Cardinal Equations** . This is an experimental fact.
In **Galilei-Newton Mechanics** the total linear momentum for and isolated system is a conserved quantity if and only if mass for an isolated system is conserved; this follows from Galilei invariance, read § 24.10.04.
linear momentum, **P**, is, by definition, a quantity additive over sub-systems. That is, additivity of linear momentum is the good way to define linear momentum such that it is a useful quantity in the formulation of **Cardinal Equations** . This is an experimental fact.

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Galilei Transformation

Given two Inertial Reference Frame , $\mathcal{I}$ and $\mathcal{I}'$, such that the velocity of $\mathcal{I}'$ with respect to $\mathcal{I}$ is $\mathbf{V}$, the linear momentum of any (sub-)system transform as:

$$\boxed{\mathbf{P} = \mathbf{P}' + M\mathbf{V} \qquad \textbf{Galilei Transformation}} \quad .$$

24.04.01.03 Angular Momentum

Angular momentum of any system is, by definition, a quantity additive over sub-systems. That is, additivity of angular momentum is the good way to define angular momentum such that it is a useful quantity in the formulation of **Cardinal Equations** . This is an experimental fact.

Total angular momentum with respect to any pole is given, in general, by an orbital angular momentum $\mathbf{L}$, plus an intrinsic, or spin, angular momentum, $\mathbf{S}$:

$$\boxed{\mathbf{J} = \mathbf{L} + \mathbf{S}} \quad .$$

Orbital angular momentum changes according to the usual law of change of momentum when changing the pole:

$$\boxed{\mathbf{L}_B = \mathbf{L}_A + \mathbf{x}_{AB} \wedge \mathbf{P} = \mathbf{L}_A + (\mathbf{x}_A - \mathbf{x}_B) \wedge \mathbf{P} = \mathbf{L}_A + (A - B) \wedge \mathbf{P}} \quad , \qquad (24.04.01)$$

→ 1394

dove $\mathbf{x}_{AB} \equiv \mathbf{x}_A - \mathbf{x}_B \equiv A - B$ è il vettore posizione del polo A rispetto al polo B.

It is seen from this formula that the angular momentum depends on the choice of origin except when the system is at rest as a whole $\mathbf{P} = 0$.

Spin is the part of the angular momentum which is invariant with respect to the choice of the pole.

Angular Momentum is, by definition, a quantity additive over sub-systems That is, additivity of angular momentum is the good way to define angular momentum such that it is a useful quantity in the formulation of **Cardinal Equations** . This is an experimental fact.

Galilei Transformation

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We may also derive a relation between the angular momenta in two inertial frames of reference K and K', of which the latter moves with velocity $\mathbf{V}$ relative to the former. We shall suppose that the origins in the frames K and K' coincide at a given instant. Then the radius vectors of the particles are the same in the two frames, while their velocities are related by $\mathbf{v}_a = \mathbf{v}_a' + \mathbf{V}$. Hence we have

$$\mathbf{M} = \sum_a m_a \mathbf{r}_a \times \mathbf{v}_a = \sum_a m_a \mathbf{r}_a \times \mathbf{v}_a' + \sum_a m_a \mathbf{r}_a \times \mathbf{V}.$$

The first sum on the right-hand side is the angular momentum $\mathbf{M}'$ in the frame K'; using in the second sum the radius vector of the centre of mass (8.3), we obtain

$$\mathbf{M} = \mathbf{M}' + \mu \mathbf{R} \times \mathbf{V}. \tag{9.5}$$

This formula gives the law of transformation of angular momentum from one frame to another, corresponding to formula (8.1) for momentum and (8.5) for energy.

If the frame K' is that in which the system considered is at rest as a whole, then $\mathbf{V}$ is the velocity of its centre of mass, $\mu\mathbf{V}$ its total momentum $\mathbf{P}$ relative to K, and

$$\mathbf{M} = \mathbf{M}' + \mathbf{R} \times \mathbf{P}. \tag{9.6}$$

In other words, the angular momentum $\mathbf{M}$ of a mechanical system consists of its "intrinsic angular momentum" in a frame in which it is at rest, and the angular momentum $\mathbf{R} \times \mathbf{P}$ due to its motion as a whole.

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Given two Inertial Reference Frame , $\mathcal{I}$ and $\mathcal{I}'$, such that the position of the origin of $\mathcal{I}'$, $\mathcal{O}'$, with respect to the origin of $\mathcal{I}$, $O \equiv \mathcal{O}$, is $\mathbf{R}$ and the velocity of $\mathcal{I}'$ with respect to $\mathcal{I}$ is $\mathbf{V} \equiv \dot{\mathbf{R}}$, the orbital angular momentum with respect to the pole $O \equiv \mathcal{O}$ of any (sub-)system transforms as:

$$\boxed{\mathbf{L}_0 = \mathbf{L}_0{}' + \mathbf{R}\wedge\mathbf{P} \qquad \textbf{Galilei Transformation}} \quad .$$

In fact:

$$\mathbf{r} = \mathbf{r}' + \mathbf{R} \equiv P - O \quad ,$$
$$\mathbf{v} = \mathbf{v}' + \mathbf{V} \quad ,$$
$$\mathbf{L}_0 \equiv \sum_a \mathbf{r}_a\wedge\mathbf{p}_a = \sum_a \mathbf{r}_a\wedge m_a\left(\mathbf{v}'_a + \mathbf{V}\right) = \sum_a m_a\mathbf{r}_a\wedge\mathbf{v}'_a + \sum_a m_a\mathbf{r}_a\wedge\mathbf{V} = \mathbf{L}_0{}' + M\,\mathbf{R}\wedge\mathbf{V} = \mathbf{L}_0{}' + \mathbf{R}\wedge\mathbf{P} \quad .$$

Note: $\mathbf{L}_0{}'$ is the same orbital angular momentum with respect to the same pole as $\mathbf{L}_0$, as seen from another Reference Frame.

24.04.01.04 Work

The work done on any system, $\mathcal{W}$, is, by definition, a quantity additive over sub-systems. That is, additivity of work is the good way to define work such that it is a useful quantity in the formulation of **Cardinal Equations** . This is an experimental fact.

24.04.01.05 Kinetic Energy

Kinetic energy of any system, $\mathcal{K}$, is, by definition, a quantity additive over sub-systems. That is, additivity of kinetic energy is the good way to define kinetic energy such that it is a useful quantity in the formulation of **Cardinal Equations** . This is an experimental fact.

24.04.01.06 Energy

©|L.D.Landau *et al.*, *Mechanics*, ..., ..., ...Ed., WEB - URL - ISBN: 978-0-7506-2896-9.|||

The energy of a mechanical system which is at rest as a whole is usually called its internal energy, $\mathcal{U}$. This includes the kinetic energy of the relative motion of the particles in the system and the potential energy of their interactions plus the rest masses of the constituents, read § 65.05.

Galilei Transformation of Energy

The total energy of a system moving as a whole with velocity $\mathbf{V}$ can be written as:

$$\mathcal{E} = \frac{1}{2}mV^2 + \mathcal{U} \quad . \tag{24.04.02}$$

→ 1299

Although his formula is fairly obvious, we may give a direct proof of it. Given two Inertial Reference Frame , $\mathcal{I}$ and $\mathcal{I}'$, such that the velocity of $\mathcal{I}'$ with respect to $\mathcal{I}$ is $\mathbf{V}$, the energy of any mechanical (sub-)system transform as:

$$\boxed{\mathcal{E} = \mathcal{E}' + \mathbf{V}\cdot\mathbf{P}' + \frac{1}{2}mu^2 \qquad \textbf{Galilei Transformation}} \quad .$$

If the Center-Of-Mass of the system is at rest in $\mathcal{I}'$, then: $\mathcal{E}' = \mathcal{U}' = \mathcal{U}$ and $\mathbf{P}' = 0$; therefore one finds equation (24.04.02). Note the invariance of the internal energy:

$$\boxed{\mathcal{U}' = \mathcal{U}} \quad .$$

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24.04.02
Dynamical Quantities for Point Particles

Common meaningful definitions for point particles are recalled here.

The definition of the dynamical quantities relevant to the description of other systems - linear momentum, angular momentum, kinetic energy - depend on the model system.

Point-particles, defined by a mass and a spin, $\mathbf{s}$, and located at $\mathbf{r}$, subject to forces and torques are considered in this section:

$$\boxed{\mathbf{f}[\mathbf{r}] \qquad \boldsymbol{\gamma}_0[\mathbf{r}] = \mathbf{r}\wedge f[\mathbf{r}] + \tilde{\boldsymbol{\gamma}}[\mathbf{s}] \equiv (P-O)\wedge\mathbf{f}[\mathbf{r}] + \tilde{\boldsymbol{\gamma}}[\mathbf{s}]} \quad .$$

24.04.02.01 Linear Momentum

$$\boxed{\mathbf{p} \equiv m\mathbf{v}} \quad .$$

24.04.02.02 Angular Momentum

$$\boxed{\mathbf{j}_O \equiv \boldsymbol{\ell}_O + \mathbf{s} \equiv (\mathbf{r}-\mathbf{r}_O)\wedge\mathbf{p} + \mathbf{s} \qquad \text{with respect to pole O}} \quad .$$

24.04.02.03 Work

Work Due to Displacement of the Point of Application of the Force for a mass-point

Il lavoro compiuto nello spostamento di un point particle da una configurazione posizionale iniziale $\mathbf{x}_1$ ad una configurazione posizionale finale $\mathbf{x}_2$, lungo una curva C_x, nello spazio della configurazione di posizione, da parte di una forza $\mathbf{f}$ è dato da:

$$\boxed{w \equiv \int_{C_x} \mathbf{f}\cdot \mathrm{d}\mathbf{x}} \quad ,$$

e la potenza sviluppata

$$\boxed{\pi \equiv \frac{\mathrm{d}w}{\mathrm{d}t} = \mathbf{f}\cdot\mathbf{v}} \quad .$$

Consider a force, $\mathbf{f}$, applied in P, and a fixed axis, $\hat{\mathbf{n}}$ passing by the pole O; let $\parallel$ and $\perp$ refer to the plane defined by P, O and $\hat{\mathbf{n}}$; let $\mathbf{x} = P - O$.

$$\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{x}_{\parallel} + \mathrm{d}\mathbf{x}_{\perp} \quad ,$$

$$\mathbf{f}\cdot\mathrm{d}\mathbf{x} = \mathbf{f}\cdot\mathrm{d}\mathbf{x}_{\parallel} + \mathbf{f}\cdot\mathrm{d}\mathbf{x}_{\perp} \quad ,$$

$$\mathbf{f}\cdot\mathrm{d}\mathbf{x} = \mathbf{f}\cdot\mathrm{d}\mathbf{x}_{\parallel} + \mathbf{f}\cdot(\mathrm{d}\boldsymbol{\theta}\wedge\mathbf{x}) = \mathbf{f}\cdot\mathrm{d}\mathbf{x}_{\parallel} + \mathrm{d}\boldsymbol{\theta}\cdot(\mathbf{x}\wedge\mathbf{f}) = \mathbf{f}\cdot\mathrm{d}\mathbf{x}_{\parallel} + \boldsymbol{\gamma}_0\cdot\mathrm{d}\boldsymbol{\theta} \quad .$$

So the **arm** part of the total torque on the particle is described by the $\mathbf{f}\cdot\mathrm{d}\mathbf{x}$ term.

Note that, here, $\mathrm{d}\boldsymbol{\theta}$ is the rotation around the external axis, $\hat{\mathbf{n}}$, of the point-particle, described by the external degree of freedom $\boldsymbol{\theta}$.

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Work Due to Intrinsic Rotation Caused by the Torque for a mass-point

Il lavoro compiuto nella rotazione intrinseca di un point particle dotata di una proprietà intrinseca direzionale (come un momento di dipolo), da una configurazione angolare iniziale $\tilde{\boldsymbol{\theta}}_1$ ad una configurazione angolare finale $\tilde{\boldsymbol{\theta}}_2$ (dove i $\tilde{\boldsymbol{\theta}}$ non sono vettori), lungo una curva $C_{\tilde{\theta}}$, da parte di un torque, $\tilde{\boldsymbol{\gamma}}$, indipendente dal polo, è dato da:

$$\boxed{w \equiv \int_{C_{\tilde{\theta}}} \tilde{\boldsymbol{\gamma}} \cdot \mathrm{d}\tilde{\boldsymbol{\theta}}} \quad ,$$

e la potenza sviluppata

$$\boxed{\pi \equiv \frac{\mathrm{d}w}{\mathrm{d}t} = \tilde{\boldsymbol{\gamma}} \cdot \tilde{\boldsymbol{\omega}}} \quad .$$

So the $\tilde{\boldsymbol{\gamma}} \cdot \mathrm{d}\tilde{\boldsymbol{\theta}}$ only includes the **intrinsic** (torque) part of the total torque on the particle.

Note that, here, $\mathrm{d}\tilde{\boldsymbol{\theta}}$ is the intrinsic rotation, described by the internal degree of freedom $\tilde{\boldsymbol{\theta}}$.

General mass-point

In generale il lavoro dipende dal cammino percorso C_x e $C_{\tilde{\theta}}$ e:

$$\boxed{\pi \equiv \frac{\mathrm{d}w}{\mathrm{d}t} = \mathbf{f} \cdot \mathbf{v} + \tilde{\boldsymbol{\gamma}} \cdot \tilde{\boldsymbol{\omega}}}$$

Se su un point particle agiscono più Force|Torque il lavoro totale compito sul point particle è, per definizione, la somma dei lavori compiuti dalle Force|Torque.

Per sistemi composti da più punti materiali, il lavoro è definito come la somma dei lavori compiuti su tutte i punti materiali.

General Work

In general, for every system, the correct and specific expression of work has to be defined.

24.04.02.04 Kinetic Energy

L'energia cinetica di una point particle è definita, in tutta generalità, dalla relazione

$$\boxed{k \equiv \frac{1}{2} m v v} \quad ,$$

in termini della velocità, v, e della massa, scalare positivo.

L'energia cinetica di una point particle dotata di una proprietà intrinseca direzionale (come un momento di dipolo), in tutta generalità, dalla relazione

$$\boxed{k \equiv \frac{1}{2} \mathbb{I}_{rs} \omega_r \omega_s} \quad ,$$

in termni della velocità angolare, $\boldsymbol{\omega}$, e del tensore simmetrico di inerzia, $\mathbb{I}$, positivo definito.

24.04.02.05 Potential Energy

Per una point particle dotata di una proprietà intrinseca direzionale (come un momento di dipolo), in un campo di Force|Torque conservative si può definire in tutta generalità, la differenza di energia potenziale tra due configurazioni (2) ed (1)

$$\boxed{-\mathrm{d}u[\mathbf{x}; \tilde{\boldsymbol{\theta}}] = \mathbf{f} \cdot \mathrm{d}\mathbf{x} + \tilde{\boldsymbol{\gamma}} \cdot \mathrm{d}\tilde{\boldsymbol{\theta}}} \quad . \tag{24.04.03}$$

→ 1302

In termini dell'energia potenziale la Force|Torque si esprime come

$$\boxed{\mathbf{f} = -\mathbf{grad}_{\mathbf{x}} u \qquad \tilde{\boldsymbol{\gamma}} = -\mathbf{grad}_{\tilde{\theta}} u}$$

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La funzione scalare energia potenziale è definita a meno di una costante additiva arbitraria in quanto in base alla (24.04.03) solo la differenza di energia potenziale è definita.

Il bilancio dell'energia meccanica, $e \equiv k + u$, è

$$\boxed{\Delta e = \Delta k + \Delta u = w_{\text{NC}}} \quad . \tag{24.04.04}$$

→ 1349

dove w_{NC} è il lavoro delle Force|Torque non conservative o, in generale, di tutte le forze il cui contributo non è incluso nell'energia potenziale u.

Per una mass point tale che $w_{\text{NC}} = 0$ si conserva l'energia meccanica

$$e \equiv k + u \quad .$$

24.04.03
Dynamical Quantities for General Systems

Start from a generic system made of n particles, in the continuum limit. Here the word particle must be interpreted with the general meaning of physical system with certain known physical properties and modeled as a geometrical point.

La massa, la quantità di moto, il momento angolare e l'energia cinetica del sistema sono definite come la somma delle rispettive quantità di tutte le particelle che compongono il sistema:

$$M \equiv \sum_{j=1}^{n} \mathbf{m}_j \to \int \mathbb{d}m \qquad \mathbf{P} \equiv \sum_{j=1}^{n} \mathbf{p}_j \to \int \mathbb{d}\mathbf{p} \qquad \mathbf{J} \equiv \sum_{j=1}^{n} (\mathbf{s}_j + \mathbf{l}_j) \to \int (\mathbb{d}\mathbf{s} + \mathbb{d}\mathbf{l}) \qquad \mathcal{K} \equiv \sum_{j=1}^{n} k_j \to \int \mathbb{d}k \quad .$$

Il lavoro compiuto sul sistema da un sistema di forze è definito come la somma dei lavori fatti su tutte le particelle, e la variazione di energia potenziale del sistema è definita come la variazione delle energie potenziali di tutte le particelle:

$$W[A\Gamma B] = \sum_{j=1}^{n} w_j[A\Gamma B] \qquad \Delta\Phi[AB] = \sum_{j=1}^{n} \Delta u_j[AB] \quad .$$

Note that fields also are typically present, which need to be accounted for in the modeling of the systems as they can carry dynamical quantities.

24.04.04
Center-Of-Mass

24.04.04.01 Definition

Si dice centro di massa di un sistema di particelle il punto:

$$\mathbf{X}_{\text{CM}} = \frac{\sum_j m_j \mathbf{x}_j}{M} \quad .$$

Dalla definizione segue:

$$\mathbf{V}_{\text{CM}} = \frac{\sum_j m_j \mathbf{v}_j}{M} \qquad \mathbf{A}_{\text{CM}} = \frac{\sum_j m_j \mathbf{a}_j}{M} \qquad \mathbf{P} = M\mathbf{V}_{\text{CM}} \quad .$$

The center of mass is a geometrical point which can be external to the body.

An equivalent definition for the Reference Frame of the Center-Of-Mass, which results a better one beyond **Galilei-Newton Mechanics** , is: the translating Reference Frame such that the sum of the momenta is zero.

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24.04.04.02 Some Properties

Proprietà Distributiva Del Center-Of-Mass

Dalla definizione stessa segue che il centro di massa di un sistema si può calcolare suddividendo il sistema in due sottosistemi arbitrari, la cui intersezione è nulla e la cui unione è il sistema originale, e calcolando i centri di massa dei due sottosistemi.

Il centro di massa del sistema complessivo si trova come la media pesata dei centri di massa dei due sottosistemi, usando come pesi le masse dei due sottosistemi stessi (cioè è il centro di massa dei due centri di massa assumendo la massa dei due sottositei concentrata nei due centri di massa).

Tale proprietà può essere usata anche per calcolare il centro di massa di un sistema per sottrazione, in presenza di buchi.

The demonstration is trivial from the definition of Center-Of-Mass.

I Teoremi Di Pappo-Guldino Sul Center-Of-Mass

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©|T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...|Vol. 2, 11.17 , 2nd edition||

Consider any homogeneous solid; the following results, being purely geometrical, are only true for homogeneous solids.

La superficie di un solido di rotazione, generato dalla rotazione di una linea piana attorno ad un asse che giace nel piano della linea e che non interseca la linea stessa, è data dal prodotto della lunghezza della linea per la lunghezza della circonferenza percorsa nella rotazione dal centro di massa della linea.

Il volume di un solido di rotazione, generato dalla rotazione di una superficie piana attorno ad un asse che giace nel piano della superficie e che non interseca la superficie stessa, è dato dal prodotto dell'area della superficie per la lunghezza della circonferenza percorsa nella rotazione dal centro di massa della superficie.

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24.05

Axiom 1 - the Inertia Principle

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There exists a class of Reference Frame, the Inertial Reference Frames , with respect to which a free mass-point moves with uniform rectilinear motion.

A free mass-point is a point far enough from all others objects with which it can interact. In fact, it is assumed that interactions are always produced by the **presence of others nearby objects** and that the **intensity of all interactions decreases when increasing the distance** away from such objects.

Obviously the concept of a free mass-point is an abstraction, but there are methods to verify the inertiality of a Reference Frame (for instance the **Focault Pendulum** , read § 24.21.15). Consequently, the concept of Reference Frame is also an abstraction.

The principle of inertia therefore defines the Reference Frames where the principles of mechanics apply.

Note that inertial frames need not have existed at all, and **the fact that they do exist is the real physical content of the First Law.**

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24.06 Axiom 2 - the Cardinal Equations

The Cardinal equations characterizes how a system interact with the universe.
Consider any system made of arbitrary objects.
Assume any interaction satisfies the **superposition principle** .

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24.06.01
Second Newton Law

Per una particella puntiforme dotata di quantità di moto $\mathbf{p}$ e momento angolare intrinseco $\mathbf{s}$, lo spin, se $\mathbf{f}$ è la forza totale che agisce sulla particella, e $\boldsymbol{\gamma}$ il momento totale che agisce sulla particella, con polo la posizione, P, della particella stessa, si ha:

$$\boxed{\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \qquad \boldsymbol{\gamma} = \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t}} \quad . \tag{24.06.01}$$

→ 1305

Tale enunciato generalizza la seconda legge della dinamica al caso in cui la particella puntiforme oltre ad essere dotata di quantità di moto è dotata anche di un momento angolare intrinseco, lo spin. Tale contributo è fondamentale, ad esempio, nella trattazione dei dipoli elettrici e magnetici o nella trattazione delle particelle elementari.

From the conceptual point of view, equation (24.06.01) defines the total Force|Torque acting on the point particle. Using a suitable Force|Torque law predictions and Falsifiability are possible.

Dalla relazione (24.06.01), fissato un polo arbitrario O, e detto P il punto in cui è posizionata la particella, si deduce la relazione equivalente

$$\boxed{\boldsymbol{\gamma}_O = \frac{\mathrm{d}\mathbf{j}_O}{\mathrm{d}t}} \quad . \tag{24.06.02}$$

In fact,

$$\mathbf{r} \equiv (P - O) \qquad \mathbf{j}_O \equiv \boldsymbol{\ell}_O + \mathbf{s} \qquad \boldsymbol{\gamma}_P = \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} \qquad \mathbf{r} \wedge \mathbf{f} = \frac{\mathrm{d}\boldsymbol{\ell}_O}{\mathrm{d}t} \qquad \boldsymbol{\gamma}_O = \boldsymbol{\gamma}_P + \mathbf{r} \wedge \mathbf{f} \quad .$$

24.06.02
First Cardinal Equation (Principle of Linear Momentum)

Let $\mathbf{F}^{(\mathrm{E})}$ be the resultant of external forces and $\mathbf{P}$ the total linear momentum of the system.

The first **Cardinal Equation** of mechanics (a postulate, in fact) reads:

$$\boxed{\mathbf{F}^{(\mathrm{E})} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}} \quad , \tag{24.06.03}$$

→ 1306 1327

that is:

$$\boxed{\mathbf{F}^{(\mathrm{E})} = 0 \qquad \Leftrightarrow \qquad \mathbf{P} = \text{constant}} \quad .$$

The total force on a system is a frame-independent quantity.

The first **Cardinal Equation** of mechanics, for closed systems, can be also written in terms of Center-Of-Mass kinematics:

$$\boxed{\mathbf{F}^{(\mathrm{E})} = M\mathbf{A}_{\mathrm{CM}}} \quad , \tag{24.06.04}$$

if and only if M is constant.

Thus, the Center-Of-Mass of any system moves as a mass point with mass equal to the total mass of the system and subject to the resultant of external forces. Equivalently, the motion of the Center-Of-Mass is only affected by the resultant of external forces.

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24.06.03
Second Cardinal Equation (Principle of Angular Momentum)

Let $\mathbf{\Gamma}_O^{(\mathrm{E})}$ be the resultant of the momenta of the external Force|Torque with respect to the pole O and and $\mathbf{J}_O^{(\mathrm{E})}$ the total angular momentum of the system with respect to the pole O.

The second **Cardinal Equation** of mechanics (a postulate, in fact) reads:

$$\boxed{\mathbf{\Gamma}_O^{(\mathrm{E})} = \frac{\mathrm{d}\mathbf{J}_O}{\mathrm{d}t} + \mathbf{v}_O \wedge \mathbf{P}} \quad . \tag{24.06.05}$$

→ 1327 1339

The equation reduces to the simpler form

$$\mathbf{\Gamma}_O^{(\mathrm{E})} = \frac{\mathrm{d}\mathbf{J}_O}{\mathrm{d}t}$$

in the case the pole is fixed in the chosen Reference Frame ($\mathbf{v}_O = 0$), in case the pole is at the Center-Of-Mass ($O = \mathrm{CM}$), or when $\mathbf{v}_O \parallel \mathbf{P}$. In all these cases, when $\mathbf{v}_O \wedge \mathbf{P} = 0$, one has:

$$\boxed{\mathbf{v}_O \wedge \mathbf{P} = 0 \implies \left\{ \mathbf{\Gamma}_O^{(\mathrm{E})} = 0 \qquad \Leftrightarrow \qquad \mathbf{J}_O = \text{constant} \right\}} \quad .$$

The total torque on a system is a frame-independent quantity.

It should be emphasized that, in the most general case, the total torque about any pole is given by the sum of the momenta of the single forces plus the sum of the momenta of all torques, schematically:

$$\boxed{\mathbf{\Gamma}_O = (\mathbf{x}_P - \mathbf{x}_O) \wedge \mathbf{f} + \tilde{\gamma}} \quad .$$

When using the Center-Of-Mass as the pole one has the second **Cardinal Equation** in the form resembling equation (24.06.03) as:

$$\boxed{\mathbf{\Gamma}_{\mathrm{CM}}^{(\mathrm{E})} = \frac{\mathrm{d}\mathbf{J}_{\mathrm{CM}}}{\mathrm{d}t}} \quad . \tag{24.06.06}$$

→ 1457

That is, the motion of any system about its Center-Of-Mass is only affected by the resultant of external Force|Torque.

24.06.04
Third Cardinal Equation (Principle of Energy)

The third **Cardinal Equation** (a postulate, in fact) is the work-energy theorem, which also includes, at variance from the first two equations, the contribution from internal interactions:

$$\boxed{\frac{\mathrm{d}W^{(\mathrm{E})}}{\mathrm{d}t} + \frac{\mathrm{d}W^{(\mathrm{I})}}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} \qquad \Leftrightarrow \qquad W^{(\mathrm{E})} + W^{(\mathrm{I})} = \Delta\mathcal{K}} \quad . \tag{24.06.07}$$

→ 1328 1330

In principle, both internal and external forces can be conservative or not. Any conservative interaction can be described either as work or as potential energy; of course one and only one of the two must be included. The third **Cardinal Equation** thus can be written as an energy balance:

$$\boxed{\Delta\mathcal{K} + \Delta\mathcal{U}^{(\mathrm{E})} + \Delta\mathcal{U}^{(\mathrm{I})} = W_{\mathrm{NC}}^{(\mathrm{E})} + W_{\mathrm{NC}}^{(\mathrm{I})} + W_{\mathrm{CNU}}^{(\mathrm{E})} + W_{\mathrm{CNU}}^{(\mathrm{I})}} \quad . \tag{24.06.08}$$

→ 1330

where NC means **non conservative** and CNU means **conservative interactions not included in potential energy**.

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24.06.05
Cardinal Equations of a Generic System

Note that the **Cardinal Equations** , seven scalar equations in total, are not, in general, equivalent to the set of all equations of motion of all particles. In general, they are a consequence of all the equations of motion of all particles, but the vice versa is not true.

It can be observed that while in some particular cases of simple systems some of the seven equations may be dependent on each other, in the case of generic systems the **Cardinal Equation** are, in general, among themselves independent. Simple cases such that less than the seven equations are are sufficient: a material point, a rigid body and any flat system.

In general, **Cardinal Equations** can be applied not only to any system but to any sub-system, while taking into account the **superposition principle** .

24.06.06
Conserved Quantities for Isolated Systems

This § is referenced at pages:
[Never referenced.]

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From the first **Cardinal Equations** for isolated system one finds two constants of motion:

$$\boxed{\mathbf{A}_{CM} = 0 \implies \mathbf{X}_{CM}[t] = \mathbf{X}^0_{CM} + t\mathbf{V}_{CM} \qquad \mathbf{X}^0_{CM} = \text{constant} \qquad \mathbf{V}_{CM} = \text{constant}} \quad , \tag{24.06.09}$$

$$\boxed{\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = 0 \implies \mathbf{P} = \text{constant}} \quad , \tag{24.06.10}$$

$\boxed{M = \text{constant}}$ from the above and from **Galilei-Invarian*** as in equation , (24.06.11)

$$\boxed{\mathbf{K} \equiv M\mathbf{X}^0_{CM} = M\mathbf{X}_{CM}[t] - t\mathbf{P} = \text{constant}} \quad . \tag{24.06.12}$$

that gives two usual constants of motion: $\mathbf{P}$ and $\mathbf{K} \propto \mathbf{X}^0_{CM}$, the sometimes called *dynamic mass moment*.

From the second **Cardinal Equations** for isolated system, whenever $\mathbf{v}_A \wedge \mathbf{P} = 0$ one finds:

$$\boxed{\frac{\mathrm{d}\mathbf{L}_A}{\mathrm{d}t} = 0 \implies \mathbf{L}_A = \text{constant}}$$

that gives the constant of motion: $\mathbf{L}_A$ for any pole such that $\mathbf{v}_A \wedge \mathbf{P} = 0$. Note that, in this second case, there is no quantity similar to $\mathbf{X}^0_{CM}$, because there is no angular variable analogous to $\mathbf{X}$. That is, **angular momentum is not the velocity of something**.

From the third **Cardinal Equation** for conservative systems one finds

$$\boxed{E = \text{constant}}$$

that gives the constant of motion mechanical energy: E. In fact, at a fundamental level, all fundamental interactions are conservative.

This gives in total the ten classical constants of motion for isolated systems:

$$\boxed{\mathbf{X}^0_{CM} \qquad \mathbf{P} \qquad \mathbf{L}_A \qquad E} \quad .$$

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24.07 Axiom 3 - the Action-Reaction Principle

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©|E.Gerjuoy|On Newton Third Law and the Conservation of Momentum, Am.J.Phys, , ..., ..., ...Ed., ... 17, 477-482 (1949)||

the Action-Reaction Principle characterizes the internal forces/torques/interactions.

An Isolated System is one system such that the internal interactions dominate the dynamics with respect to the interactions with the rest of the universe.

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E. Gerjuoy, On Newton Third Law and the Conservation of Momentum, Am.J.Phys, , ..., ..., ...Ed., ... 17, 477-482 (1949).
For this reason it is of interest to note that the conservation laws of linear and angular momentum can be justified by an argument which does not depend on special assumptions regarding the nature of the forces between the elementary particles composing material bodies. In particular, it is not necessary specifically to invoke Newton third law. Naturally this argument also involves a physical assumption. It is sufficient to assume that the internal forces in material bodies do no work in any virtual displacement which is merely an infinitesimal translation or rotation. This assumption is in many respects more in keeping with our present understanding of the nature of matter than is the assumption that material bodies consist of **Galilei-Newton Mechanics** particles.

Denote by $T_O \equiv \{\mathbf{F}, \mathbf{\Gamma}_0\}$ the set made of the total force and the total torque, with respect to the arbitrary pole O, of a system of forces.

For any system one can write the first two **Cardinal Equations** :

$$\mathbb{T}_O^{(E)} \equiv \left\{\mathbf{F}^{(E)}, \mathbf{\Gamma}_O^{(E)}\right\} = \left\{\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t},\ \mathbf{v}_0 \wedge \mathbf{P} + \frac{\mathrm{d}\mathbf{J}_O}{\mathrm{d}t}\right\} \quad ,$$

in terms of the external forces.

24.07.01 Nullity of the System of Internal Interactions

Consider a generic system and divide it conceptually into two parts, a and b.

Apply the first two **Cardinal Equations** either to the entire system and to both parts. Using the **superposition principle** one finds:

$$\left\{\mathbf{F}^{(E)}[A \oplus B], \mathbf{\Gamma}_O^{(E)}[A \oplus B]\right\} = \quad ,$$

$$\left\{\mathbf{F}^{(E)}[A], \mathbf{\Gamma}_O^{(E)}[A]\right\} + \left\{\mathbf{F}^{(E)}[B], \mathbf{\Gamma}_O^{(E)}[B]\right\} = \quad ,$$

$$= \left\{\frac{\mathrm{d}\mathbf{P}[A \oplus B]}{\mathrm{d}t},\ \mathbf{v}_0 \wedge \mathbf{P}[A \oplus B] + \frac{\mathrm{d}\mathbf{J}_O[A \oplus B]}{\mathrm{d}t}\right\} \quad ,$$

$$= \left\{\frac{\mathrm{d}\mathbf{P}[A]}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{P}[B]}{\mathrm{d}t},\ \mathbf{v}_0 \wedge \mathbf{P}[A] + \mathbf{P}[B] + \frac{\mathrm{d}\mathbf{J}_O[A] + \mathbf{J}_O[B]}{\mathrm{d}t}\right\} \quad ,$$

$$\left\{\mathbf{F}^{(E)}[A] + \mathbf{F}[B \to A], \mathbf{\Gamma}_O^{(E)}[A] + \mathbf{\Gamma}_0[B \to A]\right\} = \left\{\frac{\mathrm{d}\mathbf{P}[A]}{\mathrm{d}t},\ \mathbf{v}_0 \wedge \mathbf{P}[A] + \frac{\mathrm{d}\mathbf{J}_O[A]}{\mathrm{d}t}\right\} \quad ,$$

$$\left\{\mathbf{F}^{(E)}[B] + \mathbf{F}[A \to B], \mathbf{\Gamma}_O^{(E)}[B] + \mathbf{\Gamma}_0[A \to B]\right\} = \left\{\frac{\mathrm{d}\mathbf{P}[B]}{\mathrm{d}t},\ \mathbf{v}_0 \wedge \mathbf{P}[B] + \frac{\mathrm{d}\mathbf{J}_O[B]}{\mathrm{d}t}\right\} \quad .$$

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The generalized form of the **Action-Reaction Principle** follows:

$$\mathbb{T}^{a\to b}_{O} + \mathbb{T}^{b\to a}_{O} = 0 \quad . \tag{24.07.01}$$

This is assumed as a postulate for the mutual interaction of any pair of systems.

The generalized **Action-Reaction Principle** is equivalent to the fact that the resultant of internal forces and the resultant of internal torques is zero for any system:

$$\mathbf{F}^{(\mathtt{I})} = 0 \qquad \mathbf{\Gamma}^{(\mathtt{I})}_{\mathtt{0}} = 0 \quad . \tag{24.07.02}$$

Note that all terms like $\mathbf{\Gamma}^{\mathtt{0}}_{(\mathtt{E})}[Z]$, $\mathbf{\Gamma}^{\mathtt{0}}_{(\mathtt{I})}[Z]$ and $\mathbf{\Gamma}^{\mathtt{0}}[X \to Y]$ may include, in general, both terms of type $(\mathbf{x}_{\mathtt{P}} - \mathbf{x}_{\mathtt{0}}) \wedge \mathbf{f}$ and intrinsic torques.

Simple and neat examples in section § 36.20.10, § 36.20.11.

24.07.02
Nullity of the Virtual Work in Any Rigid Translation/Rotation

An equivalent formulation of the **Action-Reaction Principle** is the Generalized Newton third law, which assumes the homogeneity and isotropy of space.

Consider a **rigid** translation/rotation of any system made of point particles. By this we mean not only that the distance among any pair of particles does not change, but also that the angle between any vector describing physical properties of the system does not change. Thus, for instance, if there are **Dipoles** , they shall be rotated; same for **Quadrupoles** or other properties; and for the spins.

The system of internal interactions (forces, torques, interaction on higher **Multipoles** , ...) of any system is such that it does not perform any virtual work in any rigid translation or rotation of the whole system.

This statement generalizes Newton's third law and is of general validity being equivalent to the conservation of the quantity of momentum and total angular momentum of an isolated system.

Simple and neat examples in section § 36.20.10, § 36.20.11.

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24.08 Consequences of the Axioms

24.08.01
Meaning of Cardinal Equations

For a many-body system, the Cardinal equations only represent a sort of summary of the system mechanics.

In fact, in principle, the Cardinal equations might be applied to any sub-system, so that the number of possible independent equations is roughly of the order of the number of the number of degrees of freedom of the system.

They represent a useful summary, providing the motion of the Center-Of-Mass and the motion about the Center-Of-Mass. In general, they provide little information about the details of the motion, except for the overall translation and rotation.

The virial theorem, § 30.07, is another summary relation similar to the cardinal equations, providing the dynamics of the overall size of the system via the polar momentum of inertia.

24.08.02
Teorema Dello Pseudo-Lavoro

Dalla prima **Cardinal Equation** si deduce il teorema dello pseudo-lavoro:

$$\int \mathbf{F}^{(\mathrm{E})} \cdot \mathrm{d}\mathbf{X}_{\mathrm{CM}} = \frac{1}{2} M \Delta V_{\mathrm{CM}}^2 \quad ,$$

in cui la quantità a primo membro è detta pseudo-lavoro.

24.08.03
Invariance of the Power of Internal Force|Torque

Thanks to the **Action-Reaction Principle** it can be shown that the power of internal forces is invariant for changes of Reference Frame in arbitrary relative motion.

It becomes a postulate.

It is of fundamental importance in thermodynamics.

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24.08.04
Statics

Condizione necessaria, ma non sufficiente, affinché un sistema sia in equilibrio è che la forza risultante esterna e il momento risultante delle forze esterne siano nulli.

A system is in equilibrium if and only if all its sub-systems are in equilibrium.

24.08.05
Impulso Lineare E Impulso Angolare

Dalle **Cardinal Equation** si definisce l'impulso lineare:

$$\mathbf{I}_P \equiv \int_{t_0}^{t} \mathbf{F}^{(E)} \, \mathrm{d}t = \Delta\mathbf{P} = \mathbf{P}[t] - \mathbf{P}[t_0] \quad . \tag{24.08.01}$$

Dalle **Cardinal Equation** si definisce l'impulso angolare, rispetto ad un polo O, supposto in quiete:

$$\mathbf{I}_{L_O} \equiv \int_{t_0}^{t} \mathbf{\Gamma}_O^{(E)} \, \mathrm{d}t = \Delta\mathbf{L}_O = \mathbf{L}_O[t] - \mathbf{L}_O[t_0] \qquad (\text{nel caso} \quad \mathbf{v}_0 \wedge \mathbf{P} = 0) \quad . \tag{24.08.02}$$

→ 1311

Si osservi che se la relazione (24.08.02) è applicata ad un fenomeno di urto, nel quale agisce una forza molto intensa per un intervallo di tempo molto breve, non è necessario imporre la condizione $\mathbf{v}_0 \wedge \mathbf{P} = 0$ per la validità della (24.08.02) stessa. In tal caso infatti, per la definizione stessa di fenomeno impulsivo, l'integrale temporale, durante l'urto, di $\mathbf{v}_0 \wedge \mathbf{P}$ è trascurabile rispetto all'integrale temporale di $\mathbf{\Gamma}_O^{(E)}$.

In generale, l'integrale temporale va esteso dall'istante di inizio dell'urto a quello di termine dell'urto in quanto in un fenomeno di urto interessa solitamente la variazione di $\mathbf{P}$ e di $\mathbf{\Gamma}_0$ tra un istante precedente e un istante seguente l'urto.

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24.09 Two-Body Three-Body and N-Body Interactions

Physics has been remarkably successful in describing complex phenomena of many-particle systems by summing two-body interactions. However, not all complex systems can be satisfactorily described by two-body interactions.

A three-body Force|Torque|Interaction is a Force|Torque|Interaction that does not exist in a system of two objects but appears in a three-body system. In general, if the behavior of a system of more than two objects cannot be described by the two-body interactions between all possible pairs of objects, as a first approximation, the deviation is mainly due to a three-body (n-body) Force|Torque|Interaction. It is known that in some cases there must exist special Force|Torque|Interaction effects when many systems are present, effects which are not there in two-body systems.

Suppose that objects A and B are subject to a two-body Force|Torque|Interaction, $\mathbf{i}_{AB}$, in absence of any other object. If, in the presence of a third object C, the Force|Torque|Interaction between A and B cannot be described anymore by the same expression, $\mathbf{i}_{AB}$, we are in presence of a three-body Force|Torque|Interaction, triggered by the third body C. One should assume that the presence of C changes some properties of A and|or B which change the Force|Torque|Interaction between A and B.

In other words: we have a three-body Force|Torque|Interaction whenever the Force|Torque|Interaction between A and B, in the presence of a third body C, also depends on the coordinates of C: $\mathbf{i}_{AB} \to \mathbf{i}_{AB|C}$

The generalisation to n-body Force|Torque|Interaction is straightforward.

Examples

- A non elementary and simple example of a three-body force can be found in the Earth-Moon-Satellite system. The position of the satellite can be calculated from the sum of the potentials between the individual objects. However, the result of this calculation does not agree with the observed position of the satellite. This is due to the presence of a three-body force: the gravitational force between the Moon and the Earth changes the shape of the Earth creating tides, which, consequently, alters the force between the Earth and the Satellite.
- A number of experimental observations seem to indicate that three-body forces exist inside atomic nuclei. Nuclei are built of nucleons (protons and neutrons), which are composite objects. When two nucleons interact, they are modified, so that they are no longer identical to a free particle. When a third nucleon is interacting with this two-nucleon system, the system can therefore not be described by summing two-body forces between free objects.
- Consider three Polarizable molecules: the force on each one depends on the coordinates of all three molecules (three-body force) because each molecules polarises the other two.

Note that the **superposition principle** is not violated by n-body forces! The force on any object is still the superposition of the forces generated by all the other objects but this same force cannot be described by the same expression one uses in absence of other objects.

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24.10 Galilei-Invarian*

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It is a postulate that the law of interactions, forces and torques, are Galilei-Invarian* . Note that for the case of Lorentz force, which depends on the velocity, this implies that the ElectroMagnetic fields must change when changing Reference Frame.

24.10.01 Inertial Reference Frame

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Nowadays we interpret the first Newton law as defining an Inertial Reference Frame :

Inertial Reference Frames are all and only the Reference Frames such that the principle of inertia applies, that is an object not subject to any force moves with constant velocity.

As a corollary:

Inertial Reference Frames do exist.

When is an object subject to no force? Experience shows that all known real forces are due to the existence of something which is the source of the force and that the intensity of all real forces decreases when increasing the distance from the source. In fact, if the forces did not fall off rapidly we could never isolate the interactions of two bodies among those of all other bodies in the universe. Therefore:

An object is subject to no force when it is far enough from any other body source of interactions.

How can one decide whether its own Reference Frame is a inertial one? Operatively: throw force-free massive objects in various directions and observe their motion; if they move at constant speed: then the Reference Frame is an inertial one; if they do not: use their motion to correct.

© - QUOTE

©|P.W. Bridgman|Am.J.Phys, , ..., ..., ...Ed., ... 29 (1961) 32|*Excellent; several excerpts below.*|

A system of three rigid orthogonal axes fixes a Galilean frame if three force-free massive particles projected along the three axes with arbitrary velocities continue to move along the axes with uniform velocities. Our terrestrial laboratories do not constitute such a frame, but we may construct such a frame in our laboratories by measuring how three arbitrarily projected masses deviate from the requirement ... and incorporating these deviations as negative corrections into our specifications for the Galilean frame. There need be no reference to the stars, but the behavior of bodies can be relevantly described in terms of such immediately observable things as the rotation of the plane of the Foucault pendulum with respect to the earth or the deviation of a falling body from the perpendicular. Even if the rocket operator who is trying to put a satellite into orbit finds it convenient to make some of his specifications in terms of observations on the pole star, it is obvious that his apparatus must eventually be described in terrestrial terms. ... In a Galilean frame a rotating body, after it has been set into rotation and the forces disconnected, preserves the orientation of its plane of rotation in the frame and, consequently, preserves the direction of its axis of rotation.

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24.10.02
Galilei Transformation of Space-Time Coordinates

This § is referenced at pages:
[2396]

©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|||

Galilei Transformation are those transformations mapping one Inertial Reference Frame into another Inertial Reference Frame .

Galilei relativity principle: the law of physics must be the same for all inertial observers linked by **Galilei Transformation** ; that is all inertial observers are equivalent.

Consider two Inertial Reference Frames , $\mathcal{I}$ and $\mathcal{I}'$ with origins at O and O'. Let $\mathbf{X}$ the position of the origin O' with respect to the Inertial Reference Frame $\mathcal{I}$. Assume that $\mathcal{I}'$ is moving, with respect to $\mathcal{I}$, with a velocity $\mathbf{U}$. Assume that a orthonormal Cartesian Coordinates Coordinate System and a time coordinate are defined in both Inertial Reference Frames . For simplicity always use a orthonormal Cartesian Coordinates Coordinate System. Assume that the rotation matrix between the two orthonormal Cartesian Coordinates Coordinate System is $\mathbb{R}$. The rotation between the two Coordinate System, if any, is time independent and therefore the rotation matrix $\mathbb{R}$, a special orthogonal matrix in three dimensions, if any, is time independent. The spatial translation of the origins of the two Coordinate System at $t = t' = 0$ is: $\mathbf{X}_0$. The time translation of the origins of the two time scales is: t_0.

Galilei Transformation in their most general form, read:

$$\begin{cases} \boxed{\mathbf{x}' = \mathbb{R}\mathbf{x} - \mathbf{X} = \mathbb{R}\mathbf{x} - \mathbf{U}t - \mathbf{X}_0} \ , \\ \boxed{t' = t - t_0} \quad \text{different time-zones are equivalent for physics} \end{cases} \ , \quad \rightarrow 2406$$

$$\begin{cases} \boxed{\mathbf{v}' = \mathbb{R}\mathbf{v} - \mathbf{U}} \ , \quad \text{if no rotation:} \quad \mathbf{v}' = \mathbf{v} - \mathbf{U} \ , \\ \boxed{\mathbf{a}' = \mathbb{R}\mathbf{a}} \ , \quad \text{if no rotation:} \quad \mathbf{a}' = \mathbf{a} \end{cases} .$$

Galilei Transformation form a group with ten parameters:

$$\boxed{\mathbf{U} \quad \mathbf{X}_0 \quad \mathbb{R}(\text{three parameters}) \quad t_0} \ .$$

24.10.03
ElectroMagnetism and Galilei Transformation

This § is referenced at pages:
[2396, 2488]

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|||

Only consider ElectroMagnetism in vacuum, in order to avoid a privileged Reference Frame given by matter.

ElectroMagnetism is not invariant with respect to **Galilei Transformation** , because **Maxwell Equations** for ElectroMagnetic waves implies that ElectroMagnetic waves have a fixed speed, $c \equiv 1/\sqrt{\varepsilon_0 \mu_0}$, unequivocally determined. So they cannot be valid in any Reference Frame if **Galilei Transformation** (in particular the law of addiction of velocities) are valid.

Therefore the following possibilities arise:

- c is measured with respect to some privileged Reference Frame: the so-called **ether frame**;
- ElectroMagnetism is wrong or, at least, is only valid in the ether Reference Frame;
- **Galilei-Newton Mechanics** is wrong.

To the physicists of the end of XIX century: the latter could be hardly accepted after centuries of successes for **Galilei-Newton Mechanics** : remember, for instance, the prediction of the existence of

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Neptune and its discovery as predicted. The second option seemed to be, anyway false, due to the many correct predictions of ElectroMagnetism.

It should be noted that at the end of the XIX century all known waves had a medium to propagate on as traveling oscillations in the material medium. It was therefore natural to think of Electric|Magnetic fields as strains in an invisible jelly-like unknown medium: the **ether**.

Therefore people started to look for the ether frame, trying to detect the **ether wind**.

See section § 58.04.01, § 61.03.05.

24.10.04
Linear Momentum and Mass Conservation With Galilei-Invarian*

This § is referenced at pages:
[1296, 1296, 1600, 1600]

It can be shown that, if linear momentum is conserved for an isolated system in some Inertial Reference Frame , it is also conserved in any other Inertial Reference Frame if and only if mass is also conserved.

See, for a deeper discussion, § 60.01.03.03, after the introduction of relativistic physics.

24.10.05
Other Transformation Laws

Kinetic energy: **Koënig theorem** (24.11.05) shows that only the Center-Of-Mass term changes.

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24.11 Dynamical Quantities With Respect to the SdRCM

The Center-Of-Mass Reference Frame, say $\mathcal{I}'$, for any system is defined as the Reference Frame translating with the Center-Of-Mass without rotation. With no loss of generality it can be assumed that the axes of the Laboratory system and the Center-Of-Mass system are parallel and the origin is at the Center-Of-Mass. In general it is a non-Inertial Reference Frame , unless the Center-Of-Mass moves with constant velocity.

In general:

$$\boxed{\mathbf{V}'_{\text{CM}} = 0 \qquad \boldsymbol{\omega} = 0} \quad .$$

More specifically:

$$\boxed{\mathbf{X}'_{\text{CM}} = 0 \qquad \boldsymbol{\omega} = 0 \qquad \mathbf{e}'_k \parallel \mathbf{e}_k} \quad .$$

Koënig theorems express the dynamical quantities of a system as the sum of two quantities: one associated to the Center-Of-Mass as a mass point having the mass of the full system; the other associated to the motion relative to the Center-Of-Mass.

In the Center-Of-Mass Reference Frame, by definition, one has:

$$\mathbf{x}' \equiv \mathbf{x} - \mathbf{X}_{\text{CM}} \quad ,$$

$$\mathbf{v}' \equiv \mathbf{v} - \mathbf{V}_{\text{CM}} \qquad \text{pure translation} \quad ,$$

$$\sum_k^{\dots} \mathbf{p}'_k = 0 \qquad \text{definition of the [glo:CMRF]Center-of-Momenta Reference Frame} \quad ,$$

$$\sum_k^{\dots} m_k \mathbf{x}'_k = 0 \qquad \text{origin of } \mathcal{I}' \text{ at the Center-Of-Mass} \quad ,$$

$$\sum_k^{\dots} m_k \mathbf{x}'_k = ...M \qquad \text{position of the Center-Of-Mass with respect to } \mathcal{I}' \quad .$$

24.11.01 Koënig Theorem for Linear Momentum

Koënig theorem for Linear Momentum is just the well-known statement that the linear momentum relative to the Center-Of-Mass is, by definition, zero:

$$\boxed{\mathbf{P} = \mathbf{P}_{\text{CM}} + \mathbf{P}' = M\mathbf{V}_{\text{CM}} + 0 \qquad \mathbf{P}' = 0} \quad , \tag{24.11.01}$$

and the total linear momentum corresponds to the Center-Of-Mass linear momentum.

24.11.02 Koënig Theorem for Angular Momentum

This § is referenced at pages:
[1321, 1326, 1609]

Koënig theorem for Angular Momentum states that il momento angolare di un sistema rispetto ad un polo arbitrario O [1] può essere decomposto in una parte legata al moto del centro di massa, $\mathbf{L}_{\text{CM}}$, (momento angolare orbitale) e una parte legata al moto relativo al centro di massa con polo il Center-Of-Mass stesso,

[1] There are two conventions for the definition of angular momentum. The first convention uses the relative velocity between the particle and the pole while the second convention uses the absolute velocity of the particle. This second convention is more in line with the general treatment of the topic of system of applied vectors. Either conventions is perfectly valid, but some results differ, depending on the definition used. Examples of books that define angular momentum using the relative velocity convention include Greenwood (1977; 1988) while examples of books that define angular momentum using the absolute velocity convention include Synge and Griffith (1959), O'Reilly (2001) and Thornton and Marion (2004).

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$\mathbf{L}'$, (momento angolare intrinseco o di spin).

$$\boxed{\mathbf{L}_O = \mathbf{L}_{CM} + \mathbf{L}' = (\mathbf{X}_{CM} - \mathbf{x}_O)\wedge\mathbf{P} + \mathbf{L}'} \quad . \tag{24.11.02}$$

→ 1317 1442 1604

Note that, if any particle has spin, this does not enter the theorem, as the spin is independent of the pole.

La decomposizione esprime il fatto che il momento angolare rispetto ad un polo arbitrario può essere espresso come somma del momento angolare che il sistema avrebbe se fosse concentrato nel centro di massa più il momento angolare rispetto al centro di massa.

24.11.02.01 Theorems on Angular Momentum

Il momento angolare intrinseco può essere espresso in modo diversi, ma equivalenti, tramite la posizione relativa al centro di massa e la velocità relativa al centro di massa, assumendo che il polo e origine of the moving frame coincidano entrambi con il Center-Of-Mass:

$$\boxed{\mathbf{L}' = \sum_k \mathbf{x}'_k\wedge\mathbf{p}_k = \sum_k \mathbf{x}_k\wedge\mathbf{p}'_k = \sum_k \mathbf{x}'_k\wedge\mathbf{p}'_k \qquad \text{CM} \equiv \text{pole} \equiv \mathcal{O}'} \quad . \tag{24.11.03}$$

Comunque, dalla relazione (24.11.02) discende il fatto fondamentale che il momento angolare nel Reference Frame del Center-Of-Mass non dipende dalla scelta del polo in quanto in tale Reference Frame, per definizione, $\mathbf{P}' = 0$:

$$\boxed{\mathbf{L}_{\mathcal{O}'} = \mathbf{L}' \qquad \text{in the [glo:CMRF]Center-of-Momenta Reference Frame, for any pole } \mathcal{O}' \text{ fixed in the moving frame}} \quad , \tag{24.11.04}$$

so that there is no ambiguity in the notation $\mathbf{L}'$ as the angular momentum in the [glo:CMRF]Center-of-Momenta Reference Fr regardless of the pole fixed in the moving [glo:CMRF]Center-of-Momenta Reference Frame. Obviously it is convenient to place the pole at the Center-Of-Mass.

24.11.03 Koënig Theorem for Kinetic Energy

This § is referenced at pages:
[1609, 2576]

Koënig theorem for Kinetic Energy states that l'energia cinetica di un sistema di particelle può essere decomposta in una parte legata al moto del centro di massa e una parte legata al moto relativo al centro di massa. Si consideri la velocità relativa al centro di massa di ogni particella, $\mathbf{v}'_k \equiv \mathbf{v}_k - \mathbf{V}_{CM}$. Si osservi che tale velocità è la velocità della particella vista da un Reference Frame solidale con il centro di massa e con gli assi paralleli a quelli del Reference Frame originale. Si ha allora

$$\boxed{\mathcal{K} = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}\sum_k m_k {\mathbf{v}'_k}^2} \quad . \tag{24.11.05}$$

→ 1315 1443 1604

The above decomposition, is often said to decompose the total kinetic energy into the sum of a translational part plus a rotational part.

La decomposizione esprime il fatto che l'energia cinetica può essere espressa come la somma dell'energia cinetica che il sistema avrebbe se fosse concentrato nel centro di massa più l'energia cinetica rispetto al centro di massa.

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24.11.03.01 Theorems on Kinetic Energy

Per un sistema di n particelle con massa m_k, l'energia cinetica relativa al centro di massa, $\mathcal{K}'$, può essere espressa tramite la relazione

$$\mathcal{K}' = \frac{1}{2}\sum_{k=1}^{n} m_k v_k'^{\,2} = \frac{1}{2}\sum_{i,j=1;i\neq j}^{n} \frac{m_i m_j v_{ij}^2}{2M} \quad . \tag{24.11.06}$$

dove $\mathbf{v}_{ij} \equiv \mathbf{v}_i - \mathbf{v}_j$ è la velocità della particella i relativa alla particella j e il fattore due a denominatore corregge per il doppio conteggio nella doppia sommatoria.

24.11.04
Separation of Rotational Motion Around the Center-Of-Mass

©|J.Jellinek and D. H. Li, Phys. Rev. Lett., 1989, 62 241||*Very clear explanation.*|

©|H.Essén|Eur. J. Phys., 1993, Vol. 14, Issue 5, Article number 002, Pages 201-205|*WEB - URL* |

The separation of rotational motion around the Center-Of-Mass is not trivial. See the references for the formulation of a general prescription for separation of the energy of the overall rotation from that of the **internal/vibrational** motion in any isolated N-body (mass points) system, without imposing any restriction on the degree of its floppiness or making any assumption regarding its symmetry.

By defining a **rigid-body** instantaneous angular velocity, a vector $\boldsymbol{\omega}^{\mathrm{rb}}[t]$, satisfying the equation

$$\boxed{\mathbf{L} \equiv \mathbb{I}[t]\cdot\boldsymbol{\omega}^{\mathrm{rb}}[t]} \quad .$$

The meaning of $\boldsymbol{\omega}^{\mathrm{rb}}[t]$, which is defined by the above equation uniquely, is easily understood: it is the angular velocity with which the non-rigid system of total angular momentum $\mathbf{L}$ and tensor of inertia $\mathbb{I}[t]$ would rotate if at instant t it were a true rigid-body. Or in other words, at any time t we can associate with the non-rigid system a rigid-body rotating with angular velocity $\boldsymbol{\omega}^{\mathrm{rb}}[t]$ and having the same $\mathbf{L}$ and $\boldsymbol{\omega}^{\mathrm{rb}}[t]$ as the non-rigid system; the motion of this rigid-body represents the instantaneous overall rotation of the non-rigid system. Thus by assigning $\boldsymbol{\omega}^{\mathrm{rb}}[t]$ to a non-rigid system we separate out its overall or **rigid-body** part rotation. This means that a different rigid-body corresponds to our non-rigid system at different times. The kinetic energy is then:

$$\boxed{\mathcal{K} = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{rb}}[t]\cdot\mathbf{L}} \quad .$$

UNDER CONSTRUCTION IN PROGRESS

24.12 Mechanics of a Two-Particle System

This § is referenced at pages:
[Never referenced.]

24.12.01 Reduction of the Two-Particle Problem

Suppose that two particles are subject to two-body forces. Ignore torques, if any, and only consider forces.

Il moto relativo al centro di massa può essere separato da quello del centro di massa. Il moto del centro di massa è descritto dalla prima equazione cardinale in termini delle sole forze esterne. Detta $\mathbf{f}_{12}$ la forza che la particella 1 esercita sulla particella 2, e $\mathbf{f}_{21}$ la forza che la particella 2 esercita sulla particella 1 il moto interno del sistema, cioè, il moto relativo delle due particelle, può essere separato dal moto del Centro di Massa. Infatti:

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{f}_{21} + \mathbf{F}_1^{\mathrm{E}} \Longrightarrow m_2 m_1 \ddot{\mathbf{r}}_1 = m_2\mathbf{f}_{21} + m_2\mathbf{F}_1^{\mathrm{E}} \quad ,$$
$$m_2\ddot{\mathbf{r}}_2 = \mathbf{f}_{12} + \mathbf{F}_2^{\mathrm{E}} \Longrightarrow m_1 m_2 \ddot{\mathbf{r}}_2 = m_1\mathbf{f}_{12} + m_1\mathbf{F}_2^{\mathrm{E}} \quad .$$

The first **Cardinal Equation** follows by summing the two equations on the left side. The equation of the relative motion follows by subtracting the equations on the right side:

$$\boxed{\mu\left(\ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1\right) \equiv \mu\left(\mathbf{a}_2 - \mathbf{a}_1\right) \equiv \mu\mathbf{a} = \mathbf{f}_{12} + \mu\left(\frac{\mathbf{F}_2^{\mathrm{E}}}{m_2} - \frac{\mathbf{F}_1^{\mathrm{E}}}{m_1}\right)} \quad , \qquad (24.12.01)$$

→ 1635

dove μ,

$$\boxed{\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2}}$$

indica la massa ridotta della coppia.

The results for a two-body problem can be obtained by considering the motion of a fictitious particle, of mass equal to the reduced mass, whose position is to be interpreted as the relative position of the two original particles and the force is the force between the two particles. In this way the problem of two particles is reduced to two independent problems for one particle, that is the two equations of motions have been decoupled into two independent equations for two different fictitious particles: the Center-Of-Mass and the **reduced particle** with mass equal to the reduced mass, position vector given by the relative position of one particle with respect to the other and force equal to the internal force between the two particles. In general, this reduction is not possible when more than two particles are present, even if the Center-Of-Mass can be always separated.

Note that, whenever $\mathbf{F}_1^{\mathrm{E}} = \mathbf{F}_2^{\mathrm{E}} = 0$ the system is isolated, but the system be isolated $\mathbf{F}_1^{\mathrm{E}} = -\mathbf{F}_2^{\mathrm{E}} \neq 0$.

Read § 24.13 for the basics of the three-body dynamics.

PRELIMINARY PROOFS

24.12.02
Jacobi Coordinates for a Two-Particle System

This § is referenced at pages:
[Never referenced.]

The Jacobi coordinates for a two-particle system are usually defined as:

$$\boldsymbol{\rho}_1 \equiv \mathbf{R}_{\text{CM}} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \quad ,$$
$$\boldsymbol{\rho}_2 \equiv \mathbf{r} \equiv (\mathbf{r}_2 - \mathbf{r}_1) \quad . \tag{24.12.02}$$

Sometimes the definition is made more general by introducing the factor k_2 to be chosen in the most appropriate way:

$$\boldsymbol{\rho}_1 \equiv \mathbf{R}_{\text{CM}} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \quad ,$$
$$\boldsymbol{\rho}_2 \equiv \mathbf{r} \equiv \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{k_2} \quad . \tag{24.12.03}$$

→ 1320

Unless stated otherwise the choice $k_2 = 1$ will be made.

The conjugate momenta can be calculated in the usual way from the kinetic energy which will be assumed to contain only a quadratic part and assuming that the potential energy in velocity independent. Read § **??**.

The inverse transformations of (24.12.03) are:

$$\mathbf{r}_1 = \mathbf{R}_{\text{CM}} - k_2\frac{m_2\mathbf{r}}{m_1 + m_2} \quad ,$$
$$\mathbf{r}_2 = \mathbf{R}_{\text{CM}} + k_2\frac{m_1\mathbf{r}}{m_1 + m_2} \quad . \tag{24.12.04}$$

→ 1320

From equations (24.12.04) one can also find the position and velocity with respect to the [glo:CMRF]Center-of-Momenta Refe

$$\mathbf{r}'_1 \equiv \mathbf{r}_1 - \mathbf{R}_{\text{CM}} = -k_2\frac{m_2\mathbf{r}}{m_1 + m_2} \quad ,$$
$$\mathbf{r}'_2 \equiv \mathbf{r}_2 - \mathbf{R}_{\text{CM}} = +k_2\frac{m_1\mathbf{r}}{m_1 + m_2} \quad , \tag{24.12.05}$$

→ 1321

and velocities

$$\mathbf{v}'_1 \equiv \mathbf{v}_1 - \mathbf{V}_{\text{CM}} = -k_2\frac{m_2\dot{\mathbf{r}}}{m_1 + m_2} \quad ,$$
$$\mathbf{v}'_2 \equiv \mathbf{v}_2 - \mathbf{V}_{\text{CM}} = +k_2\frac{m_1\dot{\mathbf{r}}}{m_1 + m_2} \quad . \tag{24.12.06}$$

→ 1321

24.12.03
Momenta of Two Point Particles

Using the methods presented in section § **??** one finds:

$$\mathbf{P} \equiv \frac{\partial\mathcal{L}}{\partial\mathbf{V}_{\text{CM}}} = (m_1 + m_2)\,\mathbf{V}_{\text{CM}} \quad ,$$
$$\mathbf{p}_r \equiv \frac{\partial\mathcal{L}}{\partial\mathbf{v}_r} = k_2^2\frac{m_1 m_2}{m_1 + m_2}\mathbf{v}_r \qquad \mathbf{v}_r \equiv \dot{\mathbf{r}} \quad . \tag{24.12.07}$$

→ 1321

Note that in the above relations, the mass coefficients multiplying the velocities are, respectively, the total mass and the reduced mass of the (2|1) pair.

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24.12.04
Angular Momenta of Two Point Particles

The angular momentum calculated in [glo:CMRF] Center-of-Momenta Reference Frame is independent from the choice of the pole. The concept of intrinsic angular momentum is therefore meaningful.

By using the Jacobi variables one can write the orbital part of the intrinsic angular momentum of a two particle system as:

$$\boxed{\mathbf{L}' = \mathbf{L}_{2|1}} \quad .$$

In fact an observer fixed on the [glo:CMRF] Center-of-Momenta Reference Frame, such that $\mathbf{P}_{\text{CM}} = \mathbf{p}'_1 + \mathbf{p}'_2 = 0$, will write:

$$\boxed{\mathbf{L}' = \sum_{k=1}^{2} \mathbf{r}'_k \wedge \mathbf{p}'_k = \mathbf{r} \wedge \mathbf{p}_r} \quad ,$$

the latter equality coming from the relations (24.12.05), (24.12.06), , (24.12.07) with $k_2 = 1$.

Therefore, the pair of variables $\{\mathbf{r}, \mathbf{p}_r\}$ provide the same form of the angular momentum with respect to the [glo:CMRF] Center-of-Momenta Reference Frame as the standard definition.

In summary, from equation § 24.11.02:

$$\boxed{\mathbf{L} = \mathbf{L}_{\text{CM}} + \mathbf{L}' = \mathbf{L}_{\text{CM}} + \mathbf{L}_{2|1}} \quad .$$

24.12.05
Kinetic Energy of Two Point Particles

Per un sistema di due particelle con massa m_1, m_2 l'energia cinetica relativa al centro di massa, che appare nel teorema di Koënig, può essere espressa tramite la relazione

$$\boxed{\mathcal{K}' = \frac{1}{2}\mu v_{\text{R}}^2} \quad ,$$

dove $\mathbf{v}_{\text{R}} \equiv \mathbf{v}_2 - \mathbf{v}_1$ è la velocità relativa delle due particelle. Quindi il teorema di Koënig si può riscrivere nella forma

$$\boxed{\mathcal{K} = \frac{1}{2}MV_{\text{CM}}^2 + \frac{1}{2}\mu v_{\text{R}}^2} \quad , \qquad (24.12.08)$$

→ 1321

dove $M = m_1 + m_2$, $\mathbf{V}_{\text{CM}}$ è la velocità del centro di massa e $\mathbf{v}_{\text{R}} \equiv \mathbf{v}_2 - \mathbf{v}_1$ la velocità relativa delle due particelle.

Dalla relazione (24.12.08) si deduce che per due particelle isolate che compiono un urto elastico la velocità relativa è invariante.

An equivalent equation is given by:

$$\boxed{\mathcal{K} = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}_r^2}{2\mu}} \quad .$$

Therefore, the pair of variables $\{\mathbf{r}, \mathbf{p}_r\}$ provide the same form of the angular momentum with respect to the [glo:CMRF] Center-of-Momenta Reference Frame as the standard definition.

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$K = \frac{1}{2} M V^2 + \frac{1}{2} \mu v^2 \Rightarrow \vec{p} \equiv \mu \vec{v}$

$\vec{v} = \vec{v}_2 - \vec{v}_1$

$\vec{p} = \mu \vec{v}$

$$\begin{cases} \vec{v}_1 = \vec{V} - \frac{\mu}{m_1}\vec{v} \\ \vec{v}_2 = \vec{V} + \frac{\mu}{m_2}\vec{v} \end{cases}$$

$$\begin{cases} \vec{r}_1 = \vec{R}_{cm} - \frac{\mu}{m_1}\vec{r} \\ \vec{r}_2 = \vec{R}_{cm} + \frac{\mu}{m_2}\vec{r} \end{cases}$$

$$\begin{cases} \vec{r}_1^{\,*} = -\frac{\mu}{m_1}\vec{r} \\ \vec{r}_2^{\,*} = +\frac{\mu}{m_2}\vec{r} \end{cases}$$

$\vec{p}_1^{\,*} = m_1 \vec{v}_1^{\,*} = m_1 (\vec{v}_1 - \vec{V}) = m_1 \left(-\frac{\mu \vec{v}}{m_1}\right) = -\mu \vec{v}$

$\vec{p}_2^{\,*} = +\mu \vec{v}$

$m_1 \vec{r}_1^{\,*} + m_2 \vec{r}_2^{\,*} = \varnothing \equiv \vec{R}_{cm}^{\,*}$

$\vec{r}_1^{\,*} + \vec{r}_2^{\,*} = \mu \vec{r} \left(\frac{1}{m_1} - \frac{1}{m_2}\right)$

$\vec{v} \equiv \vec{v}_2 - \vec{v}_1 \equiv \vec{v}_2^{\,*} - \vec{v}_1^{\,*} \equiv \vec{v}_0^{\,*}$

$\vec{r} \equiv \vec{r}_2 - \vec{r}_1 \equiv \vec{r}_2^{\,*} - \vec{r}_1^{\,*}$

$\vec{l}^{\,*} = \vec{r}_1^{\,*} \wedge \vec{p}_1^{\,*} + \vec{r}_2^{\,*} \wedge \vec{p}_2^{\,*} = \vec{r}_1^{\,*} \wedge \vec{p}_1^{\,*} + \left(+\frac{m_1 \vec{r}_1^{\,*}}{m_2}\right) \wedge \vec{p}_1^{\,*} = \vec{r}_1^{\,*} \left(1 + \frac{m_1}{m_2}\right) \wedge \vec{p}_1^{\,*}$

$= \vec{r}_1^{\,*} \frac{\prod}{m_2} \wedge \vec{p}_1^{\,*} = m_1 \vec{r}_1^{\,*} \frac{\perp}{\mu} \wedge \vec{p}_1^{\,*} = \left(-\mu \vec{r}\right) \frac{\perp}{\mu} \wedge \left(-\mu \vec{v}\right) = \vec{r} \wedge \vec{v}$

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24.13 Mechanics of a Three-Particle System

This § is referenced at pages:
[1319]

24.13.01 Reduction of the Three-Particle Problem

Suppose that three particles are subject to two-body forces. Ignore torques, if any, and only consider forces.

The separation of the relative and Center-Of-Mass motion in a two-particle system cannot be in general extended to more than two particles. While the Center-Of-Mass motion can be always separated, the internal motions cannot, in general.

24.13.02 Jacobi Coordinates for a Three-Particle System

This § is referenced at pages:
[Never referenced.]

The Jacobi coordinates for a three-particle system are usually defined as:

$$\begin{aligned} \boldsymbol{\rho}_1 \equiv \mathbf{R}_{\text{CM}} &\equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3} \\ \boldsymbol{\rho}_2 \equiv \mathbf{r} &\equiv (\mathbf{r}_2 - \mathbf{r}_1) \quad , \\ \boldsymbol{\rho}_3 \equiv \boldsymbol{\rho} &\equiv \left(\mathbf{r}_3 - \left(\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}\right)\right) \quad . \end{aligned} \tag{24.13.01}$$

Sometimes the definition is made more general by introducing the factors k_2 and k_3 to be chosen in the most appropriate way:

$$\begin{aligned} \boldsymbol{\rho}_1 \equiv \mathbf{R}_{\text{CM}} &\equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3} \quad , \\ \boldsymbol{\rho}_2 \equiv \mathbf{r} &\equiv \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{k_2} \quad , \\ \boldsymbol{\rho}_3 \equiv \boldsymbol{\rho} &\equiv k_3\left(\mathbf{r}_3 - \left(\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}\right)\right) \quad . \end{aligned} \tag{24.13.02}$$

→ 1325

Unless stated otherwise the choice $k_2 = 1$ and $k_3 = 1$ will be made. Another used choice is:

$$k_2^2 = k_3^2 \equiv \frac{m_3}{\mu}\left(\frac{m_1 + m_2}{m_1 + m_2 + m_3}\right) \qquad \text{and} \qquad \mu^2 \equiv \frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}$$

The conjugate momenta can be calculated in the usual way from the kinetic energy which will be assumed to contain only a quadratic part and assuming that the potential energy in velocity independent. Read **§ ??**.

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The inverse transformations of (24.13.02) are:

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{R}_{\text{CM}} - k_2 \frac{m_2 \mathbf{r}}{m_1 + m_2} - \frac{m_3 \boldsymbol{\rho}}{k_3 (m_1 + m_2 + m_3)} \\
\mathbf{r}_2 &= \mathbf{R}_{\text{CM}} + k_2 \frac{m_1 \mathbf{r}}{m_1 + m_2} - \frac{m_3 \boldsymbol{\rho}}{k_3 (m_1 + m_2 + m_3)} \\
\mathbf{r}_3 &= \mathbf{R}_{\text{CM}} + \frac{\boldsymbol{\rho} (m_1 + m_2)}{k_3 (m_1 + m_2 + m_3)} \quad .
\end{aligned} \tag{24.13.03}$$

→ 1325

From equations (24.13.03) one can also find the position and velocity with respect to the [glo:CMRF]Center-of-Momenta Refe positions

$$\begin{aligned}
\mathbf{r}'_1 &\equiv \mathbf{r}_1 - \mathbf{R}_{\text{CM}} = -k_2 \frac{m_2 \mathbf{r}}{m_1 + m_2} - \frac{m_3 \boldsymbol{\rho}}{k_3 (m_1 + m_2 + m_3)} \quad , \\
\mathbf{r}'_2 &\equiv \mathbf{r}_2 - \mathbf{R}_{\text{CM}} = +k_2 \frac{m_1 \mathbf{r}}{m_1 + m_2} - \frac{m_3 \boldsymbol{\rho}}{k_3 (m_1 + m_2 + m_3)} \quad , \\
\mathbf{r}'_3 &\equiv \mathbf{r}_3 - \mathbf{R}_{\text{CM}} = +\frac{\boldsymbol{\rho} (m_1 + m_2)}{k_3 (m_1 + m_2 + m_3)} \quad ,
\end{aligned} \tag{24.13.04}$$

→ 1326

and velocities

$$\begin{aligned}
\mathbf{v}'_1 &\equiv \mathbf{v}_1 - \mathbf{V}_{\text{CM}} = -k_2 \frac{m_2 \dot{\mathbf{r}}}{m_1 + m_2} - \frac{m_3 \dot{\boldsymbol{\rho}}}{k_3 (m_1 + m_2 + m_3)} \quad , \\
\mathbf{v}'_2 &\equiv \mathbf{v}_2 - \mathbf{V}_{\text{CM}} = +k_2 \frac{m_1 \dot{\mathbf{r}}}{m_1 + m_2} - \frac{m_3 \dot{\boldsymbol{\rho}}}{k_3 (m_1 + m_2 + m_3)} \quad , \\
\mathbf{v}'_3 &\equiv \mathbf{v}_3 - \mathbf{V}_{\text{CM}} = +\frac{\dot{\boldsymbol{\rho}} (m_1 + m_2)}{k_3 (m_1 + m_2 + m_3)} \quad .
\end{aligned} \tag{24.13.05}$$

→ 1326

24.13.03
Momenta of Three Point Particles

Using the methods presented in section § ?? one finds:

$$\begin{aligned}
\mathbf{P} &\equiv \frac{\partial \mathcal{L}}{\partial \mathbf{V}_{\text{CM}}} = (m_1 + m_2 + m_3)\, \mathbf{V}_{\text{CM}} \\
\mathbf{p}_r &\equiv \frac{\partial \mathcal{L}}{\partial \mathbf{v}_r} = k_2^2 \frac{m_1 m_2}{m_1 + m_2} \mathbf{v}_r \qquad \mathbf{v}_r \equiv \dot{\mathbf{r}} \\
\mathbf{p}_\rho &\equiv \frac{\partial \mathcal{L}}{\partial \mathbf{v}_\rho} = \frac{m_3 (m_1 + m_2)}{k_3^2 (m_1 + m_2 + m_3)} \mathbf{v}_\rho \qquad \mathbf{v}_\rho \equiv \dot{\boldsymbol{\rho}} \quad .
\end{aligned} \tag{24.13.06}$$

→ 1326

Note that in the above relations, the mass coefficients multiplying the velocities are, respectively, the total mass, the reduced mass of the (2|1) pair, $\mu_{(2|1)}$, and the reduced mass of the (3|(1+2)) mass, $\mu_{(3|(1+2))}$.

24.13.04
Angular Momentum of Three Point Particles

The angular momentum calculated in [glo:CMRF]Center-of-Momenta Reference Frame is independent from the choice of the pole. The concept of intrinsic angular momentum is therefore meaningful.

By using the Jacobi variables one can write the orbital part of the intrinsic angular momentum of a three particle system as:

$$\boxed{\mathbf{L}' = \mathbf{L}_{2|1} + \mathbf{L}_{3|12}} \quad .$$

In fact an observer fixed on the [glo:CMRF]Center-of-Momenta Reference Frame, such that $\mathbf{P}_{\text{CM}} = \mathbf{p}'_1 + \mathbf{p}'_2 + \mathbf{p}'_3 = 0$, will write:

$$\boxed{\mathbf{L}' = \sum_{k=1}^{3} \mathbf{r}'_k \wedge \mathbf{p}'_k = \mathbf{r} \wedge \mathbf{p}_r + \boldsymbol{\rho} \wedge \mathbf{p}_\rho} \quad ,$$

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the latter equality coming from the relations (24.13.04), (24.13.05), , (24.13.06) with $k_2 = 1$ and $k_3 = 1$.

Therefore, the pair of variables $\{\mathbf{r}, \mathbf{p}_r\}$ and $\{\boldsymbol{\rho}, \mathbf{p}_\rho\}$ provide the same form of the angular momentum with respect to the [glo:CMRF]Center-of-Momenta Reference Frame as the standard definition.

In summary, from equation § 24.11.02:

$$\boxed{\mathbf{L} = \mathbf{L}_{\text{CM}} + \mathbf{L}' = \mathbf{L}_{\text{CM}} + \mathbf{L}_{2|1} + \mathbf{L}_{3|12}} \quad .$$

24.13.05

Kinetic Energy of Three Point Particles

Per un sistema di tre particelle con massa m_1, m_2, m_3 l'energia cinetica relativa al centro di massa, che appare nel teorema di Koënig, può essere espressa tramite la relazione

$$\mathcal{K}' = \frac{1}{2}\sum_{k=1}^{3} m_k v_k'^{\,2} = \frac{1}{2}\left[\frac{m_1 m_2 v_{12}^2 + m_2 m_3 v_{23}^2 + m_3 m_1 v_{31}^2}{m_1 + m_2 + m_3}\right] \quad . \tag{24.13.07}$$

An equivalent equation is given by:

$$\boxed{\mathcal{K} = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}_r^2}{2\mu_{(2|1)}} + \frac{\mathbf{p}_\rho^2}{2\mu_{(3|(1+2))}}} \quad .$$

Therefore, the pair of variables $\{\mathbf{r}, \mathbf{p}_r\}$ and $\{\boldsymbol{\rho}, \mathbf{p}_\rho\}$ provide the same form of the angular momentum with respect to the [glo:CMRF]Center-of-Momenta Reference Frame as the standard definition.

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24.14 Variable Mass - Open Systems

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Variable Mass Systems (open systems) refer to systems with mass entering/exiting the system, that is crossing its - well defined - boundary. Read also § 65.05.08.

Consider an open system with some mass entering, $\left(\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{\mathrm{P}} > 0\right)$, and|or exiting, $\left(\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{\mathrm{P}} < 0\right)$, the system, at various places, P, located at $\mathbf{x}_{\mathrm{P}}$, with velocities $\mathbf{v}_{\mathrm{P}}$, with respect to the given Reference Frame (absolute velocity), and velocities $\mathbf{u}_{\mathrm{P}}$, with respect to the (absolute) velocity $\mathbf{V}$ of the point P where mass enters/exits the system:

$$\boxed{\mathbf{v}_{\mathrm{P}} \equiv \mathbf{u}_{\mathrm{P}} + \mathbf{V}} \quad .$$

Note that it is assumed that the absolute Reference Frame is a inertial one and that the control surface defining the variable mass system is fixed and non deformable in this Reference Frame. In case the Reference Frame is not a inertial one, inertial forces must be added.

24.14.01 Theorem of Linear Momentum for Open Systems

The first **Cardinal Equation** (24.06.03) reads:

$$\boxed{\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \mathbf{F}^{(\mathrm{E})} + \sum_P \left(\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{\mathrm{P}} \mathbf{v}_{\mathrm{P}}\right)} \quad . \tag{24.14.01}$$

→ 1327

Note that the velocity $\mathbf{v}_{\mathrm{P}}$ is the absolute velocity and the correctness of the sign of the last term to be a thrust.

For a stationary system (which includes the notable case of isolated systems):

$$\boxed{\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = 0 \implies \mathbf{F}^{(\mathrm{E})} + \sum_P \left(\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{\mathrm{P}} \mathbf{v}_{\mathrm{P}}\right) = 0} \quad .$$

The equation (24.14.01) can be cast in the equivalent form:

$$\boxed{M[t]\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{F}^{(\mathrm{E})} + \sum_P \left(\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{\mathrm{P}} \mathbf{u}_{\mathrm{P}}\right)} \quad , \tag{24.14.02}$$

in terms of the relative velocity of the entering/exiting mass.

24.14.02 Theorem of Angular Momentum for Open Systems

Assume the case of a fixed pole, $\mathbf{v}_0 \wedge \mathbf{P} = \mathbf{0}$.

The second **Cardinal Equation** (24.06.05) reads:

$$\boxed{\frac{\mathrm{d}\mathbf{L}_0}{\mathrm{d}t} = \mathbf{\Gamma}_0^{(\mathrm{E})} + \sum_P \left(\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{\mathrm{P}} (\mathbf{x}_{\mathrm{P}} - \mathbf{x}_0) \wedge \mathbf{v}_{\mathrm{P}}\right)} \quad . \tag{24.14.03}$$

Note that the velocity $\mathbf{v}_{\mathrm{P}}$ is the absolute velocity and the correctness of the sign of the last term to be a thrust.

For a stationary system (which includes the notable case of isolated systems):

$$\boxed{\frac{\mathrm{d}\mathbf{L}_0}{\mathrm{d}t} = 0 \implies \mathbf{\Gamma}_0^{(\mathrm{E})} + \sum_P \left(\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{\mathrm{P}} (\mathbf{x}_{\mathrm{P}} - \mathbf{x}_0) \wedge \mathbf{v}_{\mathrm{P}}\right) = 0} \quad .$$

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24.14.03
Theorem of Kinetic Energy for Open Systems

The third **Cardinal Equation** (24.06.07) reads:

$$\boxed{\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = \Pi^{(\mathrm{E})} + \Pi^{(\mathrm{I})} + \frac{1}{2}\sum_P \left(\frac{\mathrm{d}M}{\mathrm{d}t}\bigg|_{\mathrm{P}} \mathbf{v}_{\mathrm{P}}^2 \right)} \quad . \tag{24.14.04}$$

Note that the velocity is the absolute velocity.

For a stationary system:

$$\boxed{\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = 0 \implies \Pi^{(\mathrm{E})} + \Pi^{(\mathrm{I})} + \frac{1}{2}\sum_P \left(\frac{\mathrm{d}M}{\mathrm{d}t}\bigg|_{\mathrm{P}} \mathbf{v}_{\mathrm{P}}^2 \right) = 0} \quad .$$

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24.15 Mechanics of a Point Particle in a non-Inertial Reference Frame

24.15.01 Inertial Reference Frame and non-Inertial Reference Frame

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24.15.02 Dynamics

Dalle leggi della dinamica e dalla legge di composizione delle accelerazioni (23.07.03) si possono ricavare le leggi della dinamica in un Reference Frame non inerziale. Se $\mathbf{f}_0$ è la forza fisica cui è soggetta una particella, la forza totale $\mathbf{f}$ di cui risente la particella in n Reference Frame non inerziale include anche i contributi delle forze inerziali: la forza di trascinamento e quella di Coriolis

$$\mathbf{f}_0 = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\mathbf{a} \quad .$$

Le forze inerziali (o fittizie) non sono legate alla presenza di corpi nelle vicinanze della particella, non soddisfano alla terza legge di Newton e non diminuiscono al crescere della distanza, diversamente da tutte le forze fisiche reali note. Dipendono in modo essenziale dal moto del Reference Frame non inerziale rispetto ad un Reference Frame inerziale.

Similarly, for a particle subject to an elementary torque, γ^0, the relative dynamics can be found from the law of transformation of angular velocities and accelerations, sections § 23.06.02.05, § 23.06.02.06, starting from **Cardinal Equations** :

$$\gamma_0 = \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \{\mathbb{I}\}\cdot\boldsymbol{\omega} \quad ,$$

where the pole in the equation above is the location of the point particle.

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24.16 Equilibrium and Stability

24.17 Some Properties of the Equation of Motion

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24.17.01 Determinism of Galilei-Newton Mechanics

The first two **Cardinal Equations** are, in general, systems of second-order differential equations in normal form; therefore their solution exists and is unique with Cauchy initial conditions.

The third **Cardinal Equation**, depending on whether it is expressed in either form (24.06.07) or (24.06.08), can be of either second-order of first order. Whenever it can be cast in the form (24.06.07) with zero right-hand side, that is, $0 = W^{(\mathrm{E})}_{\mathrm{NC}} = W^{(\mathrm{I})}_{\mathrm{NC}} = W^{(\mathrm{E})}_{\mathrm{CNU}} = W^{(\mathrm{I})}_{\mathrm{CNU}}$, it is first-order, thus greatly simplifying the mathematical solution, which is the great advantage of energy conservation from the computational point of view.

24.17.02 Mathematical Properties of the Equations of Motion

Consider mass-points subject to Force|Torque only.

24.17.02.01 Finite Force|Torque

Assume that Force|Torque are limited functions, as a function of time, but possibly with discontinuities of the first kind. Thus, impulsive phenomena are excluded here.

Consider the first **Cardinal Equation**; integration with respect to time provides continuous velocity functions and thus also continuous position functions.

Consider the second **Cardinal Equation**; integration with respect to time provides continuous angular momentum functions, assuming that the velocity of the pole is a limited function of time.

Similar considerations apply to the third equation, which can be either a second-order or a first-order system of differential equations.

24.17.02.02 Infinite Force|Torque (Impulsive Dynamics)

Assume possible impulsive phenomena on the first two **Cardinal Equations**.

Momenta and angular momenta can have discontinuities of the first-kind, in presence of impulsive forces. Positions are still continuous functions of time. Angles can be integrated from angular momenta/velocities, in general only locally; in this case they are continuous functions of time.

Read § 29.06.20, § 26.07.10, § 26.07.11.

24.18 Continuous Systems

24.18.01 Homogeneity and Isotropy

A medium is homogeneous when its physical properties do not depend on the position inside the medium.

A medium is isotropic when its physical properties at any point do not depend on the direction around that point.

For example: a liquid under hydrostatic pressure in a gravity field is isotropic but not necessarily homogeneous, because pressure and density, for instance, might depend on the height; a non-cubic crystal, is homogeneous but, possibly, not isotropic.

24.18.02 Nomenclature for Force|Torque Inside a Continuous Medium

This § is referenced at pages:
[1455]

Consider a three-dimensional material medium, with an ideal plane surface inside the medium and an ideal straight line perpendicular to the plane.

- The component of the force per unit area perpendicular to the surface is called the **normal stress** (also called pressure or tension).
- The components of the forces per unit area parallel to the surface are called **shear stresses**.
- The component of the torque along the axis is called **twisting torque**.
- The components of the torques perpendicular to the axis are called **bending torques**.

Read also § 27.

24.18.02.01 Volume Force|Torque

Le forze e i momenti delle forze agenti su un oggetto continuo (fluido o solido) possono essere classificate come Force|Torque di volume e Force|Torque di superficie, a seconda che l'intensità sia proprozionale al volume (il punto di applicazione delle forze è distribuito nell'intero volume) o alla superficie (il punto di applicazione delle forze è distribuito nell'intera superficie che forma il bordo del volume), della porizione di sistema continuo.

Supponiamo, per semplicità, di usare un Coordinate System cartesiano nel Reference Frame prescelto. Si consideri un elemento infinitesimo di volume, macroscopicamente piccolo e microscopicamente grande.

Le Force|Torque di volume sono proporzionali al volume della porzione di oggetto considerato:

$$\boxed{\mathrm{d}\mathbf{F}^{\mathrm{V}}[\mathbf{x}] \equiv \mathbf{f}^{\mathrm{V}}[\mathbf{x}]\ \mathrm{d}V} \quad ,$$

$$\boxed{\mathrm{d}\boldsymbol{\Gamma}^{\mathrm{V}}[\mathbf{x}] \equiv \boldsymbol{\gamma}^{\mathrm{V}}[\mathbf{x}]\ \mathrm{d}V = \left(\mathbf{x}\wedge\mathbf{f}^{\mathrm{V}}[\mathbf{x}] + \boldsymbol{\tau}^{\mathrm{V}}[\mathbf{x}]\right)\ \mathrm{d}V} \quad ,$$

where the fixed but arbitrary pole is not explicitly written and total torque is divided into the pole dependent/independent parts.

Instead of unit volume one can use the density per unit mass:

$$\boxed{\mathrm{d}\mathbf{F}^{\mathrm{V}} \equiv \mathbf{f}^{\mathrm{M}}\,\mathrm{d}M \equiv \rho\mathbf{f}^{\mathrm{M}}\,\mathrm{d}V \equiv \mathbf{f}^{\mathrm{V}}\,\mathrm{d}V \qquad \mathrm{d}\boldsymbol{\Gamma}^{\mathrm{V}} \equiv \boldsymbol{\gamma}^{\mathrm{M}}\,\mathrm{d}M \equiv \rho\boldsymbol{\gamma}^{\mathrm{M}}\,\mathrm{d}V \equiv \boldsymbol{\gamma}^{\mathrm{V}}\,\mathrm{d}V \qquad \mathrm{d}\mathbf{T}^{\mathrm{V}} \equiv \boldsymbol{\tau}^{\mathrm{M}}\,\mathrm{d}M \equiv \rho\boldsymbol{\tau}^{\mathrm{M}}\,\mathrm{d}V \equiv \boldsymbol{\tau}^{\mathrm{V}}\,\mathrm{d}V} \quad .$$

Volume Force|Torque include inertial forces in non-Inertial Reference Frame .

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24.18.02.02 Force|Torque Di Superficie

Le Force|Torque di superficie, che agiscono su una superficie infinitesima della frotniera del volume, hanno in generale una componente perpendicolare, normal stress e normal (twisting) torque-stress, ed una parallela alla superficie, shear stress e shear (bending) torque-stress.

La componente k della Force|Torque di superficie che agisce su una superficie infinitesima ideale di area $\mathbb{d}\mathbf{S}$ e orientata perpendicolarmente al unit vector $\widehat{\mathbf{n}}$, $\mathbb{d}\mathbf{S} \equiv \widehat{\mathbf{n}}\, \mathbb{d}\mathbf{S}$, si può esprimere come:

$$\boxed{\mathbb{d}F^{\mathsf{S}}_k[\mathbf{x}] = \mathcal{S}^{\cdot\cdot}_{kj}[\mathbf{x}]\, \mathbb{d}\mathbf{S}_j[\mathbf{x}]} \quad ,$$

$$\boxed{\mathbb{d}\Gamma^{\mathsf{S}}_k[\mathbf{x}] = \mathcal{M}^{\cdot\cdot}_{kj}[\mathbf{x}]\, \mathbb{d}\mathbf{S}_j[\mathbf{x}]} \quad ,$$

dove $\mathbb{d}\mathbf{S}_j[\mathbf{x}]$ è l'area della proiezione sull'asse j della superficie $\mathbb{d}\mathbf{S}[\mathbf{x}]$ ad $[\mathbf{x}]$.

La grandezza $\mathcal{S}^{\cdot\cdot}_{kj}$ è il **force-stress tensor** e descrive la forza per unità di superficie in direzione k che agisce su una superficie infinitesima perpendicolare alla direzione j.

La grandezza $\mathcal{M}^{\cdot\cdot}_{kj}$ è il **torque-stress tensor** e descrive la coppia per unità di superficie in direzione k che agisce su una superficie infinitesima perpendicolare alla direzione j.

Surface forces include forces due pressure and forces due to rigid-bodies in contact wit the fluid.

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24.19 Modeling Forces and Torques on Particles/Systems

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Le interazioni (forze e momenti) che agiscono su un sistema sono in generale schematizzabili in tre categorie:

1. interazioni a contatto;
2. interazioni a distanza;
3. interazioni inerziali.

As far as their dependence important cases arise:

- Force|Torque dependent on position.
- Force|Torque dependent on time.
- Force|Torque dependent on velocity.

24.19.01 Some Force|Torque Laws for Point Particles

24.19.01.01 Gravitational Force

This is one of the fundamental laws of Nature.

La forza gravitazionale attrattiva che una mass point (1) di massa gravitazionale m_1 esercita su una mass point (2) di massa gravitazionale m_2 posta ad $\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1$ vale:

$$\mathbf{f}_{12} = -G\frac{m_1 m_2}{r^2}\widehat{\mathbf{e}}_r \quad ,$$

dove $\widehat{\mathbf{e}}_r$ il unit vector diretto dalla mass point (1) verso la mass point (2).

È un assioma fondamentale della fisica relativistica che massa inerziale e massa gravitazionale siano identiche. Read § 24.20.

24.19.01.02 Reazioni Vincolari

Quando un sistema è vincolato i vincoli esercitano sul sistema delle forze, dette reazioni vincolari, necessarie a mantenere il sistema vincolato.

Le reazioni vincolari non sono, in generale, note a priori ma vanno determinate tramite le equazioni del moto; sono sempre incognite del problema.

Una point particle può essere vincolata a muoversi lungo una linea oppure una superficie; i vincoli possono essere bi-laterali (definiti da una eguaglianza) o uni-laterali (definiti da una diseguaglianza).

La reazione vincolare a cui è soggetto una point particle vincolata ad una curva si può sempre scomporre in una componente perpendicolare alla curva (componente normale) e in una componente tangente alla curva (forza di attrito).

La reazione vincolare a cui è soggetto una point particle vincolata ad una superficie si può sempre scomporre in una componente perpendicolare alla superficie (componente normale) e in una componente tangente alla superficie (forza di attrito).

Nei casi reali, qualora la reazione vincolare fornita dalla soluzione del problema con la configurazione prevista o ipotizzata, sia eccessiva rispetto alla capacità meccaniche del vincolo, il vincolo può deformarsi, permanentemente o no, o rompersi e il problema deve essere reimpostato.

Nel caso di attrito statico, la eventualmente ipotizzata condizione di staticità non può essere soddisfatta, e il problema deve essere reimpostato con la point particle in movimento.

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24.19.01.03 Reazioni Vincolari Normali - Vincoli Ideali

Normal constraints cannot do any work whenever they are fixed. On the other hand, moving normal constraints can do work: see, for instance, § 24-038.

24.19.01.04 Reazioni Vincolari Tangenti: Attrito Radente

La forza di attrito statico, $\mathbf{f}_s$, presente nel caso il point particle sia in quiete a contatto con una superficie, è diretta tangenzialmente alla superficie e, detto $N = |\mathbf{N}|$ il modulo della reazione normale, soddisfa alla relazione

$$\boxed{|\mathbf{f}_s| \leq \mu_s N \qquad \mathbf{f}_s \propto -\mathbf{v}} \quad . \tag{24.19.01}$$

→ 1334 1363

In generale la forza di attrito statico si oppone al moto (cioè alla velocità) che ci sarebbe se l'attrito statico non ci fosse.

La forza di attrito cinetico $\mathbf{f}_c$, presente nel caso il point particle sia in movimento rispetto alla superficie, è tale che

$$\boxed{\mathbf{f}_c = -\mu_c N \frac{\mathbf{v}}{v}} \tag{24.19.02}$$

→ 1334 1363

ed è quindi sempre diretta in senso opposto alla velocità relativa alla superficie.

Le relazioni (24.19.01), (24.19.02) sono relazioni tra i moduli dei vettori $\mathbf{f}_s$, $\mathbf{f}_c$ e $\mathbf{N}$, essendo le direzioni dei vettori diverse tra loro. Infatti $\mathbf{N}$ è sempre normale alla superficie del vincolo mentre $\mathbf{f}_s$ e $\mathbf{f}_c$ sono sempre tangenziali alla superficie.

Il coefficiente di attrito non dipende dall'estensione della superficie di contatto. Questo perché la superficie di contatto effettiva, a causa delle deformazioni della superficie, è in prima approssimazione indipendente dalla superficie di contatto e proporzionale alla pressione.

La forza di attrito statico è determinata dalle condizioni di equilibrio. Si suppone il sistema in equilibrio, si determina il valore della forza di attrito statico necessaria per l'equilibrio e si confronta con la relazione (24.19.01) per verificare se l'attrito statico può fornire o meno la forza richiesta affinché il point particle sia in equilibrio. Il fatto che la forza di attrito statico così determinata soddisfi alla relazione (24.19.01) è condizione necessaria e sufficiente all'equilibrio. Nel caso la relazione non sia verificata si deve reimpostare il problema tenendo conto che il sistema si mette in moto ed usare, eventualmente, la relazione (24.19.02).

Le proprietà descritte dell'attrito, sia statico che cinetico, giustificano il principio in base al quale se si trascura l'attrito in un problema di statica si agisce in favore della sicurezza in quanto l'attrito si oppone sempre al moto attuale del sistema (attrito cinetico) oppure al moto che ci sarebbe in assenza di attrito (attrito statico).

In generale deve essere

$$\boxed{\mu_c \leq \mu_s} \quad ,$$

come si dimostra per assurdo.

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24.19.01.05 Kinematic Constraints

A kinematic constrain is a constraint on the kinematics of the system to its possible motions and/ot motion of its parts. Read § 24-038, § 24-019.

24.19.01.06 Legge Di Hooke

Per deformazioni elastiche attorno ad una posizione di equilibrio si applica la legge di Hooke

$$\mathbf{f} = -k\,(\mathbf{r} - \mathbf{r}_0) = -k\Delta\mathbf{r} \quad , \tag{24.19.03}$$

dove k è la costante elastica e $\Delta\mathbf{r}$ lo spostamento dalla posizione di equilibrio.

24.19.01.07 Resistenza Del Mezzo

La resistenza al moto che incontra un point particle che si muove entro un fluido dipende dalla velcità.
A piccole velocità la resistenza del mezzo tende ad essere proporzionale alla velocità:

$$\mathbf{f} = -a\mathbf{v} \qquad a \text{ costante} \quad .$$

A grandi velocità la resistenza del mezzo tende ad essere proporzionale al quadrato della velocità:

$$\mathbf{f} = -b\hat{\mathbf{v}}v^2 = -bv\mathbf{v} \qquad \text{con } b \text{ costante} \quad .$$

24.19.01.08 ElectroStatic Coulomb Force)

La forza coulombiana che un charged mass point (1) di carica elettrica q_1 esercita su un charged mass point (2) di carica elettrica q_2 posta ad $\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1$ vale:

$$\mathbf{f}_{12} = +\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{e}}_r \quad ,$$

dove $\hat{\mathbf{e}}_r$ il unit vector diretto dal charged mass point (1) verso il charged mass point (2).
See equation (36.06).

24.19.01.09 MagnetoStatic Biot-Savart Force)

See equation (36.06).

24.19.01.10 Lorentz Force

Lorentz force, equation (36.05.01) is actually the definition of the Electric|Magnetic fields.
Note that Lorentz force violates Newton third law, read § 58.03.04.

[16]The charged particles in a rigid body are presumably held together by electromagnetic forces, but the magnetic part of the **Lorentz** force law does not obey Newton's 3rd law, with the violation being of order v^2/c^2, where v is the velocity of the charges and c is the speed of light in vacuum. That is, for electric charges q_i at positions $\mathbf{x}_i$ with velocities $\mathbf{v}_i$ with $v_i \ll c$, the Lorentz force is, in Gaussian units,

$$\begin{aligned}
\mathbf{F}_{1,2} &= q_1\mathbf{E_2} + q_1\frac{\mathbf{v}_1}{c}\times\mathbf{B}_2 = q_1 q_2\frac{\mathbf{x}_{12}}{x_{12}^3} + q_1\frac{\mathbf{v}_1}{c}\times\left(q_2\frac{\mathbf{v}_2}{c}\times\frac{\mathbf{x}_{12}}{x_{12}^3}\right) \\
&= q_1 q_2\left[\frac{\mathbf{x}_{12}}{x_{12}^3}\left(1 - \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right) + \frac{\mathbf{v}_1\cdot\mathbf{x}_{12}}{x_{12}^3}\mathbf{v}_2\right], \qquad (20) \\
\mathbf{F}_{2,1} &= q_1 q_2\left[\frac{\mathbf{x}_{21}}{x_{21}^3}\left(1 - \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right) + \frac{\mathbf{v}_2\cdot\mathbf{x}_{21}}{x_{21}^3}\mathbf{v}_1\right] = -q_1 q_2\left[\frac{\mathbf{x}_{12}}{x_{12}^3}\left(1 - \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right) + \frac{\mathbf{v}_2\cdot\mathbf{x}_{12}}{x_{12}^3}\mathbf{v}_1\right] \\
&= -\mathbf{F}_{1,2} + q_1 q_2\frac{(\mathbf{v}_2\cdot\mathbf{x}_{12})\mathbf{v}_1 - (\mathbf{v}_1\cdot\mathbf{x}_{12})\mathbf{v}_2}{x_{12}^3}, \qquad (21)
\end{aligned}$$

where $\mathbf{x}_{12} = \mathbf{x}_1 - \mathbf{x_2} = -\mathbf{x}_{21}$. Thus, when $\mathbf{v}_1$ is different from $\mathbf{v}_2$ and neither is along the line of centers $\mathbf{x}_{12}$, we have that $\mathbf{F}_{12} \neq -\mathbf{F}_{12}$.

DRAFT - IN - PROGRESS

24.19.02
Other Examples of Force|Torque

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In general the **Cardinal Equations** of mechanics can be applied to any system after a description of the forces, torques and work on the components of the system are modeled. The kind of interaction any component has is part of the characteristics of the system.

Typical examples follows.

- A point electric fixed **Dipole** , which is affected by a force, in any non-homogeneous external electric field, plus a pole-independent torque in any external electric field. Read § 36.05.01, § 41.02.02, § 36.05.04.
- A point magnetic fixed **Dipole** , which is affected by a force, in any non-homogeneous external magnetic field, plus a pole-independent torque in any external magnetic field. Read § 36.05.01, § 41.02.02, § 36.05.04.
- A point electric induced **Dipole** .
- A point magnetic induced **Dipole** .
- The **Dipole -Dipole** force, which is a non-central force. However, the resultant moment of internal interactions sums to zero.
- Darwin interaction Lagrangian gives the lowest order relativistic correction to the interaction of point charge particles. It neglects radiation of ElectroMagnetic waves. It gives a non-central force.

$$L_{\text{int}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\left(-1+\frac{1}{2c^2}\left(\mathbf{v}_1\cdot\mathbf{v}_2+\frac{(\mathbf{v}_1\cdot\mathbf{r})(\mathbf{v}_2\cdot\mathbf{r})}{r^2}\right)\right) \quad .$$

- The tensor force is found in systems of nucleons: it is given by an interaction Hamiltonian depending on the relative position, $\mathbf{r}$, and spins, $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, of the two particles:

$$H_{\text{int}} \propto \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \frac{3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})}{r^2} \quad .$$

  It gives a non-central force.
- A spin-orbit coupling is found in many systems: it is given by an Hamiltonian like:

$$H_{\text{int}} \propto \mathbf{L}\cdot\mathbf{S} \quad .$$

Note that any vector linked to some of the properties of the particles, such as **Electric Dipole** | **Magnetic Dipole** moments, spin or velocity, can be used to provide a non-central force without breaking the space isotropy.

24.19.03
Interactions in Macroscopic Systems

Interactions in macroscopic systems arise from microscopic interactions summed on large number of particles.

- Static and dynamic sliding friction.
- Normal reaction forces.
- Elastic forces, in one, two and three dimensions.

DRAFT - IN PROGRESS

24.19.03.01 Body Force|Torque

Example: weight force.

24.19.03.02 Surface Force|Torque

Example: pressure forces.

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24.20 Mass - Inertial and Gravitational

This § is referenced at pages:
[1333]

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DRAFT-IN-PROGRESS

24.21 Examples and Physical Applications

24.21.01
Second Cardinal Equation and law of transformation of momenta

Suppose one has derived the second cardinal equation (24.06.05) without the term $\mathbf{v}_0 \wedge \mathbf{P}$. Show, by applying it to two different poles, that the missing term makes it incompatible with the law of transformation of momenta for force and angular momentum.

24.21.02
Arnold Proposition 1.7

Show that:

$$L^2 \leq 2I\mathcal{K} \quad .$$

24.21.03
Internal Torque Between Two Particles

The force of attraction/repulsion between two particles is given by:

$$\mathbf{f}_{21} = k\left(\mathbf{r}_2 - \mathbf{r}_1 + \frac{r}{v_0}(\mathbf{v}_2 - \mathbf{v}_1)\right) \quad ,$$

where k is a constant, v_0 is a constant velocity, and $r = |\mathbf{r}_2 - \mathbf{r}_1|$. Calculate the internal torque for the system; why does this quantity not vanish? Is the system conservative?

24.21.04
On the Cardinal Equations

Show that if the second **Cardinal Equation** is valid with respect to two different poles, then it implies the first **Cardinal Equation** .

24.21.05
System of Forces

Consider a system of applied Force|Torque.

- Resultant external force: it affects the Center-Of-Mass motion.
- Resultant external torque: it affects motion about the Center-Of-Mass.
- A force: ...
- A pure torque: the resultant is zero; many cases may arise:
  - a couple;
  - a n-couple: any set of identical vectors, applied in a circle, tangentially with the same handedness and applied at equi-distant points.
  - ... infinite other cases....
- a zero force and zero torque system; many cases may arise:
  - any two opposite torques;
  - any number of identical vectors applied on a circle, at equi-distant points, radially pointing; they provoke a dilatation/compression; the virial is non-zero;
  - four identical forces, applied at the vertexes of a square in a **Quadrupole** -like arrangement.

DRAFT-IN-PROGRESS

24.21.06
Direzione Della Verticale E Peso Apparenti Sulla Terra

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This § is referenced at pages:
[1342]

Si supponga che il Reference Frame solidale con il Sole sia un Reference Frame inerziale e si considerino i moti di rivoluzione e di rotazione della Terra supponendo l'orbita del moto di rivoluzione circolare con centro nel Sole ed entrambi i moti di rivoluzione e rotazione uniformi. Introducendo un Reference Frame $Oxyz$ solidale con la Terra e avente origine nel suo centro discutere i vari contributi dell'accelerazione di trascinamento del Reference Frame. Si supponga la Terra di forma sferica e con una distribuzione interna di massa a simmetria sferica e si supponga inoltre che la forza peso che agisce su un punto materiale di massa m alla superficie della Terra valga

$$\mathbf{F}_p = -\frac{GM_T m}{r^3}\mathbf{r}$$

dove $M_T = 5.98{\cdot}10^{24}$ kg è la massa della Terra, $G = 6.67{\cdot}10^{-11}$ $\mathrm{m^3 s^{-2} kg^{-1}}$ è la costante di gravitazione universale, e $\mathbf{r}$ è il vettore posizione del punto materiale rispetto al centro della Terra, supponendo $r < R_T$ e sapendo che $R_T = 6.38{\cdot}10^3$ km. Determinare la direzione della verticale apparente alla superficie della Terra e il valore del peso apparente alla latitudine di 45°. Si assuma il periodo di rivoluzione della Terra $T_{\mathtt{riv}} = 365.25$ giorni (periodo siderale), il periodo di rotazione della Terra $T_{\mathtt{rot}} = 23.93$ ore (giorno siderale pari a 86160 s, diverso dal giorno solare medio) e il raggio dell'orbita terrestre pari a $R = 1.496{\cdot}10^{11}$ m.

WORKED SOLUTION

Si denoti con $\mathbf{R}$ il vettore posizione della Terra rispetto al Reference Frame inerziale solidale con il Sole e avente origine nel centro del Sole e con $\mathbf{r}$ il vettore posizione del punto materiale rispetto al Reference Frame solidale con la Terra e avente origine nel centro della Terra. Sia Ω la velocità angolare, supposta costante, di rivoluzione attorno al Sole e ω la velocità angolare, supposta costante, di rotazione della Terra su sé stessa. L'accelerazione di trascinamento ha, per le ipotesi fatte, le due componenti di modulo

$$\left|\frac{d^2\mathbf{R}}{dt^2}\right| = \left|-\Omega^2\mathbf{R}\right| = \Omega^2 R \quad ,$$

$$|\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r})| \approx \omega^2 R_T \qquad \text{vale esattamente } \omega^2 R_T \text{ all'equatore}$$

Le velocità angolari di rivoluzione e di rotazione della Terra sono

$$\Omega = 1.99{\cdot}10^{-7} \text{ rad/s} \quad ,$$
$$\omega = 7.29{\cdot}10^{-5} \text{ rad/s} \quad .$$

Da queste si deducono le accelerazioni centripete

$$\Omega^2 R = 5.93{\cdot}10^{-3} \text{ m/s}^2 \quad ,$$
$$\omega^2 R_T = 3.39{\cdot}10^{-2} \text{ m/s}^2 \qquad \text{all'equatore}$$

da cui risulta che l'accelerazione legata al moto di rivoluzione è un ordine di grandezza inferiore a quella legata al moto di rotazione e può essere trascurata in prima approssimazione. Per confronto l'accelerazione centripeta legata al moto del Sole attorno al centro della galassia può essere stimata pari a

$$\Omega_G^2 R_G \approx 3{\cdot}10^{-10} \text{ m/s}^2$$

ed è quindi di molto inferiore alle due precedenti.

Nel Reference Frame non inerziale solidale con la Terra la direzione della verticale apparente si determina dalla risultante delle forze agenti su un punto materiale in quiete (peso e forza centrifuga, senza la forza di Coriolis):

$$\mathbf{F} = \mathbf{F}_p + \mathbf{F}_c = -\frac{GM_T m}{r^3}\mathbf{r} - m\,\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) = +m\mathbf{g} + m\omega^2\,[\mathbf{r} - \widehat{\mathbf{n}}\,(\mathbf{n}\cdot\mathbf{r})]$$

dove, essendo alla superficie della Terra, si è introdotta l'accelerazione di gravità $g = GM_T/R_T^2$ e la forza centrifuga è stata decomposta nelle componenti radiale e parallela ad $\boldsymbol{\omega}$ introducendo il unit vector $\widehat{\mathbf{n}} \equiv \boldsymbol{\omega}/\omega$. La forza centrifuga risulta dunque diretta perpendicolarmente all'asse di rotazione terrestre e diretta radialmente verso l'esterno. Il modulo della forza centrifuga è, detta λ la latitudine,

$$|\mathbf{F}_c| = m\omega^2 R_T \cos[\lambda] = (3.39{\cdot}10^{-2}\ \mathrm{m/s^2})m\cos[\lambda]$$

La relazione sopra permette di introdurre una accelerazione di gravità efficace alla superficie della Terra definita da

$$\mathbf{g}_{\mathtt{eff}} \equiv \mathbf{g} - \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) \tag{24.21.01}$$

Una relazione approssimata per $g_{\mathtt{eff}}$ si può ricavare applicando il teorema di Carnot e sviluppando al primo ordine in $\omega^2 R_T/g = 3.46{\cdot}10^{-3} \ll 1$

$$g_{\mathtt{eff}} \simeq g - \omega^2 R_T \cos^2\lambda \tag{24.21.02}$$

che si riduce a $g_{\mathtt{eff}} = g - \omega^2 R_T$ (valore esatto) per $\lambda = 0$ e $g_{\mathtt{eff}} = g$ (valore esatto) per $\lambda = \pm 90°$. Si noti che il parametro di sviluppo, $\omega^2 R_T/g$ altro non è che il rapporto F_c/F_p tra la forza centrifuga e la forza peso.

Detta $\Delta\theta$ la deviazione dalla direzione radiale, cioè dalla direzione di $\mathbf{r}$, si ha dal teorema dei seni

$$\frac{\sin[\Delta\theta]}{F_c} = \frac{\sin[(\Delta\theta+\lambda)]}{F_p}$$

da cui si deduce

$$\sin[\Delta\theta] = \frac{\omega^2 R_T}{g}\left[\sin[\lambda]\cos[\lambda]\cos[\Delta\theta] + \cos^2\lambda\sin[\Delta\theta]\right] \ll 1 \tag{24.21.03}$$

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essendo $\omega^2 R_T/g \ll 1$ (forza centrifuga molto minore della forza peso). Raccogliendo nella (24.21.03) a fattore comune $\sin[\Delta\theta]$ si può scrivere

$$\sin[\Delta\theta]\left[1 - \frac{\omega^2 R_T}{g}\cos^2\lambda\sin[\Delta\theta]\right] = \frac{\omega^2 R_T}{g}\sin[\lambda]\cos[\lambda]\cos[\Delta\theta]$$

da cui segue

$$\sin[\Delta\theta] \simeq \frac{\omega^2 R_T}{g}\sin[\lambda]\cos[\lambda]\cos[\Delta\theta] \ll 1$$

Si ha allora $\sin[\Delta\theta] \simeq \Delta\theta \ll 1$ e $\cos[\Delta\theta] \simeq 1$ e la relazione precedente si può approssimare a

$$\Delta\theta \simeq \frac{\omega^2 R_T}{2g}\sin[2\lambda] = 1.73{\cdot}10^{-3}\ \mathrm{rad} \simeq 0.1°$$

che fornisce la deviazione voluta con il segno corretto se si conviene di misurare l'angolo $\Delta\theta$ a partire da $\mathbf{g}$ verso $\mathbf{g}_{\mathtt{eff}}$ e positivo se in senso antiorario. Tale deviazione è nulla al polo (dove $F_c = 0$), è nulla all'equatore (dove $\mathbf{F}_c \parallel \mathbf{F}_p$) ed è massima alla latitudine di $\pm 45°$. Il calcolo dell'angolo $\Delta\theta$ poteva anche essere effettuato osservando che $\Delta\theta$ è l'angolo fra il vettore forza risultante, $\mathbf{F}$, e il vettore $-\mathbf{r}$. Ne segue, usando l'approssimazione $|\mathbf{F}| \simeq |\mathbf{F}_p| = mg$ (giustificata da $\omega^2 R_T/g \ll 1$)

$$|\sin[\Delta\theta]| = \frac{|-\mathbf{r}\wedge\mathbf{F}|}{|\mathbf{F}||\mathbf{r}|} \simeq \frac{\omega^2 R_T}{2g}|\sin[2\lambda]|$$

La relazione esatta per $\Delta\theta$ è

$$\tan[\Delta\theta] = \frac{\omega^2 R_T \sin[\lambda]\cos[\lambda]}{g - \omega^2 R_T\cos^2\lambda} \tag{24.21.04}$$

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che si riduce alla (24.21.03) nel caso $\omega^2 R_T/g \ll 1$.

24.21.07
Deviazione Verso Est/Ovest Della Caduta Dei Gravi Alla Superficie Della Terra

This § is referenced at pages:
[1375]

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Si stimi approssimativamente l'entità della deviazione verso est in una (breve) caduta libera di un grave che parte da fermo alla superficie terrestre nell'emisfero nord. Si assuma la terra a simmetria sferica e ruotante con velocità angolare costante, $\boldsymbol{\omega}$ e che la zona alla superficie della Terra in cui avviene uil moto del proiettile sia piccola rispetto alla distanza della zona stessa dall'asse di rotazione terrestre.

Di quanto devia verso est un grave lasciato cadere da un'altezza $h = 100$ m alla latitudine di $45°$?

WORKED SOLUTION

Si consideri, come da ipotesi, la zona di caduta così piccola da poter trascurare le variazioni di accelerazione di gravità e le variazioni della forza centrifuga durante la caduta.

Sia $\mathbf{x}$ la posizione relativa al Reference Frame terrestre e la posizione iniziale del grave sia: $\mathbf{x}[t=0] = \mathbf{x}_S$.

L'accelerazione relativa al Reference Frame terrestre, non inerziale, è data da

$$\mathbf{a}[\mathbf{x},\mathbf{v}] = \mathbf{g}[\mathbf{x}] - \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{x}) - 2\,\boldsymbol{\omega}\wedge\mathbf{v} \simeq \mathbf{g}[\mathbf{x}_0] - \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{x}_0) - 2\,\boldsymbol{\omega}\wedge\mathbf{v} \equiv \mathbf{g}_E[\mathbf{x}_0] - 2\,\boldsymbol{\omega}\wedge\mathbf{v} \quad ,$$

dove $\mathbf{g}_E[\mathbf{x}_0]$ è l'accelerazione efficace alla superficie della Terra, nelle vicinanze del posto dove si muove il corpo, che include il contributo della forza centrifuga come discusso nel problema § 24.21.06.

L'accelerazione $\mathbf{g}_E$ giace nel piano meridiano della Terra e quindi una deviazione trasversale può solo derivare dal termine di Coriolis: $-2\,\boldsymbol{\omega}\wedge\mathbf{v}$.

Possiamo supporre, approssimativamente, che sia $\mathbf{g}_{\mathtt{eff}} \parallel \mathbf{g} \parallel \mathbf{r}$, trascurando sia il fatto che la Terra non ha una distribuzione di massa a simmetria sferica, per cui in generale $\mathbf{g}$ non è parallelo ad $\mathbf{r}$, sia l'effetto della forza centrifiga, per cui in generale $\mathbf{g}_{\mathtt{eff}}$ non è paralelo a $\mathbf{g}$ (e quindi ad $\mathbf{r}$).

In fact in the usual case, at the Earth surface, $|\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{x}_S)| \ll |\mathbf{g}|$ and therefore the centrifugal force can be neglected with respect to the weight. The driving parameter is $R\omega^2/g \approx 3\cdot 10^{-3}$. Note that in the generic case of a rotating object it might be nont possible to neglect the effect of centrifugal forces, if the object is rotating with a large angular velocity.

Adottiamo un metodo approssimato che consiste nel considerare l'effetto della forza di Coriolis come una piccola perturbazione al moto (imperturbato) che si avrebbe in assenza di forza di Coriolis. Si determina quindi prima il moto imperturbato in assenza di forza di Coriolis e poi si valuta l'effetto della forza di Coriolis usando l'equazione oraria del moto imperturbato.

Detta v_M la componente della velocità del grave nel piano meridiano l'accelerazione trasversale a tale piano, diretta verso est nell'emisfero nord, ha modulo

$$|a_T| = 2\omega|v_M\cos[\lambda]| \qquad \text{avendo assunto } \mathbf{g}_{\mathtt{eff}} \parallel \mathbf{r} \quad , \tag{24.21.05}$$

e indicando con λ la latitudine, perchè, per ipotesi, il proiettile cade radialmente nel moto imperturbato con l'approssiamzione fatta $\mathbf{g}_{\mathtt{eff}} \parallel \mathbf{r}$.

Il moto imperturbato è una caduta libera radiale verso il centro della Terra con accelerazione costante. In base all'approccio che si vuole usare, si osserva che, in prima approssimazione, per un tragitto breve cosicché v_M resta sufficientemente piccola in modulo, si può sostituire v_M con il valore della velocità che si calcolerebbe trascurando l'effetto della forza di Coriolis (moto imperturbato).

Ciò è giustificato dal fatto che, nelle ipotesi fatte, l'accelerazione di Coriolis rappresenta solo una piccola correzione alla $\mathbf{g}_{\mathrm{E}}$, essendo $2\omega|v_{\mathrm{M}}\cos[\lambda]| \ll g$ per $|\mathbf{v}_{\mathrm{M}}|$ abbastanza piccolo.
Si scelga un Coordinate System cartesiane ortoginali con l'asse z radiale uscente dalla Terra, l'asse x tangente al meridiano terrestre e orientato da nord a sud (corentemente con il fatto che l'angolo polare si intende contato a partire dal polo nord, e l'asse y orinetato di conseguenza (risulta orientato da ovest verso est).
Note that the Coriolis force for a falling body is oriented, in the northern hemisphere, from west to east.
Si può allora porre, essendo $v = -gt$,

$$a_{\mathrm{T}} \equiv \ddot{y} \simeq 2\omega v\cos[\lambda] \simeq 2\omega g t\cos[\lambda] \quad ,$$

con il segno corretto per l'emisfero nord perchè l'asse y è orientato verso est.
L'equazione, integrata con la condizione iniziale $\dot{y}(t=0)=0$, essendo la velocità trasversa iniziale nulla, fornisce

$$\boxed{\Delta y = \frac{1}{3}g\omega\cos[\lambda]\,t^3} \quad . \tag{24.21.06}$$

Nelle approssimazioni fatte la deviazione si calcola usando la relazione

$$h = \frac{g}{2}t^2 \quad ,$$

e risulta

$$\boxed{\Delta y \simeq \frac{2}{3}h\omega\cos[\lambda]\sqrt{\frac{2h}{g}}} = 1.55\ \mathrm{cm} \quad .$$

Si noti che la velocità trasversale da origine, tramite la forza di Coriolis, ad una correzione alla velocità nel piano meridiano che è però una correzione di secondo ordine.

24.21.08
Motion Relative to the Earth

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Show that the motion of a particle of mass m relative to the Earth is described by the equation

$$\ddot{\mathbf{x}} + 2\,\boldsymbol{\omega}\wedge\dot{\mathbf{x}} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{x}) = \mathbf{g}[\mathbf{x}] + \frac{\mathbf{F}[\mathbf{x}]}{m} \equiv \hat{\mathbf{g}}[\mathbf{x}] \quad , \tag{24.21.07}$$

→ 1344 1345

where $\mathbf{x}$ is the position of the particle relative to the Center of the Earth, $|\boldsymbol{\omega}| = 7.3{\cdot}10^{-5}$ rad/s is the angular velocity of the Earth, assumed to be constant, $\mathbf{g}[\mathbf{x}]$ is the acceleration of gravity and $\mathbf{F}[\mathbf{x}]$ is any other positional force acting on the particle.

Use the method of successive approximation to show that when the particle starts at $\mathbf{x}[t=0] \equiv \mathbf{x}_{\mathrm{S}}$ with velocity $\mathbf{v}[t=0] \equiv \mathbf{v}_{\mathrm{S}}$, in a neighboring of $\mathbf{x}_{\mathrm{S}}$ such that both $\mathbf{g}[\mathbf{x}]$ and $\mathbf{F}[\mathbf{x}]$ can be considered as uniform, the position as a function of time, to third order in time, is given by:

$$\mathbf{x}[t] = \mathbf{x}_{\mathrm{S}} + \mathbf{v}_{\mathrm{S}}t + \frac{t^2}{2}(\hat{\mathbf{g}} - \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{x}_{\mathrm{S}}) - 2\,\omega\wedge\mathbf{v}_{\mathrm{S}}) - \frac{1}{3}(\boldsymbol{\omega}\wedge\hat{\mathbf{g}})\,t^3 \quad , \tag{24.21.08}$$

→ 1344

where

$$\hat{\mathbf{g}}[\mathbf{x}] \equiv \mathbf{g}[\mathbf{x}] + \frac{\mathbf{F}[\mathbf{x}]}{m} \qquad \text{and} \qquad \hat{\mathbf{g}} \equiv \hat{\mathbf{g}}[\mathbf{x}_{\mathrm{S}}] \quad .$$

If either $\mathbf{g}[\mathbf{x}]$ or $\mathbf{F}[\mathbf{x}]$ cannot be considered uniform then the method must be applied by taking into account $\mathbf{g}[\mathbf{x}[t]]$ or $\mathbf{F}[\mathbf{x}[t]]$.

Note that if the object is not a point particle one must interpret the motion as the motion of the Center-Of-Mass and the forces as the resultant of ecternal forces. In this case the object might also have an internal motion.

PROGRESS

WORKED SOLUTION

Note that the solution (24.21.08) includes a constant acceleration part, with acceleration given by

$$\hat{\mathbf{g}} - \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{x}_S) - 2\,\omega\wedge\mathbf{v}_S \quad ,$$

plus a corrective term, depending of the third power of t.

To solve the problem first note that, according to the hypothesis that both $\mathbf{g}[\mathbf{x}]$ and $\mathbf{F}[\mathbf{x}]$ can be considered as uniform in the region of motion, the effect of $\mathbf{F}[\mathbf{x}]$ can be trivially accounted for by defining an effective gravity field:

$$\hat{\mathbf{g}}[\mathbf{x}] \equiv \mathbf{g}[\mathbf{x}] + \frac{\mathbf{F}[\mathbf{x}]}{m} \equiv \hat{\mathbf{g}} \quad .$$

Assume therefore that it is possible to neglect the variations of $\hat{\mathbf{g}}[\mathbf{x}]$ in the region of motion.

Solve the differential equation by means of Picard method of successive approximations. Note that the series development is a series development in t, the independent variabile of the differential equation. Even better, from a physical point of view, one should think of a series development in the dimensionless parameter $|\boldsymbol{\omega}|t$.

The zero-order approximation is obtained by replacing in equation (24.21.07), written in normal form

$$\ddot{\mathbf{x}} = \hat{\mathbf{g}} - 2\,\boldsymbol{\omega}\wedge\dot{\mathbf{x}} - \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{x}) \quad , \tag{24.21.09}$$

→ 1344

the initial conditions

$$\mathbf{x}[t] \equiv \mathbf{x}_S = \mathbf{x}[t=0] \equiv \mathbf{x}_S \qquad \mathbf{v}[t] \equiv \mathbf{v}_S = \mathbf{v}[t=0] \equiv \mathbf{v}_S \quad . \tag{24.21.10}$$

→ 1344

Note that the method of successive approximations (Picard method) is normally used with first-order systems of differential equations. However, the single second-order equation of motion (24.21.09) can be reduced in a standard fashion to a first-order systems of differential equations by defining the new variable:

$$\mathbf{y} \equiv \dot{\mathbf{x}} \quad .$$

After doing that the two variables, $\mathbf{x}$ and $\mathbf{y}$, must be considered as independent variables, and the relation $\mathbf{y} \equiv \dot{\mathbf{x}}$ emerging only as a result of the solution of the system. Therefore the method of successive approximations implies that one chooses as zero order approximation the initial value of the two variables, that is equation (24.21.10). This explains the apparently inconsistent initial conditions (24.21.10) with a non-zero constant velocity corresponding to a constant position.

This gives:

$$\ddot{\mathbf{x}} = \hat{\mathbf{g}} - 2\,\omega\wedge\mathbf{v}_S - \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{x}_S) \equiv \mathbf{G} = \text{constant} \tag{24.21.11}$$

$$\Longrightarrow \mathbf{x}_0[t] = \mathbf{x}_S + \mathbf{v}_S t + \frac{t^2}{2}\mathbf{G} \quad .$$

The zero-order approximation is then replaced into the right-hand side of the full equation (24.21.09) to get the first-order approximation:

$$\ddot{\mathbf{x}} = \hat{\mathbf{g}} - 2\,\omega\wedge\mathbf{v}_S - \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{x}_S) - t\,(2\,\boldsymbol{\omega}\wedge\mathbf{G} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{v}_S)) - \frac{t^2}{2}\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{G}) \quad ,$$

$$= \mathbf{G} - t\,(2\,\boldsymbol{\omega}\wedge\mathbf{G} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{v}_S)) - \frac{t^2}{2}\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{G}) \quad ,$$

$$\Longrightarrow \mathbf{x}_1[t] = \mathbf{x}_S + \mathbf{v}_S t + \frac{t^2}{2}\mathbf{G} - \frac{t^3}{6}(2\,\boldsymbol{\omega}\wedge\mathbf{G} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{v}_S)) - \frac{t^4}{24}\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{G}) \quad .$$

Note that the second-order term of the first-order solution $\mathbf{x}_1[t]$ is the same as the second-order term of the zero-order solution $\mathbf{x}_0[t]$. Therefore one might expect that it will not change in higher-order approximations.

The third-order coefficient gives, in its exact expression and the truncated expression to second-order in ω,

$$2\,\boldsymbol{\omega}\wedge\mathbf{G} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{v}_S) = 2\,\boldsymbol{\omega}\wedge\mathbf{g} - 3\,\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{v}_S) + 2\omega^2\,\boldsymbol{\omega}\wedge\mathbf{x}_S \quad ,$$

$$2\,\boldsymbol{\omega}\wedge\mathbf{G} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{v}_S) = 2\,\boldsymbol{\omega}\wedge\mathbf{g} - 3\,\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{v}_S) + \mathcal{O}\,(\omega)^3 \quad .$$

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It is expected that the third-order coefficient will not change in higher-order approximations, while the fourth-order term will change. This can be indeed verified by carrying on the next iteration.
In the usual case, at the Earth surface, $|\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{x}_S)| \ll |\hat{\mathbf{g}}|$ and therefore the centrifugal force can be neglected with respect to the weight. The driving parameter is $R\omega^2/g \approx 3.5\cdot 10^{-5}$.
By using this simplification the first-order approximation is simplified by $\mathbf{G} \simeq \hat{\mathbf{g}} - 2\,\boldsymbol{\omega}\wedge\mathbf{v}_S$, showing that, for small times, the acceleration of the particle is given by the acceleration of gravity corrected by the term $-2\,\boldsymbol{\omega}\wedge\mathbf{v}_S$, whose direction is, a priori, arbitrary. In the case of an object starting with zero velocity the initial velocity is directed along $\mathbf{g}$ and therefore the corrective term to the acceleration of gravity is perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{g}$, that is eastwards/westwards.
Note that the corrective term to the acceleration of gravity given by the Coriolis force is of the same order in ω as the acceleration of gravity, because $v \approx gt$. As a matter of fact the Coriolis acceleration is small with respect to the acceleration of gravity as long a the velocity of the object is $v \lesssim 10^5$ m/s.
The result is a development in series of time, or better, in the dimensionless variable $|\boldsymbol{\omega}|t$. Therefore it applies, for small centrifugal and Coriolis force, for small enough times.
Note finally that the equation (24.21.07) can actually be solved exactly, as it is a linear system.
Note that the Coriolis force is of first-order in $\boldsymbol{\omega}$ while the centrifugal force is of second-order in $\boldsymbol{\omega}$. Therefore, as long as the angular velocity is small enough, the centrifugal force can be neglected with respect to the Coriolis force. Actually the balance of the two terms is given by the relative magnitude of the two quantities ωR and v, in term of the distance R of the object from the Earth axis and the velocity v of the object. Note, however, that when the velocity of the body is zero with respect to the non inertial system the Coriolis force is zero. This happens, in particular, when the object starts with zero velocity. Therefore, due to the latter two arguments, it is not possible to ignore the centrifugal force with respect to the Coriolis force only in the grounds that the former is of second-order with respect to the small angular velocity.
Note that the centrifugal force at the Earth surface is always small, in module, with respect to the force of gravity, as long as the body is close to the Earth surface (not too far from the Earth rotation axis). Therefor, under the latter conditions, one might just neglect it with respect to the weight as long as the body does not move so much that the position dependence of the weight and centrifugal force is negligible.
The eastwards deviation for an object falling from $h = 100$ m with zero start velocity at a latitude of λ 45 °, is found to be: $\Delta y = 1.5$ cm.
The final formula for an object falling starting with zero velocity ($\mathbf{v}_S = 0$) can be obtained as follows. Let us neglect in the formula the centrifugal force with respcet to the force of gravity, as it is appropriate at the surface of the Earth. Then:

$$\mathbf{G} \simeq \mathbf{g} \quad .$$

Therefore the first approximation $\mathbf{x}_1[t]$ does not carry any effect of the Earth rotation, under the current hypotesis.
The third order term becomes, to first-order in ω:

$$2\,\boldsymbol{\omega}\wedge\mathbf{g} + \mathcal{O}\left(\omega\right)^2 \quad .$$

The (eastwards) deviation of the real motion with respect to the case of no roation of the Earth is thus:

$$-\frac{t^3}{6}\left(2\,\boldsymbol{\omega}\wedge\mathbf{g}\right) = -\frac{t^3}{3}\boldsymbol{\omega}\wedge\mathbf{g} \quad .$$

Compare with problem § 24-050.

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24.21.09
Oggetto Fermo a Terra Visto Da Una Giostra in Moto

Un bambino si trova fermo al centro di una giostra che ruota con velocità angolare costante Ω. Egli osserva i suoi genitori che sono fermi a terra ma che, dal suo punto di vista, si muovono di moto circolare uniforme rispetto al centro della giostra. Nella descrizione del bambino chi fornisce la forza centripeta necessaria per mantenere i genitori in moto circolare uniforme?

WORKED SOLUTION

Si considerino le sole forze agenti sul piano orizzontale. Sui genitori non agisce alcuna forza reale con componenti nel piano orizzontale. Dal punto di vista del bambino, osservatore non inerziale, sui genitori agiscono però le forze (fittizie) centrifuga e di Coriolis. La loro risultante fornisce la forze centripeta richiesta.

24.21.10
Il Blocco Che Scivola Su Un Piano Inclinato Libero - Leggi Di Conservazione

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|5.68 9.43||

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|8.27||

This § is referenced at pages:
[1369]

Risolvere il problema § 24-038 utilizzando le equazioni cardinali della meccanica dei sistemi.

WORKED SOLUTION

Si usino le notazioni del problema § 24-038. Consideriamo come sistema l'insieme del blocco e del piano inclinato. Il sistema è conservativo e inoltre, poiché il tavolo è liscio, la risultante delle forze esterne sul sistema è nulla lungo la direzione orizzontale. L'accelerazione del centro di massa è dunque nulla nella direzione orizzontale. Ne segue

$$MA_x + ma_x = 0 \implies MV_x + mv_x = costante = 0 \tag{24.21.12}$$

→ 1346 1347

in cui il valore della costante è stato posto uguale a zero grazie alle condizioni iniziali. Introduciamo la coordinata s che misura la posizione del blocco rispetto alla cima del piano inclinato. La relazione tra la coordinata s e le coordinate cartesiane x' e y' che individuano la posizione del blocco relativa al piano inclinato è

$$\begin{aligned} x' &= +s\cos[\theta] \quad , \\ y' &= -s\sin[\theta] \end{aligned}$$

La relazione tra le velocità è

$$v_x = V_x + \dot{x}' = V_x + \dot{s}\cos[\theta] \tag{24.21.13}$$

→ 1346 1347

$$v_y = V_y + \dot{y}' = -\dot{s}\sin[\theta] \tag{24.21.14}$$

→ 1346 1347

essendo $V_y = 0$. Il principio di conservazione dell'energia meccanica applicato al sistema fornisce allora l'equazione

$$0 = \frac{1}{2}m\left(v_x^2 + v_y^2\right) + \frac{1}{2}MV_x^2 - mgs\sin[\theta] \tag{24.21.15}$$

→ 1346

avendo scelto come riferimento per l'energia potenziale gravitazionale la configurazione in cui il blocco è al vertice del piano inclinato. Si noti che l'energia potenziale gravitazionale del piano inclinato non cambia. Le equazioni (24.21.12), (24.21.13), (24.21.14) permettono di esprimere le velocità v_x, v_y e V_x in termini di $s(t)$ e delle sue derivate. Sostituendo tali espressioni in (24.21.15)

si perviene ad una equazione differenziale per $s(t)$:

$$\dot{s}^2 = \frac{2g\sin[\theta]\,(M+m)}{M+m\sin^2\theta}s \tag{24.21.16}$$

→ 1347

Integrando l'equazione differenziale per $s(t)$, avendo scelto il segno positivo per $\dot{s}(t)$ in quanto s è positiva nel problema, ed osservando che $\dot{s}$ non è identicamente nulla, si ottiene

$$s(t) = \frac{k}{4}t^2$$

con

$$k = \frac{2g\sin[\theta]\,(M+m)}{M+m\sin^2\theta}$$

L'espressione per $s(t)$ non dipende dalla posizione iniziale e dalle velocità iniziali perché si è posto in partenza velocità iniziali nulle e posizione iniziale $s_0 = 0$. da cui segue l'accelerazione lungo il piano inclinato

$$\ddot{s} = \frac{g\sin[\theta]\,(M+m)}{M+m\sin^2\theta}$$

Alternativamente l'accelerazione può essere determinata dalla (24.21.16) derivando entrambi i membri rispetto al tempo e semplificando la velocità, cosa possibile in quanto la velocità non è identicamente nulla come risulta evidente dalla (24.21.16) stessa. Un ulteriore metodo per derivare l'accelerazione dalla (24.21.16) è l'uso della (23.03.09.02) dato che il moto è un moto rettilineo uniformemente accelerato. Le componenti cartesiane dell'accelerazione possono essere ricavate dalle equazioni (24.21.12), (24.21.13), (24.21.14), derivate rispetto la tempo, e si ottiene

$$\begin{aligned} a_x &= \frac{M}{M+m}\cos[\theta]\,\ddot{s} \quad , \\ a_y &= -\ddot{s}\sin[\theta] \quad , \\ A_x &= -\frac{m}{M+m}\cos[\theta]\,\ddot{s} \end{aligned}$$

Si osservi che dalle stesse equazioni (24.21.12), (24.21.13), (24.21.14), si può ricavare il vincolo cinematico:

$$-a_y = a_x\left(1+\frac{m}{M}\right)\tan[\theta]$$

Si noti che in questo problema la coppia di reazioni vincolari tra il blocco e il piano inclinato, $\mathbf{N}$ e $-\mathbf{N}$, coppia di azione e reazione, compiono lavoro, nel giudizio dell'osservatore a terra. Infatti è la forza che il blocco esercita sul piano che accelera il piano inclinato e il suo ruolo è dunque quello di far in modo che una parte dell'energia potenziale gravitazionale del blocco vada in energia cinetica del piano inclinato anziché tutta in energia cinetica del blocco. La coppia di reazioni vincolari tra il blocco e il piano inclinato, $\mathbf{N}$ e $-\mathbf{N}$, coppia di azione e reazione, è una coppia di forze interne al sistema blocco-piano che compie lavoro. Il lavoro totale delle due forze è però nullo in quanto le forze sono uguali ed opposte ed hanno lo stesso punto di applicazione (sono forze a contatto). Quindi non contribuiscono al bilancio globale del sistema (teorema lavoro-energia). Infatti sul blocco agiscono due forze che compiono lavoro, la forza di gravità e la reazione normale, e si ha

$$\begin{aligned} \mathrm{d}L_B^{\mathtt{peso}} &= m\mathbf{g}\cdot\,\mathrm{d}\mathbf{X} + m\mathbf{g}\cdot\,\mathrm{d}\mathbf{x}' = m\mathbf{g}\cdot\,\mathrm{d}\mathbf{x}' \quad , \\ \mathrm{d}L_B^{\mathtt{normale}} &= \mathbf{N}\cdot\,\mathrm{d}\mathbf{X} + \mathbf{N}\cdot\,\mathrm{d}\mathbf{x}' = +\mathbf{N}\cdot\,\mathrm{d}\mathbf{x}' \end{aligned}$$

mentre sul piano inclinato la sola reazione normale $-\mathbf{N}$ compie lavoro

$$\mathrm{d}L_P^{\mathtt{normale}} = -\,\mathbf{N}\cdot\,\mathrm{d}\mathbf{X} - \mathbf{N}\cdot\,\mathrm{d}\mathbf{x}' = -\mathbf{N}\cdot\,\mathrm{d}\mathbf{x}' > 0 \quad .$$

Infatti il vettore $\mathbf{N}$ è perpendicolare allo spostamento relativo al piano inclinato $\mathrm{d}x'$. I due contributi, uguali ed opposti, $\mathrm{d}L_B^{\mathtt{normale}}$ e $\mathrm{d}L_P^{\mathtt{normale}}$, esistono se e solo se il piano inclinato si muove.
Si confronti con il problema § 24-038.

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24.21.11
Il Nastro Trasportatore

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|E9.13 9.55||

Un nastro trasportatore orizzontale si muove a velocità costante **v** sotto l'azione del motore. Della sabbia viene caricata sul nastro ad un ritmo costante con velocità nulla relativa al suolo. Determinare la potenza che deve applicare il motore per mantenere il nastro in moto a velocità costante. Si assuma che la sabbia venga scaricata dall'altro estremo del nastro con velocità nulla relativa al nastro.

24.21.12
La Macchina Di Atwood (2)

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|12.E6||

Si riconsideri il moto della puleggia del problema § 24-019 nel caso in cui la puleggia abbia massa M non trascurabile rispetto alle masse in gioco, momento di inerzia I e raggio R. si supponga che la fune, ideale e priva di massa, non strisci sulla carrucola.

1. Determinare l'accelerazione delle due masse.
2. Determinare il moto del centro di massa delle due masse. Risolvere il problema sia usando le **Cardinal Equation** che considerazioni energetiche.

WORKED SOLUTION

Poiché la massa della puleggia non è trascurabile la tensione della fune non è uguale ai due capi della stessa. La caduta di tensione serve a fornire un'accelerazione angolare alla carrucola. Le equazioni del moto per i due oggetti e per la carrucola si scrivono, avendo scelto un sistema di coordinate con asse x rivolto verso l'alto e scelto per l'angolo di rotazione della carrucola come positivo il verso antiorario,

$$\begin{aligned} T_1 - m_1 g &= m_1 a_1 \quad , \\ T_2 - m_2 g &= m_2 a_2 \quad , \\ F - T_1 - T_2 - M g &= 0 \quad , \\ R\,(T_1 - T_2) &= I\alpha \quad . \end{aligned}$$

ed avendo scelto la tensione T_1, corrispondente alla massa m_1 come quella che tende a far ruotare la carrucola in senso antiorario, $\Delta\theta > 0$. Il vincolo cinematico che i due oggetti sono legati ai due capi della corda impone le condizioni

$$\Delta x_1 = -\Delta x_2 \implies a \equiv a_1 = -a_2$$

Il vincolo di non strisciamento della corda sulla carrucola impone la condizione

$$\Delta x \equiv \Delta x_1 = -\Delta x_2 = -R\Delta\theta \implies v = -\omega R$$

i cui segni sono dettati dalle condizioni sui versi di x e θ scelti in precedenza. Le due relazioni ricavate costituiscono due vincoli cinematici per il sistema. Ne risulta un sistema di sei equazioni

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nelle sei incognite a_1, a_2, T_1, T_2, F e α che risolto da

$$
\begin{aligned}
a_1 = -\, a_2 = -R\alpha = g\left[\frac{m_2 - m_1}{m_2 + m_1 + I/R^2}\right] \quad , \\
T_1 = m_1 g\left[\frac{2m_2 + I/R^2}{m_1 + m_2 + I/R^2}\right] \quad , \\
T_2 = m_2 g\left[\frac{2m_1 + I/R^2}{m_1 + m_2 + I/R^2}\right] \quad , \\
F = g\left[\frac{4m_1 m_2 + (m_1 + m_2) I/R^2}{m_1 + m_2 + I/R^2}\right] + Mg \quad .
\end{aligned}
$$

Ne risulta che il modulo dell'accelerazione delle due masse è inferiore rispetto al caso in cui $I = 0$. Parte dell'energia potenziale gravitazionale delle due masse è convertita infatti in energia cinetica di rotazione della carrucola tramite la forza di attrito statico tra la carrucola e la fune.
Il problema può pure essere risolto applicando il bilancio dell'energia meccanica alle due masse e alla carrucola. Il bilancio dell'energia meccanica, equazione (24.04.04), è

$$\boxed{\Delta E = \Delta \mathcal{K} + \Delta \Phi = L_{\text{NC}}}$$

dove L_{NC} è il lavoro delle forze non conservative o, in generale, di tutte le forze il cui contributo non è incluso nell'energia potenziale Φ. Partendo da una configurazione iniziale di quiete e applicando il bilancio dell'energia meccanica (24.04.04) rispettivamente alla massa m_1, alla massa m_2 e alla carrucola si ottengono le tre equazioni

$$
\begin{aligned}
\frac{1}{2} m_1 v^2 + m_1 g \Delta x_1 = \frac{1}{2} m_1 v^2 + m_1 g \Delta x = T_1 \Delta x_1 = +T_1 \Delta x \quad , \\
\frac{1}{2} m_2 v^2 + m_2 g \Delta x_2 = \frac{1}{2} m_2 v^2 - m_2 g \Delta x = T_2 \Delta x_2 = -T_2 \Delta x \quad , \\
\frac{1}{2} I \left[\frac{v}{R}\right]^2 = \Gamma \Delta\theta = (T_2 - T_1)\, \Delta x = R\,(T_1 - T_2)\, \frac{\Delta x}{R}
\end{aligned}
$$

Il secondo membro della terza equazione è semplicemente il lavoro fatto sulla carrucola dalle forze esterne, $\Gamma\Delta\theta$. Il suo segno può anche essere determinato dal fatto che, in base alle convenzioni fatte, $\Delta x_1 = \Delta x$ per cui se $\Delta x > 0$ si ha che $T_2 > T_1$ e questo capita se e solo se $m_2 > m_1$. Il sistema di tre equazioni permette di ricavare T_1, T_2 e v in funzione di Δx. Il bilancio dell'energia meccanica (24.04.04) può pure essere applicato al sistema complessivo delle due masse più la carrucola e si ottiene la relazione

$$\frac{1}{2}(m_1 + m_2)\, v^2 + g\Delta x\,(m_1 - m_2) + \frac{I v^2}{R^2} = \Delta E = 0$$

→ 1349

da cui si ricava

$$v^2 = 2\Delta x \left(\frac{g\,(m_2 - m_1)}{m_1 + m_2 + I/R^2}\right)$$

Notare che nel bilancio energetico globale non compare il contributo delle tensioni T_1 e T_2, forze interne, in quanto il lavoro complessivo di tutte le forze di tensione è nullo. Questo approccio è sufficiente a determinare la dinamica ma se si vogliono ricavare le tensioni della fune bisogna ricorrere al bilancio energetico dei tre sottosistemi ricavato sopra mentre per determinare la forza di reazione che tiene sospesa la carrucola bisogna ricorrere alle **Cardinal Equation** . Infatti il bilancio globale elimina le tensioni interne T_1 e T_2 (forze interne che compiono lavoro complessivo nullo). Per determinare tali tensioni occorre allora usare il bilancio separato delle due masse e della carrucola (24.21.12) in cui le tensioni appaiono.

WORKED SOLUTION

1) $a_1 = -a_2 = -R\alpha = g\,[m_2 - m_1]\,/\,[m_2 + m_1 + I/R^2]$; 2) $a_{CM} = -g\,[m_2 - m_1]^2\,/\,[m_2 + m_1]^2$.

24.21.13
La Corsa Sul Vagone Ferroviario

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|9.31||

This § is referenced at pages:
[1391, 1391, 1391]

Un vagone ferroviario di massa M si muove su un binario rettilineo con velocità costante u_0 in assenza di attrito apprezzabile. Sul vagone si trova un uomo, inizialmente fermo rispetto al vagone. L'uomo si mette a correre lungo il vagone in verso opposto a quello del moto del vagone e quando salta giú dalla coda del vagone la sua velocità rispetto al vagone ha modulo v_R.

1. Determinare, subito dopo che l'uomo è saltato giú, la velocità del vagone, V, e dell'uomo, v, rispetto al suolo.

WORKED SOLUTION

Sul sistema vagone più uomo non agiscono forze esterne con componente orizzontale quindi la componente della quantità di moto del sistema nella direzione orizzontale si conserva. Denotando con v e V rispettivamente le componenti delle velocità finali dell'uomo e del vagone, rispetto al Reference Frame solidale con il suolo, e con v_R il modulo della velocità relativa, per la legge di trasformazione delle velocità si ha $v = V - v_R$, in cui il segno meno davanti a v_R è dovuto la fatto che v_R ha verso opposto ad u_0 e che con v_R si è indicato il modulo della velocità relativa dell'uomo rispetto al vagone. La conservazione della quantità di moto fornisce allora

$$(m + M)\,u_0 = P_{0x} = P_x = MV + m\,(V - v_R)$$

Ne segue

$$V = u_0 + \frac{m}{m + M} v_R \quad ,$$

$$v = u_0 - \frac{m}{m + M} v_R \quad .$$

Il risultato ottenuto è del tutto indipendente dai dettagli della corsa dell'uomo sul vagone. La sola cosa che conta è la sua velocità finale.

WORKED SOLUTION

1) $V = u_0 + m/\,(m + M)\,v_R$; $v = u_0 - m/\,(m + M)\,v_R$.

24.21.14
Some Applications of the Galilei-Newton Mechanics of a Particle

©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|§ 4||

24.21.15
Motion Relative to the Earth - the Focault Pendulum

This § is referenced at pages:
[1304]

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|§ 6.18 6.19 6.20 621 6.53 ...

24.21.16
Ballistic Dynamics

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|6.25 6.26 6.44-6.51||

24.21.17
Change of Constant Potential Energy

©|L.D.Landau *et al.*, *Mechanics*, ..., ..., ...Ed., WEB - URL - ISBN: 978-0-7506-2896-9.|§ 7|*problem 1*|

A particle of mass m moving with velocity v_1 leaves a half-space in which its potential energy is a constant Φ_1 and enters another in which its potential energy is a different constant Φ_2. Determine the change in the direction of motion of the particle.

24.21.18
Terminal Speed

Find the terminal speed of a mass point subject to a viscous friction force proportional to v^α.

24.21.19
Foucault Pendulum

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|6.18-6.21|*full solution*|

DRAFT-IN-PROGRESS

24.21.20
Simple Toy Model for Bike Braking

Simple 2D model. Bike going in the negative x (horizontal) direction with velocity, at time t_0, $\mathbf{V}_0$ and $V_{x0} < 0$. Assume the wheels are blocked while braking, so that they are slipping without rolling. Assume that the trailing wheel is attached to the road. Start solving the problem in the Inertial Reference Frame of the ground.

1. Solve the problem with pole for the 2nd Cardinal Equation at the Center-Of-Mass.
2. Solve the problem with pole for the 2nd Cardinal Equation at the two contact points.
3. Solve the problem with pole for the 2nd Cardinal Equation at any point on the road.
4. Write the expression for the angular momentum of the bike with respect to both the two moving contact points of the wheels with the road; note that they are moving poles.
5. Extend the problem to the case of rotating without slipping wheels.
6. Solve the problem in the non-Inertial Reference Frame attached to the bike.

$$
\begin{aligned}
&F_x = f_{1x} + f_{2x} = MA_x \geq 0 \qquad f_{1x} \geq 0 \qquad f_{2x} \geq 0 \quad , \\
&N_{1y} \equiv N_1 \geq 0 \qquad N_{2y} \equiv N_2 \geq 0 \qquad N_1 + N_2 = Mg \quad , \\
&f_{1x} = \pm\mu_1 N_1 \implies f_{1x} = +\mu_1 N_1 \geq 0 \qquad V_x < 0 \quad , \\
&f_{2x} = \pm\mu_2 N_2 \implies f_{2x} = +\mu_2 N_2 \geq 0 \qquad V_x < 0 \quad ,
\end{aligned}
$$

another equation is needed: 2nd C.E. $h\,(f_{1x} + f_{2x}) + L_2 N_2 - L_1 N_1 = 0$ pole at the Center-Of-Mass ,

unkowns: $A_x \quad N_1 \quad N_2 \quad f_{1x} \quad f_{2x}$,

$$
N_1 = Mg\left(\frac{L_2 + h\mu_2}{L_1 + L_2 + h\,(\mu_2 - \mu_1)}\right) \geq 0 \quad ,
$$

$$
N_2 = Mg\left(\frac{L_1 - h\mu_1}{L_1 + L_2 + h\,(\mu_2 - \mu_1)}\right) \qquad N_2 \geq 0 \qquad \Leftrightarrow \qquad h\mu_1 \leq L_1 \quad . \tag{24.21.17}
$$

CREATIVE-PROGRESS

24.22 Exercises Problems and Physical Applications

24-001 Recoil From a Moving Wall

This § is referenced at pages:
[2184]

24-002 La Nave a Vela Solare

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|5.9||

24-003 Il Dinamometro E La Tensione Della Fune

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|5.24||

Due pesi di massa $10\ \mathrm{kg}$ sono connessi da una fune e pendono ai due lati di una puleggia, in equilibrio. Sulla fune è inserito un dinamometro e si suppone trascurabile la massa della fune e del dinamometro.

1. Quale è la lettura del dinamometro? Uno dei due pesi viene rimosso e la fune legata al suolo.
2. Quale è la lettura del dinamometro in questa seconda configurazione?

WORKED SOLUTION

1) $98.1\ \mathrm{N}$; 2) $98.1\ \mathrm{N}$.

24-004 L'ascensore

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|5.28||

Un ascensore di massa $2812\ \mathrm{kg}$ è sollevato dal suo cavo con un'accelerazione che, ad un certo istante, vale $0.91\ \mathrm{m/s^2}$.

1. Quale è la tensione del cavo?
2. Quale è la tensione quando l'ascensore è accelerato verso il basso di $0.91\ \mathrm{m/s^2}$ ma si muove verso l'alto?

WORKED SOLUTION

1) $30{\cdot}10^3\ \mathrm{N}$; 2) $25{\cdot}10^3\ \mathrm{N}$.

DRAFT - IN PROGRESS

24-005 Il Salto Sul Pavimento

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Un uomo di massa $83\ \mathrm{kg}$ salta giú da un tavolo su un pavimento di cemento da un'altezza di $0.48\ \mathrm{m}$. Arrivando al suolo non flette le ginocchia e il suo moto si arresta perciò in circa $d = 2.2\ \mathrm{cm}$.

1. Quale è l'accelerazione media dell'uomo dall'istante in cui tocca il suolo all'istante in cui è fermo?
2. Quale è la forza media esercitata dal pavimento sulla struttura ossea dell'uomo per fermarlo?

WORKED SOLUTION

1) $210\ \mathrm{m/s^2}$; 2) $17{\cdot}10^3\ \mathrm{N}$.

24-006 Il Blocco Trascinato

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Un blocco di massa $5.1\ \mathrm{kg}$ viene trascinato su un piano orizzontale privo di attrito da una fune che esercita una forza di $12\ \mathrm{N}$ ad un angolo di $12\ ^\circ$ rispetto all'orizzontale.

1. Quale è l'accelerazione del blocco? Il modulo della forza viene lentamente aumentato.
2. Quale è il valore della forza all'istante prima che il blocco si sollevi dal suolo?
3. Quale è il valore dell'accelerazione all'istante prima che il blocco si sollevi dal suolo?

WORKED SOLUTION

1) $2.1\ \mathrm{m/s^2}$; 2) $120\ \mathrm{N}$; 3) $21\ \mathrm{m/s^2}$.

24-007 La Fune per Calare Un Oggetto

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|5.34||

Come può essere calato da un tetto un oggetto di massa $100\ \mathrm{kg}$ usando una fune che ha una carico di rottura di $500\ \mathrm{N}$ senza che la corda si spezzi?

24-008 Il Blocco Sospeso a Due Corde

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|§ 5.10E4||

24-009 Due Blocchi Che Scivolano Giù Dal Tavolo

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SOFTEL IN PROGRESS

24-010 L'auto Trainata Sul Pendio

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24-011 L'aereo a Reazione

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24-012 Il Piano Inclinato Sull'ascensore

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24-013 La Scimmia Sulla Carrucola

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|5.D.25 5.54||

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|3.88||

Una scimmia è appesa ad un capo di una fune ideale che passa attorno ad una carrucola supposta priva di massa e di attrito. All'altro capo, all'altezza della scimmia, è appeso uno specchio che ha la stessa massa della scimmia.

1. Può la scimmia allontanarsi dalla propria immagine riflessa nello specchio arrampicandosi sulla corda, scendendo lungo la corda o lasciandosi cadere?
2. Se la scimmia si arrampica ad una velocità di 1 ms quale è il moto dello specchio?

24-014 I Tre Blocchi Trascinati

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|5.55||

Tre blocchi di massa $m_1 = 1.2$ kg, $m_2 = 2.4$ kg e $m_3 = 3.1$ kg giacciono su un piano orizzontale senza attrito e sono connessi da due funi aventi, rispettivamente, la tensione T_{12} (quella che connette i blocchi 1 e 2) e T_{23} (quella che connette i blocchi 2 e 3). Il blocco 3 è tirato da una terza fune la cui tensione è $T_3 = 6.5$ N.

1. Determinare le due tensioni T_{12} e T_{23}.
2. Determinare l'accelerazione del sistema.

WORKED SOLUTION

1) $T_{12} = 1.2$ N, $T_{23} = 3.5$ N; 2) 0.97 m/s^2.

DRAFT - IN PROGRESS

24-015 I Due Blocchi a Contatto

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|5.56||

24-016 Tensione Di Una Fune Con Massa

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This § is referenced at pages:
[1357]

Una fune inestensibile e omogenea di lunghezza L e massa m è appesa al soffitto e sorregge un peso di massa M.

1. Determinare, in condizioni di equilibrio, l'andamento della tensione lungo la fune.
2. Determinare la condizione sotto la quale si può trascurare la variazione della tensione della fune e il valore della tensione costante in tale caso.
3. Si supponga di avere a che fare con una catena costituita da 5 anelli di massa $100\ \mathrm{g}$ sollevata con un'accelerazione costante pari ad $a = 2.50\ \mathrm{m/s^2}$ Si determini la forza risultante agente su ciascun anello e la forza esterna che agisce sul primo anello.

WORKED SOLUTION

La densità lineare di massa della fune vale $\lambda = m/L$. Si scelga un sistema di coordinate con asse x verticale, orientato verso il basso e con origine nel punto in cui la fune è attaccata al soffitto. Sia $T(x)$ il modulo della tensione della fune all'ascissa x. La condizione di equilibrio per l'elemento Δx della fune si scrive:

$$T(x + \Delta x) + \lambda g \Delta x = T(x)$$

da cui segue, nel limite $\Delta x \to 0$,

$$\boxed{\frac{dT(x)}{dx} = -\lambda g}$$

che integrata fornisce

$$T(x) = -\lambda g x + c$$

La condizione di equilibrio applicata alla massa M fornisce l'equazione

$$Mg - T(L) = 0$$

da cui si determina la costante c che interviene in $T(x)$. Risulta:

$$T(x) = Mg + \lambda g(L - x)$$

che fornisce la tensione della corda in ogni punto e determina, in particolare, la tensione massima della corda nel punto di attaccamento al soffitto $T(0) = g(M + m)$.
Si noti che se la fune è supposta senza massa ($\lambda = 0$) l'equazione differenziale trovata per $T(x)$ si riduce a $T(x)$ costante, indipendente da x e $T(x) = Mg$, dalla condizione di equilibrio della massa appesa.
La condizione di tensione costante lungo la fune si può determinare osservando che

$$\boxed{\frac{|T(L) - T(0)|}{T(L)} = \frac{m}{M}}$$

Tensione costante lungo la fune significa allora massa della fune trascurabile rispetto alla massa del corpo appeso. Si noti che la variazione di tensione ai due capi della fune, $|T(L) - T(0)|$, conseguenza della massa non nulla della fune, è stata confrontata con il valore della tensione che la fune assume nel caso in cui se ne trascura la massa, $T = Mg$.

La condizione di tensione costante lungo la fune si può determinare alternativamente imponendo la condizione

$$\frac{|T(L)-T(0)|}{T(0)} = \frac{m}{M+m} \ll 1$$

che può essere risolta nel modo seguente

$$\frac{|T(L)-T(0)|}{T(0)} = \frac{m}{M+m} \equiv \epsilon \ll 1 \implies \frac{m}{M} = \frac{\epsilon}{1-\epsilon} \simeq \epsilon \ll 1$$

ottenendo la stessa condizione $m/M \ll 1$ con l'approssimazione fatta nell'ultima equazione giustificata dal fatto che $\epsilon \ll 1$.

24-017 Piano Inclinato

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24-018 Dinamica Dell'ascensore

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|5.61||

24-019 La Macchina Di Atwood (1)

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This § is referenced at pages:
[1334, 1348]

Una forza **F** diretta verso l'alto è applicata all'asse di una puleggia. Si supponga che la carrucola e la fune siano di massa trascurabile, che la fune sia perfettamente flessibile e inestensibile e che non ci siano attriti. Agli estremi della fune, passante sulla carrucola, sono appesi due corpi di massa $m_1 = 1.2$ kg e $m_2 = 1.9$ kg.

1. Determinare l'accelerazione delle due masse e la tensione della fune nel caso la forza **F** sia quella necessaria a sorreggere la carrucola (ad esempio nel caso che la carrucola sia appesa al soffitto). Si supponga poi che l'oggetto di massa m_2 sia a contatto col suolo.
2. Determinare quale è il valore massimo del modulo della forza **F** che consente all'oggetto di restare attaccato al suolo senza sollevarsi.
3. Quale è la tensione della fune se $F = 110$ N e quanto vale l'accelerazione delle due masse?
4. Se è data l'accelerazione verso l'alto della carrucola, **A**, quale è l'accelerazione massima che consente alla massa m_2 di non sollevarsi dal suolo ?

WORKED SOLUTION

Essendo le masse della fune e della puleggia trascurabili la tensione della fune è la stessa in ogni punto (problema § 24-016). Le equazioni del moto per i due oggetti si scrivono, avendo scelto un sistema di coordinate con asse x rivolto verso l'alto

$$\begin{aligned} T_1 - m_1 g &= m_1 a_1 \quad , \\ T_2 - m_2 g &= m_2 a_2 \quad , \\ F - T_1 - T_2 &= 0 \quad , \\ T_1 - T_2 &= 0 \quad . \end{aligned}$$

Il vincolo cinematico che i due oggetti sono legati ai due capi della corda impone le condizioni

$$\Delta x_1 = -\Delta x_2 \implies a \equiv a_1 = -a_2$$

che fornisce la quinta equazione necessaria a risolvere il sistema nelle cinque incognite a_1, a_2, T_1, T_2 ed F. La soluzione del sistema fornisce

$$\begin{aligned} a = a_1 = -a_2 &= g\frac{m_2 - m_1}{m_2 + m_1} \quad , \\ T_1 = T_2 = T &= \frac{2m_1 m_2}{m_1 + m_2} g \quad , \\ F &= \frac{4m_1 m_2}{m_1 + m_2} g \quad . \end{aligned}$$

L'accelerazione del corpo di massa m_1 è positiva (cioè diretta verso l'alto) nel caso $m_2 > m_1$. Si noti che si ha

$$F = \frac{4m_1 m_2}{m_1 + m_2} g < (m_1 + m_2) g \qquad \text{se } m_1 \neq m_2$$

cioè in tutti i casi in cui $a \neq 0$ la forza che si deve esercitare per sorreggere la carrucola è minore della somma dei pesi dei due oggetti che è la forza che si deve esercitare nel caso la carrucola sia in equilibrio ($m_2 = m_1$). Inoltre, essendo $m_1 < m_2$, si ha sempre

$$m_1 g < T = \frac{2m_1 m_2}{m_1 + m_2} g < m_2 g$$

come deve essere in quanto la massa m_1 è sollevata dalla fune mentre il peso della massa m_2 è maggiore della tensione della fune stessa. Nei casi limite $m_1 \ll m_2$ oppure $m_1 \gg m_2$ l'accelerazione ha modulo pari a g.

Nel caso alla carrucola sia applicata una forza $\mathbf{F}$ diretta verso l'alto di valore assegnato le equazioni del moto sono ancora (la massa della carrucola è supposta trascurabile)

$$\begin{aligned} T_1 - m_1 g &= m_1 a_1 \quad , \\ T_2 - m_2 g &= m_2 a_2 \quad , \\ F - T_1 - T_2 &= 0 \quad , \\ T_1 - T_2 &= 0 \quad . \end{aligned}$$

ma adesso F è assegnata anziché incognita e le accelerazioni a_1 e a_2 non sono più legate dal vincolo cinematico $a_1 = -a_2$ in quanto le due masse oltre a cadere lungo la carrucola partecipano del moto di insieme della stessa. La soluzione del sistema fornisce ora

$$\begin{aligned} a_1 &= \frac{F}{2m_1} - g \quad , \\ a_2 &= \frac{F}{2m_2} - g \quad , \\ T_1 = T_2 &= F/2 \quad . \end{aligned}$$

Si osservi che, denotando con $\Delta x_1'$ e $\Delta x_2'$ gli spostamenti dei corpi relativi alla puleggia e con Δx lo spostamento della puleggia, si ha

$$\Delta x_1 = \Delta x_1' + \Delta x \qquad \Delta x_2 = \Delta x_2' + \Delta x \tag{24.22.01}$$

→ 1359

da cui si deduce

$$\begin{aligned} a_1 &= a_1' + a \quad , \\ a_2 &= a_2' + a \quad . \end{aligned}$$

e il vincolo cinematico diventa ora

$$\Delta x_1' = -\Delta x_2' \implies a_1' = -a_2' \tag{24.22.02}$$

In questa forma il vincolo cinematico è di validità generale, qualunque sia lo stato di moto della carrucola. Nel Reference Frame solidale con la carrucola, in moto con accelerazione a, le due masse hanno accelerazioni a_1' e a_2' e risentono delle forze inerziali $-m_1 a$ e $-m_2 a$. Le accelerazioni incognite a_1 e a_2 potrebbero essere sostituite, come variabili dinamiche, dall'accelerazione della

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carrucola a e dalle accelerazioni interne $a'_1 = -a'_2$. Si ricava dalle equazioni del moto

$$a = \frac{F}{4}\left(\frac{m_1 + m_2}{m_1 m_2}\right) - g \quad ,$$

$$a'_1 = -a'_2 = \frac{F}{4}\left(\frac{m_2 - m_1}{m_1 m_2}\right) \quad .$$

Supponiamo ora che la massa m_2 sia appoggiata. Le equazioni del moto diventano (imponendo subito $T_1 = T_2 = T$)

$$\begin{aligned} T - m_1 g &= m_1 a_1 \quad , \\ T - m_2 g + N_x &= 0 \quad , \\ F - 2T &= 0 \quad . \end{aligned}$$

dove si è indicato con N_x la componente della reazione vincolare del tavolo. Le incognite sono ora T, a ed N_x. La reazione vincolare $\mathbf{N}$ si determina dalle equazioni del moto. Il problema ha ora una incognita in più, $\mathbf{N}$, ma la presenza del vincolo stesso ha eliminato l'incognita $\mathbf{a}_2$. In generale i vincoli aggiungono incognite ma eliminano gradi di libertà. Per la sua caratteristica di vincolo unilatero la componente di $\mathbf{N}$ lungo l'asse x deve essere non negativa: $N_x \geq 0$. Si deduce

$$\begin{aligned} N_x &= m_2 g - F/2 \quad , \\ a_1 &= \frac{F}{2m_1} - g \quad , \\ T &= \frac{F}{2} \quad . \end{aligned}$$

Il corpo resta appoggiato in equilibrio sul tavolo se e solo se

$$N_x > 0 \qquad \Leftrightarrow \qquad F < 2m_2 g = 37.3 \text{ N}$$

Lo stacco dal tavolo avviene per $N_x = 0$. Il risultato $F < 2m_2 g$ mette in chiaro come solo metà della forza applicata $\mathbf{F}$ viene usata per sollevare la massa m_2. Una condizione equivalente che poteva essere applicata per determinare a quale forza avviene lo stacco dal tavolo era imporre che l'accelerazione a_2, determinata nel caso libero, fosse uguale a zero. Infatti, dal confronto dei due sistemi di equazioni del moto nel caso libero e in presenza del tavolo si osserva che la sola equazione che cambia è quella relativa alla massa m_2 in cui $m_2 a_2$ a secondo membro viene rimpiazzato da N_x a primo membro. Porre $N_x = 0$ equivale allora a porre $a_2 = 0$ nel caso libero.
Nel caso richiesto, $F = 110$ N la massa m_2 si solleva e si ha:

$$\begin{aligned} T &= F/2 = 55 \text{ N} \quad , \\ a_1 &= \frac{F}{2m_1} - g = 36.0 \text{ m/s}^2 \quad , \\ a_2 &= \frac{F}{2m_2} - g = 19.1 \text{ m/s}^2 \quad . \end{aligned}$$

Dalle equazioni (24.22.01) e dal vincolo cinematico (24.22.04) si deduce ora

$$\Delta x_1 = 2\Delta x \implies a_1 = 2a$$

da cui si deduce

$$N = g\,(m_2 - m_1) - 2m_1 A \geq 0$$

da cui la condizione di sollevamento del blocco

$$a \geq \frac{g}{2m_1}(m_2 - m_1)$$

Si osservi che il risultato fornisce, come deve, la condizione che, se $m_2 < m_1$ la massa m_2 si stacca anche per $a = 0$ Tale risultato è coerente con quello ottenuto sopra in termini della forza $\mathbf{F}$.

WORKED SOLUTION

1) $a_1 = -a_2 = g\dfrac{m_2 - m_1}{m_2 + m_1}$, $T_1 = T_2 = T = \dfrac{2m_1 m_2}{m_1 + m_2}g$, $F = \dfrac{4m_1 m_2}{m_1 + m_2}g$; 2) $F < 2m_2 g = 37.3$ N; 3) $T = F/2 = 55$ N; $a_1 = \dfrac{F}{2m_1} - g = 36.0\ \mathrm{m/s^2}$, $a_2 = \dfrac{F}{2m_2} - g = 19.1\ \mathrm{m/s^2}$.

24-020 Due Particelle Legate

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24-021 La Sbarra

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Una sbarra orizzontale è usata per sostenere tra due muri un peso di massa $75\ \mathrm{kg}$ appeso al centro della sbarra. Le due forze, di uguale modulo, che la sbarra esercita contro i due muri possono essere variate cambiando la lunghezza della sbarra tramite un sistema di vite e molla. Solo l'attrito tra le estremità della sbarra e i muri sorregge la sbarra e il coefficiente di attrito statico vale 0.41. Quale è il minimo valore della forza che la sbarra deve esercitare contro il muro affinché si abbia equilibrio?

24-022 Il Pendio

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24-023 Il Libro Schiacciato Contro Il Muro

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|6.9||

©|H.C.Ohanian, , ..., ..., ...Ed., ...|6.39||

Una forza orizzontale $F = 53.4\ \mathrm{N}$ spinge un blocco di massa $m = 2.27\ \mathrm{kg}$ contro un muro verticale. Il coefficiente di attrito statico tra il muro ed il blocco vale $\mu_s = 0.60$ e quello di attrito cinetico vale 0.40. Il blocco è inizialmente fermo.

1. Il blocco inizierà a muoversi?
2. Quale è la forza esercitata sul blocco dal muro? Si cambia la direzione della forza esercitata sul libro in modo da formare un angolo θ rispetto alla parete (misurato a partire dall'alto).
3. Quale è il modulo della forza minimo affinché il libro non cada?
4. Per quale valore dell'angolo θ il modulo di tale forza assume il valore minimo possibile e quanto vale tale valore minimo?
5. Se l'angolo θ è maggiore dell'angolo retto (cioè se la forza spinge verso il basso) occorrerà esercitare una forza notevole per impedire al libro di cadere. Per quale valore dell'angolo θ risulta comunque impossibile sostenere il libro?

DRAFT-IN-PROGRESS

24-024 La Cassa Da Spostare

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|6.10||

Una cassa di $136\ \mathrm{kg}$ di massa è appoggiata al pavimento. Un uomo tenta di spostarla spingendola con una forza orizzontale di $412\ \mathrm{N}$. Il coefficiente di attrito statico tra la cassa e il suolo vale 0.37.

1. Dimostrare che la cassa non si muove. Un secondo uomo aiuta il primo sollevando la cassa.
2. Quale minima forza verticale deve applicare perché la cassa si muova?
3. Se il secondo uomo applica una forza orizzontale invece che verticale quale minima forza, oltre a quella applicata dal primo uomo, deve applicare perché la cassa si muova?

24-025 Il Mucchio Di Sabbia

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|6.13||

24-026 Il Piano Inclinato Sulla Bilancia

Un blocco di massa m è appoggiato senza attrito su un piano inclinato di angolo θ e massa M che appoggia su una bilancia orizzontale. Il coefficiente di attrito statico tra il piano inclinato e la bilancia vale μ_s.

1. A quale condizione deve soddisfare μ_s affinché il piano inclinato non scivoli durante la discesa del blocco?
2. Quale è il peso R letto della bilancia?

WORKED SOLUTION

Supponiamo che il piano inclinato non scivoli e sia quindi fermo. Si scelga un Coordinate System con asse x parallelo al piano inclinato. Le equazioni del moto del blocco sono

$$mg\sin[\theta] = ma \quad ,$$
$$N - mg\cos[\theta] = 0 \quad .$$

Si scelga un secondo Coordinate System con asse x' parallelo al piano orizzontale della bilancia per descrivere il moto (in questo caso la condizione di equilibrio) del piano inclinato. Le equazioni di equilibrio del piano inclinato sono

$$-N\sin[\theta] + f_s = 0 \quad ,$$
$$R - Mg - N\cos[\theta] = 0 \quad .$$

Ne segue

$$f_s = mg\sin[\theta]\cos[\theta] \quad ,$$
$$R = Mg + mg\cos^2\theta$$

Il blocco resta allora fermo se e solo se

$$m\sin[\theta]\cos[\theta] \le \mu_s(M + m\cos^2\theta)$$

essendo $\sin[\theta] \ge 0$ e $\cos[\theta] \ge 0$. Il peso letto dalla bilancia vale

$$R = Mg + mg\cos^2\theta \le g(M + m)$$

ed è massimo, $R = g(M + m)$, per $\theta = 0$. La forza di attrito statico richiesta per mantenere fermo il piano inclinato è massima per $\theta = \pi/4$.

WORKED SOLUTION

1) $m\sin[\theta]\cos[\theta] \le \mu_s(M + m\cos^2\theta)$; 2) $R = Mg + mg\cos^2\theta \le g(M + m)$.

24-027 Due Blocchi Che Scivolano Dal Pendio

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Una cassa è appoggiata sul pianale scabro di un autocarro. Il coefficiente di attrito statico tra la cassa e il pianale è μ_s. Se l'autocarro frena bruscamente la cassa scivola in avanti. Quanto vale la decelerazione massima che l'autocarro può avere in frenata se la cassa deve restare ferma?

24-028 Moto Di Un Oggetto Soggetto a Forza Crescente in Presenza Di Attrito

Un punto materiale di massa m è fermo su un piano orizzontale. I coefficienti di attrito statico e cinetico sono rispettivamente μ_s e μ_c. Il punto materiale è soggetto ad una forza orizzontale, dipendente dal tempo, $F = kt$. Descrivere il moto del punto materiale.

1. A quale istante t_0 inizia a muoversi?
2. Quale è l'andamento dell'accelerazione in funzione del tempo?

WORKED SOLUTION

1) $t_0 = \mu_s mg/k$; 2) $a(t) = 0$ per $t < t_0$, $a(t) = kt/m - \mu_c g$ per $t > t_0$.

24-029 Blocco Lanciato Su per Un Piano Inclinato Con Attrito

Un blocco di massa m viene lanciato su per un piano inclinato di angolo θ con velocità iniziale v_0. Se i coefficienti di attrito statico e cinetico tra il piano inclinato e il blocco sono rispettivamente μ_s e μ_c (con $\mu_s \geq \mu_c$) si determini il moto del blocco.

WORKED SOLUTION

Scegliamo un sistema di coordinate con asse x parallelo al piano inclinato e diretto verso il basso, asse y uscente dal piano inclinato e origine nel punto più basso del piano inclinato. Si noti che essendo $0 \leq \theta \leq 90°$ si ha $\cos[\theta] \geq 0$ e $\sin[\theta] \geq 0$. Le forze agenti sul blocco sono la forza peso $m\mathbf{g}$, la reazione normale del piano $\mathbf{N}$ e la forza di attrito. L'equazione del moto si scrive

$$m\mathbf{g} + \mathbf{N} + \mathbf{f} = m\mathbf{a}$$

Nella fase di salita si ha che la forza di attrito cinetico è diretta verso il basso, opposta alla velocità. Proiettando l'equazione si ha dunque

$$\begin{aligned} mg\sin[\theta] + f_c &= ma_x^S \quad , \\ N - mg\cos[\theta] &= ma_y = 0 \quad . \end{aligned}$$

con

$$f_c = \mu_c N > 0$$

Ne segue per l'accelerazione nella fase di salita

$$a_x^S = g\left(\sin[\theta] + \mu_c \cos[\theta]\right)$$

Durante la fase di salita l'effetto della forza di attrito cinetico si somma all'accelerazione di gravità nel rallentare il moto. Nel caso $\mu_c = 0$ si riduce correttamente all'accelerazione per un piano inclinato senza attrito, $g\sin[\theta]$. Nel caso $\theta = 0$ si riduce correttamente all'accelerazione sul piano in presenza di attrito, $g\mu_c$. Data l'accelerazione costante del blocco lo spazio percorso prima di fermarsi si deduce dalla relazione (23.03.09.02)

$$v^2 = v_0^2 + 2a(x - x_0)$$

imponendo $v = 0$ e data la condizione iniziale $x_0 = 0$ si determina la coordinata x_f a cui il blocco si ferma:

$$0 = v_0^2 + 2a(x_f - x_0) \implies x_f = -\frac{v_0^2}{2a_x^S} < 0$$

Occorre ora stabilire se il blocco, dopo essersi fermato, ridiscende oppure se l'attrito statico è sufficiente a mantenerlo fermo. Nel caso il blocco resti in equilibrio si hanno le equazioni

$$\begin{aligned} mg\sin[\theta] + f_s &= 0 \quad , \\ N - mg\cos[\theta] &= 0 \quad . \end{aligned}$$

da cui si deduce il valore della componente f_s, ora incognita, della forza di attrito statico necessario a mantenere il blocco in equilibrio

$$f_s = -mg\sin[\theta] < 0$$

negativa in quanto deve opporsi alla forza di gravità la cui componente lungo l'asse x è positiva. La forza di attrito statico può fornire la componente richiesta a mantenere l'equilibrio se e solo se, in base alla (24.19.01),

$$|f_s| = mg\sin[\theta] \leq \mu_s N = \mu_s mg\cos[\theta]$$

cioè se e solo se

$$\tan[\theta] \leq \mu_s$$

Questa relazione definisce l'angolo limite per un piano inclinato al di sopra del quale il blocco scende e può essere usata per misurare il coefficiente di attrito statico μ_s. Si noti che la forza di attrito statico è determinata dalle equazioni del moto.
Se la relazione sopra non è soddisfatta il blocco riprende a scendere. Le equazioni del moto sono ora

$$\begin{aligned} mg\sin[\theta] + f_c &= ma_x^D \quad , \\ N - mg\cos[\theta] &= ma_y = 0 \quad . \end{aligned}$$

ma ora la componente della forza di attrito cinetico f_c, essendo sempre opposta alla velocità risulta negativa:

$$f_c = -\mu_c N = -\mu_c mg\cos[\theta] < 0$$

Ne segue per l'accelerazione nella fase di discesa

$$a_x^D = g(\sin[\theta] - \mu_c\cos[\theta]) > 0$$

Si ha che $a_x^D \geq 0$ perché il blocco ridiscende se e solo se $\tan[\theta] > \mu_s \geq \mu_c$. In modulo l'accelerazione nella discesa risulta minore a quella durante la salita perché questa volta la forza di attrito cinetico ha verso opposto alla componente della forza di gravità lungo l'asse x coerentemente con la relazione (24.19.02) per cui la forza di attrito cinetico è sempre opposta al moto.

24-030 Blocchi E Carrucola

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24-031 trascinare una cassa su un piano con attrito

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Si supponga di dover trascinare una cassa al suolo tramite una fune. La massa della cassa vale m e il coefficiente di attrito cinetico tra cassa e suolo vale $\mu_c = 0.25$. Si schematizzi la cassa come un punto materiale.

1. Determinare la direzione ottimale a cui bisogna dirigere la fune per trascinare la cassa di moto rettilineo uniforme applicando la forza minima e il valore di tale forza minima. Se la fune ha una tensione di rottura pari a $T_{\max} = 1.22\cdot 10^3$ N e se il coefficiente di attrito statico tra la cassa e il suolo vale $\mu_s = 0.35$ quale è il valore massimo della massa della cassa che si può trascinare?

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WORKED SOLUTION

Si scelga un sistema di coordinate con asse x orizzontale e asse y verticale. Sia $\mathbf{F}$ la forza che bisogna applicare ad un angolo θ con l'orizzontale e sia $\mathbf{f}$ la forza di attrito e $\mathbf{N}$ la reazione normale del piano. Le equazioni del moto sono

$$F\cos[\theta] - f = 0 \quad ,$$
$$F\sin[\theta] + N - mg = 0 \quad .$$

con $0 \leq \theta \leq 90°$. Si ricava il valore della forza $\mathbf{F}$ necessaria in funzione dell'angolo θ a cui è applicata:

$$\boxed{F = \frac{\mu_c mg}{\cos[\theta] + \mu_c \sin[\theta]}}$$

che è una funzione crescente di μ_c. Nel caso limite $\theta = 0$ si riduce a $F = \mu_c mg$, come deve. Nel caso limite $\theta = 90°$ si riduce a $F = mg$, come deve perché in tal caso solo la forza di attrito cinetico ha una componente orizzontale che non può essere equilibrata da $\mathbf{F}$ per cui la forza di attrito cinetico va azzerata cosa possibile se e solo se $\mathbf{N} = 0$ che implica $\mathbf{F} = -m\mathbf{g}$
Il valore minimo di $|\mathbf{F}|$ al variare di θ si ha per

$$\tan[\theta] = \mu_c$$

e risulta

$$F = \frac{\mu_c mg}{\sqrt{1+\mu_c^2}}$$

che è una funzione crescente del coefficiente di attrito cinetico.
Essendo il coefficiente di attrito statico maggiore o uguale di quello di attrito cinetico il momento critico si ha alla partenza quando è necessario applicare la massima forza. Le relazioni che si applicano sono le stesse determinate sopra pur di sostituire il coefficiente cinetico con quello statico. Applicando la forza all'angolo ottimale, dato dalla relazione $\tan[\theta] = \mu_s$, si ottiene il valore massimo del peso trasportabile come

$$mg \leq \frac{\sqrt{1+\mu_s^2}}{\mu_s} T_{\max}$$

Mantenendo la stessa direzione e la stessa intensità della forza $T_{\max}$ applicata all'istante iniziale il moto della cassa sarà uniformemente accelerato in quanto la forza di attrito diminuisce subito dopo la partenza. Volendo proseguire in modo ottimale si potrà, una volta che la cassa è in moto, riaggiustare l'angolo, in base alla $\tan[\theta] = \mu_c$, e diminuire l'intensità della forza.
Questo problema mostra che per trascinare con la minor forza possibile una cassa in presenza di attrito occorre applicare una componente della forza verso l'alto in modo da diminuire il valore della reazione normale. Questo spiega anche perché sia più semplice trascinare una cassa invece che spingerla. Nel primo caso è infatti relativamente semplice applicare una componente verticale della forza, mentre nel secondo lo è meno, anzi si rischia di applicare una componente verso il basso che aumenta la forza di attrito.

WORKED SOLUTION

1. $\tan[\theta] = \mu_c$; $F = \mu_c mg/\sqrt{1+\mu_c^2}$;
2. $mg \leq T_{\max}\sqrt{1+\mu_s^2}/\mu_s$.

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24-032 Problema Del Ristagno in Presenza Di Attrito Statico

Un blocco di massa m è posizionato su un piano inclinato di angolo θ. Il coefficiente di attrito statico tra il piano e il blocco vale μ_s. Una molla di costante elastica k e lunghezza a riposo x_0 è disposta parallelamente alla superficie del piano inclinato e collega il blocco al vertice superiore del piano inclinato. Determinare per quale intervallo di posizioni il blocco resta in equilibrio.

WORKED SOLUTION

Si scelga un sistema di coordinate con asse x parallelo al piano inclinato, orientato verso il basso e avente origine alla sommità del blocco e asse y diretto perpendicolarmente al piano e verso l'alto. Detta $\mathbf{N}$ la reazione normale del piano, $\mathbf{f}$ la forza di attrito statico tra il blocco e il piano e $\mathbf{F}$ la forza che la molla esercita sul blocco la condizione di equilibrio si scrive

$$m\mathbf{g} + \mathbf{F} + \mathbf{N} + \mathbf{f} = \mathbf{0}$$

Proiettando sugli assi x e y si ottengono le equazioni scalari

$$mg\sin[\theta] + f_x + F_x = 0 \quad ,$$
$$N - mg\cos[\theta] = 0 \quad .$$

avendo denotato f_x e F_x rispettivamente le componenti lungo l'asse x delle due forze $\mathbf{f}$ e $\mathbf{F}$. Si ha inoltre, dalla legge di Hooke,

$$\boxed{F_x = -k(x - x_0) \equiv -k\Delta x}$$

Ne segue

$$N = mg\cos[\theta] \quad ,$$
$$f_x = k\Delta x - mg\sin[\theta]$$

La condizione di equilibrio può essere soddisfatta se e solo se

$$|f_x| \leq \mu_s N$$

relazione che coinvolge il valore assoluto della componente f_x, e che equivale a

$$|k\Delta x - mg\sin[\theta]| \leq \mu_s mg\cos[\theta]$$

che risolta da

$$mg\left(\sin[\theta] - \mu_s\cos[\theta]\right) \leq k\Delta x \leq mg\left(\sin[\theta] + \mu_s\cos[\theta]\right)$$

Nel caso $\mu_s = 0$ la diseguaglianza sopra fornisce la condizione di equilibrio $k\Delta x = mg\sin[\theta]$, da cui risulta pure $\Delta x > 0$. Nel caso $\theta = 0$ la condizione di equilibrio si riduce a $|k\Delta x| \leq \mu_s mg$, coerente con il fatto che in tal caso si ha $f \leq \mu_s N = \mu_s mg$. Nel caso $\theta = \pi/2$ si ha $k\Delta x = mg$, che esprime il fatto che in tal caso solo la molla può opporsi alla forza peso in quanto $\mathbf{N}$ (e quindi $\mathbf{f}$) deve valere zero perché non esistono altre forze nella direzione dell'asse y.

WORKED SOLUTION

$mg\left(\sin[\theta] - \mu_s\cos[\theta]\right) \leq k\Delta x \leq mg\left(\sin[\theta] + \mu_s\cos[\theta]\right)$.

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24-033 Le Due Masse Sul Piano Inclinato

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Due oggetti di massa $m_1 = 1.65$ kg e $m_2 = 3.22$ kg sono collegati da un'asta di massa trascurabile parallela al piano inclinato su cui i due oggetti scivolano con la massa m_2 posta in bassa rispetto alla massa m_1. L'angolo del piano inclinato è $\theta = 29.5\,^\circ$ e i coefficienti di attrito cinetico tra le due mese e il piano sono rispettivamente $\mu_1 = 0.226$ e $\mu_1 = 0.127$.

1. Calcolare la comune accelerazione dei due oggetti.
2. Calcolare la tensione della sbarra.
3. Come si modificano le due risposte precedenti se le posizioni delle due masse sono invertite?

24-034 Il Traino Della Slitta

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Si supponga di dover trascinare una slitta di massa M il cui coefficiente di attrito cinetico con il suolo sia μ_c. Sopra la slitta è appoggiata una cassa di massa m e il coefficiente di attrito statico tra la cassa e la slitta vale μ_s. Se alla slitta è applicata una forza orizzontale $\mathbf{F}$ determinare il valore massimo di tale forza affinché la cassa non scivoli sopra la slitta e l'accelerazione del sistema in questa condizione.

WORKED SOLUTION

Si scelga un Coordinate System con asse x orizzontale e concorde con il verso della forza $\mathbf{F}$ e asse y diretto verso l'alto. Le forze agenti sulla cassa sono il peso $m\mathbf{g}$, la forza di attrito statico tra cassa e slitta $\mathbf{f}$, diretta nello stesso verso della forza $\mathbf{F}$, e la reazione normale della slitta sulla cassa $\widehat{\mathbf{n}}$. Le forze che agiscono sulla slitta sono le forze corrispondenti alle coppie azione-reazione con la cassa, $-\mathbf{f}$ e $-\widehat{\mathbf{n}}$, la forza esterna $\mathbf{F}$, il peso $M\mathbf{g}$, la reazione normale del suolo $\mathbf{N}$ e la forza di attrito cinetico con il suolo $\mathbf{f}_0$. Supponendo che la cassa resti solidale con la slitta e chiamando $\mathbf{a}$ l'accelerazione del sistema slitta-cassa, e supponendo che la slitta si muova nella direzione della forza $\mathbf{F}$, le equazioni del moto di cassa e slitta, proiettate sugli assi x e y, si scrivono:

$$\begin{aligned} f &= ma \quad , \\ n &= mg \quad , \\ F - f_0 - f &= Ma \quad , \\ N - n - Mg &= 0 \quad , \\ f_0 &= \mu_c N \quad . \end{aligned}$$

dove le incognite sono a, f, f_0, n ed N. Risolvendo il sistema si trova

$$\begin{aligned} a &= \frac{F - \mu_c(M+m)g}{M+m} = \frac{F}{M+m} - g\mu_c \quad , \\ n &= mg \quad , \\ N &= g(M+m) \quad , \\ f &= m\frac{F - \mu_c(M+m)g}{M+m} \quad , \\ f_0 &= \mu_c(M+m)g \quad . \end{aligned}$$

Si osservi che l'accelerazione del sistema è quella che si calcolerebbe considerando solo le forze esterne, $m\mathbf{g}$, $M\mathbf{g}$, $\mathbf{F}$, $\mathbf{N}$ e $\mathbf{f}_0$, agenti sul sistema complessivo di massa $M+m$. Nel caso di assenza di attrito con il suolo, $\mu_c = 0$ l'accelerazione si riduce ad $a = F/(M+m)$ come deve. Si noti anche che si è supposto che il sistema si muova nella direzione della forza $\mathbf{F}$, cioè si è usato per la componente di $\mathbf{f}_0$ lungo l'asse x $-f_0 < 0$. Se la slitta si muovesse, ad un certo istante nella direzione negativa

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dell'asse x, essendo decelerata dalla forza $\mathbf{F}$, si sarebbe dovuto usare per la componente di $\mathbf{f}_0$ lungo l'asse x $f_0 > 0$. Affinché la cassa resti solidale con la slitta occorre e basta che sia soddisfatta la condizione

$$0 \leq f = m\frac{F - \mu_c(M+m)g}{M+m} \leq \mu_s mg \implies F \leq g(M+m)(\mu_s + \mu_c)$$

Questo risultato sulla forza massima può essere applicato all'analogo problema di un foglio estratto velocemente da sotto un peso: se la forza è troppo elevata il peso resta fermo.

24-035 L'auto in Curva

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Si deve progettare una strada con un tratto di curva circolare il cui raggio di curvatura è $R = 100$ m. Si determini l'angolo di inclinazione ottimale della strada nel tratto di curva affinché un'auto con velocità $v_0 = 70$ km/h possa percorrerla senza dipendere dall'aderenza dei pneumatici in modo cioè che la sola reazione normale della strada fornisca la forza centripeta necessaria. Supponendo che la curva sia costruita con questo angolo ottimale determinare la massima velocità che un auto può tenere senza perdere aderenza nel caso in cui il coefficiente di attrito statico tra i pneumatici e la strada vale $\mu_{aa} = 0.75$ (asfalto asciutto), $\mu_{ab} = 0.50$ (asfalto bagnato), $\mu_g = 0.25$ (ghiaccio).

WORKED SOLUTION

Si schematizzi l'auto come un punto materiale e si scelga un Coordinate System con asse x orizzontale diretto verso il centro di curvatura della strada e asse y verticale. La condizione affinché l'auto percorra il tratto di curva circolare è

$$m\mathbf{g} + \mathbf{N} + \mathbf{f} = m\mathbf{a}_c$$

che proiettata sulle componenti da il sistema

$$N\sin[\theta] + f\cos[\theta] = m\frac{v_0^2}{R} \quad ,$$
$$N\cos[\theta] - f\sin[\theta] - mg = 0 \quad .$$

dove si è supposto che la forza di attrito statico sia diretta verso l'interno della curva. Nel caso la forza di attrito statico necessaria sia diretta in verso opposto a quello convenzionalmente scelto (cioè verso l'esterno della curva) si otterrà un valore di f negativo. Volendo sempre interpretare f come il modulo del vettore $\mathbf{f}$, cioè un numero positivo, il segno meno andrebbe riassorbito nei coseni direttori del vettore $\mathbf{f}$. Le incognite sono dunque i moduli delle forze $N \equiv |\mathbf{N}|$ e $f \equiv |\mathbf{f}|$ tenendo conto della precisazione fatta sopra circa il caso in cui f risultasse negativo. Il problema non sussiste per N in quanto, essendo il vincolo unilatero, il verso di $\mathbf{N}$ è noto a priori. La soluzione del sistema fornisce

$$f = m\frac{v_0^2}{R}\cos[\theta] - mg\sin[\theta] \quad ,$$
$$N = mg\cos[\theta] + m\frac{v_0^2}{R}\sin[\theta]$$

La forza di attrito vale zero se e solo se

$$\boxed{\tan[\theta] = \frac{v_0^2}{Rg}} = 0.385 \implies \theta = 21.1^\circ$$

che fornisce, in funzione di R e θ l'angolo di cui occorre inclinare la curva affinché la sola reazione normale della strada fornisca la forza centripeta necessaria. Nel caso $\theta = 0$ la relazione fornisce correttamente $f = mv_0^2/R$ e $N = mg$. La relazione ricavata per f mostra che se v_0 cresce il modulo

di **f** cresce e f risulta positivo, concordemente con l'orientamento convenzionale scelto per il vettore **f**. Il caso in cui f, che é un modulo, diventa negativo (per v_0 piccola) va interpretato come il fatto che in tali condizioni il vettore **f** è diretto in verso opposto a quello convenzionalmente scelto. In tal caso il segno negativo di f potrebbe essere riassorbito nei $\cos[\theta]$ e $\sin[\theta]$ delle equazioni del moto proiettate.
La velocità massima, v, per un'auto sulla curva si determina dalla condizione

$$0 \le f \le \mu_s N \implies \frac{v^2}{gR}(\cos[\theta] - \mu_s \sin[\theta]) \le (\sin[\theta] + \mu_s \cos[\theta]) \qquad \text{per } f \ge 0$$

con **f** diretta verso l'interno della curva.
Nel caso particolare $\theta = 0$ la relazione diventa $v^2 \le \mu_s R g$.
Essendo sempre $\cos[\theta] \ge 0$ e $\sin[\theta] \ge 0$ si ha che nel caso in cui $\tan[\theta] > 1/\mu_s$ la condizione è sempre verificata cioè in tal caso la pendenza della curva e l'attrito statico sono in grado di impedire all'auto di uscire dalla curva verso l'esterno per quanto grande sia la sua velocità. Ciò è possibile in quanto più θ è grande e più la reazione normale **N** è inclinata verso l'orizzontale fornendo la forza centripeta necessaria.
Si noti che questa condizione limita l'uscita verso l'esterno dell'auto e non la possibile caduta verso l'interno nel caso in cui v è piccolo e/o θ è grande. Con i valori del problema si ottengono per le velocità massime nei tre casi

$$v_{aa} = 142 \text{ km/h} \qquad v_{ab} = 118 \text{ km/h} \qquad v_g = 94.5 \text{ km/h}$$

La relazione ricavata per $\theta = 0$ può essere invertita per ottenere la condizione su R data una certa velocità: $R > v_0^2/g\mu_s$ che da il raggio di curvatura minimo ottimale a cui può essere percorsa una curva circolare orizzontale ad una velocità fissata. Si osservi che tale raggio di curvatura è doppio della distanza ($v_0^2/2g\mu_s$) in cui l'auto potrebbe fermarsi se frenasse in condizioni ottimali.
La condizione affinché l'auto non cada verso l'interno della curva si determina imponendo che

$$0 \le -f \le \mu_s N \implies \frac{v^2}{gR}(\cos[\theta] + \mu_s \sin[\theta]) \ge (\sin[\theta] - \mu_s \cos[\theta]) \qquad \text{per } f \le 0$$

con **f** diretta verso l'esterno della curva.
Nel caso particolare $\theta = \pi/2$ la relazione diventa $v^2 \ge Rg/\mu_s$.
Essendo sempre $\cos[\theta] \ge 0$ e $\sin[\theta] \ge 0$ si ha che nel caso in cui $\tan[\theta] < \mu_s$ la condizione è sempre verificata cioè in tal caso l'attrito statico è in grado di impedire all'auto di uscire dalla curva verso l'interno.
Se la velocità è sufficientemente alta dunque l'auto potrebbe restare sulla curva circolare anche nel caso in cui questa sia verticale. Tale effetto è sfruttato anche in certi giochi da Luna-Park ("Il Rotore") § 24-047.
Si noti che se $\mu_s > 1$ si può avere, per un θ opportuno $1/\mu_s < \tan[\theta] < \mu_s$ e l'auto può restare in curva qualunque sia la sua velocità.

24-036 L'auto Sulla Strada Con Dossi E Avvalamenti

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24-037 Misuratore Di Velocità Angolare a Forza Centrifuga

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This § is referenced at pages:
[1380]

Un disco orizzontale ruota attorno al proprio asse con velocità angolare costante e ignota ω. Al bordo del disco, che ha raggio R, è appeso un pendolo di lunghezza l che può ruotare solo nel piano verticale passante per l'asse di rotazione del disco. L'angolo α che il pendolo forma con la verticale dipende perciò dalla velocità angolare del disco e può quindi essere usato per misurare la velocità angolare. Si determini la relazione che lega la velocità angolare del disco ω, il suo raggio R, la lunghezza l del pendolo e l'angolo α. Si veda anche il problema § 24-068

24-038 Il Blocco Che Scivola Su Un Piano Inclinato Libero - Cardinal Equation

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|5.68||

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This § is referenced at pages:
[1334, 1334, 1346, 1346, 1346]

Un piano inclinato di massa M e angolo θ è appoggiato su un tavolo e sorregge un blocco di massa m. Si supponga che tra il piano inclinato e il tavolo e tra il piano inclinato e il blocco non ci sia attrito. Il sistema viene lasciato libero con blocco e piano inclinato in quiete.

1. Quale è l'accelerazione del piano inclinato e quali sono le componenti orizzontale e verticale dell'accelerazione del blocco? Queste accelerazioni si riducono ai valori corretti nel caso in cui $M \gg m$? Quale accelerazione orizzontale del piano è necessaria, rispetto al tavolo, affinché il corpo resti fermo rispetto al piano? Quale forza orizzontale bisogna applicare al sistema per ottenere questo risultato?

WORKED SOLUTION

Si scelga un sistema di coordinate con asse x orizzontale e asse y verticale e origine sul vertice superiore del piano inclinato. Si supponga il piano inclinato orientato con l'angolo $\theta < 90°$ nel verso positivo dell'asse x e, di conseguenza, con l'angolo di $90°$ orientato nel verso negativo. Detta $\mathbf{N}$ la reazione vincolare del piano inclinato sul corpo e $\mathbf{R}$ quella del tavolo sul piano inclinato le equazioni del moto del sistema si scrivono come

$$\begin{aligned} -N\sin[\theta] &= MA_x \quad , \\ R - Mg - N\cos[\theta] &= MA_y = 0 \qquad \text{(il piano è vincolato sul tavolo)} \quad , \\ N\sin[\theta] &= ma_x \quad , \\ N\cos[\theta] - mg &= ma_y \quad . \end{aligned}$$

Le incognite sono R, N, A_x, a_x e a_y con sole quattro equazioni. Esiste tuttavia un vincolo cinematico. Detti infatti $\Delta x'$ e $\Delta y'$ gli spostamenti del blocco rispetto al piano inclinato, ΔX e ΔY gli spostamenti

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del piano rispetto al tavolo e Δx e Δy gli spostamenti del corpo rispetto al tavolo si ha la condizione cinematica:

$$-\Delta y' = \Delta x' \tan[\theta]$$

che lega i cambiamenti di coordinate del corpo sul piano inclinato. Dalle relazioni che legano il moto in due sistemi di coordinate diversi si ottiene:

$$\begin{aligned} x &= X + x' \implies \Delta x = \Delta X + \Delta x' \quad , \\ y &= y' \implies \Delta y = \Delta y' \end{aligned}$$

Ne segue per le accelerazioni

$$\begin{aligned} a_x &= A_x + a'_x \quad , \\ a_y &= a'_y \end{aligned} \tag{24.22.03}$$

da cui si ottiene il legame voluto

$$-a_y = \tan[\theta]\,(a_x - A_x)$$

che fornisce la quinta equazione necessaria per risolvere le equazioni del moto. Si noti che dal sistema (24-038) si deduce

$$-\frac{A_x}{a_x} = \frac{m}{M}$$

e quindi

$$-a_y = a_x\left(1 + \frac{m}{M}\right)\tan[\theta] \tag{24.22.04}$$

Notare che si ha

$$\begin{aligned} \frac{d}{dt}\Delta x &= \frac{dx}{dt} \quad , \\ \frac{d}{dt}\Delta x' &= \frac{dx'}{dt} \quad , \\ \frac{d}{dt}\Delta X &= \frac{dX}{dt} \end{aligned}$$

E che la derivata temporale nei due sistemi di coordinate coincide essendo questi animati da moto reciproco traslatorio ($\boldsymbol{\omega} = 0$). Notare che la seconda delle equazioni del sistema (24-038) è la sola che contiene R e sarà quindi usata solo per determinare tale reazione vincolare. La soluzione del sistema fornisce

$$\begin{aligned} a_x &= +\,g\left[\frac{M\tan[\theta]}{M(1+\tan^2\theta) + m\tan^2\theta}\right] = +g\left[\frac{M\sin[\theta]\cos[\theta]}{M + m\sin^2\theta}\right] \quad , \\ a_y &= -\,g\left[\frac{(M+m)\tan^2\theta}{M(1+\tan^2\theta) + m\tan^2\theta}\right] = -g\left[\frac{(M+m)\sin^2\theta}{M + m\sin^2\theta}\right] \quad , \\ A_x &= -\,g\left[\frac{m\tan[\theta]}{M(1+\tan^2\theta) + m\tan^2\theta}\right] = -g\left[\frac{m\sin[\theta]\cos[\theta]}{M + m\sin^2\theta}\right] \quad . \end{aligned}$$

Analizziamo alcuni casi limite.
Per $\theta = 0$ risulta $a_x = a_y = A_x = 0$.
Per $\theta = \pi/2$ risulta $a_x = A_x = 0$ e $a_y = -g$.
Per $M \gg m$ risulta $a_x = g\sin[\theta]\cos[\theta]$, $a_y = -g\sin^2\theta$ e $A_x = 0$.
Per $m \gg M$ risulta $a_x = 0$, $a_y = -g$ e $A_X = -g/\tan[\theta]$.
Si veda pure § 24.21.10.

→ 1370
→ 1359

24-039 Il Piano Oscillante Con La Massa Sopra

Un piano orizzontale può oscillare lungo la verticale con legge del moto armonico di pulsazione ω grazie ad un molla opportuna. Il piano parte da fermo dalla quota $x = -x_0 < 0$ rispetto alla posizione di riposo della molla, posta alla quota $x = 0$. Una massa m giace sul piano, in quiete rispetto allo stesso nel momento in cui il piano viene rilasciato.

1. Determinare sotto quali condizioni la massa si stacca dal piano e in quale posizione.
2. Determinare la velocità della massa all'istante dello stacco.

WORKED SOLUTION

1) Lo stacco si ha per $x(t)$ tale che $x(t)\omega^2 \geq g$. 2) $v = \omega x_0 \sqrt{1 - g^2/\omega^4 x_0^2}$ quando $x = g/\omega^2$.

24-040 Resistenza proporzionale alla velocità - Il cannone - La pioggia

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|6.70 6.71 6.72 6.73||

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This § is referenced at pages:
[1260]

Discutere il moto di un punto materiale soggetto ad una forza costante (per esempio la forza di gravità) e ad una resistenza del mezzo proporzionale alla velocità $\mathbf{F_R} = -k\mathbf{v}$ dove k è una costante positiva che dipende dalle caratteristiche dell'oggetto e del mezzo. Si consideri il moto di un proiettile sparato con velocità iniziale $\mathbf{v}_0$. Si introduca Coordinate System con asse x parallelo al suolo e nella direzione verso cui è sparato il proiettile e asse y verticale diretto verso l'alto. Si mostri che la legge oraria

$$x(t) - x_0(t) = \frac{v_{x0}}{b}[1 - \exp -b\Delta t] \quad ,$$

$$y(t) - y_0(t) = -\frac{g}{b}\Delta t + \frac{1}{b^2}(gv_{y0} + b)[1 - \exp -b\Delta t]$$

in cui $k \equiv mb$ e $\Delta t \equiv t - t_0$, soddisfa l'equazione del moto. Si mostri che la velocità è data dalle relazioni

$$v_x(t) = v_{x0} \exp -b\Delta t \quad ,$$

$$v_y(t) = -\frac{g}{b} + \frac{1}{b}(gv_{y0} + b) \exp -b\Delta t$$

WORKED SOLUTION

Sviluppando le (24-040) in serie di b con $b \simeq 0$, si ritrova l'equazione del moto senza resistenza dell'aria. L'equazione della traiettoria si può ricavare dalle (24-040) sostituendo la grandezza in parentesi quadra nella equazione della componente x nella equazione per la componente y e ricavando Δt dalla equazione della componente x e sostituendola nella equazione per la componente y. Si trova

$$y - y_0 = \frac{g}{b^2}\log 1 - \frac{b(x - x_0)}{v_{x0}} + \frac{1}{b}(gv_{y0} + b)\left(\frac{x - x_0}{v_{x0}}\right)$$

→ 1371

Per ottenere l'equazione della traiettoria occorre sviluppare per $b \simeq 0$ all'ordine b^2. Si riottiene così, per esempio, la formula per la gittata. Lo sviluppo all'ordine b^3 (o superiori) fornisce le correzioni dovute alla resistenza dell'aria.

24-041 Resistenza proporzionale al quadrato della velocità - Il paracadute

©|Alonso & Finn, , ..., ..., ...Ed., ...|7.59 7.86||
©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|1.104||

Un oggetto si muove sotto l'azione di una forza costante $\mathbf{F}$ in un mezzo che presenta una forza resistente $\mathbf{F_R} = -bv\mathbf{v}$ dove b è una costante positiva che dipende dalle caratteristiche dell'oggetto e del mezzo. Supponendo che l'oggetto parta all'istante $t = 0$ dalla posizione $\mathbf{x} = \mathbf{x}_0 = 0$ con velocità $\mathbf{v}_0$ parallela a $\mathbf{F}$ dimostrare che la velocità limite è data da

$$v_L = \sqrt{F/b}$$

che l'andamento della velocità in funzione del tempo è dato da

$$v = v_L \frac{(v_0 + v_L)\exp bv_L t/m + (v_0 - v_L)\exp -bv_L t/m}{(v_0 + v_L)\exp bv_L t/m - (v_0 - v_L)\exp -bv_L t/m}$$

e che la relazione tra velocità e spazio percorso è

$$v^2 = F/b + \left[v_0^2 - F/b\right]\exp -2b(x - x_0)/m$$

PRELIM

WORKED SOLUTION

L'equazione del moto, supposto unidimensionale in quanto $\mathbf{F} \parallel \mathbf{v}_0$ implica $\mathbf{F} \parallel \mathbf{v}$, è

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} - bv\mathbf{v}$$

che proiettata sull'asse x fornisce (supponendo $\mathbf{F}\cdot\mathbf{v_0} > 0$)

$$m\frac{dv}{dt} = F - bv^2 \qquad (24.22.05)$$

→ 1372

Se $F - bv_0^2 > 0$ l'accelerazione è positiva e la velocità cresce fino ad una velocità limite, v_L,

$$\frac{dv}{dt} = 0 \implies v_L = \sqrt{F/b}$$

Se $F - bv_0^2 < 0$ l'accelerazione è negativa e la velocità decresce fino alla stessa velocità limite. Nel caso $F - bv_0^2 = 0$ l'accelerazione è nulla e la velocità resta costante e uguale alla velocità limite. L'equazione del moto può essere integrata per separazione delle variabili. Supponiamo $v_0 < v_L$.

$$\frac{dv}{1 - bv^2/F} = \frac{Fdt}{m}$$

Integrando e tenendo conto della condizione iniziale $v_0 < v_L$ si trova

$$\log\left(\frac{v_L + v}{v_L - v}\right) = \frac{2\sqrt{Fb}}{m}t + cost \quad ,$$

$$cost = \frac{v_L + v_0}{v_L - v_0}$$

da cui si deduce il risultato richiesto per $v(t)$.
Per ricavare la dipendenza della velocità dalla posizione, $v(x)$, si parte dalla stessa equazione (24.22.05) e la si riscrive come

$$mv\frac{dv}{dx} = F - bv^2$$

e procedendo per separazione delle variabili come sopra si ottiene, nel caso $v_0 < v_L$,

$$v^2 = F/b + \left[v_0^2 - F/b\right]\exp -2b(x - x_0)/m$$

come richiesto.

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24-042 La Catena Che Scivola Dal Bordo Del Tavolo

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©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|3.85 3.86 3.111||

Una catena ideale, perfettamente flessibile e inestensibile, di lunghezza L e densità per unità di lunghezza λ, scivola dal bordo di un tavolo.

1. Supponendo che non ci sia attrito, che la catena parta da ferma e che all'istante iniziale una parte lunga x_0 di essa penda dal bordo del tavolo dimostrare che il tempo occorrente alla catena per scivolare dalla tavola vale
$$\sqrt{\frac{L}{g x_0^2}} \log L + \sqrt{L^2 - x_0^2}$$
Si supponga ora che il coefficiente di attrito cinetico tra la catena e il piano del tavolo valga μ_c.
2. Determinare l'accelerazione della catena in funzione della lunghezza x della parte di catena che pende dal bordo del tavolo.
3. Dimostrare che in tal caso, supponendo che nella posizione iniziale la forza di attrito statico non sia in grado di tenere ferma la catena, il tempo occorrente alla catena per scivolare dalla tavola vale
$$\sqrt{\frac{L}{g\,(1+\mu)}} \log \left\{ \frac{L + \sqrt{L^2 - [x_0\,(1+\mu) - L\mu]^2}}{x_0\,(1+\mu) - L\mu} \right\}$$
4. Se il coefficiente di attrito statico vale μ_s dimostrare che la catena non cade dal tavolo se e solo se
$$x_0\,(1+\mu_s) \leq \mu_s L$$

WORKED SOLUTION

2) $a = (g/L)\,(x\,(1+\mu_c) - \mu_c L)$

24-043 La Catenaria

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|7.32||

The fastest route is not always the straight one.

24-044 Il Peso Apparente in Ascensore

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24-045 La Cassa Sull'autocarro Che Frena

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DRAFT IN PROGRESS

24-046 La Moneta Sul Piatto Del Giradischi

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Una moneta è sistemata a $r = 13$ cm dall'asse del piatto di un giradischi. Si osserva che la moneta non si muove quando il giradischi ruota alla velocità angolare di 33.3 giri/s mentre scivola via quando il giradischi ruota alla velocità angolare di 45.0 giri/s. Determinare entro quali limiti è compreso il coefficiente di attrito statico tra la moneta e il piatto del giradischi.

24-047 Il Rotore Del Luna Park

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This § is referenced at pages:
[1367]

24-048 L'imbuto Ruotante

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|6.53||

Un piccolo cubo di massa m è deposto all'interno di un imbuto che ruota con velocità angolare costante ω. La parete dell'imbuto forma un angolo θ con l'orizzontale, il coefficiente di attrito statico tra il cubo e la parete dell'imbuto vale μ_s e il cubo si trova a distanza r dall'asse di rotazione.

1. Determinare il valore massimo della velocità angolare per cui il cubo non si muove rispetto all'imbuto.
2. Determinare il valore minimo della velocità angolare per cui il cubo non si muove rispetto all'imbuto.

WORKED SOLUTION

$$1)\ \omega_{\max} = \sqrt{\frac{g\left(\tan\left[\theta\right] + \mu_s\right)}{r\left(1 - \mu_s \tan\left[\theta\right]\right)}};\ 2)\ \omega_{\min} = \sqrt{\frac{g\left(\tan\left[\theta\right] - \mu_s\right)}{r\left(1 + \mu_s \tan\left[\theta\right]\right)}};$$

24-049 Periodo Di Un Pendolo in Un Treno in Curva

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|6.80||

Il periodo di un pendolo semplice vale T. Dimostrare che il periodo T' dello stesso pendolo quando si trova appeso al soffitto di un treno che percorre un binario circolare di raggio R con velocità di modulo v_0 vale

$$T' = T\frac{\sqrt{Rg}}{\sqrt{v_0^4 + R^2 g^2 4}}$$

PRELIMINARY IN PROGRESS

24-050 Calcolo Approssimato Della Deviazione Verso Est Della Caduta Dei Gravi

This § is referenced at pages:
[1343]

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©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|6.15 6.16 6.17||

Si stimi approssimativamente l'entità della deviazione verso est di un grave in una (breve) caduta libera da fermo alla superficie terrestre nell'emisfero nord. Di quanto devia verso est un grave lasciato cadere da un'altezza $h = 100$ m alla latitudine di $45°$?

Compare with problem § 24.21.07.

WORKED SOLUTION

L'accelerazione relativa al Reference Frame terrestre, non inerziale, è data da

$$\mathbf{a} = \mathbf{g}_{\mathtt{eff}} - 2\,\boldsymbol{\omega}\wedge\mathbf{v}$$

dove $\mathbf{g}_{\mathtt{eff}}$ è l'accelerazione effettiva alla superficie della Terra che include il contributo della forza centrifuga. L'accelerazione $\mathbf{g}_{\mathtt{eff}}$ giace nel piano meridiano della Terra e quindi una deviazione trasversale può solo derivare dal termine di Coriolis. Possiamo supporre, approssimativamente, che sia $\mathbf{g} = \mathbf{g}_{\mathtt{eff}}$ in modo che sia $\mathbf{g} \parallel \mathbf{r}$. Questo equivale a trascurare la forza centrifuga rispetto alla forza peso. Detta $\mathbf{v}_m$ la componente della velocità del grave nel piano meridiano l'accelerazione trasversale a tale piano, e diretta verso est nell'emisfero nord, ha modulo

$$a_{\mathtt{trasv}} = 2\omega v_m \cos[\lambda] \tag{24.22.06}$$

→ 1376

avendo indicato con λ la latitudine. In prima approssimazione, per un tragitto breve cosicché v_m resta sufficientemente piccola in modulo, si può sostituire v_m con il valore della velocità che si calcolerebbe trascurando l'effetto della forza di Coriolis. Ciò è giustificato dal fatto che nelle ipotesi fatte l'accelerazione di Coriolis rappresenta solo una piccola correzione alla $\mathbf{g}_{\mathtt{eff}}$ essendo $2\omega v_m \cos[\lambda] \ll g$ per v_m abbastanza piccolo. Allora si può porre, essendo $v = gt$,

$$\boxed{a_{\mathtt{trasv}} \equiv \ddot{y} \simeq 2\omega\cos[\lambda]\, v \simeq 2\omega\cos[\lambda]\, gt}$$

che integrata con la condizione iniziale $\dot{y}(t=0) = 0$, essendo la velocità trasversa iniziale nulla, fornisce

$$\boxed{\Delta y = \frac{1}{3} g\omega\cos[\lambda]\, t^3} \tag{24.22.07}$$

Nelle approssimazioni fatte la deviazione si calcola usando la relazione

$$h = \frac{g}{2}t^2$$

e risulta

$$\boxed{\Delta y \simeq \frac{2}{3} h\omega\cos[\lambda]\sqrt{\frac{2h}{g}}} = 1.55 \text{ cm}$$

Si noti che la velocità trasversale da origine, tramite la forza di Coriolis, ad una correzione alla velocità nel piano meridiano che è però una correzione di secondo ordine.

24-051 Deviazione Verso Ovest Di Un Oggetto Lanciato Verso L'alto

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Un punto materiale è lanciato verticalmente verso l'alto con velocità v_0. Dimostrare che ricadrà in un punto spostato verso ovest di una distanza pari a

$$\Delta y = \frac{4}{3}\omega \cos[\lambda]\sqrt{\frac{8h^3}{g}}$$

dove h è l'altezza a cui sale il punto materiale.

WORKED SOLUTION

In generale, risolvendo l'equazione (24.22.06), si trova

$$\Delta y = \dot{y}(t=0)t + \omega v_0 \cos[\lambda]\, t^2 + \frac{1}{3}\omega a \cos[\lambda]\, t^3$$

dove a è l'accelerazione nel piano meridiano. Si noti che il segno della forza di Coriolis in salita ed in discesa è diverso.

24-052 Oggetto Fermo Visto Da Una Giostra in Moto

Un bambino si trova fermo al centro di una giostra che ruota con velocità angolare costante Ω. Egli osserva i suoi genitori che sono fermi a terra ma che, dal suo punto di vista, si muovono di moto circolare uniforme rispetto al centro della giostra. Nella descrizione del bambino chi fornisce la forza centripeta necessaria per mantenere i genitori in moto circolare uniforme?

WORKED SOLUTION

Si considerino le sole forze agenti sul piano orizzontale. Sui genitori non agisce alcuna forza con componenti nel piano orizzontale. Dal punto di vista del bambino, osservatore non inerziale, sui genitori agiscono però le forze (fittizie) centrifuga e di Coriolis. La loro risultante fornisce la forze centripeta richiesta.

24-053 Equazione Dei Grandi Moti Di Un Proiettile

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|6.25 6.26 6.89 6.90||

24-054 Equazione Della Traiettoria Del Grande Moto Di Un Proiettile

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DRAFT - UNDER PEER'S

24-055 Deviazioni Di Ordine Superiore Del Moto Alla Superficie Terrestre

©|Sch, , ..., ..., ...Ed., ...|1.22||

24-056 Accelerazione Di Protoni

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Un protone (nucleo dell'isotopo più leggero dell'atomo di idrogeno) può essere accelerato da un acceleratore lineare lungo un tratto rettilineo con accelerazione $a = 3.60 \cdot 10^{15}\ \mathrm{m/s^2}$. Un protone entra in un tratto accelerante lungo $l = 3.50\ \mathrm{cm}$ con una velocità iniziale $v_0 = 2.40 \cdot 10^{7}$ ms.

1. Si calcoli la velocità del protone al termine dell'accelerazione.
2. Si calcoli il guadagno di energia cinetica del protone. La massa del protone vale $m = 1.67 \cdot 10^{-27}\ \mathrm{kg}$. Si ignorino gli effetti relativistici sul moto del protone.

24-057 Collisioni Tra Comete E La Terra E Crateri Prodotti

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Una cometa di massa $m = 8.30 \cdot 10^{11}\ \mathrm{kg}$ urta la Terra con velocità relativa $v = 30\ \mathrm{km/s}$.

1. Si calcoli l'energia cinetica della cometa espressa in megatoni di TNT. Un megatone di TNT corrisponde all'energia liberata nell'esplosione di un milione di tonnellate di tritolo ed è pari a $4.2 \cdot 10^{15}$ J. Il diametro del cratere prodotto da una grande esplosione è proporzionale alla radice cubica dell'energia sviluppata; con un megatone di TNT si produce un cratere di diametro circa pari a $D = 1\ \mathrm{km}$.
2. Quanto misura il diametro D del cratere prodotto dall'impatto della cometa sulla Terra?

WORKED SOLUTION

1) $\mathcal{K} = 9.0 \cdot 10^{4}$ megatoni di TNT; 2) $D = 45\ \mathrm{km}$.

24-058 La Catena Sul Tavolo

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Una catena è trattenuta ad un estremo su un tavolo privo di attrito con una quarto della sua lunghezza che pende dal bordo. Se la catena è lunga l ed ha massa m quale è il lavoro L che occorre fare per portare sul tavolo la parte pendente della catena?

WORKED SOLUTION

$L = mgl/32$.

DRAFT - IN PROGRESS

24-059 La Palla Non Elastica

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Una palla, rimbalzando su un selciato di cemento, perde una frazione $f = 15\%$ della sua energia cinetica. Con quale velocità v_0 bisogna lanciare la palla verticalmente verso il basso dall'altezza $h = 12.4$ m affinché risalga alla stessa quota h? Si trascuri la resistenza dell'aria.

WORKED SOLUTION

$v_0 = 6.55\,\text{ms}$.

24-060 Il Blocco Che Cade Sulla Molla

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|7.33||

Un blocco di massa $m = 263$ g è lasciato cadere verticalmente sopra una molla di costante elastica $k = 2.52{\cdot}10^2$ N/m. Il blocco colpisce la molla che si accorcia di $\Delta x = 11.8$ cm prima di fermarsi momentaneamente. Quanto lavoro viene compiuto nella compressione della molla

1. dalla forza di gravità L_g,
2. dalla molla L_m?
3. Quanto vale la velocità del blocco v_1 un istante prima di colpire la molla?
4. Se la velocità iniziale della molla viene raddoppiata quale è il valore della compressione $\Delta x'$ della molla? Si ignorino gli attriti.

WORKED SOLUTION

1) $L_g = 304$ mJ; 2) $L_m = -1.75$ J; 3) $v_1 = 3.32\,\text{ms}$; 4) $\Delta x' = 22.5$ cm.

24-061 Potenza Sviluppata Da Un Transatlantico

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|7.42||

Il transatlantico Queen Elizabeth II è equipaggiato con un motore diesel che sviluppa una potenza massima di $P = 32$ kW alla velocità di crociera di $v = 32.5$ nodi. Quanto vale la forza esercitata dalle eliche sull'acqua in queste condizioni?

24-062 La Centrale Idroelettrica

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|7.44||

Da una cascata alta $h = 96.3$ m scendono ogni minuto $V = 7.38{\cdot}10^5$ m^3 di acqua. Supponendo che un generatore idroelettrico sia in grado di convertire in energia elettrica una frazione $f = 58.0\%$ dell'energia cinetica acquistata dall'acqua nella caduta calcolare la potenza sviluppata dal generatore. La densità dell'acqua è $\rho = 1$ g/cm^3.

DRAFT IN PROGRESS

24-063 L'argano

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|7.47||

Un blocco di granito di massa $m = 1380$ kg viene trascinato su per un piano inclinato alla velocità costante di $v = 1.34$ ms da un argano. Se il coefficiente di attrito cinetico tra il blocco e il piano vale $\mu_s = 0.41$ e se l'altezza e la lunghezza della base del piano inclinato valgono rispettivamente $h = 28.2$ m e $b = 39.4$ m, quale potenza W deve sviluppare l'argano?

WORKED SOLUTION

$W = 16.6$ kW.

24-064 La Potenza Dell'automobile

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|7.52||

Si dimostri che la velocità v raggiunta da un'automobile di massa m che viaggia sviluppando una potenza costante P vale, in funzione dello spazio x percorso dall'auto partendo da ferma, $v = [3xP/m]^{1/3}$.

24-065 La Potenza Dell'aereoplano

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|7.53||

Si supponga che la resistenza aereodinamica al moto di un aereoplano sia $D = bv^2$, con b una costante che dipende dalle proprietà del mezzo e dell'aereoplano ma non dalla velocità.

1. Si dimostri che la potenza sviluppata da un aereoplano che naviga orizzontalmente a velocità costante v è proporzionale a v^3.
2. Di quale fattore p' occorre aumentare la potenza dei motori per far crescere la velocità di $v' = 25\%$?

WORKED SOLUTION

2) $p' = 1.95$

24-066 L'automobile

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|7.57||

La resistenza al moto di un'automobile dipende dall'attrito al suolo, che è in prima approssimazione indipendente dalla velocità, e dalla resistenza dell'aria, che dipende dal quadrato della velocità. Per una particolare automobile, che pesa $P = 1.2{\cdot}10^4$ N, la resistenza totale al moto è data dalla relazione $R = a + bv^2$ con $a = 300$ N e $b = 1.8$ N/s^2/ m^2. Si calcoli la potenza P necessaria per accelerare la macchina di $A = 0.92$ m/s^2 quando la velocità vale $v = 80$ km/h.

WORKED SOLUTION

$P = 69$ hp.

DRAFT. NOT FOR DISTRIBUTION

24-067 Covarianza Del Teorema Lavoro-Energia

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|7.59||

Si considerino due osservatori uno solidale con il suolo e l'altro su un treno che si muove con velocità **u** costante rispetto al suolo. Entrambi osservano una particella, inizialmente ferma rispetto al treno, accelerare per effetto di una forza applicata in avanti per un tempo t.

1. Si dimostri che per entrambi gli osservatori il lavoro fatto dalla forza uguaglia la variazione di energia cinetica della particella e che tale lavoro vale, rispettivamente per l'osservatore, al suolo e quello sul treno, $L_s = 1/2ma^2t^2 + maut$ $L_t = 1/2ma^2t^2$
2. Si spieghi la diversità del lavoro fatto dalla stessa forza nei due sistemi in termini del diverso percorso della particella nel tempo t visto dai due sistemi.
3. Si spieghi la diversità delle energie cinetiche della particella nei due sistemi in termini dell'energia necessaria per fermare la particella nei due sistemi.

24-068 Regolatore a Forza Centrifuga

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|7.58||

This § is referenced at pages:
[1369]

Un regolatore consiste in due sferette di massa $m = 200$ g attaccate ad un asse verticale ruotante mediante due asticelle rigide, di lunghezza $l = 10$ cm, vincolate a muoversi nel piano dell'asse di rotazione. Quando l'angolo θ formato dalle due sferette con la verticale raggiunge il valore $\theta_0 = 45^\circ$ le sfere toccano le pareti del cilindro entro il quale è inserito il regolatore.

1. Quale è la velocità minima di rotazione che consente alle sfere di toccare le pareti del cilindro?
2. Se il coefficiente di attrito cinetico tra le sferette e le pareti vale $\mu_c = 0.35$ quanto vale la potenza dissipata quando l'asse ruota con velocità angolare $\omega = 300$ giri/s?

Si veda anche il problema § 24-037

24-069 Il Cubetto Entro La Calotta Sferica

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|8.5||

Un piccolo cubetto di ghiaccio viene lasciato scivolare all'interno di un contenitore a forma di semisfera di raggio $r = 23.6$ cm partendo dal bordo. Se l'attrito è trascurabile quanto vale la velocità v del cubetto quando raggiunge il fondo del contenitore?

WORKED SOLUTION

$v = 2.15$ ms.

DRAFT - IN PROGRESS

24-070 Il Pendolo

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|8.8||

This § is referenced at pages:
[Never referenced.]

Un pendolo è formato da una massa m fissata all'estremo di una sbarra leggera di lunghezza L vincolata all'altro estremo a ruotare nel piano verticale. La massa può quindi muoversi solo lungo una circonferenza nel piano verticale. Il pendolo partendo dalla posizione orizzontale riceve una spinta verso il basso in modo tale da arrivare a fermarsi nella posizione verticale. Supponendo trascurabili gli attriti determinare la velocità iniziale da fornire alla palla.

24-071 Il Blocco Spinto Su per Un Piano Inclinato Da Una Molla

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|8.19||

Un blocco di massa $m = 1.93\ \mathrm{kg}$ è posto a contatto con una molla compressa che è situata alla base di un piano inclinato di angolo $\theta = 27.0^\circ$ e privo di attrito. La molla, di costante elastica $k = 20.8\ \mathrm{N/cm}$, viene compressa di $\Delta x_0 = 18.7\ \mathrm{cm}$ e quindi lasciata andare. Quanta strada d percorre il blocco lungo il piano inclinato prima di fermarsi? Si misuri la posizione finale del blocco rispetto a quella iniziale.

WORKED SOLUTION

$d = 4.24\ \mathrm{m}$.

24-072 Il Blocco Frenato Dalla Molla in Fondo Ad Un Piano Inclinato

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|8.20||

Una molla ideale priva di massa se sottoposta ad una forza $F = 268\ \mathrm{N}$ si comprime di $d = 2.33\ \mathrm{cm}$. Un blocco di massa $m = 3.18\ \mathrm{kg}$ è lasciato andare da fermo dall'alto di un piano inclinato che ha un angolo $\theta = 32.0^\circ$ alla cui base è posta la molla. Il blocco si ferma momentaneamente dopo aver compresso la molla di $\Delta x = 5.48\ \mathrm{cm}$.

1. Quanto strada ha percorso il blocco fino a questo istante?
2. Quale è la velocità del blocco un istante prima di toccare la molla?

24-073 Tarzan E La Liana

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Tarzan, la cui massa è $m = 81.63\ \mathrm{kg}$, salta da una collinetta appeso ad una liana di lunghezza $L = 15.24\ \mathrm{m}$. Dalla cima della collinetta da cui si lancia al punto più basso della traiettoria Tarzan scende di $h = 2.60\ \mathrm{m}$. Se la liana ha un carico di rottura di $T = 1112.1\ \mathrm{N}$ riesce a resistere senza spezzarsi?

DRAFT - IN PROGRESS

24-074 Il Punto Materiale Che Cade Da Una Semisfera Rovesciata

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|3.82||
©|Alonso & Finn, , ..., ..., ...Ed., ...|8.31||
©|H.C.Ohanian, , ..., ..., ...Ed., ...|7.55||
©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|8.36||
©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|1.87||

This § is referenced at pages:
[1383, 1383]

Un ragazzo di massa m è seduto in cima ad una grande calotta di ghiaccio di forma semisferica con raggio R. Si supponga nullo l'attrito tra il ragazzo e il ghiaccio. Se il ragazzo incomincia a scivolare partendo da fermo in quale punto si stacca dalla calotta? Per quale valore della velocità iniziale $\mathbf{v}_0$, orizzontale, il ragazzo si stacca subito? Quale è la sua velocità all'istante dello stacco? Quale è la sua traiettoria dopo lo stacco?

WORKED SOLUTION

Si schematizzi il ragazzo come un punto materiale. Si introduca un Coordinate System polari, centrato al centro della semisfera di ghiaccio, che individua la posizione del ragazzo, e si misuri l'angolo θ a partire dal punto più alto della semisfera. In tal modo $\theta(t=0)=0$. Detto N il modulo della reazione normale ed $\mathbf{F}$ la risultante delle forze che agiscono sul ragazzo, le equazioni del moto del ragazzo si scrivono, in coordinate polari,

$$F_r = N - mg\cos[\theta] = m\left(\ddot{r} - r\dot{\theta}^2\right) \tag{24.22.08}$$

→ 1383

$$F_\theta = mg\sin[\theta] = m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) \tag{24.22.09}$$

→ 1383

L'equazione per F_θ è equivalente ad applicare la legge della variazione del momento angolare rispetto al polo fisso O coincidente con il centro della semisfera

$$\boxed{\boldsymbol{\gamma} = \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t} + \mathbf{v}_0 \wedge \mathbf{p}} \tag{24.22.10}$$

Finché il ragazzo resta a contatto con la semisfera le incognite sono $\theta(t)$ ed N, essendo $r(t)=R$ noto. Il vincolo cioè, finché è effettivo, aggiunge una incognita (il valore della reazione vincolare) ma elimina un grado di libertà (la coordinata radiale). Dopo che il ragazzo si è staccato N non è più una grandezza fisica del problema e le due incognite diventano $r(t)$ e $\theta(t)$. Finché il ragazzo resta a contatto con la semisfera la seconda equazione permette di determinare il moto $\theta(t)$ mentre la prima determina la reazione vincolare. In tale condizione la seconda equazione può essere integrata moltiplicando entrambi i membri per $\dot{\theta}$ e sostituendo $r(t)=R$. Questo è equivalente ad applicare il principio di conservazione dell'energia meccanica. Scegliendo il livello di riferimento dell'energia potenziale gravitazionale al livello del suolo si ha

$$mgR + \frac{m}{2}v_0^2 = mgR\cos[\theta] + \frac{m}{2}v^2 \equiv E$$

da cui si deduce

$$v = R\dot{\theta} = \sqrt{v_0^2 + 2gR\left(1-\cos[\theta]\right)}$$

che fornisce la velocità in funzione della posizione sulla semisfera. Ora il valore della reazione vincolare può essere ricavato sostituendo l'espressione di $\dot{\theta}$ nella prima equazione

$$N = mg\left(3\cos[\theta] - 2\right) - \frac{mv_0^2}{R}$$

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La reazione normale è effettiva finché $N > 0$ e si ha $N(\theta = 0) = mg - \frac{mv_0^2}{R}$. Se $v_0 = 0$ lo stacco avviene cioè per $\cos[\theta] = 2/3$.

Se la velocità iniziale è $v_0 \neq 0$ lo stacco avviene subito se e solo se $N(\theta = 0) = mg - \frac{mv_0^2}{R} \leq 0$ cioè se e solo se $v_0^2 \geq Rg$. Tale risultato può anche essere ricavato utilizzando le componenti intrinseche delle quantità cinematiche.
La velocità del ragazzo all'istante dello stacco si ricava dall'espressione per v e vale $v = R\dot{\theta} = \sqrt{2gR/3}$.
Dopo lo stacco il moto del ragazzo è quello di un grave in caduta libera. Introduciamo un Coordinate System cartesiane centrato nel centro della semisfera, con asse x orizzontale e asse y verticale. La posizione iniziale, $\mathbf{r}_0$, e la velocità iniziale, $\mathbf{v}_0$, sono date dalle relazioni

$$\begin{aligned}\mathbf{r}_0 &= R\sin[\theta]\,\widehat{\mathbf{e}}_1 + R\cos[\theta]\,\widehat{\mathbf{e}}_2 \quad ,\\ \mathbf{v}_0 &= v\cos[\theta]\,\widehat{\mathbf{e}}_1 - v\sin[\theta]\,\widehat{\mathbf{e}}_2 \quad .\end{aligned}$$

e l'equazione del moto del ragazzo dopo lo stacco, se lo stacco avviene al tempo t_0, è

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0\,(t - t_0) - \frac{g}{2}\widehat{\mathbf{e}}_2(t - t_0)^2$$

Si veda la generalizzazione in section § 24-075
In presenza di attrito l'equazione per la componente radiale della forza (24.22.08) resta immutata mentre nell'equazione per la componente tangenziale (24.22.09) entra il contributo della forza di attrito. L'espressione della reazione normale che si ricava dalla prima equazione è dunque la stessa ma cambia la velocità che deve essere introdotta in tale equazione. L'energia meccanica non è più conservata ma occorre applicare il bilancio energetico. In ogni caso, per la presenza dell'attrito, la particella avrà, a parità di posizione, una velocità inferiore al caso senza attrito per cui lo stacco avverrà più tardi (cioè a θ maggiore).

24-075 Il Punto Materiale Che Cade Lungo Un Profilo Qualunque

©|BBM, , ..., ..., ...Ed., ...|2.3.7||

This § is referenced at pages:
[1382]

Si consideri una generalizzazione del problema § 24-074. Un punto materiale di massa m scivola lungo una guida piana posta in un piano verticale con la concavità rivolta verso il basso. La guida è descritta dalla funzione $g(x, y) = 0$ dove l'asse x è orizzontale e l'asse y è diretto verticalmente verso il basso. Il raggio di curvatura della guida, variabile, è la funzione $\rho(x, y)$.

1. Determinare in funzione di θ, v e ρ la condizione di distacco del punto dalla superficie.
2. Sapendo che il punto materiale viene lanciato dalla quota y_0 con velocità v_0 tangente alla guida determinare la quota y a cui avviene il distacco dalla guida in funzione di y, y_0, ρ, θ e v_0.

WORKED SOLUTION

Supponiamo $\partial g/\partial y \neq 0$. è possibile in tal caso scegliere il segno di $g(x, y)$ in modo tale che $\partial g/\partial y > 0$. In tal caso il versore normale è dato da

$$\widehat{\mathbf{n}} = \frac{\frac{\partial g}{\partial x}\widehat{\mathbf{e}}_1 + \frac{\partial g}{\partial y}\widehat{\mathbf{e}}_2}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}$$

Conviene utilizzare il Coordinate System intrinseco al profilo della guida dato che, fino a che il punto non si stacca, il moto del punto materiale coincide con il profilo della stessa. Si denoti con θ l'angolo formato tra la forza peso (cioè il versore $\hat{\mathbf{e}}_2$) e il versore normale alla guida. Le equazioni del moto proiettate sulla terna intrinseca sono

$$-N + mg\cos[\theta] = m\frac{v^2}{\rho} \quad ,$$

$$mg\sin[\theta] = m\frac{dv}{dt}$$

Da cui si ricava il valore di N

$$N = m\left(g\cos[\theta] - \frac{v^2}{\rho}\right)$$

Il punto si stacca dalla guida quando $N = 0$. Di conseguenza il punto resta attaccato alla guida finché

$$g\cos[\theta] - \frac{v^2}{\rho} > 0$$

e si stacca allorché

$$g\cos[\theta] - \frac{v^2}{\rho} = 0$$

In particolare all'istante iniziale il punto resta attaccato o si stacca a seconda che le condizioni iniziali siano soddisfatte o no all'istante iniziale. L'energia è conservata per cui si può scrivere

$$mgy_0 + \frac{m}{2}v_0^2 = mgy + \frac{m}{2}v^2 \equiv E$$

e quindi la condizione di distacco diventa

$$\rho(x,y)g\cos[\theta] - v_0^2 + 2g(y - y_0) = 0$$

che permette, insieme all'equazione della curva $g(x,y) =$ di determinare le coordinate del punto di distacco. Dalla soluzione trovata si deduce che il distacco avviene sempre con $\cos[\theta] > 0$, cioè il punto materiale lascia sempre la guida prima che diventi verticale. La formula trovata può essere applicata al caso della semisfera discussa in section § 24-074.

WORKED SOLUTION

1) Il punto si stacca quando $g\cos[\theta] - v^2/\rho = 0$. 2) Il punto si stacca quando $\rho(x,y)g\cos[\theta] - v_0^2 + 2g(y - y_0) = 0$.

24-076 Il Pendolo Balistico

©|H.C.Ohanian, , ..., ..., ...Ed., ...|7.51 7.52||

Un punto materiale è appeso in quiete ad una fune ideale di lunghezza R (un pendolo). Si colpisce il punto materiale con una forza impulsiva in modo tale da fornirgli, all'istante iniziale, una velocità $\mathbf{v}_0$ in direzione orizzontale. Quale deve essere il valore minimo della velocità iniziale affinché il punto materiale compia tutta la circonferenza? Quanto vale il valore massimo della tensione assunto dalla fune?

WORKED SOLUTION

Si supponga di misurare la posizione del punto con l'angolo θ a partire dalla posizione più bassa. Supponiamo che il punto si muova lungo una circonferenza di raggio R. Ciò è possibile se e solo se la tensione della fune è diretta verso il centro della circonferenza in quanto la fune non può esercitare che una trazione sul punto materiale. L'equazione del moto del punto in coordinate

intrinseche diventa, in tale condizione,

$$-T + mg\cos[\theta] = -m\frac{v^2}{R} \quad ,$$
$$-mg\sin[\theta] = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

che sono le equazioni del moto di un pendolo. Il principio di conservazione dell'energia meccanica si scrive, scegliendo come punto di riferimento per l'energia potenziale gravitazionale il punto più basso,

$$\frac{m}{2}v_0^2 = \frac{m}{2}v^2 + mgR(1-\cos[\theta])$$

da cui si ricava l'espressione per T

$$T = \frac{mv_0^2}{R} + mg(3\cos[\theta] - 2)$$

Finché $T > 0$ il pendolo si muove sulla circonferenza. Il valore minimo di T si ha per $\theta_{\mathtt{min}} = -\pi$ e si deve quindi avere

$$T_{\mathtt{min}} = \frac{mv_0^2}{R} - 5mg > 0 \implies v_0^2 > 5gR$$

affinché il pendolo percorra tutta la circonferenza.
La tensione della corda assume il suo valore massimo per $\theta_{\mathtt{max}} = 0$ e vale

$$T_{\mathtt{max}} = \frac{mv_0^2}{R} + mg$$

e con la velocità minima per percorrere tutta la circonferenza si ha $T_{\mathtt{max}} = 6mg$.

24-077 Il Giro Della Morte

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|8.37||

This § is referenced at pages:
[1468]

Una particella di massa m si muove all'interno di una guida circolare di raggio R disposta nel piano verticale. La velocità della particella nel punto più basso della sua traiettoria vale v_0.

1. Quale è il valore minimo $v_{\mathtt{min}}$ della velocità v_0 che consente alla particella di effettuare un giro completo senza perdere contatto con la guida?
2. La velocità v_0 vale $v_0 = 0.775v_{\mathtt{min}}$. Sia θ l'angolo che descrive la posizione della particella entro la guida, a partire dalla posizione orizzontale. In tale modo la posizione iniziale della particella è $\theta_0 = -\pi/2$ rad. Per quale valore di θ la particella perde contatto con la guida?

WORKED SOLUTION

1) $v_{\mathtt{min}} = \sqrt{5gR}$; 2) $\sin[\theta] = 1/3$.

24-078 Punto Materiale Attaccato Alla Molla Ruotante

Un punto materiale di massa m ruota in una traiettoria circolare con velocità angolare costante ω attorno ad un punto cui è collegato da una molla di costante elastica k e lunghezza a riposo d. Determinare la distanza di equilibrio, r_0 sia usando le **Cardinal Equation** che considerazioni energetiche.

WORKED SOLUTION

Consideriamo la situazione dal punto di vista del Reference Frame ruotante solidalmente con il punto materiale. La condizione di equilibrio tra la forza reale della molla e la forza centrifuga fittizia si scrive

$$m\omega^2 r_0 - k(r_0 - d) = 0$$

da cui segue

$$r_0 = \frac{kd}{k - m\omega^2}$$

Esaminiamo alcuni casi limite. Nel caso $\omega = 0$ si ha $r_0 = d$ come deve (molla a riposo). Nel caso $\omega^2 < k/m$ e $\omega^2 \longrightarrow k/m$ si ha $r_0 \longrightarrow \infty$: se la velocità angolare cresce a sufficienza la posizione di equilibrio tende all'infinito in quanto la forza elastica della molla non è più sufficiente ad equilibrare la forza centrifuga. Naturalmente in tal caso una molla reale smetterà di comportarsi idealmente secondo la legge di Hooke. Nel caso $\omega^2 \geq k/m$ non è possibile equilibrio.
La posizione di equilibrio può essere pure determinata dal minimo dell'energia potenziale (nel Reference Frame ruotante)

$$\Phi(r) = \frac{k}{2}(r - d)^2 - \frac{m\omega^2}{2} r^2$$

WORKED SOLUTION

$r_0 = kd/\left(k - m\omega^2\right)$

24-079 Il Potenziale Di Yukawa

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|8.45||

Il cosidetto potenziale di Yukawa

$$\Phi(r) = \frac{r_0}{r}\Phi_0 \exp\left[-r/r_0\right] \tag{24.22.11}$$

fornisce una descrizione approssimata dell'interazione tra due nucleoni (cioè protoni e neutroni), i costituenti del nucleo atomico. La costante r_0 vale circa $r_0 \simeq 1.5{\cdot}10^{-15}$ m mentre la costante Φ_0 vale circa $\Phi_0 \simeq 50$ MeV.

1. Si determini la corrispondente espressione della forza $F(r)$ tra due nucleoni.
2. Per rendersi conto del suo corto raggio di azione si calcoli il rapporto del modulo di tale forze per $r = 2r_0, 4r_0, 10r_0$ con il valore $F(r = r_0)$.

WORKED SOLUTION

1) $F(r) = -\Phi_0\left(r_0 r^{-2} + r^{-1}\right)\exp -r/r_0$; 2) 0.14, 0.0078, $6.8{\cdot}10^{-6}$.

PROGRESS

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24-080 L'atomo Di Idrogeno

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In un atomo di idrogeno il modulo della forza di attrazione tra il protone del nucleo, carico positivamente, e l'elettrone che orbita attorno al nucleo, carico negativamente, vale

$$F(r) = k\frac{e^2}{r^2} \tag{24.22.12}$$

dove e è la carica dell'elettrone, k una costante universale ed r la distanza tra protone ed elettrone. Si supponga il protone fisso. Si immagini quindi che l'elettrone si muova inizialmente lungo una circonferenza di raggio r_1 attorno al protone e che successivamente salti improvvisamente su un orbita di raggio r_2.

1. Si calcoli la variazione dell'energia cinetica dell'elettrone.
2. Si calcoli la variazione di energia potenziale dell'atomo.
3. Di quanto è cambiata in questo processo l'energia totale dell'atomo? Tale energia è spesso liberata oppure assorbita sotto forma di radiazione elettromagnetica.

24-081 Scala Richter Della Magnitudo Di Un Terremoto

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|8.62||

Nella scala Richter la magnitudo M di un terremoto è legata all'energia liberata E, espressa in Joule, dall'equazione

$$\log E = 1.44M + 5.24 \tag{24.22.13}$$

1. Il terremoto di San Francisco del 1989 fu di magnitudo $M = 7.1$. Quanta energia venne liberata?
2. A quale massa è equivalente, secondo la relazione relativistica tra massa ed energia
$$E = mc^2 \tag{24.22.14}$$
tale energia?

24-082 Momento Angolare Di Una Particella in Moto Piano

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|13.1 13.3||

Dimostrare che per un punto materiale che si muove in un piano il momento angolare è perpendicolare al piano del moto.

24-083 Momento Angolare Di Una Particella in Moto Rettilineo Uniforme

Dimostrare che il momento angolare rispetto ad un qualunque polo di una particella in moto rettilineo uniforme è costante.

24-084 Momento Angolare Nel Sistema Solare

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©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|§ 6.6||

Calcolare il momento angolare totale di tutti i pianeti associato al loro moto di rivoluzione attorno al Sole. Quale è la frazione del momento angolare totale associata al pianeta Giove?

DRAFT IN PROGRESS

24-085 Legge Di Variazione Del Momento Angolare Al Variare Del Polo

This § is referenced at pages:
[1394]

Sia l_O il momento angolare di un punto materiale rispetto al polo O. Dimostrare che il momento angolare rispetto al polo X è dato dalla relazione

$$\mathbf{l}_X = \mathbf{l}_O + [\mathbf{X} - \mathbf{O}] \wedge \mathbf{p} \tag{24.22.15}$$

Si veda anche il problema § 24-108.

24-086 Accelerazione Angolare Che Accompagna La Contrazione

©|Alonso & Finn, , ..., ..., ...Ed., ...|8.43||

©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|§ 6.4 § 6.5||

Un punto materiale di massa m si muove con velocità v_0 lungo una traiettoria circolare di raggio r_0 su un tavolo orizzontale privo di attrito. Il punto è mantenuto nella sua orbita circolare da una fune che passa attraverso un'apertura senza attrito al centro del tavolo e trattenuta all'altro capo da una forza costante $\mathbf{F}_0$. Determinare l'espressione di $\mathbf{F}_0$ in funzione dei parametri dati. Supponendo che la forza venga aumentata molto lentamente fino al valore $\mathbf{F}$ e che l'orbita finale sia circolare determinare le caratteristiche dell'orbita finale. Determinare il lavoro fatto dalla forza $\mathbf{F}$ per cambiare il raggio dell'orbita e il valore finale della velocità angolare.

WORKED SOLUTION

L'equazione del moto si scrive, nelle due situazioni iniziale e finale, essendo per ipotesi le orbite circolari,

$$F_0 = m\frac{v_0^2}{r_0} \quad ,$$
$$F = m\frac{v^2}{r}$$

Essendo la forza $\mathbf{F}$ sempre diretta lungo il raggio il suo momento rispetto al centro è nullo e quindi il momento angolare è conservato, per cui ne segue

$$mr_0v_0 = mrv$$

Usando tale relazione è possibile determinare il legame tra F_0 ed F, la forza che bisogna applicare affinché il punto materiale si muova sull'orbita circolare di raggio r in funzione dei parametri F_0 ed r_0,

$$F = F_0\left(\frac{r_0}{r}\right)^3$$

La variazione di energia cinetica è

$$\Delta\mathcal{K} = \mathcal{K} - \mathcal{K}_0 = \frac{m}{2}\left(v^2 - v_0^2\right) = \mathcal{K}_0\left[\left(\frac{r_0}{r}\right)^2 - 1\right]$$

e le due energie cinetiche sono legate da

$$\mathcal{K} = \mathcal{K}_0\left(\frac{r_0}{r}\right)^2$$

La variazione di energia cinetica risulta dunque positiva se $r < r_0$ per cui la forza fa lavoro positivo. Considerando che la forza aumenti molto lentamente in modo tale che il moto può essere considerato in ogni istante circolare uniforme con raggio lentamente variabile si può anche calcolare

direttamente il lavoro fatto dalla forza $\mathbf{F}$

$$\mathrm{d}L \equiv \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -F(r)\widehat{\mathbf{e}}_r \cdot \mathrm{d}r\widehat{\mathbf{e}}_r = -F(r)\,\mathrm{d}r$$

che integrata da

$$L = \int_{r_0}^{r} -F(r)\,\mathrm{d}r = -F_0 \int_{r_0}^{r} \left(\frac{r_0}{r}\right)^3 \mathrm{d}r = \frac{F_0 r_0^3}{2}\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right) = \frac{m v_0^2 r_0^2}{2}\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right)$$

che coincide con l'espressione della variazione di energia cinetica calcolata sopra, come deve per il teorema lavoro energia.
La velocità angolare finale

$$\omega = \frac{v}{r} = \omega_0 \left(\frac{r_0}{r}\right)^2$$

La velocità angolare dunque aumenta se il raggio dell'orbita diminuisce.

24-087 Centro Di Massa Del Sistema Terra-Luna

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|9.3||

Quanto dista dal centro della Terra il centro di massa del sistema Terra-Luna?

24-088 Distanza Di Due Particelle Dal Loro Centro Di Massa

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|9.4||

Dimostrare che il rapporto tra le distanze x_1 e x_2 di due particelle dal loro centro di massa è uguale all'inverso del rapporto delle masse m_1 ed m_2: $x_1/x_2 = m_2/m_1$.

24-089 L'uomo Che Sale Sulla Mongolfiera

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|9.7||

Un uomo di massa m è appeso ad una scala di corda sospesa ad una mongolfiera di massa M. Il sistema è fermo rispetto al suolo.

1. Se l'uomo inizia ad arrampicarsi con velocità v rispetto alla scala, quale è la direzione e il modulo della velocità che la mongolfiera acquista rispetto al suolo?
2. Cosa succede quando l'uomo si ferma?

WORKED SOLUTION

1) $mv/(M+m)$ verso il basso; 2) il pallone si mette di nuovo in quiete.

IN-PROGRESS

24-090 Il Cannone Sul Vagone Ferroviario

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Un piccolo cannoncino si trova entro un vagone ferroviario chiuso di lunghezza L con una riserva di proiettili. Il cannone viene fatto sparare verso destra e il vagone rincula a sinistra. I proiettili sparati dal cannone restano entro il vagone dopo aver urtato le pareti.

1. Dopo che tutti i proiettili sono stati sparati quale è il massimo spostamento teoricamente possibile che il vagone può avere avuto verso sinistra rispetto alla sua posizione iniziale?
2. Quale è la velocità del vagone dopo che tutti i proiettili sono stati sparati?

WORKED SOLUTION

1) L; 2) zero.

24-091 La Molecola Dell'ammoniaca

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24-092 La Catena Che Scivola

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24-093 Il Peso Di Judy

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24-094 Centro Di Massa Di Un Oggetto Composto

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24-095 Centro Di Massa Di Una Lastra Composta

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24-096 Centro Di Massa Del Liquido in Un Serbatoio

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24-097 Il Primo Teorema Di Pappo-Guldino per Il Calcolo Del Centro Di Massa

Si consideri un filo piano a forma di semicirconferenza con raggio R e di massa M. Si usi un Coordinate System Cartesiane ortonormali con centro nel centro di curvatura del semicirconferenza, asse x coincidente con il diametro e asse y nel piano del filo e con verso positivo dalla parte del filo.

1. Determinare la posizione del centro di massa per calcolo diretto.
2. Determinare la posizione del centro di massa usando il primo teorema di Pappo-Guldino: "La superficie di un solido di rotazione, generato dalla rotazione di una linea piana attorno ad un asse che giace nel piano della linea e che non interseca la linea stessa, è data dal prodotto della lunghezza della linea per la lunghezza della circonferenza percorsa dal centro di massa della linea".

WORKED SOLUTION

1) $x_{\text{CM}} = 0$ e $y_{\text{CM}} = 2R/\pi$.

24-098 Il Secondo Teorema Di Pappo-Guldino per Il Calcolo Del Centro Di Massa

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|9.21||

Si consideri una lamina piana a forma di semicerchio con raggio R e di massa M. Si usi un Coordinate System Cartesiane ortonormali con centro nel centro di curvatura del semicerchio, asse x coincidente con il diametro e asse y nel piano della lamina e con verso positivo dalla parte della lamina.

1. Determinare la posizione del centro di massa per calcolo diretto.
2. Determinare la posizione del centro di massa usando il secondo teorema di Pappo-Guldino: "Il volume di un solido di rotazione, generato dalla rotazione di una superficie piana attorno ad un asse che giace nel piano della superficie e che non interseca la superficie stessa, è dato dal prodotto dell'area della superficie per la lunghezza della circonferenza percorsa dal centro di massa della superficie".

WORKED SOLUTION

1) $x_{\text{CM}} = 0$ e $y_{\text{CM}} = 4R/3\pi$.

24-099 La Mitragliatrice

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|9.33||

24-100 Il Distacco Dell'ultimo Stadio Del Missile

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L'ultimo stadio di un missile viaggia a velocità $u_0 = 7600\ \text{km/s}$ ed è composto dalla sonda spaziale di massa $m = 150\ \text{kg}$ e dal serbatoio vuoto dell'ultimo stadio, di massa $M = 290\ \text{kg}$. Le due parti sono bloccate insieme da un fermo che mantiene compressa una molla. All'istante in cui si decide il distacco dell'ultimo stadio il fermo viene rimosso e la molla allontana così le due parti impartendo loro una velocità relativa $v_R = 910\ \text{km/s}$. Si supponga che tutte le velocità giacciano sulla stessa direzione.

1. Quali sono le velocità finali del serbatoio, V, e della capsula, v?
2. Quale è l'energia cinetica del sistema prima e dopo la separazione? Se sono diverse spiegare perché.

UNDER EDITORIAL REVIEW

WORKED SOLUTION

La soluzione è del tutto analoga al problema § 24.21.13. Supponendo, a differenza del problema § 24.21.13, che sia la sonda spaziale a muoversi con velocità $\mathbf{v}$ concorde ad $\mathbf{u}_0$ si ha

$$V = u_0 - \frac{m}{m+M} v_R = 7290 \text{ km/s} \quad ,$$
$$v = u_0 + \frac{M}{m+M} v_R = 8200 \text{ km/s} \quad .$$

in cui i segni sono opposti al quelli della soluzione del problema § 24.21.13.
Il calcolo dell'energia cinetica iniziale e finale fornisce

$$\mathcal{K}_0 = \frac{1}{2}(m+M)\, u_0^2 = 1.271 \cdot 10^{10} \text{ J} \quad ,$$
$$\mathcal{K}_f = \frac{1}{2} M V^2 + \frac{1}{2} m v^2 = 1.275 \cdot 10^{10} \text{ J} \quad .$$

L'aumento di energia cinetica del sistema è dovuto al lavoro delle forze interne che provocano lo sganciamento della capsula (la molla).

WORKED SOLUTION

1) $V = 7290$ km/s; $v = 8200$ km/s. 2) $\mathcal{K}_0 = 1.271 \cdot 10^{10}$ J; $\mathcal{K}_f = 1.275 \cdot 10^{10}$ J.

24-101 Decadimento Di Un Nucleo Radioattivo

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24-102 Le Persone Che Corrono Sul Vagone

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24-103 Teorema Di Koenig per Due Particelle

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Dimostrare che, nel caso di due particelle di massa m_1 ed m_2, il teorema di Koenig si può riscrivere nella forma

$$\mathcal{K} = \frac{1}{2} M V_{\text{CM}}^2 + \frac{1}{2} \mu v_{\text{REL}}^2 \tag{24.22.16}$$

dove $M = m_1 + m_2$, V_{CM} è la velocità del centro di massa, μ indica la massa ridotta della coppia e v_{REL} è la velocità relativa delle due particelle.

WORK-IN-PROGRESS

24-104 Teorema Di Koënig per Tre O Più Particelle

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|7.97||

Dimostrare che per un sistema di tre particelle con massa m_1, m_2, m_3 l'energia cinetica relativa al centro di massa, come risulta dal teorema di Koenig, può essere espressa tramite la relazione

$$\frac{1}{2}\sum_{k=1}^{3} m_k {v'_k}^2 = \frac{1}{2}\left[\frac{m_1 m_2 v_{12}^2 + m_2 m_3 v_{23}^2 + m_3 m_1 v_{31}^2}{m_1 + m_2 + m_3}\right] \tag{24.22.17}$$

WORKED SOLUTION

Sia $M \equiv m_1 + m_2 + m_3$.

$$v'_1 = v_1 - V_{CM} = \frac{(m_2 + m_3)\, v_1 - m_2 v_2 - m_3 v_3}{M} \quad ,$$
$$v'_2 = v_2 - V_{CM} = \frac{(m_3 + m_1)\, v_2 - m_3 v_3 - m_1 v_1}{M} \quad ,$$
$$v'_3 = v_3 - V_{CM} = \frac{(m_1 + m_2)\, v_3 - m_1 v_1 - m_2 v_2}{M}$$

Ne segue

$$\frac{1}{2} m_1 {v'_1}^2 = \frac{m_1}{2M^2}\left[(m_2 + m_3)^2 v_1^2 + m_2^2 v_2^2 + m_3^2 v_3^2 - 2m_2 (m_2 + m_3) v_1 v_2 - 2m_3 (m_2 + m_3) v_1 v_3 + 2m_2 m_3 v_2 v_3\right] \quad ,$$
$$\frac{1}{2} m_2 {v'_2}^2 = \frac{m_2}{2M^2}\left[(m_3 + m_1)^2 v_2^2 + m_3^2 v_3^2 + m_1^2 v_1^2 - 2m_3 (m_3 + m_1) v_2 v_3 - 2m_1 (m_3 + m_1) v_2 v_1 + 2m_3 m_1 v_3 v_1\right] \quad ,$$
$$\frac{1}{2} m_3 {v'_3}^2 = \frac{m_3}{2M^2}\left[(m_1 + m_2)^2 v_3^2 + m_1^2 v_1^2 + m_2^2 v_2^2 - 2m_1 (m_1 + m_2) v_3 v_1 - 2m_2 (m_1 + m_2) v_3 v_2 + 2m_1 m_2 v_1 v_2\right]$$

Sviluppando poi v_{12}^2, v_{23}^2, v_{31}^2, a secondo membro e raccogliendo i termini analoghi si verifica l'uguaglianza.

24-105 Teorema Dello Pseudo-Lavoro

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24-106 Il Vagone Merci Sotto La Pioggia

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Un vagone ferroviario merci aperto, che pesa 9.75 ton, si sta muovendo a velocità costante $v = 1.36$ ms lungo un binario piano rettilineo, con attrito trascurabile. Inizia a piovere violentemente e la pioggia cade verticalmente rispetto al suolo.

1. Determinare la velocità del vagone dopo che ha raccolto 0.5 ton di acqua.
2. Occorre fare delle ipotesi addizionali per poter rispondere?

DRAFT IN PROGRESS

24-107 Il Jet

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Un jet viaggia alla velocità di $V = 184\,\text{ms}$. Il motore prende $V_{\text{IN}} = 68.2\ \text{m}^3$ di aria, pari a una massa $M_{\text{IN}} = 70.2\ \text{kg}$, ogni secondo. L'aria è usata per bruciare $m = 2.92\ \text{kg}$ di combustibile al secondo. L'energia prodotta è usata per comprimere i prodotti di combustione ed espellerli all'indietro alla velocità di $v = 497\,\text{ms}$ rispetto all'aereo.

1. Determinare la spinta del motore.
2. Determinare la potenza sviluppata dal motore.

WORKED SOLUTION

Dall'equazione del moto di un sistema a massa variabile

$$M(t)\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{F}_{\text{ext}} + \mathbf{u}_{\text{REL}}\frac{\mathrm{d}M}{\mathrm{d}t} \tag{24.22.18}$$

dove $\mathbf{u}_{\text{REL}}$ rappresenta la velocità con cui la massa viene espulsa relativa al sistema. Nel caso del jet si hanno due contributi. Uno dovuto all'aria che entra nel motore e l'altro dovuto all'espulsione dei gas combusti.

WORKED SOLUTION

1) $F = 23.4{\cdot}10^3\ \text{N}$; 2) $W = 4.31{\cdot}10^6\ \text{W}$.

24-108 Momento Angolare Di Un Sistema: Variazione Al Variare Del Polo

This § is referenced at pages:
[1388]

Si dimostri che il momento angolare di un sistema varia al variare del polo secondo la stessa legge con cui varia il momento angolare di una particella (problema § 24-085, equazione (24.04.01))

$$\mathbf{L}_X = \mathbf{L}_O + [\mathbf{O} - \mathbf{X}]\wedge\mathbf{P}$$

24-109 Momento Angolare Di Un Sistema: Componente Orbitale E Di Spin

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|13.7 13.8||

Il momento angolare totale di un sistema di particelle rispetto ad un polo O in un Reference Frame inerziale è definito da

$$\mathbf{L} = \sum_k \mathbf{r}_k\wedge\mathbf{p}_k \tag{24.22.19}$$

1. Si usino le relazioni
$$\begin{aligned}\mathbf{r}_k &= \mathbf{r}_{\text{CM}} + \mathbf{r'}_k \quad ,\\ \mathbf{p}_k &= m_k\mathbf{V}_{\text{CM}} + \mathbf{p'}_k \quad .\end{aligned}$$
per esprimere il momento angolare totale in termini delle grandezze $\mathbf{r'}_k$ e $\mathbf{p'}_k$ relative al centro di massa.
2. Si dimostri come il momento angolare totale può essere decomposto in una parte orbitale ed una parte di momento angolare rispetto al centro di massa, $\mathbf{L'}$, detta anche momento angolare di spin, tramite la relazione
$$\mathbf{L} = M\,\mathbf{r}_{\text{cm}}\wedge\mathbf{V}_{\text{CM}} + \mathbf{L'} \tag{24.22.20}$$

24-110 Momento Angolare Nel Reference Frame Del Centro Di Massa

©|Am.J.Phys, , ..., ..., ...Ed., ...|???||

Si consideri il momento angolare di un sistema di particelle relativo al Reference Frame del centro di massa, il Reference Frame che ha origine nel centro di massa del sistema e si muove solidalmente con esso senza ruotare rispetto al Reference Frame del Laboratorio. Si denotino, rispettivamente, con $\mathbf{r}_k$ e $\mathbf{p}_k$ posizione e quantità di moto della k-esima particella rispetto al Reference Frame del Laboratorio e $\mathbf{r'}_k$ e $\mathbf{p'}_k$ le stesse quantità rispetto al Reference Frame del centro di massa. Si dimostri che il momento angolare del sistema rispetto al Reference Frame del centro di massa si può esprimere con le tre espressioni equivalenti

$$\mathbf{L}_{CM} = \sum_k \mathbf{r}'_k \wedge \mathbf{p}_k \quad ,$$

$$\mathbf{L}_{CM} = \sum_k \mathbf{r}_k \wedge \mathbf{p}'_k \quad ,$$

$$\mathbf{L}_{CM} = \sum_k \mathbf{r}'_k \wedge \mathbf{p}'_k$$

24-111 Il Razzo Multistadio

©|Barger & Olsson, , ..., ..., ...Ed., ...|||

©|H.C.Ohanian, , ..., ..., ...Ed., ...|10.55||

24-112 Free Mass Point Motion Solved via the Second Cardinal Equation

24-113 Example

Show an example of a system for which the total external Force|Torque is zero but it is not necessarily in equilibrium.

24-114 Rotation Around a Fixed Axis

Consider rotation around a fixed axis and calculate the angular velocity from its general definition.

DRAFT - IN - PROGRESS

24-115 Elliptical Motion

Consider a plane motion described, in the xy plane, by

$$x[t] = a\cos[\omega t + \phi] \quad ,$$
$$y[t] = b\cos[\omega t + \phi] \quad ,$$

with ω, a, b and ϕ given constants.

Determine the trajectory of the motion, the velocity and acceleration as a function of time and the curvature and torsion as a function of time.

24-116 A Ladder Resting Against a Wall

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|1.46||

24-117 A Piston Rod

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|1.143||

24-118 A Boat Crossing a River

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24-119 Velocity and Acceleration in Cylindrical Coordinates

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24-120 Velocity and Acceleration in Spherical Coordinates

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24-121 Rotation Without Slipping and Instantaneous Center of Rotation

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|9.34 9.35 9.36||

24-122 A Sphere Rolling Down a Sphere

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|9.42||

Read § 26-030.

DRAFT - IN-PROGRESS

24-123 A Car Passing By

A car moves on a straight trajectory at constant velocity having module v. Someone is standing still at a distance D from the trajectory of the car. The car passes at the minimum distance from the observer at time t_0. Write the expressions of angular velocity and angular acceleration of the car as seen by the observer.

24-124 Free Particle

Discuss the motion of a free particle using only the second **Cardinal Equation** , applied at any generic pole, either on the particle straight line trajectory and outside it.

24-125 Oscillation Period of a Rigid Semi-Sphere With Vertex Facing Down

24-126 Dynamics of mass point in a non-Inertial Reference Frame

©|D.V.Sivuchin, , ..., ..., ...Ed., ...|§ 64|*All nice problems at the end.*|

24-127 EseFisGen 15-07-2005

Un disco cilindrico omogeneo, di massa $M = 80\ \mathrm{g}$ e raggio $R = 15\ \mathrm{cm}$, è imperniato su un asse orizzontale che passa per il suo centro O ed è inizialmente in quiete.

Un piccolo oggetto di massa $m = 5\ \mathrm{g}$, che viaggia in orizzontale con velocità $v_0 = 0.8\ \mathrm{m/s}$ perpendicolare all'asse del disco, urta contro il disco conficcandosi in esso al bordo, nella sua estremità inferiore. Tra l'asse ed il disco agisce una molla a di torsione avente costante di torsione $k = 2{\cdot}10^{-3}\ \mathrm{N \cdot m \cdot rad^{-1}}$ ed è anche presente è attrito radente con momento $\tau_a = 10^{-3}\ \mathrm{N \cdot m}$.

Si calcoli:

1. la velocità angolare del sistema dopo l'urto nonchè l'impulso sull'asse conseguente all'urto;
2. dove, ed a quale tempo, si arresta per la prima volta il sistema dopo l'urto;
3. come cambia la risposta al punto precedente se anziché l'attrito radente è presente l'attrito viscoso dell'aria proporzionale alla velocità angolare che genera un momento $\tau_v = -c\omega$ con $c = 10^{-3}\ \mathrm{N \cdot m \cdot s}$?

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WORKED SOLUTION

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- Nell'urto si conserva il momento angolare rispetto al centro del disco. Si ha quindi:

$$mv_0 R = I\Omega \qquad I = \frac{MR^2}{2} + mR^2 = 10^{-3} kg\, m^2$$

da cui

$$\Omega = \frac{mv_0}{(m + M/2)R} = 0.59\ rad\ s^{-1}.$$

Il CM del disco non si muove e quindi $\Delta P_x^{\texttt{disco}} = 0$. La conservazione della quantità di moto,

$$\Delta P_x^{\texttt{asse}} + \Delta P_x^{\texttt{sfera}} + \Delta P_x^{\texttt{disco}} = 0,$$

fornisce quindi:

$$\Delta P_x^{\texttt{asse}} = mv_0 - mR\Omega = 3.6 \times 10^{-3} N\ s^{-1}.$$

- La seconda **Cardinal Equation** è:

$$I\ddot{\theta} = -k\theta - mgR\sin\theta \pm \tau_a$$

(segno + se $\dot{\theta} < 0$), che nel limite di piccole oscillazioni diventa:

$$\ddot{\theta} = -\omega^2(\theta \mp \delta)$$

dove:

$$\omega = \sqrt{\frac{k + mgR}{I}} = 3.06\ rad\ s^{-1} \qquad \delta = \frac{\tau_a}{k + mgR} = 0.11\ rad$$

Con le condizioni iniziali :

$$\theta(0) = 0 \qquad \dot{\theta}(0) = \Omega$$

dalla soluzione generale

$$\theta(t) = A\cos(\omega t + \phi) \pm \delta$$

e tenendo conto che nella prima fase del moto abbiamo $\dot{\theta} > 0$, otteniamo:

$$\theta(t) = \sqrt{\frac{\Omega^2}{\omega^2} + \delta^2}\cos(\omega t + \phi) - \delta$$

con $\cos\phi = \dfrac{\delta}{\sqrt{\delta^2 + (\Omega/\omega)^2}}$ e $\tan\phi = -\Omega/(\delta\omega)$.
Abbiamo quindi l'arresto quando $\cos(\omega t + \phi) = 1$ e quindi:

$$\theta = \sqrt{\frac{\Omega^2}{\omega^2} + \delta^2} - \delta = 0.11\ rad$$

che si verifica al tempo

$$t = -\frac{1}{\omega}\arctan\left(-\frac{\Omega}{\omega\delta}\right) = 0.34\ s.$$

- L'attrito viscoso produce un momento $\tau_v = -c\dot{\theta}$. L'**Cardinal Equation** diventa quindi:

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2\theta = 0$$

con $\gamma = c/(2I) = 0.5 s^{-1}$.
Posto $\omega' = \sqrt{\omega^2 - \gamma^2} = 3.02\ rad\ s^{-1}$, le condizioni iniziali nella soluzione generale

$$\theta(t) = A\exp -\gamma t\cos(\omega' t + \phi)$$

forniscono:

$$\theta(t) = \frac{\Omega}{\omega'}\exp -\gamma t\sin(\omega' t).$$

Il sistema si arresta per la prima volta quando $\dot{\theta} = 0$, vale a dire:

$$\frac{\Omega}{\omega'}\exp -\gamma t(-\gamma\sin(\omega' t) + \omega'\cos(\omega' t)) = 0$$

e quindi:

$$t = \frac{1}{\omega'}\arctan\omega'/\gamma = 0.47\ s$$

Relativamente alla posizione

$$\theta = \frac{\Omega}{\omega'}\exp -\gamma t\sin(\omega' t) = 0.15\ rad$$

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24.23 External Documents

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CHAPTER 25

Oscillations of a few mass points

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Oscillations, that is periodic motions; mainly periodic.

This § is referenced at pages:
[1990, 2025, 2061]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.21; 1.22; 1.23; 1.24; 1.25; ...
©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|1|*Good text.*|
©|K.R. Symon - Mechanics 3rd Ed., § 12|-|*Extensive and detailed treatment*|
©|WEB - URL |Nicholas Wheeler Physics Lectures|*advanced - also multi-DOF*|
©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|§ 7|*Excellent*|
©|J.P.Pérez *et al.*, *Mécanique*, ..., DUNOD, ...Ed., WEB - URL.|10, 11, 27||

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25.01 Overview

Oscillation is the name of a periodic or almost-periodic motion; almost-periodic motion is a motion which can be considered periodic over a small number of periods, that is whose period is changing slowly, over a large number of periods.

Many different systems and phenomena are described by the damped, forced harmonic oscillator equation. These include many kind of different systems including, in classical physics, many types of mechanical as well as electric systems. This explains the outermost importance of second-order linear systems in physics, deserving an accurate study of them.

In realistic cases one must almost always add a damping term, because damping almost always exists, even if it is sometimes negligible.

Moreover, in most real cases, one may have an external forcing, keeping the oscillations from fading away.

A damped, forced harmonic oscillator system/equation is a linear system. In real cases one must always check whether the system can be treated as a linear system ad what conditions need to be satisfied for that. In particular one must check what happens when the oscillations are big, and the usual small oscillations approximation breaks down. At some point, in fact, the restoring generalized force might stop to be linear.

It might be also possible that the damping force might stop to be linear. An important case, for instance, is the case of aero-dynamic drag when the viscous friction can be modeled as a force linear in the velocity at small velocities but must be modeled as a force quadratic in the velocities at higher velocities.

25.01.01 One Versus Many Degrees of Freedom - Normal Modes

One important point is the fundamental result that any un-damped mechanical system with an arbitrary number of degrees of freedom, near an equilibrium configuration, can be described, for sufficient small oscillations, in terms of **normal modes of oscillations** characterized by the fact that all coordinates oscillate with the same frequency and phases but possibly different amplitudes. The number of normal modes is exactly equal to the number of degrees of freedom of the system.

Therefore while studying the case of a single degree-of-freedom system one should keep in mind that many results can extended to an arbitrary number of degrees of freedom by simply considering the system having many frequencies of resonance, instead of a single one.

25.01.02 Small Oscillations Around a Stable Equilibrium Position

Si consideri, per concretezza, il caso di oscillazioni meccaniche. Per piccole oscillazioni attorno ad una posizione di equilibrio stabile, ξ_0, la Force|Torque di richiamo, σ, può essere sempre approssimata, tramite la formula di Taylor, con una forza armonica, proporzionale e di verso opposto allo spostamento dalla posizione di equilibrio, $\Delta\xi \equiv \xi - \xi_0$, schematicamente, per un solo grado di libertà:

$$\sigma[\xi] = -\alpha\,(\xi - \xi_0) + \mathcal{O}\left[(\xi - \xi_0)^2\right] \simeq -\alpha\Delta\xi \quad .$$

Starting from any stable equilibrium configuration of the system, let the system be described by the set of variables ξ_k and let $\xi_k[t] = \xi_{k|0}$ the equilibrium configuration.

Whatever the variable describing the system always use as a variable the **change with respect to the equilibrium configuration**:

$$\xi_k[t] \equiv \eta_k[t] - \eta_{k|0} \quad .$$

This choice is useful to remove constant terms from the equations, without changing the physics involved.

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25.01.03 Common Types of Damping

$$\mathbf{f} = -\mu|\mathbf{N}|\,(\mathbf{v}/v) \qquad \text{sliding friction damping (Coulomb)} \quad ,$$
$$\mathbf{f} = -a\mathbf{v} \qquad \text{viscous fluid damping} \quad ,$$
$$\mathbf{f} = -bv\mathbf{v} \qquad \text{aerodynamic fluid damping} \quad .$$

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A good representation of the viscous damping force is also provided by a flat plate moving normal to its plane through a gas at very low pressure, as in Fig. 7.11, provided the speed of the plate V is much slower than the average speed of the molecules v of the gas. The pressure must be sufficiently low that we can neglect the collisions of the molecules with each other. The rate at which molecules strike the plate is proportional to the relative velocity of the incoming molecules and the plate. Suppose that the molecules move only in one dimension. On one side of the plate the relative velocity is $v + V$; on the other side it is $v - V$. The pressure is proportional to the rate at which molecules strike the plate times the average momentum transfer to the plate per molecule. The average momentum transfer is itself proportional to the relative velocity, so that the pressures p_1 and p_2 on the two sides of the plate are $p_1 \propto (v + V)^2$ and $p_2 \propto (v - V)^2$. The net pressure is the difference of the pressures on the opposite sides of the plate, $p_1 - p_2 = 4vV$, so that the drag (the net force on the moving plate) is directly proportional to the speed V of the plate. The direction of the drag can be seen to oppose the motion of the plate.

25.01.04 Examples of Oscillating Systems

25.01.04.01 Spring-Mass System With Coulomb Damping

25.01.04.02 A Forced Mass-Spring System With Linear Damping Term

This § is referenced at pages:
[Never referenced.]

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = G[t] \qquad \omega_0^2 \equiv \frac{k}{m} \quad .$$

25.01.04.03 A Mass-Spring System Hanging in the Gravity Field

The constant gravity force just shifts the equilibrium position by the amount mg/k with respect to the case of zero gravity. The use of the displacement with respect to the equilibrium position as a variable removes the constant term in the equation.

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25.01.04.04 Simple and the Physical Pendulum

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The exact equation is not linear. It can be linearised for small oscillations with respect to the equilibrium position.

$$\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = G[t] \qquad \omega_0^2 \equiv \frac{k}{m} \quad .$$

25.01.04.05 Torsion Pendulum

The equation is linear, in the linear regime of the torsion of the elastic wire.

25.01.04.06 A Forced RLC Series Circuit

This § is referenced at pages:
[Never referenced.]

$$\ddot{q} + \frac{R}{L}\dot{q} + \omega_0^2 q = G[t] \qquad \omega_0^2 \equiv \frac{1}{LC} \quad .$$

25.01.04.07 Spherical Pendulum

25.01.04.08 Foucault Pendulum

25.01.04.09 Harmonic Oscillator in Three Dimensional Space

A mass particle under the 3D linear restoring force

$$\mathbf{f}[\mathbf{r}] = -k\mathbf{r} \quad ,$$

is subject to a force trying to restore its position near the origin. It is a 3D harmonic oscillator. Read also § 29.06.07.

25.01.04.10 Motion Stability: Radial Oscillations Around a Circular Orbit in a Keplerian Field

Read § 29.06.11.

25.01.04.11 Motion Stability: Free Motion of a Rigid-Body

Read § 26.06.07.

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25.02 Harmonic Oscillations

Una grandezza fisica armonica function of a variable w and with period $\mathcal{P}$, è una grandezza fisica, $\xi = \xi[w]$, la cui dipendenza è Sinusoidal|Cosinusoidal :

$$\xi[w] = C\cos\left[\frac{2\pi}{\mathcal{P}}(w-w_0)-\delta\right] \equiv A\cos\left[\frac{2\pi}{\mathcal{P}}(w-w_0)\right] + B\sin\left[\frac{2\pi}{\mathcal{P}}(w-w_0)\right] \quad , \qquad (25.02.01)$$

→ 1407

dove le due coppie di costanti $\{C,\delta\}$ e $\{A,B\}$ sono due coppie alternative di costanti determinate dalle (condizioni iniziali)|(condizioni al contorno) e tra loro collegate tramite le relazioni:

$$A = \xi[w_0] \equiv \xi_0 \qquad \frac{2\pi}{\mathcal{P}}B = \frac{\mathrm{d}\xi}{\mathrm{d}w}[w_0] \equiv \left.\frac{\mathrm{d}\xi}{\mathrm{d}w}\right|_{w_0} \quad ,$$

$$A = C\cos[\delta] \qquad B = C\sin[\delta] \quad ,$$

$$\begin{cases} C = \sqrt{A^2+B^2} \geq 0 \quad , \\ \delta = \arctan(B/A) \qquad \text{se } A > 0 \quad , \\ \delta = \arctan(B/A) + \pi \qquad \text{se } A < 0 \quad , \\ \delta = -\pi/2 \qquad \text{se } A = 0 \text{ e } B > 0 \quad , \\ \delta = +\pi/2 \qquad \text{se } A = 0 \text{ e } B < 0 \end{cases} .$$

In the two most important cases, time and space oscillation, we have the following definitions.

$$T \quad \text{period of the oscillation} \quad \Leftrightarrow \quad \lambda \quad \text{wavelength of the oscillation} \quad ,$$

$$\omega_0 \equiv \frac{2\pi}{T} = 2\pi\nu \quad \text{angular frequency} \quad \Leftrightarrow \quad k_0 \equiv \frac{2\pi}{L} = 2\pi\sigma \quad \text{wave-number} \quad ,$$

$$C \quad \text{amplitude of the oscillation} \quad ,$$

$$\delta \quad \text{phase of the oscillation} \quad .$$

Also read § 18.02.

Read § 44.02.02.02 for the case of waves, doubly-periodic in time and space.

Read § 19.02 for the important and useful relations about the sum of harmonic functions with the same frequency.

In the rest of the chapter only the case of time dependence will be considered.

PRELIMINARY PROGRESS

25.03 Free Undamped Harmonic Oscillator

L'equazione differenziale che fornisce una dipendenza temporale armonica (equazione (25.02.01)) è:

$$\boxed{\ddot{\xi} + \omega_0^2 \xi = 0} \quad .$$

Note that the interpretation depends on the specific meaning of the involved physical quantity, ξ. It might be a real motion of a point, a current in an electric circuit, ... In what follows, for definiteness, when it is required to specify the system, we will always implicitly refer to a real motion of a point particle except when specified otherwise.

L'energia cinetica e potenziale, nel caso delle oscillazioni meccaniche di un oggetto di massa m, sono date dalle relazioni:

$$k = \frac{m}{2}\omega_0^2 C^2 \sin^2\left[\omega_0\left(t - t_0\right) - \delta\right] \Longrightarrow \langle k \rangle = \frac{m}{4}\omega_0^2 C^2 \quad ,$$

$$u = \frac{m}{2}\omega_0^2 C^2 \cos^2\left[\omega_0\left(t - t_0\right) - \delta\right] \Longrightarrow \langle u \rangle = \frac{m}{4}\omega_0^2 C^2 \quad .$$

L'energia totale è costante e pari a:

$$e = k + u = \frac{m}{2}\omega_0^2 C^2 = \langle k \rangle + \langle u \rangle \quad .$$

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25.04 Free Damped Harmonic Oscillator

L'equazione del moto armonico smorzato libero è

$$\ddot{\xi} + 2\beta\dot{\xi} + \omega_0^2\xi = 0 \quad \Leftrightarrow \quad \ddot{\xi} + \Gamma\dot{\xi} + \omega_0^2\xi = 0 \quad . \tag{25.04.01}$$

→ 1411

As for every linear system the superposition of solutions apply (read § 19.01).

L'uso dell'uno o dell'altro dei due fattori (entrambi costanti e positivi, essendo il moto smorzato)

$$\Gamma \equiv 2\beta \geq 0 \quad ,$$

è dettato da pura semplicità delle espressioni per ovviare a vari fattori 2 (esatti) che intervengono in espressioni diverse.

Altra notazione usata:

$$\Gamma \equiv 2\beta \equiv \tau^{-1} \quad . \tag{25.04.02}$$

→ 1408

La soluzione dipende dal tipo di soluzioni dell'equazione caratteristica di secondo grado:

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0 \implies \lambda = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

che implica:

$\beta^2 - \omega_0^2 < 0 \quad \omega_1 \equiv \sqrt{\omega_0^2 - \beta^2} \quad \implies \xi[t] = C\exp[-\beta t]\cos[\omega_1 t - \delta]$ moto debolmente smorzato

(25.04.03) → 1408 1409 1410 1412

$\beta^2 - \omega_0^2 = 0 \quad \implies \xi[t] = (A + Bt)\exp[-\beta t]$ moto criticamente smorzato

(25.04.04) → 1408 1410 1412

$\beta^2 - \omega_0^2 > 0 \quad \alpha \equiv \sqrt{\beta^2 - \omega_0^2} \quad \implies \xi[t] = \exp[-\beta t]\,(c_1\exp[+\alpha t] + c_2\exp[-\alpha t])$ moto ultra-smorzato .

con $\{C, \delta\}$, $\{A, B\}$ e $\{c_1, c_2\}$ costanti determinate dalle condizioni iniziali.

Note that in the usual physical cases when $\beta > 0$, that is a system dissipating energy, the real part of λ is negative. In fact in the case of two distinct real roots the characteristic equation shows that the sum of the roots has to be negative ($\lambda_1 + \lambda_2 = -\beta < 0$) while their product has to be positive ($\lambda_1\lambda_2 = \omega_0^2 > 0$). In the case of two identical real roots one has $\lambda_0 = -\beta$; in the case of two complex conjugate solutions the real part of the roots is $-\beta$. Therefore in any case the exponential function goes to zero for large times.

The results, equations (25.04.03), (25.04.04), (25.04) show the meaning of the constant τ defined in equation (25.04.02): it is one-half the time constant for time decay of the amplitude of the oscillation as it is shown by equation:

$$\exp[-\beta t] = \exp\left[-\frac{t}{2\tau}\right] \quad .$$

→ 1408 1410 1412

La frequenza di oscillazione nel caso debolmente smorzato (25.04.03) vale

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} \leq \omega_0 \quad . \tag{25.04.05}$$

→ 1410

Il coefficiente α nel moto ultra-smorzato vale

$$\alpha = \sqrt{\beta^2 - \omega_0^2} \leq \beta \quad . \tag{25.04.06}$$

→ 1410

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25.04.01
Weakly Damped Oscillations

Consider a weakly damped oscillator: weakly damped means that the exponential factor is about constant during one oscillation period. Neglecting numerical factors of the order of the unity one can then deduce the following condition for weak damping:

$$\frac{1}{\beta} \gg \frac{2\pi}{\omega_1} \Longrightarrow \beta \ll \omega_0 \quad .$$

Energies are always quadratic functions of either coordinates or velocities. Therefore when averaging over one oscillation period a weakly damped oscillation, taking into account that the exponential factor might be considered about constant in the derivative, one has:

$$\xi[t] = C \exp[-\beta t] \cos[\omega_1 t - \delta] \quad \Longrightarrow \xi^2[t] \propto \exp[-2\beta t] \cos[\omega_1 t - \delta] \quad \Longrightarrow \langle \xi^2[t] \rangle \propto \exp[-2\beta t] \quad ,$$

$$\dot{\xi}[t] \simeq -C\omega_1 \exp[-\beta t] \sin[\omega_1 t - \delta] \quad \Longrightarrow \dot{\xi}^2[t] \propto \omega_1^2 \exp[-2\beta t] \sin[\omega_1 t - \delta] \quad \Longrightarrow \langle \dot{\xi}^2[t] \rangle \propto \omega_1^2 \exp[-2\beta t] \quad .$$

Therefore the time-averaged energies are:

$$\langle k \rangle = \langle k \rangle_0 \exp[-2\beta t] \quad ,$$
$$\langle u \rangle = \langle u \rangle_0 \exp[-2\beta t] \quad ,$$
$$\langle e \rangle = \langle e \rangle_0 \exp[-2\beta t] \quad .$$

25.04.02
Critically Damped and Over-Damped Oscillations

For a given set of initial conditions and a fixed ω_0, while varying β, the critically damped harmonic oscillator ($\beta = \omega_0$) is the one going fastest to zero.

In fact it is physically obvious that a weakly damped oscillator will not go to zero quickly and also that a strongly damped one will not go to zero quickly. There is, therefore, a certain value of β corresponding to the fastest settling to zero, for a fixed ω_0.

Start from a weakly damped oscillations: increasing β will give a faster settling to zero, due the presence of the $\exp[-\beta t]$ factor, until the critically damped, $\beta \equiv \beta_{\text{CR}} = \omega_0$, case is reached. Increasing β even more will not improve the settling to zero because the term $\exp[-\beta t] \exp[+\alpha t]$, which is the slowest of the two exponential terms of the over-damped case, will be slower than the critically damped case to settle to zero; in fact $\beta - \alpha < \omega_0 \equiv \beta_{\text{CR}}$, because $\beta > \omega_0$.

This fact is often used in finding the optimal damping when designing systems which must settle to zero as fast as possible after a perturbation.

25.04.03
Initial Constants and Full Solution

This § is referenced at pages:
[1046]

The explicit expression in terms of the initial constants for the motion (25.04.03) is:

$$\xi[t] = \exp[-\beta t] \left(\xi_0 \cos[\omega_1 t] + (v_0 + \beta \xi_0) \frac{\sin[\omega_1 t]}{\omega_1} \right) \quad . \qquad (25.04.07)$$

→ 1409 1410 1430

In the case of weak damping, $\beta \ll \omega_0$, the exponential in (25.04.07) can be considered about constant during one cycle. The velocity can thus be calculated by taking the exponential factor constant, to a good approximation. It turns out that the mechanical energy is about constant during one cycle and it decays exponentially over many cycles with time decay constant $\tau = \beta^{-1}$.

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The explicit expression in terms of the initial constants for the motion (25.04.04) is:

$$\boxed{\xi[t] = \exp\left[-\beta t\right]\left(\xi_0 + \left(v_0 + \beta\xi_0\right)t\right)} \quad . \tag{25.04.08}$$

→ 1430

The explicit expression in terms of the initial constants for the motion (25.04) is:

$$\boxed{\xi[t] = \exp\left[-\beta t\right]\left(\xi_0 \operatorname{coh}\left[\alpha t\right] + \left(v_0 + \beta\xi_0\right)\frac{\operatorname{sih}\left[\alpha t\right]}{\alpha}\right)} \quad . \tag{25.04.09}$$

→ 1410 1430

Note that the two equations (25.04.07), (25.04.09) can be actually considered as fully equivalent formulas, in any case, if one allows ω_1 and α to become pure imaginary numbers and takes into account the relations between circular and hyperbolic functions with pure imaginary arguments (equation (14.03.03)).

25.04.04
Q-Factor

This § is referenced at pages:
[1419, 1420]

The Q-factor for a free damped harmonic oscillator is defined by equation

$$\boxed{Q \equiv \frac{\omega_0}{2\beta} \equiv \frac{\omega_0}{\Gamma}} \quad . \tag{25.04.10}$$

→ 1419 1420

Weak damping, therefore, means $Q \gg 1$.

It can be shown that (see problem (25-002)), in the case of small damping ($Q \gg 1/2$), the evolution of the total energy is described by the relations:

$$\frac{\mathrm{d}\langle e\rangle}{\mathrm{d}t} = -m\Gamma\langle v^2\rangle \qquad \Longrightarrow \qquad \boxed{\frac{\mathrm{d}\langle e\rangle}{\mathrm{d}t} \simeq -\Gamma\langle e\rangle \equiv -\frac{\langle e\rangle}{\tau} \equiv -\omega_0\frac{\langle e\rangle}{Q}} \quad . \tag{25.04.11}$$

→ 1420

La frequenza di oscillazione nel caso debolmente smorzato (25.04.03) si può scrivere in termini del Q come (equazione (25.04.05)):

$$\boxed{\omega_1 = \sqrt{\omega_0^2 - \beta^2} = \omega_0\sqrt{1 - \frac{1}{4Q^2}} \leq \omega_0 \qquad \text{per } Q > \frac{1}{2}} \quad . \tag{25.04.12}$$

Il coefficiente α nel moto ultra-smorzato (25.04) si può scirvere in termini del Q come (equazione (25.04.06)):

$$\boxed{\alpha = \sqrt{\beta^2 - \omega_0^2} = \omega_0\sqrt{\frac{1}{4Q^2} - 1} \leq \beta \qquad \text{per } Q < \frac{1}{2}} \quad . \tag{25.04.13}$$

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25.05 Forced Damped Harmonic Oscillator

This § is referenced at pages:
[1420]

All is already known from Calculus: just re-phrase in physical terms

We will consider a generic system described by the oscillating variable ξ. All the arguments can be applied to any system described by a second-order linear differential equation with constant coefficients when the equation after the appropriate equation has been written.

Si può supporre che la forza esterna vari sinusoidalmente con il tempo. Poiché il sistema è lineare, infatti, la risposta del sistema ad una generica forza esterna (periodica o non periodica) può essere ottenuta sovrapponendo le risposte alle singole componenti armoniche (principio di sovrapposizione delle sorgenti, read § 19.01). In generale:

$$F[t] = \sum_k G_k \cos\left[\omega_k t + \phi_k\right] \qquad F[t] = \int G[\omega] \cos\left[\omega t + \phi[\omega]\right] \, \mathrm{d}\omega \quad ,$$

dove la somma è una **Fourier Series** , nel caso di $F[t]$ periodica, mentre è un integrale di Fourier, nel caso di $F[t]$ non periodica.

25.05.01
General Solution

Let us consider the problem:

$$\ddot{\xi} + 2\beta\dot{\xi} + \omega_0^2\xi \equiv \ddot{\xi} + \Gamma\dot{\xi} + \omega_0^2\xi = G_0[\omega]\cos\left[\omega t + \phi_0\right] \qquad G_0[\omega] > 0 \quad . \tag{25.05.01}$$

→ 1411 1412

Common cases include:

$$G[\omega] = G_0 \quad ,$$
$$G[\omega] = G_0\omega^2 \quad .$$

Senza perdere di generalità si può considerare il problema di una forzante cosinusoidale di frequenza assegnata ω e fase nulla ($\phi_0 = 0$):

$$\ddot{\xi} + 2\beta\dot{\xi} + \omega_0^2\xi \equiv \ddot{\xi} + \Gamma\dot{\xi} + \omega_0^2\xi = G_0[\omega]\cos\left[\omega t\right] \qquad G_0[\omega] > 0 \quad , \tag{25.05.02}$$

→ 1421

where, in general, the amplitude of the external force might depend on its angular frequency: $G_0 = G_0[\omega]$.

Infatti, nel caso di forzante Sinusoidal | Cosinusoidal quando sia nota la soluzione del problema (25.05.01), $\xi[t]$, è sempre possibile ricavare la soluzione al problema in cui la forzante sia $G_0[\omega]\cos\left[\omega t + \phi_0\right]$ come $\xi[t + \phi_0/\omega]$. Ciò, infatti, equivale ad effettuare un cambio di variabile sui tempi: $t \longrightarrow t' = t + \phi_0/\omega$:

$$G_0[\omega]\cos\left[\omega t\right] \to G_0[\omega]\cos\left[\omega t + \phi_0\right] \quad ,$$
$$\xi[t] \to \xi[t'] \equiv \xi[t + \phi_0/\omega] \quad .$$

Ovviamente quando si sovrappongono forzanti diverse non è più possibile ignorare le fasi relative perchè non è più possibile, in generale, scegliere un tempo che annulla tutte le fasi contemporaneamente. A simple way to re-introduce in the formulas the phase ϕ_0 whenever necessary will be presented in section § 25.05.02:

$$\phi_0 = 0 \longleftrightarrow \delta \qquad \phi_0 \longleftrightarrow \delta + \phi_0$$

La soluzione generale è la somma della soluzione generale dell'equazione omogenea (25.04.01), ξ_{G}, più una soluzione particolare dell'equazione completa (25.05.01), ξ_{P}:

$$\xi[t] = \xi_{\mathrm{G}}[t] + \xi_{\mathrm{P}}[t] \quad .$$

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25.05.01.01 Steady Solution for the Position

The part of $\xi[t]$ surviving at very large times, when the exponential terms in the solution of the homogeneous equation have faded away, is called the steady solution.

Si ha:

$$\xi_{\mathrm{S}} = \frac{G_0[\omega]}{\sqrt{\left(\omega^2-\omega_0^2\right)^2+\Gamma^2\omega^2}}\cos\left[\omega t-\delta[\omega]\right] \quad , \tag{25.05.03}$$

→ 1416 1417

$$\cos\left[\delta[\omega]\right] = \frac{\omega_0^2-\omega^2}{\sqrt{\left(\omega^2-\omega_0^2\right)^2+\Gamma^2\omega^2}} \Longrightarrow \begin{cases} 0 \le \delta[\omega] < \pi/2 & \text{for } \omega < \omega_0 \\ \pi/2 < \delta[\omega] \le \pi & \text{for } \omega > \omega_0 \end{cases} \quad , \tag{25.05.04}$$

→ 1416

$$\sin\left[\delta[\omega]\right] = \frac{\Gamma\omega}{\sqrt{\left(\omega^2-\omega_0^2\right)^2+\Gamma^2\omega^2}} \geq 0 \Longrightarrow 0 \le \delta[\omega] \le \pi \quad . \tag{25.05.05}$$

→ 1416

The amplitude of the response is thus:

$$\max\left[|\xi_{\mathrm{P}}|\right] = \frac{G_0[\omega]}{\sqrt{\left(\omega^2-\omega_0^2\right)^2+\Gamma^2\omega^2}} \quad .$$

As $\sin\left[\delta[\omega]\right] \geq 0$ while $\cos\left[\delta[\omega]\right]$ can have both signs sign the solution can be written in terms of the $\arctan$, provided the solution is always chosen such that $0 \le \delta[\omega] \le \pi$:

$$\delta[\omega] = \arctan\left(\frac{\Gamma\omega}{\omega_0^2-\omega^2}\right) \qquad \text{the solution such that } 0 \le \delta[\omega] \le \pi \quad ,$$

and the only ambiguity from the formula, between $\delta = 0$ and $\delta = \pi$, might arise when the argument of the arc-tangent, $\Gamma\omega/\left(\omega_0^2-\omega^2\right)$, is zero.

Even better:

$$\delta[\omega] = \arccos\left(\frac{\omega_0^2-\omega^2}{\sqrt{\left(\omega^2-\omega_0^2\right)^2+\Gamma^2\omega^2}}\right) \quad .$$

Note that in the case $\phi_0 \neq 0$ the phase-angle $\delta[\omega]$ must be interpreted as the phase-angle with respect to the phase of the forcing term, ϕ_0.

La soluzione generale dell'equazione omogenea è una soluzione transiente in quanto in ogni caso è presente un esponenziale negativo che fa tendere la soluzione a zero per $t \longrightarrow \infty$, come mostrato dalle equazioni (25.04.03), (25.04.04), (25.04).

La soluzione stazionaria rappresenta il termine permanente per tempi grandi e per questo è solitamente detta soluzione stazionaria, ξ_{S}. La soluzione stazionaria si determina sottraendo da una soluzione particolare la eventuale parte che risolve l'omogenea. Infatti la derivata prima e seconda rispetto al tempo della soluzione generale, a causa della presenza dell'esponenziale decrescente, tendono a zero per $t \longrightarrow \infty$. è quindi chiaro dalla equazione (25.05.01) che, almeno nel caso di forzante Sinusoidal | Cosinusoidal , l'andamento per tempi grandi della soluzione deve avere la stessa dipendenza temporale della forzante estern e quindi oscillare con ampiezza costante.

25.05.01.02 Steady Solution for the Velocity

The steady-state solution for $\dot{\xi}$ is:

$$\dot{\xi}_{\mathrm{S}}[t] = \frac{-G_0[\omega]\omega}{\sqrt{\left(\omega^2-\omega_0^2\right)^2+\Gamma^2\omega^2}}\sin\left[\omega t-\delta[\omega]\right] = \frac{G_0[\omega]\omega}{\sqrt{\left(\omega^2-\omega_0^2\right)^2+\Gamma^2\omega^2}}\cos\left[\omega t-\delta[\omega]+\pi/2\right] \quad . \quad (25.05.06)$$

→ 1418

The velocity is out-of-phase by $\pi/2$, with respect to the position.

25.05.01.03 Steady Solution for the Acceleration

The steady-state solution for $\ddot{\xi}$ is:

$$\ddot{\xi}_{\mathrm{S}}[t] = \frac{-G_0[\omega]\omega^2}{\sqrt{\left(\omega^2-\omega_0^2\right)^2+\Gamma^2\omega^2}}\cos\left[\omega t-\delta[\omega]\right] = \frac{G_0[\omega]\omega^2}{\sqrt{\left(\omega^2-\omega_0^2\right)^2+\Gamma^2\omega^2}}\cos\left[\omega t-\delta[\omega]+\pi\right] \quad . \quad (25.05.07)$$

The acceleration is out-of-phase by π, with respect to the position.

25.05.01.04 Limiting Cases for the Response in Terms of the Position

- If $0 \leq \omega \ll \omega_0$:

$$\mathcal{H}[\omega] \simeq \frac{G_0[\omega]}{\omega_0^2} \qquad \delta[\omega] \simeq 0 \qquad \xi_{\mathrm{P}} \sim +\frac{G_0[\omega]}{\omega_0^2}\cos\left[\omega t\right] \quad . \quad (25.05.08)$$

→ 1419

  The response is in-phase with the forcing.
- If $\omega \approx \omega_0$ (it is equivalent to $\omega \approx \omega_{\mathrm{R}}$ if and only if the oscillations are weakly damped):

$$\mathcal{H}[\omega] \simeq \frac{G_0[\omega]}{\Gamma\omega_0} \qquad \delta[\omega] \simeq \frac{\pi}{2} \qquad \xi_{\mathrm{P}} \sim \frac{G_0[\omega]}{\Gamma\omega_0}\sin\left[\omega t\right] \quad . \quad (25.05.09)$$

→ 1419

  The response is in-quadrature with the forcing.
- If $\omega \gg \omega_0$:

$$\mathcal{H}[\omega] \simeq \frac{G_0[\omega]}{\omega^2} \qquad \delta[\omega] \simeq \pi \qquad \xi_{\mathrm{P}} \sim -\frac{G_0[\omega]}{\omega^2}\cos\left[\omega t\right] \quad . \quad (25.05.10)$$

→ 1419

  The response is out-of-phase by π rad with respect to the forcing.

25.05.01.05 Limiting Cases for the Response in Terms of the Velocity

25.05.01.06 Limiting Cases for the Response in Terms of the Acceleration

25.05.01.07 Graphs

Consider the common case when there is no dependence on the frequency in $G_0[\omega] = G_0$.

The amplitude and the phase of the response in position as a function of the frequency of the forcing are shown in figures 25.11, 25.12.

The amplitude and the phase of the response in velocity as a function of the frequency of the forcing are shown in figures 25.13, 25.14.

All the plots refer to a fixed ω_0 and different values of Γ, from $\omega_0/\Gamma = 0.1$ (strong damping) to $\omega_0/\Gamma = 10$ (weak damping).

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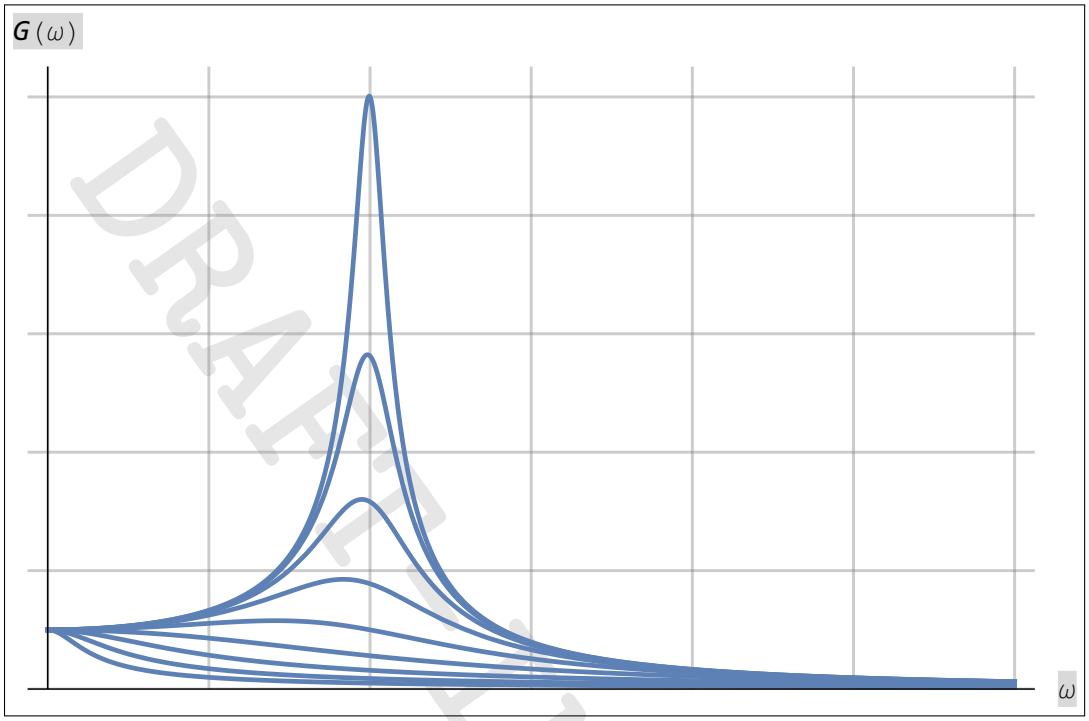

Figure 25.11: Amplitudes,ξ: the weaker the damping, the sharpest the peak.

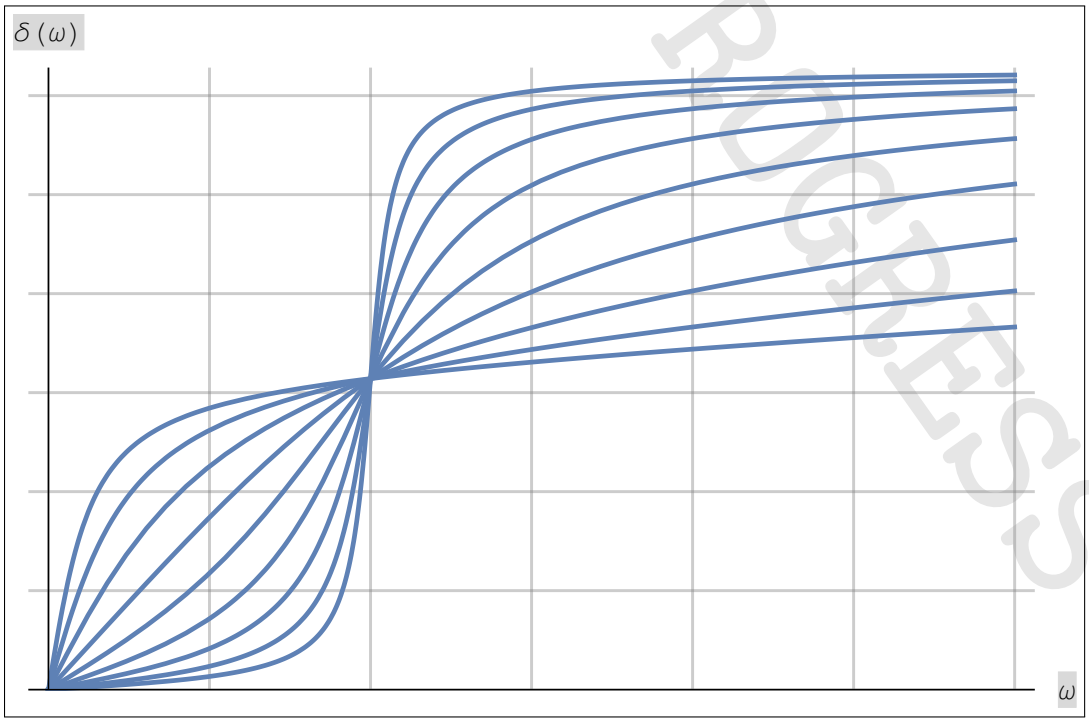

Figure 25.12: Phases,ξ: the weaker the damping, the sharpest the transition.

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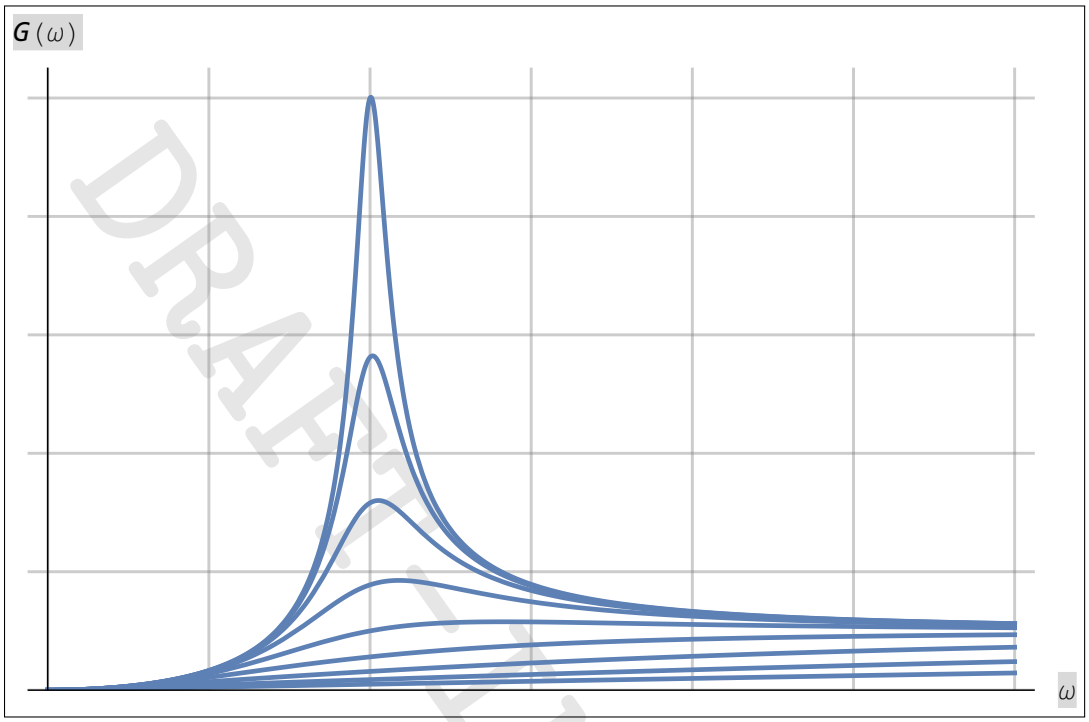

Figure 25.13: Amplitudes, $\dot{\xi}$: the weaker the damping, the sharpest the peak.

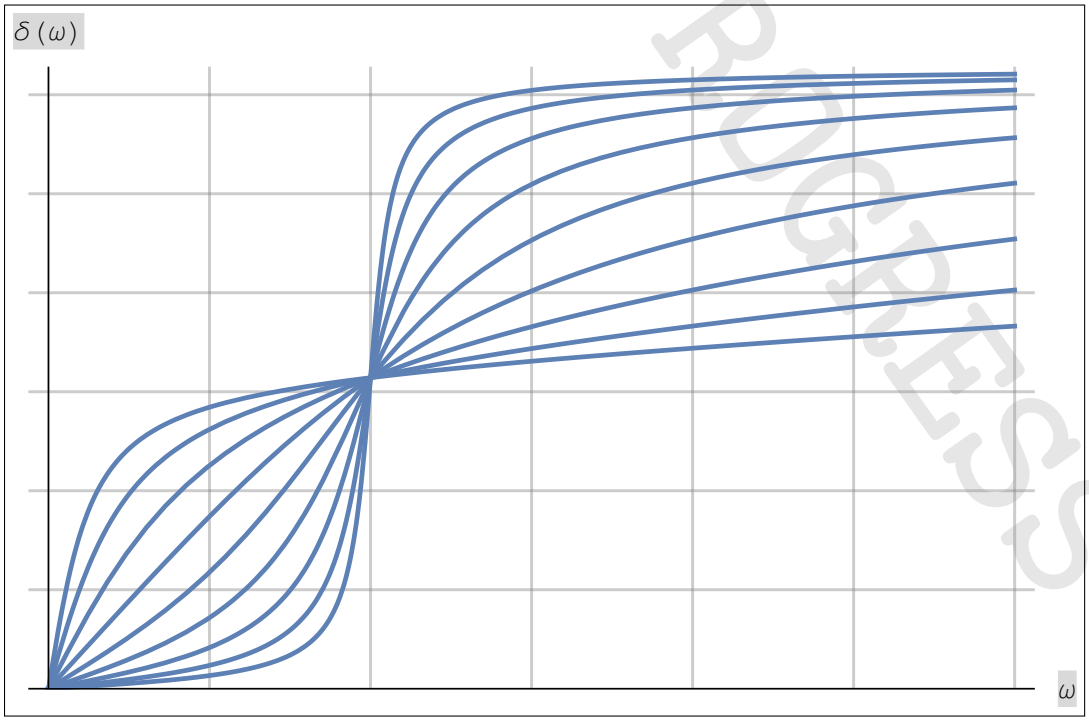

Figure 25.14: Phases, $\dot{\xi}$: the weaker the damping, the sharpest the transition.

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25.05.02
Absorptive and Elastic Amplitudes

This § is referenced at pages:
[1411, 2322]

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|3.2||

25.05.02.01 Absorptive and Elastic Amplitudes for Zero Phase of the External Forcing

An alternative equivalent way to write the steady-state solution of equation (25.05.03) is based on the so-called **absorptive** and **elastic** (sometimes called **dispersive**) amplitudes:

$$\boxed{\xi_{\mathrm{S}}[t] = \mathcal{A}_{\mathrm{A}} \sin[\omega t] + \mathcal{A}_{\mathrm{E}} \cos[\omega t] \qquad \text{for } \phi_0 = 0} \quad , \quad ,$$

$$\boxed{\mathcal{A}_{\mathrm{E}} = G_0[\omega] \left(\frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \Gamma^2 \omega^2} \right)} \quad , \quad ,$$

$$\boxed{\mathcal{A}_{\mathrm{A}} = G_0[\omega] \left(\frac{\Gamma \omega}{\left(\omega_0^2 - \omega^2\right)^2 + \Gamma^2 \omega^2} \right)} \quad , \quad ,$$

$$\mathcal{A}_{\mathrm{E}}^2 + \mathcal{A}_{\mathrm{A}}^2 = G_0^2[\omega] \left(\frac{1}{\left(\omega_0^2 - \omega^2\right)^2 + \Gamma^2 \omega^2} \right) \quad .$$

Note that the elastic and absorptive amplitudes are actually strictly related to the expressions (25.05.04), (25.05.05) for the sinus and cosinus of the phase-lag, as it is obvious from trigonometry.

The steady-state solution for $\dot{\xi}$ in terms of absorptive and elastic amplitudes is:

$$\boxed{\dot{\xi}[t] = \omega \mathcal{A}_{\mathrm{A}} \cos[\omega t] - \omega \mathcal{A}_{\mathrm{E}} \sin[\omega t]} \quad . \tag{25.05.11}$$

The names come from the fact that the time-averaged power given by the external force to the systems (and dissipated by the system in steady conditions) comes exclusively from the absorptive term $\mathcal{A}_{\mathrm{A}} \sin[\omega t]$. The term $\mathcal{A}_{\mathrm{E}} \cos[\omega t]$, on the other hand, only contributes to the instantaneous power, which, however, averages to zero over one full cycle of a steady-state oscillation. Note also that at the frequency $\omega = \omega_0$ the elastic part is zero showing that absorption is dominating.

In fact the instantaneous power is given by the external force times the velocity:

$$P[t] = m G_0[\omega] \cos[\omega t] \, \dot{\xi}[t] \quad ,$$

where a mechanical system of mass m has been assumed.

The instantaneous velocity has one component in phase and one component which is $\pi/2$ rad out of phase with respect to the external force. Only the velocity component in phase with the external force contributes to the time-averaged power. The in-phase component of the velocity is obviously produced by the out-of-phase component of the displacement.

25.05.02.02 Absorptive and Elastic Amplitudes for Arbitrary Phase of the External Forcing

The concepts of elastic and absorptive amplitudes allows to reintroduce the phase ϕ_0 of the external forcing very easily as:

$$\boxed{\xi_{\mathrm{S}}[t] = \mathcal{A}_{\mathrm{A}} \sin[\omega t + \phi_0] + \mathcal{A}_{\mathrm{E}} \cos[\omega t + \phi_0]} \quad .$$

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25.05.03
Power and Energy

The time-averaged steady-state power is then:

$$\langle P\rangle = \frac{1}{2} m G_0[\omega]\omega \mathcal{A}_{\mathtt{A}} = \frac{1}{2}\left(\frac{m G_0^2[\omega]\Gamma\omega^2}{\left(\omega_0^2-\omega^2\right)^2+\Gamma^2\omega^2}\right) \quad , \tag{25.05.12}$$

→ 1417

from which the name absorptive amplitude follows.

As a cross-check one can calculate the time-averaged power dissipated into friction to find that it is identical to the above expression for P (equation (25.05.12)):

$$\langle P_{\mathtt{F}}\rangle = m\Gamma\langle \dot{\xi}_{\mathtt{S}}{}^2\rangle = \frac{1}{2} m\Gamma\omega^2\left(\mathcal{A}_{\mathtt{A}}^2+\mathcal{A}_{\mathtt{E}}^2\right) = \frac{1}{2}\left(\frac{m G_0^2[\omega]\Gamma\omega^2}{\left(\omega_0^2-\omega^2\right)^2+\Gamma^2\omega^2}\right) \quad . \tag{25.05.13}$$

In steady-state oscillation the time-averaged stored mechanical energy is given by:

$$\langle e\rangle = \langle k\rangle + \langle u\rangle = \frac{m}{2}\langle \dot{\xi}_{\mathtt{S}}{}^2\rangle + \frac{m\omega_0^2}{2}\langle \xi_{\mathtt{S}}{}^2\rangle = \frac{m}{4}\left(\omega^2+\omega_0^2\right)\left(\mathcal{A}_{\mathtt{A}}^2+\mathcal{A}_{\mathtt{E}}^2\right) = \frac{1}{4} m G_0^2[\omega]\left(\omega^2+\omega_0^2\right)\left(\frac{1}{\left(\omega_0^2-\omega^2\right)^2+\Gamma^2\omega^2}\right) \quad ,$$

in terms of the time-averaged kinetic and potential energies. The two terms, kinetic and potential energies, are equal if and only if $\omega = \omega_0$. This should be compared to the free damped harmonic oscillator, where the two terms are equal.

25.05.04
External Forcing Independent From the Frequency

This § is referenced at pages:
[Never referenced.]

Let us consider the common case when

$$G_0[\omega] = G_0 \quad ,$$

that is the amplitude of the external forcing is independent of the frequency.

Read § 25.05.05 for the case $G_0[\omega] = G_0\omega^2$.

25.05.04.01 Resonance Curves for the Position

Amplitude

La frequenza di risonanza è quella frequenza della forza esterna in corrispondenza della quale la risposta forzata del sistema ha l'ampiezza massima. Dalla (25.05.03) si ricava la frequenza di risonanza

$$\omega_{\mathtt{R}} = \sqrt{\omega_0^2-2\beta^2} = \sqrt{\omega_0^2-\frac{\Gamma^2}{2}} \qquad \text{for either } \omega_0^2-2\beta^2>0 \text{ or } \omega_0^2-\frac{\Gamma^2}{2}>0 \quad .$$

Se lo smorzamento β è così grande che la condizione non è soddisfatta l'ampiezza massima si ha per $\omega \longrightarrow 0$ con la forza esterna che tende a diventare costante così come la posizione $\xi[t]$. Il valore della ampiezza in condizione di risonanza vale ($\omega = \omega_{\mathtt{R}}$)

$$\xi[\omega=\omega_{\mathtt{R}}] \equiv \xi_{\mathtt{M}} = \frac{G_0}{2\beta\sqrt{\omega_0^2-\beta^2}} = \frac{2G_0}{\Gamma\sqrt{4\omega_0^2-\Gamma^2}} \quad .$$

L'ampiezza della risonanza nel caso di piccolo smorzamento vale

$$\xi[\omega=\omega_{\mathtt{R}}] \equiv \xi_{\mathtt{M}} = \frac{G_0}{2\omega_0\beta} = \frac{G_0}{\omega_0\Gamma} \quad .$$

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Phase

The phase of the response is always $\pi/2$ rad at ω_0: $\delta[\omega = \omega_0] = \pi/2$

25.05.04.02 Resonance Curves for the Velocity

Amplitude

When one considers the response of the system in terms of the velocity, $\dot{\xi}$, instead of the coordinate, ξ, the response curve is different, as it is clearly shown by the fact that in equation (25.05.06) there is one more factor ω in the numerator.

This is particularly important in ElectroMagnetic oscillating circuits, for in that case one is often interested in the response in terms of the current, not charge.

Therefore, for instance, the maximum of the response of the system in terms of velocity is at the exact frequency ω_0 while the phase-angle is still $\pi/2$ rad at the exact frequency ω_0:

$$\dot{\xi}[\omega = \omega_0] = \frac{G_0}{\Gamma} \quad \text{value of the maximum} \quad , \quad \delta[\omega = \omega_0] = \pi/2 \quad .$$

Note, however, that the presence of the $-\sin[\ldots]$ function gives an additional phase of $+\pi/2$ of the velocity with respect to the external forcing, as shown from equation (25.05.06): the velocity is in advance of phase by $+\pi/2$ with respect to the coordinate and has a phase $-\delta[\omega] + \pi/2$ with respect to the external forcing.

The frequencies corresponding to $1/\sqrt{2}$ with respect to the maximum, ω_1 and ω_2, satisfy the following relations

$$\Delta\omega \equiv \omega_2 - \omega_1 = \Gamma \qquad \omega_1\omega_2 = \omega_0^2 \qquad \frac{\omega_1 + \omega_2}{2} = \sqrt{\omega_0^2 + \frac{\Gamma^2}{4}} \quad ,$$

$$\tan[\delta[\omega_1]] = +\pi/4 \qquad \tan[\delta[\omega_2]] = +3\pi/4 \quad .$$

Phase

25.05.05 External Forcing Proportional to the Square of the Frequency

This § is referenced at pages:
[1417]

©|J.P.Pérez *et al.*, *Mécanique*, ..., DUNOD, ...Ed., WEB - URL.|11.1.4, 11.4.1||

A common case is when $G_0[\omega] \propto \omega^2$. In this case, amplitude and phase for the position response are shown in figures 25.13, 25.14.

The equation of motion of a mass point relative to a non-Inertial Reference Frame , $\mathcal{I}'$, moving in a straight line with time dependence $Z[t]$,

$$z \equiv Z + z' \quad ,$$

with $\bar{z}$ the displacement with respect to the relative equilibrium position,

$$\bar{z} \equiv z' - z'_0 \quad ,$$

gives:

$$\ddot{\bar{z}} + \Gamma\dot{\bar{z}} + \omega_0^2\bar{z} = -\ddot{Z} \quad .$$

Note that both the restoring force and the viscous force are in the relative coordinate system, as for a damped spring inside a box:

$$f_{\text{R}} = -m\omega^2\left(z' - z'_0\right) \equiv -m\omega^2\bar{z} \quad ,$$

$$f_{\text{V}} = -2m\beta\dot{z}' \equiv -2m\beta\dot{\bar{z}}$$

Preliminary Progress

Assume $Z[t] = Z_0 \cos[\omega t]$:

$$\ddot{\bar{z}} + \Gamma\dot{\bar{z}} + \omega_0^2 = +Z_0\omega^2 \cos[\omega t] \quad .$$

One would like a fast damping of the transient part of the solution, therefore choosing $\beta = \omega_0$. One then finds:

$$\mathcal{H}[\omega] = Z_0 \frac{\omega^2}{\omega^2 + \omega_0^2} \quad .$$

The expression

$$\frac{\bar{z}_{\text{MAX}}}{Z_0} = \frac{\omega^2}{\omega^2 + \omega_0^2} \quad ,$$

is thus the amplification of the instrument, that is the ratio between the amplitude of the oscillations of the mass, relative to $\mathcal{I}'$, with respect to the amplitude of the oscillations of the **box**.

25.05.05.01 Seismograph

Choose $\omega_0 \ll \omega$: then $\mathcal{H}[\omega] \simeq Z_0$.

The system responds to all frequencies with the same amplitude: faithful representation of different frequencies.

See equations (25.05.08), (25.05.09), (25.05.10).

25.05.05.02 Accelerometer Shock-Absorber and Insulation From Vibrations

Choose $\omega \ll \omega_0$: then $\mathcal{H}[\omega] \simeq Z_0\omega^2/\omega_0^2$

See equations (25.05.08), (25.05.09), (25.05.10).

- The system response is proportional to the acceleration of the non-inertial Reference Frame. It is then measuring the acceleration.
- The system has a vanishingly small response, that is its response damps the oscillations of the non-Inertial Reference Frame : it is a shock-absorber.
- Anther example include mechanical insulation from vibrations: when one assembly is attached to another assembly which can vibrate, transmitting the vibrations to the assembly, one should make sure that the amplification is small in the involved frequency range. One possibility is to make sure that the lowest frequency of the assembly, like ω_0, is much larger than the maximum frequency of the external excitation, thus working in a regime similar to the one of the shock-absorber, so that the system has a vanishingly small response.

25.05.06 Q-Factor in the Forced Case

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.24.1|*On the definition of ...*

The Q-factor was defined in equation (25.04.10) and discussed section § 25.04.04 for a free damped motion.

One must generalize the definition to the forced case. It should be noted that in the literature two different definitions might encountered, which are presented below.

It is worth remembering that the small damping condition reads:

$$Q \gg 1 \quad \Leftrightarrow \quad \Gamma \ll \omega_0 \quad \text{small-damping condition} \quad .$$

In the weak-damping case of a system near resonance the two definitions given below are basically the same. As the Q-factor is mostly useful when it is large, that is the system is weakly-damped, all the definitions are basically the same on practical grounds.

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25.05.06.01 Energetic-Based Definition

Equation (25.04.11) can be used to generalize the definition to the forced case as follows:

$$Q^{-1} \equiv -\frac{1}{\omega\langle e\rangle}\frac{\mathrm{d}\langle e\rangle}{\mathrm{d}t} \simeq -\frac{\Delta e}{e}\frac{1}{\omega T} = -\frac{1}{2\pi}\frac{\Delta e}{e} \qquad \text{for } Q > \frac{1}{2} \quad ,$$

in terms of Δe, the mechanical energy dissipated in one period of the external forcing entity, which equals the period of the oscillations.

This definition is expressed in terms of the external forcing frequency, because the steady-state solution will pulsate at the external oscillation frequency (read § 25.05). It measures the entity of the damping or, equivalently, the amount of energy required to keep the oscillations going on.

In the weak damping case it can be shown that

$$Q \simeq \frac{\omega_0}{\Delta\omega} \quad ,$$

where $\Delta\omega$ represents the width of the resonance curve at $1/\sqrt{2}$ height with respect to the maximum.

This definition, therefore, can be also written in the equivalent form

$$Q \equiv 2\pi\left(\frac{\text{Total Energy}}{\text{Energy dissipated in one oscillation}}\right) = \frac{\omega^2+\omega_0^2}{2\Gamma\omega} \quad .$$

This definition, therefore, emphasizes the energy balance of the system. When remaining near the proper frequency of the system, ω_0, one has:

$$Q = \frac{\omega^2+\omega_0^2}{2\Gamma\omega} \simeq \frac{\omega_0}{\Gamma} \qquad \text{for } \omega \simeq \omega_0 \quad .$$

25.05.06.02 Amplitude-Based Definition

In this case the same identical definition as for the free case, as discussed in section § 25.04.04, is used, equation (25.04.10):

$$Q \equiv \frac{\omega_0}{2\beta} \equiv \frac{\omega_0}{\Gamma} \quad . \tag{25.05.14}$$

This definition, therefore, emphasizes the amplitude response of the system, without considering the frequency of the actual forcing, that is the real period of the oscillations.

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25.06 Complex Formalism

This § is referenced at pages:
[2018, 2067]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.22|*Excellent discussion*|
©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|§ 7 - MATHEMATICAL NOTES|*Excellent ...*

The calculations about linear second-order forced systems are often made easier by using complex numbers. Complex numbers have no real significance, but are just used as a tool for calculations. In fact we are dealing with sinusoids at fixed frequency characterized by an amplitude and a phase. A single complex number can keep both information, thus simplifying the formalism.

As all terms of the linear second-order forced system are real, if ξ_1 and ξ_2 are solutions of the homogeneous equation then also $\xi_1 \pm \mathtt{i}\xi_2$ are solutions of the homogeneous equation. Therefore it is possible to look for complex solutions of the homogeneous equation: two complex conjugate solutions will be found, from which two real solutions can be extracted by taking, for instance, the real and imaginary part.

Moreover, if one takes the forcing term to be complex and one finds a particular complex solution of the equation then the full complex equation can be separated into its two real and imaginary parts, both of them correct equations in terms of real numbers with a real forcing term. Therefore one can look for complex solutions with a complex forcing term, solve the equation and then take the real parts of both the solutions and the forcing term, to recover the physical quantities.

The previous two points are the basis for the complex formalism.

The convention will be assumed that the real physical entity is described by the real part of the complex number.

Consider a complex particular solution of the form:

$$\overline{\xi}[t] = \overline{\xi}0_{\exp}\left[\pm\mathtt{i}\omega t\right] \qquad \left(\longrightarrow \overline{\xi}0_{\exp}\left[-\mathtt{i}\omega t\right]\right) \quad .$$

The convention for the sign of the exponent varies in the literature. We will use the **minus sign** convention, because that is consistent with the standard choice in Quantum Physics:

$$\overline{\xi}[t] = \overline{\xi}_0 \exp\left[-\mathtt{i}\omega t\right] \qquad \Leftrightarrow \qquad , \qquad \rightarrow 1421$$

$$\boxed{\mathfrak{Re}\left[\overline{\xi}_0 \exp\left[-\mathtt{i}\omega t\right]\right] = \mathfrak{Re}\left[\overline{\xi}_0\right]\mathfrak{Re}\left[\exp\left[-\mathtt{i}\omega t\right]\right] - \mathfrak{Im}\left[\overline{\xi}_0\right]\mathfrak{Im}\left[\exp\left[-\mathtt{i}\omega t\right]\right] = \mathfrak{Re}\left[\overline{\xi}_0\right]\cos\left[\omega t\right] + \mathfrak{Im}\left[\overline{\xi}_0\right]\sin\left[\omega t\right]} \quad .$$

Let us use the trial solution (25.06) in equation (25.05.02) after making the forcing term a complex function:

$$G_0[\omega]\cos\left[\omega t + \phi_0\right] = G_0[\omega]\cos\left[-\omega t - \phi_0\right] \longrightarrow \boxed{G_0[\omega]\exp\left[-\mathtt{i}\left(\omega t + \phi_0\right)\right] \equiv \overline{G}_0[\omega]\exp\left[-\mathtt{i}\omega t\right]} \quad ,$$

$$\overline{G}_0[\omega] : \qquad \begin{cases} \text{module} : G_0[\omega] \quad , \\ \text{phase} : -\phi_0 \end{cases} \qquad ,$$

$$\boxed{\overline{\xi}_0 = \frac{\overline{G}_0[\omega]}{-\omega^2 - \mathtt{i}\Gamma\omega + \omega_0^2} \qquad \overline{\xi}[t] = \overline{\xi}_0 \exp\left[-\mathtt{i}\omega t\right]} \quad .$$

A common time dependence factor, $\exp\left[-\mathtt{i}\omega t\right]$, appears in all the terms as the derivative of an exponential is still an exponential. Therefore the use of complex numbers allows to factorize the time dependence leaving an algebraic expression only.

Moreover, it is clear that the complex formalism allows to account easily for a phase in the external forcing, ϕ_0.

WORK IN PROGRESS

25.07 Multi-Degrees of Freedom Free Undamped Coupled Harmonic Oscillators

©|H.C. Corben and P. Stehle|Classical Mechanics|*§ 45*|

Consider a conservative system with n degrees of freedom and any equilibrium configuration. As the gradient of the potential energy is zero, the potential energy can be developed in a Taylor series around the equilibrium configuration. The equation of motion then become a system of second-order linear differential equations with real and constant coefficients.

Normal modes of any oscillating system is a motion such that all degrees of freedom move sinusoidally with the same frequency and the same phase. The normal modes take place at the fixed frequencies.

It can be shown that any system with n degrees of freedom has n normal modes, that is n possible proper frequencies. The general small motion around an equilibrium configuration is a linear combination of the normal modes.

25.07.01
Normal Modes of Two Coupled Identical Oscillators

Consider two identical masses, of mass m connected by three springs, one with elastic constant K, the middle one which provides the coupling between the two identical oscillators, the other two with elastic constant k. The masses and springs form a linear chain whose extremes are kept fixed. See figure 25.15. Assume that the length at rest of the springs is equal to zero and that friction is negligible.

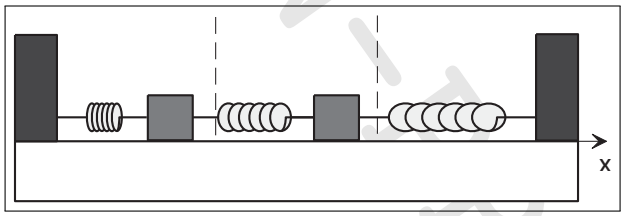

Figure 25.15: Two masses and three springs linear chain

Let $\overline{x}_1$ and $\overline{x}_2$ the position of the two masses, let $\overline{x}_0$ the position of the left-most extreme of the chain and let $\overline{x}_0 + L$ the position of the right-most extreme of the chain.

Equations:

$$m\ddot{\overline{x}}_1 = -k\left(\overline{x}_1 - \overline{x}_0\right) + K\left(\overline{x}_2 - \overline{x}_1\right) = -\overline{x}_1\left(k + K\right) + K\overline{x}_2 + k\overline{x}_0 \quad ,$$
$$m\ddot{\overline{x}}_2 = -K\left(\overline{x}_2 - \overline{x}_1\right) + k\left(\overline{x}_0 + L - \overline{x}_2\right) = +K\overline{x}_1 - \overline{x}_2\left(k + K\right) + k\left(\overline{x}_0 + L\right) \quad .$$

The middle spring, of constant K, couples two otherwise decoupled oscillators.

Transform to new variables describing the displacement with respect to the equilibrium position:

$$x_1^\star = \overline{x}_0 + \frac{KL}{k + 2K} \qquad \left(x_1^\star = \overline{x}_0 + \frac{1}{3}L \qquad \text{for } k = K\right) \quad ,$$
$$x_2^\star = \overline{x}_0 + \frac{(k + K)\,L}{k + 2K} \qquad \left(x_2^\star = \overline{x}_0 + \frac{2}{3}L \qquad \text{for } k = K\right) \quad ,$$

$$\overline{x}_1 \longrightarrow x_1 \equiv \overline{x}_1 - x_1^\star \qquad \overline{x}_2 \longrightarrow x_2 \equiv \overline{x}_2 - x_2^\star \quad .$$

In case the rest length of the spring is not zero there would be additional constant terms in the equations which would cancel in the final equations, after changing variables to the displacement with respect to the equilibrium position. Read also § 25.07.04.

The special motions of a system with many degrees of freedom when all degrees of freedom oscillate by harmonic motion with the same frequency and same phase (but possibly different amplitudes) are called normal modes.

DRAFT - IN PROGRESS

Let try to guess the normal modes: we look for solutions such that both x_1 and x_2 oscillate with the same frequency and phase, but allowing for different amplitudes:

$$x_1[t] = x_{1|0} \cos[\omega t + \phi] \quad ,$$
$$x_2[t] = x_{2|0} \cos[\omega t + \phi] \quad .$$

By substitution one can find the ratio of the amplitudes from both equations and equate them:

$$\frac{x_{1|0}}{x_{2|0}} = \frac{K}{k + K - m\omega^2} = \frac{k + K - m\omega^2}{K} \quad .$$

The characteristic equation then gives the two proper frequencies:

$$\left(k - m\omega^2\right)\left(k + 2K - m\omega^2\right) = 0 \qquad \Longrightarrow \left\{\omega_1^2 = \frac{k}{m} \qquad \omega_2^2 = \frac{k + 2K}{m}\right\} \quad .$$

It follows:

$$\omega_1^2 = \frac{k}{m} \Longrightarrow x_{1|0} = +x_{2|0} \quad ,$$
$$\omega_2^2 = \frac{k + 2K}{m} \Longrightarrow x_{1|0} = -x_{2|0} \quad .$$

The middle spring couples two otherwise decoupled oscillators and, as a consequence of that, the overall system has tow frequencies which, in general, are different from the original frequencies of the un-coupled system.

The first normal mode is such that the middle spring does not change its length. Therefore, in this mode, even if the middle spring is extended it applies just a constant force which does not change the proper frequency. Therefore the first proper frequency of the system is the same as the proper frequency of the single oscillators.

On the other hand the second normal mode is such that the middle spring changes its length, and changes the oscillation frequency.

The general solution is a superposition of the two modes:

$$x_1[t] = C_1 \cos[\omega_1 t + \phi_1] + C_2 \cos[\omega_2 t + \phi_2] \quad ,$$
$$x_2[t] = C_1 \cos[\omega_1 t + \phi_1] - C_2 \cos[\omega_2 t + \phi_2] \quad ,$$

where the constants of integration are the four constants: $\{C_1, C_2\}$ and $\{\phi_1, \phi_2\}$.

25.07.02
Normal Modes of Two Coupled Oscillators With the Same Mass

Generalize the previous system to arbitrary spring constants: k_1 and k_3 for the two oscillators, and k_2 as a coupling. Let the masses still be identical: $m_1 = m_2 = m$.

The proper frequencies turn out to be:

$$\Omega = \begin{pmatrix} \dfrac{-k_1 - k_2}{m} & \dfrac{k_2}{m} \\ \dfrac{k_2}{m} & \dfrac{-k_2 - k_3}{m} \end{pmatrix} \quad ,$$
$$\omega_1^2 = \frac{k_1 + 2k_2 + k_3 - \sqrt{4k_2^2 + (k_1 - k_3)^2}}{2m} \quad ,$$
$$\omega_2^2 = \frac{k_1 + 2k_2 + k_3 + \sqrt{4k_2^2 + (k_1 - k_3)^2}}{2m}$$

The general relation provides limiting cases for other simpler systems:

- No coupling: $k_2 \to 0$ implies $\omega^2 = \left\{\dfrac{k_1}{m}, \dfrac{k_3}{m}\right\}$;

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- No spring at the left/right walls:

$$k_1 \to 0 \qquad \omega_L^2 = \left\{ \frac{2k_2 + k_3 + \sqrt{4k_2^2 + k_3^2}}{2m}, \frac{2k_2 + k_3 - \sqrt{4k_2^2 + k_3^2}}{2m} \right\} \quad ,$$

$$k_3 \to 0 \qquad \omega_R^2 = \left\{ \frac{k_1 + 2k_2 + \sqrt{k_1^2 + 4k_2^2}}{2m}, \frac{k_1 + 2k_2 - \sqrt{k_1^2 + 4k_2^2}}{2m} \right\}$$

- No walls: $k_1 \to 0$ and $k_3 \to 0$ implies $\omega^2 = \left\{ \frac{2k_2}{m}, 0 \right\}$;
- Rigidly fixed mass (1): $k_1 \to \infty$ implies $\omega^2 = \left\{ \frac{\infty}{m}, \frac{k_2 + k_3}{m} \right\}$;
- Rigidly fixed mass (2): $k_3 \to \infty$ implies $\omega^2 = \left\{ \frac{\infty}{m}, \frac{k_2 + k_1}{m} \right\}$.

25.07.03
Normal Modes of Two Coupled Different Oscillators

$$\Omega = \begin{pmatrix} \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2 - k_3}{m_2} \end{pmatrix} \quad ,$$

$$\omega_1^2 = \frac{k_3 m_1 + k_1 m_2 + k_2 (m_1 + m_2) + \sqrt{((k_2 + k_3) m_1 + (k_1 + k_2) m_2)^2 - 4 (k_2 k_3 + k_1 (k_2 + k_3)) m_1 m_2}}{2 m_1 m_2} \quad ,$$

$$\omega_2^2 = \frac{k_3 m_1 + k_1 m_2 + k_2 (m_1 + m_2) - \sqrt{((k_2 + k_3) m_1 + (k_1 + k_2) m_2)^2 - 4 (k_2 k_3 + k_1 (k_2 + k_3)) m_1 m_2}}{2 m_1 m_2}$$

25.07.04
Longitudinal Oscillations/Vibrations of a Linear Spring-Mass Chain

This § is referenced at pages:
[1422, 1425, 1991, 1994, 2012, 2013]

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|3.5|*Detailed discussion*|

©|A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...|§ 7|*Detailed discussion*|

©|D.Morin|WEB - URL |*Detailed discussion*|

©|J.P.Pérez *et al.*, *Mécanique*, ..., DUNOD, ...Ed., WEB - URL.|||

A linear mass-spring chain can be used as a very simplified model for a one-dimensional chain of atoms inside a solid. It is in fact a linear mass-spring chain of N coupled identical oscillators.

Consider a straight linear chain of N identical masses m forming a chain with $N + 1$ identical springs of elastic constant k and the rest length d. The two extremes are kept in a fixed position at a distance $L \equiv (N + 1)\, a$, where a is the distance between the masses at rest. The only external forces acting on the system (masses plus springs) is the constraint at the two extremes of the chain. Denote the absolute position of the i-mass as x_i and the position relative to equilibrium as ψ_i. The two fixed extreme points can be considered as a fictitious (0)-mass and (N+1)-mass, at positions, respectively, x_0 and $x_{N+1} = x_0 + L$. Then:

$$x_i = \psi_i + ia + x_0 \quad .$$

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The force on the i-mass is:

$$\begin{aligned} m\ddot{x}_i &= -k\left((x_i - x_{i-1}) - d\right) + k\left((x_{i+1} - x_i) - d\right) \quad , \\ &= -k\left((\psi_i - \psi_{i-1}) + (a-d)\right) + k\left((\psi_{i+1} - \psi_i) + (a-d)\right) \quad , \\ &= k\left(\psi_{i+1} + \psi_{i-1} - 2\psi_i\right) \qquad \text{for } i = 1, \ldots N \quad . \end{aligned}$$

The latter equation shows that in terms of the displacements, ψ_i, the rest length of the springs and the distance between the masses at equilibrium do not show up in the final result.

The system is described by the set of coupled equations of motion:

$$m\ddot{\psi}_i = k\left(\psi_{i-1} - 2\psi_i + \psi_{i+1}\right) \qquad \text{for } i = 1, \ldots N \quad ,$$

or, equivalently, by the set:

$$\ddot{\psi}_i + 2\omega_0^2\psi_i = \omega_0^2\left(\psi_{i-1} + \psi_{i+1}\right) \qquad \text{for } i = 1, \ldots N \quad .$$

25.07.04.01 Solution

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|3.5|*Detailed discussion*|

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It can be shown that the proper frequency, ω_p, and wave-number, k_p, for the p-mode ($p = 1, \ldots N$), and the **Dispersion Relation** , $\omega_p[k_p]$ (read § 44.03.08) for the system of N coupled identical oscillators described in section § 25.07.04, with spring elastic constant k and mass m, on a linear chain with spacing a and length L, are:

$$\omega_p = 2\omega_0 \sin\left[\frac{p\pi}{2\left(N+1\right)}\right] \qquad (25.07.01) \quad \rightarrow \text{1426 2012}$$

$$k_p = \frac{p\pi}{\left(N+1\right)a} \qquad (25.07.02) \quad \rightarrow \text{2012}$$

$$\omega_p[k_p] = 2\omega_0 \sin\left[\frac{ak_p}{2}\right] \qquad (25.07.03) \quad \rightarrow \text{2012}$$

$$\text{for } p = 1, \ldots N \quad .$$

Due to the bounds for p the argument of the sinus is always strictly smaller than $\pi/2$, and therefore $0 < \omega_p < 2\omega_0$.

25.07.04.02 Continuum Limit for Infinite Masses/Springs

Suppose that we want to study the limiting case of infinitely many masses/springs. We then look for the limit when the number of masses and springs increases, $N \longrightarrow \infty$, by keeping the same total length, L, of the chain, (implies $a \longrightarrow 0$), the same total mass, $M \equiv Nm$, (implies $m \longrightarrow 0$), and the same total elastic constant of the whole chain, $K\left(N+1\right) \equiv k$.

In the limit for $a \longrightarrow 0$ the set of functions $\psi_i[t]$ is replaced by the interpolating function $\psi[x,t]$:

$$\psi_i[t] \longrightarrow \psi[x,t] \qquad \text{such that} \qquad \psi[x = x_0 + ia] \equiv \psi_i[t] \quad .$$

Therefore one can write the expressions:

$$\frac{\psi_{i+1} - \psi_i}{a} \simeq \frac{\partial\psi}{\partial x}[x = x_0 + (i + 1/2)\,a] \qquad \text{for } i = 1, \ldots N \quad ,$$

$$\frac{\psi_i - \psi_{i-1}}{a} \simeq \frac{\partial\psi}{\partial x}[x = x_0 + (i - 1/2)\,a] \qquad \text{for } i = 1, \ldots N \quad ,$$

$$\frac{\frac{\partial\psi}{\partial x}[x = x_0 + (i + 1/2)\,a] - \frac{\partial\psi}{\partial x}[x = x_0 + (i - 1/2)\,a]}{a} \simeq \frac{\partial^2\psi}{\partial x^2}[x = x_0 + ia] \qquad \text{for } i = 1, \ldots N \quad .$$

Moreover:

$$a^2\omega_0^2 = a^2\frac{k}{m} = a^2\frac{K\left(N+1\right)}{\frac{M}{N}} \longrightarrow \frac{L^2K}{M} \quad .$$

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The system can then be transformed as:

$$\ddot{\psi}_i = \omega_0^2 \left((\psi_{i+1} - \psi_i) - (\psi_i - \psi_{i-1}) \right) \qquad \text{for } i = 1, \dots N \quad ,$$

$$\frac{\partial^2 \psi}{\partial t^2}[x,t] \simeq a^2 \omega_0^2 \frac{\partial^2 \psi}{\partial x^2}[x,t] \simeq \frac{L^2 K}{M} \frac{\partial^2 \psi}{\partial x^2}[x,t] \quad .$$

The equation is the **D'Alembert non-dispersive wave-equation** and it is normally written as:

$$\boxed{\frac{\partial^2 \psi}{\partial t^2}[x,t] = v^2 \frac{\partial^2 \psi}{\partial x^2}[x,t]} \quad ,$$

in term of a velocity, which takes the meaning of speed of propagation of the wave. Read § 44. The speed of the continuum limit, v, is given by equation (45.02.04), (45.13.01), (45.13.02).

The meaning of the interpolating function $\psi[x,t]$ is: the longitudinal displacement with respect to the equilibrium position of the element of the chain whose longitudinal position at equilibrium is at x.

Cut-Off Frequency

Note that equation (25.07.01) shows that, when N increases the maximum frequency goes as:

$$\omega_N \simeq 2\omega_0 = 2N \sqrt{\frac{K}{M}} \quad .$$

The fact that the solutions for large N are, up to a large cut-off frequency, similar to the continuum-wave one is good to tell us that we can use the continuum approximation even if we know we are describing a large but finite number of atoms.

25.07.05
Transverse Oscillations/Vibrations of a Linear Spring-Mass Chain

This § is referenced at pages:
[1994]

25.07.05.01 Solution

25.07.05.02 Continuum Limit for Infinite Masses/Springs

The equation is the **D'Alembert non-dispersive wave-equation** and it is normally written as:

$$\boxed{\frac{\partial^2 \psi}{\partial t^2}[x,t] = v^2 \frac{\partial^2 \psi}{\partial x^2}[x,t]} \quad ,$$

in term of a velocity, which takes the meaning of speed of propagation of the wave. Read § 44. The speed of the continuum limit, v, is given by equation (45.03.01).

The meaning of the interpolating function $\psi[x,t]$ is: the transverse displacement with respect to the equilibrium position of the element of the chain whose longitudinal position is at x.

Cut-Off Frequency

25.08
Parametric Resonance

This § is referenced at pages:
[Never referenced.]

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Some oscillatory systems which are not closed, are such that the external action amounts only to a time variation of some of the parameters. One simple case is a simple pendulum whose length is changed by some external agent.

Several systems can be described by a second-order differential similar to the damped linear oscillator equation but with time-dependent coefficients:

$$\ddot{\xi} + \Gamma\dot{\xi} + \omega_0^2[t]\xi = 0 \quad .$$

25.08.01
Simple Model for a See-Saw

©|J.P.Pérez *et al.*, *Mécanique*, ..., DUNOD, ...Ed., WEB - URL.|10.6||
©|D.Garanin - Analytical Dynamics - Graduate Center CUNY - Parametric resonance|WEB - URL |*Full detailed solution*|

A very simple model of the oscillation amplitude increase in a **see-saw** goes as follow.

This example emphasizes:

- non conservation of mechanical energy;
- internal work.

Consider a pendulum, of mass m and length l whose length can be changed by pushing or pulling the string. The string tension, $\mathbf{T}$, does no work if the length of the pendulum is kept constant.

Suppose the length is slightly decreased every time the pendulum passes by the vertical position, $\theta = 0$, by an amount $\Delta l_+ \ll l$. During this process the tension does a positive work, L_+, on the mass. Suppose the length is slightly increased every time the pendulum passes at the maximum elongation, $\theta = \pm\theta_{\text{M}}$, by an equal and opposite amount $\Delta l_- = -\Delta l_+ \ll l$. During this process the tension does a negative work, L_-, on the mass. Let us calculate the overall balance as total work per full cycle.

Assume:

$$0 \le \Delta l \equiv |\Delta l_+| = |\Delta l_-| \ll l \quad .$$

Solution by the Cardinal Equations

$$mgl\,(1 - \cos[\pm\theta_{\text{M}}]) = \frac{mv^2}{2} \qquad \text{energy conservation} \quad ,$$

$$L_+ = +2|T[\theta = 0]|\Delta l = +2\left(mg + m\frac{v^2[\theta = 0]}{l}\right)\Delta l \ge 0 \quad ,$$

$$L_- = -2|T[\theta = \pm\theta_{\text{M}}]|\Delta l = -2\left(mg\cos[\pm\theta_{\text{M}}] + m\frac{v^2[\theta = \pm\theta_{\text{M}}]}{l}\right)\Delta l = -2\,(mg\cos[\pm\theta_{\text{M}}])\,\Delta l \le 0 \quad ,$$

$$L_+ + L_- = 2\left(mg + m\frac{v^2[\theta = 0]}{l} - mg\cos[\pm\theta_{\text{M}}]\right)\Delta l = 6mg\,(1 - \cos[\pm\theta_{\text{M}}])\,\Delta l > 0 \quad .$$

Note the factor 2, taking into account that the elongate/shorten operation is carried on twice per cycle.

The result shows that a positive energy is given to the system per cycle.

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Solution by Energy Balance

25.09 Examples and Physical Applications

25.09.01 Harmonic Motion From Circular Motion

Consider a circular motion with constant angular velocity. The projected motion along any axis passing by the center of the circumference is an harmonic motion.

25.09.02 Anharmonic Oscillations

©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|§ 7 - ADVANCED TOPICS||

25.09.03 Tides in Closed Basins - Laguna Di Venezia

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25.10 Exercises Problems and Physical Applications

25-001 Trigonometric Identities

Show that the following identity

$$\boxed{G_0 \cos[\omega_0 t - \delta_0] = A \cos[\omega_0 t] + B \sin[\omega_0 t]} \quad ,$$

in terms of the two pairs of constants $\{G_0, \delta_0\}$ and $\{A, B\}$ imply the relations

$$A = G_0 \cos[\delta_0] \qquad B = G_0 \sin[\delta_0] \quad ,$$

$$G_0 = \sqrt{A^2 + B^2} \geq 0 \quad ,$$

$$\delta_0 = \arctan(B/A) \qquad \text{if } A > 0 \quad ,$$

$$\delta_0 = \arctan(B/A) + \pi \qquad \text{if } A < 0 \quad ,$$

$$\delta_0 = -\pi/2 \qquad \text{if } A = 0 \text{ and } B > 0 \quad ,$$

$$\delta_0 = +\pi/2 \qquad \text{if } A = 0 \text{ and } B < 0 \quad .$$

25-002 Perdita Di Energia Meccanica Nel Moto Smorzato Libero

1. Dimostrare che, nel moto smorzato libero, descritto dall'equazione

$$\ddot{x} + \Gamma \dot{x} + \omega_0^2 x = 0 \quad ,$$

la perdita di energia meccania per unità di tempo vale

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -m\Gamma v^2 \quad .$$

2. Dimostare che, nel caso debolmente smorzato, ($Q > 1/2$), si può scrivere l'espressione per l'energia meccanica media in un periodo, $\langle e \rangle$, come

$$\frac{\mathrm{d}\langle e \rangle}{\mathrm{d}t} \simeq -\Gamma \langle e \rangle \quad .$$

3. Dimostrare che in un periodo si ha:

$$\frac{\Delta e}{e} = -\frac{2\pi}{Q} \quad .$$

25-003 Moto Armonico Semplice E Condizioni Iniziali

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|15.31||

Dimostrare che il legame generale tra la soluzione del moto armonico semplice

$$x[t] = A \cos[(\omega t - \phi)] \quad ,$$

e le condizioni iniziali $x_0 = x[t = 0]$ e $v_0 = v[t = 0]$ sono

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

$$\begin{cases} \tan[\phi] = \dfrac{v_0}{\omega x_0} & \text{se } x_0 > 0 \quad , \\ \tan[\phi] = \dfrac{v_0}{\omega x_0} + \pi \quad , & \\ \quad \text{se } x_0 < 0 \ \tan[\phi] = +\pi/2 \quad , & \\ \quad \text{se } x_0 = 0 \text{ e } v_0 > 0 \ \tan[\phi] = -\pi/2 \quad , & \\ \quad \text{se } x_0 = 0 \text{ e } v_0 < 0 & \end{cases} \quad .$$

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25-004 Moto Armonico Semplice E Risposta in Velocità

©|D.Halliday & R.Resnick, *Fisica*, 1978, C.E.A., 3rdEd., ...|P-15.55||

Si consideri il moto armonico smorzato forzato

$$\ddot{\phi} + \Gamma\dot{\phi} + \omega_0^2\phi \equiv G_0 \cos[\omega t + \alpha] \qquad G_0 > 0 \quad .$$

1. Si calcoli la risposta in velocità, cioè la velocità in funzione del tempo, $\dot{\phi}[t]$, e si studino l'ampiezza e la fase delle oscillazioni della velocità in fuzione della frequenza esterna ω.
2. Confrontare la frequenza della risonanza in velocità con la frequenza della risonanza in posizione.

25-005 Critical Damped Motion

Show that both equations (25.04.07), (25.04.09) give the same expression in the limit towards critical damping and determine expression (25.04.08).

25-006 Molle in Serie E in Parallelo

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Due molle di costante elastica k_1 e k_2 sono collegate in serie, cioè con un estremo di una molla collegato ad un estremo dell'altra. I due estremi liberi sono così gli estremi di una nuova molla. Quanto vale la costante elastica equivalente della molla complessiva?

Le due molle sono poi collegate in parallelo, cioè con entrambi gli estremi collegati tra loro. Le due coppie di estremi collegate tra loro sono così gli estremi di una nuova molla. Quanto vale la costante elastica equivalente della molla complessiva?

WORKED SOLUTION

$k_{\text{SER}} = k_1 k_2 / (k_1 + k_2)$; $k_{\text{PAR}} = k_1 + k_2$.

25-007 A Mass Particle Kept by Two Springs

Consider (see figure 25.16) two identical springs having one extreme attached to a mass particle of mass m and their second extreme fixed in such a way that at equilibrium the springs and mass are on straight line (the x axis, with origin at the middle, that is the equilibrium position of the mass). All the parts are on a horizontal plane with negligible friction.

Consider the oscillations on the median line (y axis).

Show under which conditions the oscillations are linear.

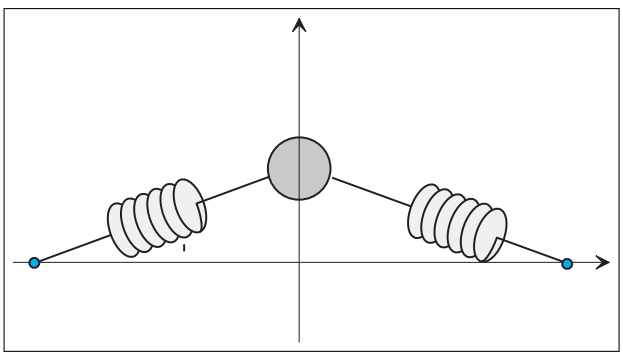

Figure 25.16: Amass particle kept by two springs - linear oscillations

DRAFT - IN - PROGRESS

25-008 Oscillazioni Verticali Di Un'automobile

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|15.7||

Per quello che riguarda le oscillazioni verticali un'automobile può essere considerata montata su quattro molle (gli ammortizzatori). Gli ammortizzatori di un'automobile di massa $M = 1460$ kg sono regolate in modo che le vibrazioni verticali dell'automobile abbiano frequenza $\nu = 2.95$ Hz.

1. Determinare la costante elastica di ciascuna delle quattro molle (supposte identiche) che formano gli ammortizzatori.
2. Quale sarà la frequenza di vibrazione se sull'auto viaggiano cinque persone aventi in media la massa $m = 73.2$ kg ?

WORKED SOLUTION

$k = 1250$ Ncm. $\nu = 2.63$ Hz.

25-009 Misura Della Massa Degli Astronauti

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|15.11||

Per misurare la massa M di un astronauta su un veicolo spaziale si usa un dispositivo costituito da un seggiolino, di massa m, montato su molle. L'astronauta misura il proprio periodo di oscillazione sulla sedia e deduce la massa dalla relazione tra massa e periodo di un sistema oscillante.

1. Dimostrare che
$$M = \frac{kT^2}{4\pi^2} - m$$
dove T è il periodo di oscillazione e k la costante elastica.
2. La costante elastica per il dispositivo usato sullo Skylab era $k = 605$ N/m e il periodo di oscillazione della sedia vuota era $T_0 = 0.90149$ sec. Calcolare la massa della sedia.
3. Il periodo di oscillazione con un astronauta sulla sedia diventava $T = 2.08832$ sec. Determinare la massa dell'astronauta.

WORKED SOLUTION

$m = 12.47$ kg. $M = 72.85$ kg.

25-010 Potenziale Di Interazione Di Due Atomi in Una Molecola Biatomica

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|15.17||

La forza di interazione tra due atomi in una molecola biatomica può essere rappresentata dalla relazione
$$F[r] = -\frac{a}{r^2} + \frac{b}{r^3}$$
dove a e b sono costanti positive ed r è la distanza tra i due atomi. Rappresentare graficamente $F[r]$ in funzione di r.

1. Dimostrare che la distanza di equilibrio è data da $r_0 = b/a$.
2. Dimostrare che per piccole oscillazioni attorno a questa distanza di equilibrio la costante della forza vale a^4/b^3.
3. Trovare il periodo T delle piccole oscillazioni attorno alla distanza di equilibrio.

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WORKED SOLUTION

$T = 2\pi\sqrt{mb^3/a^4}$.

25-011 Proiettile Fermato Da Un Sistema Blocco-Molla

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|15.34||

Un blocco di massa M è in quiete su un piano orizzontale privo di attrito ed è connesso ad un supporto rigido mediante una molla di costante elastica k. Un proiettile di massa m e velocità v_0 viaggia orizzontalmente colpisce il blocco e vi rimane conficcato. Determinare l'ampiezza del moto armonico semplice risultante in funzione di m, M, v_0 e k.

25-012 Sistema Blocco-Molla Ed Effetto Della Forza Di Gravità

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|15.35||

Si consideri una molla di massa trascurabile e di costante elastica k in un campo gravitazionale uniforme. si supponga di fissare alla molla un oggetto di massa m.

1. Se $x = 0$ è la posizione di riposo della molla dimostrare che la posizione di equilibrio statico è $x_e = mg/k$.
2. Dimostrare che l'equazione del moto del sistema è
$$m\ddot{x} + kx = mg \quad , \tag{25.10.01}$$
→ 1432
e che la soluzione di tale equazione è
$$x[t] = A\cos\left[(\omega t + \phi)\right] + \frac{mg}{k} \quad ,$$
con $\omega = k/m$, come nel caso di assenza di gravità.
3. Dimostrare che nel campo di gravità costante ω, v, a e T sono gli stessi che si ottengono in assenza di gravità con la sola differenza che la posizione di equilibrio è spostata di mg/k.
4. Si consideri l'energia totale del sistema, costante, e si dimostri che differenziando rispetto al tempo si ottiene l'equazione del moto (25.10.01).

25-013 Cilindro E Molla

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|15.37||

Un cilindro di massa M rotola senza strisciare su un piano orizzontale ed è collegato al suo centro con una molla, di costante elastica $k = 2.94$ Ncm,
il cui secondo estremo è fissato ad un muro. Se il sistema viene lasciato libero da fermo quando la molla è allungata di 23.9 cm determinare

1. l'energia cinetica di traslazione;
2. l'energia cinetica di rotazione del cilindro quando passa per la posizione di equilibrio.
3. Dimostrare che il centro di massa del cilindro si muove di moto armonico semplice con periodo $T = 2\pi\sqrt{3M/2k}$.

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25-014 Il Pendolo Di Kater

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|15.53||

Il pendolo di Kater può essere utilizzato per misurare l'accelerazione di gravità. Si immagini di avere un pendolo fisico che può essere imperniato in due punti. La posizione del primo punto è fissa mentre quella del secondo punto può essere spostata lungo il pendolo. Quando il pendolo è sospeso dalla posizione fissa il suo periodo è T. Successivamente il pendolo viene capovolto e sospeso al perno mobile. Per successivi tentativi la posizione del perno mobile viene variata fino a che il periodo è uguale a quello determinato quando il pendolo è sospeso dal perno fisso, T. Dimostrare che l'accelerazione di gravità vale $g = 4\pi^2 L/T^2$ essendo L la distanza tra i due punti di oscillazione. Si osservi che con questo metodo si può determinare g senza conoscere nulla delle proprietà del pendolo eccetto la distanza, misurata, L.

25-015 Oscillazioni a Due Corpi

©|D.Halliday & R.Resnick, *Fisica*, 1978, C.E.A., 3rdEd., ...|§ 15.8 P-15.53||

Si consideri un sistema formato da due masse, m_1 e m_2, legate da una molla di costante elastica k.

1. Determinare le frequenze proprie di oscillazione del sistema e le ampiezze relative della due coordinate posizione delle due masse nei due modi normali.
2. Determinare il moto del centro di massa delle due masse.
3. Scrivere l'espressione dell'energia cinetica del sistema.
4. Scrivere le equazioni del moto in termini della massa ridotta del sistema.
5. Dimostrare che nel caso in cui $m_1 \gg m_2$ la massa ridotta tende ad assumere il valore della massa m_2 e il moto del sistema diventa essenzialmente quello del moto ad un solo grado di libertà della massa m_2.

25-016 Coupled RLC Circuits

©|J.P.Pérez *et al.*, *Mécanique*, ..., DUNOD, ...Ed., WEB - URL.|27.3.3||

- Consider two independent RLC series circuits, made of a resistor R, a capacitance C and an inductance L. The two circuits are coupled in the following way: the two circuits are opened and, afterwards, they are both closed on another capacitor, C_0, to form a circuit with two loops, symmetrical with respect to the coupling capacitor. This circuit has capacitive coupling.
  1. Write down the equations of the circuit.
  2. Write down the proper frequencies and the amplitudes of the normal modes.
- Consider two independent RLC series circuits, made of a resistor R, a capacitance C and an inductance L. The two circuits are coupled in the following way: the inductances of the two circuits put one near the other is such a way that they couple by their mutual inductance M. This circuit has inductive coupling.
  1. Write down the equations of the circuit.
  2. Write down the proper frequencies and the amplitudes of the normal modes.

DRAFT - PRE-PRESS

25-017 Molecule of Carbon Dioxide

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Consider the longitudinal vibrations of a molecule of CO_2, that is assume that the three atoms move on a straight line. The two oxygen atoms have mass m (oxygen molar mass $16\ \mathrm{g/mol}$) and the Carbon atom has mass M (carbon molar mass $12\ \mathrm{g/mol}$). Each oxygen atom is supposed to be linked to the central carbon atom by a force which can be described by a linear restoring force with elastic constant k, for small displacement. The system has three degrees of freedom.

1. Write the equations of motion in the Center-Of-Mass Reference Frame. Note that in this Reference Frame the system has only two degrees of freedom.
2. Determine the proper frequencies and the amplitudes of the normal modes.

DRAFT-IN-PROGRESS

CHAPTER 26

Introduction to Mechanics of Rigid-Bodies

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This § is referenced at pages:
[Never referenced.]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.19; 1.20|*Elementary* ...

©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|8|*Good introduction*|

©|H.Goldstein *et al.*, *Classical Mechanics*, 2014, Pearson Education, 3rdEd., ...|8|*Advanced introduction*|

©|M.Fabrizio, *La Meccanica Razionale E I Suoi Metodi Matematici*, 1992, Zanichelli, ...Ed., ...||*Advanced introduction*|

©|R.D.Gregory|Classical Mechanics|*Nice*|

©|W.B.Heard|rigid-body Mechanics|*Extensive treatise*|

©|WEB - URL |Dynamics of rigid-bodies - advanced topics|*Nicholas Wheeler Physics Lectures*|

SHEET-IN-PROGRESS

26.01 Introduction

By definition, a rigid-body is a model a physical system such that the distance between any two particles, k and j, never changes:

$$|\mathbf{x}_k - \mathbf{x}_j| = \text{costant} \quad . \tag{26.01.01}$$

→ 1439

Let us model a rigid-body as a continuous system made of particles of known properties, whose relative positions are immutable, by integrating overs macroscopically small and microscopically large integration volumes.

When considering the internal structure of the particles making the rigid-body, a few different cases may arise:

- The classical rigid-body, made of mass points with no internal structure, that is spinless.
- A rigid-body made of rigidly fixed particles: let us call it a rigid-rigid-body.
  The particles making the rigid-body, must be endowed with some intrinsic vector properties (such as spin, **Dipole** moment, ...) and may have to be subject to a torque, $\boldsymbol{\gamma}[\mathbf{x}]$. A rigid-body made of rigidly fixed particles is an extension of the rigid-body concept to include the requirement that each particle has its intrinsic constant vector property, fixed relative to all the others.
  In this case the angular velocity $\boldsymbol{\omega}$ of the rigid-body is the same for every particle making the rigid body as well as for the intrinsic vector property, that is it does not depend on $\mathbf{x}$. This model, for instance, can apply to a ferroelectric/ferromagnetic material with fixed Polarization/Magnetization in external Electric|Magnetic field.
- A rigid-body made of particles with fixed positions but possibly rotating orientation of the intrinsic vector property: let us call it a rigid-nonrigid-body.
  In this case the angular velocity $\boldsymbol{\omega}$ of the rigid-body is the same for every particle making the rigid body, that is it does not depend on $\mathbf{x}$; However, the intrinsic vector property, at any point, $\mathbf{x}$, can rotate independently of the overall angular velocity $\boldsymbol{\omega}$ of the rigid-body.

Legenda:

- Q_P: Q is a property of the point P, e.g.: $\mathbf{R}_{\mathtt{CM}}$, $\mathbf{v}_{\mathtt{P}}$. ...;
- Q' a property relative to the Center-Of-Mass, measured in the Reference Frame of the Center-Of-Mass, e.g.: $\mathbf{L}'$, ... A **prime** is used, in agreement with the general convention to use $\mathcal{I}$ for the **observer/laboratory** Reference Frame and $\mathcal{I}'$ for the **observed/moving** Reference Frame.

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26.02 Coordinate Systems

When studying the mechanics of rigid-bodies, it is useful to use different Reference Frame and Coordinate System, depending on which one is more suitable.

Assume to describe the motion from an Inertial Reference Frame , $\mathcal{I}$, and introduce a orthonormal Cartesian Coordinates Coordinate System: $\mathcal{I} \equiv \{\mathcal{O}; \mathbf{e}_k\}$.

A second useful Reference Frame, as for the general case, is the non-Inertial Reference Frame moving with the Center-Of-Mass of the body, with origin at the Center-Of-Mass of the body and with axes parallel to the axes of the inertial frame $\mathcal{I}$: $\mathcal{I}' \equiv \mathcal{I}_{\mathtt{CM}}$. Note that this Reference Frame is non-rotating with respect to $\mathcal{I}_0$.

A third useful Reference Frame is any non-Inertial Reference Frame rigidly fixed to the body. This Reference Frame is usually, but not necessarily, centered at the Center-Of-Mass of the body, with, a priori, arbitrary orientation: $\mathcal{I}'' \equiv \mathcal{I}_{\mathtt{RB}}$. It will be shown later that setting the origin at the Center-Of-Mass and setting some specific orientation of the axes can provide a much simpler analysis of motion. Note that this Reference Frame is, in general, rotating with respect to both $\mathcal{I}_0$ and $\mathcal{I}_{\mathtt{CM}}$ and in general not inertial. This Reference Frame is useful because, with respect to it, the mass distribution and all static properties derived from it are described in the most simple way. For instance, the mass density is a given time-independent function, fixed once and for ever, irrespective of the motion of the body.

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26.03 Rigid-Body Kinematics

This § is referenced at pages:
[1149, 1192, 1241]

©|Uniqueness of the angular velocity of a rigid body: Correction of two faulty proofs|Am.J.Phys, , ..., ..., ...Ed., ... 68, 668 ...

Read also sections § 17.03, § 20.05, § 23.06. This section presents the implementation to the kinematics of rigid-bodies.

Note that a Reference Frame is by definition based on some rigid-body, to the define the rigid axes.

The definition of rigid-body, equation (26.01.01), implies that:

$$\text{constant} = |\mathbf{x}_k - \mathbf{x}_j|^2 = (\mathbf{x}_k - \mathbf{x}_j)\cdot(\mathbf{x}_k - \mathbf{x}_j) \implies (\mathbf{x}_k - \mathbf{x}_j)\cdot(\mathbf{v}_k - \mathbf{v}_j) = 0 \quad .$$

The interpretation of the above relation (26.03.02) is that, any infinitesimal displacement of the rigid-body is given by a translation of any body-fixed point O plus a rotation around it. In fact equation (26.01.01),

$$(\mathbf{x}_P - \mathbf{x}_0)\cdot(\mathbf{v}_P - \mathbf{v}_0) = 0 \quad ,$$

when it is seen as a vector equation in the unknown $(\mathbf{v}_P - \mathbf{v}_0)$ implies

$$\mathbf{v}_P = \mathbf{v}_0 + \mathbf{A}\wedge(\mathbf{x}_P - \mathbf{x}_0) \quad , \tag{26.03.01}$$

→ 1439 1449

for any body-fixed point O and any other arbitrary body-fixed point P, and where $\mathbf{A}$ is an arbitrary vector, which can depend on time, a priori. It is clear that the vector $\mathbf{A}$ is actually the angular velocity of any Reference Frame rigidly fixed to the rigid-body relative to the fixed Reference Frame. This is obvious by applying equations (23.07.02).

Conversely, it is immediate to show that whenever equation (26.03.01) is valid then equation (26.01.01) is satisfied:

$$(\mathbf{x}_A - \mathbf{x}_B)\cdot(\mathbf{v}_A - \mathbf{v}_B) = (\mathbf{x}_A - \mathbf{x}_B)\cdot(\mathbf{A}\wedge(\mathbf{x}_A - \mathbf{x}_B)) = 0$$

Therefore, the velocity, $\mathbf{v}_P$, of any body-fixed point P, can be expressed as

$$\boxed{\mathbf{v}_P = \mathbf{v}_0 + \boldsymbol{\omega}\wedge(\mathbf{x}_P - \mathbf{x}_0)} \quad , \tag{26.03.02}$$

→ 1128 1439 1440 1441 1442 1443

as a function of the angular velocity of the rigid-body, $\boldsymbol{\omega}$, of the position, $\mathbf{x}_0$, and velocity, $\mathbf{v}_0$, of any other body-fixed point O. The point O is usually named **center of reduction of the velocities of the rigid-body**, abbreviated CRV.

It follows, therefore, that all and only motions described by equation (26.03.01) with any arbitrary time-dependent vector $\mathbf{A}$ is compatible with the rigidity constraint (26.01.01) and therefore describe rigid motions.

Similarly, by deriving equation (26.03.02) with respect to time, one finds the law of accelerations:

$$\boxed{\mathbf{a}_P = \mathbf{a}_0 + \dot{\boldsymbol{\omega}}\wedge(\mathbf{x}_P - \mathbf{x}_0) + \boldsymbol{\omega}\wedge(\mathbf{v}_P - \mathbf{v}_0) = \mathbf{a}_0 + \dot{\boldsymbol{\omega}}\wedge(\mathbf{x}_P - \mathbf{x}_0) + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge(\mathbf{x}_P - \mathbf{x}_0))} \quad , \tag{26.03.03}$$

as a function of the angular velocity and angular acceleration of the rigid-body, $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$, of the position, $\mathbf{x}_0$, velocity, $\mathbf{v}_0$, and acceleration, $\mathbf{a}_0$, of any other body-fixed point O.

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26.03.01
Most General Motion of a Rigid-Body

The most general motion of a rigid-body is the same as the general motion of a Reference Frame: a combination of a translation, described by the motion of the origin of any Reference Frame rigidly fixed to the body, plus a rotation, described by the angular velocity, $\boldsymbol{\omega}$, that is a roto-translation.

26.03.02
Instantaneous Axis of Rotation

At any instant time, the rigid motion in 3D is such that the moving Reference Frame is instantaneously rotating about an axis, the instantaneous axis of rotation; this axis need not be the same at different times.

It can be determined from equation (26.03.02),

$$\boxed{\mathbf{v}_P = \mathbf{v}_0 + \boldsymbol{\omega}\wedge(\mathbf{x}_P - \mathbf{x}_0)} \quad ,$$

by imposing $\mathbf{v}_P = 0$, and using the results of § 17.02, to obtain:

$$\boxed{\mathbf{x}_P = \mathbf{x}_0 + \frac{\boldsymbol{\omega}\wedge\mathbf{v}_0}{\omega^2} + \lambda\boldsymbol{\omega} \qquad \lambda \text{ arbitrary real number}} \quad .$$

This equation defines all the points that have zero velocity at any instant, which is then a straight line parallel to the angular velocity, called axis of instantaneous rotation. This axis, in general, varies with time.

26.03.03
Euler Theorem

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Euler theorem states that any sequence of arbitrary displacements of a rigid-body such that one point on the rigid-body, say S, remains fixed, is equivalent to a suitable rotation about a suitable fixed axis through the point S. In algebraic terms, it expresses the fact that rotations form a group; that is, any sequence of rotations is equivalent to one suitable single rotation.

Read § 20.05.06

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26.03.04
Mozzi/Chasles Theorem

©|W.B. Heard|rigid-body Mechanics|*2.1.5*|

Mozzi/Chasles theorem states that the most general rigid-body displacement can be composed by a translation along an axis plus a rotation about that same axis. This is the transformation executed by a turning screw and is referred to as screw transformation; the axis is named screw axis.

26.03.05
Rolling and Slipping

Consider two rigid surfaces in contact at a single point. Pure rolling happens when the relative velocity of the two contact points on the two rigid surfaces is zero.

If one of the two surfaces is fixed, the condition of pure rolling can be obtained imposing that the velocity of the contact point of the moving surface is zero, using equation (26.03.02).

In case the two surfaces are in contact at multiple points, the zero relative velocity condition must apply to all contact points.

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26.04 Dynamics of Rigid-Bodies

Legenda: $Q_{\mathtt{CM}}$ is a property of the Center-Of-Mass, such as $\mathbf{R}_{\mathtt{CM}}$; Q' a property relative to the Center-Of-Mass, such as $\mathbf{x}'$.

26.04.01 Linear Momentum of Rigid-Bodies

$$\boxed{\mathbf{P} = M\mathbf{V}_{\mathtt{CM}}} \quad .$$

26.04.02 Angular Momentum of Rigid-Bodies

It is useful to distinguish three cases: use, as a pole, either a fixed and body-fixed pole (if it does exist!), or a generic body-fixed pole, or any arbitrary pole, via **Koënig theorem** .

Moreover, the CRV of the rigid-body is another, a priori different, point which must be considered: it may be either a fixed and body-fixed pole (if it does exist!), or a generic body-fixed pole, as the CRV of a rigid-body must be a body-fixed point. The pole and the CRV may be different but when using a body-fixed pole it is normally easier to use the same as the CRV.

26.04.02.01 Angular Momentum With Respect to a Fixed and Body-Fixed CRV

If the rigid-body has a fixed point, S, such that $\mathbf{v}_{\mathtt{S}} = \mathbf{0}$, then:

$$\boxed{\mathbf{L}_{\mathtt{S}} = \underset{\mathtt{S}}{\mathbb{I}}\cdot\boldsymbol{\omega}} \quad . \tag{26.04.01}$$

26.04.02.02 Angular Momentum With Respect to a Generic Body-Fixed CRV

If S is a body-point, such that in general $\mathbf{v}_{\mathtt{S}} \neq \mathbf{0}$, then, using equation (26.03.02) one finds:

$$\boxed{\mathbf{L}_{\mathtt{S}} = M\,(\mathbf{X}_{\mathtt{CM}} - \mathbf{x}_{\mathtt{S}})\wedge\mathbf{v}_{\mathtt{S}} + \underset{\mathtt{S}}{\mathbb{I}}\cdot\boldsymbol{\omega}} \tag{26.04.02}$$

The equation can be also applied, as a particular case, to $S = CM$ and it reduces to equation (26.04.03).

26.04.02.03 Angular Momentum via Koënig theorem

Il momento angolare di un corpo rigido rispetto ad un polo O qualunque può essere espresso tramite la (24.11.02)

$$\boxed{\mathbf{L}_{\mathtt{O}} = (\mathbf{X}_{\mathtt{CM}} - \mathbf{x}_{\mathtt{O}})\wedge\mathbf{P} + \mathbf{L}' = (\mathbf{X}_{\mathtt{CM}} - \mathbf{x}_{\mathtt{O}})\wedge\mathbf{P} + \underset{\mathtt{CM}}{\mathbb{I}}\cdot\boldsymbol{\omega}} \quad . \tag{26.04.03}$$

→ 1442

26.04.02.04 Angular Momentum via the Law of Transformation of Momentum

For the system of applied vectors $\mathbf{v}[\mathbf{x}, t]$, defining the field of velocities of the rigid-body, the law of of transformation of angular momentum when varying the pole is, from equation (15.04.07):

$$\boxed{\mathbf{L}_{\mathtt{B}} = \mathbf{L}_{\mathtt{A}} + (\mathbf{x}_{\mathtt{A}} - \mathbf{x}_{\mathtt{B}})\wedge\mathbf{P}} \quad , \tag{26.04.04}$$

of which equations (26.04.03) is a particular case when $B \equiv O$ and $A \equiv CM$.

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26.04.03
Kinetic Energy of Rigid-Bodies

It is useful to distinguish three cases: use, as a CRV, either a fixed and body-fixed point (if it does exist!), or a generic body-fixed point, or the Center-Of-Mass, via **Koënig theorem** .

In this case there is no pole, so that the different cases refer to the CRV.

26.04.03.01 Kinetic Energy With Respect to a Fixed and Body-Fixed CRV

If the rigid-body has a fixed point, S, such that $\mathbf{v}_{\mathrm{S}} = \mathbf{0}$, then:

$$\mathcal{K} = \frac{1}{2}\left(\boldsymbol{\omega}\cdot\underset{\mathrm{S}}{\mathbb{I}}\cdot\boldsymbol{\omega}\right) \quad . \tag{26.04.05}$$

26.04.03.02 Kinetic Energy With Respect to a Generic Body-Fixed CRV

If S is a body-point, such that in general $\mathbf{v}_{\mathrm{S}} \neq \mathbf{0}$, then, using equation (26.03.02) one finds:

$$\mathcal{K} = \frac{1}{2}M\mathbf{v}_{\mathrm{S}}^2 + M\,\mathbf{v}_{\mathrm{S}}\cdot(\boldsymbol{\omega}\wedge(\mathbf{X}_{\mathrm{CM}} - \mathbf{x}_{\mathrm{S}})) + \frac{1}{2}\left(\boldsymbol{\omega}\cdot\underset{\mathrm{S}}{\mathbb{I}}\cdot\boldsymbol{\omega}\right) \quad . \tag{26.04.06}$$

→ 1448

It can be also applied, as a particular case, to $S = CM$, and it reduces to equation (26.04.07).

26.04.03.03 Kinetic Energy via Koënig Theorem

The kinetic energy of rigid-bodies can be expressed by means of equation (24.11.05)

$$\mathcal{K} = \frac{1}{2}MV_{\mathrm{CM}}^2 + \frac{1}{2}\left(\boldsymbol{\omega}\cdot\underset{\mathrm{CM}}{\mathbb{I}}\cdot\boldsymbol{\omega}\right) \quad . \tag{26.04.07}$$

→ 1443

26.04.04
Inertia Matrix|Tensor

The inertia Matrix|Tensor can be considered as the representation of the linear transformation from the angular velocity to the angular momentum for a rigid body. It is actually a second-order tensor.

For a fixed rigid-body the inertia Matrix|Tensor depends on:

- the origin of the body-fixed Coordinate System;
- the orientation of the body-fixed Coordinate System.

Therefore, the notation $\underset{\mathrm{S}}{\mathbb{I}}$ implies that both the position of the origin of the body-fixed Coordinate System, $\mathbf{x}_{\mathrm{S}}$, and the orientation of the body-fixed Coordinate System, $\mathbf{e}_k^{(\mathrm{S})}$, are given:

$$\mathrm{S} \equiv \left\{\mathbf{x}_{\mathrm{S}}, \mathbf{e}_k^{(\mathrm{S})}\right\} \quad .$$

It is important to emphasize that use of any base fixed in the Inertial Reference Frame would give time-dependent elements of the inertia Matrix|Tensor which make the successive time-derivations complex. On the contrary, if a base rigidly fixed to the rigid-body is used, the elements of the inertia Matrix|Tensor are time-independent and only depending on the mass geometry of the rigid-body.

Introduce a Reference Frame fixed to the rigid-body, with origin $\mathcal{O}'$ and Coordinate System $\mathbf{x}'$. For ease of writing and reading $\mathbf{y} \equiv \mathbf{x}'$ will be used. La matrice di inerzia di un corpo rigido continuo avente densità $\rho[\mathbf{y}]$, rispetto ad un punto $\mathbf{y}_{\mathrm{S}}$ solidale con il corpo rigido, è definito dalla

$$\underset{\mathrm{S}}{\mathbb{I}}_{rs} \equiv \iiint \rho[\mathbf{y}]\left[y^2\delta_{rs} - y_r y_s\right]\,\mathrm{d}V_{\mathbf{y}} \tag{26.04.08}$$

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dove le $\mathbf{y}$ sono le coordinate di un Coordinate System cartesiane, solidale con il corpo rigido, centrato in $\mathbf{y}_{\mathrm{S}}$ e definito dalla terna intrinseca $\mathbf{e}_k^{(\mathrm{S})}$.

As a consequence of its definition, with respect ot a body-fixed Coordinate System, the inertia Matrix|Tensor elements are time-independent and only dependent of the geometry of the masses.

The integral can be considered to be extended either to the rigid-body or the full space, as the mass density is zero outside the body.

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La matrice è reale e simmetrica per costruzione. I termini diagonali sono detti momenti di inerzia, quelli non diagonali prodotti di inerzia.

The following property can be demonstrated by direct calculation:

$$\mathrm{Tr}\, \mathbb{I}_{\mathrm{S}} \equiv \mathbb{I}_{\mathrm{S}}^{ij}\delta_{ij} = \mathbb{I}_{\mathrm{S}}^{ij}\delta_{ij} = \mathbb{I}_{\mathrm{S}}^{ii} = 2\iiint \rho[\mathbf{y}]y^2 \,\mathrm{d}V_{\mathbf{y}} \qquad \text{invariant under rotation of the Coordinate System } \mathbf{y} \quad .$$

26.04.04.01 Tensor Character

This § is referenced at pages:
[1570]

The second-order tensor character can be immediately demonstrated by the fact that it links angular velocity and angular momentum, which are vectors, thanks to the **Quotient Law** § 20.03.07.

The second-order tensor character can be also directly demonstrated from the definition. Consider a second rotated orthonormal Cartesian Coordinates Coordinate System with the same origin as the first one, both in the Reference Frame fixed to the rigid-body. The rotation does not change the volume element in the integral and it changes the components of the position vector with the rotation matrix. Using the fact that the rotation matrix is orthogonal, the **Kronecker tensor** can be re-written factoring the same two rotation matrix elements as for the product of two coordinates. The common factor given by the product of two rotation matrix elements is exactly the one needed for the inertia Matrix|Tensor to behave as a second-order tensor.

26.04.04.02 Additivity of the Geometry of Masses

From its definition, the inertia Matrix|Tensor of a any system is the sum of the inertia Matrix|Tensor of the subsystems composing it.

In fact, the inertia Matrix|Tensor, for a given Coordinate System rigidly fixed to a rigid body, is additive in the mass density; in other words it is an extensive quantity.

26.04.04.03 Principal Axes of Inertia of a Rigid-Body

La matrice di inerzia è reale e simmetrica e può quindi essere sempre diagonalizzata; its eigenvalues are always non-negative and there exists a basis of orthogonal eigenvectors, possibly not a unique one.

Gli assi, che necessariamente esistono ma non sono necessariamente unici, del Coordinate System in cui la matrice è diagonale sono detti assi principali di inerzia.

Equivalently, a principal axis of inertia is any axis of the rigid-body such that whenever the angular velocity is along the direction of the axis then the angular momentum is along the direction of the axis as well.

If z' is a principal axis of inertia (that is one eigenvector of the inertia Matrix|Tensor): $L'_z = I_{z'z'}\omega'_z$ but, in general: $\mathbf{L}_{\mathrm{S}} \nparallel \boldsymbol{\omega}$. In order to have: $\mathbf{L}_{\mathrm{S}} \parallel \boldsymbol{\omega}$ it is necessary and sufficient that either $I_{x'x'} = I_{y'y'} = I_{z'z'}$ or $\boldsymbol{\omega}$ parallel to any principal axis of inertia. In summary, $\mathbf{L}_{\mathrm{S}} \parallel \boldsymbol{\omega}$ if and only if $\boldsymbol{\omega}$ is an eigenvector of the inertia Matrix|Tensor.

For the kinetic energy, after choosing a Coordinate System made of principal axes of inertia, one has:

$$\mathcal{K}' = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) = \frac{1}{2}\left(\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right) \quad .$$

Whenever $I_{x'x'} = I_{y'y'} = I_{z'z'}$ one uses to name the rigid-body spherical top. However, this does not imply spherical symmetry (but the reverse is clearly true); consider for example the inertia Matrix|Tensor of any homogeneous cube with respect to its center.

Whenever there exists a set of axes such that $I_{x'x'} = I_{y'y'} \neq I_{z'z'}$ one uses to name the rigid-body symmetrical top. In this case, clearly, all other axes perpendicular to z' are principal axes with the same inertia moment as $I_{x'x'} = I_{y'y'}$.

Whenever $I_{x'x'} \neq I_{y'y'} \neq I_{z'z'}$ one uses to name the rigid-body asymmetrical top.

After choosing a Coordinate System made of principal axes of inertia, the inertia Matrix|Tensor becomes diagonal. From the definition of the elements of the inertia Matrix|Tensor it immediately follows that:

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- $I_{x'x'} \geq 0$,
- $I_{y'y'} \geq 0$,
- $I_{z'z'} \geq 0$;
- $I_{x'x'} + I_{y'y'} \geq I_{z'z'}$,
- $I_{y'y'} + I_{z'z'} \geq I_{x'x'}$,
- $I_{z'z'} + I_{x'x'} \geq I_{y'y'}$.

26.04.04.04 Variazione Della Matrice Di Inerzia Al Variare Dell'origine

Dati due Coordinate System cartesiani ortogonali solidali con un corpo rigido, con assi tra loro paralleli e con origine uno nel centro di massa e l'altro nel punto S solidale al copro rigido, la relazione tra i due tensori di inerzia relativi ai due punti è

$$\boxed{\mathbb{I}^{\cdot\cdot}_{S\,ij} = \mathbb{I}^{\cdot\cdot}_{CM\,ij} + M\left[a^2\delta^{\cdot\cdot}_{ij} - a_i a_j\right]} \tag{26.04.09}$$

dove $\pm\mathbf{a}$ è il vettore posizione del centro di massa rispetto ad S, e l'espressione non dipende dal segno di $\pm\mathbf{a}$.

It is important to stress that the theorem is only valid whenever one of the origins is the Center-Of-Mass.

$$\mathbf{y} \equiv P - O' = (P - CM) + \left(CM - O'\right) \equiv \mathbf{y}' + \mathbf{a} \quad ,$$

$$\mathbb{I}^{\cdot\cdot}_{S\,rs} \equiv \iiint \rho[\mathbf{y}]\left[\delta^{\cdot\cdot}_{rs}|\mathbf{y}|^2 - y_r y_s\right] \mathrm{d}V_{\mathbf{y}} = \iiint \rho[\mathbf{y}]\left[\delta^{\cdot\cdot}_{rs} y_k y_k - y_r y_s\right] \mathrm{d}V_{\mathbf{y}} = \quad ,$$

$$\int \rho'[\mathbf{y}']\left[\delta^{\cdot\cdot}_{rs}\left(y'_k + a_k\right)\left(y'_k + a_k\right) - \left(y'_r + a_r\right)\left(y'_s + a_s\right)\right] \mathrm{d}\mathbf{y}' = \quad ,$$

$$\int \rho'[\mathbf{y}']\left[\delta^{\cdot\cdot}_{rs}|\mathbf{y}'|^2 - y'_r y'_s\right] \mathrm{d}\mathbf{y}' + M\left(\delta^{\cdot\cdot}_{rs}|\mathbf{a}|^2 - a_r a_s\right) \quad ,$$

$$\rho'[\mathbf{y}'] = \rho[\mathbf{y}] \qquad \mathrm{d}\mathbf{y}' = \mathrm{d}\mathbf{y} \quad ,$$

$$\iiint \rho'[\mathbf{y}'] y'_p a_q \, \mathrm{d}V'_{\mathbf{y}} = 0 \qquad \text{as } \mathbf{y}' \text{ is the position relative to the Center-Of-Mass} \quad .$$

Teorema Degli Assi Paralleli (Di Huygens-Steiner)

Come caso particolare si ha il teorema di Huygens-Steiner. Il momento di inerzia di un corpo rigido di massa M rispetto ad un asse qualunque, I, è legato al momento di inerzia rispetto all'asse parallelo al precedente e passante per il centro di massa del corpo rigido, I_{CM}, dalla relazione

$$\boxed{I = I_{CM} + Md^2} \quad ,$$

dove d è la distanza tra i due assi.

26.04.04.05 Teorema Degli Assi Ortogonali

Dato un sistema rigido piano (una lamina bidimensionale omogenea) il momento di inerzia rispetto ad un asse perpendicolare alla lamina, z', e passante per il punto S della lamina è uguale alla somma dei momenti di inerzia rispetto a due assi perpendicolari tra loro, x' e y', giacenti sul piano e passanti per S, può essere espresso come

$$\boxed{I_{z'} = I_{x'} + I_{y'}} \quad .$$

26.04.04.06 Inertia Moment With Respect to an Arbitrary Axis

Il momento di inerzia rispetto ad un asse qualunque passante per S e definito dal unit vector $\hat{\mathbf{n}}$, vale:

$$\boxed{\mathbb{I}_{S}[\hat{\mathbf{n}}] = n_i n_j \mathbb{I}^{\cdot\cdot}_{S\,ij}} \quad , \tag{26.04.10}$$

as can be shown by direct calculation from the definition, as the expression gives the distance from the axis of any point.

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26.04.04.07 Raggio Di Girazione

Per un sistema di massa M che ruota attorno ad un asse fisso con momento di inerzia I, si definisce il raggio di girazione

$$r_g \equiv \sqrt{\frac{I}{M}} \quad .$$

Rappresenta la distanza dall'asse a cui occorrerebbe concentrare la massa del sistema per avere momento di inerzia I.

26.04.04.08 Properties of the Inertia Matrix|Tensor Deriving From Symmetries of the Mass Geometry

1. If the $x'y'$ plane is a symmetry plane, then the mass distribution is invariant for: $z' \to -z'$ and therefore $I_{z'x'} = I_{y'z'} = 0$. Therefore, by definition, z' is a principal axis.
2. If the z' axis is a symmetry axis, then the mass distribution is invariant for: $x' \to -x'$ and $y' \to -y'$ (at the same time). and therefore $I_{z'x'} = I_{y'z'} = 0$ but, in general, $I_{x'y'} \neq 0$. Therefore, by definition, z' is a principal axis.
3. If the z' axis is a revolution axis, then the mass distribution is invariant for any rotation around the z' axis, $\theta \to \theta + \delta$. Therefore $I_{x'x'} = I_{y'y'}$, in addition to $I_{z'x'} = I_{y'z'} = 0$, as the z' axis is also a symmetry axis. Moreover, as $z'x'$ and $y'z'$ planes are also symmetry planes, it follows that: $I_{z'x'} = I_{y'z'} = I_{x'y'} = 0$. Therefore, by definition, z' is a principal axis and also x' and y' are principal axes of inertia (and the inertia Matrix|Tensor is diagonal). Any axis in the $x'y'$ plane is a principal axis.
4. If the shape and mass distribution of a rigid-body is symmetric with respect to a plane, its Center-Of-Mass and two of its principal axes lie in that plane. The third principal axis is perpendicular to it.
5. When the body possesses axial symmetry, that is if it is symmetric under rotations about a certain axis, then the Center-Of-Mass lies on the symmetry axis and that axis is a principal axis. The remaining two are perpendicular to it. The corresponding moments of inertia being degenerate, they must be chosen by hand in the plane through the Center-Of-Mass that is perpendicular to the symmetry axis.

26.04.05 Third Cardinal Equation for a Rigid-Body

26.04.05.01 Work and Power on a Rigid-Rigid-Body

Let us evaluate the power of an arbitrary system of Force|Torque, $\mathbf{f}[\mathbf{x}]$ and $\boldsymbol{\gamma}[\mathbf{x}]$, modeling the system as a continuous rigid-body.

Let S be any arbitrary point rigidly fixed to the body. The total power can be expressed in terms the total force, $\mathbf{F}$, the total torque with respect to the pole S, $\boldsymbol{\Gamma}_S$, angular velocity $\boldsymbol{\omega}$, and velocity $\mathbf{v}_S$ as:

$$\begin{aligned}\Pi \equiv \frac{\mathrm{d}W}{\mathrm{d}t} &= \int \mathbf{v}\cdot \mathrm{d}\mathbf{f}[\mathbf{x}] + \int \boldsymbol{\omega}\cdot \mathrm{d}\boldsymbol{\gamma}[\mathbf{x}] \quad , \\ &= \int (\mathbf{v}_S + \boldsymbol{\omega}\wedge(\mathbf{x}-\mathbf{x}_S))\cdot \mathrm{d}\mathbf{f} + \int \boldsymbol{\omega}\cdot \mathrm{d}\boldsymbol{\gamma} = \int \mathbf{v}_S\cdot \mathrm{d}\mathbf{f} + \int \boldsymbol{\omega}\cdot(\,(\mathbf{x}-\mathbf{x}_S)\wedge \mathrm{d}\mathbf{f}\,) + \int \boldsymbol{\omega}\cdot \mathrm{d}\boldsymbol{\gamma} \quad , \\ &= \mathbf{v}_S\cdot\mathbf{F} + \boldsymbol{\omega}\cdot\boldsymbol{\Gamma}_S \quad .\end{aligned}$$

Therefore:

$$\Pi \equiv \frac{\mathrm{d}W}{\mathrm{d}t} = \mathbf{v}_S\cdot\mathbf{F} + \boldsymbol{\omega}\cdot\boldsymbol{\Gamma}_S \quad . \tag{26.04.11}$$

It immediately follows that the total power of internal forces is always zero, because the total force, $\mathbf{F}$, and the total torque with respect to any pole S, $\boldsymbol{\Gamma}_S$, of a system of internal Force|Torque is zero:

$$\mathbf{F}^{(\mathrm{I})} = \boldsymbol{\Gamma}_S^{(\mathrm{I})} = 0 \implies \Pi^{(\mathrm{I})} = 0 \quad .$$

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Therefore, for any body-fixed point S:

$$\Pi \equiv \frac{\mathrm{d}W}{\mathrm{d}t} = \mathbf{v}_S \cdot \mathbf{F}^{(\mathrm{E})} + \boldsymbol{\omega} \cdot \boldsymbol{\Gamma}_S^{(\mathrm{E})} \quad . \tag{26.04.12}$$

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Equation (26.04.12) shows the important results that, for a rigid-body, the work/power only depend on the external Force|Torque and, actually, only on total Force|Torque from external agents.

The classical rigid-body is a special case of the above with $\boldsymbol{\gamma}[\mathrm{x}] = 0$.

26.04.05.02 Work and Power on a Rigid-Nonrigid-Body

The Einsten-deHaas effect involves the magnetic moment vector of the molecules changing direction.

26.04.05.03 Third Cardinal Equation

Equation (26.04.12), when used with the Center-Of-Mass, $S = CM$, gives:

$$\Pi = \mathbf{V}_{\mathrm{CM}} \cdot \mathbf{F}^{(\mathrm{E})} + \boldsymbol{\omega} \cdot \boldsymbol{\Gamma}_{\mathrm{CM}}^{(\mathrm{E})} \quad . \tag{26.04.13}$$

The first and second **Cardinal Equations** , and the result discussed in section § 26.07.01, imply:

$$\mathbf{V}_{\mathrm{CM}} \cdot \mathbf{F}^{(\mathrm{E})} + \boldsymbol{\omega} \cdot \boldsymbol{\Gamma}_{\mathrm{CM}}^{(\mathrm{E})} = M\, \mathbf{V}_{\mathrm{CM}} \cdot \frac{\mathrm{d}\mathbf{V}_{\mathrm{CM}}}{\mathrm{d}t} + \boldsymbol{\omega} \cdot \frac{\mathrm{d}\mathbf{L}'}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{M\mathbf{V}_{\mathrm{CM}}^2}{2}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\left(\boldsymbol{\omega} \cdot \mathbb{I}_{\mathrm{CM}} \cdot \boldsymbol{\omega}\right)}{2}\right) = \frac{\mathrm{d}\mathcal{K}_{\mathrm{CM}}}{\mathrm{d}t} + \frac{\mathrm{d}\mathcal{K}'}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} \equiv \Pi \quad , \tag{26.04.14}$$

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where the last equality is actually the third **Cardinal Equation** for a rigid body, deduced by only using the first and second equations, as well as the peculiar properties of the rigid-body.

Note the importance of the results discussed in section § 26.07.01 for the above derivation, because the inertia Matrix|Tensor is time-independent in the body-fixed Reference Frame while the time-derivative in the **Cardinal Equations** are in the absolute Reference Frame.

Clearly, in the case of a rigid-body, the third **Cardinal Equation** is a consequence of the first and second **Cardinal Equations** . In fact, equation (26.04.14), the third **Cardinal Equation** , has been derived only from the first and second **Cardinal Equations** . Therefore, for a rigid-body, the third **Cardinal Equation** is a consequence of the first and second **Cardinal Equations** , not an independent one.

The derivation of the latter equation also shows that the total force changes the kinetic energy of the Center-Of-Mass, and only that one, while the total torque with respect to the Center-Of-Mass changes the kinetic energy relative to the Center-Of-Mass, and only that one. In fact, the above derivation shows that the two addenda in the third **Cardinal Equation** give separately the change of the kinetic energy of the Center-Of-Mass and relative to the center of mass:

$$\frac{\mathrm{d}\mathcal{K}_{\mathrm{CM}}}{\mathrm{d}t} = \mathbf{V}_{\mathrm{CM}} \cdot \mathbf{F}^{(\mathrm{E})} \qquad \frac{\mathrm{d}\mathcal{K}'}{\mathrm{d}t} = \boldsymbol{\omega} \cdot \boldsymbol{\Gamma}_{\mathrm{CM}}^{(\mathrm{E})} \quad . \tag{26.04.15}$$

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26.04.05.04 Free Rigid-Body

In case the rigid-body is free, $\mathbf{0} = \mathbf{F}^{(\mathrm{E})} = \boldsymbol{\Gamma}_{\mathrm{S}}^{(\mathrm{E})}$:

- the Center-Of-Mass moves with constant velocity, and therefore there is a Reference Frame such that the Center-Of-Mass is fixed;
- in any Reference Frame such that the Center-Of-Mass is fixed, the rigid-body rotates around the Center-Of-Mass.

26.04.05.05 Pole at the Center-Of-Mass of the Rigid-Body

In case the resultant external torque with respect to the Center-Of-Mass is zero, $\boldsymbol{\Gamma}_{\mathrm{CM}} = \mathbf{0}$, equation (26.04.15) implies:

$$\boldsymbol{\Gamma}_{\mathrm{CM}} = \mathbf{0} \implies \frac{\mathrm{d}\mathcal{K}'}{\mathrm{d}t} = 0 \implies \mathcal{K}' = \text{constant} \quad , \tag{26.04.16}$$

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that is, the kinetic energy about the Center-Of-Mass is constant.

26.04.05.06 Pole at a Fixed and Body-Fixed Point of the Rigid-Body

In case the rigid-body has a fixed and body-fixed point, S, and the resultant external torque with respect to S is zero, $\boldsymbol{\Gamma}_{\mathrm{S}}^{(\mathrm{E})} = \mathbf{0}$, equations (26.04.06), (26.04.12), (26.04.15) imply, as $\mathbf{v}_{\mathrm{S}} = \mathbf{v}_{\mathrm{S}}{}' = 0$:

$$\mathbf{0} = \boldsymbol{\omega} \cdot \boldsymbol{\Gamma}_{\mathrm{S}}^{(\mathrm{E})} = \boldsymbol{\omega} \cdot \frac{\mathrm{d}\mathbf{L}_{\mathrm{S}}}{\mathrm{d}t} = \Pi = \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\omega} \cdot \mathbb{I}_{\mathrm{S}} \cdot \boldsymbol{\omega}\right) = \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} \implies \mathcal{K} = \text{constant} \quad , \tag{26.04.17}$$

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that is, the total kinetic energy is constant.

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26.05 Statics of Rigid-Bodies

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It is important to note that the equilibrium equations are three in number, for plane problems. This means that, in any problem concerning the plane statics of a rigid-body, we can find three unknowns and no more; actually three maximum, and less than three in case the forces are unidimensional. If there are more than three unknowns, the problem is a statically indeterminate problem, which means that it cannot be solved by means of the conditions of equilibrium alone. Similarly, equilibrium equations are six in number, for space problems and one can get at most six unknowns.
Apparently our mathematical methods have failed us; they have not provided us with the answer to a question of physical interest. The fault actually lies in the selection of a mathematical model. rigid-bodies do not exist in nature, and this problem is one where the use of a rigid-body as a mathematical model is not justified. We should take an elastic bar as a model instead

The basic principle of static equilibrium states that a system is in equilibrium if and only if all its parts are in equilibrium as well; it obviously applies to rigid-bodies as well.

For general systems,

$$\mathbf{F}^{(\mathrm{E})} = 0 \qquad \boldsymbol{\Gamma}_{\mathrm{S}}^{(\mathrm{E})} = 0 \quad ,$$

is a necessary but non sufficient condition for equilibrium. For rigid-bodies, it is a sufficient condition for equilibrium.

A rigid-body has at most six degrees of freedom, for instance, the Center-Of-Mass position and the three Euler angles; if it is constrained it may have less than six.

A rigid-body is in static equilibrium if and only if, in some Inertial Reference Frame , the velocity of all the points of the rigid-body are zero.

Whenever at least three not-aligned points of a rigid-body have zero velocity, the rigid-body is in equilibrium. In fact, if three not-aligned points of a rigid-body have zero velocity, then all the points have zero velocity. Moreover, the angular velocity, thanks to equation (26.03.01), must be aligned with the three sides of the triangle defined by the three not-aligned points at rest; therefore the angular velocity is zero.

A rigid-body is in static equilibrium if and only if, at some time: the velocity of any one of its points, $\mathbf{v}_{\mathrm{S}}$, is zero, in some Inertial Reference Frame ; the angular velocity, $\boldsymbol{\omega}$, is zero; the resultant of external forces, $\mathbf{F}^{(\mathrm{E})}$, is zero; the resultant external torque with respect to the pole S, $\boldsymbol{\Gamma}_{\mathrm{S}}^{(\mathrm{E})}$, is zero. In fact, in this case,

$$\begin{aligned}
&\mathbf{v}_{\mathrm{S}} = 0 \quad \boldsymbol{\omega} = 0 \quad \longrightarrow \quad \mathbf{v}_{\mathrm{P}} = 0 && \text{all points of the rigid-body have zero velocity;} \quad ,\\
&\qquad\qquad \longrightarrow \quad \mathbf{V}_{\mathrm{CM}} = 0 && \text{stays zero, due to lack of unbalanced external forces;} \quad ,\\
&\mathbf{F}^{(\mathrm{E})} = 0 \quad \boldsymbol{\Gamma}_{\mathrm{S}}^{(\mathrm{E})} = 0 \quad \longrightarrow \quad \boldsymbol{\Gamma}_{\mathrm{CM}}^{(\mathrm{E})} = 0 && \text{law of transformation of momentum;} \quad ,\\
&\boldsymbol{\Gamma}_{\mathrm{CM}}^{(\mathrm{E})} = 0 \quad \boldsymbol{\omega} = 0 \quad \longrightarrow \quad \mathbf{L}' = 0 && \text{stays zero, due lack of unbalanced external Force|Torque} \quad .
\end{aligned}$$

For 3D problems, the first and second **Cardinal Equations** provide, in general, six scalar equations, and therefore equating to zero the sum of external forces and torques the necessary equilibrium condition becomes, for rigid-bodies, also a sufficient one. Therefore no more than six unknowns can be determined by requiring static equilibrium for a space rigid-body.

For 2D problems, the first and second **Cardinal Equations** provide, in general, three scalar equations, therefore no more than three unknowns can be determined by requiring static equilibrium for a planar rigid-body problem.

Law Problems

For 1D problems, the first and second **Cardinal Equations** provide, in general, one scalar equation, therefore no more than one unknown can be determined by requiring static equilibrium for a linear rigid-body problem.

Obviously, for rigid-bodies, the third **Cardinal Equation** does not change the conclusions above as it is dependent on the first two.

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26.06 Euler Equations

Consider the second **Cardinal Equation** with pole at the Center-Of-Mass of a rigid-body:

$$\boxed{\boldsymbol{\Gamma}^{(\mathrm{E})}_{\mathrm{CM}} = \frac{\mathrm{d}\mathbf{L}'}{\mathrm{d}t}} \quad . \tag{26.06.01}$$

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It is convenient to project equation (26.06.01) onto an orthonormal Coordinate System rigidly fixed with the rigid-body, in order to have time-independent elements of the inertia Matrix|Tensor. The inertia Matrix|Tensor elements thus represent just mass-geometrical properties of the rigid-body, calculated once for all. Moreover, among all the possible Coordinate System, it is useful to use a Coordinate System made of principal axes of inertia rigidly fixed to the rigid-body, unless stated otherwise. In other words, the **Cardinal Equations** are projected onto the moving principal axes.

One can then project the vector equation (26.06.01),

$$\boxed{\boldsymbol{\Gamma}^{(\mathrm{E})}_{\mathrm{CM}} = \frac{\mathrm{d}\mathbf{L}'}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\mathrm{A}}\mathbf{L}' = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\mathrm{R}}\mathbf{L}' + \boldsymbol{\omega}\wedge\mathbf{L}'} \quad , \tag{26.06.02}$$

→ 1451

onto the principal axes of inertia of the rigid-body.

Note that vector equation (26.06.01) is written in an Inertial Reference Frame , and no inertial forces appear. The second **Cardinal Equation** written in an Inertial Reference Frame is then projected onto the principal axes of inertia of the body, which is a orthonormal Coordinate System moving in the inertial Laboratory Reference Frame, which causes the equation to be transformed into equation (26.06.02). The orthonormal Coordinate System rigidly fixed to the rigid-body defines a non-Inertial Reference Frame , but the second cardinal equation is not written in this non-Inertial Reference Frame .

Therefore: Euler equations use a moving orthonormal Coordinate System, fixed to the rigid-body, to describe the physics in the inertial Laboratory frame.

The following notations will be used in this section:

$$I \equiv \mathbb{I}_{\mathrm{CM}} \qquad I_k \equiv \mathbb{I}_{\mathrm{CM}}\dot{k}\dot{k} \quad .$$

The relation between angular velocity and angular momentum is:

$$\dot{\mathbf{e}'}_i \cdot \mathbf{L}' = \mathbb{I}_{\mathrm{CM}}\dot{i}\dot{j}\dot{\omega'}_j \quad .$$

The Euler equations follow, in terms of the components with respect to a Coordinate System fixed on the rigid-body:

$$\mathbb{I}_{\mathrm{CM}}{}_{11}\dot{\omega}'_1 + \omega'_2\omega'_3\left(\mathbb{I}_{\mathrm{CM}}{}_{33} - \mathbb{I}_{\mathrm{CM}}{}_{22}\right) = \mathbf{e}'_1 \cdot \boldsymbol{\Gamma}^{(\mathrm{E})}_{\mathrm{CM}} \tag{26.06.03}$$

→ 1453

$$\mathbb{I}_{\mathrm{CM}}{}_{22}\dot{\omega}'_2 + \omega'_3\omega'_1\left(\mathbb{I}_{\mathrm{CM}}{}_{11} - \mathbb{I}_{\mathrm{CM}}{}_{33}\right) = \mathbf{e}'_2 \cdot \boldsymbol{\Gamma}^{(\mathrm{E})}_{\mathrm{CM}} \tag{26.06.04}$$

$$\mathbb{I}_{\mathrm{CM}}{}_{33}\dot{\omega}'_3 + \omega'_1\omega'_2\left(\mathbb{I}_{\mathrm{CM}}{}_{22} - \mathbb{I}_{\mathrm{CM}}{}_{11}\right) = \mathbf{e}'_3 \cdot \boldsymbol{\Gamma}^{(\mathrm{E})}_{\mathrm{CM}} \quad .$$

Euler equations are a set of three non-linear equations, whose solution, in general, is rather difficult.

Clearly, the same procedure of using a basis fixed to the rigid-body can be applied to the first **Cardinal Equation** .

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26.06.01
Free Tops

A rigid, rotating body is usually called a top. A top is called symmetric top if two of its principal moments of inertia are equal. If the three principal moments of inertia are all equal the top is called spherical top. If the three principal moments of inertia are all different the top is called asymmetrical top.

Two different cases arise which can be treated, formally, in the same way, because in both cases the second **Cardinal Equation** reduces to the short-form:

$$\boldsymbol{\Gamma}_{\mathrm{P}}^{(\mathrm{E})} = \frac{\mathrm{d}\mathbf{L}_{\mathrm{P}}}{\mathrm{d}t} \quad ,$$

with $P \equiv CM$, the Center-Of-Mass, or $P \equiv S$, any fixed and body-fixed point.

One can distinguish between the free top and the heavy top:

- the free top assumes that no external Force|Torque act on the body, so that the torque with respect to some fixed point vanishes;
- the heavy top is acted on by external Force|Torque, for example gravity, centrifugal forces, friction forces, ...

The use of any other pole must take into account the second **Cardinal Equation** for a generic pole.

26.06.01.01 Zero Resultant External Force|Torque With Respect to the Center-Of-Mass

In this case:

$$\boxed{\mathbf{L}' = \underset{\mathrm{CM}}{\mathbb{I}} \cdot \boldsymbol{\omega} \qquad \mathcal{K}' = \frac{1}{2}\left(\boldsymbol{\omega} \cdot \underset{\mathrm{CM}}{\mathbb{I}} \cdot \boldsymbol{\omega}\right)} \quad .$$

Moreover, from the second **Cardinal Equations** and from the third **Cardinal Equation** in the form of equation (26.04.16), one has, whenever $\boldsymbol{\Gamma}_{\mathrm{CM}}^{(\mathrm{E})} = 0$; there are two constants of motion of Euler equations:

$$\boxed{\mathbf{L}' = \underset{\mathrm{CM}}{\mathbb{I}} \cdot \boldsymbol{\omega} = \text{constant} \qquad \mathcal{K}' = \frac{1}{2}\left(\boldsymbol{\omega} \cdot \underset{\mathrm{CM}}{\mathbb{I}} \cdot \boldsymbol{\omega}\right) = \text{constant}} \quad .$$

Note that $\mathbf{L}'$ is constant in the Laboratory Frame, not necessarily in the rigid-body frame. On the other hand, the scalar $\mathcal{K}'$ is the same in both Reference Frames, as well as $\mathbf{L}' \cdot \mathbf{L}'$. In the body-fixed Reference Frame the angular momentum vector, in general, precesses.

The integrals of motions are not four in number, because for a rigid body the third **Cardinal Equation** is not independent from the second one.

Note that the total kinetic energy might be non constant, as $\mathbf{F}^{(\mathrm{E})}$, in general, might be different from zero.

Example: a rigid-body falling in free rotation in a uniform gravitational field.

The Center-Of-Mass system might be a non-inertial one but this is not relevant.

26.06.01.02 Zero Resultant External Force|Torque With Respect to a Fixed and Body-Fixed Point

In this case, let S be a fixed and body-fixed point, $\mathbf{v}_{\mathrm{S}} = \mathbf{v}_{\mathrm{S}}{}' = 0$, and:

$$\boxed{\mathbf{L}_{\mathrm{S}} = \underset{\mathrm{S}}{\mathbb{I}} \cdot \boldsymbol{\omega} \qquad \mathcal{K} = \frac{1}{2}\left(\boldsymbol{\omega} \cdot \underset{\mathrm{S}}{\mathbb{I}} \cdot \boldsymbol{\omega}\right)} \quad .$$

Moreover, from the second **Cardinal Equations** and from the third **Cardinal Equation** in the form of equation (26.04.17), one has, whenever $\boldsymbol{\Gamma}_{\mathrm{S}}^{(\mathrm{E})} = 0$:

$$\boxed{\mathbf{L}_{\mathrm{S}} = \underset{\mathrm{S}}{\mathbb{I}} \cdot \boldsymbol{\omega} = \text{constant} \qquad \mathcal{K} = \frac{1}{2}\left(\boldsymbol{\omega} \cdot \underset{\mathrm{S}}{\mathbb{I}} \cdot \boldsymbol{\omega}\right) = \text{constant}} \quad ,$$

that is there are two constants of motion of Euler equations.

Note that $\mathbf{L}_{\mathrm{S}}$ is constant in the Laboratory Frame, not necessarily in the rigid-body frame. On the other hand, the scalar $\mathcal{K}$ is the same in both Reference Frames, as well as $\mathbf{L}_{\mathrm{S}} \cdot \mathbf{L}_{\mathrm{S}}$. In the body-fixed Reference Frame the angular momentum vector, in general, precesses.

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The integrals of motions are not four in number, because for a rigid body the third **Cardinal Equation** is not independent from the second one.

Note that it might be $\mathbf{F}^{(\mathrm{E})} \neq 0$, in general, because the point might be constrained at S and $\mathcal{K}_S$ is the total kinetic energy, a constant.

26.06.02
Euler Equations: Permanent Rotations

In case of a free body, $\boldsymbol{\Gamma}^{(\mathrm{E})}_{\mathrm{CM}} = 0$, Euler equations (equations (26.06.03), (26.06.03), (26.06.03)) show that, a motion with constant angular velocity, a permanent rotation, is possible if and only if, $\boldsymbol{\omega}$ it is parallel to one of the principal axes of inertia.

In fact, under the previous hypothesis, $\dot{\boldsymbol{\omega}} = 0$, implies:

- for $I_1 \neq I_2 \neq I_3$, that at least two of the components of $\boldsymbol{\omega}$ must be zero;
- for $I_1 = I_2 \neq I_3$, that either $\omega_3 = 0$, that is ω lies in the $1-2$ plane, where all axes are principal axes, or $\omega_1 = \omega_2 = 0$, that is ω is parallel to the 3 axis;
- for $I_1 = I_2 = I_3$, any constant $\boldsymbol{\omega}$ can be a solution, but also any axis is a principal axis.

An alternative way to demonstrate the same result is to observe that the constancy of $\boldsymbol{\omega}$, in the Laboratory Frame, implies (from Poisson relations for the angular velocity)

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{A}} \boldsymbol{\omega} = \mathbf{0} \Longrightarrow \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{R}} \boldsymbol{\omega} = \mathbf{0} \Longrightarrow \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{R}} \mathbf{L}' = \mathbf{0} \quad ,$$

because the time derivative of the angular velocity vector is the same in both the laboratory and rigid-body Reference Frame, and therefore

$$\mathbf{0} = \boldsymbol{\Gamma}^{(\mathrm{E})}_{\mathrm{CM}} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{A}} \mathbf{L}' = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{R}} \mathbf{L}' + \boldsymbol{\omega} \wedge \mathbf{L}' = \boldsymbol{\omega} \wedge \mathbf{L}' \Longrightarrow \boldsymbol{\omega} \parallel \mathbf{L}' \quad .$$

26.06.03
Euler Equations: The Free Spherical Top

One first case when Euler equations are easily solved is the case of a free spherical top

$$\boxed{I_1 = I_2 = I_3 \equiv I} \quad .$$

In this case Euler equations are trivially solved:

$$\boldsymbol{\Gamma}^{(\mathrm{E})}_{\mathrm{CM}} = \frac{\mathrm{d}\mathbf{L}'}{\mathrm{d}t} = I \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \quad ,$$

$$\boldsymbol{\Gamma}^{(\mathrm{E})}_{\mathrm{CM}} = 0 \Longrightarrow \boldsymbol{\omega} = \text{constant} \quad .$$

26.06.04
Euler Equations: The Free Symmetrical Top

One second case when Euler equations are easily solved is the case of the free symmetrical top. A symmetrical top is a rigid-body with two principal moment of inertia equal and different from the third one:

$$\boxed{I_1 = I_2 \equiv I \neq I_3} \quad .$$

Note that a sufficient condition for a body to be a symmetrical top is that the body has an axis of revolution, but this condition is by no means necessary (e.g. one homogeneous cube).

Read § 26.07.16 for the detailed solution.

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26.06.05
Euler Equations: The Free Asymmetrical Top

Whenever

$$\boxed{I_1 \neq I_2 \neq I_3} \quad ,$$

the system is called an asymmetrical, or triaxial top.

A general solution, in the body-fixed Reference Frame, in this case is complex but can be found by quadrature by making use of the conservation of kinetic energy and module of the angular momentum. In fact, by using the two constants of motion, the three Euler equations can be reduced to one equation, which can be integrated via elliptic integrals.

From the solution in the body-fixed Reference Frame, the solution in the Laboratory Frame can be also determined.

26.06.06
Euler Equations: Deficiencies

The solutions of Euler equations yield the time variation of the angular velocity as seen from the frame fixed with the rigid-body. The position of the body is unknown: Euler equations do not tell us where the body is at time t.

A second but related deficiency appears when the applied moment of external forces is not zero: in fact its components will not generally be known, since the orientation of the body is unknown. In such a case, one cannot even start the solution.

These difficulties disappear when the motion of the body is given, and we wish to find the moments that are being applied to the body to permit that motion: in such a case, Euler equations give the answer immediately.

26.06.07
Stability of the Motion of a Free Rigid-Body Around a Principal Axis

This § is referenced at pages:
[1405]

Any rigid-body with three different principal moments of inertia is stable to small perturbations when rotating about the principal axes with the largest and smallest moments; it is unstable when rotating about the axis with the intermediate moment. If two of the principal moments are equal the body is only stable to small perturbations when rotating about the principal axis whose moment is different from the other two.

WORKED SOLUTION

Assume $I_z > I_y > I_x$ and let $\{\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3\}$ be the Reference Frame rigidly fixed to the rigid-body.
Start with the body rotating around the z axis: $\omega = \omega_0 \widehat{\mathbf{e}}_3$. The system is given a small perturbation such that $\omega = \omega_{x0}\widehat{\mathbf{e}}_1 + \omega_{y0}\widehat{\mathbf{e}}_2 + \omega_0\widehat{\mathbf{e}}_3$ with $\omega_{y0} \ll \omega_0$ and $\omega_{x0} \ll \omega_0$.
Replace the perturbed ω into the free Euler equations and retain only first-order terms. It follows that ω_0 stays constant while ω_{x0} and ω_{y0} satisfy harmonic motions equations. Therefore the motion is stable.
The same conclusion is obtained starting from a rotation around the x axis.
On the other hand, when starting from a rotation around the y axis, one find an exponentially growing solution. Therefore the motion is unstable.
A similar reasoning leads to the conclusion when two of the principal moments are equal: the body is only stable to small perturbations when rotating about the principal axis whose moment is different from the other two.
Clearly, when all three principal moments are equal, the motion is stable as every axis is a possible rotational axis with constant angular velocity.

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26.07 Examples and Physical Applications

26.07.01 Absolute and Relative Derivative

This § is referenced at pages:
[1447, 1447]

Let us show that:

$$\boldsymbol{\omega}\cdot\frac{\mathrm{d}\mathbf{L}'}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\left(\boldsymbol{\omega}\cdot\mathbb{I}_{\mathrm{CM}}\cdot\boldsymbol{\omega}\right)}{2}\right)\quad .$$

The left-hand side has the same value whether one uses the absolute of relative time-derivative, because the scalar product with $\boldsymbol{\omega}$ sets to zero the $\boldsymbol{\omega}\wedge\mathbf{L}'$ term in the Poisson relation. Also, for both terms either the absolute or the relative time-derivative can be used, because they are scalars.

Therefore the relative time-derivative and the symmetry of $\mathbb{I}_{\mathrm{CM}}$ imply:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\left(\boldsymbol{\omega}\cdot\mathbb{I}_{\mathrm{CM}}\cdot\boldsymbol{\omega}\right)}{2}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega'_k\,\mathbb{I}_{\mathrm{CM}kj}\,\omega'_j}{2}\right)=\frac{1}{2}\left(\dot{\omega}'_k\,\mathbb{I}_{\mathrm{CM}kj}\,\omega'_j+\omega'_k\,\mathbb{I}_{\mathrm{CM}kj}\,\dot{\omega}'_j\right)=\omega'_k\,\mathbb{I}_{\mathrm{CM}kj}\,\dot{\omega}'_j=\omega'_k\,\frac{\mathrm{d}L'_k}{\mathrm{d}t}\quad .$$

26.07.02 Rigid-Bodies With Mass

Even if a rigid-body is ideally rigid, Force|Torque must be present internally to keep the body rigid. Ideally, these Force|Torque can have arbitrary strength, but in real life, when modeling any body as a rigid-body one should take into account that internal Force|Torque, above certain limits, cannot keep the body rigid anymore, eventually leading to breaking.

Determine the distribution of internal Force|Torque for a rigid thin cylindrical bar of mass M and length L, under its own weight, in the following cases:

- bar resting horizontally on two vertical supports, sustaining the bar at its two extremes;
- as above with an additional support at the middle of the bar;
- as above but with the support at a generic position between the two extremes;
- the bar as an horizontal cantilevered beam (read § 27.07.11);
- as above, with a mass m hanging at the free end (read § 27.07.11);
- as above but with the mass hanging at an arbitrary point;
- bar statically hanging vertically, sustained at one end (read § 27.07.09);
- bar hanging vertically, sustained at one end, and oscillating as a physical pendulum;
- the falling chimney (read § 26.07.09).

At any geometrical section of the bar, perpendicular to the axis of the bar, normal and tangential forces, as well as torsion and flexing torques are acting (read § 24.18.02).

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26.07.03
Momento D'inerzia Di Un'asta Sottile

Determinare, usando solo le proprietà generali dei momenti di inerzia, il momento d'inerzia di un'asta sottile di massa M e lunghezza L rispetto ad un asse perpendicolare all'asta passante per il suo centro.

WORKED SOLUTION

Sia I_c il momento di inerzia rispetto ad un asse perpendicolare all'asta e passante per il centro. Sia I_e il momento di inerzia rispetto ad un asse perpendicolare all'asta e passante per un estremo. Si ha

$$I_c(M,L) = 2I_e(\frac{M}{2},\frac{L}{2}) = \frac{1}{4}I_e(M,L)$$

per l'additività dei momenti di inerzia rispetto ad un asse e per la proporzionalità del momento di inerzia alla massa e al quadrato delle dimensioni. Inoltre dal teorema di Steiner

$$I_e(M,L) = I_c(M,L) + \frac{ML^2}{4}$$

Mettendo a sistema le due equazioni si trova

$$I_c(M,L) = \frac{ML^2}{12} \qquad I_e(M,L) = \frac{ML^2}{3}$$

26.07.04
Properties of the Inertia Matrix|Tensor

Consider a diagonal inertia Matrix|Tensor; show that at most one of the diagonal elements can be zero. Describe such a system.

26.07.05
Oggetto in Rotazione Attorno Ad Un Asse

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26.07.06
Momento Angolare Di Un Manubrio Ruotante

This § is referenced at pages:
[1459]

Read also § 26.07.07.03.

Un manubrio è formato da un'asta rigida di massa trascurabile e lunghezza $2r$ che connette due masse puntiformi di massa m. L'asta ruota con velocità angolare costante ω attorno ad un asse verticale passante per il centro dell'asta formando un angolo costante θ con l'asse di rotazione.

1. Determinare l'espressione del momento angolare dell'asta rispetto al suo centro.
2. Determinare il momento richiesto per mantenere l'asta in rotazione a velocità angolare costante.

WORKED SOLUTION

Let $\widehat{\mathbf{n}}$ be the oriented unit vector along the rotation axis.
Sia ℓ il momento angolare di una delle due masse rispetto al centro dell'asta ed $\mathbf{r}$ il vettore posizione della massa rispetto al centro dell'asta.

$$\widehat{\mathbf{n}} \equiv \widehat{\mathbf{e}}_3 \qquad |\widehat{\mathbf{n}}| = 1 \qquad \widehat{\mathbf{e}}_r \equiv \frac{\mathbf{r}}{r} \qquad \widehat{\mathbf{u}}_r \equiv \mathbf{n}\wedge(\mathbf{r}\wedge\mathbf{n}) \qquad |\widehat{\mathbf{u}}_r| = 1 \quad .$$

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Si ha

$$\boldsymbol{\ell} = \mathbf{r}\wedge\mathbf{p} = m\,\mathbf{r}\wedge\mathbf{v} = m\,\mathbf{r}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) = m\left[\boldsymbol{\omega}r^2 - (\boldsymbol{\omega}\cdot\mathbf{r})\,\mathbf{r}\right] = m\left[\boldsymbol{\omega}r^2 - \omega r\cos[\theta]\,\mathbf{r}\right]$$

Il momento angolare del sistema è allora un vettore che ha una componente parallela alla velocità angolare ω ed una componente proporzionale al vettore posizione $\mathbf{r}$ che ruota attorno all'asse di rotazione di moto circolare uniforme. La componente ortogonale all'asse di rotazione si annulla se e solo se $\theta = \pi/2$. Dalla relazione ricavata si deduce il modulo al quadrato del momento angolare ℓ

$$\ell^2 = \mathbf{l}\cdot\mathbf{l} = \left(m\omega r^2\sin[\theta]\right)^2$$

Il modulo di $\boldsymbol{\ell}$ è quindi costante mentre la direzione, se $\theta \neq \pi/2$, varia. Si possono anche ricavare le componenti del momento angolare parallela e perpendicolare all'asse di rotazione (componente radiale)

$$\boldsymbol{\ell}_{\parallel} = \boldsymbol{\ell}\cdot\widehat{\mathbf{e}}_z = m\omega r^2\sin^2\theta$$

$$\boldsymbol{\ell}_{\perp} = \boldsymbol{\ell}\cdot\widehat{\mathbf{u}}_r = -m\omega r^2\sin[\theta]\cos[\theta]$$

Si osservi che il vettore momento angolare va considerato un vettore applicato nella posizione del punto materiale $\mathbf{r}$. Il segno di $\boldsymbol{\ell}_{\perp}$ dipende dal valore di θ. Il momento angolare della seconda massa è uguale a quello della prima massa in quanto per passare da una all'altra occorre effettuare le sostituzioni $\mathbf{r} \to -\mathbf{r}$ e $\mathbf{v} \to -\mathbf{v}$. Quindi per il momento totale si ha $\mathbf{L} = 2\boldsymbol{\ell}$. Si verifica anche che $\mathbf{L}$ è ortogonale ad $\mathbf{r}$ cioè all'asta, essendo $\mathbf{l}\cdot\mathbf{r} = 0$. Solo nel caso $\theta = \pi/2$ si ha $\mathbf{L} \parallel \boldsymbol{\omega}$.
La seconda **Cardinal Equation** (24.06.06), essendo il polo fisso, fornisce

$$\boldsymbol{\Gamma}^{(\mathrm{E})} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = -2m\omega r\cos[\theta]\,\mathbf{v} = -2m\omega r\cos[\theta]\,(\boldsymbol{\omega}\wedge\mathbf{r})$$

Da questa relazione risulta che non è necessario alcun momento per la rotazione con $\boldsymbol{\omega}$ e θ costante se e solo se $\theta = \pi/2$. Per ogni altro angolo $\mathbf{L}$ non è costante ma lo è il suo modulo. Per $\theta \neq \pi/2$ è necessario applicare un momento parallelo alla velocità $\mathbf{v}$, cioè perpendicolare all'asse di rotazione, che può essere fornito, ad esempio, da una coppia di forze che giacciano nel piano definito dall'asse di rotazione e dall'asta.
Se il giunto si rompe improvvisamente non esiste più alcun momento esterno esercitato e il momento angolare si conserva. L'asta si mette a ruotare con velocità angolare parallela al momento angolare e di modulo $\omega' = \omega\cos[\theta]$.
Nel caso θ è costante ma $\boldsymbol{\omega}$ cambia modulo mantenendo però direzione costante.
Come caso particolare delle relazioni generali dedotte sopra si ha che, per $\theta = \pi/2$ per variare il modulo di $\boldsymbol{\ell}$ occorre variare ω e serve un momento parallelo all'asse. Per variare la direzione di $\boldsymbol{\ell}$ senza variare il modulo occorre un momento perpendicolare all'asse di rotazione.

26.07.07
Motion Around a Fixed Axis

Consider a rigid-body constrained to rotate around a fixed axis.

26.07.07.01 Kinematics

Take an infinitesimal mass element, P, with mass, $\mathrm{d}m \equiv \rho\,\mathrm{d}V$, whose position is described by the position vector $\mathbf{r}$ with respect to any arbitrary point, O, on the rotations axis and having a velocity $\mathbf{v}$. Let the z axis be along the angular velocity $\boldsymbol{\omega} \equiv \omega\widehat{\mathbf{n}}$. The velocity of the point P is:

$$\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\omega}\wedge\mathbf{r} = \boldsymbol{\omega}\wedge\mathbf{r} \quad .$$

We assume that the angle θ between the position vector $\mathbf{r}$ and the angular velocity $\boldsymbol{\omega}$ is fixed for any infinitesimal mass element, as it is appropriate for a rigid-body.

Its angular momentum is:

$$\mathrm{d}\mathbf{L} \equiv \mathbf{r}\wedge\mathrm{d}\mathbf{p} = \mathrm{d}m\,\mathbf{r}\wedge\mathbf{v} = \mathrm{d}m\,\mathbf{r}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) = \mathrm{d}m\left(\boldsymbol{\omega}r^2 - (\boldsymbol{\omega}\cdot\mathbf{r})\,\mathbf{r}\right) \quad .$$

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The projection of the angular momentum along the z axis, is:

$$\mathrm{d}\mathbf{L}\cdot\hat{\mathbf{n}} \equiv \frac{\boldsymbol{\omega}\cdot\,\mathrm{d}\mathbf{L}}{\omega} \equiv \frac{\mathrm{d}m}{\omega}\left(\omega^2 r^2 - (\,\boldsymbol{\omega}\cdot\mathbf{r})^2\right) = \mathrm{d}m\omega r^2\sin^2[\theta] = \mathrm{d}m\omega r_\perp^2 \quad .$$

We also have:

$$\begin{aligned}
|\boldsymbol{\ell}| &= \mathrm{d}mrv = \mathrm{d}mr^2\omega\sin[\theta] \quad , \\
|\boldsymbol{\ell}| = 0 \qquad &\Leftrightarrow \qquad \sin[\theta] = 0 \quad , \\
\ell_z = |\boldsymbol{\ell}|\sin[\theta] &= \mathrm{d}mr^2\omega\sin^2[\theta] \quad , \\
\ell_\perp &= \mathrm{d}mr^2\omega\sin[\theta]\cos[\theta] \quad , \\
\ell_\perp = 0 \qquad &\Leftrightarrow \qquad \theta = \{0, \pi/2, \pi\} \quad .
\end{aligned}$$

The integration over all the rigid-body gives:

$$\mathbf{n}\cdot\mathbf{L} = \iiint \omega r_\perp^2\,\mathrm{d}m = \iiint \rho\omega r_\perp^2\,\mathrm{d}V = \omega\iiint \rho r_\perp^2\,\mathrm{d}V \equiv I_{zz}\omega \quad .$$

Therefore the component of the angular momentum along the angular velocity is proportional to the angular velocity and to a factor depending only on the geometry and mass distribution of the rigid-body and on the location of the fixed rotation axis with respect to the rigid-body. The quantity

$$I_{zz} \equiv \iiint \rho r_\perp^2\,\mathrm{d}V \quad ,$$

is called the (axial) moment of inertia of the rigid-body with respect to the z axis.

Note that, in general the angular momentum is not parallel to the angular velocity:

$$\mathbf{L} \nparallel \boldsymbol{\omega} \quad .$$

However, if the rotation axis is a symmetry axis of the rigid-body then the angular momentum is parallel to the angular velocity, because for any mass element there is an identical symmetrical mass element such that the pair gives an angular momentum parallel to the angular velocity.

Actually it is not necessary that the axis is fixed in space: it is sufficient that the axis moves parallel to itself in such a way that the angular velocity does not change its direction.

The kinetic energy is:

$$k \equiv \frac{1}{2}\mathrm{d}mv^2 = \frac{1}{2}\mathrm{d}m\,(\,\boldsymbol{\omega}\wedge\mathbf{r})^2 = \frac{1}{2}\mathrm{d}m\left(\omega^2 r^2 - (\,\boldsymbol{\omega}\cdot\mathbf{r})^2\right) = \frac{1}{2}\mathrm{d}m\omega^2 r_\perp^2 \quad .$$

The integration over all the rigid-body gives:

$$\mathcal{K} = \frac{1}{2}\iiint \omega^2 r_\perp^2\,\mathrm{d}m = \frac{1}{2}I_{zz}\omega^2 \quad .$$

The kinetic energy is thus expressed in terms of the angular velocity and the (axial) moment of inertia.

26.07.07.02 Dynamics

Applying the second **Cardinal Equation** of dynamics with any pole on the rotation axis, as we can choose any arbitrary pole on the axis, the pole will not be explicitly written in the following equations. As the rotation axis is supposed to be fixed, we can write:

$$\boldsymbol{\Gamma}^{(\mathrm{E})} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \Longrightarrow \Gamma_z^{(\mathrm{E})} = \frac{\mathrm{d}L_z}{\mathrm{d}t} = I_{zz}\dot{\omega} \quad .$$

Even when the angular velocity is constant, if $\ell_\perp \neq 0$, the angular momentum vector will change, because the component of the angular momentum perpendicular to rotation axis will rotate in the xy plane. However, its module, $\ell_\perp = mr^2\omega\sin[\theta]\cos[\theta]$, will remain constant.

Note that, the two angular momenta of the two masses are vectors applied at the position of the masses; the total angular momentum is a free vector.

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26.07.07.03 Toy Model: A Rotating Dumbbell

This § is referenced at pages:
[1456]

Read also § 26.07.06.

Consider a dumbbell made of two identical masses of mass m, joined by an ideal rigid bar of length $d = 2r$ and negligible mass, whose center is fixed to the vertical fixed rotation axis, z, at a fixed angle θ. Let $\Delta\phi$ be the angle of rotation of the dumbbell-axis plane around the axis and $\omega = \dot{\phi}$ be constant. Use, as a pole, the point where the dumbbell is constrained to the rotation axis; while always using this pole, it will not be explicitly mentioned in the symbols.

Let the positions be $\mathbf{r}_\pm$, with $\pm\mathbf{r}_+ = \mp\mathbf{r}_-$. The angle $\phi = 0$ shall correspond to the position of the mass at $\mathbf{r}_+$. Let $\mathbf{L} \equiv \boldsymbol{\ell}_+ + \boldsymbol{\ell}_- = 2\boldsymbol{\ell}_+ = 2\boldsymbol{\ell}_-$, as the two angular momenta are identical:

$$\boldsymbol{\ell}_\pm = m\,\mathbf{r}_\pm \wedge \mathbf{v}_\pm = m\,\mathbf{r}_\pm \wedge (\boldsymbol{\omega} \wedge \mathbf{r}_\pm) = m\left(\boldsymbol{\omega} r^2 - \mathbf{r}_\pm\,(\boldsymbol{\omega}\cdot\mathbf{r}_\pm)\right) \quad ,$$

$$\widehat{\mathbf{n}} \equiv \boldsymbol{\omega}/\omega \quad ,$$

$$\widehat{\mathbf{n}}\cdot\boldsymbol{\ell}_\pm = m\omega r^2\left(1 - \cos^2[\theta]\right) = m\omega r^2 \sin^2[\theta] \quad ,$$

$$[\boldsymbol{\ell}_\pm]_\perp = [-m\mathbf{r}_\pm\,(\boldsymbol{\omega}\cdot\mathbf{r}_\pm)]_\perp = -m\,(\boldsymbol{\omega}\cdot\mathbf{r}_\pm)\,[\mathbf{r}_\pm]_\perp = -m\,(\boldsymbol{\omega}\cdot\mathbf{r}_\pm)\left(\widehat{\mathbf{n}} \wedge (\mathbf{r}_\pm \wedge \widehat{\mathbf{n}})\right) = -m\,(\boldsymbol{\omega}\cdot\mathbf{r}_\pm)\,(\mathbf{r}_\pm - \widehat{\mathbf{n}}\,(\widehat{\mathbf{n}}\cdot\mathbf{r}_\pm)) \quad .$$

There is one very tricky issue here, related to vector operations while using bases which depend on the point, as for cylindrical coordinates. When the total angular momentum is calculated, $\mathbf{L} \equiv \boldsymbol{\ell}_+ + \boldsymbol{\ell}_-$, two vectors applied at different points, the two masses, are added. Geometrically, this implies translating either one or both vectors to let them have the same origin. At this point, and only at this point, operations in components can be done.

Only the component perpendicular to the axis changes:

$$\frac{\mathrm{d}\boldsymbol{\ell}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{\ell}_\perp}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{\ell}_\parallel}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{\ell}_\perp}{\mathrm{d}t} \quad .$$

The total angular momentum vector, $\mathbf{L}$ of the dumbbell is perpendicular to the dumbbell bar. As it is a resultant momentum, it can be considered as an applied vector on the pole, but this is not at all relevant here. The component perpendicular to the rotation axis has constant module:

$$L_\perp \equiv |\mathbf{L}_\perp| = 2mr^2\omega \sin[\theta]\cos[\theta] \quad ,$$

and it is rotating with angular velocity $\boldsymbol{\omega}$ around the axis. Therefore:

$$\left|\frac{\mathrm{d}\mathbf{L}_\perp}{\mathrm{d}t}\right| \equiv \left|\lim_{\Delta t\to 0}\frac{\mathbf{L}[t+\Delta t] - \mathbf{L}[t]}{\Delta t}\right| = \left|\lim_{\Delta t\to 0}\frac{2L_\perp \sin[\Delta\phi/2]}{\Delta t}\right| = L_\perp\omega \quad ,$$

$$\left|\frac{\mathrm{d}\mathbf{L}_\perp}{\mathrm{d}t}\right| = \omega|\mathbf{L}_\perp| \quad .$$

The direction is clearly the direction of $\widehat{\mathbf{e}}_\psi$, but it is not trivial to determine the orientation, geometrically.

In a algebraic way, translating all the vectors to the position of the particle at $\mathbf{r}_+$, in order to use, at that point, the same base in cylindrical coordinates, $\{\widehat{\mathbf{e}}_r, \widehat{\mathbf{e}}_\psi, \widehat{\mathbf{e}}_3\}$, for all vectors, one has:

$$\mathbf{L}_\perp = -L_\perp\widehat{\mathbf{e}}_r \implies \boldsymbol{\Gamma}^{(\mathrm{E})} = \frac{\mathrm{d}\mathbf{L}_\perp}{\mathrm{d}t} = -L_\perp\omega\widehat{\mathbf{e}}_\psi \qquad \left|\Gamma^{(\mathrm{E})}\right| = L_\perp|\omega| \quad .$$

Therefore in the case $\mathbf{L}_\perp \neq 0$ a torque is required to make the angular momentum precess around the rotation axis, and the direction of the torque is perpendicular to the plane defined by the angular velocity and $\mathbf{r}$.

The torque is provided by the centripetal forces which keeps the two masses rotating, finally provided by the joint on the axis, and transmitted to the two masses by the rigid bar.

UNDER CONSTRUCTION

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26.07.07.04 Static and Dynamical Balancing of Car Wheels

When the motion of the rigid-body is given, one can find the moments that are being applied to the rigid-body to permit that motion by using Euler equations.

Consider an axially symmetric tire, considered as a rigid-body, constrained to rotate about the wheel rotation axis, which is inaccurately mounted so that the rotation axis is not exactly coincident with the symmetry axis and|or the Center-Of-Mass does not lie exactly on the wheel rotation axis.

Static balancing refers to adding small masses at suitable places in order to get the Center-Of-Mass to lie on the wheel rotation axis.

Dynamic balancing refers to adding small masses at suitable places in order to get the wheel rotation axis to be a principal axis.

Replacing the above expressions into Euler equations one finds zero, for the two components of the external moment in the plane containing the dumbbell and the rotation axis and, letting I_3 be the moment of inertia along the axis of the dumbbell and $I > I_3$ be the moment of inertia along any axis perpendicular to the axis of the dumbbell, one finds the absolute value of the component of the external moment perpendicular to the plane plane containing the dumbbell and the rotation axis:

$$\Gamma_\perp = (I - I_3)\,\omega^2 \cos[\theta] \sin[\theta] \quad .$$

26.07.08
La Sfera/Cilindro Che Rotola Sul Piano Inclinato

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Una sfera/cilindro omogenei di massa M e raggio R inizia a rotolare da ferma giú da un piano inclinato di angolo α, $0 \leq \alpha \leq 90°$.

1. Supponendo che rotoli senza strisciare, determinare l'accelerazione del centro di massa, l'accelerazione angolare e la condizione sotto la quale rotola senza strisciare.
2. Risolvere il problema usando come polo il Centro di Massa, prima, e il punto mobile di contatto con il piano, dopo.
3. Si risolva inoltre il problema usando sia le **Cardinal Equation** che il principio di conservazione dell'energia meccanica.

WORKED SOLUTION

Si scelga un Reference Frame con asse x parallelo al piano e diretto verso il basso e asse y perpendicolare al piano e diretto verso l'alto. Sia θ l'angolo di rotazione di un punto fisso rispetto ad una direzione fissata, considerato positivo/negativo in base all'orientamento di $\hat{\mathbf{e}}_3$. Le equazioni per il moto del centro di massa e l'equazione dei momenti con polo il centro di massa sono, indicando con I il momento di inerzia rispetto al suo centro

$$Mg\sin\alpha + f_x = MA_x \tag{26.07.01}$$ → 1462

$$N - Mg\cos\alpha = MA_y = 0 \qquad \text{vincolo cinematico: la sfera/cilindro resta appoggiata al piano inclinato} \tag{26.07.02}$$ → 1462

$$f_x R = I\ddot{\theta} \equiv I\dot{\omega} \tag{26.07.03}$$ → 1462

con le quattro incognite A_x, f_x, N e $\dot{\omega}$. Il vincolo di puro rotolamento, che significa che il punto di contatto non striscia e cioè che la sua velocità relativa al piano è nulla, si scrive

$$\Delta x = -R\Delta\theta \qquad \text{notare il segno} - \text{dovuto alla convenzione sugli assi cartesiani} \quad . \tag{26.07.04}$$ → 1461

La soluzione del sistema di equazioni porta a

$$\boxed{A_x = \frac{gMR^2 \sin\alpha}{I + MR^2}} = \begin{cases} A_x = \frac{5}{7} g \sin\alpha \leq g \sin\alpha & \text{per una sfera, minore che per il punto materiale} \\ A_x = \frac{2}{3} g \sin\alpha g \sin\alpha & \text{per un cilindro, minore che per il punto materiale} \end{cases} ,$$

$$\boxed{f_x = -\frac{MgI \sin\alpha}{I + MR^2}} \quad ,$$

$$\boxed{N = Mg\cos\alpha}$$

In ogni caso si ha $|A_x| \leq g \sin\alpha$, l'accelerazione del centro di massa della sfera è minore di quella del punto materiale avente la stessa massa e si riduce a questa nel caso $I = 0$. Si osservi che nel caso $\alpha = 0$ (puro rotolamento nel piano) si ha $f_x = 0$ cioè il puro rotolamento su un piano non richiede alcuna forza di attrito. Ciò è altresì evidente se si considera che un puro rotolamento ha momento della quantità di moto costante e momento angolare costante. Dalle espressioni per N ed f_x si ricava la condizione che deve essere soddisfatta per aver il puro rotolamento:

$$\boxed{\tan[\alpha] \leq \mu_s \left(\frac{I + MR^2}{I} \right)} = \begin{cases} \frac{7\mu_s}{2} & \text{per una sfera} \\ 3\mu_s & \text{per un cilindro} \end{cases} ,$$

Notare il valore maggiore dell'angolo limite per la sfera rispetto al cilindro. Infatti, a parità di altre condizioni, il cilindro, con momento di inerzia maggiore, richiede una maggiore forza di attrito rispetto alla sfera.

Una soluzione alternativa si può ottenere scegliendo come polo per l'equazione dei momenti il punto di contatto istantaneo, inteso come il punto mobile che si muove rispetto al suolo parallelamente al Centro di Massa per cui, quindi, il termine $\mathbf{v} \wedge \mathbf{P}$ della seconda **Cardinal Equation** si annulla. Notare che non si tratta del **centro di rotazione istantanea**.

L'espressione della componente z del momento angolare rispetto al polo mobile può essere ricavata dalla decomposizione del momento angolare di un sistema in una parte di spin e una parte orbitale

$$\mathbf{L} = \mathbf{R} \wedge \mathbf{P} + \mathbf{L}' \tag{26.07.05}$$

da cui si deduce

$$L_z = -MRV_x + I\omega \quad ,$$

che, per il vincolo di puro rotolamento (26.07.04) diventa $L_z = \omega \left(I + MR^2 \right) \equiv I'\omega$.

Nel caso di rotolamento con strisciamento, invece, l'espressione $L_z = -MRV_x + I\omega$ non può essere ridotta ad una espressione che dipende solo da V_x oppure ω, che restano due variabili indipendenti.

Il problema può pure essere risolto utilizzando la conservazione dell'energia meccanica, nel caso in cui ci sia puro rotolamento. Infatti, in tal caso, la forza di attrito statico non compie lavoro e l'energia meccanica è conservata in quanto la reazione normale non compie lavoro e la forza peso è conservativa. La conservazione dell'energia meccanica si scrive allora, denotando con D lo spazio percorso lungo il piano inclinato

$$\frac{1}{2} M V_x^2 + \frac{1}{2} I \omega^2 = MgD \sin\alpha$$

ed essendo, per il vincolo di puro rotolamento,

$$V_x = -R\omega$$

si ricava

$$\boxed{V_x^2 = \frac{2gDMR^2 \sin\alpha}{I + MR^2}}$$

coerentemente con quello che si otterrebbe dalla relazione (23.03.09.02) usando l'accelerazione del centro di massa A_x calcolata in precedenza.

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In questo approccio si determina solo il moto della sfera, mentre per determinare le reazioni vincolari occorre usare le equazioni del moto (26.07.01), (26.07.02), (26.07.03).
Si noti che la forza di attrito statico non fa lavoro cioè non dissipa energia. La sua presenza è però indispensabile per la rotazione. La sua presenza converte parte dell'energia potenziale gravitazionale in energia cinetica rotazionale, anziché trasformare tutta dell'energia potenziale gravitazionale in energia cinetica di traslazione del centro di massa.

26.07.09
La Ciminiera Che Cade E Si Rompe

This § is referenced at pages:
[1455]

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©|Toy models for the falling chimney |||
©|A photo of a falling chimney |||
©|Am.J.Phys, , ..., ..., ...Ed., ...|45(2), 1977, 182||

Una ciminiera di forma cilindrica, massa M e altezza L molto maggiore del suo diametro, viene abbattuta alla base da una carica esplosiva e inizia a cadere rigidamente. Il centro di massa della ciminiera si trova ad un'altezza H rispetto alla base B.

1. Determinare l'accelerazione radiale e tangenziale della cima della ciminiera C in funzione dell'angolo θ che la ciminiera forma con la verticale.
2. Scrivere le equazioni del moto della cima della ciminiera.
3. Determinare l'accelerazione radiale e tangenziale di un punto generico della ciminiera in funzione dell'angolo θ che la ciminiera forma con la verticale.
4. Scrivere le equazioni del moto di un punto generico della ciminiera.
5. Determinare lo sforzo in direzione perpendicolare alla ciminiera in funzione della posizione lungo la ciminiera.
6. Determinare la reazione vincolare del suolo sulla base della ciminiera, necessaria a tenere la base ferma.

WORKED SOLUTION

Si misuri l'angolo θ tra la ciminiera e la verticale partendo dalla verticale; quindi θ cresce mentre la ciminiera cade. La seconda **Cardinal Equation** della dinamica con polo la base B della ciminiera si scrive:

$$MgH\sin[\theta] = I_{\mathrm{B}}\ddot{\theta} \quad . \tag{26.07.06}$$

→ 1463

Tale equazione è analoga a quella del pendolo fisico tranne che il segno relativo dei due membri è opposto nei due casi. La scelta della base B come polo permette di eliminare la reazione vincolare incognita che agisce alla base della ciminiera e dovuta al suolo. Ovviamente tale scelta non permette di determinare la reazione vincolare alla base.
Per determinare la reazione vincolare al suolo occorre utilizzare la prima **Cardinal Equation** , dopo aver risolto il moto della ciminiera.
Alternativamente, si applichi il principio di conservazione dell'energia meccanica alla ciminiera considerata come un corpo rigido. Si trova

$$MgH = \frac{1}{2}MV_{\mathrm{CM}}^2 + \frac{1}{2}I_{\mathrm{CM}}\omega^2 + MgH\cos[\theta] = \frac{1}{2}I_{\mathrm{B}}\omega^2 + MgH\cos[\theta] \qquad I_{\mathrm{B}} = I_{\mathrm{CM}} + MH^2$$

in cui l'energia cinetica della ciminiera è stata espressa in due possibili modi, attraverso le coordinate del centro di massa e attraverso le coordinate della base fissa B. La reazione del suolo non interviene

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perch,` fino a che la base è fissa, non fa lavoro siccome non sposta il punto di applicazione.
Vale la relazione

$$V_{\mathtt{CM}} = H\omega \quad ,$$

tra la componente tangenziale di $\mathbf{V}_{\mathtt{CM}}$ (modulo e segno) e la componente z della velocità angolare, ω (modulo e segno). Risolvendo rispetto ad ω^2:

$$\omega = \dot{\theta} = \sqrt{\frac{2MgH\,(1-\cos[\theta])}{I_{\mathtt{CM}} + MH^2}} = \sqrt{\frac{2MgH\,(1-\cos[\theta])}{I_{\mathtt{B}}}}$$

L'accelerazione angolare si può ricavare sia derivando l'espressione per ω, che direttamente dalla relazione (26.07.06).
Nel caso di asta sottile omogenea si hanno per i momenti di inerzia i valori

$$I_{\mathtt{CM}} = \frac{ML^2}{12} \qquad I_{\mathtt{B}} = \frac{ML^2}{3}$$

Ne segue l'espressione per l'accelerazione radiale della cima C della ciminiera

$$a_{\mathtt{R}} = -\omega^2 L = -\frac{2MgHL\,(1-\cos[\theta])}{I_{\mathtt{B}}} < 0$$

che è negativa essendo un'accelerazione centripeta. Per un'asta omogenea ($H = L/2$) si ottiene

$$a_{\mathtt{R}} = -3g\,(1-\cos[\theta])$$

L'espressione per l'accelerazione tangenziale della cima C della ciminiera vale invece

$$a_{\mathtt{T}} = \frac{\mathrm{d}v_{\mathtt{C}}}{\mathrm{d}t} = L\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{MgHL\sin[\theta]}{I_{\mathtt{B}}} > 0$$

Per un'asta omogenea si ottiene

$$a_{\mathtt{T}} = \frac{3}{2}g\sin[\theta]$$

Si osservi che in tal caso si ha

$$a_{\mathtt{T}}[\theta = \frac{\pi}{2}] = \frac{3}{2}g > g$$

Su ogni mattone agiscono, oltre alla forza peso, le forze dovute agli altri mattoni vicini la cui risultante può essere scomposta in uno sforzo di taglio e uno sforzo parallelo alla ciminiera.
L'esistenza di un'accelerazione tangenziale significa che i vari punti della ciminiera devono essere soggetti ad una forza tangenziale per essere accelerati tangenzialmente. Tale forza può essere fornita dalla componente tangenziale della forza peso oppure dallo sforzo di taglio. Nel caso la componente verticale dell'accelerazione tangenziale risulti maggiore dell'accelerazione di gravità tale accelerazione addizionale deve essere fornita dallo sforzo di taglio. In particolare il fatto che l'accelerazione tangenziale possa diventare maggiore di g a $\theta = \pi/2$ significa che lo sforzo di taglio in quella posizione deve fornire l'accelerazione addizionale di $g/2$ oltre a quella fornita al mattone dalla forza peso.
Finché la ciminiera cade rigidamente le accelerazioni sono determinate dalle relazioni ricavate sopra per il corpo rigido. Tali accelerazioni devono essere fornite ad ogni mattone dalla forza peso e dalle forze interne dovute ai mattoni adiacenti. Se i mattoni adiacenti non sono in grado di fornire le forze interne necessarie per l'accelerazione richiesta il moto non può essere quello di un corpo rigido e la ciminiera si rompe cadendo non appena in qualche punto viene richiesta una forza superiore alla forza massima che può essere fornita. Nel caso di caduta rigida essendo note le accelerazioni le forze richieste possono essere calcolate tramite le **Cardinal Equation** .
Sia $T[r] \equiv SF_T[r]$ lo sforzo in funzione della distanza r dalla base della ciminiera, considerato positivo nella direzione di θ crescente. L'equazione del moto per l'elemento di ciminiera posto a distanza r rispetto alla base, di spessore Δr e sezione di area A, è

$$A\,(T[r] - T[r+\Delta r]) + (\rho[r]A\Delta r)\,g\sin[\theta] = (\rho[r]A\Delta r)\,a_{\mathtt{T}}[r] = (\rho[r]A\Delta r)\,\frac{MgHr\sin[\theta]}{I_{\mathtt{B}}} > 0$$

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Passando al limite per $\Delta r \to 0$ si trova

$$\frac{\mathrm{d}T}{\mathrm{d}r}[r,\theta] = \rho[r] g \sin[\theta] \left[1 - \frac{MHr}{I_{\mathrm{B}}}\right] \qquad \text{con} \qquad T[r=L] = 0 \quad .$$

Se la ciminiera non è omogenea, $\rho = \rho[r]$, che interviene nell'equazione differenziale. Nel seguito si considererà quindi ciminiera omogenea in r.
La derivata della funzione $T[r]$ si annulla per $I_{\mathrm{B}} = MHr$ che corrisponde ad un punto di massimo relativo della funzione come si verifica dal fatto che la derivata seconda della $T[r]$ è negativa. Tuttavia il punto di massimo assoluto del valore assoluto di $T[r]$ è in $r = 0$, ai bordi dell'insieme di definizione della funzione.
Risolvendo,

$$T[r] = \rho g \sin[\theta] (L-r) \left(\frac{MH(L+r)}{2I_{\mathrm{B}}} - 1\right) \longrightarrow T[r] = \rho g \sin[\theta] (L-r) \left(\frac{3r-L}{4L} - 1\right) \qquad \text{per ciminiera omogenea} \quad .$$

Ne segue che i punti in alto devono essere accelerati tangenzialmente, mentre quelli vicini alla base devono essere decelerati tangenzialmente. Naturalmente, per ogni angolo θ esiste una distribuzione di sforzo di taglio dalla dalla precedente relazione.
Analogamente, ogni mattone della ciminiera deve soddisfare la seconda **Cardinal Equation** , in quanto ha anche una accelerazione angolare, la stessa della ciminiera nel complesso. Ne segue:

.....

La base della ciminiera può anche muoversi, se la reazione del suolo non è sufficiente.

WORKED SOLUTION

1) $a_{\mathrm{R}} = -\dfrac{2MgHL(1-\cos[\theta])}{I_{\mathrm{B}}}$; $a_{\mathrm{T}} = \dfrac{MgHL\sin[\theta]}{I_{\mathrm{B}}}$.

26.07.10
Il Tiro Della Palla Al Biliardo

This § is referenced at pages:
[1330]

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©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|12.60 13.19 13.23 13.24 13.25||

Una palla da biliardo di massa M e raggio R è appoggiata a riposo sul tavolo da biliardo. Un giocatore la colpisce con la stecca disposta orizzontalmente, a distanza h dal piano del biliardo e nel piano mediano verticale della sfera.

- Determinare la relazione tra la velocità iniziale v_0 e la velocità angolare iniziale ω_0.
- Determinare sotto quali condizioni la palla inizia a rotolare senza strisciare.
- Nel caso la palla parta strisciando determinare dopo quanto tempo smette di strisciare e per iniziare un moto di puro rotolamento.

WORKED SOLUTION

La palla è soggetta alla forza impulsiva che deriva dal colpo che riceve dalla stecca, alla forza peso, alla reazione normale del piano e alla forza di attrito del piano del tavolo.
Si scelga un Coordinate System con asse x parallelo al piano del tavolo e diretto nel verso della forza impressa dalla stecca, e asse y verticale. Si scelga come verso positivo per le rotazioni quello antiorario.

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La prima **Cardinal Equation** della dinamica, proiettata sugli assi x e y, e la proiezione sull'asse z della seconda **Cardinal Equation** con polo nel centro di massa della palla forniscono, detta **F** la forza esercitata dalla stecca e **f** la forza di attrito

$$F + f_x = Ma \tag{26.07.07}$$
→ 1466

$$N - Mg = 0 \tag{26.07.08}$$

$$Rf_x - F\,(h - R) = I_{\text{CM}}\ddot{\theta} \tag{26.07.09}$$
→ 1466

Si è denotato con I_{CM} il momento di inerzia della sfera, che può anche non essere omogenea in generale.
Durante l'azione della forza impulsiva **F** si può trascurare l'effetto della forza di attrito a condizione che il modulo della forza impulsiva sia molto maggiore della forza di attrito ($F \gg \mu_s N$) e che la durata dell'azione della forza impulsiva sia così breve da poter considerare la palla ferma in quell'intervallo di tempo.
In tal caso, considerando la forma impulsiva delle equazioni del moto, si ricava il valore della velocità del centro di massa, v, e della velocità angolare della palla, ω, a seguito dell'impulso

$$M\,(v - v_0) = \int F\,\mathrm{d}t \qquad \text{con } v_0 = 0 \tag{26.07.10}$$
→ 1466

$$I_{\text{CM}}\,(\omega - \omega_0) = (R - h)\int F\,\mathrm{d}t \qquad \text{con } \omega_0 = 0 \tag{26.07.11}$$
→ 1466

da cui, eliminando l'impulso $\int F\,\mathrm{d}t$, si ricava la relazione tra la velocità del centro di massa e la velocità angolare

$$\boxed{\omega = \frac{M\,(R-h)}{I_{\text{CM}}}v} = \frac{5}{2}\left(\frac{R-h}{R^2}\right)v$$

avendo usato il valore $I_{\text{CM}} = 2MR^2/5$ del momento di inerzia di una sfera omogenea. La relazione vale in generale per una sfera non omogenea purché a simmetria sferica. Nel caso $h = R$ si ha $\omega = 0$.
Nell'ipotesi di poter trascurare l'effetto della forza di attrito durante l'azione della forza impulsiva e nell'ipotesi che l'impulso sia di durata sufficientemente breve, dunque, la velocità iniziale del centro di massa e la velocità angolare iniziale si determinano in funzione del valore dell'impulso ricevuto. La relazione tra le due è invece indipendente dal valore dell'impulso ricevuto. Al termine dell'azione della forza impulsiva, la palla si muove con la velocità del centro di massa e la velocità angolare legate dalla relazione trovata. L'evoluzione successiva dipende da queste condizioni iniziali.
Per determinare sotto quali condizioni il moto è di puro rotolamento occorre calcolare la velocità del punto di contatto, v_p, che vale

$$\boxed{v_p = v + \omega R = v\left[\frac{I_{\text{CM}} + MR\,(R-h)}{I_{\text{CM}}}\right]} = \left[\frac{7R-5h}{2R}\right]v = \frac{\omega R}{5}\left[\frac{7R-5h}{R-h}\right] \qquad \omega = 0 \text{ se } h = R \tag{26.07.12}$$
→ 1466

Si trova

$$v_p = 0 \qquad \Leftrightarrow \qquad v = -\omega R \qquad \Leftrightarrow \qquad h = \frac{I_{\text{CM}} + MR^2}{MR} = \frac{7}{5}R$$

In tali condizioni il punto di contatto istantaneo è in quiete relativamente al piano, quindi non striscia e la forza di attrito è quella di attrito statico. Tale forza deve essere nulla in quanto una forza, ad esempio, diretta nel verso negativo dell'asse x, diminuirebbe la velocità del centro di massa e aumenterebbe il valore assoluto della velocità angolare che è negativa, producendo un valore di v_p negativo. Ma questo implicherebbe a sua volta una forza di attrito cinetico diretta nel verso positivo dell'asse x, in contraddizione con quanto supposto. Nel caso di puro rotolamento,

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infatti, le **Cardinal Equation** della dinamica si scrivono (dalle (26.07.07) ed (26.07.09) con $\mathbf{F} = 0$)

$$f_x = Ma \tag{26.07.13}$$

$$Rf_x = I_{\text{CM}}\ddot{\theta} \tag{26.07.14}$$

che, con il vincolo di puro rotolamento, $a = -R\ddot{\theta}$, implicano $f_x = 0$. Si possono riassumere i segni di v_p e di ω (v è sempre positivo in questo problema)

$$\begin{aligned} 0 \le h < R &\Longrightarrow \omega > 0, \qquad v_p > 0 \quad , \\ h = R &\Longrightarrow \omega = 0, \qquad v_p > 0 \quad , \\ R < h < 7R/5 &\Longrightarrow \omega < 0, \qquad v_p > 0 \quad , \\ h = 7R/5 &\Longrightarrow \omega < 0, \qquad v_p = 0 \quad , \\ 7R/5 < h < 2R &\Longrightarrow \omega < 0, \qquad v_p < 0 \quad . \end{aligned}$$

Il verso di v_p determina il verso della forza di attrito cinetico f_x. Si può anche vedere come, nel caso $h \neq 7R/5$, la forza di attrito fa sempre in modo che il moto della palla evolva verso un moto di puro rotolamento. Infatti supponiamo, per esempio, che sia inizialmente $v > 0$, $\omega < 0$ e $v_p < 0$. Il verso di v_p implica $f_x > 0$ (attrito cinetico radente). Quindi la forza di attrito accelera il centro di massa e produce un'accelerazione angolare positiva. Ne segue che nella (26.07.12) entrambi i termini aumentano con il tempo e questo aumento procede fino quando non si arriva a $v_p = 0$ in quanto il verso della forza di attrito dipende solo dal segno di v_p. Una volta raggiunto il puro rotolamento questo si mantiene come descritto sopra e la palla non ricomincia a strisciare. Si osservi, per inciso, che non è possibile in presenza di strisciamento applicare la seconda **Cardinal Equation** della dinamica usando come polo il punto di contatto nella forma $\Gamma = I'\ddot{\theta}$ in quanto, in presenza di strisciamento il punto di contatto non è istantaneamente fermo, quindi il polo non è un punto di quiete istantanea del corpo rigido e quindi il momento angolare non può essere posto nella forma $I'\ddot{\theta}$.

Determiniamo il moto nel caso $7R/5 < h < 2R$. In tal caso si ha $f_x > 0$ essendo $v_p < 0$. Dalle equazioni (26.07.13), (26.07.14) il moto del centro di massa è uniformemente accelerato con $a = f_x/M$ e il moto rotatorio attorno al centro di massa è uniformemente accelerato con accelerazione angolare $\ddot{\theta} = Rf_x/I$. Denotando con v_0 e ω_0 le velocità del centro di massa e angolare iniziali, cioè quelle acquisite a seguito del colpo della stecca, e che sono determinate dalle (26.07.10), (26.07.11) si ha

$$v(t) = v_0 + \frac{f_x t}{M} \tag{26.07.15}$$

$$\omega(t) = \omega_0 + \frac{Rf_x t}{I}$$

Come già discusso il moto evolve verso un moto di puro rotolamento che viene raggiunto al tempo $\bar{t}$ determinato dalla condizione di puro rotolamento $v(t) = -R\omega(t)$

$$\boxed{\bar{t} = -\frac{MIv_{p0}}{f_x\,(I + MR^2)} = -\frac{Mv_0\,[I + MR\,(R-h)]}{f_x\,(I + MR^2)} > 0} \tag{26.07.16}$$

da cui si deduce

$$v(\bar{t}) = -\omega(\bar{t})R = \left[\frac{MRh}{I + MR^2}\right] v_0$$

Il valore massimo di tale velocità finale si ottiene per $h = 2R$ e vale $v = 10/7v_0 > v_0$

La velocità del punto di contatto in funzione del tempo si determina come nella (26.07.12) usando le (26.07.15), (26.07.16), e vale

$$v_p(t) = v(t) + \omega(t)R = v(t)\left[\frac{I_{\text{CM}} + MR\,(R-h)}{I_{\text{CM}}}\right] = v_0 + \omega_0 R + tf_x\left[\frac{I + MR^2}{MI}\right] = v_p(t=0)\left[1 - \frac{t}{\bar{t}}\right]$$

Da questa relazione si deduce che $v(\bar{t}) \lessgtr v_0$ a seconda che $h \lessgtr 7R/5$

26.07.11
Centro Di Percussione Di Una Mazza Da Baseball

This § is referenced at pages:
[1330]

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DRAFT

Una mazza da baseball di massa M è appoggiata su un piano orizzontale, diretta lungo l'asse y crescente, con l'estremo dalla parte del manico coincidente con l'origine. Il centro di massa giace all'ordinata y_{CM}. Una forza impulsiva, diretta lungo l'asse x nel verso positivo, agisce sulla mazza ad un'ordinata $y_{\text{CM}} + h$.

Determinare se esiste un punto della mazza che resta inizialmente fisso a seguito dell'impulso in funzione del valore di h.

WORKED SOLUTION

Scegliendo come positivo il verso antiorario per le rotazioni la prima **Cardinal Equation** della dinamica e la seconda applicata con polo il centro di massa dell'asta si scrivono

$$\begin{aligned} F &= Ma_x \\ N - Mg &= Ma_y = 0 \quad , \\ -Fh &= I_{\text{CM}}\ddot{\theta} \end{aligned} \tag{26.07.17}$$

→ 1467

Dalle equazioni (26.07.17) si deduce, integrando rispetto al tempo,

$$I_{\text{CM}}\omega + Mvh = 0 \tag{26.07.18}$$

→ 1467 1468

Si osservi che la forza **F** dipende dal tempo ma la relazione tra v ed ω è indipendente dal tempo. La velocità v_p del punto generico del corpo rigido che dista b dal centro di massa, con b crescente nel verso negativo dell'asse y, è data da

$$v_p = v + \omega b = v\left[\frac{I - Mab}{I}\right] \tag{26.07.19}$$

→ 1467

da cui

$$v_p = 0 \qquad \Leftrightarrow \qquad I = Mhb$$

Conseguenza immediata della (26.07.19) è che h e b hanno lo stesso segno, e cioè, date le convenzioni scelte, si trovano da parti opposte rispetto al centro di massa. Il punto così determinato (centro di percussione) è dunque istantaneamente in quiete all'istante dell'urto. Ad istanti successivi la velocità del punto non sarà però nulla come evidente dalla forma vettoriale dall'atto di moto del corpo rigido in quanto dopo l'urto $\mathbf{v}$ non è più parallelo a $\boldsymbol{\omega}\wedge\mathbf{b}$. Inizialmente la velocità del entro di percussione è nulla mentre la velocità di qualunque altro punto (ad esempio il centro di massa) varia senza continuità da zero ad una valore diverso da zero. Altra conseguenza della (26.07.19) è che se $h = 0$ allora non esiste soluzione per b per cui la mazza trasla senza ruotare. Si osservi che la seconda **Cardinal Equation** assume la stessa forma (26.07.17) se come polo si sceglie il punto fisso solidale con il piano che coincide con il punto in cui l'asta è colpita. Infatti il momento della forza impulsiva rispetto a questo polo vale zero, e vale zero pure il momento totale delle forze esterne rispetto a tale polo, e il polo è fisso. Ne segue che il momento angolare dell'asta rispetto a questo polo è costante e vale zero perché è nullo prima del colpo. D'altra parte scomponendo il momento angolare in una parte orbitale ed una parte di spin si ha che il momento angolare orbitale dell'asta vale $+Mvh$ mentre il momento angolare di spin vale $I_{\text{CM}}\omega$. Se ne deduce ancora l'equazione (26.07.18). Alternativamente si può usare come polo il punto fisso solidale con il piano che coincide con il centro di massa dell'asta. In tale caso il momento risultante delle forze esterne rispetto al polo vale $-Fh$ e scomponendo il momento angolare il parte orbitale e di spin si ha che il momento angolare orbitale è nullo perché l'impulso totale dell'asta è un vettore che passa per

il polo scelto. Resta solo il momento angolare di spin che si esprime come $I_{CM}\omega$. Si deduce allora anche in questo modo la relazione (26.07.18).

26.07.12
La Scheggia Che Salta Dal Disco in Rotazione

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|13.35||

26.07.13
Il Giro Della Morte per Una Sfera

Svolgere il problema § 24-077 per una sferetta di raggio e massa date.

26.07.14
La Biglia Che Oscilla Entro La Calotta Semisferica

Read § 26-030.

A small sphere of radius r is left with zero initial velocity inside a larger semi-sphere of radius R and vertex downward and starts oscillating.

Determine the oscillation period, assuming rolling without slipping.

26.07.15
A Rectangular Homogeneous Lamina

A rectangular homogeneous flat lamina, with sides a and b and mass M, is constrained to rotate around a fixed axis with constant angular velocity $\boldsymbol{\omega}$ by some external engine.

Determine the Force|Torque applied by the axis to the lamina.

26.07.16
Free Symmetrical Top

This § is referenced at pages:
[1453]

26.07.16.01 Euler Equations for a Free Symmetrical Top: Body System

Consider a free symmetrical top and describe the motion in the body Reference Frame, according to Euler equations.

The kinetic energy is constant, as it follows from the third **Cardinal Equation** .

The third component of the angular velocity is constant:

$$\omega_3[t] = \omega_3 = \text{constant} \quad .$$

The use of the latter result allows to write:

$$\dot{\omega}_1 + \Omega\omega_2 = 0 \quad ,$$
$$\dot{\omega}_2 - \Omega\omega_1 = 0 \quad ,$$

in terms of the quantity:

$$\Omega \equiv \omega_3 \left(\frac{I_3 - I}{I} \right) \quad .$$

The solution for the other two components of the angular velocity is then:

$$\omega_1[t] = C \cos\left[\Omega t - \phi\right] \quad ,$$
$$\omega_2[t] = C \sin\left[\Omega t - \phi\right] \quad ,$$

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where C and ϕ are two arbitrary integration constants.

Note that, as it is consistent with a set a two first-order differential, the solution depends on two arbitrary integration constants, C and ϕ, to be determined by the initial conditions.

The solution shows that, in the body system, the angular velocity vector, $\boldsymbol{\omega}$, precesses around the third axis, $\widehat{\mathbf{k}}_3$, (rigidly fixed to the rigid-body) with an angular velocity of precession Ω. As $\mathbf{L}'_1 = I\omega_1$ and $\mathbf{L}'_2 = I\omega_2$ the angular momentum also precess around the $\widehat{\mathbf{k}}_3$ axis.

Free Precession of the Earth as a Symmetrical Top

This problem describes, for instance, how an observer on the Earth sees the direction of the angular velocity of the Earth changing. The latter can be detected by observing the north/south poles on the sky. In the case of the Earth predictions are not accurate has the Earth is not free and non axially symmetric.

If the angular momentum axis and the figure axis do not coincide, the figure axis performs precession about the angular momentum axis.

The Earth is expected to undergo a free precession with a period of about 450 days and an amplitude of precession of about $0.2''$. On the contrary, the free precession is known to have a period of about 300 days. This is due to the meant differences of the Earth with a rigid-body (non rigidity, ...)

26.07.16.02 Euler Equations for a Free Symmetrical Top: Laboratory System

Consider a free symmetrical top and describe the motion from the laboratory system, according to Euler equation (26.06.01).

Both angular momentum and kinetic energy are conserved:

$$0 = \frac{\mathrm{d}\mathbf{L}'}{\mathrm{d}t} \quad ,$$

$$0 = \frac{\mathrm{d}\mathcal{K}'}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\omega}\cdot\mathbf{L}'\right) \implies \boldsymbol{\omega}\cdot\mathbf{L}' = \text{constant} \quad .$$

The vector ω describe a fixed plane in space, as a consequence of equation $\boldsymbol{\omega}\cdot\mathbf{L}' = \text{constant}$ and of the fact that $\mathbf{L}'$ is a fixed vector in the laboratory system.

In the rigid-body frame all the axes in the $\left\{\widehat{\mathbf{k}}_1, \widehat{\mathbf{k}}_2\right\}$ plane are equivalent. Choose the $\widehat{\mathbf{k}}_1$ axis in the plane determined by $\mathbf{L}'$ and $\widehat{\mathbf{k}}_3$. Note that this construction is only possible because the top is symmetrical, otherwise the axes in the $\left\{\widehat{\mathbf{k}}_1, \widehat{\mathbf{k}}_2\right\}$ plane are not equivalent.

By construction $\mathbf{L}'_2 = 0$, which implies, as $\left\{\widehat{\mathbf{k}}_1, \widehat{\mathbf{k}}_2, \widehat{\mathbf{k}}_3\right\}$ is a set of principal axes of inertia, that $\omega_2 = 0$. That is ω lies in the plane $\left\{\widehat{\mathbf{k}}_1, \widehat{\mathbf{k}}_3\right\}$. Therefore, as both the $\widehat{\mathbf{k}}_3$ axis and the angular velocity lye in the plane $\left\{\widehat{\mathbf{k}}_1, \widehat{\mathbf{k}}_3\right\}$, the velocity of any point on the $\widehat{\mathbf{k}}_3$ axis is:

$$\mathbf{v} = \boldsymbol{\omega}\wedge\mathbf{x} \implies \delta\mathbf{x} \perp \left\{\widehat{\mathbf{k}}_1, \widehat{\mathbf{k}}_3\right\} \quad .$$

The above construction can be repeated at any instant of time without nay change. At any instant, therefore, the points on the $\widehat{\mathbf{k}}_3$ axis have a velocity perpendicular to the $\left\{\widehat{\mathbf{k}}_1, \widehat{\mathbf{k}}_3\right\}$ plane, which also contains the angular velocity vector. Therefore the $\left\{\widehat{\mathbf{k}}_1, \widehat{\mathbf{k}}_3\right\}$ plane and, in particular, the $\widehat{\mathbf{k}}_3$ axis rotates around the fixed angular momentum vector, $\mathbf{L}'$, as the points on the $\widehat{\mathbf{k}}_3$ axis always have a velocity perpendicular to the $\left\{\widehat{\mathbf{k}}_1, \widehat{\mathbf{k}}_3\right\}$ plane.

The precession angular velocity is given by the projection of the angular velocity on the rotation axis, that is the angular momentum vector, $\mathbf{L}'$. It can be calculated as follows. Let θ be the angle between the angular momentum vector, $\mathbf{L}'$, and the axis $\widehat{\mathbf{k}}_3$. This angle is constant because it has been shown that the velocity of the points of the axis $\widehat{\mathbf{k}}_3$ are orthogonal to the $\left\{\widehat{\mathbf{k}}_1, \widehat{\mathbf{k}}_3\right\}$ plane and zero component on the plane itself. The same conclusion can be found observing that:

$$\omega_3 = \text{constant} = \frac{\mathbf{L}'_3}{I_3} = \frac{|\mathbf{L}'|\cos[\theta]}{I_3} \implies \theta = \text{constant} \qquad \left(\text{because} \qquad |\mathbf{L}'| = \text{constant}\right) \quad .$$

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One finds:

$$\frac{|\mathbf{L}'|\sin[\theta]}{I_1} = \frac{\mathbf{L}'_1}{I_1} = \omega_1 = \Omega_P \sin[\theta] \quad ,$$
$$\Omega_P = \frac{|\mathbf{L}'|}{I_1} \quad ,$$
$$\omega_3 = \frac{\mathbf{L}'_3}{I_3} = \frac{|\mathbf{L}'|\cos[\theta]}{I_3} \quad .$$

In the laboratory system, therefore, both the angular velocity and the $\widehat{\mathbf{k}}_3$ axis precess around the fixed angular momentum vector, $\mathbf{L}'$.

Free Symmetrical Top: Constancy of the Module of the Angular Velocity

The constancy of the module of the angular velocity can be alternatively deduced as follows, by only using first integral of motion and not using Euler equations.

$$|\mathbf{L}'|^2 = I_1^2\left(\omega_1^2+\omega_2^2\right) + I_3^2\omega_3^2 = \text{constant} \quad ,$$
$$2|\mathcal{K}'|^2 = I_1\left(\omega_1^2+\omega_2^2\right) + I_3\omega_3^2 = \text{constant} \quad .$$

Solving the system for $\left(\omega_1^2+\omega_2^2\right)$ and ω_3^2:

$$\left(\omega_1^2+\omega_2^2\right) = \text{constant} \quad ,$$
$$\omega_3^2 = \text{constant} \quad ,$$

implying

$$|\boldsymbol{\omega}|^2 = \text{constant} \quad .$$

Free Symmetrical Top: Spin Plus Precession

One can write:

$$\begin{aligned}\mathbf{L}' &= I\omega_1\widehat{\mathbf{k}}_1 + I\omega_2\widehat{\mathbf{k}}_2 + I_3\omega_3\widehat{\mathbf{k}}_3 \quad ,\\ &= I\omega_1\widehat{\mathbf{k}}_1 + I\omega_2\widehat{\mathbf{k}}_2 + I\omega_3\widehat{\mathbf{k}}_3 + (I_3\omega_3 - I\omega_3)\,\widehat{\mathbf{k}}_3 \quad ,\\ &= I\boldsymbol{\omega} + (I_3 - I)\,\omega_3\widehat{\mathbf{k}}_3 \quad .\end{aligned}$$

Therefore:

$$\boldsymbol{\omega} = \frac{\mathbf{L}'}{I} + \frac{(I-I_3)}{I}\omega_3\widehat{\mathbf{k}}_3 \quad . \tag{26.07.20}$$

→ 1470

Both $\mathbf{L}'$ and ω_3 are constant. Equation (26.07.20) shows that the motion of the top is a precession and the angular velocity is the sum of two (non-orthogonal) angular velocities:

$$\boldsymbol{\omega} = \Omega_L + \Omega_S \quad , \tag{26.07.21}$$

one rotation around the constant angular momentum $\mathbf{L}'$ (precession axis) with constant angular velocity $\Omega_L = \mathbf{L}'/I$, and one rotation around the axis $\widehat{\mathbf{k}}_3$, with angular velocity of constant module and directed around the solidal axis $\widehat{\mathbf{k}}_3$. As the angular velocity ω is constant in module (not in direction) the motion is a regular precession.

26.07.16.03 Gyroscope and Gyrocompass

©|WEB - URL |||

A gyroscope is a device made of a rigid-body with a point which is kept fixed by a suitable set of constraints. The gyroscope undergoes a very fast rotation around an axis passing by the fixed point, but this axis, in general, changes with time. In absence of external Force|Torque it keeps the same rotation axis in an Inertial Reference Frame .

A gyrocompass ia an instrument always pointing to the north. One important component of a gyrocompass is a gyroscope, but a gyrocompass is built to use the effect of gyroscopic precession. Gyrocompasses are widely used for navigation on ships, because they have two significant advantages over magnetic

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compasses: they find true north as determined by Earth rotation, which is different from, and navigationally more useful than, magnetic north, and they are unaffected by ferromagnetic materials, such as a ship steel hull, which change the magnetic field.

Free Gyroscope

If the fixed point is the Center-Of-Mass, if the resultant external torque with respect to the Center-Of-Mass is zero, and the body rotates around a principal axis of inertia it turns out that the angular velocity is constant. In fact $\mathbf{L}'$ is constant and $\mathbf{L} \parallel \boldsymbol{\omega}$ because the body rotates around a principal axis of inertia. Therefore the rotation axis keeps the same direction in space. This is used in dynamic stabilizers: as the axis of the gyroscope keeps the same direction in space its direction relative to the host device measures the deviation of the host device from a fixed direction in space.

Typical applications are in artificial satellites and airplanes as well ship stabilisers.

Another typical application is in projectiles which are imparted a rotation around their axis.

Non-Free Gyroscope (Heavy Gyroscope)

Assume the fixed point is the Center-Of-Mass, the body rotates around a principal axis of inertia but the resultant external torque with respect to the Center-Of-Mass is not zero.

Assume that a force acts perpendicular to the rotation axis, applied, for instance, on the rotation axis (but any other application point would be good as well). The change of the angular momentum with respect to the Center-Of-Mass is perpendicular to the rotation axis that is perpendicular to the force. The module of the angular momentum with respect to the Center-Of-Mass is constant, because the angular momentum with respect to the Center-Of-Mass and its variation are perpendicular. Therefore the angular momentum with respect to the Center-Of-Mass undergoes a precession.

26.07.17
Free Asymmetrical Top

©|H.Goldstein *et al.*, *Classical Mechanics*, 2014, Pearson Education, 3rdEd., ...|||

The motion of the asymmetrical top is more complex.

In the case of a free asymmetrical top the motion of precession is superimposed to a motion of nutation.

26.07.18
Why Do Satellites Spin?

Why do satellites spin?

WORKED SOLUTION

Satellites are given a spinning motion around an axis of rotation passing through their own center of mass, in order to stabilize them in orbit. When they rotate, they acquire an angular momentum. If no external disturbances are present then the vector angular momentum is constant in time. Therefore it is constant both in magnitude and direction. A constant direction of angular momentum means a constant orientation of the axis of rotation, therefore a stable attitude of the satellite in orbit.

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26.07.19
A Rolling Cone

©|D.Morin, *Introduction to Classical Mechanics With Problems and Solutions*, ..., ..., ...Ed., TO ADD:.|problem 9.3||

A cone rolls without slipping on a table. The half-angle at the vertex is α, and the axis has length h. Let the speed of the center of the base, point P, be $\mathbf{v}$. What is the angular velocity of the cone with respect to the lab frame at the instant shown?

26.07.20
Spinning Symmetrical Top in the Gravity Field (Heavy Symmetrical Top)

©|D.Morin, *Introduction to Classical Mechanics With Problems and Solutions*, ..., ..., ...Ed., TO ADD:.|problem 9.- ...

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|problem 10.25|*solved* ...

Assume that initial conditions have been set up so that the heavy symmetrical top spins around its symmetry axis, the Center-Of-Mass trajectory is a circle around the vertical axis and the symmetry axis makes a constant angle with the vertical. Note that these are special initial conditions to make solution simpler, but more general solutions are possible.

1. Assuming that the angular momentum due to the angular velocity around the symmetry axis is much larger than any other angular momentum in the problem, and that the angular velocity around the symmetry axis is much larger than any other angular velocity, find the expression for the frequency of precession.
2. Solve the problem without the previous assumption.
3. Determine the motion of the Center-Of-Mass.
4. Determine the reaction of the table on the tip, both the vertical component and the horizontal one (static friction).

$$\boldsymbol{\gamma}_S = Mgh \sin[\theta]\, \widehat{\mathbf{K}} \quad ,$$

$$\mathbf{L}_S \simeq \dot{\psi} I_3 \widehat{\mathbf{n}} = \dot{\psi} I_3 \left(\cos[\theta]\, \widehat{\mathbf{e}}_3 + \sin[\theta]\, \widehat{\mathbf{N}} \right) \quad ,$$

$$\mathbf{L}_{S\perp} = \dot{\psi} I_3 \sin[\theta]\, \widehat{\mathbf{N}} \quad ,$$

$$\frac{\mathrm{d}\mathbf{L}_{S\perp}}{\mathrm{d}t} = \dot{\psi} I_3 \sin[\theta]\, \dot{\phi} \widehat{\mathbf{K}} \quad ,$$

$$\dot{\psi} = \frac{Mgh}{\dot{\phi} I_3}$$

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Step by step solution follows.

$$\widehat{\mathbf{n}} \equiv \mathbf{e}'_z \qquad \text{the top symmetry axis} \quad ,$$

$$\widehat{\mathbf{N}} = +\widehat{\mathbf{e}}_1 \cos[\phi] + \widehat{\mathbf{e}}_2 \sin[\phi] \qquad \text{unit vector directed along the projection of } \widehat{\mathbf{n}} \text{ on the } xy \text{ plane} \quad ,$$

$$\widehat{\mathbf{n}} = \mathbf{e}'_z = \widehat{\mathbf{e}}_3 \cos[\theta] + \widehat{\mathbf{N}} \sin[\theta] \quad ,$$

$$\widehat{\mathbf{K}} \equiv (\widehat{\mathbf{e}}_3 \wedge \widehat{\mathbf{n}}) / \sin[\theta] = -\widehat{\mathbf{e}}_1 \sin[\phi] + \widehat{\mathbf{e}}_2 \cos[\phi] \qquad \text{on the } xy \text{ plane and perpendicular to the top-vertical axes} \quad ,$$

$$\mathbf{U} \equiv \mathbf{n} \wedge \mathbf{K} = (\widehat{\mathbf{e}}_3 - \cos[\theta]\, \widehat{\mathbf{n}}) / \sin[\theta] \qquad \text{on the top-vertical axes plane} \quad ,$$

$$\left\{ \widehat{\mathbf{K}}; \mathbf{U}; \widehat{\mathbf{n}} \right\} \qquad \text{orthonormal basis} \quad ,$$

$$\widehat{\mathbf{e}}_3 = \widehat{\mathbf{n}} \cos[\theta] + \mathbf{U} \sin[\theta] \quad ,$$

$$\boldsymbol{\omega} = \dot{\psi}\widehat{\mathbf{n}} + \dot{\phi}\widehat{\mathbf{e}}_3 + \dot{\theta}\widehat{\mathbf{K}} = \dot{\psi}\widehat{\mathbf{n}} + \dot{\phi}\,(\widehat{\mathbf{n}} \cos[\theta] + \mathbf{U} \sin[\theta]) + \dot{\theta}\widehat{\mathbf{K}} = \dot{\theta}\widehat{\mathbf{K}} + \sin[\theta]\, \dot{\phi}\mathbf{U} + \left(\dot{\psi} + \dot{\phi}\cos[\theta]\right) \widehat{\mathbf{n}} \quad ,$$

$$\mathbf{L}_{\mathsf{S}} = \mathbb{I}_{\mathsf{S}} \cdot \boldsymbol{\omega} = \mathbb{I}_{\mathsf{S}} \cdot \left(\dot{\theta}\widehat{\mathbf{K}} + \sin[\theta]\, \dot{\phi}\mathbf{U} + \left(\dot{\psi} + \dot{\phi}\cos[\theta]\right) \widehat{\mathbf{n}} \right) = \dot{\theta} I_\perp \widehat{\mathbf{K}} + \sin[\theta]\, \dot{\phi} I_\perp \mathbf{U} + \left(\dot{\psi} + \dot{\phi}\cos[\theta]\right) I_3 \widehat{\mathbf{n}} \quad ,$$

$$\boldsymbol{\gamma}_{\mathsf{S}} = Mgh \sin[\theta]\, \widehat{\mathbf{K}} \quad ,$$

$$\frac{\mathrm{d}\widehat{\mathbf{N}}}{\mathrm{d}t} = +\dot{\phi}\,(-\widehat{\mathbf{e}}_1 \sin[\phi] + \widehat{\mathbf{e}}_2 \cos[\phi]) = +\dot{\phi}\widehat{\mathbf{K}} \quad ,$$

$$\frac{\mathrm{d}\widehat{\mathbf{K}}}{\mathrm{d}t} = -\dot{\phi}\,(+\widehat{\mathbf{e}}_1 \cos[\phi] + \widehat{\mathbf{e}}_2 \sin[\phi]) = -\dot{\phi}\widehat{\mathbf{N}} \quad ,$$

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \ldots \quad ,$$

$$\frac{\mathrm{d}\widehat{\mathbf{n}}}{\mathrm{d}t} = -\widehat{\mathbf{e}}_3 \dot{\theta} \sin[\theta] + \sin[\theta]\, \frac{\mathrm{d}\widehat{\mathbf{N}}}{\mathrm{d}t} + \widehat{\mathbf{N}} \dot{\theta} \cos[\theta] = -\dot{\theta} \sin[\theta]\, \widehat{\mathbf{e}}_3 + \sin[\theta]\, \dot{\phi}\widehat{\mathbf{K}} + \widehat{\mathbf{N}} \dot{\theta} \cos[\theta] \quad ,$$

$$\dot{\theta} = 0 \qquad \ddot{\phi} = 0 \qquad \ddot{\psi} = 0 \implies \frac{\mathrm{d}\mathbf{L}_{\mathsf{S}}}{\mathrm{d}t} = \sin[\theta]\, \dot{\phi} I_\perp \dot{\mathbf{U}} + \left(\dot{\psi} + \dot{\phi}\cos[\theta]\right) I_3 \frac{\mathrm{d}\widehat{\mathbf{n}}}{\mathrm{d}t} \quad ,$$

$$\left| \mathbf{L}_{\mathsf{S}} \cdot \widehat{\mathbf{K}} \right| \ll |\mathbf{L}_{\mathsf{S}} \cdot \widehat{\mathbf{n}}| \qquad |\mathbf{L}_{\mathsf{S}} \cdot \mathbf{U}| \ll |\mathbf{L}_{\mathsf{S}} \cdot \widehat{\mathbf{n}}| \qquad \left|\dot{\phi}\right| \ll \left|\dot{\psi}\right| \implies \mathbf{L}_{\mathsf{S}} \simeq \dot{\psi} I_3 \widehat{\mathbf{n}} \qquad \frac{\mathrm{d}\widehat{\mathbf{n}}}{\mathrm{d}t} \simeq \sin[\theta]\, \dot{\phi}\widehat{\mathbf{K}} \quad ,$$

$$\dot{\psi}\dot{\phi} I_3 \sin[\theta] = Mgh \sin[\theta] \implies \dot{\phi} = \frac{Mgh}{\dot{\psi} I_3}$$

26.07.21
Precession of the Equinoxes

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26.07.22
SAROS Cycle

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26.07.23
A Pulsating Neutron Star

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26.07.24
Gyrocompass

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26.07.25
Orbit of the Moon

Consider an observer on the Earth: what is the angular velocity of the Moon, with respect to the Earth-fixed Reference Frame, with z axis aligned with the Earth rotation axis and xy plane coincident with the equatorial plane.

Assume the Moon is orbiting around the center of the Earth in a circular orbit with the average orbital parameters (see, for instance: WEB - URL).

26.07.26
Rolling Lollipop

©|D.Morin, *Introduction to Classical Mechanics With Problems and Solutions*, ..., ..., ...Ed., TO ADD:.|problem 9.51||

Consider a lollipop made of a solid sphere of mass m and radius r that is radially pierced by a massless horizontal stick. The free end of the stick is pivoted on a pole (see Fig. 9.65), and the sphere rolls on the ground without slipping, with its center moving in a circle of radius R with frequency Omega. What is the normal force between the ground and the sphere?

26.07.27
Trave Incernierata Al Muro E Sorretta Da Una Corda

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|E14.5||

Una trave rigida uniforme di lunghezza $L = 3.3\ \mathrm{m}$ e massa $M = 8.5\ \mathrm{kg}$ è incernierata ad un muro ad un estremo. All'altro estremo è collegata una fune ideale attaccata al muro che la sorregge. L'angolo tra la trave e l'orizzontale è $0 \leq \beta \leq 90°$ e l'angolo tra la fune e l'orizzontale è α. La distanza tra la cerniera e il punto in cui la fune è attaccata al muro vale $d = 2.1\ \mathrm{m}$. Un peso di massa $m = 56\ \mathrm{kg}$ è attaccato all'estremo della trave.

1. Determinare la reazione della cerniera, $\mathbf{R}$, e la tensione della fune, T, nel caso in cui $\beta = 30°$.

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WORKED SOLUTION

Si scelga un Coordinate System con asse x orizzontale e verso uscente dal muro e asse y verticale. Siano R_x ed R_y le componenti della reazione della cerniera e T il modulo della tensione della fune. Si noti che la tensione della fune è necessariamente diretta lungo la fune mentre la reazione vincolare sulla cerniera non ha necessariamente la direzione della trave. Infatti questa è un corpo rigido e quindi è in grado di esercitare idealmente sulla cerniera una forza avente direzione arbitraria e non necessariamente parallela a sé stessa. La reazione che la cerniera esercita sulla trave non ha di conseguenza necessariamente la direzione della trave. Occorre anzitutto determinare l'angolo α che è dato da

$$\tan[\alpha] = \frac{d - L\sin\beta}{L\cos\beta} = 0.157 \implies \alpha = 8.94^\circ$$

La prima **Cardinal Equation** e la seconda equazione cardinale usando come polo la cerniera si scrivono

$$\begin{aligned} &R_x - T\cos\alpha = 0 \quad , \\ &R_y + T\sin\alpha - mg - Mg = 0 \quad , \\ &TL\sin(\alpha+\beta) - mgL\cos\beta - MgL/2\cos\beta = 0 \quad . \end{aligned}$$

La scelta di un polo rispetto al quale entrambe le componenti di $\mathbf{R}$ sono nulle permette di ottenere un'equazione con la sola incognita T.

$$T = g\cos\beta\left[\frac{m + M/2}{\sin(\alpha+\beta)}\right] = 814\ \text{N} \quad ,$$

$$R_x = g\cos\beta\cos\alpha\left[\frac{m + M/2}{\sin(\alpha+\beta)}\right] = 804\ \text{N} \quad ,$$

$$R_y = -g\cos\beta\sin\alpha\left[\frac{m + M/2}{\sin(\alpha+\beta)}\right] + (m+M)\,g = 506\ \text{N} \quad .$$

La reazione esercitata dalla cerniera sulla trave vale

$$R = \sqrt{R_x^2 + R_y^2} = 950\ \text{N}$$

Si noti che sia R che T sono considerabilmente più grandi dei pesi combinati della trave e del peso. L'angolo che la reazione della cerniera forma con l'orizzontale vale

$$\tan[\phi] = \frac{R_y}{R_x} = 0.629 \implies \phi = 32.2^\circ$$

Per un valore di β fissato, cioè per una inclinazione della trave fissata, la tensione minima, al variare di α occorre quando $\alpha + \beta = 90^\circ$.

WORKED SOLUTION

1) $T = 814$ N; $R_x = 804$ N; $R_y = 506$ N.

26.07.28
La Scala Appoggiata Al Muro

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|E14.3 E14.4||

26.07.29
Euler Equations for Non-Rigid-Bodies - Liouville equations

See: WEB - URL.

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26.08 Exercises Problems and Physical Applications

26-001 Il Volano

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|12.10||

26-002 Raggio Di Girazione

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|12.11||

26-003 Momento Di Inerzia Come Limite Di Una Sommatoria

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|12.12||

26-004 Rotazione Della Terra

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26-005 Blocchi Più Carrucola

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|12.29||

26-006 Il Disco Ruotante Appoggiato Al Tavolo Con Attrito

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|12.40||

26-007 Lo Yo-Yo

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26-008 Caduta Lungo Una Guida

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26-009 Variazione Sul Problema Dello Yo-Yo

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|12.51||

26-010 Il Tappeto Che Si Srotola

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|12.52||

26-011 L'oggetto Incognito Che Rotola

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|12.54||

26-012 La Ruota Dell'aereoplano Che Striscia Al Suolo

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|12.56||

©|H.C.Ohanian, , ..., ..., ...Ed., ...|10.49||

26-013 Impulso Angolare

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|13.09||

26-014 Evoluzione Del Sole in Una Nana Bianca

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|13.27||

26-015 Lo Scioglimento Delle Calotte Polari

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|13.40||

26-016 Dinamica Della Nascita Della Terra

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|13.41||

26-017 Quantizzazione Del Momento Angolare

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26-018 L'automobile Nel Fango

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26-019 Equilibrio Di Un Sistema

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26-020 La Massa Di Un Regolo Graduato

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26-021 La Ruota Che Sale Il Gradino

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26-022 La Carrucola

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26-023 L'insegna Appesa

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26-024 Equilibrio Di Una Asta

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26-025 La Mensola

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|14.29 14.30||

26-026 L'auto Che Frena

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26-027 Lo Scalatore

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26-028 II Tunnel

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26-029 A Cylinder Rolling Outside A Cylinder

This § is referenced at pages:
[1067, 1480]

Read § 26-030.

Consider a fixed rigid circular cylinder of radius R and a second rigid circular cylinder of radius $r < R$. The two cylinders have parallel axes. The small cylinder rolls without slipping on the large cylinder. Introduce the two angles: θ, defining the position of the center of the small cylinder with respect to the center of large cylinder; ϕ, defining the absolute rotation of small cylinder.

Determine the relation between θ and ϕ such that the rolling is without slipping.

26-030 A Sphere Rolling Inside a Sphere

This § is referenced at pages:
[1067, 1396, 1468, 1480]

Read § 26-029.

A sphere of radius r is rolling without slipping inside a larger semi-sphere of radius $R > 2r$, with vertex facing down. The motion is a plane motion, in a plane containing the center of both spheres. In this plane, measure all angles from the downward vertical direction and let θ be the angle of the center of the small sphere and ϕ the angle of rotation of the small sphere, both measured as positive in the counter-clockwise direction. One specific point of the small sphere, fixed with the small sphere, makes an angle ϕ_0 with the downward vertical direction when the small sphere is in the lowest position, $\theta = 0$.

1. Determine the rolling without slipping condition.
2. In case of rolling without slipping, determine the axis of instantaneous rotation.
3. In case of rolling without slipping, determine the acceleration of all the points of the edge.
   Let C be the center of the small sphere; P be the generic point on the edge of the small sphere; H be the generic point of contact of the two spheres; O be the center of the large semi-sphere and global origin.

WORKED SOLUTION

$$\mathbf{R}_C - \mathbf{R}_0 = (R-r)\left(+\widehat{\mathbf{e}}_1 \sin[\theta] - \widehat{\mathbf{e}}_2 \cos[\theta]\right) \quad ,$$

$$\mathbf{R}_P - \mathbf{R}_C = r\left(+\widehat{\mathbf{e}}_1 \sin[\phi+\phi_0] - \widehat{\mathbf{e}}_2 \cos[\phi+\phi_0]\right) \quad ,$$

$$\mathbf{v}_P = \dot{\theta}(R-r)\left(+\widehat{\mathbf{e}}_1 \cos[\theta] - \widehat{\mathbf{e}}_2 \sin[\theta]\right) + \dot{\phi} r\left(+\widehat{\mathbf{e}}_1 \cos[\phi+\phi_0] - \widehat{\mathbf{e}}_2 \sin[\phi+\phi_0]\right) \quad ,$$

$$\text{P is in contact with the semi-sphere} \quad \Leftrightarrow \quad P \to H \quad \Leftrightarrow \quad \phi + \phi_0 = \theta \quad ,$$

$$\Longrightarrow \mathbf{v}_H = +\widehat{\mathbf{e}}_1 \cos[\theta]\left(\dot{\theta}(R-r) + \dot{\phi} r\right) - \widehat{\mathbf{e}}_2 \sin[\theta]\left(\dot{\theta}(R-r) + \dot{\phi} r\right) \quad ,$$

$$\mathbf{v}_H = 0 \quad \Leftrightarrow \quad \dot{\theta}(R-r) + \dot{\phi} r = 0 \quad .$$

Note that the contact condition, $\phi + \phi_0 = \theta$, cannot be used to imply $\dot{\theta} = \dot{\phi}$, because angle and

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angular velocity must be treated as independent variables.

26-031 Rotation Around a Fixed Axis

Consider the rotation around a fixed axis, z. Determine, by explicit calculation from its definition, the angular velocity.

26-032 A Spinning Top

A spinning top rotates around its axis with angular velocity $\boldsymbol{\omega}$, while the axis stays always at a constant angle θ with respect to the vertical and rotates around the vertical with angular velocity $\boldsymbol{\Omega}$. Determine the angular velocity vector of the top.

26-033 Inertia Matrix|Tensor for a Homogeneous Cube

Determine the inertia Matrix|Tensor for a homogeneous cube with origin at one vertex and axes along the sides, by:

1. direct integration;
2. from the generalized Huygens-Steiner theorem and a minimal number of integrations.

WORKED SOLUTION

$$\mathbb{I} = ML^2 \begin{pmatrix} +2/3 & -1/4 & -1/4 \\ -1/4 & +2/3 & -1/4 \\ -1/4 & -1/4 & +2/3 \end{pmatrix},$$

26-034 Matrix of Inertia of a Thin Bar Without Integrations

Only use general theorems for the inertia Matrix|Tensor to calculate the moment of inertia with respect to the center of a thin cylindrical bar of mass M and length L.

26-035 A Special Cylinder

What must the ratio of height to radius of a cylinder be so that every axis is a principal axis (with the Center-Of-Mass as the origin)?

26-036 Matrix of Inertia of a Truncated Cone With Respect to Its Axis

Calculate the matrix of inertia of a truncated homogeneous cone, of height H and base radii a and b, with respect to its axis.

WORKED SOLUTION

$$\frac{\pi\rho H}{10}\frac{b^5 - a^5}{b - a} \quad .$$

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26-037 Moments of Inertia of Some Molecules

©|L.D.Landau *et al.*, *Mechanics*, ..., ..., ...Ed., WEB - URL - ISBN: 978-0-7506-2896-9.|§ 32 problem 1||

Determine the principal moments of inertia, with origin at the Center-Of-Mass, for the following types of molecule, regarded as systems of point particles at fixed distances apart: a molecule of collinear atoms; a tri-atomic molecule which is an isosceles triangle; a tetra-atomic molecule which is an equilateral-based tetrahedron.

26-038 Moments of Inertia of Some Solids

©|L.D.Landau *et al.*, *Mechanics*, ..., ..., ...Ed., WEB - URL - ISBN: 978-0-7506-2896-9.|§ 32 problem 2||

Determine the principal moments of inertia for the following homogeneous bodies: a thin rod of length L; a sphere of radius R; a circular cylinder of radius R and height H; a rectangular parallelepiped of sides a, b and c; a circular cone of height H and base radius R; an ellipsoid of semi-axes a, b and c.

26-039 Kinetic Energy of a Non-Homogeneous Cylinder

©|L.D.Landau *et al.*, *Mechanics*, ..., ..., ...Ed., WEB - URL - ISBN: 978-0-7506-2896-9.|§ 32 problem 5||

Find the kinetic energy of a cylinder of radius R rolling on a plane, if the mass of the cylinder is so distributed that one of the principal axes of inertia is parallel to the axis of the cylinder and at a distance a from it, and the moment of inertia about that principal axis is I.

26-040 A Cylinder Rolling Inside a Cylinder

©|L.D.Landau *et al.*, *Mechanics*, ..., ..., ...Ed., WEB - URL - ISBN: 978-0-7506-2896-9.|§ 32 problem 6||

Find the kinetic energy of a homogeneous cylinder of radius r rolling inside a cylindrical surface of radius R.

26-041 Rolling Cone

©|L.D.Landau *et al.*, *Mechanics*, ..., ..., ...Ed., WEB - URL - ISBN: 978-0-7506-2896-9.|§ 32 problem 7||

Find the kinetic energy of a homogeneous cone rolling on a plane.

26-042 Another Rolling Cone

©|L.D.Landau *et al.*, *Mechanics*, ..., ..., ...Ed., WEB - URL - ISBN: 978-0-7506-2896-9.|§ 32 problem 8||

Find the kinetic energy of a homogeneous cone whose base rolls on a plane and whose vertex is fixed at a height above the plane equal to the radius of the base, so that the axis of the cone is parallel to the plane.

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26-043 A Rolling Sphere

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|1.267|*Adapted*|

A uniform sphere of mass M and radius R rolls, with angular velocity $\boldsymbol{\omega}$ and without sliding, over a horizontal plane, rotating about a horizontal axle, OC, passing by its center, C; $\mathbf{v}_0 = 0$. In the process, the center of the sphere moves along a circle on the horizontal plane of radius D, with velocity angular velocity $\boldsymbol{\Omega}$.

1. Determine the rolling without slipping condition.
2. Find the kinetic energy of the sphere.
3. Determine the reaction at O (Force|Torque).

26-044 A Deformed Disk

A homogeneous circular disk of mass M, radius R and height H has been slightly deformed in such a way that its inertia matrix with respect to the Center-Of-Mass has become:

$$\mathbb{I} = \begin{pmatrix} \frac{1}{4}MR^2 & \epsilon & 0 \\ \epsilon & \frac{1}{4}MR^2 & 0 \\ 0 & 0 & \frac{1}{2}MR^2 + \delta \end{pmatrix} ,$$

where the xy plane is the average disk plane, passing by the Center-Of-Mass, and the z axis is the axis perpendicular to the xy plane passing by the Center-Of-Mass.

Try to model the deformed disk with a homogeneous circular disk of mass M', radius R and height H plus a point mass, $m \equiv M - M'$ giving a inertia matrix equal to the one of the deformed cylinder.

26-045 Angular Velocity of a Moving Point

©|D.Morin, *Introduction to Classical Mechanics With Problems and Solutions*, ..., ..., ...Ed., TO ADD:.|problem 9.1||

Consider a particle at the point $P = \{d, 0, 0\}$, with velocity $\mathbf{v} = v\widehat{\mathbf{e}}_3$ at a certain instant. At this instant, the particle may be considered to be rotating around many different axes passing through the origin. Find all the possible angular velocity vectors compatible with the motion of the particle.

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26.09 Additional Problems

26.09.01 Plane Motion of Rigid-Bodies

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|§ 9|*A lot of*...

26.09.02 Space Motion of Rigid-Bodies

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|§ 10|*A lot of*...

26.09.03 Problems on Rigid-Bodies

©|D.Morin, *Introduction to Classical Mechanics With Problems and Solutions*, ..., ..., ...Ed., TO ADD:.||*A lot of*...

DRAFT IN PROGRESS

CHAPTER 27

Introduction to Mechanics of Elastic Solids

27.01 Introduction 1487
27.02 Uniform and Isotropic Deformations of Homogeneous Isotropic Elastic Solids 1489
27.03 Properties of Materials 1493
27.04 General Non-Uniform Deformation in One Dimension 1494
27.05 Statics of Continuous Media 1495
27.06 Some Related Concepts From Thermodynamics 1496
27.07 Examples and Physical Applications 1497
27.08 Exercises Problems and Physical Applications 1510
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This § is referenced at pages:
[1331]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.38 2.39||

©|L.E.Malvern|Introduction to the Mechanics of a Continuous Medium|*1969 - § 5.3 - WEB - URL* |

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27.01 Introduction

In this section the study of **solid elastic materials which are both homogeneous and isotropic** is introduced. While both solids and fluids can deform, the study of elasticity theory is typically targeted to solids, while the elastic properties of fluids are typically studied in the realm of fluid mechanics and ThermoDynamics.

This introductory discussion is a purely mechanical one, while a full treatment should be a thermo-mechanical one, in order to deal with transfers of mechanical energy from/to internal energy. As long as one studies purely mechanical processes, it must be assumed that mechanical energy is conserved, and there is no heat transfer. Read also § 32.

Under the effect of Force|Torque a material medium may deform in many ways. Whenever **the Force|Torque and deformations are small enough**, the deformation is proportional to the Force|Torque, as it follows while considering the first-order approximation of the functions relating Force|Torque and deformation. In these cases the process is said to be a **linear or elastic process**. Linearity is a combined property of the Force|Torque values, of the material, as well as of any other condition affecting the response of the material. **The superposition principle can be applied in elasticity theory, as it deals with linear systems.** Linearity and small deformations will always be assumed. In fact the discussion of large deformations, with possibly non-linear effects, is much more complex. However, most materials show, under small deformations, an elastic behavior, that is a deformation proportional to the applied Force|Torque. This behavior is observed in many common processes.

Non-elastic (non-linear) behavior includes: **non-linear response**; **hysteresis**, that is dependence of the response not only on the actual values but also on the history; **permanent deformations**; **breaking**.

We consider only continuous media made of non-dipolar particles.

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...
Up to this point we have assumed the non-dipolar case with no distributed assigned torques per unit area on the boundary surfaces and no distributed assigned body-torques (per unit volume or per unit mass), but only external surface traction forces and body forces; and with no torque-stress es, i.e., no distributed torques per unit area across internal surfaces.
Assigned external torque distributions could result from the action of an external magnetic field on Magnetized particles of the material or the action of an electric field on Polarized matter. Even without assigned external torques, torque-stress can arise from interactions between adjacent parts of the material other than central-force interactions. Torque|Stress were included in continuum mechanics by Voigt (1887, 1894) and the Cosserats (1909), but these formulations were not much applied. More recently Truesdell and Toupin (1960) have discussed them; Aero and Kuvshinskii (1960), Toupin (1962), and Mindlin and Tiersten (1962) have considered elastic materials with torque-stress; and several authors have considered possible effects on stress concentration.
We assume that the interaction across any internal surface consists of equal, opposite, and collinear forces plus equal and opposite torques.
...

PRELIMINARY

27.01.01
Small Force|Torque and Small Deformations

This § is referenced at pages:
[1032]

In general, a generic infinitesimal deformation $\mathfrak{d}\xi$ of a generic systems will depend on the applied Force|Torque, $\mathbf{f}/\gamma$, and the linear/elastic regime is given by approximating the displacement to first-order.

Specific examples, for special simple case, are defined in equations (27.02.01), (27.02.02), (27.02.03), (27.02.04).

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27.02 Uniform and Isotropic Deformations of Homogeneous Isotropic Elastic Solids

This § is referenced at pages:
[1645]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.38||

A configuration is a set defining the positions of all particles of the body.

Deformation, in continuum mechanics, is the transformation of a body from a reference configuration to a current configuration.

Uniform and Isotropic deformations are such that, for every direction, every element of the solid shows the same relative change in length at all points, for any direction:

$$\frac{\Delta \ell_w}{\ell_w} = \text{the same at any point for any direction } w \quad .$$

As the small elastic deformations are linear, the **superposition principle** can be applied to determine the overall deformation by **summing the deformations induced by all the Forces|Torques**.

Two independent constants, out of the total of four basic constants which are usually defined (the Young modulus, the Poisson module, the shear modulus and the bulk modulus), completely describe the elastic properties of any LHI solid.

27.02.01 Change of Size of a Cylindrical Bar under longitudinal force

©|W.C.Elmore & M.A.Heald, , ..., ..., ...Ed., ...|3.1||

Consider a cylindrical bar made of LHI solid material, with rest length given by l_0, and cross-section, at rest, with surface S_l.

27.02.01.01 Elongation and Compression

Suppose that the deformation is small enough to treat it as a linear process.

If the bar undergoes either a traction or a compression, via the uniform application at its bases of a force F_l parallel to its axis, one can define the normal stress:

$$f_l \equiv \frac{F_l}{S_l} \quad .$$

By convention the external force F_l is positive for a force pulling the bar (a traction) while it is negative for a force pushing the bar (a compression):

$$F_l > 0 \quad \Leftrightarrow \quad \text{traction} \quad ,$$

$$F_l < 0 \quad \Leftrightarrow \quad \text{compression} \quad .$$

Note that this convention is the opposite of the one normally used in elementary treatments of the pressure, when the convention is that a positive pressure compresses the system.

The fractional stretch (stretch per unit length) is called the longitudinal strain:

$$\epsilon_l \equiv \frac{\Delta l}{l_0} \quad .$$

UNDER-PROGRESS

The Young modulus is defined as the ratio between the normal stress and the longitudinal strain:

$$f_l \equiv Y \epsilon_l \qquad \Leftrightarrow \qquad \frac{F_l}{S_l} \equiv Y \frac{\Delta l}{l_0} \quad . \tag{27.02.01}$$

→ 1488 1490 1497

The definition of the Young modulus allows one to introduce a coefficient of proportionality, the Young modulus, which does not depend on the dimensions of the block of material and it is thus a property of the material itself.

Equation (27.02.01), in case the Young modulus is a constant, is called Hooke law.

Heuristically it is clear that the elastic constant of the elementary Hooke law must be proportional to the cross-sectional area of the bar and inversely proportional to the length of the bar. Read § 27.07.01.

Hooke law is actually a constitutive equation for the material.

Young modulus must be non negative, read § 27.07.02.

$$Y \geq 0 \quad .$$

A perfectly rigid material is such that: $Y \to +\infty$, while $Y = 0$ means a structure-less material: any force can give any deformation.

27.02.01.02 Lateral Shrinking and Expansion

Suppose that the deformation is small enough to treat it as a linear process.

A change of length of a material, as just described, is accompanied, in general, by a change in the transverse dimensions. Let d be any transverse dimension of the bar.

The Poisson module, Σ, is defined as the negative ratio between the transverse strain, ϵ_d, and the longitudinal strain, ϵ_l:

$$\epsilon_d \equiv \frac{\Delta d}{d_0} \equiv -\Sigma \frac{\Delta l}{l_0} \equiv -\Sigma \epsilon_l = -\frac{\Sigma}{Y} \frac{F_l}{A_l} \quad . \tag{27.02.02}$$

→ 1488 1490

Equation (27.02.02), in case the Poisson modulus is a constant, is called Poisson law.

It can be shown that Poisson module is constrained to be

$$-1 \leq \Sigma \leq \frac{1}{2} \quad ,$$

read § 27-004.

No common material is known with a negative Poisson ratio (called auxetics) even if this is not forbidden by any law of Nature. However, complex materials with negative Poisson ratio do exist. One common example is Gore-TeX (see **WEB - URL**).

Note that Poisson law is actually a constitutive equation for the material.

A material such that $\Sigma = 0$ does not show any change in its lateral dimensions for a longitudinal traction/compression.

It can be shown that materials such that $\Sigma = 1/2$ may change shape but not volume, see equation (27.02.07) and § 27.07.06.

27.02.02
Plane Shear Distortions

This § is referenced at pages:
[1649]

©|W.C.Elmore & M.A.Heald, , ..., ..., ...Ed., ...|3.3||
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.38.2||

R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...- 2.38

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Read also section § 16.12.18, § 16.12.17.

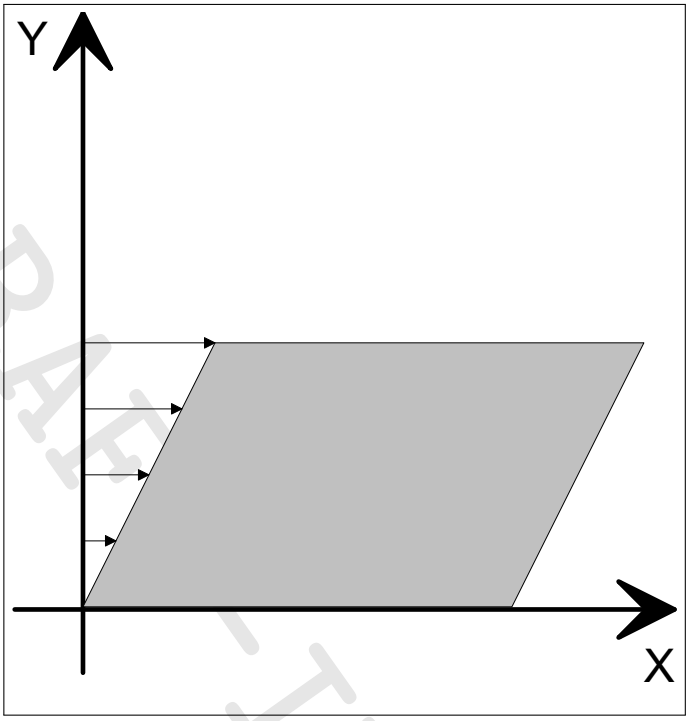

Figure 27.11: ShearStrain

Suppose that the deformation is small enough to treat it as a linear process.

Suppose to apply to a parallelepiped of LHI material a shearing stress, with forces along the $\pm x$ axis, causing a change of shape without change of volume, such that the two opposite faces where the forces are applied move parallel to themselves. All displacements of all the points are parallel each other. Any section, which at fixed z was a rectangle, deforms into a parallelogram.

The shear modulus is defined as the ratio between the shear stress and the shear strain, that is the displacement along x divided by the height of the parallelepiped, y_0:

$$\boxed{\frac{F_{xy}}{S_{xz}} \equiv \frac{F_{xy}}{x_0 z_0} \equiv g = G\theta = G\frac{\Delta x}{y_0}} \quad . \tag{27.02.03}$$

→ 1488

- F_{xy}: force along the x axis applied to a surface perpendicular to the y axis.
- S_{xz}: area of the element of surface perpendicular to the y axis.

As the deformation is small the angle can be approximated with the tangent.

The shear modulus must be positive or otherwise one could get work out of a self-shearing medium:

$$\boxed{G \geq 0} \quad .$$

27.02.03
Volume Changes

Suppose that the deformation is small enough to treat it as a linear process.

A fourth constant is usually introduced. It is the bulk modulus, B, defined in terms of the **volume change under a uniform pressure** all around the material:

$$\boxed{\frac{1}{B} \equiv -\frac{1}{V}\frac{\partial V}{\partial p}} \quad . \tag{27.02.04}$$

→ 1488

Note that this concept is only useful for uniform pressure, that is normal stress, which is typically the case for a fluid, but not necessarily for a solid.

It can be shown that the bulk modulus is necessarily non-negative:

$$\boxed{B \geq 0} \quad .$$

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A material such that $B \to \infty$ is incompressible.
Read § 67.05.02 for the extension of the definition in thermodynamics.

27.02.04
Constants of an Elastic Homogeneous and Isotropic Solid

This § is referenced at pages:
[1659, 1659, 2034, 2035]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.38||

For any LHI process the following relations apply:

- $$B = (2G(\Sigma+1))/(3(1-2\Sigma)) \quad ,$$
$$B = (GY)/(3(3G-Y)) \quad , \qquad (27.02.05)$$
$$B = (Y)/(3(1-2\Sigma)) \quad .$$
- $$G = (3BY)/(9B-Y) \quad ,$$
$$G = (3B(1-2\Sigma))/(2(\Sigma+1)) \quad ,$$
$$G = (Y)/(2(\Sigma+1)) \quad .$$
- $$\Sigma = (3B-2G)/(2(3B+G)) \quad ,$$
$$\Sigma = (3B-Y)/(6B) \quad ,$$
$$\Sigma = (Y-2G)/(2G) \quad .$$
- $$Y = 2G(\Sigma+1) \quad , \qquad (27.02.06)$$
$$Y = 3B(1-2\Sigma) \quad , \qquad (27.02.07)$$
$$Y = (9BG)/(3B+G) \quad .$$

→ 1667
→ 1490

showing that any pair out of the four constants Y, Σ, G and B can be used to fully describe the elastic properties of a homogeneous and isotropic solid.

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27.03 Properties of Materials

©|M.Ashby|Material property data for engineering materials|*WEB - URL* |

Density versus Young modulus: WEB - URL
For a typical behavior of a stress-strain relation of real materials see, for instance: WEB - URL

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27.04 General Non-Uniform Deformation in One Dimension

This § is referenced at pages:
[1502]

©|W.C.Elmore & M.A.Heald, , ..., ..., ...Ed., ...|3||

A more general approach is required to treat general non-uniform deformation in one dimension.

See figure 27.12. We designate the bar axis as the z axis and introduce the function $\xi[z]$ to measure the displacement with respect to the equilibrium position (or to any other predefined reference position) of the point which is at z in the equilibrium (or any other reference) position. Consider a small section of length Δz of the bar. It follows that:

$$\epsilon_z \equiv \frac{((z+\Delta z+\xi[z+\Delta z])-(z+\xi[z]))-\Delta z}{\Delta z} = \frac{\Delta\xi}{\Delta z} \longrightarrow \frac{\partial\xi}{\partial z} \quad .$$

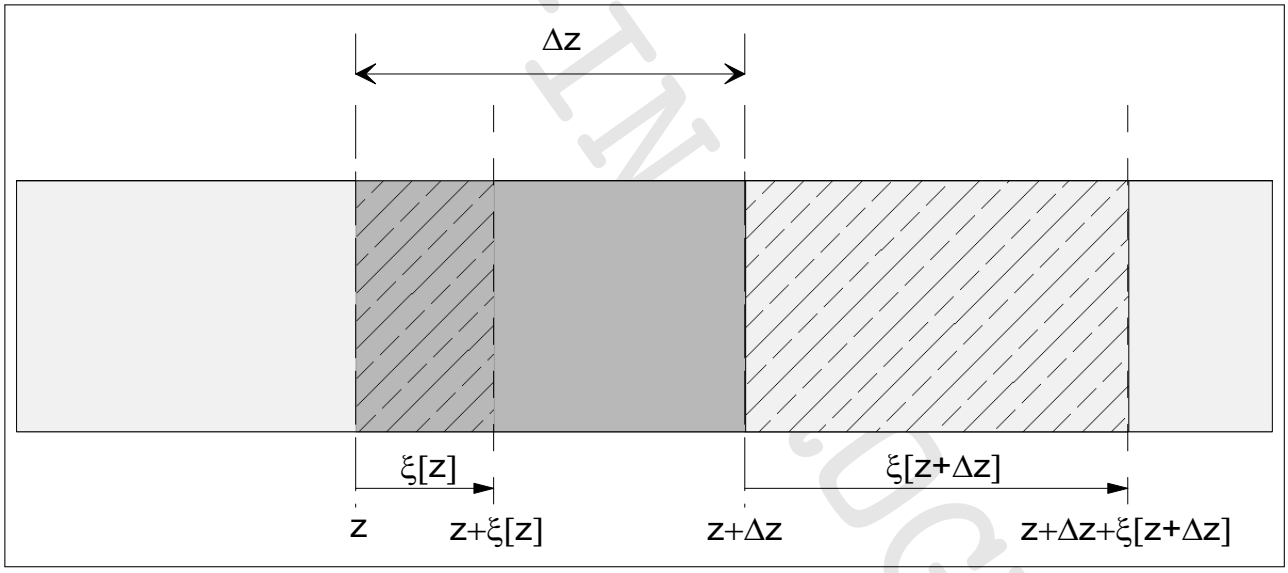

Figure 27.12: Kinematics of longitudinal waves

Read § 45.02 for the use in dynamics (mechanical waves).

PRELIMINARY PROGRESS

27.05 Statics of Continuous Media

Any continuous medium as a whole is in equilibrium if and only if every part of it is in equilibrium.

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27.06 Some Related Concepts From Thermodynamics

Strictly speaking, all the definitions of the four parameters should apply under some thermodynamical condition, such as: fixed temperature (isothermal); fixed entropy (iso-entropic), that is non exchange of heat..... All the parameters, a priori, depend on thermodynamical parameters, such as temperature.

27.06.01 Coefficienti Di Dilatazione Lineare Ed Espansione Volumetrica

Il coefficiente di dilatazione termica lineare è definito come

$$\boxed{\alpha \equiv \frac{1}{L}\left(\frac{\partial L}{\partial T}\right)_{\mathcal{X}} = \alpha[T,\mathcal{X}]} \quad .$$

Il coefficiente di espansione termica volumetrica è definito come

$$\boxed{\beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{\mathcal{X}} = \beta[T,\mathcal{X}]} \quad .$$

Nelle definizioni si assume che le eventuali altre variabili, $\mathcal{X}$, da cui dipende lo stato del sistema, oltre alla temperatura, siano tenute fisse. In generale i coefficienti così definiti dipendono quindi anche da tali variabili.

Per una sostanza isotropa il coefficiente di dilatazione termica lineare non dipende dalla direzione. In tal caso si ha

$$\beta = 3\alpha \quad .$$

In generale per una sostanza non isotropa, detti α_1, α_2 ed α_3 i coefficienti di dilatazione lineare lungo tre assi ortogonali si ha:

$$\boxed{\beta = \alpha_1 + \alpha_2 + \alpha_3} \quad .$$

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27.07 Examples and Physical Applications

27.07.01 Definition of the Young Modulus

This § is referenced at pages:
[1490]

Show how the Young module definition, for a homogeneous and isotropic material, makes the definition independent of the geometry of the block.

WORKED SOLUTION

The definition of the Young modulus for a homogeneous and isotropic material, equation (27.02.01), is motivated by two facts. Start from Hooke law:

$$F = k\Delta l \quad .$$

First of all one expects the elongation Δl to be proportional to the length of the bar. This can be understood by imagining a single bar divided in two parts along its length. If the bar is in equilibrium the stress is the same at any section but the elongation of any of the two parts will be proportional to its length at rest. In fact:

$$\Delta l = \Delta l_1 + \Delta l_2 = F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \equiv \frac{F}{k} \quad .$$

One has that doubling the (rest) length of the bar the elastic constant is halved:

$$\left\{ \Delta l_1^{(0)} = \Delta l_2^{(0)} = \frac{\Delta l^{(0)}}{2} \equiv \Delta l_{1/2}^{(0)} \qquad k = \frac{k_{1/2}}{2} \right\} \qquad \Longrightarrow \qquad \boxed{k \propto \frac{1}{\Delta l^{(0)}}} \quad .$$

Secondly one expects that the elongation Δl will be inversely proportional to the section of the bar. This can be understood by imagining a single bar divided longitudinally in two identical parts. In order to obtain a certain Δl the applied force has to be the sum of the two forces to be applied to each of the two longitudinal sections. Therefore, proceeding as in the previous case,

$$F = F_1 + F_2 = (k_1 + k_2)\,\Delta l \equiv k\Delta l \qquad S = S_1 + S_2 \quad .$$

One has that doubling the cross-sectional area of the bar the elastic constant is doubled:

$$\left\{ S_1 = S_2 = \frac{S}{2} \equiv S_{1/2} \qquad k = 2k_{1/2} \right\} \qquad \Longrightarrow \qquad \boxed{k \propto S} \quad .$$

The Young module is thus related to the elastic constant by the equation:

$$\boxed{k \equiv Y\frac{S}{l^{(0)}}} \quad .$$

27.07.02
Positivity of the Young Modulus

This § is referenced at pages:
[1490, 1498]

Show that the Young modulus must be positive.

WORKED SOLUTION

Suppose that the Young modulus is negative, by absurd.
Consider a bar of material immersed in a gas at constant pressure enclosed in a container with rigid walls. Due to random fluctuations there will be instants when the pressure on the two opposite sides of the base increases and the normal force, due to the gas pressure, acting on the two bases would then become more negative than at equilibrium, tending to compress the bar. The length of the bar would increase, as $Y < 0$, compressing the surrounding gas and thus increasing its pressure once more. The increase of pressure of the gas would increase the force on the bases of the bar which would expand and so on: the situation would be unstable.

27.07.03
Positivity of the Bulk Modulus

This § is referenced at pages:
[Never referenced.]

As in section § 27.07.02.

27.07.04
Positivity of the Shear Modulus

Show that the shear modulus must be positive.

WORKED SOLUTION

Consider, for instance, a block, kept, from one extreme, in the horizontal direction in a gravity field (a cantilevered beam, for instance) with a mass hanging at the free end. Suppose that the shear modulus is negative. In this case the beam would deform rising its free end in the gravity field. This behavior would make any solid object unstable under random fluctuation in its shear deformations.

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27.07.05
Examples of shears

1. Shearing along the x axis:

$$\begin{pmatrix} 1 & \tan(\theta) \\ 0 & 1 \end{pmatrix} \tag{27.07.01}$$

2. Shearing along the y axis:

$$\begin{pmatrix} 1 & 0 \\ \tan(\theta) & 1 \end{pmatrix} \tag{27.07.02}$$

3. Shearing in the $x-y$ plane along the x axis:

$$\begin{pmatrix} 1 & 0 & \tan(\theta) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{27.07.03}$$

4. Shearing the $x-z$ plane along the x axis:

$$\begin{pmatrix} 1 & \tan(\theta) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{27.07.04}$$

5. A shearing by angle θ in the $1,1$ direction in the line $1,-1\cdot p == 0$:

$$\begin{pmatrix} \frac{1}{2}(\tan(\theta)+2) & -\frac{\tan(\theta)}{2} \\ \frac{\tan(\theta)}{2} & \frac{1}{2}(2-\tan(\theta)) \end{pmatrix} \tag{27.07.05}$$

27.07.06
A Block under uniform Pressure

This § is referenced at pages:
[1490, 1659, 2057]

R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...- 2.38.2

The two constants Y and Σ completely define the elastic properties of any homogeneous and isotropic solid. The relation between Y, Σ and B is derived.

Materials are subject to uniform stresses when the stresses are uniform across the medium. For arbitrary stresses the **superposition principle** can be applied as elasticity theory deals with linear systems.

Apply a uniform change of pressure, Δp, to a parallelepiped block, starting from some equilibrium condition. The stress is then uniform. By applying the **superposition principle** we have that any pair of opposite faces is subject to the elongation|compression, as described by the Young modulus, and by shrinking, due to the elongation|compression on the two other pairs of opposite faces. Consider in fact each of the three axes: the elongation|compression along any of them is caused by the elongation|compression on the two opposite faces perpendicular to the axis plus the effect of the Poisson shrinking on the other two pairs of opposite faces. The problem is obviously symmetrical in all the three faces.

Fix one of the three directions, for instance z. Let the l_z variable denote the lengths and change of lengths along the chosen fixed direction caused by stresses on the two faces perpendicular to the chosen

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direction. Let the d_x and d_y variables denote the lengths and change of lengths, along the same fixed direction z, caused by the stresses on the two faces perpendicular to the other two directions, x and y. Then one has, for the longitudinal strain along the z direction:

$$\epsilon_z \equiv \frac{\Delta l_z}{l_z} + \frac{\Delta d_x}{d_x} + \frac{\Delta d_y}{d_y} = -\frac{\Delta p}{Y}(1-2\Sigma) \quad . \tag{27.07.06}$$

→ 1500

Note that the result does not depend on the chosen fixed direction z.

The volume change follows:

$$\boxed{\frac{\Delta V}{V_0} = \epsilon_x + \epsilon_y + \epsilon_z = -3\frac{\Delta p}{Y}(1-2\Sigma)} \quad . \tag{27.07.07}$$

→ 1500

From equation (27.07.07) the upper limit of the Poisson module follows:

$$\Sigma \leq \frac{1}{2} \quad . \tag{27.07.08}$$

→ 1510

An incompressible solid is therefore characterized by a value of the Poisson module equal to $1/2$.

27.07.07
A Constrained Bar

This § is referenced at pages:
[2034]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.38.2||

This problem involves non isotropic deformations.

Consider a cylindrical bar with rectangular cross-section which is compressed/elongated along one of the directions of its sides (call it the longitudinal direction) while its sides are prevented from expanding/contracting. Note that on the practical side it is technically easier to setup the problem in such a way that the bar is compressed while its sides are prevented from expanding.

Calculate the relation between the longitudinal stress and the longitudinal strain under these conditions.

WORKED SOLUTION

To avoid the expansion/contraction of the sides we need some kind of forces to prevent the sides from contracting. The magnitude of these forces needs to be calculated.

Let z be the longitudinal axis and let the sides of the rectangular cross-section be parallel to the x and y axes. Let F_x, F_y and F_z be the external forces applied to the system along the x, y and z axes. Thanks to the assumed linearity for small deformations we can sum the changes in length due to all the forces. Following equation (27.07.06), we then have:

$$\epsilon_z \equiv \frac{\Delta l_z}{l_z} = \frac{1}{Y}\frac{F_z}{A_z} - \frac{\Sigma}{Y}\left(\frac{F_x}{A_x} + \frac{F_y}{A_y}\right) = \frac{1}{Y}\left(\frac{F_z}{A_z} - \Sigma\left(\frac{F_x}{A_x} + \frac{F_y}{A_y}\right)\right) \equiv \frac{1}{Y}(\sigma_z - \Sigma(\sigma_x + \sigma_y)) \quad , \tag{27.07.09}$$

→ 1501

$$\epsilon_x \equiv \frac{\Delta l_x}{l_x} = \frac{1}{Y}\frac{F_x}{A_x} - \frac{\Sigma}{Y}\left(\frac{F_y}{A_y} + \frac{F_z}{A_z}\right) = \frac{1}{Y}\left(\frac{F_x}{A_x} - \Sigma\left(\frac{F_y}{A_y} + \frac{F_z}{A_z}\right)\right) \equiv \frac{1}{Y}(\sigma_x - \Sigma(\sigma_y + \sigma_z)) \quad , \tag{27.07.10}$$

$$\epsilon_y \equiv \frac{\Delta l_y}{l_y} = \frac{1}{Y}\frac{F_y}{A_y} - \frac{\Sigma}{Y}\left(\frac{F_z}{A_z} + \frac{F_x}{A_x}\right) = \frac{1}{Y}\left(\frac{F_y}{A_y} - \Sigma\left(\frac{F_z}{A_z} + \frac{F_x}{A_x}\right)\right) \equiv \frac{1}{Y}(\sigma_y - \Sigma(\sigma_z + \sigma_x)) \quad . \tag{27.07.11}$$

It is assumed that the sides are kept from changing their direction by suitable forces. Therefore we

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must have:

$$\epsilon_x = \epsilon_y = 0 \quad .$$

We are left with three equations with four unknowns: σ_x, σ_y, σ_z, ϵ_z. This implies a well-defined relation among F_z, F_x and F_y:

$$\sigma_x = \sigma_y = \frac{\Sigma}{1-\Sigma}\sigma_z \qquad \left\{ \frac{\Sigma}{1-\Sigma} > 0 \qquad \Leftrightarrow \qquad \Sigma > 0 \right\} \quad .$$

The previous relation correctly implies that if the force F_z is a traction force then also the forces F_x and F_y must be positive, that is a traction must be exerted on the lateral sides to keep it from contracting as well if and only if $\Sigma > 0$.
By substituting back into the above expression (27.07.09) for $\Delta l_z / l_z$:

$$\epsilon_z = \frac{1}{Y}\left(\frac{1-\Sigma-2\Sigma^2}{1-\Sigma}\right)\sigma_z \quad ,$$

$$\sigma_z = \left(\frac{1-\Sigma}{(1+\Sigma)(1-2\Sigma)}\right) Y \frac{\Delta l_z}{l_z} \quad .$$

That is when the sides are constrained the Young modulus gets multiplied by a complex factor of the Poisson module. The factor is always larger than one, and therefore the block becomes stronger in case the sides are constrained. This factor acts in fact as a effective Young modulus for a constrained bar.

Refer to § 45.04.01 for the general longitudinal motion, where the same factor $\left(\frac{1-\Sigma}{(1+\Sigma)(1-2\Sigma)}\right)$ as above shows up.

27.07.08
The Ideal String|Bar

This § is referenced at pages:
[1502, 1991, 1994]

An ideal String|Bar is, by definition, inextensible (that is its length never changes) with no mass (that is mass negligible with respect to other masses in the problem) and perfectly flexible.

- **Inextensible**: the kinematic constraint that the distance between any two points along the string never changes is effective.
- **Massless**: the tension along the string does not change, as the mass is assumed to be zero.
- **Perfectly flexible**: changing the curvature and|or the torsion of the string at any point requires no work.

In practice, the above characteristics of the ideal string are often just approximated.

PROGRESS

27.07.09
An Inextensible Massive String|Bar Hanging in the Gravity Field

This § is referenced at pages:
[1455, 1502]

Consider an ideal string (read § 27.07.08) of length L fixed to the ceiling and with a small mass M at the free extreme. Find the tension along the string as a function of the position. Note that, in fact, the problem would have the same solution if the string were replaced by a rigid cylindrical bar.

Determine the equation of equilibrium assuming no strain and no lateral change of dimensions.

WORKED SOLUTION

Introduce a vertical z axis, oriented downwards, with origin at the point where the string is hanging. Let $T[z]$ be the force that the piece of String|Bar before z exerts on the piece after z. The first **Cardinal Equation** for the piece of String|Bar which, in the unstretched String|Bar, is between z and $z+\Delta z$ reads:

$$(T[z]-T[z+\Delta z])+\rho S\Delta z g=0 \quad ,$$

$$\frac{\mathrm{d}T}{\mathrm{d}z}=\rho S g>0 \quad ,$$

$$T[L]=-Mg \quad ,$$

$$T[z]=mg\left(\frac{z}{L}-1\right)-Mg \quad .$$

We expect the tension to be negative and increasing in value, while deceasing in module when z increases. However, the tension is the unknown to be determined for this problem.

In fact, it turns out that the tension has maximum absolute value at $z=0$, where the derivative is maximum as it has to sustain the mass and the proper mass o the String|Bar.

Note that the tension is not zero at $z=L$, as the mass M must be sustained,

Compare with problem § 27.07.10: in this probllem the partial derivative of ξ with respect to z is zero.

The condition of constant tension, typically used in elementary problem, is related to the mass of the String|Bar, via the tension one would calculate neglecting the mass of the String|Bar, $|T[L]|$:

$$\frac{|T[0]-T[L]|}{|T[L]|}=\frac{m}{M}\ll 1 \quad .$$

27.07.10
A LHI String|Bar Statically Hanging in the Gravity Field

This § is referenced at pages:
[1502, 2049]

Read § 45.12.06 for the dynamics.

Consider an elastic String|Bar, made of an homogeneous and isotropic material, hanging in the gravity field from one extreme. The rest length is L, its cross-sectional area is S, the sound speed is c and its total mass m; a mass M is hanging on the String|Bar.

Determine the configuration of equilibrium assuming small strain and no lateral change of dimensions.

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WORKED SOLUTION

Note: the **reference configuration is not, here, equilibrium; it is equilibrium when there are no external forces on the String|Bar, that is the un-stretched String|Bar.**
Introduce a vertical z axis, oriented downwards, with origin at the point where the String|Bar is hanging.
Read § 27.04 for the general setup of the problem and let $\xi[z]$ be the displacement with respect to the equilibrium condition of the parts of the string at z. Let $T[z^\star]$ the force, positive when directed along the z axis, that any part at $z < z^\star$ exerts on the part at $z > z^\star$.

Elementary solution

From the first **Cardinal Equation** applied to a small disk of thickness Δz one has:

$$(T[z] - T[z + \Delta z]) + \rho A g \Delta z = 0 \quad ,$$

$$\frac{\mathrm{d}T}{\mathrm{d}z}[z] = \rho A g \quad ,$$

$$T[z] = \rho A g z - g\,(m + M) = \rho A g\,(z - L) - M g = -Y A \frac{\mathrm{d}\xi}{\mathrm{d}z} < 0 \quad .$$

The solution is formally the same as in section § 27.07.09 but the meaning is quite different. In fact for the inextensible string z correspond in any case to the position along the string, while for the extensible string z denotes the position in the equilibrium condition while the real position changes after deformation.

General solution from d'Alembert (classical) wave equation

The equation is equation (45.02.03). Therefore, the equation is:

$$\frac{\partial^2 \xi}{\partial t^2}[z,t] - c^2 \frac{\partial^2 \xi}{\partial z^2}[z,t] = \frac{f_{\mathrm{v}}}{\rho_0} = g \qquad c^2 = \frac{Y}{\rho} \quad .$$

At equilibrium the acceleration is zero. Therefore:

$$\frac{\partial^2 \xi}{\partial t^2}[z,t] = 0 \implies -c^2 \frac{\partial^2 \xi}{\partial z^2}[z,t] = g \quad .$$

The general solution, in terms of two integration constants, β_0 and β_1, is:

$$\xi[z,t] \longrightarrow \xi[z] = -\frac{g}{2c^2} z^2 + \beta_1 z + \beta_0 \quad .$$

Assume $M = 0$. The boundary conditions for the specific problem are:

- the String|Bar has its fixed extreme attached to a fixed point, therefore it cannot undergo any displacement: $\xi[z = 0] = 0$;
- the free extreme of the String|Bar, being free, does not exert any force, its stress is null, because it has nothing to sustain: $\dfrac{\partial \xi}{\partial z}[z = L] = 0$

It follows:

$$\xi[z] = -\frac{g}{2c^2} z^2 + \frac{gL}{c^2} z \quad .$$

The plot of $\xi[z]$ is a parabola whose vertex is a maximum located at $z = L$ and passing by the origin at $z = 0$. The displacement is maximum at $z = L$: $\xi[z = L] = \dfrac{gL^2}{2c^2}$.
The stress along the String|Bar is:

$$S_{zz}[z] = \frac{F_\ominus}{S} = -Y \frac{\partial \xi}{\partial z} = \frac{gY}{c^2}\,(z - L) < 0 \quad .$$

Its maximum value is:

$$S_{zz}[z = 0] = -gL \frac{Y}{c^2} = -gL\rho = -\frac{mg}{S} \quad ,$$

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The stress, that is the first derivative, $\frac{\partial \xi}{\partial z}$, is maximum at $z = 0$, because there the String|Bar must support the weight of whole String|Bar. That is also the most likely point where the String|Bar might break.
Read § 45.12.06 for $M \neq 0$.

27.07.11
A Fixed Horizontal Cylindrical Rigid Cantilevered Beam

This § is referenced at pages:
[1455, 1455, 1508]

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Consider a fixed horizontal cylindrical rigid cantilevered beam. Determine the distribution along the beam of forces and torques assuming that the beam is exactly rigid, that is its geometrical shape is not altered.

Consider the same problem with an additional vertical downward force, $\mathbf{F}_0$, applied at $x = h$.

WORKED SOLUTION

When the entire beam is in equilibrium, any portion of it must be in equilibrium as well.
Let the x axis be along the beam, exiting the wall with origin at the point where the beam enters the wall.
Assume the beam is perfectly rigid, so that any deformation is negligible. Let the uniform cross-sectional area of the beam be A and let ρ be the mass density. Let M be the total mass and L be the length of the beam.
Consider any section perpendicular to the beam axis, at x_0: $\mathcal{S}[x_0]$. The system of forces that the part of the beam at $x < x_0$ applies on the part at $x > x_0$ can be reduced to a total force applied at any point on $\mathcal{S}[x_0]$, $\mathbf{F}[x_0]$, plus the total torque, with pole at the point where the resultant force is applied, $\mathbf{\Gamma}[x_0]$. Once the point of application of the total force is chosen and the resultant external torque with respect to the pole is calculated, the pole used is not relevant, as the torque is independent of the pole. Therefore we will omit to mention that the torques are evaluated with respect to a pole, O, and just write: $\mathbf{\Gamma} \equiv \mathbf{\Gamma}_0$.
All the forces and torques we will use are those applied by the part of the beam at $x < x_0$ on the part at $x > x_0$. These are often named $\mathbf{F}^-[x_0]$ and $\mathbf{\Gamma}^-[x_0]$. The forces applied by the part of the beam at $x > x_0$ on the part at $x < x_0$ are often named: These are often named $\mathbf{F}^+[x_0]$ and $\mathbf{\Gamma}^+[x_0]$.
Both $\mathbf{F}[x_0]$ and $\mathbf{\Gamma}[x_0]$ can be decomposed into their components perpendicular to $\mathcal{S}[x_0]$ and parallel to $\mathcal{S}[x_0]$:

- the component of $\mathbf{F}[x_0]$ perpendicular to $\mathcal{S}[x_0]$, F_x, is the normal stress (also known as pressure);
- the component of $\mathbf{F}[x_0]$ parallel to $\mathcal{S}[x_0]$, $\mathbf{F}_\perp$, is the shear stress;
- the component of $\mathbf{\Gamma}[x_0]$ perpendicular to $\mathcal{S}[x_0]$, Γ_x, is the twisting moment (torsional moment due to torsional stress);
- the component of $\mathbf{\Gamma}[x_0]$ parallel to $\mathcal{S}[x_0]$, $\mathbf{\Gamma}_\perp$, is the bending moment.

We can write the **Cardinal Equations** along the x direction. let the y axis oriented vertically up.
One should note that $F_\perp[x]$ is required to balance the weight and $\Gamma_\perp$ is required to balance the bending action of the weight. Therefore $F_\perp[x]$ will be along the vertical direction (y axis) while $\Gamma_\perp$ will be along the horizontal direction perpendicular to x, that is along the z axis.
The first and second **Cardinal Equations** applied to a small cylinder of the beam of length Δx,

taking into account that the generalized third Newton law implies:

$$\mathbf{F}[x_0] \equiv \mathbf{F}^-[x_0] = -\mathbf{F}^+[x_0] \qquad \mathbf{\Gamma}[x_0] \equiv \mathbf{\Gamma}^-[x_0] = -\mathbf{\Gamma}^+[x_0]$$

read:

$$A\,(F[x] - F[x+\Delta x]) - \rho A \Delta x g = 0 \qquad (27.07.12)$$

→ 1505

$$A\,(\Gamma[x] - \Gamma[x+\Delta x]) - \rho A \Delta x g \frac{\Delta x}{2} - AF[x+\Delta x]\Delta x = 0 \qquad \text{pole at } x \quad .$$

→ 1505

Note that the first two terms of the second **Cardinal Equation** above do not depend on the choice of the pole, while the last two terms do depend on it. It is easy to check that the result would not change by choosing the pole, for instance, at $x + \Delta x$. The limit for $\Delta x \longrightarrow 0$ gives:

$$\frac{\mathrm{d}F}{\mathrm{d}x} = -\rho[x]g \quad ,$$

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}x} = -F[x] \quad .$$

At the free end, both F and Γ must vanish.
At the clamped end, $x = 0$, both F and Γ must have the value required to support the full beam:

$$F[x=0] = Mg \qquad \Gamma[x=0] = MgL/2$$

The solution of the system gives:

$$F[x] = -\rho g A x + F[x=0] \implies F[x] = \frac{Mg}{L}(L-x) \geq 0 \quad ,$$

$$\Gamma[x] = \frac{Mg}{2L}(x-L)^2 + \Gamma[x=L] \implies \Gamma[x] = \frac{Mg}{2L}(x-L)^2 \geq 0 \quad .$$

Note that, in case the beam is sustained at the second extreme too, at $z = L$, the problem of a rigid beam would be, a priori, under-determined, even if a reasonable assumption, in some cases, might be the equality of the forces of the two supports.
Consider the case when the additional force is applied. The **Cardinal Equations** of motion (27.07.12), (27.07.11) become, in terms of the generic external vertical force per unit volume $w[x]$:

$$\boxed{\frac{\mathrm{d}F}{\mathrm{d}x} = -w[x] \qquad \frac{\mathrm{d}\Gamma}{\mathrm{d}x} = -F[x]} \quad ,$$

which implies

$$\boxed{\frac{d^2\Gamma[x]}{\mathrm{d}x^2} = w[x]} \quad .$$

In case of a discrete force, applied at $x = h$, the shear stress has a discontinuity at $x = h$ of amplitude F_0 which translates into a non derivable bending moment at $x = h$ (first specie discontinuity of its first derivative).

27.07.12
Torsion of a Cylindrical Bar

This § is referenced at pages:
[1666, 1666]

R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...- 2.38.3

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Consider a cylindrical bar, made of an homogeneous and isotropic material, having length L and a uniform circular section of radius R. The bar is in static equilibrium under a uniform torsion by an angle ϕ, kept by a suitable set of external forces. Calculate the required torsional torque along the bar as a function of the torsion angle ϕ.

WORKED SOLUTION

Let the z axis be coincident with the axis of the cylinder, let its origin be located at one of the ends and let its orientation be such that the bar is defined by $0 \leq z \leq L$.
Just consider the equilibrium of the bar under torsion along and around its axis, the z axis. That is assume that a suitable set of forces exist to guarantee both the translational equilibrium and the rotational equilibrium around the x and y axes which are outside the interests of the present problem. The equilibrium condition is thus driven by the components of the torques along the z axis only. Only the components of the forces perpendicular to the z axis contribute to the components of the torques along the z axis.
Consider any section $\mathcal{S}[z_0]$ of the bar defined by a plane perpendicular to the z axis at the position z_0. Let $\Gamma_O^-[z_0]$ be the component along z of the total torque with respect to the point $O = \{0, 0, z_0\}$ (the point of $\mathcal{S}[z_0]$ lying on the z axis) of all the forces that the part of the bar at $z < z_0$ applies to the part of the bar at $z > z_0$. Moreover, let $\Gamma_O^+[z_0]$ be the component along z of the total torque with respect to the point $O = \{0, 0, z_0\}$ (the point of $\mathcal{S}[z_0]$ lying on the z axis) of all the forces that the part of the bar at $z > z_0$ applies to the part of the bar at $z < z_0$. The forces involved are applied at the section $z = z_0$ separating the two parts (they are contact forces) and, as it was already said, only the components of those forces perpendicular to the z axis contribute to $\Gamma_O[z_0]$.
At equilibrium, for any section $\mathcal{S}[z_0]$, we must have:

$$\Gamma_O^-[z_0] + \Gamma_O^+[z_0] = 0 \quad ,$$

because the two parts apply equal and opposite torques on each other.
Consider any slice of the bar defined by the two sections $\mathcal{S}[z_0]$ and $\mathcal{S}[z_0 + \Delta z]$, perpendicular to the axis of the bar and separated by a distance Δz. At equilibrium, for any pair of sections $\mathcal{S}[z_0]$ and $\mathcal{S}[z_0 + \Delta z]$, we must have:

$$\Gamma_O^-[z_0] + \Gamma_O^+[z_0 + \Delta z] = 0 \quad ,$$

because any slice of the bar is in equilibrium.
It follows that:

$$\boxed{\Gamma_O^+[z_0] = \Gamma_O^+[z_0 + \Delta z] \qquad \Gamma_O^-[z_0] = \Gamma_O^-[z_0 + \Delta z]} \quad ,$$

that is both $\Gamma_O^-[z]$ and $\Gamma_O^+[z]$ do not depend on z:

$$\boxed{\frac{\mathrm{d}\Gamma_O^-[z]}{\mathrm{d}z} = \frac{\mathrm{d}\Gamma_O^+[z]}{\mathrm{d}z} = 0} \quad .$$

One can therefore calculate either $\Gamma_O^-[z]$ or $\Gamma_O^+[z]$ at any section $\mathcal{S}[z]$.
In order to calculate $\Gamma_O^-[z]$ consider the torsion of a small slice of length Δz, located between z and $z + \Delta z$. Assume, for more generality, that the cylindrical bar is a hollow bar with internal radius r_1 and external radius r_2.
Introduce polar coordinates, $\{r, \theta\}$, on the circular basis of the bar. Consider an infinitesimal element of the small slice of the bar spanning an angle $\mathrm{d}\theta$, with a radial span $\mathrm{d}r$ and height Δz. This element, when considered as an infinitesimal element, is actually an infinitesimal parallelepiped.
Let $\mathrm{d}\mathbf{F}_\theta$ the infinitesimal shear force acting on an infinitesimal element of the basis of the small slice, spanning an angle $\mathrm{d}\theta$ and with a radial span $\mathrm{d}r$. Only the tangential component of the shear force on the basis contributes to $\Gamma_O^-[z]$. Radial components, if any, would contribute to the compression or expansion of the bar in the transverse direction.
The infinitesimal parallelepiped is deformed, as a consequence of the shear forces on its infinitesimal

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bases. Its deformation angle, γ, is linked to the total angle of rotation of the bar, ϕ by the relation:

$$\gamma = \frac{r\phi}{L} \quad .$$

In fact, for a given angle ϕ, the deformation angle of the infinitesimal parallelepiped depends linearly on its distance from the axis. The tangential component of the shear stress, f_θ, is thus:

$$\mathrm{d}F_\theta = f_\theta \, (r \, \mathrm{d}r \, \mathrm{d}\theta) = G\gamma \, (r \, \mathrm{d}r \, \mathrm{d}\theta) \quad .$$

Note that the radial component of the shear stress will just try to cause either a lateral expansion of contraction of the bar.
The value of $\Gamma_O^-[z]$, the component along z of the total torque with respect to the point O of the external shear forces on the section $\mathcal{S}[z]$ acting on the slice and caused by the part of the bar at the left of z, is finally given by the integral:

$$\Gamma_O^-[z] = \int\limits_{\text{BASE}} \mathrm{d}\Gamma_O^- = \int\limits_{\text{BASE}} r \, \mathrm{d}F_\theta = \int\limits_{r=r_1}^{r_2} \int\limits_{\theta=0}^{2\pi} (r f_\theta) \, r \, \mathrm{d}r \, \mathrm{d}\theta = \int\limits_{r=r_1}^{r_2} \int\limits_{\theta=0}^{2\pi} G \left(r \frac{r\phi}{L} \right) r \, \mathrm{d}r \, \mathrm{d}\theta = \quad , \qquad \rightarrow 1507$$

$$\int\limits_{r=r_1}^{r_2} \int\limits_{\theta=0}^{2\pi} G \left(\frac{\phi}{L} \right) r^3 \, \mathrm{d}r \, \mathrm{d}\theta = \frac{\pi G}{2L} \left(r_2^4 - r_1^4 \right) \phi \quad ,$$

where the z dependence has been dropped because there is no dependence on z.
Note that in fact equation (27.07.12) just applies to the absolute values. In fact the full relation must express the fact that, as long as $\phi[z=0] = 0$ and $\phi[z=L] \equiv \Delta\phi$,

$$\boxed{\Gamma_O^-[z] = -\Gamma_O^+[z] - \frac{\pi G}{2L} \left(r_2^4 - r_1^4 \right) \Delta\phi} \quad . \qquad \rightarrow 1507\ 1510\ 1666\ 1667$$

In fact the positive orientation for both the angle $\Delta\phi$ and $\Gamma_O^-[z]$ are chosen to be the same.
It should be emphasised that all the internal stresses between shells cancel out, as far as the equilibrium of the bar is concerned, in agreement with the **Cardinal Equations** .
Note also that the rotational equilibrium is guaranteed by the fact that a pair of equal and opposite torques act on the two basis of any slice of the bar, due to the neighboring material. Therefore the torque on one basis of the shell gives the uniform torque along any section of the bar, equal, in the end, to the required external torque. Any external torque applied to any of the extremes of the bar has an absolute value equal, at equilibrium, to the internal torques $\Gamma_O^-[z]$ and $\Gamma_O^+[z]$.
The coefficient of ϕ is the torsional constant of the bar:

$$\Gamma_O^-[z] = \chi\phi \quad .$$

27.07.13
Torsional Constant and Shear Modulus

See equation (27.07.12)

$$\boxed{k = \frac{\pi G r^4}{2L}} \quad .$$

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27.07.14
Bending of a Simple Beam

R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...- 38.4

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DRAFT

Consider a cylindrical beam whose length is large with respect to its section. Assume that the bending radius of the beam is much larger than the transverse dimensions of the beam (its thickness). Assume the beam is bent in such a way that its deformed shape still is a planar shape. Furthermore assume that the beam is subject to simple bending, that is no stretching nor compression is applied at the two extremes (obviously at the internal points there will be stretching/compression). It is known that there exists a neutral surface all along the beam, whose points are not stretched nor compressed. Determine the deflection of the beam if it is cantilevered and a weight W is hanging at its free extreme. Assume the weight W is much higher than the weight of the beam itself.

WORKED SOLUTION

It is quite obvious that a neutral surface exists because the upper part of the beam will be stretched, while the lower part will be compressed. What is not self-evident is where the neutral surface lies. Small local curvature will be considered because in any case the linear theory of elasticity would not be applicable for large deformations. Note, however, that small local curvatures on a long beam might result in large deflections anyway. This does not contradict the necessity to stay in the linear regime, because locally the deformations will be small.
For a pure bending a slice of the beam is deformed in such a way that the material above/below the neutral surface has a stretching/compression which is proportional to the distance from the neutral surface the constant of proportionality being the radius of curvature:

$$\frac{\Delta l}{l_0} = \frac{y}{R[x]} \quad .$$

As a consequence the normal stress in a small strip at a distance y from the neutral surface is given by:

$$f = \frac{F}{A} = Y\frac{y}{R[x]} \qquad \Longrightarrow \qquad \mathrm{d}F = \frac{Y}{R[x]} y \, \mathrm{d}\mathbf{S} \quad .$$

This amount to assume that the bending is due only to the normal internal stresses.
The torque of the bending force with respect to any point P on the neutral surface is:

$$\Gamma_P[x] = \int\limits_{\text{SECTION}} y \, \mathrm{d}F = \frac{Y}{R[x]} \int\limits_{\text{SECTION}} y^2 \, \mathrm{d}\mathbf{S} \frac{YI}{R[x]} \quad .$$

Therefore the torque acting at any section of the beam is proportional to the moment of inertia, I, of the geometric section about an horizontal axis through its neutral surface.
In other words the stiffness of the beam is proportional to Y and to I. That is why for a given mass one tries to increase I as much as possible by designing **I** shaped beams. Of course one cannot go too much to the extreme along this way because the beam would twist and buckle. The formula shows that the product YI gives the strength of the beam, because when the factor YI increases/decreases for the same $R[x]$ the torque Γ_P increases/decreases. Moreover, for any given mass the value of Γ_P is determined by the value required at the fixed extreme, which has to sustain all the beam. However, Γ_P is not uniform, if the mass of the beam is not negligible.
Let x denote the coordinate along the beam, when it is not bent. Let $z[x]$ be the shape of the bent

beam. We have:

$$\frac{1}{R[x]} = \frac{\frac{d^2 z}{\mathrm{d}x^2}}{\left(1+\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\right)^{3/2}} \simeq \frac{d^2 z}{\mathrm{d}x^2} \quad .$$

One needs to know the bending moment as a function of x. Neglecting the weight of the beam the torque at x is:

$$\Gamma_P[x] = W\,(L-x) \quad ,$$

because the part of beam in between x and L is in static equilibrium and thus it just transfers to the element at x the torque at its boundary, because the mass of the beam is negligible with respect to the hanging weight. Consider in fact the part of beam located between x and L. That system is in equilibrium under the effect of the torque $\Gamma_P[x] = W\,(L-x)$, applied at the free end, and the internal moment due to the internal normal forces at the section at x (always use the point P on the neutral surface at x as a pole).
Note that the equation $\Gamma_P[x] = W\,(L-x)$ actually fixes the condition at the free extreme by saying that $\Gamma_P[x=L] = 0$.
It follows:

$$\frac{d^2 z}{\mathrm{d}x^2} = \frac{W}{YI}(L-x) \quad ,$$

$$z[x=0] = 0 \qquad \frac{\mathrm{d}z}{\mathrm{d}x}[x=0] = 0 \quad ,$$

$$z[x] = \frac{W}{YI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \qquad z[x=L] = \frac{W}{YI}\frac{L^3}{3} \quad .$$

Note that the function $z[x]$ describes the geometrical position of the neutral surface. As the mass of the beam is negligible with respect to the hanging weight, only W enters the equation. If the Young modulus increases one finds, in the limit of very large Y, the rigid cantilevered beam is found (read § 27.07.11).
The derivation was based on the assumption that the section of the beam did not change under bending. When the thickness of the beam is small with respect to the radius of curvature this is a good assumption. In general this is not true, for instance with a soft-rubber eraser, which expands on the compressed side die to the Poisson module effect. Rubber is almost incompressible, this is expressed by the fact that the Poisson module is close to $1/2$.

PRESS

27.08 Exercises Problems and Physical Applications

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27-003 I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...

1.0290, 1.0295, 1.0296, 1.0297, 1.0298, 1.0299, 1.0309, 1.0310.

27-004 The Poisson Ratio Is Less Than 1/2

This § is referenced at pages:
[1490]

Demonstrate equation (27.07.08).

27-005 Elongation of a rotating Steel Bar

Data for steel: $\rho = 8{\cdot}10^3$ kg/m^3; $Y = 2{\cdot}10^{11}$ N/m^2; $v_s = 5$ km/s.

1. A steel cylindrical rod of length L and sectional area A is rotated about one end with angular speed ω. Find out the expression for the elongation of the rod.
2. Input reasonable values, and find the elongation of the rod.

27-006 Torsional Constant

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Show that the dependence on the geometry in equation (27.07.12) can be written in terms of the second momentum of area of the bar about its axis, I, as:

$$\Gamma L = GI\phi \quad .$$

27-007 Buckling

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27-008 Catenary

27-009 La Palla Che Cade E Rimbalza Anaelasticamente Sul Pavimento

27-010 Equilibrio Di Un Tavolo a Quattro Gambe Elastiche

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|Esempio ?.?||

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27-011 Deformation of a cube

27-012 Variazione Con La Profondità Della Densità Dell'acqua Nell'oceano

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...|17.20||

Read § 28.07.07.

Find the depth of a lake at which the density of water is 0.01 fractionally greater than the surface. The compressibility is $k = 50{\cdot}10^{-6}\ \mathrm{atm}^{-1}$.

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27.09 Additional Problems

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|§ 1.6|*A lot of problems*|

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CHAPTER 28

Introduction to Mechanics of Fluids

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28.01 Fluids

This § is referenced at pages:
[1663]

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28.01.01 Classical Fluids

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A fluid substance is a substance that continually deforms and flows, under an applied shear force-stress and|or twisting torque-stress; depending on the container it can also deform and flow under an applied normal force/stress (e.g. put a finger into a glass of water) Fluids include liquids (hardly compressible) and gases (easily compressible).

Si dimostra in generale che il tensore force-stress è simmetrico se e solo se sono assenti body-torques and surface-torques.

I fluidi che obbediscono a tale caratterizzazione sono definiti **fluidi classici**, in breve semplicmente fluidi.

Read § 32.03.01 for the more general discussion.

Classical fluids only will be discussed here.

28.01.02 Ideal Versus Real Fluids

Ideal fluids, by definition, cannot resist any shear force-stress and|or twisting torque-stress applied to them without flowing, when they are in static equilibrium in any Reference Frame, possibly a non-Inertial Reference Frame ; they have zero shear modulus.

While the shear modulus of an ideal fluid is zero, the shear-modulus of a real fluid might be non zero, implying that the fluid will deform and flow but with non zero shear force-stress and|or twisting torque-stress; this is the concept of viscosity, read § 28.05.

28.01.03 Pressure - Surface Forces

For any continuous system, given an infinitesimal oriented flat surface $\mathbb{d}\mathbf{S}$, the normal force due to pressure exerted along $\mathbb{d}\mathbf{S}$, that is with the same orientation, by the subsystem "below" $\mathbb{d}\mathbf{S}$ on the subsystem "above" $\mathbb{d}\mathbf{S}$ is:

$$\mathbb{d}\mathbf{F} \equiv p[\mathbf{x}]\,\mathbb{d}\mathbf{S} \equiv p[\mathbf{x}]\widehat{\mathbf{n}}\,\mathbb{d}\mathbf{S} \qquad \text{most often, but not necessarily, } p[\mathbf{x}] \geq 0 \quad .$$

Pressure is omni-present in fluid mechanics. Both the total force and the total torque of the pressure forces, which are surface forces, on an arbitrary volume of fluid, the system, can be expressed as a volume integral over the volume force, by a mathematical theorem (see section § 16.12.15):

$$\mathbf{f}^{\mathrm{V}}_{\mathrm{p}}[\mathbf{x}] = -\mathbf{grad}p[\mathbf{x}] \quad . \tag{28.01.01}$$

→ 1514

Note that the above expression (28.01.01) is just a mathematical equivalence, not a physical equivalence at all. It means that one can calculate the total force and the total torque of the pressure forces, as a volume integral, instead of a surface integral, the former being, most often, easier to calculate than the latter.

28.02 Elements of Fluid Statics

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A fluid is in static mechanical equilibrium if and only if all its parts are, with respect to some, in general non inertial, Reference Frame. This is just the same definition as for a general continuous medium.

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28.02.01
Static Equilibrium - Hydrostatics

Un fluido a riposo in un certo Reference Frame, anche non inerziale, è in equilibrio sotto l'azione delle forze di volume e di quelle di superficie, che si riducono solitamente alla sola pressione essendo il fluido a riposo.

Se si denota con $\mathbf{f}^{\mathtt{M}}$ la risultante, per unità di massa, di tutte le forze di volume agenti sul fluido, e con ρ la densità, la condizione di equilibrio è data dalla relazione:

$$\mathbb{d}\mathbf{F}[\mathbf{x}] \equiv \mathbf{f}^{\mathtt{M}}[\mathbf{x}]\ \mathbb{d}M \equiv \rho[\mathbf{x}]\mathbf{f}^{\mathtt{M}}[\mathbf{x}]\ \mathbb{d}V \equiv \mathbf{f}^{\mathtt{V}}[\mathbf{x}]\ \mathbb{d}V \qquad 0 = \mathbf{f}^{\mathtt{V}} + \mathbf{f}^{\mathtt{V}}_{\mathtt{p}} = \rho\mathbf{f}^{\mathtt{M}} - \mathbf{grad}p \quad . \tag{28.02.01}$$

→ 1534

Nel caso le forze di volume siano conservative, introduciamo l'energia potenziale:

- energia potenziale, $\Phi[\mathbf{x}]$;
- energia potenziale per unità di massa, $\phi[\mathbf{x}]$.

Allora si ha la condizione di equilibrio:

$$\mathbb{d}\Phi[\mathbf{x}] \equiv \phi[\mathbf{x}]\ \mathbb{d}M \qquad \mathbf{f} = -\mathbf{grad}\phi[\mathbf{x}] \qquad \mathbf{grad}p[\mathbf{x}] + \rho[\mathbf{x}]\mathbf{grad}\phi[\mathbf{x}] = 0 \quad ,$$

da cui l'equazione fondamentele dell'idrostatica in presenza di sole forze conservative:

$$\mathbf{grad}p[\mathbf{x}] + \rho[\mathbf{x}]\mathbf{grad}\phi[\mathbf{x}] = 0 \quad .$$

Note that, conservative forces may include inertial conservative forces, in any non-Inertial Reference Frame .

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The equation of hydrostatics, in general, has no solution. If the density varies in space in an arbitrary way, there is no way for the forces to be in balance, and the fluid cannot be in static equilibrium. Convection currents will start up. We can see this from the equation since the pressure term is a pure gradient, whereas for variable ρ the other term is not. Only when ρ is a constant is the potential term a pure gradient. Then the equation has a solution $p + \rho\phi =$ constant. Another possibility which allows hydrostatic equilibrium is for ρ to be a function only of p.

28.02.02
Pascal Principle

Se il fluido è incompressibile, cioè se la densità $\rho[\mathbf{x}, t] = \rho$ è uniforme e costante, si può scrivere il principio di Pascal:

$$\boxed{p[\mathbf{x}] + \rho\phi[\mathbf{x}] = \text{constant}} \quad . \tag{28.02.02}$$

→ 1516

e quindi per un fluido incompressibile in un campo di forze conservativo le superfici isobare coincidono con le superfici equipotenziali.

From the previous property (28.02.02), since potential energy changes are functions only of the place and the pressure variations are equal to potential energy variations, the Pascal principle follows: **every pressure variation applied to an enclosed incompressible fluid at equilibrium is transmitted unaltered to any other point in the fluid and to any walls of the container**. In fact, at fixed geometry, given two any fixed points:

$$\boxed{\Delta p[\mathbf{x}_2] = \Delta p[\mathbf{x}_1]} \quad .$$

28.02.03
Archimedes Principle

Un corpo immerso in un fluido nel campo gravitazionale riceve una spinta verso l'alto pari al peso della massa di fluido spostata. La forza è applicata al baricentro della massa di fluido spostata. In presenza di altre forze di volume oltre alla forza peso (per esempio forze fittizie) il principio di Archimede è ancora valido pur di considerare la forza totale di volume agente sul fluido spostato.

La dimostrazione del principio di Archimede si fonda sulla condizione di equilibrio del fluido per cui, data una qualunque porzione di fluido, la risultante delle forze di volume sullo stesso è uguale ed opposta alla risultante delle forze di superficie. Quando un oggetto è immerso nel fluido la risultante delle forze e il momento risultante di volume sarà quella che compete all'oggetto in questione, mentre la risultante delle forze di superficie sarà la stessa che si avrebbe se la posto dell'oggetto ci fosse il fluido.

Read § 16.12.15 for the demonstration.

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28.03 Elements of Fluid Kinematics

La cinematica del moto di un fluido può essere descritta con due approcci:

- la descrizione lagrangiana,
- la descrizione euleriana,

read § 32.01.

Nel primo caso ogni porzione infinitesima di fluido viene seguita nel suo moto, come un qualunque sistema materiale chiuso, mentre nel secondo caso si descrive il moto del fluido fornendo, per ogni punto dello spazio e ad ogni istante, come varia nel tempo la velocità delle diverse porzioni di fluido che all'istante dato transitano per il punto dato. La descrizione euleriana è ottenuta dunque attraverso un campo vettoriale, il campo di velocità, $\mathbf{v}[\mathbf{x},t]$ funzione dello spazio e del tempo, in generale. Ad un dato istante il valore del campo vettoriale in un punto è la velocità della porzione di fluido che all'istante dato transita per quel punto.

La velocità di un elemento di fluido che si muove con legge $\mathbf{x}[t]$ sarà data da $\mathbf{v}[\mathbf{x}(t),t]$.

L'accelerazione si ottiene prendendo la derivata totale rispetto al tempo della velocità.

Le quantità cinematiche di un elemento di fluido che si muove con legge $\mathbf{x}[t]$ nell'approccio euleriano sono dunque

$$\mathbf{r}_{\mathrm{P}}[t] \implies \mathbf{v}_{\mathrm{P}}[t] \equiv \frac{\mathrm{d}\mathbf{r}_{\mathrm{P}}}{\mathrm{d}t} = \mathbf{v}[\mathbf{x}[t],t] \implies \mathbf{a}_{\mathrm{P}}[t] \equiv \frac{\mathrm{d}\mathbf{v}_{\mathrm{P}}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\cdot\mathbf{grad}\right)\mathbf{v} + \partial_t\mathbf{v} \quad .$$

Important kinematics definitions are as follows.

- Velocity field: $\mathbf{v}[\mathbf{x},t]$.
- steady flow: $\mathbf{v}[\mathbf{x},t] = \mathbf{v}[\mathbf{x}]$.
- Divergence-less fluid flow: $\mathbf{div}\mathbf{v} = 0$.
- Irrotational fluid flow: $\mathbf{rot}\mathbf{v} = 0$.
- The field lines of the velocity field define the stream-line (or stream filament).
- All stream-lines passing through any open surface form a stream-tube. Since, by definition, the fluid has zero velocity perpendicularly to the stream-line there is no leak of liquid in/out the walls of a stream-tube.
- A stream-line is an infinitesimally thin stream-tube.
- For a steady flow, the trajectory of any infinitesimal volumes follows the flow field lines.

28.03.01 Types of Motions of Fluids

Types of fluid motions are as follows.

- Fluid flows such that stream-lines stay side by side, in a orderly fashion, without mixing are called laminar flows, which are always realized whenever the frictional forces are large enough.
- Turbulent flows are characterized by large random fluctuations of the velocity field, with vortices which can intermix the stream-lines completely.

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28.03.02
Equazione Di Continuità Della Massa

L'equazione di continuità della massa, che descrive la conservazione della massa, nel caso generale si scrive:

$$\boxed{\mathbf{div}\mathbf{j} + \partial_t \rho = 0} \quad . \tag{28.03.01}$$

→ 2738

Nel caso di un moto puramente convettivo (in contrasto ad un moto diffusivo, in cui non c'è moto di insieme dei singoli elementi infinitesimi di massa):

$$\boxed{\mathbf{j}[\mathbf{x}, t] = \rho[\mathbf{x}, t]\mathbf{v}[\mathbf{x}, t]} \quad .$$

Nel caso di densità omogenea e costante l'equazione di continuità della massa per un fluido si riduce a:

$$\boxed{\mathbf{div}\mathbf{v} = 0 \qquad \Leftrightarrow \qquad \rho[\mathbf{x}, t] = \rho = \text{fixed}} \quad .$$

In general, density will also depend on temperature and pressure.

Nel caso di moto stazionario, l'equazione di continuità lungo un tubo di flusso di sezione A si riduce alla costanza del flusso di massa:

$$\boxed{\text{steady motion} \implies \frac{\mathrm{d}M}{\mathrm{d}t} = \rho A v = \text{constant}} \quad .$$

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28.04 Elements of Fluid Dynamics

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Let us consider infinitesimal fluid volumes of fluid such that its overall linear momentum is null: $\mathbf{P} = 0$. The momenta of the individual molecules, in general, are not zero but show a velocity distribution with directions uniformly spread over all directions (typically, the Maxwell distribution).

The macroscopic treatment of fluids motion neglects the thermal motion of the individual molecules considering instead the average motion of the infinitesimal fluid volume elements.

Let us now consider infinitesimal fluid volumes of fluid such that its overall linear momentum is not null: $\mathbf{P} \neq 0$. In this case the macroscopic velocity of the infinitesimal fluid volume must be interpreted as the Center-Of-Mass velocity of the microscopic particles constituting it (**Koënig theorem**).

Definitions:

- Hydrodynamics: fluids with zero or negligible compressibility;
- Aerodynamics: fluids with non negligible compressibility.

Interactions determining the dynamics include:

- pressure differences between different local positions, which are equivalent to a volume force: $\mathbf{f}[\mathbf{x}] = -\mathbf{grad}p[\mathbf{x}]$.
- pressure forces induce surface forces normal to every surface, either real surfaces (solid surfaces such as walls) or purely geometrical surfaces;
- the gravity force provides a volume force density: $\mathbb{d}\mathbf{F} = g[\mathbf{x}]\ \mathbb{d}m = g[\mathbf{x}]\rho[\mathbf{x}]\ \mathbb{d}V$.
- whenever the flow velocity depends on the position, friction forces, between layers of the fluid with different values of velocities, are generated (internal friction);
- friction forces between the fluid and the solid surfaces with which it is in contact exist (external frictions);
- in presence of external ElectroMagnetic fields, as well as any other external potential field, additional Force|Torque may exist;
- in non-Inertial Reference Frame , inertial forces exist;
- other external sources of interaction may exist, such as those due to a moving solid surface in a pump or in a fan.

Whenever the friction forces are negligible compared to the other forces, the approximation of ideal liquid helps.

Consider a fluid of density ρ and any open surface of area A and define (across the surface):

- the mass flow rate, $\dot{M}$;
- the volumetric flow rate: $Q \equiv \dot{M}/\rho$;
- the average velocity: $\langle v \rangle \equiv v \equiv Q/A$; it averages out the random velocities at the molecular level and it corresponds to the Center-Of-Mass velocity of any infinitesimal fluid volume.

There are pressure and density changes along the flow path, velocity changes, and possibly elevation changes and|or effects from other external potential fields.

Friction due to fluid contact with the wall and|or internal fluid friction causes mechanical energy to be converted into internal energy, with possible changes of temperatures.

Possibly, external mechanical devices put energy into or extract energy from the system, in the form of the so-called **shaft work**.

The macroscopic energy balance that may be applied to a single-input, single-output system with no chemical reactions, no phase changes, and little heat loss or heat generation is the mechanical energy balance,

28.04.01
Cardinal Equations

In flowing systems, the laws of conservation of mass, linear momentum and energy allow to calculate the dynamics, studying the conservation laws that apply to microscopic systems (in control volumes). **The conservation of angular momentum is implemented, for classical fluids, by the symmetry of the force-stress tensor**. In fact, as the stress tensor is symmetrical for a classical fluid/solid, it can be shown that the first and second **Cardinal Equations** are equivalent. For fluids with non zero body-torques and|or surface torques, the conservation of angular momentum is a necessary and independent equation.

28.04.02
Energy Balance - Bernoulli Equation

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Introduction to Fluid Mechanics Edward J. Shaughnessy, Jr. Ira M. Katz James P. Schaffer
The state of stress in an inviscid fluid is characterized solely by the pressure distribution. Since there is no fluid whose viscosity is zero, one might ask **why we study the flow of an inviscid fluid, i.e., frictionless flow**.
One answer, which is particularly relevant in engineering design, is that **the highest performance that can be achieved from a fluid-handling device is that which would occur in the absence of friction**. Thus the study of frictionless flow through a device may allow us to optimize a preliminary design.
A second answer, which is of great importance in aerodynamics, is that the principal phenomenon that arises from the existence of viscosity, i.e. **shear stress, is not significant in all flows**. To better understand this comment, consider the flow of the two fluids of greatest engineering interest, air and water. The viscosity of these fluids is not zero, but it is small in comparison to many other fluids. Now from the relationship between shear stress, viscosity, and velocity gradient is given it is easy to see that if the viscosity is zero, then the shear stress must also be zero. The equation also shows, however, that if the velocity gradient at a point in a flow of air or water is small, then the local shear stress may be negligible (from an engineering point of view) even though the viscosity is non-zero. A number of important flows in engineering contain regions in which shear stresses are in fact negligible. In these regions the flow is approximately frictionless and indistinguishable from the flow of an inviscid fluid.
...
The Bernoulli equation is an approximate relation between pressure, velocity, and elevation, and is valid in regions of steady, incompressible flow where net frictional forces are negligible. The Bernoulli equation is an approximate equation that is valid only in inviscid regions of flow where net viscous forces are negligibly small compared to inertial, gravitational, or pressure forces. Such regions occur outside of boundary layers and wakes.

Il **moto laminare e stazionario di un fluido incomprimibile e non viscoso nel campo di gravità** è descritto dal Bernoulli equation

$$p + \rho g h + \frac{1}{2}\rho v^2 = \text{costante lungo ogni linea di flusso} \quad ,$$

in assenza di scambi di calore, lavoro di altro tipo rispetto al lavoro della forze di pressione, o modifiche del fluido come reazioni chimiche o cambiamenti di fase.

Nel caso di moto irrotazionale il valore della costante è indipendente dalla linea di flusso considerata.

Un secondo caso in cui la costante non dipende dalla linea di flusso è il caso in cui tutte le linee di flusso originano da una regione in cui la costante è la stessa per tutte le linee di flusso.

Anche quando il fluido è viscoso l'equazione di Bernoulli può servire come buona approssimazione del moto reale.

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Notare che in opportune condizioni l'equazione di Bernoulli può prevedere pressioni arbitrariamente basse (e anche negative). In realtà un gas non può avere pressione negativa e un liquido bolle quando la pressione si abbassa al di sotto del valore della tensione di vapore (cavitazione). L'equazione di Bernoulli non può quindi essere applicata quando predice pressioni prossime alla tensione di vapore, o peggio pressioni negative.

The Bernoulli equation is one of the most frequently used and mis-used equations in fluid mechanics. In order to apply Bernoulli equation:

- the flow must be steady;
- viscous effects should be negligible, as well as any other friction effect between the liquid and any other object;
- surface tension effects are negligible;
- incompressible flow (but there exists a simple more general formulation for compressible flow, see equation (28.04.01);
- no shaft work (that is, only work done by pressure forces);
- no heat exchanges;
- any other irreversible effects are negligible;
- unless the flow is irrotational, flow along a stream-line and the value of the constant is different for different stream-lines.

28.04.03
Generalized Bernoulli Equation

The general form of the Bernoulli equation in the gravity field applies to the frictionless flow of a compressible or incompressible fluid reads:

$$\frac{1}{2}\left(v_2^2 - v_1^2\right) + g\left(h_2 - h_1\right) + \int_1^2 \frac{\mathrm{d}p}{\rho} + \int_1^2 \partial_t v \, \mathrm{d}L = 0 \quad . \tag{28.04.01}$$

→ 1521

The only restriction is that the path connecting the two points must be an instantaneous stream-line.

DRAFT - IN PROGRESS

28.05 Shear Force|Torque - Viscosity

This § is referenced at pages:
[1202, 1514]

28.05.01 Elementary Definition

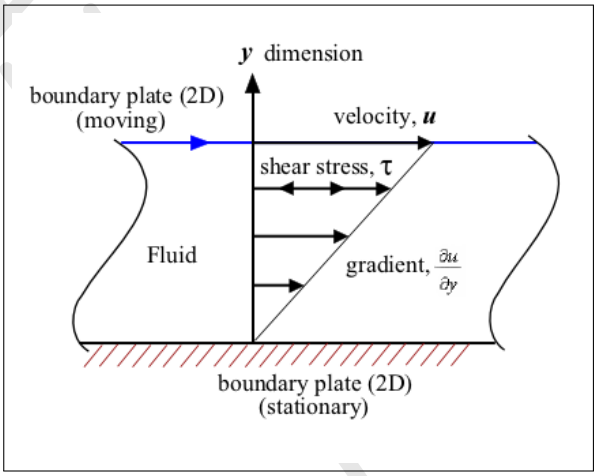

Figure 28.11: Elementary definition of viscosity (from WEB - URL)

Viscous fluid obey the co-called no-slip condition: the velocity of any viscous fluid in contact with a solid buondary is zero.

Examples: dust on the blades of a fan does not fly away when the blades rotate.

Si osserva sperimentalmente che un fluido viscoso a contatto con una superficie solida ha sempre velocità relativa alla superficie nulla a contatto con la superficie. This is an experimental fact.

Si supponga un moto rettilineo (e quindi laminare) di un fluido: le velocità di tutti gli elementi del fluido sono nella stessa direzione. Per un gran numero di fluidi la relazione tra lo sforzo di taglio e il gradiente di velocità è lineare. Il coefficiente di proporzionalità definisce la viscosità η.

Nel caso partcolare di un moto di un fluido con velocità lungo una direzione fissa, z, di modulo variabile lungo la direzione s, perpendiclare a z, lo sforzo di taglio che occorre esercitare sul fluido per mantenerlo in moto a velocità costante è dato da

$$\tau_{zs} \equiv \pm\eta\frac{\partial v_z}{\partial s} \qquad \eta \geq 0 \qquad \text{1D motion along the } z \text{ axis with velocity changing along } s \text{ perpendicular to } z \quad , \tag{28.05.01}$$

→ 1202 1523

che definisce la viscosità e dove la scelta del segno deve essere accuratamente definita definendo origine e recettore della forza.

A fluid flow between two parallel surfaces with relative motion is called **Couette flow**.

28.05.02
General Definition of Viscosity

This § is referenced at pages:
[1203]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||

Nel caso di moto generale di un fluido classico in 3D il tensore degli sforzi di viscosity deve essere generalizzato alla forma forma simmetrica, in quanto il tensore delgi sforzi è simmetrico. Lo sforzo di taglio che occorre esercitare sul fluido per mantenerlo in moto a velocità costante è dato quindi da:

$$\tau_{ij}^{\cdot\cdot} \equiv \pm\eta\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right) \qquad \eta \geq 0 \quad . \tag{28.05.02}$$

In fact, the anti-symmetrical part of the second-order tensor, $\frac{\partial v_i}{\partial x_j}$, must be taken out, as rotations are rigid motions and do non involve any shear stress. Therefore, using the general method in section § 20.03.06 the particular definition in equation (28.05.01) is written in covariant form via a second-order tensor, but with its symmetrical part only. In fact, also the trace could be retained, and in fact, a second viscosity coefficient is in general necessary, for compressible fluids.

$$\tau_{ij}^{\cdot\cdot} = \eta'\delta_{ij}^{\cdot\cdot}\mathbf{div}\,\mathbf{v} + \eta\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right) \quad . \tag{28.05.03}$$

→ 1202 1523 1530 2742

28.05.03
Units of Measure and Common Values

The SI (read § B) physical unit of dynamic viscosity, η, is the pascal-second $\mathrm{Pa\,s}$

The CGS physical unit for dynamic viscosity is the poise P named after Jean Louis Marie Poiseuille. It is more commonly expressed, particularly, as centipoise cP. The centipoise is commonly used because water has a viscosity of $1.0020\ \mathrm{cP}$ at $20\,^{\circ}\mathrm{C}$; the closeness to one is a convenient coincidence.

Kinematic viscosity, η/ρ, has SI units $\mathrm{m^2 s^{-1}}$ (read § B). The CGS physical unit for kinematic viscosity is the stokes, named after George Gabriel Stokes. It is sometimes expressed in terms of centistokes.

28.05.04
Newtonian Fluids

I fluidi caratterizzati dalla relazione (28.05.03), che definisce la viscosità, sono detti fluidi newtoniani, di cui la stessa relazione (28.05.03) è la **relazione costitutiva**.

I fluidi per i quali la relazione tra lo sforzo di taglio e il gradiente di velocità non soddisfa la relazione lineare sono detti fluidi non-Newtoniani.

Note that viscous forces are dissipative forces, which transfer mechanical energy into internal energy. The study of viscous fluids, therefore, must be carried on in the realm of thermo-mechanics. Viscosity is in fact internal friction caused by relative velocities of different parts inside the fluid. Viscous forces, dissipating mechanical energy, cause, among other things damping of sound waves.

DRAFT - In Progress

28.05.05
Non-Newtonian Fluids

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FAMorrison § 5.3
As discussed previously in this section, experiments on the parallel-plate apparatus confirm the validity of Newton law of viscosity and the Newtonian constitutive equation for many materials, including water, oil, honey, milk, and solvents. Many materials, however, do not follow the Newtonian constitutive equation. Fluids that are compressible, for example, have an additional contribution to stress other than what we described; Chapter 10 presents modifications to the Newtonian constitutive equation that account for compressibility. Many incompressible fluids, including foods such as mayonnaise, peanut butter, and ketchup, do not follow the Newtonian constitutive equation. Many industrial materials, including molten plastics, asphalt, and concrete, are non- Newtonian. Biological fluids, including blood and mucus, are almost universally non-Newtonian. In addition, the stresses in geological flows, such as those involving soil and lava, fail to follow the Newtonian constitutive equation.
The field of study that addresses the many effects seen in non-Newtonian fluids is called rheology, and there is considerable literature on the subject. In this section, we introduce the basic nature of non-Newtonian flows and constitutive models for non-Newtonian fluids; in-depth information on non-Newtonian fluids is in the cited literature. Although there is a single Newtonian constitutive equation, for non-Newtonian fluids, we have many different constitutive equations.
We need a variety of constitutive equations for non-Newtonian fluids because of what the constitutive equation is. Recall that the stress tensor accounts for the molecular-force contributions to the momentum balance. Molecular contact forces are different in every fluid because they arise from chemical interactions, and the atoms and molecules are different in every substance (see Figure 4.3). We take the continuum approach in our modeling of fluid motion but, ultimately, we must match our models to the actual chemical behavior of the fluid systems under study. The constitutive equation is the link between the continuum model and the chemical properties of the molecules that comprise the fluid. There could be as many constitutive models as there are chemicals, but it turns out to be less complicated. For thousands of fluids, including water, oil, and even gases under most circumstances, the Newtonian constitutive equation is the stress-velocity relationship. For many materials that do not follow the Newtonian constitutive equation, simple modifications often are adequate. Other materials, such as polymer melts and solutions, require complex visco-elastic constitutive equations; a detailed study of such equations is beyond the scope of this book.

PRESS

28.06 Shear Force|Torque - Surface Tension

This § is referenced at pages:
[2572]

28.06.00.01 Definition

Consideriamo il caso di **una fase liquida a contatto con una fase gassosa** a livello esemplificativo; **anche nel contatto liquido-liquido si ha lo stesso fenomeno**.

Le forze inter-molecolari entro un liquido danno origine a forze debolmente attrattive tra le molecole. Una molecola all'interno di un liquido risente di una forza risultante mediamente nulla mentre le molecole prossime alla superficie (**a distanza dalla superficie inferiore al raggio d'azione delle forze inter-molecolari**) risentono di una forza risultante media perpendicolare alla superficie del liquido e diretta verso l'interno. Di conseguenza **per portare una molecola dall'interno di un liquido verso la sua superficie occorre fare lavoro**. Occorre quindi fare lavoro per aumentare la superficie di un liquido. E viceversa un liquido tende ad assumere la configurazione corrispondente alla superficie minima compatibile con i vincoli, perchè le forze sulle molecole alla superficie tendono ad attirarle verso la massa del liquido, riducendo la superficie.

Analogamente al caso della tensione e della pressione si può definire il lavoro per aumentare una superficie A come

$$\boxed{\mathrm{d}W \equiv \gamma \, \mathrm{d}S} \quad , \tag{28.06.01}$$

→ 1525

dove γ è la tensione superficiale, definita dalla stessa relazione (28.06.01).

Equivalentemente, dalla relazione (28.06.01), si ha che il lavoro adiabatico necessario in considizoni quasi-statiche corrisponde all'aumento di energia del liquido, conformente alla definiozne di energia intenra e primi principo della termodinamica, per cui:

$$\boxed{\mathrm{d}\mathcal{U} \equiv \gamma \, \mathrm{d}S} \quad . \tag{28.06.02}$$

Si consideri una linea, anche ideale, alla superficie del liquido. La tensione superficiale origina una forza tangente alla superficie del liquido, perpendicolare alla linea e tendente ad allontanare le particelle da parti opposte della linea. Il modulo di tale forza per unità di lunghezza vale

$$\boxed{\mathrm{d}F = \gamma \, \mathrm{d}L} \quad .$$

L'analogia con la tensione superficiale di una membrana elastica è limitata dal fatto che la forza dovuta alla tensione superficiale di un liquido non dipende dalla superficie.

La forza dovuta alla tensione superficiale di un liquido dipende a rigore, oltre che dalla temperatura, anche dalle caratteristiche della fase gassosa con cui il liquido è a contatto in quanto dipende dal bilancio tra le forze di coesione delle molecole del liquido e le forze di adesione tra le molecole del liquido e quelle del gas a contatto.

Surface tension is thus surface energy as well.

DRAFT IN PROGRESS

28.06.00.02 Units of Measure and Common Values

For many liquids and solid, surface energy / surface tension is of the order of $\approx 1\ \mathrm{J/m^2} = 1\ \mathrm{N/m}$.
For water at NTP: $\gamma \simeq 0.07\ \mathrm{J/m^2} = 0.07\ \mathrm{N/m}$.
For mercury at NTP: $\gamma \simeq 0.4\ \mathrm{J/m^2} = 0.4\ \mathrm{N/m}$.
For ethanol at NTP: $\gamma \simeq 0.02\ \mathrm{J/m^2} = 0.02\ \mathrm{N/m}$.

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D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 1994, C.E.A., 4thEd., ...
Leaves and insects can be observed to float on the surface of a body of water. They are not partially submerged and thus not buoyed up because of Archimedes' principle. In this case the object is completely on the surface and none of it is submerged. The object is kept afloat by the surface tension of the liquid. You can demonstrate the surface tension of water by carefully floating a steel needle or a razor blade. There is of course no way for steel to float by Archimedes' principle, since its density is greater than that of water. If you submerge the needle or the razor blade, it will sink as Archimedes' principle predicts. Only when it is entirely on the surface can it float. You can add to the water a chemical, called a surface-active agent or surfactant, which reduces the surface tension (by reducing the cohesive force between molecules) and makes it more difficult to float the object. Detergents are common surfactants. If you carefully introduce detergent into the water on which a razor blade is floating, the surface tension suddenly decreases and the razor blade sinks to the bottom. A floating object depresses the surface layer of the fluid slightly which stretches the surface layer and thus tends to increase its potential energy. Somewhat like a trampoline, the stretched surface layer exerts a restoring force, the vertical component of which can maintain equilibrium with the weight of the object. However, this analogy of the behavior of the surface layer is not strictly correct.
...
A razor blade floating on the surface of water, supported only by surface tension. The surface is distorted by the floating object, which is kept afloat by the vertical components of the surface force.

28.06.01
La Legge Di Laplace

La differenza di pressione Δp tra le due parti di una superficie avente tensione superficiale γ e i cui raggi di curvatura principali sono R_1 e R_2 è data dalla legge di Laplace

$$\Delta p = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = \gamma C \quad . \tag{28.06.03}$$

→ 1532 1533

PROGRESS

28.07 Examples and Physical Applications

28.07.01 Hydrostatic Paradox

28.07.02 Force|Torque on a Dam

28.07.03 Stability of a Floating Body

28.07.04 Hydrostatic Equilibrium in Uniform Gravity for Incompressible Fluid

Nel campo di gravità, scegliendo l'asse z diretto verticalmente verso l'alto, l'equilibrio idrostatico è descritto dalla equazione?

$$\boxed{\frac{\mathrm{d}p[z]}{\mathrm{d}z} = -\rho[z]g[z]} \quad .$$

Nel caso sia la densità che il campo di gravità siano fissate si trova la legge di Stevino:

$$\boxed{p[z] = p_0 - \rho g\,(z - z_0) \qquad p_0 \equiv p[z_0]} \quad . \tag{28.07.01}$$

→ 1534

Note that, in general, density and pressure are linked to temperature by the equation of state.

28.07.05 Hydrostatic Equilibrium in Uniform Gravity for Ideal Gas at Uniform Temperature

Per un gas ideale in un campo gravitazionale fissato e temperatura fissate, la pressione varia con l'altezza secondo la formula barometrica:

$$\boxed{p[z] = p_0 \exp\left[-\frac{\mathcal{M}_0 g}{RT}(z - z_0)\right]} \quad . \tag{28.07.02}$$

28.07.06 Hydrostatic Equilibrium in Uniform Gravity for Rotating Fluid at Uniform Temperature

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|2.20||

This a a centrifuge for either a liquid or a gas.

Un recipiente rigido cilindrico, di raggio r_0 e altezza h, contiene una certa quantità di moli di un fluido di massa molecolare $\mathcal{M}_0$, a temperatua uniforme, e ruota con velocità angolare costante ω attorno al proprio asse. Si suppne che la rotazione sia stata accesa da un tempo sufficiente tale che il moto del fluido sia diventato rigido e il fluido abbia raggiunto una configurazione idrostatica, a seguito degli attriti viscosi interni. Read § 71.04.

1. Assume a container with an incompressible liquid. The cylindrical container has its axis along the vertical direction. Its upper base is open in atmosphere.
   a) Consider the equilibrium situation under the effects of the gravity and centrifugal forces and determine the shaper of the equipotential surfaces and the complete equation of the surface of separation between liquid and atmosphere. The number of moles of liquid is fixed, n.
2. Assume a container with and ideal gas. Consider first as negligible the effect of gravity on the gas. Determinare l'andamento della pressione del gas all'interno del recipiente in funzione di r supponendo il gas ideale e in condizioni di equilibrio termodinamico a temperatura uniforme. Si considerino i due casi seguenti.

DRAFT NEW PROGRESS

a) Si supponga che il recipiente, chiuso, contenga un piccolo foro al centro delle due basi in contatto con l'atmosfera esterna in modo che la pressione del fluido sull'asse del contenitore sia uguale a quella dello stesso fluido all'esterno, p_0, pressione atmosferica. Il numero di moli dentro al contenitore non è dunque fissato.
b) Si supponga che il contenitore sia sigillato. Il numero di moli dentro al contenitore è dunque fissato, n.

3. Assume again a container with and ideal gas but do not ignore the effect of the gravity field on the gas. The cylindrical container has its axis along the vertical direction.

28.07.07
Ocean Water Density Change With Depth

This § is referenced at pages:
[1511]

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|17.20||

Determinare la legge di variazione della densità dell'acqua in funzione della profondità nell'oceano, conoscendo il coefficiente di compressibilità k dell'acqua: $1/k \equiv B = 2.2{\cdot}10^4$ atm. Si supponga che la densità dipenda solo dalla pressione ma non dalla temperatura e che B sia costante nell'intervallo di interesse.

WORKED SOLUTION

Per definizione il coefficiente di compressibilità k è legato al bulk modulus

$$\frac{1}{k} \equiv B \equiv -V\frac{\partial p}{\partial V} = \rho\frac{\partial p}{\partial \rho} \tag{28.07.03}$$

dove le derivate parziali vanno calcolate con il vincolo opportuno al processo considerato: isotermo, adiabatico o altro. Dato che per ipotesi $\rho = \rho(p)$ le derivate parziali possono essere sostituite dalla derivata semplice. L'equivalenza delle due espressioni si deduce considerando una massa fisssata, Δm, in un volume, ΔV, fissato, e quindi costanti, da cui si deduce

$$\Delta m \equiv \rho\Delta V \implies 0 = \mathrm{d}\Delta m = \rho\,\mathrm{d}\Delta V + \Delta V\,\mathrm{d}\rho$$

Dall'equazione dell'equilibrio idrostatico si ha

$$\boxed{\frac{\mathrm{d}p[z]}{\mathrm{d}z} = -\rho[z]g} \tag{28.07.04}$$

che può essere riscritta come

$$\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z} = -\rho[z]g = \frac{B}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z}$$

da cui separando le variabili

$$\frac{\mathrm{d}\rho}{\rho^2} = -\frac{g}{B}\,\mathrm{d}z$$

che integrata, con B costante da

$$\rho[z] = \rho_0\left[1 + \frac{\rho_0 g}{B}(z - z_0)\right]^{-1} \simeq \rho_0\left[1 - \frac{\rho_0 g}{B}(z - z_0)\right] \simeq \rho_0\left[1 + \frac{\rho_0 g}{B}h\right]$$

avendo denotato con $h \equiv z_0 - z$ la profondità. Per $h = 10$ km si ha $\rho/\rho_0 = 1.05$.
The pressure as a function of depth is:

$$\frac{\mathrm{d}p[z]}{\mathrm{d}z} = -\rho[z]g = -g\rho_0\left[1 - \frac{\rho_0 g}{B}(z - z_0)\right] \quad ,$$

$$p[z] = -g\rho_0 \ldots\ldots \quad .$$

28.07.08
Equazione Di Poiseuille

Descrive il moto stazionario di un fluido incomprimibile viscoso in un condotto rettilineo a sezione circolare costante di raggio R lungo un tratto Δz. Il profilo di velocità, in funzione della distanza dall'asse del tubo, r, è:

$$v[r] = \frac{1}{4\eta}\frac{|\Delta p|}{\Delta z}\left(R^2 - r^2\right) \quad ,$$

dove Δp è la differenza di pressione ai capi di Δz.

La portata volumetrica (volume di fluido per unità di tempo) risultante vale:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi R^4}{8\eta}\frac{|\Delta p|}{\Delta z} \quad .$$

La velocità media vale

$$\langle v \rangle = \frac{R^2}{8\eta}\frac{|\Delta p|}{\Delta z} \quad .$$

The larger the viscosity the larger the pressure difference required to get the same volumetric rate.

Note that the flow is a rotational one.

28.07.09
Viscous Fluid Between Two Circular Coaxial Rotating Cylinders

This § is referenced at pages:
[1530]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||

Read also § 28.07.10.

Consider two long circular cylindrical coaxial rotating cylinders with a viscous fluid in between (viscosity η). Let the radii and angular velocities of the two cylinders be R_1 (with angular velocity ω_1) and R_2 (with angular velocity ω_2), with $R_2 > R_1$. Let the length be $L \gg R_2$ The fluid in between the cylinders is rotating in a steady flow. Let $\omega[r]$ be the angular velocity as a function of the distance from the axis.

The problem is best analysed in cylindrical coordinates. This would imply to convert the expression of the gradient velocity tensor into cylindrical coordinates.

In this simple case, where only a radial dependence exists, a simplified heuristic approach goes as follows. Let us assume ($v[r] = r\omega[r]$)

$$\frac{\Delta F}{\Delta \mathbf{S}} = \eta\frac{\mathrm{d}v[r]}{\mathrm{d}r} = \eta\left(r\frac{\mathrm{d}\omega[r]}{\mathrm{d}r} + \omega[r]\right) \quad .$$

This expression cannot be accepted without changes. In fact the second term, $\eta\omega[r]$, is non zero for a rigid rotation also. We know that in a rigid rotation viscosity cannot have any effect and therefore the second term must be subtracted out. In fact it looks reasonable that the shear stress depends on the derivative of the angular velocity with respect to the radial distance. Finally, we then have the heuristic result:

$$\frac{\mathrm{d}F}{\mathrm{d}A} = \eta r\frac{\mathrm{d}\omega[r]}{\mathrm{d}r} \quad .$$

The total torque on an infinitesimal cylindrical shell at radius r is calculated from the shear torque. let us calculate, for instance, $\Gamma^-[r]$, the total torque that the part of fluid at distance less that r applies on the fluid at distance greater than r:

$$\Gamma^- = -2\pi\eta r^3 L\frac{\mathrm{d}\omega[r]}{\mathrm{d}r} \quad .$$

DRAFT - In Progress

Note the minus sign coming from the fact that if, for instance, $\frac{\mathrm{d}\omega[r]}{\mathrm{d}r} > 0$ the interior part of the fluid tend to slow down a shell of fluid rotating with $\omega[r] > 0$.

As the situation is a steady one, the angular momentum of any shell of fluid is constant and therefore the torques on its internal and external layers must be equal and opposite. Therefore one finds the equilibrium condition:

$$-2\pi\eta r^3 L\frac{\mathrm{d}\omega[r]}{\mathrm{d}r} = \text{constant} = \Gamma_0 \quad .$$

The solution gives of the differential equation with the appropriate boundary conditions give:

$$\omega[r] = -\frac{K_0}{2r^2} + K_1 \qquad K_0 = \frac{2R_1^2R_2^2}{R_2^2 - R_1^2}(\omega_2 - \omega_1) \qquad K_1 = \frac{R_2^2\omega_2 - R_1^2\omega_1}{R_2^2 - R_1^2} \quad .$$

The torque can be calculated from:

$$\Gamma^- = 2\pi\eta L K_0 \quad .$$

If the two angular velocities are given, the required torque can be calculated. If the torque and one angular velocity are given, the second angular velocity can be calculated.

Viscosity is often measured by using such a kind of apparatus.

28.07.10
Viscous Fluid Between Two Circular Coaxial Rotating Cylinders (2)

This § is referenced at pages:
[1529]

Read also § 28.07.09.

Un fluido di viscosità η si trova tra due cilindri coassiali di raggio, rispettivamente, r_1 ed r_2 con $r_1 < r_2$. Il cilindro interno è vincolato a ruotare con velocità angolare costante ω_1 mentre quello esterno viene fatto ruotare con velocità angolare costante ω_2. Mostrare come dalla misura delle velocità angolari e dei momenti da applicare ai due gusci si possa determinare la viscosità del fluido.

WORKED SOLUTION

Lo sforzo di taglio, per fluidi newtoniani, è legato alla viscosità dalla relazione (28.05.03), (sforzo applicato ad una superficie del fluido dall'esterno).
Fissiamo un Coordinate System con asse z coincidente con l'asse dei due cilindri. In situazione stazionaria si avrà il campo di velocità tale che:

$$x[r,t] = r\cos[\omega[r]t] \implies v_x = -\omega y \quad ,$$
$$y[r,t] = r\sin[\omega[r]t] \implies v_y = +\omega x \quad ,$$

con:

$$\omega[r,t] = \omega[r] \quad .$$

Quindi si ha, per τ_{xy},

$$\tau_{xy} = \tau_{yx} = \eta\left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right] = \eta\left[x\frac{\partial\omega}{\partial x} - y\frac{\partial\omega}{\partial y}\right] = \eta\frac{\partial\omega}{\partial r}\left[x\frac{\partial r}{\partial x} - y\frac{\partial r}{\partial y}\right] = \eta\frac{\partial\omega}{\partial r}\left[x\frac{x}{r} - y\frac{y}{r}\right] = \quad ,$$
$$\implies \{f_x, f_y\} = \tau_{ij}\{\cos\theta, \sin\theta\} \quad ,$$
$$\tau_{xx} = \tau_{yy} = 2\eta\frac{\partial v_x}{\partial x} = 2\eta\frac{\partial v_y}{\partial y} = -2\eta\frac{xy}{r}\frac{\partial\omega}{\partial r} \quad ,$$
$$f_x = -\eta y\frac{\partial\omega}{\partial r}????? \quad .$$

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Lo sforzo di taglio calcolato per $y = 0$, quando $x = r$, diventa allora

$$\tau_{r\theta} = \eta r \frac{\partial \omega}{\partial r} = \eta r \frac{\mathrm{d}\omega}{\mathrm{d}r} \tag{28.07.05}$$

→ 1531

essendo ω funzione di r ma non di θ.
è ragionevole che lo sforzo di taglio sia non nullo solo nel caso in cui la velocità angolare varia con r. Nel caso opposto, in cui la rotazione è rigida, non c'è scorrimento tra gli strati adiacenti ad r diverso per cui non c'è sforzo di taglio.
La relazione (28.07.05) può anche essere ricavata osservando che, in coordinate cilindriche, la velocità tangenziale vale $\omega(r)r$. Ne segue

$$\tau'_{r\theta} \approx \eta \frac{\Delta(\omega(r)r)}{\Delta r} \Longrightarrow \tau'_{r\theta} = \eta \frac{\mathrm{d}(\omega(r)r)}{\mathrm{d}r}$$

L'espressione ricavata non è tuttavia accettabile perché prevede uno sforzo di taglio anche nel caso di rotazione rigida del fluido con velocità angolare costante. Occorre sottrarre tale termine dall'espressione e si deduce allora, svolgendo la derivata, l'espressione (28.07.05).
Se la velocità angolare di ogni guscio di fluido è costante il momento complessivo delle forze che agiscono sul guscio è nullo e quindi il momento complessivo sulla faccia interna del guscio deve essere uguale ed opposto a quello che agisce sulla faccia esterna. Cioè il modulo del momento è indipendente da r. In particolare il momento da applicare ai due gusci ha lo stesso modulo. Si ha

$$\Gamma = 2\eta\pi L r^3 \frac{\mathrm{d}\omega}{\mathrm{d}r} = \Gamma_0$$

Allora l'equazione differenziale per ω

$$\frac{\mathrm{d}\omega}{\mathrm{d}r} = \frac{\Gamma_0}{2\pi\eta L r^3} \tag{28.07.06}$$

→ 1531

può essere integrata

$$\omega(r) = \frac{\Gamma_0}{4\pi\eta L r^2} + K$$

è un fatto sperimentale che un fluido a contatto con una parete ha velocità relativa nulla rispetto alla parete. Quindi la velocità del fluido nel punto di contatto con i due cilindri sarà la stessa velocità dei cilindri. Imponendo le condizioni $\omega[r_1] = \omega_1$ e $\omega[r_2] = \omega_2$ si determinano le costanti Γ_0 e K

$$\Gamma_0 = \frac{4\pi\eta L\,(\omega_2 - \omega_1)\, r_1^2 r_2^2}{r_2^2 - r_1^2} \quad ,$$
$$K = \frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2}$$

da cui segue l'espressione per la velocità angolare

$$\omega(r) = \frac{(\omega_2 - \omega_1)\, r_1^2 r_2^2}{\left(r_2^2 - r_1^2\right) r^2} + \frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2}$$

Non essendo interessati all'andamento di $\omega(r)$ ma volendo determinare solo l'espressione del momento si poteva integrare direttamente la (28.07.06) tra i due estremi, $r_1,\ \omega_1$ e $r_2,\ \omega_2$, ottenendo direttamente

$$\omega_2 - \omega_1 = \frac{\Gamma_0}{4\eta\pi L} \frac{r_2^2 - r_1^2}{r_1^2 r_2^2}$$

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28.07.11
Pressione All'interno Di Una Goccia Sferica D'acqua

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This § is referenced at pages:
[1532, 1532, 1532, 1532]

Si consideri una goccia d'acqua sferica di raggio R in sospensione nell'atmosfera.

1. Determinare la differenza tra la pressione atmosferica esterna, p_e, e quella interna alla goccia, p_i, in termini della tensione superficiale del liquido γ.

Si trascuri l'effetto della forza di gravità. Si veda pure il problema § 28.07.12.

WORKED SOLUTION

Si consideri metà della goccia. Su tale sistema agiscono la risultante delle forze di pressione esterne (pressione atmosferica), la risultante delle forze di pressione interne (pressione esercitata dall'altra semisfera) e la risultante delle forze di tensione superficiale sul contorno circolare della semisfera. Si trascura il peso. La risultante di tutte queste tre forze è diretta parallelamente all'asse della semisfera (per simmetria). In condizioni di equilibrio si ha

$$(p_i - p_e)\,\pi R^2 - 2\pi R\gamma = 0$$

da cui si deduce

$$\boxed{p_i - p_e = \frac{2\gamma}{R}} \qquad (28.07.07)$$

e la pressione interna è maggiore di quella esterna per effetto della tensione superficiale.
Le relazioni ricavate nei problemi § 28.07.12 , § 28.07.11 possono essere ricavate direttamente dalla legge di Laplace (28.06.03), che da la differenza di pressione Δp dalle due parti della superficie di tensione superficiale γ e i cui raggi di curvatura principali sono R_1 e R_2.

WORKED SOLUTION

1. $p_i - p_e = 2\gamma/R$

28.07.12
Pressione All'interno Di Una Bolla Sferica Di Acqua Saponata

This § is referenced at pages:
[1532, 1532]

Si consideri una bolla di sapone sferica in sospensione nell'atmosfera di raggio R. Si supponga la bolla sottile rispetto al raggio R trascurando quindi la differenza tra i raggi interno ed esterno della bolla.

1. Determinare la differenza tra la pressione atmosferica esterna p_e e quella interna alla goccia p_i in termini della tensione superficiale del liquido γ.
2. Determinare il raggio della bolla se il gas all'interno può essere considerato un gas ideale a temperatura uguale alla temperatura esterna T_0. Si veda pure il paragrafo § 28.07.11.

WORKED SOLUTION

Rispetto al paragrafo § 28.07.11 ora le superfici sono due: quella interna e quella esterna. Ragionando come nel paragrafo § 28.07.11 e trascurando il peso si ha allora

$$(p_i - p_e)\,\pi R^2 - 4\pi R\gamma = 0$$

da cui si deduce

$$p_i - p_e = \frac{4\gamma}{R} \tag{28.07.08}$$

e la pressione interna è maggiore di quella esterna per effetto della tensione superficiale.
L'equazione di stato dei gas perfetti applicata al gas all'interno della bolla diventa

$$\left[p_e + \frac{4\gamma}{R}\right]\frac{4\pi}{3}R^3 = Nk_{\text{B}}T$$

da cui si può ricavare il raggio della bolla R.
Read also **Laplace equation** (28.06.03).

WORKED SOLUTION

1. $p_i - p_e = 4\gamma/R$.

28.07.13
Bolla D'aria Sommersa E Tensione Superficiale

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28.07.14
Equilibrium of a Gravitating Non-Rotating Spherical Fluid Mass

This § is referenced at pages:
[1543, 1543, 1543, 1667]

Equilibrium is give by the balance between gravity and thermal pressure.
Spherical cap of radius r, height h and polar angle θ, and limits for small θ.

$$A_c[\theta] = 2\pi r^2\left(1-\cos\theta\right) \to \pi r^2\theta^2 \qquad \text{surface of the cap} \quad ,$$
$$A_b[\theta] = \pi r^2 \sin^2\theta \to \pi r^2\theta^2 \qquad \text{surface of the plane base of the cap} \quad ,$$
$$L[\theta] = 2\pi r\sin\theta \to 2\pi r\theta \qquad \text{length of the circumference} \quad .$$

The mass of fluid contained at a radius smaller than r is:

$$\boxed{\frac{\mathrm{d}M[r]}{\mathrm{d}r} = 4\pi r^2\rho[r] \qquad \text{first stellar structure equation}} \quad .$$

Spherical and spherically symmetric fluid mass. Consider a right circular cone of radius, centered at the center of the spherical mass, of fixed angle θ.

$$p[r]A_b[r] - p[r+\Delta r]A_b[r+\Delta r] + L[r]\Delta r p[r+\Delta r/2]\sin\theta = \mathcal{G}\rho[r]A_c\Delta r\frac{M[r]}{r^2} \quad , \quad ,$$

$$p[r]\pi r^2\sin^2\theta - p[r+\Delta r]\pi\left(r+\Delta r\right)^2\sin^2\theta + 2\pi r\sin\theta\Delta r p[r+\Delta r/2]\sin\theta = \mathcal{G}\rho[r]\Delta r\frac{M[r]}{r^2}2\pi r^2\left(1-\cos\theta\right) \quad , \quad ,$$

$$p[r]r^2 - p[r+\Delta r]\left(r+\Delta r\right)^2 + 2r\Delta r p[r+\Delta r/2] = \mathcal{G}\rho[r]\Delta r\frac{M[r]}{r^2}r^2 \qquad \text{limit for } \theta\to 0 \quad , \quad ,$$

$$-\frac{\mathrm{d}p[r]r^2}{\mathrm{d}r} + 2rp[r] = -r^2\frac{\mathrm{d}p[r]}{\mathrm{d}r} = \mathcal{G}\rho[r]M[r] \qquad \text{limit for } \Delta r\to 0 \quad , \quad ,$$

$$\boxed{-\frac{\mathrm{d}p[r]}{\mathrm{d}r} = \frac{\mathcal{G}\rho[r]M[r]}{r^2}} \qquad \text{second stellar structure equation} \quad .$$

Note that $\theta = \Delta\theta$ in this problem, as the polar angle is counted from zero.

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Application of formula (28.02.01) would give directly:

$$\boxed{\frac{\mathrm{d}p[r]}{\mathrm{d}r}\widehat{\mathbf{e}}_r = -\frac{\mathcal{G}\rho[r]M[r]}{r^2}\widehat{\mathbf{e}}_r \equiv -\rho[r]\mathbf{g}} \quad .$$

The pressure is the sum of the gas pressure (molecules, ions and electrons) and the photon radiation pressure:

$$\boxed{p = \eta_{\mathrm{N}} k_{\mathrm{B}} T + \frac{4\sigma_{\mathrm{SB}}}{3c}T^4} \qquad \text{treating the molecular gas as a ideal gas} \quad .$$

Gas pressure is most important in low mass stars; radiation pressure is most important in high mass stars.

This system of two equations may be solved by specifying an equation of state relating these quantities. The simplest model is obtained by assuming that the pressure is only a function of the density so that one can solve these two equations independently from the equations involving temperature.

Read also § 32.08.08.

28.07.15
Torricelli Equation

28.07.16
Siphoning Out Gasoline From a Fuel Tank

28.08
Exercises Problems and Physical Applications

28-001 Icebergs

1. Un iceberg di volume $1000\ \mathrm{km}^3$ si scioglie. Di quanto sale il livello del mare?
2. Di quanto emerge dall'acqua un iceberg di forma sferica?

28-002 Spinta Di Archimede Su Una Sfera Sommersa

Una oggetto sferico omogeneo di raggio R è immerso in un liquido di densità ρ.

1. Se il centro della sfera si trova ad un profondità H sotto il livello del liquido, $H > R$. Mostrare, con il calcolo esplicito, che la spinta di Archimede risultante è diretta verso l'alto e ha modulo $F_A = m_a g$ dove m_a è la massa di liquido spostata.
2. Se la sfera non è completamente immersa e $0 \leq H \leq R$ calcolare la forza di Archimede.

WORKED SOLUTION

Si scelga un Coordinate System con origine nel centro della sfera sommersa e asse z rivolto verso l'alto. Si introduca l'angolo polare θ, che misura gli angoli a partire dall'asse z, ($0 \leq \theta \leq \pi$) e l'angolo azimuthale ϕ ($0 \leq \phi \leq 2\pi$). Sia $\widehat{\mathbf{e}}_r$ il versore radiale. Si ponga l'origine degli assi al centro della sfera.

L'andamento della pressione del liquido, all'equilibrio, è dato dalla legge di Stevino (28.07.01). La pressione di riferimento, $p_{\mathrm{A}} + \rho g H$, è la pressione alla quota a cui si trova il centro della sfera.

La forza infinitesima che agisce sull'elemento infinitesimo della superficie della sfera di area $\mathrm{d}\mathbf{S}$ e dovuta alla pressione vale

$$\mathrm{d}\mathbf{F} = -p[z]\ \mathrm{d}\mathbf{S}\widehat{\mathbf{e}}_r \quad .$$

La componente orizzontale di tale forza è tale che il suo integrale sull'angolo azimuthale ϕ fa zero, per ragioni di simmetria. La componente verticale vale

$$\mathrm{d}F_z = -p[z]\cos[\theta]\ \ \mathrm{d}\mathbf{S} \quad ,$$

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con

$$\mathbb{d}\mathbf{S} = r^2\, \mathbb{d}\Omega = r^2 \sin[\theta]\ \mathbb{d}\theta\, \mathbb{d}\phi \quad ,$$

e ha integrale non nullo sull'angolo azimuthale ϕ. Inoltre

$$p[z] = p_{\text{A}} + \rho g H - \rho g r \cos[\theta] \quad .$$

Ne segue

$$\mathbb{d}F_z = -\left(p_{\text{A}} + \rho g H - \rho g r \cos[\theta]\right) r^2 \cos[\theta] \sin[\theta]\ \mathbb{d}\theta\, \mathbb{d}\phi \quad .$$

1. La forza risultante vale:
$$F_z = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \mathbb{d}F_z = \frac{4\pi}{3}\rho g r^3 = m_a g \quad .$$
il cui valore e modulo coincidono con quanto previsto dalla spinta di Archimede.
La forza risultante non dipende dalla profondità H a cui è immerso l'oggetto, purché esso sia completamente immerso.
2. La forza risultante vale:
$$F_z = \int_{\cos\theta=H/R}^{\pi} \int_{\phi=0}^{2\pi} \mathbb{d}F_z = \frac{4\pi}{3}\rho g r^3 = m_a g \quad .$$
La forza risultante dipende dalla profondità H a cui è parzialmente immerso l'oggetto.

28-003 Forze E Momenti Agenti Sulla Parete Di Una Diga

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Una diga ha un muraglione di forma rettangolare, verticale, di larghezza L ed il livello dell'acqua è H.
1. Calcolare la forza risultante che agisce sul muro dovuta alla pressione della massa d'acqua.
2. Si consideri il momento risultante delle forze di pressione che agiscono sul muro rispetto ad un polo, F, posto alla base della parete della diga, al livello del fondo. Calcolare la componente di tale momento rispetto alla direzione orizzontaLE della linea di intersezione tra la superficie libera dell'acqua e la parete verticale della diga.

WORKED SOLUTION

Si scelga un Coordinate System con asse z verticale coincidente con la facciata del muro della diga rivolta verso l'acqua, asse x orizzontale, perpendicilarmente alla parte della diga e diretto dall'acqua verso il muro e coincidente con il livello dell'acqua. Con tale scelta il livello del fondo è $z = -H < 0$. La pressione idrostatica si esprime con la relazione

$$p = p_0 - \rho_a g z$$

dove p_0 è la pressione atmosferica.

1. La forza infinitesima che agisce sul muro della diga dovuta all'acqua posta alla quota z e diretta lungo la direzione positiva dell'asse x vale
$$\mathbb{d}F = p\ \mathbb{d}\mathbf{S} = (p_0 - \rho_a g z)\, L\, \mathbb{d}z$$
Ne segue l'espressione per la forza risultante
$$F = \int_{-H}^{0} \mathbb{d}F = LH\left(p_0 + \rho_a g \frac{H}{2}\right) = LHp(z = -\frac{H}{2})$$
che risulta essere uguale all'area della superficie immersa moltiplicata per il valore della pressione alla quota del centroide della superficie immersa stessa. Il valore della forza risultante sul muro dovuta alla sola pressione differenziale dell'acqua si ottiene dalla relazione

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precedente ponendo $p_0 = 0$:

$$F^{\Delta} = \rho_a g \frac{LH^2}{2}$$

2. Per la scelta fatta del Coordinate System occorre calcolare la componente del momento delle forze di pressione rispetto all'asse y. Consideriamo nel seguito implicitamente le componenti dei momenti rispetto a tale asse. Essendo il sistema delle forze di pressione applicato dall'acqua al muro un sistema di forze parallele possiamo calcolare la quota del centro di tale sistema di forze, z_c:

$$z_c \equiv \frac{1}{F}\int_{-H}^{0} z \, \mathrm{d}F = -\frac{LH^2}{F}\left[\frac{p_0}{2} + \frac{\rho_a g H}{3}\right] = -H\left[\frac{\frac{p_0}{2} + \frac{\rho_a g H}{3}}{p_0 + \frac{\rho_a g H}{2}}\right] = -\frac{\Gamma_S}{F} \qquad (28.08.01)$$

→ 1536

dove $\mathbf{\Gamma}_S$ è il momento risultante sul muro dovuto alle forze di pressione rispetto ad un polo posto sul muro al livello della superficie. La quota del centro del sistema di forze sul muro dovute alla sola pressione differenziale dell'acqua si ottiene dalla relazione precedente ponendo $p_0 = 0$

$$z_c = -\frac{2}{3}H$$

Il sistema di forze sul muro dovute alla sola pressione differenziale dell'acqua è dunque equivalente (ha cioè la stessa risultante e lo stesso momento rispetto ad un polo qualunque) al sistema costituito dalla forza risultante applicata al centroide della superficie immersa che ha coordinate $x_c = 0$ e z_c dato dalla relazione (28.08.01). Il momento totale delle forze di pressione rispetto ad un polo a livello del fondo si calcola allora tenendo conto che il braccio della forza rispetto al fondo vale

$$b = z_c + H = \frac{H}{2}\left[\frac{p_0 + \frac{\rho_a g H}{3}}{p_0 + \frac{\rho_a g H}{2}}\right]$$

per cui il momento risulta

$$\Gamma_F = -\frac{LH^2}{2}\left[p_0 + \rho_a g \frac{H}{3}\right]$$

e quello differenziale

$$\Gamma_\Delta = \rho_a g \frac{LH^3}{6}$$

la relazione tra $\mathbf{\Gamma}_F$ e $\mathbf{\Gamma}_S$ può essere anche ricavata dalla legge di trasformazione di un momento al variare del polo (15.04.03).

WORKED SOLUTION

1. $F = LH\,(p_0 + \rho_a g H/2)$;
2. $\Gamma_F = -LH^2\,[p_0/2 + \rho_a g H/6]$.

28-004 Parete Inclinata Di Una Diga

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Una vasca ha una delle pareti laterali di lunghezza L inclinata di un angolo α rispetto all'orizzontale. Se il livello di acqua nella vasca è H determinare la forza totale agente sulla parete della vasca dovuta alla pressione differenziale dell'acqua. Utilizzare tale risultato per spiegare il peso di un recipiente con acqua le cui pareti non sono verticali.

WORKED SOLUTION

Si scelga un Coordinate System con asse z verticale rivolto verso l'alto passante per e avente origine sullo spigolo inferiore della vasca. Si introduca poi un altro Coordinate System avente la stessa origine e asse z' coincidente con la parete inclinata. Con tale scelta il livello dell'acqua vale $z = H > 0$ oppure $z' = H/\sin[\alpha] > 0$. Essendo $z = z'\sin[\alpha]$ la pressione idrostatica si esprime con la relazione

$$p[z] = p_0 - \rho_a g\,(z - H) = p_0 - \rho_a g\,(z' - H')\sin[\alpha]$$

dove p_0 è la pressione atmosferica. La forza infinitesima che agisce sul muro della diga dovuta all'acqua posta alla quota z è diretta perpendicolarmente alla parete, verso l'esterno ed ha modulo

$$\mathrm{d}F = pL\,\mathrm{d}z' = \left(p_0 - \rho_a g\,(z' - H')\sin[\alpha]\right) L\,\mathrm{d}z'$$

da cui

$$F = \int_0^{H'} \mathrm{d}F = LH'\left(p_0 + \rho_a g\frac{H'}{2}\sin[\alpha]\right) = \frac{LH}{\sin[\alpha]}\left(p_0 + \rho_a g\frac{H}{2}\right) = LH' p(\frac{H}{2})$$

che coincide con la superficie della parete LH' moltiplicato per la pressione calcolata alla metà profondità $z = H/2$. Il valore della forza risultante sul muro dovuta alla sola pressione differenziale dell'acqua si ottiene dalla relazione precedente ponendo $p_0 = 0$

$$F^{\Delta} = \rho_a g\frac{LH^2}{2\sin[\alpha]}$$

Il momento risultante rispetto al punto in cui la parete giunge al livello dell'acqua si calcola usando il sistema risultante delle forze applicate e vale

$$\Gamma_0 = |\,\mathbf{F}\wedge\mathbf{r}_c| = \frac{LH^2}{\sin^2\alpha}\left(\frac{p_0}{2} + \rho_a g\frac{H}{3}\right)$$

che coincide con il momento della forza risultante F, calcolato sopra, moltiplicato per il braccio della retta di applicazione della forza risultante del sistema di forze applicate, $H'/3$.
La componente verticale della forza risultante sul muro dovuta alla sola pressione differenziale dell'acqua ha modulo

$$F_v = F\cos[\alpha] = \frac{\rho_a g H^2 L}{2\tan[\alpha]}$$

Tale forza coincide con il peso della massa di acqua che sta sopra/sotto la parete inclinata. Tale forza spiega il valore del peso di un recipiente a pareti inclinate. Si noti che se $-\pi/2 < \alpha < 0$ tale forza è diretta verso l'alto, mentre se $0 < \alpha < +\pi/2 <$ tale forza è diretta verso il basso.
Si noti che il contributo dovuto alla pressione atmosferica p_0 è compensato, nel caso il lato esterno della parete sia esposto alla pressione atmosferica, da un contributo uguale ed opposto alla forza risultante dovuto alla pressione esterna.

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28-005 A Floating Sphere

A homogeneous sphere of radius R and density ρ is floating on a fluid of density $\rho_0 > \rho$. A vertical force acting on the sphere and applied on a straight line passing by the center, presses downward the sphere. Determine the depth of immersion of the sphere as a function of the force.

28-006 Liquido in Un Recipiente in Rotazione Uniforme

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Un fluido incomprimibile ruota con velocità angolare costante ω attorno all'asse verticale di un recipiente cilindrico. Dopo una fase transitoria, il liquiodo si trova a ruotare come un corpo rigido.

Determinare la forma della superficie del liquido; determinare la variazione di pressione entro il fluido nella direzione radiale; determinare la variazione di pressione entro il fluido nella direzione verticale.

WORKED SOLUTION

Si scelga un Reference Frame non inerziale solidale con la massa di fluido ruotante e in esso un Coordinate System cilindrico con asse z coincidente con l'asse del cilindro e orientato verso l'alto. La forza di volume infinitesima che agisce sull'elemento di massa $\mathrm{d}m$ è la somma della forza peso e della forza centrifuga

$$\mathrm{d}\mathbf{F} = \left(\omega^2 r \widehat{\mathbf{e}}_r - g\widehat{\mathbf{e}}_3\right)\,\mathrm{d}m \quad ,$$

ovvero una forza per unità di volume

$$\mathbf{f}_{\mathrm{V}} = \rho\left(\omega^2 r \widehat{\mathbf{e}}_r - g\widehat{\mathbf{e}}_3\right) \quad .$$

Tale forza, all'equilibrio, deve essere compensata da una forza di pressione uguale ed opposta. Alla forza di volume corrisponde un'energia potenziale

$$\mathrm{d}\Phi(r,z) = \left(-\frac{1}{2}\omega^2 r^2 + gz\right)\,\mathrm{d}m \quad ,$$

cioè un energia potenziale per unità di volume

$$\phi(r,z) = \rho\left(-\frac{1}{2}\omega^2 r^2 + gz\right) \quad .$$

Le superfici equipotenziali sono date da

$$u(r,z) = \text{costante} \implies -\frac{1}{2}\rho\omega^2 r^2 + \rho g z = \text{costante} \quad .$$

Da cui segue l'equazione delle superfici equipotenziali

$$z = \frac{\omega^2 r^2}{2g} + z_0$$

La forza di volume per unità di volume vale:

$$\mathbf{f}_{\mathrm{V}} = -\mathbf{grad}\phi \quad ,$$

e la forza di superficie dovuta alle forze di pressione vale, per unità di volume:

$$\mathbf{f}_{\mathrm{p}} = -\mathbf{grad}p \quad ,$$

e la condizione di equilibrio statico fornisce:

$$p = -u + \text{constant} \quad ,$$

e quindi

$$p = -u(r,z) = \frac{1}{2}\rho\omega^2 r^2 - \rho g z + p_0$$

Alternativamente si possono risolvere le equazioni di equilibrio:

$$\frac{\partial p}{\partial r} = \mathbf{f}_V \cdot \hat{\mathbf{e}}_r = \rho\omega^2 r \quad ,$$
$$\frac{\partial p}{\partial z} = \mathbf{f}_V \cdot \hat{\mathbf{e}}_3 = -\rho g \quad ,$$

e, integrando per $p = p(r, z)$ si deduce:

$$p(r, z) = -\rho g z + \frac{1}{2}\rho\omega^2 r^2 + \text{costante} \quad .$$

28-007 Total Force From a Uniform Pressure

Consider an open surface whose boundary lies on a plane, while all the surface is on the same side of the plane. The open surface is subjected to a uniform external pressure. Show that the total force perpendicular to the plane, due to pressure, equals the product of the pressure times the projected area of the surface on the plane.

28-008 Piccole Oscillazioni Di Un Oggetto Galleggiante

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Si supponga di avere un'asta di sezione uniforme A, lunghezza L e massa M con una piccola massa puntiforme m appesa ad un estremo. Immersa in acqua l'asta galleggia con la massa m rivolta verso il basso.

1. Determinare la frequenza delle piccole oscillazioni dell'asta trascurando le perturbazioni al fluido circostante causate dal movimento dell'asta.
2. Stimare la frequenza delle piccole oscillazioni verticali per una petroliera.

WORKED SOLUTION

Detta x la lunghezza della parte di asta sommersa $0 \le x \le L$ l'equazione del moto delle piccole oscillazioni, detta ρ_a la densità dell'acqua e detta ρ la densità dell'asta e trascurando le perturbazioni al fluido circostante causate dal movimento dell'asta, è

$$(m + M)\,\ddot{x} = -\rho_a A g x + mg + \rho A g L$$

La posizione di equilibrio è dunque

$$x_{EQ} = \frac{m + M}{\rho A}$$

Se $m = 0$

$$x_{EQ} = \frac{\rho}{\rho_a} L$$

da cui è evidente che l'asta galleggia se e solo se $\rho < \rho_a$. La frequenza delle piccole oscillazioni vale

$$\omega^2 = \frac{\rho A g}{m + M} = \frac{g}{x_{EQ}}$$

Con i dati del testo $T = 3.17$ s.
Per una petroliera $T \approx 1$ s.

WORKED SOLUTION

1. $\omega^2 = \rho A g / (m + M)$;
2. $T \approx 1$ s.

28-009 Lead an Cork

There is a well known riddle: "What is heavier, a ton of lead or a ton of cork?" Determine how much the actual weight of the cork which is 1 ton in air, exceeds the actual weight of the lead, which is also 1 ton in air. The temperature of the air is 17 °C and the pressure 760 mm Hg.

28-010 Pallone Aerostatico

What should the weight of the rubber skin of a balloon 25 cm in diameter filled with hydrogen be for the resulting lifting force of the balloon to equal zero, i.e., for the balloon to be suspended? The air and hydrogen are in standard conditions. The pressure inside the balloon is equal to the external pressure.

28-011 Vasi Comunicanti Con Mercurio

Find the difference in the levels of mercury in two identical communicating glass tubes if the left-hand tube is maintained at a temperature of 0°Cand the right-hand one is heated to 100°C. The height of the left-hand tube is 90 cm. The coefficient of volume expansion of mercury is $1.82 \cdot 10^{-4}$ deg^{-1} Disregard the expansion of the glass.

28-012 Pressione Atmosferica - Relazione Politropica Di Gas Ideale

This § is referenced at pages:
[2708]

©|Mic, , ..., ..., ...Ed., ...|§ 1.5||
©|Lim Yung-Kuo, *Problems And Solutions On Thermodynamics And Statistical Physics*, 1990, World Scientific, ...Ed., ...

Determinare l'andamento della pressione atmosferica supponendo che l'equazione di stato del gas atmosferico sia $p = K\rho^{\delta}$. Si supponga di poter trattare l'atmosfera come un gas ideale di massa molecolare media $\mathcal{M}_0 = 28.8$ g/mol e descritto dall'equazione di stato del gas ideale, $pV = nRT$, in cui $R = 8.31$ J mol^{-1} K^{-1} è la costante universale del gas ideale, T la temperatura assoluta ed $\mathcal{M}_0$ la massa molare del gas.

1. Si determini l'andamento della pressione atmosferica nel caso T sia uniforme (atmosfera isoterma): $\delta = 1$ e $K = RT/\mathcal{M}_0$.
2. Si determini l'andamento della pressione atmosferica nel caso la temperatura assoluta in funzione della quota z sia assegnata e pari a $T[z] = T_0 - \alpha z$: $\delta \neq 1$ e K costante. Verificare il caso $\delta = 1$. Show that: $\alpha = \dfrac{g\mathcal{M}_0}{R}\dfrac{\delta - 1}{\delta}$.
3. Show that for this case no atmosphere exists above a $z_{\mathtt{MAX}}$ and find $z_{\mathtt{MAX}}$.

Si assuma l'accelerazione di gravità g indipendente dalla quota.

Read also § 67.14.21.

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WORKED SOLUTION

- In condizioni statiche si ha

$$\frac{\mathrm{d}p[z]}{\mathrm{d}z} = -g[z]\rho[z] \tag{28.08.02}$$

Utilizzando l'equazione di stato $p = K\rho$ e integrando si ha

$$p[z] = p_0 \exp\left[-\frac{g\,(z-z_0)}{K}\right] \equiv p[z] = p_0 \exp\left[-\frac{(z-z_0)}{\ell}\right] \qquad \ell \equiv K/g \tag{28.08.03}$$

→ 1541

La quantità ℓ assume allora il significato di lunghezza caratteristica di variazione della pressione atmosferica. L'equazione di stato dei gas perfetti fornisce l'espressione

$$K = \frac{RT}{\mathcal{M}_0}$$

Nel caso dell'atmosfera, assumendo una massa molecolare media $\mathcal{M}_0 = 28.8\ \mathrm{g/mol}$, e temperatura assoluta $T = 300\ \mathrm{K}$ si trova $L = 8.4\ \mathrm{km}$.

- Nel caso la temperatura decresca linearmente con la quota si ha

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -g[z]\rho[z] = -\frac{g\mathcal{M}_0}{R}\left(\frac{p}{T_0 - \alpha z}\right)$$

L'integrazione fornisce

$$p[z] = p_0\left(\frac{T_0 - \alpha z}{T_0 - \alpha z_0}\right)^{\frac{g\mathcal{M}_0}{\alpha R}} \tag{28.08.04}$$

Nel limite $\alpha \to 0$ tale equazione si riduce correttamente alla (28.08.03).
Nel caso $\delta \neq 1$ e K costante l'integrazione fornisce

$$p[z] = p_0^{1/(1-\delta)}\left(p_0 + \frac{g\rho_0\,(z-z_0)\,\delta}{(1-\delta)}\right)^{\delta/(\delta-1)} \tag{28.08.05}$$

Questa equazione è stata ricavata integrando sotto la condizione che $\delta \neq 1$. Per ottenere il risultato calcolato in precedenza occorre calcolare il limite per $\delta \to 1$ dell'espressione ottenuta.

- $$z_{\mathtt{MAX}} = \frac{\delta}{\delta - 1}\frac{RT}{\mathcal{M}_0 g} \quad .$$

Read also § 67.14.21.

28-013 Serbatoio Bucato E Coefficiente Di Efflusso

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Un serbatoio di forma cilindrica con altezza H ha un piccolo buco, di area A da cui fuoriesce un piccolo getto di liquido. Si supponga il buco così piccolo da poter considerare il moto laminare, stazionario, e si supponga di poter trascurare la viscosità.

1. Se il serbatoio è aperto all'atmosfera e ha il buco al centro della base, si calcoli la velocità del getto all'uscita dal buco.
2. Se il buco è ad un'altezza h dal fondo e fuoriesce un piccolo getto di liquido in direzione orizzontale e il serbatoio è aperto all'atmosfera, si calcoli per quale posizione del buco il getto che fuoriesce arriva a terra alla massima distanza.
3. Se il serbatoio è chiuso e contiene gas a pressione p_1, quanto vale la velocità di efflusso?

Si supponga che la pressione del gas all'interno sia così elevata da poter trascurare il contributo di energia potenziale gravitazionale del gas.

WORKED SOLUTION

Si applichi il Bernoulli equation ad una linea di flusso tra un punto alla superficie del liquido del serbatoio ed un punto all'orifizio. Si ha:

$$p_0 + \rho g z_0 + \frac{\rho v_0^2}{2} = p_1 + \rho g z_1 + \frac{\rho v_1^2}{2} \quad . \tag{28.08.06}$$

→ 1542

L'equazione di continuità fornisce:

$$v_0 A_0 = v_1 A_1 \quad .$$

Però all'uscita del foro il tubo di flusso è in fase di restringimento e solo dopo essere uscita un poco le velocità tendono a diventare parallele e la sezione costante (vena contracta) ed è tale valore che dovrebbe essere usato per la sezione nell'equazione $v_0 A_0 = v_1 A_1$, non la sezione dell'orifizio. è a questo punto che l'equazione di continuità può essere applicata. Se però $A_0 \gg A_1$ si può trascurare v_0 rispetto a v_1. Il flusso uscente non può essere calcolato moltiplicando la velocità v_1 per l'area del foro A_1. Infatti, in generale, le velocità del liquido che esce dal buco avranno anche una componente di velocità radiale, all'uscita. Il flusso reale sarà allora inferiore a $A_1 v_1$ e dipende dalla forma dell'ugello. Il flusso sarà dato dalla velocità moltiplicata per la sezione del tubo di flusso nel punto in cui le velocità sono diventate parallele.

1. ...
2. Essendo il serbatoio aperto $p_0 = p_1$, la pressione atmosferica. Allora la velocità di efflusso vale
$$v_1 = \sqrt{2g\,(H-h)} \quad .$$
Il punto in cui il getto arriva a terra, rispetto al cilindro, è dato dalla relazione
$$\Delta x = \sqrt{\frac{2v_1^2 h}{g}} = 2\sqrt{h\,(H-h)}$$
la distanza massima si ha allora quando è massimo il prodotto $h\,(H-h)$ cioè per $h = H/2$. Si noti che la velocità dipende dall'altezza della colonna di liquido nel serbatoio.
3. Dall'equazione (28.08.06) ora, supponendo ancora di poter trascurare v_0 rispetto a v_1, e trascurando il termine di energia potenziale gravitazionale, si ha
$$v_1 = \sqrt{\frac{2\,(p_1 - p_0)}{\rho}}$$
dove la densità è quella del fluido uscente dal foro. La velocità di efflusso di un gas può essere grande, anche per basse pressioni, perché la densità è piccola. Occorre tuttavia tener presente che se la pressione è troppo grande può non essere lecito trattare il fluido come incomprimibile. Inoltre se la velocità è troppo grande il moto del fluido può diventare turbolento. In entrambi i casi l'equazione di Bernoulli non può essere applicata. Nel caso di applicabilità del Bernoulli equation si può calcolare la spinta F prodotta dall'uscita del gas ad alta velocità:
$$F = 2A\,(p_1 - p_0) \quad .$$

28-014 Condizione Di Applicabilità Del Concetto Di Fluido Incompressibile

©|D.V.Sivuchin, , ..., ..., ...Ed., ...|§ 94||

Dimostrare la condizione per cui un fluido non viscoso in moto stazionario e laminare può essere considerato incompressibile:

$$\boxed{\frac{|\Delta\rho|}{\rho} \simeq \frac{|\Delta v^2|}{2c^2} \ll 1} \quad . \tag{28.08.07}$$

28-015 Stabilità Statica Dell'atmosfera

©|Mic, , ..., ..., ...Ed., ...|§ 1.s2||

28-016 Freni Idraulici Dell'auto

28-017 Differenza Di Livello Tra Le Due Spende Del Canale Di Panama

28-018 Equilibrio Di Una Massa Fluida Non Routante a Simmetria Sferica

©|Mic, , ..., ..., ...Ed., ...|§ 1.6||

Read § 28.07.14.

28-019 Pressione Al Centro Della Terra

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This § is referenced at pages:
[Never referenced.]

Read § 28.07.14.

28-020 Forma Di Equilibrio Di Un Pianeta Fluido Omogeneo Non Ruotante

©|Berkeley Physics Course, in 5 Vol., , ..., McGraw-Hill, ...Ed., ...|9.17||

Read § 28.07.14.

28-021 Moto Alla Poiseuille

1. Evaluate the rotor of a Poiseuille motion.
2. Verificare che la costante nella Bernoulli equation è diversa a seconda della linea di flusso.

28-022 Flotation in Little Fluid

A 4 in diameter solid cylinder of height 3.75 in weighing 0.85 lb is immersed in water contained in a tall, upright metal cylinder having a diameter of 5 in. Before immersion the liquid was 3.0 in deep. At what level will the solid cylinder float?

28-023 Stabilità Di Una Nave Con Centro Di Massa Più Alto Del Centro Di Spinta

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CHAPTER 29

Introduction to Central Force Fields and Gravitation

29.01 Conics 1546
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This § is referenced at pages:
[1136]

©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|9|*Good introduction*|

©|**WEB - URL** |Nicholas Wheeler Physics Lectures|*advanced*|

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29.01 Conics

This § is referenced at pages:
[1142]

29.01.01 Conics in Cartesian Coordinates Coordinate System

©|T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...|§ 13|*Clear and complete*|
©|T.M.Apostol, *Calcolo, Vol. 2, Geometria*, 1985, Bollati-Boringhieri, ...Ed., ...||*Clear and complete*|

29.01.02 Conics in Polar Coordinates

©|T.M.Apostol, *Calculus Vol. 1,2*, 1967/1969, J.Wiley & Sons, ...Ed., ...|§ 13|*Clear and complete*|
©|T.M.Apostol, *Calcolo, Vol. 2, Geometria*, 1985, Bollati-Boringhieri, ...Ed., ...||*Clear and complete*|

Consider, in a plane, a point P, a fixed point, F, the focus, and a straight line, D, the directrix, such that F does not belong to D. Let Q be one of the two points such that FP is parallel to the directrix. Let K be the intersection between the directrix and the perpendicular to it passing by P.

Let:

- h be the distance between the focus and the directrix;
- $f \equiv \overline{PF}$ be the distance between the point P and the focus;
- $d \equiv \overline{PK}$ be the distance between the point P and the directrix;
- $p \equiv \overline{FQ}$.

A conic is the locus of the points, P, in the plane such that the ratio

$$e \equiv \frac{f \equiv \overline{PF}}{d \equiv \overline{PK}} \geq 0 \quad ,$$

is a fixed non negative constant e (the eccentricity); see figure 29.11.

Introduce polar coordinates (r and φ) in the plane, centered in F and with $\varphi = 0$ being the line passing by the focus and perpendicular to the directrix; positive angles counter-clockwise.

Note that as long as $0 \leq e \leq 1$ the point P can only be in the half-plane on the same side as F with respect to the directrix. On the other hand, if $e > 1$, the point P can be on both sides of the directrix.

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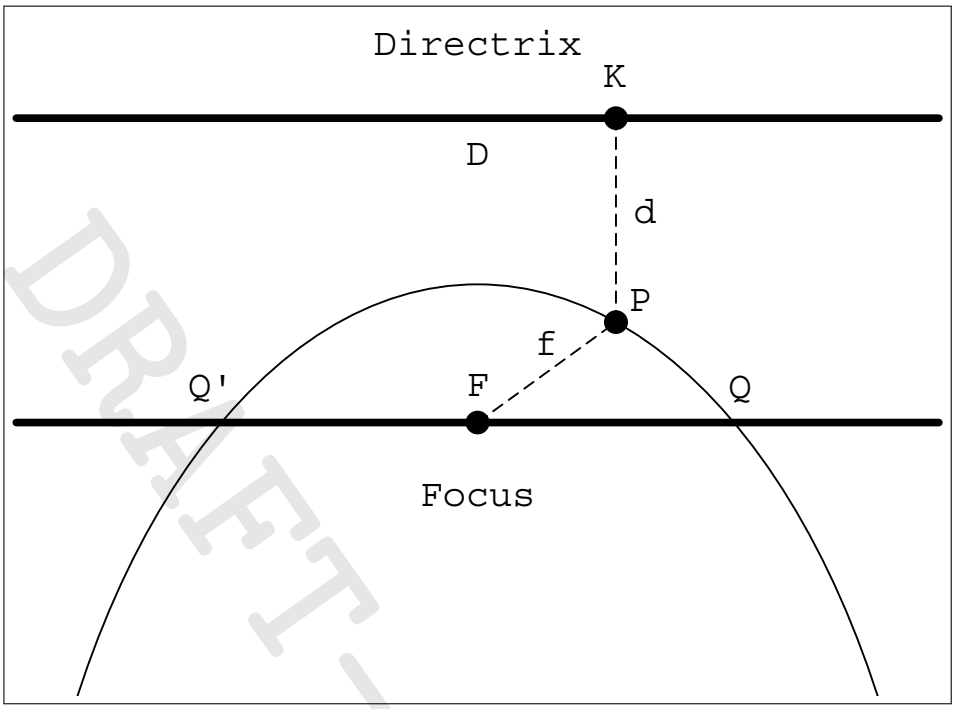

Figure 29.11: FIGURE

The following relations apply:

$$\boxed{0 \leq e = \frac{f}{d} \equiv \frac{\rho}{d}} \quad , \qquad \rightarrow 1547$$

$$e = \text{constant} = \frac{f}{d} \equiv \frac{\overline{FQ}}{\overline{KQ}} \equiv \frac{p}{h} \quad ,$$

$$\boxed{d = |h - \rho \cos[\varphi]|} \quad , \qquad \rightarrow 1547$$

$$\begin{cases} h = d + \rho \cos[\varphi] & \text{for } 0 \leq e \leq 1 \\ h = \pm d + \rho \cos[\varphi] & \text{for } e > 1 \end{cases} \quad ,$$

$$h = \alpha d + \rho \cos[\varphi] \qquad \begin{cases} \alpha \equiv +1 & \text{for } 0 \leq e \leq 1 \\ \alpha \equiv \pm 1 & \text{for } e > 1 \end{cases} \quad ,$$

$$h = \frac{\rho}{e}(\alpha + e \cos[\varphi]) \quad ,$$

$$\boxed{\rho[\varphi] = \frac{p}{\alpha + e \cos[\varphi]} \qquad \begin{cases} \alpha \equiv +1 & \text{for } 0 \leq e \leq 1 \\ \alpha \equiv \pm 1 & \text{for } e > 1 \end{cases}} \quad . \qquad \rightarrow 1548$$

One defines:

- the ellipse for $0 \leq e < 1$, defined for $0 \leq \varphi \leq 2\pi$;
- the parabola for $e = 1$, defined for $0 \leq \varphi < 2\pi$;
- the hyperbola for $e > 1$:
  - the branch $\alpha \equiv +1$ is defined for $-\arccos[-1/e] < \varphi < +\arccos[-1/e]$;
  - the branch $\alpha \equiv -1$ is defined for $-\arccos[+1/e] < \varphi < +\arccos[+1/e]$.

Note that the $\pm$ sign applies to, and only to, the case of $e > 1$, when the conic has two branches on the two different sides of the directrix with respect to the focus (the two branches of the hyperbola). In this case, and only in this case, the absolute value in equation (29.01.02) is mandatory. In the other cases: $\rho \leq d$, as from equation (29.01.02).

In physics problems both branches of the hyperbola must be considered, a priori, because depending

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of the specific physic problem one or the other might intervene, read § 29.04.04.02.

In the case $e = 1$ (parabola) one can write:

$$\rho[\varphi] = \frac{p}{2\cos^2[\varphi/2]} \quad .$$

Note that the case $e = 0$ gives the circle. This is obtained when the directrix is located at infinity.

The curve is limited if and only if $0 \leq e < 1$ (ellipse).

In the other cases, $e \geq 1$, (parabola and hyperbola) the asymptotes, where the distance goes to infinity, are given by: $\alpha + e\cos[\varphi] = 0$, showing that for the parabola one should consider two degenerate asymptotes at $\varphi = \pi$.

There are (at least) six geometric parameters defining the conics: h, e, p, a, c and b, (to be defined below). The starting geometric parameters are h and e. The first one, h, is immediately replaced by p via

$$\boxed{e \equiv \frac{p}{h}} \quad .$$

The semi-major axis is defined as half of the distance between the two vertexes, relative extrema of the distance as a function of φ:

$$2a \equiv \frac{p}{+1-e} + \frac{p}{+1+e} = \boxed{\frac{2p}{1-e^2} = 2a > 0 \implies p = a\left(1-e^2\right)} \qquad \text{for } 0 \leq e < 1 \quad ,$$

$$2a \equiv \frac{p}{-1+e} - \frac{p}{+1+e} = \boxed{\frac{2p}{e^2-1} = 2a > 0 \implies p = a\left(e^2-1\right)} \qquad \text{for } e > 1 \quad ,$$

$$a = +\infty \qquad \boxed{p \text{ is the meaningful parameter for the parabola}} \qquad \text{for } e = 1; \quad .$$

The last one, the parabola, is found as the limiting case of the other two cases.

The distance from the center of the conic to the foci is:

$$\boxed{c = ea} \quad ,$$

which is $+\infty$ for $e = 1$.

The semi-minor axis (except for $e = 1$) is:

$$\boxed{b = \sqrt{|a^2 - c^2|} = a\sqrt{|1-e^2|}} \quad ,$$

which is $+\infty$ for $e = 1$.

Moreover:

$$\boxed{p = a|1-e^2| = |a - ce| \implies ap = b^2} \quad , \quad ,$$

$$\boxed{(2a-p)^2 = 4c^2 + p^2 \implies p = a\left(1-e^2\right) \implies ap = b^2} \quad .$$

29.01.02.01 Cartesian Equations for Conics With One Focus at the Origin

Starting from the conic equation (29.01.02) one can deduce the Cartesian equation. In fact

$$\alpha\rho = p - \rho e\cos[\varphi] \quad ,$$

and therefore, taking the square of both members, one finds:

$$\boxed{x^2\left(1-e^2\right) + y^2 + 2pex = p^2} \quad . \qquad (29.01.01) \quad \rightarrow 1548$$

The equation (29.01.01) shows that the conic is symmetrical with respect to the y axis.

Except for the case $e = 1$ one can do the change of variables:

$$x = X + \frac{pe}{e^2-1} \qquad y = Y \quad ,$$

to find the equation:

$$X^2\left(1-e^2\right) + Y^2 = \frac{p^2}{1-e^2} \quad . \qquad (29.01.02) \quad \rightarrow 1549$$

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The equation (29.01.02) shows that the conic (except for the case $e = 1$) is symmetrical not only with respect to the y axis but also with respect to the X axis: $X = x - \dfrac{pe}{e^2 - 1}$. Therefore the center of symmetry of the conic (except for the case $e = 1$) is given by:

$$x = x_0 = \frac{pe}{e^2 - 1} \qquad \begin{cases} x_0 \equiv -c < 0 & \text{for } 0 \leq e < 1 \\ x_0 \equiv +c > 0 & \text{for } e > 1 \end{cases} , \qquad c \equiv \frac{pe}{|e^2 - 1|} .$$

Equation (29.01.02) can be written in the standard Cartesian form:

$$\boxed{X^2 \frac{\left(1 - e^2\right)^2}{p^2} + Y^2 \frac{\left(1 - e^2\right)}{p^2} = 1} . \tag{29.01.03}$$

The comparison with the standard Cartesian equation provides the identification with the major and minor axes:

$$a \equiv \frac{p}{|1 - e^2|} ,$$

$$b \equiv \frac{p}{\sqrt{|1 - e^2|}} ,$$

$$p = \frac{b^2}{a} .$$

The relations among the quantities a, b and c follow:

$$a^2 = c^2 + b^2 \qquad \text{for } 0 \leq e < 1 ,$$
$$a^2 = c^2 - b^2 \qquad \text{for } e > 1 .$$

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29.02 General Properties of Central Force Fields

This § is referenced at pages:
[1552]

©|H.Pollard, *Mathematical Introduction To Celestial Mechanics*, 1966, Prentice-Hall, ...Ed., ...|1||

DRAFT IN PROGRESS

The interest of the study of the general Central Force Fields comes from the fact that different central force fields appear in many applications. Moreover, most often, real problems can be solved by approximate methods based on central force fields. Central force fields are therefore the basis for the study of a great many of problems.

Some examples are: the motion in a pure gravitational or ElectroStatic field, the motion of a point mass in the equatorial plane of a gravitational field with cylindrical symmetry (having a **Quadrupole** , and possibly higher-order **Multipoles** , potential), the motion of a star in a galaxy or the motion of a galaxy in a cluster of galaxies, the general motion of an artificial satellite around the Earth, the classical motion of an electron in the field of the nucleus plus the mean field of other electrons. Moreover, any mass distribution at large enough distances, appears approximately, as a mass point.

Consider a point particle with mass m subject to the interaction with an external field and assume that the interaction of the particle with the field is fully described by a force field only (no torques, for instance, nor any other effect of the interaction due, for instance, to higher order **Multipole** fields).

By definition, a force field is named a radial force field (around the pole $\mathbf{r}_0$) if and only if

$$(\mathbf{r}-\mathbf{r}_0)\wedge\mathbf{f}[\mathbf{r}]=0 \quad , \tag{29.02.01}$$

for every point $\mathbf{r}$ in one Inertial Reference Frame . A radial force field has zero momentum with respect to the pole $\mathbf{r}_0$.

Usually one chooses: $\mathbf{r}_0=0$; this is always possible as long as the pole has constant velocity in some Inertial Reference Frame thanks to the homogeneity of space.

When dealing with radial force fields it is therefore assumed that the only interaction is the force and there is no torque nor anything else.

A radial and conservative force field is named a central force field. A Central Field is necessarily a field with spherical symmetry, that is has the most general expression

$$\boxed{\mathbf{f}=f_r[r]\widehat{\mathbf{e}}_r} \quad ,$$

as discussed in sections § 16-013 and § 16-017. Conversely: a non spherically symmetric field is either non radial or non conservative (quite unlikely for physical fields).

Let us consider the central force acting on a point mass. As the torque of the central force is null, the angular momentum of the point mass with respect to the pole is constant:

$$\boldsymbol{\ell}\equiv\mathbf{r}\wedge\mathbf{p}=m\,\mathbf{r}\wedge\dot{\mathbf{r}}\equiv m\mathbf{c} \quad , \qquad \text{with} \qquad \boxed{\dot{\boldsymbol{\ell}}=0} \quad . \tag{29.02.02}$$

→ 1550

Therefore the mass point lies in a fixed plane passing by the pole. In fact equation (29.02.02) implies, by taking the dot product with $\mathbf{r}$,

$$\mathbf{r}\cdot\mathbf{c}=0 \quad ,$$

that is the position vector is always perpendicular to the constant vector $m\mathbf{c}\equiv\boldsymbol{\ell}$. If $\mathbf{c}=\mathbf{0}$ then the trajectory becomes a straight line. The vectors $\mathbf{c}$ and $\boldsymbol{\ell}$ are determined by the initial conditions.

The motion lies in the plane orthogonal to the angular momentum $\boldsymbol{\ell}$, for $\boldsymbol{\ell}\neq 0$, and passing by the center of the force. Therefore one can consider the corresponding plane motion and use a polar Coordinate System in this plane, centered on the force center. Let the z axis be perpendicular to the plane of the orbit. Due to the previous considerations the simplified notation $\ell\equiv\ell_z$ will be always used.

From the discussion in section § 23.04.04.02 the constancy of the angular momentum can be stated in terms of areolar velocity:

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} \equiv \frac{1}{2}|\,\mathbf{x}\wedge\mathbf{v}\,| = \frac{1}{2}r^2\dot{\varphi} \equiv \frac{c}{2}} \quad . \tag{29.02.03}$$

→ 1135 1591

Quindi la conservazione del momento angolare equivale alla costanza della velocità areolare (equation (23.04.03)), fatto che si suole esprimere dicendo che il raggio vettore spazza aree uguali in tempi uguali. This is in fact **the second Kepler law**, which is thus valid for any central force field.

It also follows that:

$$\boxed{\mathbf{r}\wedge\ddot{\mathbf{r}} = 0} \quad . \tag{29.02.04}$$

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29.03 Motion in a Central Force Field

All the previous discussions only depend on the central character of the force and not on the specific form of its radial dependence. The latter only enters in the shape of the trajectory. While the determination of the time dependence $r[t]$ and $\varphi[t]$ is normally complex the determination of the trajectory or orbit, $r[\varphi]$, is often easier.

A motion in a central force field,

$$\mathbf{f}[\mathbf{r}] = f_r[r]\widehat{\mathbf{e}}_r \equiv f[r]\widehat{\mathbf{e}}_r \qquad \begin{cases} f_r[r] > 0 & \text{repulsive force} \\ f_r[r] < 0 & \text{attractive force} \end{cases} ,$$

is a plane motion (section § 29.02). Let z be the axis perpendicular to the plane of motion, that is parallel to $\boldsymbol{\ell}$, oriented as $\boldsymbol{\ell}$ and passing by the center of force. With this choice the motion is in the counter-clockwise direction in the xy plane.

The equation of motion in the plane of the orbit can be written in polar coordinates with origin at the pole of the force field:

$$ma_r = m\left(\ddot{r} - r\dot{\varphi}^2\right) = f[r] \tag{29.03.01}$$

→ 1552 1577

$$ma_\varphi = m\left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right) = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\varphi}\right) = 0 \Longrightarrow \boxed{\ell_z = \text{constant}} \quad .$$

29.03.01 Conservation of the Angular Momentum

Note that the use of the conservation of angular momentum, equation (29.03.01), allows to reduce the system of two differential equations in two variables, $r[t]$ and $\varphi[t]$, to one single differential equation in the variable $r[t]$.

Note that:

$$\boldsymbol{\ell} \equiv \mathbf{r}\wedge\mathbf{p} = m\,\mathbf{r}\wedge\mathbf{v} = m\,(r\widehat{\mathbf{e}}_r)\wedge(\dot{r}\widehat{\mathbf{e}}_r + r\dot{\varphi}\widehat{\mathbf{e}}_\theta) = mr^2\dot{\varphi}\widehat{\mathbf{e}}_3 \quad .$$

→ 1552 1556 1577

From equations (29.03.01), (29.03.01) one can derive:

$$m\ddot{r} = mr\dot{\varphi}^2 + f[r] = \frac{\ell^2}{mr^3} + f[r] \equiv f_{\mathrm{E}}[r] \quad ,$$

in terms of the effective radial force, $f_{\mathrm{E}}[r]$, taking into account the real force and the centrifugal force seen by the observer rotating with the point particle. Note that the additional force appearing in the radial equation, the centrifugal force $\dfrac{\ell^2}{mr^3} \geq 0$ is always repulsive.

29.03.02 Conservation of Mechanical Energy

Le equazioni del moto piano, espresse in coordinate polari, (29.03.01), (29.03.01), si possono combinare come

$$\boxed{m\ddot{r} - \frac{\ell^2}{mr^3} = f[r]} \quad . \tag{29.03.02}$$

→ 1556

The central force, $f[r]$, gives rise to a radial potential energy, $\Phi[r]$:

$$\Phi[r] - \Phi[r_0] \equiv -\int_{r_0}^{r} f[s]\,\mathrm{d}s \qquad \Leftrightarrow \qquad f[r] = -\frac{\mathrm{d}\Phi[r]}{\mathrm{d}r} \quad . \tag{29.03.03}$$

→ 1553 1582

Similarly, the effective central force, $f_{\mathrm{E}}[r]$, gives rise to an effective radial potential energy, $\Phi_{\mathrm{E}}[r]$:

$$\Phi_{\mathrm{E}}[r] - \Phi_{\mathrm{E}}[r_0] \equiv -\int_{r_0}^{r} f_{\mathrm{E}}[s]\,\mathrm{d}s = \Phi[r] - \Phi[r_0] + \frac{\ell^2}{2mr^2} - \frac{\ell^2}{2mr_0^2} \qquad \Leftrightarrow \qquad f_{\mathrm{E}}[r] = -\frac{\mathrm{d}\Phi_{\mathrm{E}}[r]}{\mathrm{d}r} \quad . \tag{29.03.04}$$

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Note that it is customary to choose:

- $r_0 \longrightarrow \infty$ when the improper integral in equation (29.03.03) exists, typically for forces like $f[r] \propto r^{-\alpha}$, with $\alpha > 1$;
- $r_0 = 0$ when the force is such that $|f[r=0]| < \infty$, typically for forces like $f[r] \propto r^{+\alpha}$, with $\alpha > 0$.

The derivative is not a partial derivative because in a central field the dependence on r only is allowed.
Si ottiene allora

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + \Phi[r] = \frac{1}{2}m\dot{r}^2 + \Phi_{\text{E}}[r] \quad , \tag{29.03.05}$$

→ 1554 1556 1560 1586

che esprime la conservazione dell'energia e permette di vedere il moto come un moto one-dimensionale con l'energia potenziale effettiva

$$\Phi_{\text{E}}[r] \equiv \frac{\ell^2}{2mr^2} + \Phi[r] \quad . \tag{29.03.06}$$

Clearly, the angular momentum provides a fictitious force, in the co-rotating Reference Frame, pushing the particle out, which is the centrifugal force, in the co-rotating Reference Frame, read § 29.06.01. This repulsive force is an inverse cube force law. It is sometimes called **the centrifugal-barrier force term**.

29.03.03
Equation of the Orbit: Binet Equation

This § is referenced at pages:
[1556, 1556]

While the integration of the equations of motion as a function of time is often difficult if not impossible in closed form, it is often easier to calculate the equation of the orbit, thanks to the Binet equation.
In order to derive it one can write for the equation of the orbit, taking into account that $r^2\dot{\varphi} = c$:

$$r[\varphi[t]] \Longrightarrow \dot{r} = \left(\frac{\mathrm{d}}{\mathrm{d}\varphi} r[\varphi]\right)\dot{\varphi} = \dot{\varphi}\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{c}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\varphi} = -c\frac{\mathrm{d}}{\mathrm{d}\varphi}\frac{1}{r} \tag{29.03.07}$$

→ 1560

$$r[\varphi[t]] \Longrightarrow \ddot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\dot{r} = -c\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}\varphi}\frac{1}{r}\right) = -c\dot{\varphi}\frac{d^2}{\mathrm{d}\varphi^2}\frac{1}{r} = -\frac{c^2}{r^2}\frac{d^2}{\mathrm{d}\varphi^2}\frac{1}{r} = -\frac{\ell^2}{m^2r^2}\frac{d^2}{\mathrm{d}\varphi^2}\frac{1}{r} \quad .$$

It follows:

$$a_r = \left(\ddot{r} - r\dot{\varphi}^2\right) = -\frac{\ell^2}{m^2r^2}\left(\frac{d^2}{\mathrm{d}\varphi^2}\frac{1}{r} + \frac{1}{r}\right) \quad ,$$

from which the Binet formula follows:

$$-\frac{\ell^2}{mr^2}\left(\frac{d^2}{\mathrm{d}\varphi^2}\frac{1}{r} + \frac{1}{r}\right) = f[r] \quad , \tag{29.03.08}$$

→ 1554

usually written in terms of the variable $u \equiv r^{-1}$ as

$$\frac{d^2u}{\mathrm{d}\varphi^2} + u = -\frac{m}{u^2\ell^2}f\left[\frac{1}{u}\right] \quad . \tag{29.03.09}$$

→ 1559 1581 1586

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29.03.04
Energy Conservation From Binet Equation

This § is referenced at pages:
[1585]

One should notice that equation (29.03.08) is equivalent to energy conservation as it can be shown by multiplying both sides by $\frac{\mathrm{d}r}{\mathrm{d}\varphi}$ and integrating with respect to φ to obtain:

$$\frac{\ell^2}{2m}\left(\frac{1}{r^2}+\left(\frac{\mathrm{d}}{\mathrm{d}\varphi}\frac{1}{r}\right)^2\right)+\Phi[r]=E \quad .$$

Note the expression of the kinetic energy:

$$K=\frac{m}{2}v^2=\frac{m}{2}(\dot{r}^2+r^2\dot{\varphi}^2)=\frac{\ell^2}{2m}\left(\frac{1}{r^2}+\left(\frac{\mathrm{d}}{\mathrm{d}\varphi}\frac{1}{r}\right)^2\right) \quad .$$

The Binet equation (29.03.08) can be reduced to quadrature. In fact multiply both members of equation (29.03.08) by $\frac{\mathrm{d}r}{\mathrm{d}\varphi}$ and then integrate with respect to φ to obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{mc^2}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2+\frac{mc^2u^2}{2}+\Phi[\frac{1}{u}]\right)=0\Longrightarrow\boxed{\frac{mc^2}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2+\frac{mc^2u^2}{2}+\Phi[\frac{1}{u}]=E} \quad . \qquad (29.03.10)$$

→ 1554

The interpretation in terms of energy is clear.

29.03.05
Determination of the Orbit in Terms of Physical Parameters

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Note that by integration of the Binet equation (29.03.08) one would determine the orbit in terms of initial parameters whose physical meaning might be non immediate.

It is better to start from the energy conservation equation (29.03.05) and use $\dot{r}=\left(\frac{\mathrm{d}}{\mathrm{d}\varphi}r[\varphi]\right)\dot{\varphi}$ to obtain:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2=\frac{m^2r^4}{\ell^2}\left(\frac{2\left(E-\Phi[r]\right)}{m}-\frac{\ell^2}{m^2r^2}\right) \quad .$$

Note that, according to problem § 29.06.08 it is sufficient to calculate the orbit in the portion $\frac{\mathrm{d}r}{\mathrm{d}\varphi}\geq 0$, for instance, and then extending it by symmetry. The equation can be solved by separation of the variables.

It is even better to start from equation (29.03.10), in terms of u instead of r:

$$\frac{mc^2}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2+\frac{mc^2u^2}{2}+\Phi[\frac{1}{u}]=E\Longrightarrow\boxed{\frac{\mathrm{d}u}{\mathrm{d}\varphi}=\pm\sqrt{\frac{2m\left(E-\Phi[1/u]\right)}{\ell^2}-u^2}} \quad . \qquad (29.03.11)$$

The $\pm$ in front of the integral means that the choice of the sign depends on the specific part of the orbit, between apsidal vectors, that one is considering.

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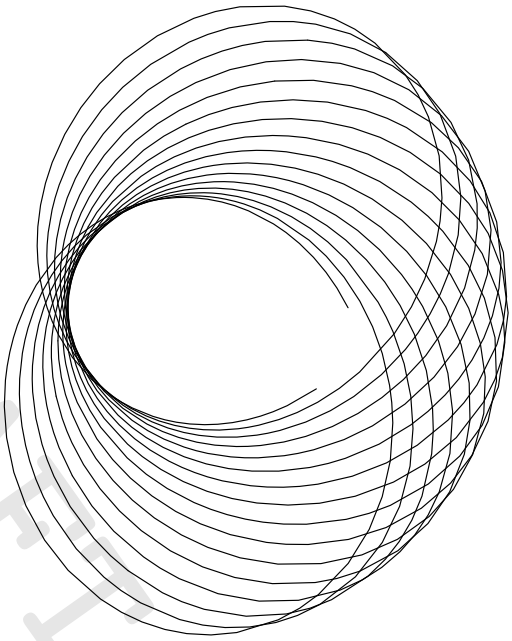
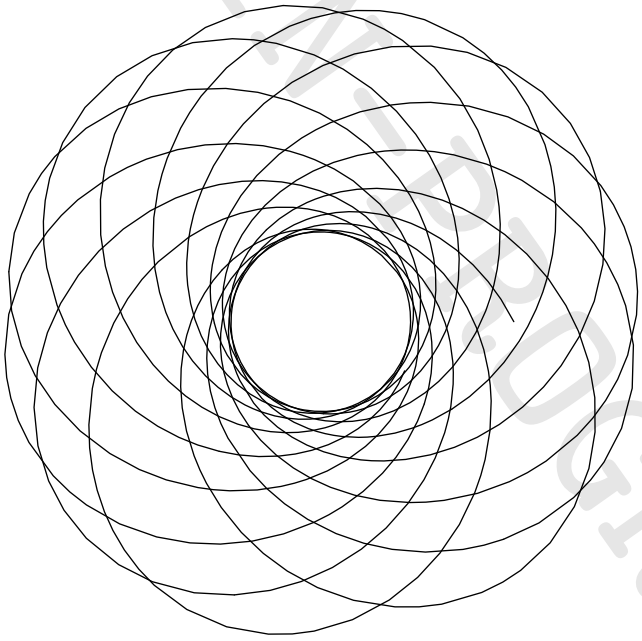


(a) Inverse square plus weak inverse quartic.

Figure 29.12: ...



(a) Inverse square plus strong inverse quartic

Figure 29.13: ...

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29.03.06
Integration of the Equation of Motion

This § is referenced at pages:
[Never referenced.]

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29.03.06.01 Determination of the Position as a Function of Time

This § is referenced at pages:
[1556]

Thanks to the conservation of angular momentum the two-dimensional problem has been reduced to a one-dimensional problem. Energy conservation allows, in general, to solve any one-dimensional problem. In fact equation (29.03.05) can be reduced to quadrature, thus determining, in principle, the function $\mathbf{r}[t]$:

$$\dot{r}^2 = \frac{2}{m}(E - \Phi_{\mathrm{E}}[r]) \implies \pm \int_{r_0}^{r} \frac{\mathrm{d}r}{\sqrt{\frac{2}{m}(E - \Phi_{\mathrm{E}}[r])}} = t - t_0 \quad . \tag{29.03.12}$$

→ 1556

The $\pm$ sign must be chosen according to whether one is considering a part of orbit with increasing or decreasing radius. However, as a consequence of the Binet equation (Read § 29.03.03, § 29.06.08), when a part of the orbit is known between the minimum distance and the maximum distance from the center of force, the full orbit can be reconstructed. It is then enough to integrate equation (29.03.12) between the minimum distance and the maximum distance from the center of force, with any one choice of the sign in front of the integral.

Once equation (29.03.12) is solved equation (29.03.01) allows to calculate $\varphi[t]$.

29.03.06.02 Determination of the Orbit

This § is referenced at pages:
[Never referenced.]

The orbit can be determined from Binet equation, read § 29.03.03.

Another way to accomplish the same result consists in starting again from equation (29.03.05) and replace:

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\dot{\varphi} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{\ell}{mr^2} \quad ,$$

to obtain:

$$\mathrm{d}\varphi = \pm \frac{\ell}{\sqrt{2m}} \frac{\mathrm{d}r}{r^2\sqrt{E - \frac{\ell^2}{2mr^2} - \Phi[r]}} \quad .$$

The $\pm$ sign must be chosen according to whether one is considering a part of orbit with increasing or decreasing radius, as discussed in section § 29.03.06.01.

29.03.07 Circular and Near-Circular Orbits in a Central Force Field (Orbit Precession)

This § is referenced at pages:
[Never referenced.]

From equation (29.03.02) and the condition for a circular orbit, $\dot{r} = 0$, one has that the condition for a circular orbit at a distance $r = r_0$, for one given value of the angular momentum, ℓ_0, is:

$$r = r_0 \implies -\frac{\ell_0^2}{mr_0^3} = f[r_0] = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}[r_0] \quad \Leftrightarrow \quad \frac{\mathrm{d}\Phi_{\mathrm{E}}}{\mathrm{d}r}[r_0] = 0 \implies f[r_0] = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}[r_0] < 0 \quad . \tag{29.03.13}$$

→ 1557 1582

All and only attractive forces admit circular orbits, because the centrifugal force need to be exactly balanced at all times.

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For each value of the angular momentum a well defined distance exists at which the circular orbit is possible, the value of which is determined by solving equation (29.03.13). Note that equation (29.03.13) corresponds to the condition for extrema of the effective potential energy, $\Phi_{\text{E}}[r]$: $\frac{\mathrm{d}\Phi_{\text{E}}}{\mathrm{d}r}[r_0] = 0$. The resulting radius of the circular orbit can be calculated in terms of the angular momentum:

$$r_0 = r_0[\ell_0] \quad .$$

Circular orbits with different radii are characterized by a different value of the angular momentum. The radius of the circular orbit corresponds to the minimum of the effective potential energy, for stable circular orbits, while it corresponds to the maximum of the effective potential energy, for unstable circular orbits).

The corresponding energy is:

$$\boxed{r = r_0 \implies E = \Phi_{\text{E}}[r_0] = \Phi[r_0] + \frac{\ell_0^2}{2mr_0^2}} \quad . \tag{29.03.14}$$

→ 1557

The elimination of r_0 between equations (29.03.13), (29.03.14) yields the unique relation between energy and angular momentum for a circular orbit. Read § 29.04 for the implementation of this in the case of an inverse square law.

29.03.07.01 Period of Circular Orbits in a Central Force Field

Consider an attractive force field: $\frac{\mathrm{d}\Phi}{\mathrm{d}r}[r_0] > 0$.

As the orbit is circular one can write:

$$\boxed{\ell_0 = m\omega_0 r_0^2} \quad ,$$

and equation (29.03.13) implies:

$$f[r_0] = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}[r_0] = -\frac{\ell_0^2}{mr_0^3} = -m\omega_0^2 r_0 \implies \boxed{\omega_0^2 = \frac{1}{mr_0}\frac{\mathrm{d}\Phi}{\mathrm{d}r}[r_0]} \quad . \tag{29.03.15}$$

→ 1557 1582 1583

The above condition (29.03.15) is actually just the radial equilibrium equation.

29.03.08
Conditions for Closed Orbits (Bertrand Theorem)

This § is referenced at pages:
[1583]

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Bertrand Theorem states: The only central forces that lead to closed orbits for all bound particles states are the inverse square and harmonic forces.

Note that it is indeed possible to find closed particular orbits for any attractive force.

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29.04 Moto in Un Campo Gravitazionale O Coulombiano

This § is referenced at pages:
[1557]

Si consideri un campo con forza dipendente dall'inverso del quadrato della distanza

$$f[r] = -\frac{\gamma}{r^2} \propto \frac{1}{r^2} \quad , \tag{29.04.01}$$

quali il campo gravitazionale e Coulombiano.

The effective potential energy is:

$$\boxed{\Phi_{\text{E}}[r] = -\frac{\gamma}{r} + \frac{\ell^2}{2mr^2}} \quad .$$

In the repulsive case, $\gamma < 0$, the effective potential energy is always positive.

The qualitative behavior of the effective potential energy for the two cases of attractive and repulsive forces are shown in figure 29.14.

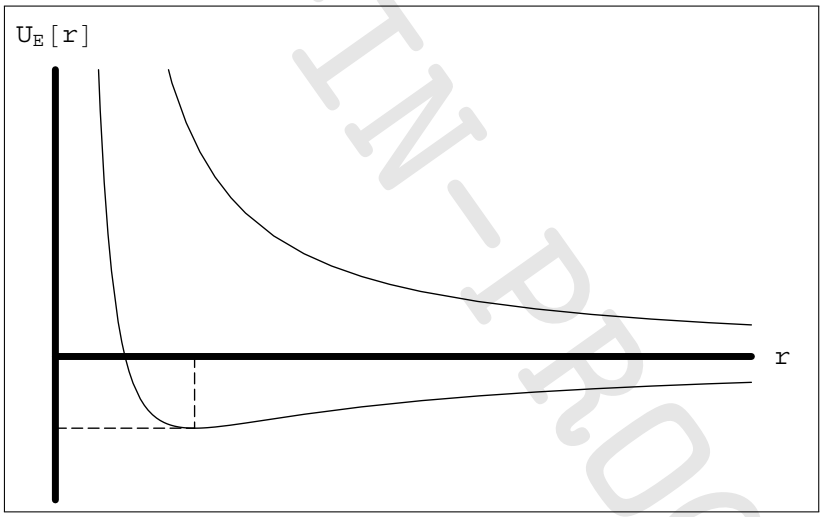

Figure 29.14: FIGURE

29.04.01 Orbite in Un Campo Proporzionale All'inverso Del Quadrato Della Distanza

This § is referenced at pages:
[1561]

We want to show that in an inverse square (either attractive or repulsive) central force field,

$$f_r[r] = -\gamma r^{-2} \quad , \tag{29.04.02}$$

→ 1558

the particle trajectory is always a conic.

The physically important cases of applicability of equation (29.04.02) are:

- gravitational force between two mass point particles: $\gamma = \mathcal{G} m_1 m_2$;
- electric force between two charged point particles: $\gamma = -\frac{1}{4\pi\varepsilon_0} q_1 q_2$.

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We can use the Binet formula (29.03.09):

$$\frac{d^2u[\varphi]}{\mathrm{d}\varphi^2} + u[\varphi] = \frac{m\gamma}{\ell^2} \quad ,$$

whose general solution is:

$$\boxed{u[\varphi] = B\cos\left[\varphi - \varphi_0\right] + \frac{m\gamma}{\ell^2}} \quad . \tag{29.04.03}$$

→ 1559 1560 1594

Two constants of integration appears, B and φ_0, to be determined from the initial conditions. The constant φ_0 can be chosen such that $\varphi_0 = 0$. In fact φ_0 is the angle where $r[\varphi]$ has an extremum, either a minimum or a maximum according to the sign of B:

- $B > 0$ implies that $r[\varphi] = 1/u$ has a minimum for $\varphi = 0$;
- $B < 0$ implies that $r[\varphi] = 1/u$ has a maximum for $\varphi = 0$ and a minimum for $\varphi = \pi$.

From the expression of effective potential energy one knows that for any orbit a minimum distance exists. One can therefore choose to count the angles form this minimum distance: $\varphi_0 =$ angle of perihelion.

The choice $\varphi_0 =$ angle of perihelion $= 0$ corresponds to fixing the initial condition:

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = 0 \qquad \text{for} \qquad \varphi_0 = \text{angle of perihelion} = 0 \quad .$$

This choice implies $B > 0$. Moreover, note that taking $B < 0$ is completely equivalent to choosing the location of the perihelion at $\varphi = \varphi_0 = \pi = \varphi_{\mathrm{P}}$. Therefore $B > 0$ will be chosen and the perihelion ia always:

$$\boxed{\varphi = \varphi_0 = 0 = \varphi_{\mathrm{P}}} \quad .$$

The stress on the location of the perihelion is motivated by the fact that the orbit is symmetrical with respect to the position of the perihelion (as well as of the aphelion) and the perihelion always exists for any orbit, while the aphelion is missing for parabolas and hyperbolas (or, alternatively, it is located at infinity in these cases).

29.04.01.01 First Method

Two kind of orbits may exist: bounded and unbounded orbits, according to the total energy, E. This can be deduced from the study of the effective potential energy:

$$\boxed{\Phi_{\mathrm{E}}[r] = -\frac{\gamma}{r} + \frac{\ell^2}{2mr^2}} \quad .$$

Solve for the extremes of the radial distance, where $\dot{r} = 0$:

$$\dot{r} = 0 \implies \Phi_{\mathrm{E}}[r] = -\frac{\gamma}{r} + \frac{\ell^2}{2mr^2} = E \implies \boxed{u[\dot{r}=0] = \frac{m\gamma}{\ell^2} \pm \sqrt{\left(\frac{m\gamma}{\ell^2}\right)^2 + \frac{2mE}{\ell^2}}} \quad . \tag{29.04.04}$$

→ 1559

The analysis of the result shows the following, taking into account that, by definition $u > 0$.

- If $\gamma > 0$ the $+$ solution is always acceptable (minimum distance), while the $-$ solution is acceptable if and only if $E < 0$ (maximum distance). This is the attractive case: there is always a minimum distance extreme, for any value of the energy and, in the case of negative energy, there is also a second extreme, at the maximum distance, such that the orbit is bounded.
- If $\gamma < 0$ only the $+$ solution is acceptable (minimum distance), provided $E > 0$. This is the repulsive case: there is only one extreme, at a minimum of the distance, and this only happens for positive energy.

Compare equation (29.04.04) with equation (29.04.03) for $\varphi = \varphi_0 = 0$ at the minimum distance. In fact we know from the study of the effective potential energy that a minimum distance exists in any case. One finds:

$$B = \pm\sqrt{\left(\frac{m\gamma}{\ell^2}\right)^2 + \frac{2mE}{\ell^2}} \quad .$$

In order to ensure that equation (29.04.03) represents the perihelion at $\varphi = \varphi_0 = 0$, as stipulated, we are forced to choose $B > 0$.

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29.04.01.02 Second Method

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We know from the study of the effective potential energy that a minimum distance exist in any case. The total energy at perihelion is:

$$E = \text{constant} = E_P = \frac{\ell^2}{2m} u_P^2 - \gamma u_P \quad ,$$

valid for any value of γ.

In equation (29.04.03) the perihelion at $\varphi_0 = 0$ is obtained for $B > 0$:

$$u_P \equiv u[\varphi = 0] = B + \frac{m\gamma}{\ell^2} \quad .$$

By eliminating u_P from the two equations one finds

$$E = E_P = \frac{B^2\ell^2}{2m} - \frac{m\gamma^2}{2\ell^2} \Longrightarrow \boxed{B = \frac{m|\gamma|}{\ell^2}\sqrt{1 + \frac{2E\ell^2}{m\gamma^2}}} \quad .$$

29.04.01.03 Third Method

One can write:

$$r[\varphi] = \left(\frac{\left(\frac{\ell^2}{m}\right)}{\gamma + \left(\frac{B\ell^2}{m}\right)\cos[\varphi]} \right) = \left(\frac{\left(\frac{\ell^2}{m|\gamma|}\right)}{\frac{\gamma}{|\gamma|} + \left(\frac{B\ell^2}{m|\gamma|}\right)\cos[\varphi]} \right) \equiv \frac{p}{\pm 1 \pm e\cos[\varphi]} \quad ,$$

with

$$\boxed{p \equiv \frac{\ell^2}{m|\gamma|} \geq 0} \qquad \boxed{e \equiv \frac{|B|\ell^2}{m|\gamma|} \geq 0} \quad .$$

The equation of the orbit is still expressed in terms of one more arbitrary constant, which can be expressed in terms of the energy (the angular momentum is already accounted for by the Binet formula). From equation (29.03.05) and from equation (29.03.07) one can write:

$$\dot{r} = -c\frac{\mathrm{d}u}{\mathrm{d}\varphi} = cB\sin[\varphi] \quad ,$$

$$E = \frac{B^2\ell^2}{2m} - \frac{m\gamma^2}{2\ell^2} \Longrightarrow B^2 = \frac{2m}{\ell^2}\left(E + \frac{m\gamma^2}{2\ell^2}\right) \Longrightarrow \boxed{e \equiv \frac{|B|\ell^2}{m|\gamma|} \equiv |B|p = \sqrt{1 + \frac{2E\ell^2}{m\gamma^2}}} \qquad (29.04.05)$$

→ 1560

$$\boxed{B = \pm\sqrt{\frac{m^2\gamma^2}{\ell^4} + \frac{2mE}{\ell^2}} = \pm\frac{m|\gamma|}{\ell^2}\sqrt{1 + \frac{2E\ell^2}{m\gamma^2}}} \quad .$$

→ 1560 1594

The choice of $B > 0$ amounts to choose the position of the perihelion at $\varphi = \varphi_0 = 0$. The choice of $B < 0$ amounts to choose the position of the perihelion at $\varphi = \varphi_0 = \pi$.

The following considerations can be carried on from equations (29.04.05), (29.04.01.03).

From the comparison with the equations of the conics in polar coordinates one can see that for $\gamma < 0$, a repulsive potential, the orbit is an hyperbola. In fact in this case one has $E > 0$ implying $e > 1$ and $\gamma/|\gamma| = -1$. The two possible choices for the sign of B give the two possible orbits, symmetric with respect to the y axis, with perihelion at $\varphi = 0$, for $B > 0$, and with perihelion at $\varphi = \pi$, for $B < 0$. In this case of repulsive force the center of force is in the one of the focii outside the branch of the hyperbola.

For $\gamma > 0$, an attractive potential, the orbit can be an ellipse, a parabola or an hyperbola, according to the sign of E and, consequently, the value of e. Moreover, $\gamma/|\gamma| = +1$. The two possible choices for the sign of B give the two possible orbits, symmetric with respect to the y axis, with perihelion at $\varphi = 0$,

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for $B > 0$, and with perihelion at $\varphi = \pi$, for $B < 0$. In this case of attractive force the center of force is in the one of the focii inside the branch of the hyperbola or parabola.

Note that the physical parameters, E and ℓ, must be related to the geometrical parameters, p, e, a, b, c in order to relate physics and geoetry, by means of the relationa derived in the present section.

29.04.02
Properties of Orbits in Inverse Square Fields

This § is referenced at pages:
[Never referenced.]

29.04.02.01 Equation of the Orbit

The different expressions of the orbits can be summarized as follows.

$$r[\varphi] = \frac{a\left(1-e^2\right)}{1 \pm e\cos[\varphi]} \qquad \text{orbita ellittica, } 0 \leq e < 1 \quad , \tag{29.04.06}$$

$$r[\varphi] = \frac{p}{1 \pm \cos[\varphi]} \qquad \text{orbita parabolica, } e = 1 \quad , \tag{29.04.07}$$

$$r[\varphi] = \frac{a\left(e^2-1\right)}{+e\cos[\varphi]+1} \qquad \text{orbita iperbolica, } e > 1\text{, attrattivo, } \varphi = 0 \text{ perielio} \quad , \tag{29.04.08}$$

→ 1561 1563

$$r[\varphi] = \frac{a\left(e^2-1\right)}{+e\cos[\varphi]-1} \qquad \text{orbita iperbolica, } e > 1\text{, repulsivo, } \varphi = 0 \text{ perielio} \quad , \tag{29.04.09}$$

→ 1561

$$r[\varphi] = \frac{a\left(e^2-1\right)}{-e\cos[\varphi]-1} \qquad \text{orbita iperbolica, } e > 1\text{, repulsivo, } \varphi = \pi \text{ perielio} \quad , \tag{29.04.10}$$

→ 1561

$$r[\varphi] = \frac{a\left(e^2-1\right)}{-e\cos[\varphi]+1} \qquad \text{orbita iperbolica, } e > 1\text{, attrattivo, } \varphi = \pi \text{ perielio} \quad . \tag{29.04.11}$$

→ 1562

The location of the perihelion, either $\varphi = 0$ or $\varphi = \pi$, is consistent with the sign of B as discussed in problem § 29.04.01.

Note that, in general, the plus sign in front of the term $e\cos[\varphi]$ implies that the perihelion is at $\varphi = 0$, while the minus sign implies that the perihelion is at $\varphi = \pi$. From this point of view it is really irrelevant to keep both versions of the equations, as they only differ by a mirror reflections around the y axis, that is $\varphi \longrightarrow \pi - \varphi$. One can only keep all and only the forms with the positive sign, assuming the convention that the perihelion is always at $\varphi = 0$.

Note also that the sign in front of the factor 1 in the denominator is always $+1$ for an attractive potential energy and it is always -1 for a repulsive potential energy (always leading to an hyperbolic orbit).

Note that reflecting the conics around the y axis is equivalent to changing φ into $\pi - \varphi$, changing the sign of $\cos[\varphi]$ without changing the sign of $\sin[\varphi]$.

Equation (29.04.08) describes an hyperbola constrained such that:

$$+e\cos[\varphi]+1 > 0 \implies \cos[\varphi] > -\frac{1}{e} \implies |\varphi| < +\arccos\left(-\frac{1}{e}\right) \quad . \tag{29.04.12}$$

Equation (29.04.09) describes an hyperbola constrained such that:

$$+e\cos[\varphi]-1 > 0 \implies \cos[\varphi] > +\frac{1}{e} \implies |\varphi| < +\arccos\left(+\frac{1}{e}\right) \quad . \tag{29.04.13}$$

Equation (29.04.10) describes an hyperbola constrained such that:

$$-e\cos[\varphi]-1 > 0 \implies \cos[\varphi] < -\frac{1}{e} \implies |\varphi-\pi| < +\arccos\left(-\frac{1}{e}\right) \quad . \tag{29.04.14}$$

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Equation (29.04.11) describes an hyperbola constrained such that:

$$-e\cos[\varphi]+1>0 \implies \cos[\varphi]<+\frac{1}{e} \implies |\varphi-\pi|<+\arccos\left(+\frac{1}{e}\right) \quad . \tag{29.04.15}$$

29.04.02.02 Relations

The relations linking the geometrical and physical parameters of the orbit are summarized.
The constant B from the integration of Binet equation is:

$$B=\pm\frac{m|\gamma|}{\ell^2}\sqrt{1+\frac{2E\ell^2}{m\gamma^2}} \qquad \text{but we can always choose } B>0 \text{ and } \varphi=\varphi_0=0 \text{ at perihelion} \quad .$$

L'eccentricità in funzione dell'energia e del momento angolare vale sempre:

$$e=\sqrt{1+\frac{2E\ell^2}{m\gamma^2}} \quad . \tag{29.04.16}$$

→ 1562 1624 1627

Si ha, per un moto ellittico,

$$p=\frac{\ell^2}{m|\gamma|}=\frac{\ell^2}{m\gamma}=a\left(1-e^2\right) \qquad \text{nel caso } 0\le e<1\text{; in this case, } \gamma>0 \quad . \tag{29.04.17}$$

→ 1562 1624

Si ha, per un moto parabolico,

$$p=\frac{\ell^2}{m|\gamma|} \qquad \text{nel caso } e=1 \quad . \tag{29.04.18}$$

→ 1562

Si ha, per un moto iperbolico,

$$p=\frac{\ell^2}{m|\gamma|}=a\left(e^2-1\right) \qquad \text{nel caso } e>1 \quad . \tag{29.04.19}$$

→ 1562

The total energy can be expressed as follows.
Equation (29.04.16) implies:

$$E=\frac{m\gamma^2\left(e^2-1\right)}{2\ell^2}=\frac{|\gamma|\left(e^2-1\right)}{2p} \quad ,$$

e dalle (29.04.17), (29.04.18), (29.04.19) si ricava

$$E=-\frac{|\gamma|}{2a} \qquad \text{nel caso } 0\le e<1\text{; in this case, } \gamma>0 \quad ,$$
$$E=0 \qquad \text{nel caso } e=1 \quad , \quad ,$$
$$E=+\frac{|\gamma|}{2a} \qquad \text{nel caso } e>1 \quad .$$

si ricava l'espressione dell'energia totale:

$$E=E_{\text{MIN}}=-\frac{m\gamma^2}{2\ell^2} \qquad \text{orbita circolare} \quad , \tag{29.04.20}$$

$$E=-\frac{|\gamma|}{2a} \qquad \text{orbita ellittica,} \quad E_{\text{MIN}}\le E<0 \quad , \tag{29.04.21}$$

→ 1586 1594 1624

$$E=0 \qquad \text{orbita parabolica} \quad , \tag{29.04.22}$$

→ 1586

$$E=+\frac{|\gamma|}{2a} \qquad \text{orbita iperbolica, } E>0 \quad .$$

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29.04.03
Leggi Di Keplero

Una particella in un campo di forza inversamente proporzionale al quadrato della distanza è soggetta alle leggi di Keplero.

1. La traiettoria è una conica in cui il centro di forza occupa uno dei due fuochi.
2. Il raggio vettore dal centro di forza alla particella copre aree uguali in tempi uguali (costanza della velocità areolare (23.04.03)) con
$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\ell}{2m}$$
→ 1565 1586
This law applies to any central motion.
3. Il periodo dell'orbita è legato al semiasse maggiore
$$\frac{T^2}{a^3} = \frac{4\pi^2}{\mathcal{G}M}$$
dove M è la massa del centro di forza, supposta molto maggiore della massa della particella. Nel caso generale di masse paragonabili, M ed m, l'equazione si generalizza
$$\frac{T^2}{a^3} = \frac{4\pi^2}{\mathcal{G}\,(M+m)}$$

29.04.04
Open (Unbounded) Orbits in Inverse Square Law Force Fields

This § is referenced at pages:
[1564]

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29.04.04.01 Scattering Angle

A particle of mass m is directed towards a fixed scattering center, described by an inverse-square force field, with impact parameter b (equation (30.01.01)).

Assume, for definiteness, that the force is attractive: $\gamma > 0$; the case of repulsive force can be treated in a similar way.

The particle is starting from a very large distance, and it has, therefore, a positive conserved mechanical energy, $E_0 > 0$, and conserved angular momentum with respect to the scattering center, ℓ_0.

One wants to determine, a long time after the interaction, the direction of the particle with respect to the initial direction: the scattering angle.

The orbit is hyperbolic, as the total conserved energy is positive. The equation for attractive hyperbolic orbits, with perihelion at $\varphi = 0$, is equation (29.04.08). The angles of the two asymptotes, φ_{A}, are given by:
$$\boxed{\cos\left[\varphi_{\mathrm{A}}\right] = -\frac{1}{e}} \quad .$$
Define, for simplicity of the formulas only, the auxiliary angle, φ_0, to obtain:
$$\boxed{\varphi_0 \equiv \pi - \varphi_{\mathrm{A}}} \Longrightarrow \boxed{\cos\left[\varphi_0\right] = \cos\left[\pi - \varphi_{\mathrm{A}}\right] = \frac{1}{e}} \quad .$$

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The scattering angle, that is the angle between the two asymptotes, is given by:

$$\boxed{\varphi_{\mathrm{S}} = \pi - 2\varphi_0 = \pi + 2\varphi_{\mathrm{A}}} \Longrightarrow \boxed{\frac{\varphi_{\mathrm{S}}}{2} = \frac{\pi}{2} - \varphi_0 \qquad \text{with } 0 \leq \frac{\varphi_{\mathrm{S}}}{2} \leq \frac{\pi}{2}} \quad .$$

On then has:

$$\boxed{\sin\left[\varphi_{\mathrm{S}}/2\right] = \cos\left[\varphi_0\right] = \frac{1}{e}}$$

From the initial conditions at large distance,

$$E = E_0 = \frac{p_{\mathrm{I}}^2}{2m} = \text{constant} \quad ,$$
$$\ell = \ell_0 = p_{\mathrm{I}} b = \text{constant} \quad ,$$
$$\text{while the linear momentum } p \text{ is not conserved} \quad ,$$

it follows, noting that $0 \leq \dfrac{\varphi_{\mathrm{S}}}{2} \leq \dfrac{\pi}{2}$ implies $\sin\left[\varphi_{\mathrm{S}}/2\right] > 0$ and $\cos\left[\varphi_{\mathrm{S}}/2\right] > 0$,

$$\cot\left[\varphi_{\mathrm{S}}/2\right] = \frac{\cos\left[\varphi_{\mathrm{S}}/2\right]}{\sin\left[\varphi_{\mathrm{S}}/2\right]} = \frac{\sqrt{1-\sin^2\left[\varphi_{\mathrm{S}}/2\right]}}{\sin\left[\varphi_{\mathrm{S}}/2\right]} = \sqrt{e^2-1} = \boxed{\sqrt{\frac{2E\ell^2}{m\gamma^2}} = \frac{2E_0 b}{|\gamma|} = \frac{\ell_0^2}{m|\gamma|b} = \cot\left[\varphi_{\mathrm{S}}/2\right]} \quad ,$$

(29.04.23) → 1564 1566

showing how the scattering angle can be expressed either in terms of the energy (plus impact parameter) or angular momentum (plus impact parameter).

In case of a repulsive potential the above result (29.04.23) is unchanged. In fact, in case of a repulsive force, at the same center of force, corresponding to the other branch of the same hyperbola, one has:

$$\boxed{\cos\left[\varphi_{\mathrm{A}}\right] = +\frac{1}{e} \qquad \varphi_{\mathrm{S}} = \pi - 2\varphi_0 = \pi + 2\varphi_{\mathrm{A}}} \quad ,$$

and all goes on the same way.

29.04.04.02 Gravitational Slingshot Effect (Gravity Assist)

This § is referenced at pages:
[1547]

Una sonda spaziale di massa m viene diretta verso un pianeta (ad esempio Giove), che ha massa M, con $m \ll M$, in modo che la sua orbita, a seguito dell'interazione con Giove, venga accelerata e deviata opportunamente per dirigerla in una direzione voluta aumentando (ovvero diminuendo) al tempo stesso l'energia cinetica della sonda rispetto al Reference Frame solidale con il Sole.

1. Determinare la variazione del modulo della velocità della sonda a seguito del suo incontro ravvicinato con Giove, sia rispetto al Reference Frame solidale con Giove (velocità relativa a Giove $\mathbf{v}'$) sia rispetto al Reference Frame solidale con il Sole (velocità relativa al Sole $\mathbf{v}$).
2. Calcolare, nel Reference Frame solidale con Giove, l'angolo di scattering (deviazione) φ_{S} della sonda in funzione della sua velocità iniziale, $\mathbf{v}'_{\mathrm{I}}$, e del parametro di impatto b rispetto a Giove.
3. Calcolare l'angolo di deviazione della sonda nel Reference Frame solidale con il Sole.

Read also § 29.04.04.

As the potential is attractive we have: $\gamma > 0$. Nel caso gravitazionale il potenziale è sempre attrattivo, mentre nel caso Coulmbiano può essere anche repulsivo.

Moreover, by hypotesis, $m \ll M$, and therefore we can neglect the reduction of the two-body problem to a one-body problem and consider Jupiter a fixed scattering center. Si assuma che quando la sonda si trova nei pressi di Giove risenta solo dell'attrazione gravitazionale del pianeta in modo da poter descrivere l'interazione come un semplice problema a due corpi (sonda entro la sfera di influenza di Giove). Si trascuri la variazione di quantità di moto di Giove durante l'interazione (essendo $M \gg m$). Si supponga il tempo di interazione così breve da poter trascurare il moto di Giove nella sua orbita durante il tempo di interazione. Sotto tali ipotesi il moto visto dal Reference Frame inerziale solidale con Giove è descritto

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dal problema gravitazionale dei due corpi e l'energia totale della sonda sarà positiva in quanto non è gravitazionalmente legata con Giove.

Nel caso specifico in questione sono solo rilevanti le orbite iperboliche, corrispondenti ad energia meccanica totale positiva:

$$r[\varphi] = \frac{p}{+1 + e\cos[\varphi]} = \frac{a\left(e^2-1\right)}{+1+e\cos[\varphi]} \qquad \text{orbita iperbolica, } e>1\text{, potenziale attrattivo, perielio a } \varphi = 0. \tag{29.04.24}$$

$$r[\varphi] = \frac{p}{+1 - e\cos[\varphi]} = \frac{a\left(e^2-1\right)}{-1-e\cos[\varphi]} \qquad \text{orbita iperbolica, } e>1\text{, potenziale attrattivo, perielio a } \varphi = \pi \quad .$$

Si ha

$$\boxed{p = \frac{\ell^2}{m|\gamma|} \geq 0} = a\left|e^2-1\right| \qquad \text{nel caso } 0 \leq e \neq 1 \tag{29.04.25}$$

→ 1565

e

$$\boxed{e^2 = 1 + \frac{2E\ell^2}{m\gamma^2}} \quad . \tag{29.04.26}$$

→ 1565

Dalle (29.04.25), (29.04.26), (2) si ricava l'energia totale

$$E = +\frac{|\gamma|}{2a} \qquad \text{orbita iperbolica, } E>0 \quad .$$

La relazione che lega le velocità nel Reference Frame solidale con il Sole, $\mathbf{v}$, e in quello solidale con Giove, $\mathbf{v}'$, denotando con $\mathbf{V}$ la velocità di Giove, sono

$$\begin{aligned} \mathbf{v}_{\mathrm{F}} &= \mathbf{V} + \mathbf{v}'_{\mathrm{F}} \quad , \\ \mathbf{v}_{\mathrm{I}} &= \mathbf{V} + \mathbf{v}'_{\mathrm{I}} \quad , \\ \Delta\mathbf{v} &\equiv \mathbf{v}_{\mathrm{F}} - \mathbf{v}_{\mathrm{I}} \quad , \\ \Delta\mathbf{v}' &\equiv \mathbf{v}'_{\mathrm{F}} - \mathbf{v}'_{\mathrm{I}} \quad , \\ \boxed{\Delta\mathbf{v}' = \Delta\mathbf{v}} & \end{aligned}$$

→ 1566

e nel Reference Frame solidale con Giove le velocità iniziale e finale della sonda sono invariate in modulo

$$\boxed{\left|\mathbf{v}'_{\mathrm{F}}\right| = \left|\mathbf{v}'_{\mathrm{I}}\right|} \quad ,$$

cioè il processo è elastico (energia cinetica immutata) nel Reference Frame solidale con il pianeta. Ne segue

$$\boxed{\frac{\Delta \mathtt{k}}{m} \equiv \frac{1}{2}\left(v_{\mathrm{F}}^2 - v_{\mathrm{I}}^2\right) = \mathbf{V}\cdot\mathbf{v}'_{\mathrm{F}} - \mathbf{V}\cdot\mathbf{v}'_{\mathrm{I}} = \mathbf{V}\cdot\mathbf{v}_{\mathrm{F}} - \mathbf{V}\cdot\mathbf{v}_{\mathrm{I}} = \mathbf{V}\cdot\Delta\mathbf{v}' = \mathbf{V}\cdot\Delta\mathbf{v}} \quad .$$

Ne segue che, a seconda della relazione tra il vettore velocità di Giove e la variazione di velocità, che è la stessa nel Reference Frame solidale con il Sole e in quello solidale con Giove, il modulo della velocità della sonda può aumentare o diminuire. In particolare dalla relazione sopra è evidente come, per una fissata velocità iniziale, la massima velocità finale si ottiene quando la velocità finale è parallela a quella di Giove. L'energia guadagnata dalla sonda è sottratta al pianeta. Il metodo può essere usato sia per accelerare che per rallentare la sonda.

Gli asintoti dell'orbita iperbolica sono determinati dall'equazione

$$\cos[\varphi_{\mathtt{A}}] = -\frac{1}{e} \quad ,$$

scegliendo sempre per $\varphi = 0$ il punto del perielio. Conviene introdurre l'angolo ausiliario

$$\varphi_0 = \pi - \varphi_{\mathtt{A}} \implies \cos[\varphi_0] = \frac{1}{e} \quad .$$

In tal modo l'angolo di deviazione, $\varphi_{\mathtt{S}}$, che risulta è

$$\varphi_{\mathtt{S}} = \pi - 2\varphi_0 \quad ,$$

da cui segue

$$\boxed{\cos[\varphi_{\mathtt{S}}] = -\cos[2\varphi_0] = 1 - 2\cos^2[\varphi_0] = 1 - \frac{2}{e^2}} \quad . \tag{29.04.27}$$

→ 1566 1613

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L'espressione dell'eccentricità dell'orbita è

$$\boxed{e^2 = 1 + \frac{2E\ell^2}{m\gamma^2}} \quad ,$$

dove $\gamma = \mathcal{G}Mm$.

Note that the result in equation (29.04.23) can be cast in the form of result of equation (29.04.27) using a trigonometric identity:

$$t \equiv \tan\left[\varphi_S/2\right] \Longrightarrow \cos\left[\varphi_S\right] = \frac{1-t^2}{1+t^2} \Longrightarrow \frac{1}{t^2} = \frac{1+\cos\left[\varphi_S\right]}{1-\cos\left[\varphi_S\right]} = e^2 - 1 \quad ,$$

$$\boxed{\frac{2E\ell^2}{m\gamma^2} = \left(\frac{2Eb}{\gamma}\right)^2 = \left(\frac{\ell^2}{m\gamma b}\right)^2 = \frac{1}{t^2} = \cot^2\left[\varphi_S/2\right]} \quad .$$

I parametri iniziali dell'orbita sono, nel Reference Frame di Giove,

$$\ell'_0 = mbv'_I \quad ,$$

$$E'_0 = \frac{1}{2}m{v'_I}^2 \quad ,$$

determinati dalla trasformazione della velocità della sonda dal Reference Frame eliocentrico a quello solidale con Giove, secondo la (29.04.04.02).

In termini dei parametri iniziali questa si scrive

$$e^2 = 1 + \left(\frac{m{v'_I}^2 b}{\gamma}\right)^2 \geq 1 \quad .$$

This equation allows to determine the eccentricity as a function of b and v'_I. Moreover, from v'_I one can determine the mechanical energy E and from that one can find the major axis of the hyperbola, thus determining the geometry of the hyperbola.

Moreover, one can write the expression for the module of the velocity change:

$$\boxed{\left|\Delta\mathbf{v}'\right| = 2\left|\mathbf{v}'_F\right|\sin\left[\varphi_S/2\right] = 2\left|\mathbf{v}'_I\right|\sin\left[\varphi_S/2\right]} \quad .$$

L'espressione per l'angolo di deviazione φ_S fornisce i seguenti casi limite.

- Nel caso $e \longrightarrow 1$, $\varphi_S \longrightarrow \pi$; questo caso si ottiene per $v'_I \longrightarrow 0$ oppure $b \longrightarrow 0$. La sonda punta molto vicino a Giove, oppure ha una energia cinetica all'infinito molto piccola.
- Nel caso $e \longrightarrow \infty$, $\varphi_S \longrightarrow 0$; questo si ottiene per $v'_I \longrightarrow \infty$ oppure $b \longrightarrow \infty$. La sonda punta molto lontano da Giove, oppure ha una energia cinetica all'infinito molto grande.

Il calcolo dell'angolo di deviazione nel Reference Frame solidale permette di calcolare anche la deviazione nel Reference Frame solidale con il Sole dalla legge di trasformazione delle velocità (29.04.04.02).

Note that, even if $b \longrightarrow 0$, the spacecraft does not fall on the center of force, in principle, but it may fall onto the planet surface in a realistic case. It will never end orbiting Jupiter, whenever the mechanical energy is conserved.

The relative velocity of the spacecraft with respect to the planet is unchanged in absolute value, but its direction changes. Therefore the velocity of the spacecraft with respect to the Sun may change in absolute value.

In the limiting case $E\ell^2 \longrightarrow 0$ the scattering angle is maximal: $\varphi_S \longrightarrow \pi$. Therefore the velocity of the spacecraft relative to the planet is reversed:

$$\mathbf{v}'_F = -\mathbf{v}'_I \Longrightarrow \Delta\mathbf{v}' = +2\mathbf{v}'_F = -2\mathbf{v}'_I \quad .$$

Depending on the direction of $\mathbf{v}'_F = -\mathbf{v}'_I$ with respect to $\mathbf{V}$ the change of kinetic energy with respect to the Sun can be positive or negative and it reaches its maximum absolute value when $\mathbf{v}'_F = -\mathbf{v}'_I$ has the same direction of $\mathbf{V}$.

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29.04.05
Laplace-Runge-Lenz Vector

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©|WEB - URL |Mysteries of the gravitational 2-body problem|*John Baez*|

It applies to both gravitational and electric forces.
The equation is:

$$m\ddot{\mathbf{r}} = -\gamma\frac{\mathbf{r}}{r^3} \quad , \tag{29.04.28}$$

→ 1567

with the convention that the constant γ is positive for attractive forces and negative for repulsive forces.
Equation (29.04.28) can be also written in the following way:

$$\frac{\mathrm{d}\widehat{\mathbf{e}}_\theta}{\mathrm{d}t} = -\dot{\varphi}\widehat{\mathbf{e}}_r \longrightarrow \frac{\mathrm{d}\widehat{\mathbf{e}}_\theta}{\mathrm{d}\varphi} = -\widehat{\mathbf{e}}_r \longrightarrow m\ddot{\mathbf{r}} = +\frac{\gamma}{r^2\dot{\varphi}}\frac{\mathrm{d}\widehat{\mathbf{e}}_\theta}{\mathrm{d}t} \quad . \tag{29.04.29}$$

The previous equation can be integrated by introducing the angular momentum:

$$\ddot{\mathbf{r}} = \frac{\gamma}{mr^2\dot{\varphi}}\frac{\mathrm{d}\widehat{\mathbf{e}}_\theta}{\mathrm{d}t} = \frac{\gamma}{\ell_z}\frac{\mathrm{d}\widehat{\mathbf{e}}_\theta}{\mathrm{d}t} \longrightarrow \mathbf{v} = \mathbf{w} + \frac{\gamma}{\ell_z}\widehat{\mathbf{e}}_\theta \longrightarrow \mathbf{w} = \mathbf{v} - \frac{\gamma}{\ell_z}\widehat{\mathbf{e}}_\theta \quad , \tag{29.04.30}$$

where $\mathbf{w}$ is a constant vector.
The following vector is therefore a constant of motion:

$$\boxed{\mathbf{w} \equiv \mathbf{v} - \frac{\gamma}{\ell_z}\widehat{\mathbf{e}}_\theta} \quad .$$

The dimensionless vector eccentricity is defined from $\mathbf{w}$ as:

$$\boxed{\mathbf{e} \equiv -\mathbf{w}\frac{\ell_z}{\gamma} = -\mathbf{v}\frac{\ell_z}{\gamma} + \widehat{\mathbf{e}}_\theta} \quad .$$

Its module is the eccentricity of the orbit.
The Runge-Lenz vector, a constant vector lying in the plane of the orbit, is defined from the two constant of motion $\boldsymbol{\ell}$ and $\mathbf{w}$, as:

$$\boxed{\mathbf{A} \equiv \frac{\mathbf{w}\wedge\boldsymbol{\ell}}{\gamma} = \frac{\mathbf{v}\wedge\boldsymbol{\ell}}{\gamma} - \widehat{\mathbf{e}}_r} \quad . \tag{29.04.31}$$

→ 1567 1568

Note that different definitions can be found in the literature. The one used here, for instance, differs by an overall factor $m\gamma$ with respect to the one used in [1]. The present definition is such that the vector is dimensionless.
By its very definition $\mathbf{A} \perp \boldsymbol{\ell}$ that is $\mathbf{A}$ lies in the plane of the orbit. As the vector is constant it can be calculated at any suitable point along the orbit, for instance at the perihelion or aphelion. It can be seen that its direction lies along the major axis of the conic.
Note that from the constancy of the vector it is immediate to demonstrate that for $\boldsymbol{\ell} = 0$ the motion is rectilinear (see problem § 29-003). In fact:

$$\boldsymbol{\ell} = 0 \implies \mathbf{r}/r = \mathbf{A} = \text{constant} \quad .$$

When $\mathbf{e} = 0$ it follows, from equation (29.04.31), by expressing $\mathbf{v}$ in terms of $\widehat{\mathbf{e}}_r$ and $\widehat{\mathbf{e}}_\theta$, that $\dot{r} = 0$ and the orbit is circular.

[1] H.Goldstein *et al.*, *Classical Mechanics*, 2014, Pearson Education, 3rdEd., ...

DRAFT: IN PROGRESS

Take the dot product of equation (29.04.31) with $m\mathbf{r}$ to obtain:

$$\cos[\varphi] = \frac{\mathbf{r}\cdot\mathbf{A}}{rA} \Longrightarrow r[\varphi] = \frac{\ell^2}{m\gamma}\left(\frac{1}{1+A\cos[\varphi]}\right) \quad ,$$

from which the interpretation of $|\mathbf{A}|$ as the eccentricity is manifest. The vector $\mathbf{A}$ points from the center of the ellipse to the perihelion as long as φ is the angle between $\mathbf{A}$ and the line from the focus to the perihelion.

The Kepler problem, in the plane, is a fourth order problem, requiring four constants of motion for the integration. It is in fact a 2-dimensional problem in the plane of the orbit, whose absolute orientation is determined by the direction of the angular momentum vector $\boldsymbol{\ell}$. By considering the motion in space one would have seven constants of motion $\boldsymbol{\ell}$, E, $\mathbf{A}$. They are linked by two equations:

$$\mathbf{A}\cdot\boldsymbol{\ell} = 0 \quad ,$$

$$A^2 = 1 + \frac{2E\ell^2}{m\gamma^2} \quad .$$

Only five constants are then independent, the exact number required by a six-degrees of freedom motion. As all the constants of motion have been found it was possible to determine the orbit equation by using algebraic methods only. The constants do not depend explicitly on time.

As we have been able to find four constants of motion the orbit can be found exclusively by algebraic methods.

29.04.06
Motion in Time for the Kepler Problem and Kepler Equation

©|H.Goldstein *et al.*, *Classical Mechanics*, 2014, Pearson Education, 3rdEd., ...|3.8|*Full description*|

©|WEB - URL |Reduced Kepler Problem in Elliptic Coordinates|*Nicholas Wheeler Physics Lectures*|

DRAFT – IN PROGRESS

29.05 Multipole Development of the Gravitational Field

This § is referenced at pages:
[1127, 1737]

It should be noted that the reasoning is closely analogous to the discussion for the ElectroMagnetic **Multipole** development in section, § 16.12.06, § 36.14, § 36.14, § 36.14.06, § 36.14.07. However, a few important differences arise in the two cases, as discussed in section § 29.05.04.

Note that, from quadrupole moments and beyond, there exist different possible definitions for the multipole moments, such as Cartesian, spherical, each on with its own properties and different use cases.

29.05.01 Multipole Cartesian Development

This § is referenced at pages:
[1737]

©|J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|1-6.4.2||

Let us study the gravitational potential produced by an arbitrary mass distribution at point $\mathbf{r}$, a point located at large distance from the mass distribution. The restriction of large distance might be relaxed to consider also points near the mass distribution but in this case some complications might arise on the convergence of the series expansion.

Choose a Coordinate System. The choice of the origin of this Coordinate System is arbitrary, but it is anticipated that it will turn out to be useful to choose the origin at the CM of the mass distribution. Therefore it is easier to choose since the beginning the origin of the orthonormal Cartesian Coordinates Coordinate System at the Center-Of-Mass of the mass distribution, but this is not strictly necessary.

Assume that the mass distribution is bounded inside a sphere of radius r_0. If $\mathbf{x}$ is the variable describing the position of the mass distribution there is therefore an upper limit to $|\mathbf{x}|$: $|\mathbf{x}| \leq r_0$.

Moreover, this hypothesis allows to choose the reference point for the gravitational potential energy at infinity:

$$\lim_{r \longrightarrow \infty} \Phi[\mathbf{r}] = 0 \quad .$$

The following notation will be used: $r \equiv |\mathbf{r}|$ and $x \equiv |\mathbf{x}|$.

Develop the gravitational potential energy at $\mathbf{r}$ up to second-order in (x/r), using

$$(1+w)^{-1/2} = 1 - \frac{1}{2}w + \frac{3}{8}w^2 + \mathcal{O}\left(w\right)^3 \qquad \text{convergent for } w < 1 \quad ,$$

to obtain:

$$\Phi[\mathbf{r}] = -\mathcal{G} \iiint \frac{\rho[\mathbf{x}]}{|\mathbf{r}-\mathbf{x}|}\, \mathrm{d}V_{\mathbf{x}} = -\mathcal{G} \iiint \frac{\rho[\mathbf{x}]}{\sqrt{r^2 + x^2 - 2\,\mathbf{r}\cdot\mathbf{x}}}\, \mathrm{d}V_{\mathbf{x}} = \quad , \qquad \rightarrow 1574$$

$$-\mathcal{G} \iiint \frac{\rho[\mathbf{x}]}{r} \left(1 - 2\frac{\mathbf{r}\cdot\mathbf{x}}{r^2} + \frac{x^2}{r^2}\right)^{-1/2} \mathrm{d}V_{\mathbf{x}} = \quad ,$$

$$-\mathcal{G} \iiint \frac{\rho[\mathbf{x}]}{r} \left(1 + \frac{\mathbf{r}\cdot\mathbf{x}}{r^2} + \frac{3}{2}\left(\frac{\mathbf{r}\cdot\mathbf{x}}{r^2}\right)^2 - \frac{x^2}{2r^2} + \mathcal{O}\left(\frac{x}{r}\right)^3\right) \mathrm{d}V_{\mathbf{x}} \quad .$$

WORK IN PROGRESS

The gravitational potential energy can be written as a sum of three terms, up to second-order:

$$\Phi[\mathbf{r}] = \Phi_0[\mathbf{r}] + \Phi_1[\mathbf{r}] + \Phi_2[\mathbf{r}] + \mathcal{O}\left(\frac{r_0}{r}\right)^3 \quad . \tag{29.05.01}$$

→ 1574

The problem of the convergence of the series expansion is not discussed here.

Developing the calculation one has, in terms of the total mass, M:

$$\boxed{\Phi_0[\mathbf{r}]} = -\frac{\mathcal{G}M}{r} \qquad \text{unipolar contribution} \quad . \tag{29.05.02}$$

→ 1575

$$\boxed{\Phi_1[\mathbf{r}]} = -\frac{\mathcal{G}M}{r^3}(\mathbf{r}\cdot\mathbf{R}_{\text{CM}}) \qquad \text{dipolar contribution: it is zero if the origin is at the CM} \quad . \tag{29.05.03}$$

$$\boxed{\Phi_2[\mathbf{r}]} = -\frac{\mathcal{G}}{r^5}\iiint \rho[\mathbf{x}]\left(\frac{3\,(\mathbf{r}\cdot\mathbf{x})^2 - r^2x^2}{2}\right) \mathrm{d}V_{\mathbf{x}} \qquad \text{quadripolar contribution} \quad . \tag{29.05.04}$$

As a general rule, note that if the term Φ_n in the development of the potential is non zero, then all the terms Φ_k with $k > n$ do depend on the choice of the origin of the Coordinate System. In the gravitational case one always has $M > 0$, therefore all the terms after the monopole do depend on the choice of the origin of the Coordinate System. This is markedly different from the ElectroStatic case, where the total charge, as well as any higher-order moment, might be zero.

The second-order term can be written with explicit tensor indexes, in terms of the **Quadrupole** moment tensor, $Q^{\cdot\cdot}_{jk}$:

$$\boxed{\Phi_2[\mathbf{r}] = -\mathcal{G}\frac{r_j r_k}{2r^5}\iiint \rho[\mathbf{x}]\left(3x_jx_k - x^2\delta^{\cdot\cdot}_{jk}\right)\mathrm{d}V_{\mathbf{x}} = -\mathcal{G}\frac{r_j r_k Q^{\cdot\cdot}_{jk}}{2r^5}} \quad . \tag{29.05.05}$$

→ 1597

The **Quadrupole** moment is obviously a second-order tensor, as it can be demonstrated as for the inertia Matrix|Tensor in section § 26.04.04.01.

The **Quadrupole** moment is a symmetric and traceless tensor:

$$\boxed{Q^{\cdot\cdot}_{jk} \equiv \iiint \rho[\mathbf{x}]\left(3x_jx_k - x^2\delta^{\cdot\cdot}_{jk}\right)\mathrm{d}V_{\mathbf{x}} \qquad \text{with} \qquad \begin{cases} Q^{\cdot\cdot}_{jk} = Q^{\cdot\cdot}_{kj} \\ \sum_i Q^{\cdot\cdot}_{ii} = 0 \end{cases},} \quad .$$

→ 1570 1571

The value of the **Quadrupole** moment tensor, $Q^{\cdot\cdot}_{jk}$, obviously depends on the choice of the origin as well as orientation of the orthonormal Cartesian Coordinates Coordinate System. The dependence on the origin, normally, is not made explicit as one normally choose the Center-Of-Mass of the mass distribution as the origin.

The relation (29.05.01) shows that the gravitational **Quadrupole** moment has only five independent components.

Note that, in presence of mass, $\Phi_0[\mathbf{r}]$ is always different from zero, in contrast to the ElectroMagnetic case where the monopole term might be zero for a globally neutral distribution. Therefore the importance of the successive terms in the expansion is comparatively more limited in the gravitational case with respect to the ElectroMagnetic case because in the gravitational case the monopole contribution is always present and dominant. Higher order terms are sometimes important in the accurate study of the motion of artificial satellites.

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29.05.01.01 Gravitational Quadrupole Moment and Inertia Matrix|Tensor

Consider the origin of the Coordinate System fixed at the Center-Of-Mass, from now on, so that the **Dipole** term is null. The inertia Matrix|Tensor is evaluated with respect to the Center-Of-Mass from equation (26.04.08), with S=CM, which will be assumed implicitly.

From definitions (29.05.01), (26.04.08) the following relation apply:

$$\boxed{Q^{\cdot\cdot}_{ij} + 3I^{\cdot\cdot}_{ij} = \delta^{\cdot\cdot}_{ij}\,\mathrm{Tr}\,\mathbb{I} = \delta^{\cdot\cdot}_{ij}\left(2\iiint x^2\rho[\mathbf{x}]\,\mathrm{d}V_{\mathbf{x}}\right) > 0} \quad .$$

It is obvious that the gravitational **Quadrupole** moment and the inertia Matrix|Tensor are not independent quantities:

$$\boxed{Q^{\cdot\cdot}_{ij} = -3\left(I^{\cdot\cdot}_{ij} - (1/3)\delta^{\cdot\cdot}_{ij}\,\mathrm{Tr}\,\mathbb{I}\right)} \quad . \tag{29.05.06}$$

→ 1572

As the **Quadrupole** tensor is traceless by definition, it has only five independent components.

Furthermore, if we define the unit vector $\widehat{\mathbf{n}} \equiv \mathbf{r}/r$ and let $\mathbb{I}[\widehat{\mathbf{n}}]$ be the moment of inertia along the line joining the Center-Of-Mass to to point where the gravitational field has to be evaluated, that is at $\mathbf{r}$, and define I_{11}, I_{22} and I_{33} the moments of inertia with respect to the principal axes of inertia, we have, exploiting equation (26.04.10):

$$\Phi_2[\mathbf{r}] = -\mathcal{G}\frac{r_j r_k Q^{\cdot\cdot}_{jk}}{2r^5} = -\frac{\mathcal{G}}{2r^5} r_j r_k\left(\delta^{\cdot\cdot}_{jk}\,\mathrm{Tr}\,\mathbb{I} - 3I^{\cdot\cdot}_{jk}\right) = -\frac{\mathcal{G}}{2r^5}\left(r^2\,\mathrm{Tr}\,\mathbb{I} - 3r^2\mathbb{I}[\widehat{\mathbf{n}}]\right) = -\frac{\mathcal{G}}{2r^3}\left(I_{11}+I_{22}+I_{33}-3\mathbb{I}[\widehat{\mathbf{n}}]\right) \quad . \tag{29.05.07}$$

The latter is known as MacCullagh formula.

29.05.01.02 Gravitational Quadrupole Moment for Systems With Cylindrical Symmetry

©|J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|§ 1-6.4.2, 2-5.3.4||

Consider a system with cylindrical symmetry around the z axis: the z axis is then a symmetry axis but the symmetry is stronger than that, as the system is invariant for rotations around the z axis. The z axis, in other words, is an axis of revolution for the system.

Let the Coordinate System be centered on the Center-Of-Mass (lying on the z axis itself). The cylindrical symmetry implies that the mass density is only dependent of z and the distance from the z axis:

$$\rho[\mathbf{r}] \longrightarrow \rho[\sqrt{x^2+y^2}, z] \qquad \text{independent of } \phi \quad .$$

As a corollary every plane passing by the z axis is a plane of symmetry [1].

The **Quadrupole** moment tensor assumes a particularly simple form in this quite common case. In fact, because of the symmetry, all the terms with an odd power of x or y are zero, as both the zx and zy plane are planes of symmetry:

$$Q_{xy} = Q_{yz} = Q_{zx} = 0 \qquad Q_{xx} = Q_{yy} \quad .$$

that is the tensor is diagonal. As the tensor is traceless it follows:

$$0 = Q_{xx} + Q_{yy} + Q_{zz} \qquad \longrightarrow \qquad Q_{xx} = Q_{yy} = -\frac{1}{2}Q_{zz} \neq 0 \quad ,$$

implying that all elements can be expressed in terms of any one of the diagonal elements.

It is common practice to define the **Quadrupole moment** of a distribution with cylindrical symmetry as $Q \equiv Q_{zz}$.

The **Quadrupole** potential becomes:

$$\Phi_2[\mathbf{r}] = -\frac{\mathcal{G}}{2r^5} r_j r_k Q^{\cdot\cdot}_{jk} = -\frac{\mathcal{G}}{2r^5}\left(r_1^2 Q_{11} + r_2^2 Q_{22} + r_3^2 Q_{33}\right) = \quad ,$$

→ 1572

$$-\frac{\mathcal{G}}{4r^5}Q_{33}\left(2r_3^2 - \left(r_1^2+r_2^2\right)\right) = -\frac{\mathcal{G}}{4r^5}Q_{33}\left(3r_3^2 - r^2\right) = -\mathcal{G}\frac{Q\left(3\cos^2[\theta]-1\right)}{4r^3} \quad .$$

[1] Note that for a simple axial symmetry it is not true that every plane passing by the z axis is a plane of symmetry: consider for instance two intersecting planes making an angle different from $\pi/2$, where the intersection line is a symmetry axis but the planes are not planes of symmetry

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Let $I_\perp \equiv I_{11} = I_{22}$. Equation (29.05.01.02) can be written in terms of the tensor of inertia using equation (29.05.06), giving:

$$\boxed{Q \equiv Q_{33} = 2\left(I_\perp - I_{33}\right)} \quad . \tag{29.05.08}$$

→ 1572

Then

$$\boxed{\Phi_2[\mathbf{r}] = -\mathcal{G}\frac{Q\left(3\cos^2[\theta]-1\right)}{4r^3} = -\mathcal{G}\frac{\left(I_\perp - I_{33}\right)\left(3\cos^2[\theta]-1\right)}{2r^3}} \quad . \tag{29.05.09}$$

The **Quadrupole** moment gravitational field becomes significant, in practice, when the body producing the field is significantly distorted with respect to a spherical shape.

Equation (29.05.08) shows that:

- $Q > 0$ implies that the body mass is closer to the z axis, on average, than a spherically symmetrical distribution: cigar-like shape along the z axis;
- $Q < 0$ implies that the body mass is closer to the xy plane, on average, than a spherically symmetrical distribution: disk-like shape along the xy plane.

The shape of the **Quadrupole** potential can be understood as follows:

- $Q > 0$ implies that on the z axis (that is $|\cos\theta| = 1$), at fixed r: $\Phi_2 < 0$; therefore the total potential is lower than it would be from the pure monopole term; therefore the equipotential surface of the pure monopole term moves at $|z| > r$, and it deforms like the mass distribution;
- $Q < 0$ implies that on the z axis (that is $|\cos\theta| = 1$), at fixed r: $\Phi_2 > 0$; therefore the total potential is higher than it would be from the pure monopole term; therefore the equipotential surface of the pure monopole term moves at $|z| < r$, and it deforms like the mass distribution.

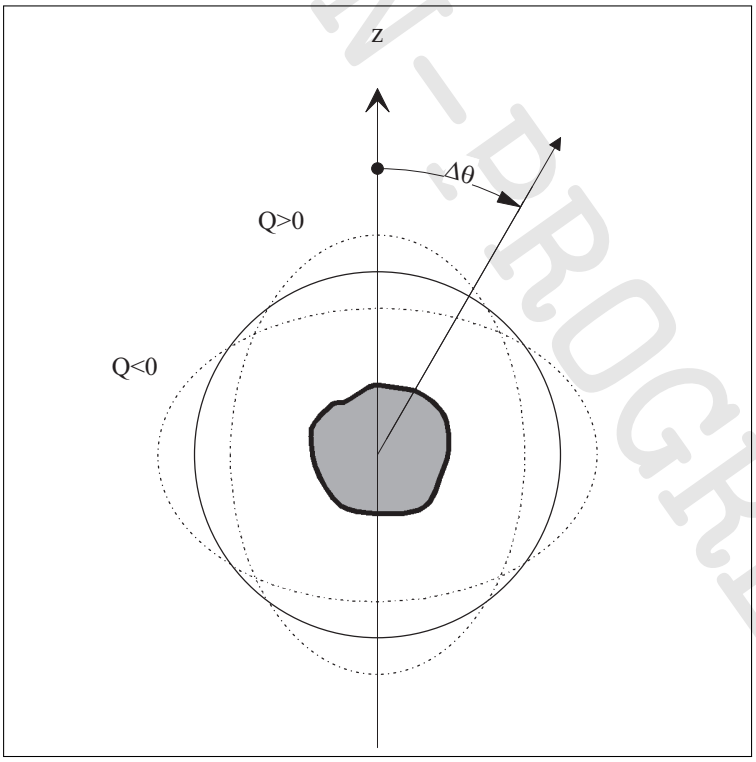

Figure 29.15: FIGURE

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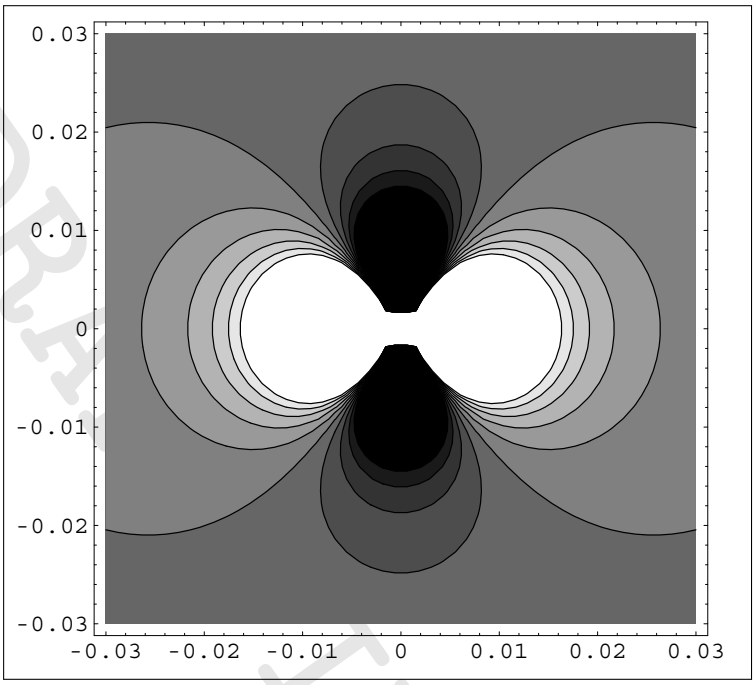

Figure 29.16: FIGURE

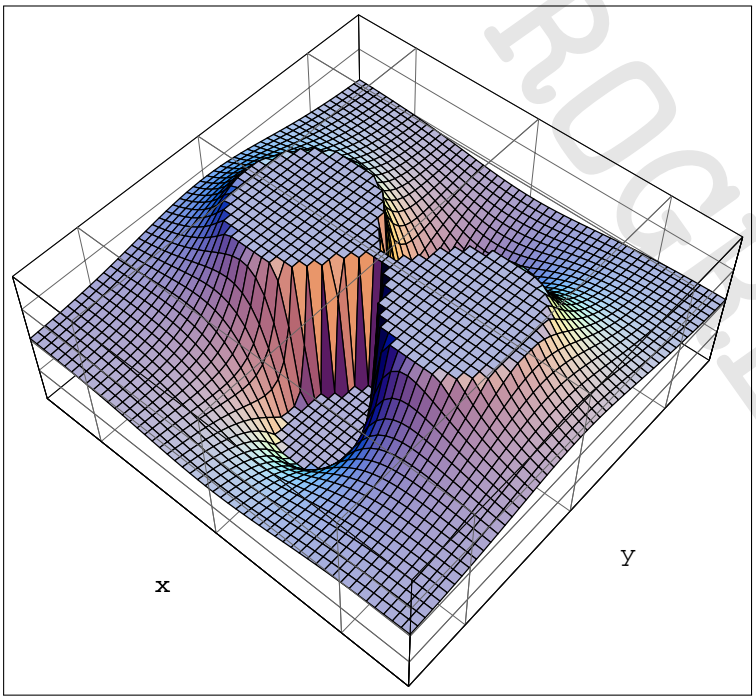

Figure 29.17: FIGURE

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Forces and Torques Produced by a Cylindrically Symmetrical Mass Distribution

This § is referenced at pages:
[1071]

Consider a main body with cylindrical symmetry around the z axis and let the Coordinate System be centered on its Center-Of-Mass and its large-distance gravitational potential truncated to second-order. A point mass m is located at $\mathbf{r}$. The components, $\{f_r; f_\theta; f_\phi\}$, of the gravitational force, $\mathbf{f}$, acting on the point mass, are calculated by taking the gradient in spherical coordinates of the potential (equation (29.05.01)) and multiplying by the test mass:

$$f_r[\mathbf{r}] \equiv -m\frac{\partial\Phi[\mathbf{r}]}{\partial r} = -\frac{\mathcal{G}Mm}{r^2} - \frac{3\mathcal{G}mQ\left(3\cos^2[\theta]-1\right)}{4r^4} \quad , \qquad \rightarrow 1574$$

$$f_\theta[\mathbf{r}] \equiv -m\frac{1}{r}\frac{\partial\Phi[\mathbf{r}]}{\partial\theta} = -\frac{3\mathcal{G}mQ\sin[2\theta]}{4r^4} \quad , \qquad \rightarrow 1574$$

$$f_\phi[\mathbf{r}] \equiv -m\frac{1}{r\sin[\theta]}\frac{\partial\Phi[\mathbf{r}]}{\partial\phi} = 0 \quad . \qquad \rightarrow 1574$$

The force field is not central with respect to the Center-Of-Mass of the main body, which is taken, since now on, as the origin of the Coordinate System. Actually, the force field is not central at all. As a consequence, **the orbit is not, in general, a plane one**; read § 29.05.03.

The force in the direction of the polar angle, $\mathbf{f}_\theta[\mathbf{r}]$, goes like $\sin[2\theta]$, that is it changes its sign every $\pi/2$ radians. This is because the mass point is attracted, at any point in its orbit, either towards the poles or towards the equator, depending on the sign of Q.

As the system (main body plus the point mass) is isolated, the sum of internal forces and torques is zero.

The (reaction) force exerted by the mass point on the planet is equal but opposite to $\mathbf{f}$ as expressed in equations (29.05.01.02), (29.05.01.02), (29.05.01.02).

While the force exerted on the mass point is just the force $\mathbf{f}$ applied at the mass point itself, the main body is subject to a distributed system of forces originating from the mass point, applied to the different infinitesimal parts of the main body, whose resultant is $-\mathbf{f}$.

The intrinsic torque exerted by the gravitational field on the mass point, $\boldsymbol{\gamma}[\mathbf{r}]$, is assumed to be zero, coherently with the properties of a gravitational field and, in particular, with the isotropy of a mass point. Because the force field acting on the mass point is non central (with respect to the Center-Of-Mass of the main body) the momentum of the force $\mathbf{f}$ on the mass point, with respect to the Center-Of-Mass of the main body, is:

$$\boldsymbol{\Gamma} = \boldsymbol{\gamma} + \mathbf{r}\wedge\mathbf{f} = \mathbf{r}\wedge\mathbf{f} \neq 0 \quad .$$

In order to balance to zero the internal torques of the planet plus satellite system, the mass point must exert an equal but opposite total torque on the planet, $\boldsymbol{\Gamma}'$ given by:

$$\boldsymbol{\Gamma}' + \boldsymbol{\Gamma} = 0 \qquad \boldsymbol{\Gamma}' = -\boldsymbol{\Gamma} \quad .$$

29.05.02
Multipole Development in Terms of Legendre Polynomials

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|4.1, P4.1, P4.2, P4.3, P4.4, P4.5, P4.6 ||

The contribution to the potential of any infinitesimal mass element is (compare with equation (29.05.01)):

$$\mathrm{d}\Phi[\mathbf{r}] = -\mathcal{G}\frac{\mathrm{d}m}{r}\left(1 - 2\frac{\mathbf{r}\cdot\mathbf{x}}{r^2} + \frac{x^2}{r^2}\right)^{-1/2} = -\mathcal{G}\frac{\mathrm{d}m}{r}\left(1 - 2\frac{x}{r}\cos[\Phi] + \frac{x^2}{r^2}\right)^{-1/2} \quad .$$

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The expansion in terms of the Legendre polynomials gives:

$$\text{d}\Phi[\mathbf{r}] = -\mathcal{G}\frac{\text{d}m}{r}\sum_{k=0}^{\infty}\left(\frac{x}{r}\right)^k P_k[\cos[\Phi]] \quad , \tag{29.05.10}$$

→ 1575

where P_k is the Legendre polynomial of order k.

The first four Legendre polynomials are:

$$P_0[s] = 1 \quad ,$$
$$P_1[s] = s \quad ,$$
$$P_2[s] = (1/2)\left(3s^2 - 1\right) \quad ,$$
$$P_3[s] = (1/2)\left(5s^3 - 3s\right)$$

Integrating term by term equation (29.05.10) up to second-order one finds:

$$\Phi[\mathbf{r}] = -\frac{\mathcal{G}M}{r} - \frac{\mathcal{G}}{r^2}\int x\cos[\Phi]\ \text{d}m - \frac{\mathcal{G}}{r^3}\int x^2\frac{\left(3\cos^2[\Phi]-1\right)}{2}\text{d}m + \quad ,$$
$$\ldots - \frac{\mathcal{G}}{r^{n+1}}\int x^n P_n[\cos[\Phi]]\ \text{d}m + \ldots \quad .$$

Let $\mathbf{r} \equiv r\left(\cos[\alpha_1], \cos[\alpha_2], \cos[\alpha_2]\right)$ and $\mathbf{x} \equiv (x_1, x_2, x_3)$. It follows that:

$$\mathbf{x}\cdot\mathbf{r} = xr\cos[\Phi] = r\left(x_1\cos[\alpha_1] + x_2\cos[\alpha_2] + x_3\cos[\alpha_2]\right) \longrightarrow x\cos[\Phi] = \left(x_1\cos[\alpha_1] + x_2\cos[\alpha_2] + x_3\cos[\alpha_2]\right) \quad .$$

The second term of the development is thus zero if the origin of the coordinates is chosen at the Center-Of-Mass of the mass distribution whose gravitational field has to be evaluated.

Consider the origin of the Coordinate System fixed at the Center-Of-Mass, since now on.

The **Quadrupole** term can be re-written, to obtain equation (29.05.02), as:

$$\cos[\Phi] = \frac{\mathbf{r}\cdot\mathbf{x}}{rx} \longrightarrow \quad ,$$
$$\Phi_2[\mathbf{r}] = -\frac{\mathcal{G}}{r^3}\int x^2\frac{\left(3\cos^2[\Phi]-1\right)}{2}\text{d}m = -\frac{\mathcal{G}}{2r^5}\int\left(3\left(x_k x_j r_k r_j\right) - x^2\delta_{kj}^{\cdot\cdot}r_k r_j\right)\ \text{d}m = \quad ,$$
$$-\frac{\mathcal{G}}{2r^5}r_k r_j\iiint\rho[\mathbf{x}]\left(3\left(x_k x_j\right) - x^2\delta_{kj}^{\cdot\cdot}\right)\ \text{d}V_{\mathbf{x}} = -\frac{\mathcal{G}}{2r^5}r_k r_j Q_{kj}^{\cdot\cdot} \quad .$$

29.05.02.01 Development in Terms of Legendre Polynomials for a System With Cylindrical Symmetry

For a system with cylindrical symmetry one can also write:

$$\Phi[\mathbf{r}] = -\frac{\mathcal{G}M}{r}\left(1 - \sum_{k=2}^{\infty}J_k\left(\frac{x}{r}\right)^k P_k[\sin[\phi]]\right) \quad , \tag{29.05.11}$$

→ 1672

where ϕ is the latitude of the point where the potential has to be calculated.

29.05.02.02 Legendre Polynomials Spherical Harmonics and Laplace equation

29.05.03 Orbit in a Gravitational Field With Non Zero Quadrupole Moment

This § is referenced at pages:
[1574]

The non-central force implies that the orbit is not a plane one, in general; special cases of plane orbits may arise, such as an equatorial orbit for a system with cylindrical symmetry and monopole plus quadrupole potential terms only.

DRAFT IN PROGRESS

29.05.04
Gravitational Versus ElectroMagnetic Multipole Development

This § is referenced at pages:
[1127, 1569, 1737]

DRAFT-IN-PROGRESS

29.06 Examples and Physical Applications

29.06.01
Central Force Fields: Equation of Motion in the Co-Rotating non-Inertial Reference Frame

This § is referenced at pages:
[1553]

Consider a non-Inertial Reference Frame , centered in the center of force, with its $\widehat{\mathbf{e}}_3$ axis oriented parallel to ℓ, and with its $\widehat{\mathbf{e}}_r$ axis always directed from the center of force to the point mass; call it the co-rotating non-Inertial Reference Frame .

Show that the equation of motion in this non-Inertial Reference Frame are the same as (29.03.01), (29.03.01).

Note that the center of force is assumed to be at rest in an Inertial Reference Frame .

In the co-rotating non-Inertial Reference Frame the three inertial forces related to the rotation of the Reference Frame with respect to the non inertial one appear, in addition to the real central force field. Note that the motion of the co-rotating non-Inertial Reference Frame is described by the same angle, φ, describing the motion of the point mass in the inertial Reference Frame.

Moreover:

$$
\begin{aligned}
r' &= r \quad , \\
\widehat{\mathbf{e}}_r &\equiv \mathbf{r}/r \quad , \\
\mathbf{r}' &= r\widehat{\mathbf{e}}_r \quad , \\
\mathbf{v}' &= \dot{r}\widehat{\mathbf{e}}_r \qquad \text{because the motion in co-rotating Reference Frame is along } \widehat{\mathbf{e}}_r \quad , \\
\mathbf{a}' &= \ddot{r}\widehat{\mathbf{e}}_r \qquad \text{because the motion in co-rotating Reference Frame is along } \widehat{\mathbf{e}}_r \quad , \\
\widehat{\mathbf{e}}_3 &\equiv \widehat{\mathbf{e}}_r \wedge \widehat{\mathbf{e}}_\psi \quad .
\end{aligned}
$$

The forces read:

$$
\begin{aligned}
\mathbf{f}[\mathbf{r}'] &= f[r]\widehat{\mathbf{e}}_r \qquad \text{forces are absolute in } \textbf{Galilei-Newton Mechanics} \quad , \\
\mathbf{f}_{\text{CEN}}[\mathbf{r}'] &= m\dot{\varphi}^2 r\widehat{\mathbf{e}}_r \quad , \\
\mathbf{f}_{\text{COR}}[\mathbf{r}'] &= -2m\dot{\varphi}\dot{r}\widehat{\mathbf{e}}_\psi \qquad \text{the Coriolis force is transverse, because the velocity is radial } \mathbf{v}' = \dot{r}\widehat{\mathbf{e}}_r \quad , \\
\mathbf{f}_{\text{ACC}}[\mathbf{r}'] &= -m\ddot{\varphi}r\widehat{\mathbf{e}}_\psi \quad .
\end{aligned}
$$

Equations (29.03.01), (29.03.01) then follow.

29.06.02
Monotonicity of the Angular Position

Show that, for any motion with non zero angular momentum, the angle is a strictly monotonic function of time.

For a non zero angular momentum, one has, from equation

$$\ell = mr^2\dot{\varphi} \neq 0 \quad ,$$

that $\dot{\varphi} \neq 0$ and therefore the angle is a strictly monotonic function of time.

PRELIMINARY-IN-PROGRESS

29.06.03
Behavior of the Force for Power-Law Potentials

Consider the effective potential energy:

$$\Phi_{\text{E}}[r] = -\gamma r^{\alpha} + \frac{\ell^2}{2mr^2} \qquad \text{with} \qquad r \geq 0 \quad , \qquad \alpha \neq 0 \quad .$$

The effective force will be given by:

$$f_{\text{E}}[r] = -\frac{\mathrm{d}\Phi_{\text{E}}[r]}{\mathrm{d}r} \quad ,$$

which has a minimum at

$$r^{(\alpha+2)} = -\frac{\ell^2}{\gamma\alpha m} \qquad \Leftrightarrow \qquad \gamma\alpha < 0 \quad .$$

Note that (from the expression of the force alone, not of the effective force):

- the force is attractive for $\alpha\gamma < 0$;
- the force is repulsive for $\alpha\gamma > 0$.

29.06.04
Behavior of the Effective Potential Energy for Power-Law Potentials

Consider a potential energy of the type:

$$\Phi[r] = -\gamma r^{\alpha} \qquad \text{with} \qquad r \geq 0 \quad , \qquad \alpha \neq 0 \quad .$$

Note that γ will be proportional to the gravitational mass, in case of gravitational forces.

Note that:

- for $\alpha > 0$ the potential energy is attractive for $\gamma < 0$ and repulsive for $\gamma > 0$ and the zero potential energy state cannot be chosen at $r \longrightarrow \infty$ but it is normally chosen at $r = 0$;
- for $\alpha < 0$ the potential energy is attractive for $\gamma > 0$ and repulsive for $\gamma < 0$ and the zero potential energy state is normally chosen at $r \longrightarrow \infty$.

Determine the behavior of $\Phi_{\text{E}}[r]$ for $r \longrightarrow 0$ and for $r \longrightarrow +\infty$.

WORKED SOLUTION

For $0 < \alpha$:

- $\gamma > 0 \implies \lim_{r\longrightarrow 0} \Phi_{\text{E}}[r] = +\infty$;
- $\gamma > 0 \implies \lim_{r\longrightarrow +\infty} \Phi_{\text{E}}[r] = -\infty$;
- $\gamma < 0 \implies \lim_{r\longrightarrow 0} \Phi_{\text{E}}[r] = +\infty$;
- $\gamma < 0 \implies \lim_{r\longrightarrow +\infty} \Phi_{\text{E}}[r] = +\infty$;

For $-2 < \alpha < 0$:

- any $\gamma \implies \lim_{r\longrightarrow 0} \Phi_{\text{E}}[r] = +\infty$;
- any $\gamma \implies \lim_{r\longrightarrow +\infty} \Phi_{\text{E}}[r] = 0$;

For $\alpha < -2$:

- $\gamma > 0 \implies \lim_{r\longrightarrow 0} \Phi_{\text{E}}[r] = -\infty$;
- $\gamma > 0 \implies \lim_{r\longrightarrow +\infty} \Phi_{\text{E}}[r] = 0$;
- $\gamma < 0 \implies \lim_{r\longrightarrow 0} \Phi_{\text{E}}[r] = +\infty$;
- $\gamma < 0 \implies \lim_{r\longrightarrow +\infty} \Phi_{\text{E}}[r] = 0$;

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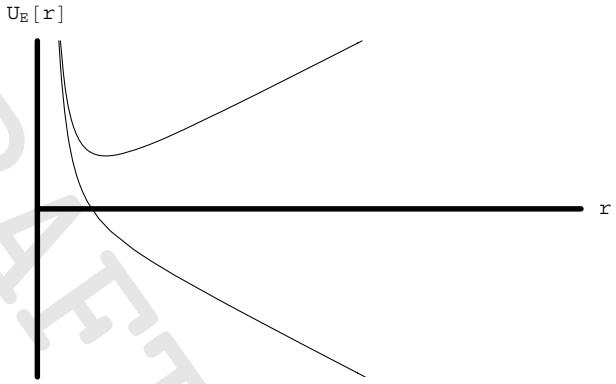
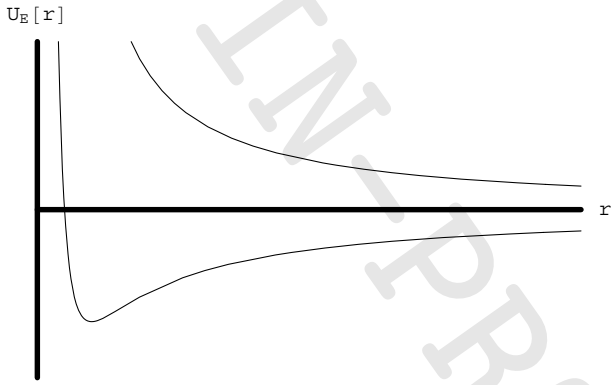
29.06.05
Plots of the Effective Potential Energy

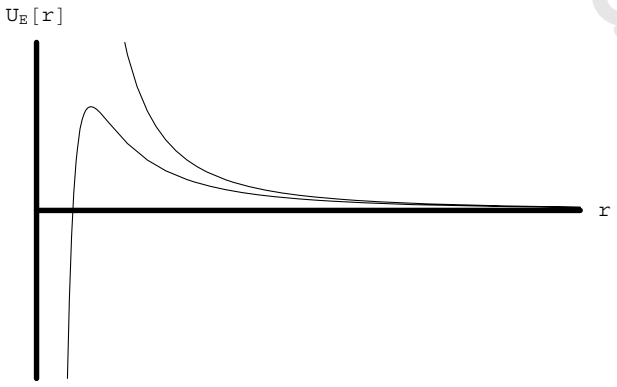
Figure 29.18 shows qualitative plots of the behavior of the effective potential energy for all the possible cases of the exponent, α, and for both signs of γ. Figure 29.19 shows qualitative plots of the behavior of



(a) Power law exponent: $0 < \alpha$ (positive and negative γ).



(b) Power law exponent: $-2 < \alpha < 0$ (positive and negative γ).



(c) Power law exponent: $\alpha < -2$ (positive and negative γ).

Figure 29.18: ....

the effective potential energy for all the possible cases of the exponent, α, and for both signs of γ.

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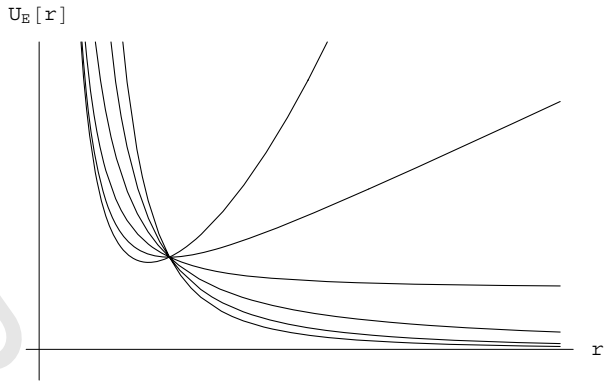
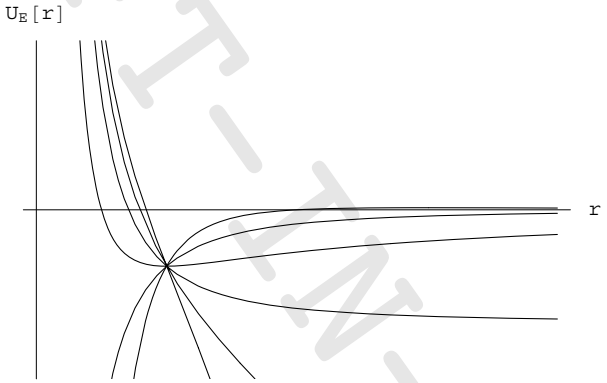


(a) Power law exponent: $\gamma < 0$ (positive and negative α).



(b) Power law exponent: $\gamma > 0$ (positive and negative α).

Figure 29.19: ....

29.06.06
Determination of the Orbit for an Inverse Square Law

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Determine the orbit for an inverse square force law.

29.06.07
Determination of the Orbit for an Elastic Restoring Force Law

This § is referenced at pages:
[1405]

Note: in this case it is easy to start from scratch and just solve the two-dimensional first cardinal law in orthonormal orthonormal Cartesian Coordinates Coordinate System.

WORKED SOLUTION

It is an ellipse whose geometrical center is at the center of force.

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29.06.08
Symmetry of the Orbit With Respect to the Apsidal Vectors

This § is referenced at pages:
[1554, 1556]

The apsidal vectors are defined as the vectors joining the center of force to the points of the orbit such that $\frac{\mathrm{d}r}{\mathrm{d}\varphi} = 0$, including the points of either maximum or minimum distance along the orbit. Choose the origin of the angles in the plane of the orbit in such way that one apsidal vector appears at $\varphi = 0$.

Show that the trajectory is an even function of φ, that is symmetric under the interchange $\varphi \longrightarrow -\varphi$. This means that it is enough to know the equation of the orbit between any two apsidal points, that is the turning points of the orbit, to be able to reconstruct the full orbit.

Solve the Binet equation with initial conditions at $\varphi = 0$. Let $u[\varphi]$ be such a solution (see equation (29.03.09)):

$$u[\varphi] \equiv \frac{1}{r[\varphi]} \quad ,$$

$$u[\varphi = 0] = \frac{1}{r_0} \tag{29.06.01}$$ → 1581

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi}[\varphi = 0] = 0 \qquad \text{because } \frac{\mathrm{d}u}{\mathrm{d}\varphi} = -\frac{1}{r^2}\frac{\mathrm{d}u}{\mathrm{d}r} \text{ implies } \frac{\mathrm{d}u}{\mathrm{d}\varphi} = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}r} = 0 \tag{29.06.02}$$ → 1581

$$\frac{d^2u}{\mathrm{d}\varphi^2} + u = -\frac{m}{u^2\ell^2} f[\frac{1}{u}] \quad .$$

Consider now the function:

$$\hat{u}[\varphi] \equiv u[-\varphi] \quad .$$ → 1581

It is obvious that such a function satisfies both the same Binet equation as $u[\varphi]$, equation (29.06.08), and the same initial conditions as $u[\varphi]$, equations (29.06.01), (29.06.02). In fact,

$$\frac{d^2}{\mathrm{d}\varphi^2}\hat{u}[\varphi] = \frac{d^2}{\mathrm{d}\varphi^2}u[-\varphi] \implies ,$$

$$\frac{d^2}{\mathrm{d}\varphi^2}\hat{u}[\varphi] + \hat{u}[\varphi] = \frac{d^2}{\mathrm{d}\varphi^2}u[-\varphi] + u[-\varphi] = \text{from Binet} = -\frac{m}{u^2\ell^2} f[\frac{1}{u[-\varphi]}] = \text{by definition} = -\frac{m}{u^2\ell^2} f[\frac{1}{\hat{u}[\varphi]}] \quad ,$$

$$\hat{u}[\varphi = 0] = u[\varphi = 0] = \frac{1}{r_0} \quad ,$$

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}\varphi}[\varphi = 0] = -\frac{\mathrm{d}u}{\mathrm{d}\varphi}[\varphi = 0] = 0 \quad ,$$

as only a second derivative appears in the equation, and the first derivative condition is set to zero (this is where the condition that $\varphi = 0$ be an apsidal vector).

Therefore the unicity of the solution of the equation allows one to conclude that:

$$\hat{u}[\varphi] \equiv u[-\varphi] = u[+\varphi] \quad ,$$

that is the desired result ($u[\varphi]$ is an even function).

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29.06.09
Orbite Circolari in Potenziale Centrale

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29.06.10
Radial Stability of a Circular Orbit in a Central Force Field

Determine the conditions for the radial stability of a circular orbit in a central force field.

Study the effective potential energy (equation (29.03.03)):

$$\Phi_{\text{E}}[r] = \Phi[r] + \frac{\ell^2}{2mr^2} \quad .$$

Take the first and second derivative with respect to r:

$$\Phi'_{\text{E}}[r] = \Phi'[r] - \frac{\ell^2}{mr^3} = 0 \quad \Longrightarrow \Phi'[r_0] = \frac{\ell^2}{mr_0^3} \qquad \text{at the extremum value } r_0 \quad , \tag{29.06.03}$$

$$\Phi''_{\text{E}}[r] = \Phi''[r] + \frac{3\ell^2}{mr^4} \qquad \Longrightarrow \Phi''_{\text{E}}[r_0] = \Phi''[r_0] + 3\frac{\Phi'[r_0]}{r_0} = \Phi''[r_0] + 3m\omega_0^2 \qquad \text{at the extremum value } r_0 \quad . \tag{29.06.04}$$

■

The extrema condition is the same determined in equation (29.03.13), as it should. 1582

A sufficient condition for the stability of the radial motion is thus:

$$\boxed{\frac{\Phi''_{\text{E}}[r_0]}{m} = \frac{\Phi''[r_0]}{m} + 3\omega_0^2 > 0} \quad .$$

In the case of a power-law potential energy:

$$\Phi[r] = -\gamma r^\alpha \qquad \text{with} \qquad r \geq 0 \quad , \alpha \neq 0 \quad ,$$

the stability condition implies $\alpha > -2$.

29.06.11
Small Radial Oscillations Around a Stable Circular Orbit

This § is referenced at pages:
[1405]

Determine the frequency of the small radial oscillations around a stable circular orbit in a central force field. Study the case of a potential energy

$$\Phi[r] = -\gamma r^\alpha \qquad \text{with} \qquad r \geq 0 \quad , \alpha \neq 0 \quad . \tag{29.06.05}$$

→ 1582

For a one-dimensional motion the frequency of the small oscillations, ω_{R}, around a stable equilibrium position, at $r = r_0$, in the potential energy $\widetilde{\Phi}[r]$, is given by:

$$\boxed{\omega_{\text{R}}^2 \equiv \frac{\widetilde{\Phi}''[r_0]}{m}} \quad .$$

Therefore the result from equation (29.06.04) is:

$$\boxed{\omega_{\text{R}}^2 \equiv \frac{\Phi''_{\text{E}}[r_0]}{m} \Longrightarrow \omega_{\text{R}}^2 = \frac{\Phi''[r_0]}{m} + 3\omega_0^2} \quad . \tag{29.06.06}$$

→ 1583 1584

Consider now the potential energy in equation (29.06.05). One finds, from equation (29.03.15):

$$\omega_0^2 = -\frac{\alpha\gamma}{m} r_0^{\alpha-2} = \frac{1}{m\,(\alpha-1)}\Phi'' \quad ,$$

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and therefore, from equation (29.06.06), one finds:

$$\boxed{\omega_{\mathrm{R}}^2 = (\alpha+2)\,\omega_0^2} \quad , \tag{29.06.07}$$

→ 1583

which, implicitly, shows that the equilibrium is stable for $\alpha > -2$.

Note that for $\alpha = 2$ the effective potential is an inverse square potential whose sign is given by the relative magnitude of the two coefficients of the physical potential and centrifugal potential. It is a monotonic function and therefore it does not admit circular orbits.

Note the following cases:

- $\alpha = -1$, the Kepler and Coulomb problem, $\omega_{\mathrm{R}} = \omega_0$;
- $\alpha = +2$, the harmonic oscillator, $\omega_{\mathrm{R}} = 2\omega_0$, that is the radial distance oscillates twice during a $\Delta\varphi = 2\pi$ turn (due to the fact the orbit is always an ellipse centered on the force field).

Equation (29.06.07) shows that the perturbed orbit is closed if and only if $\sqrt{\alpha+2}$ is a rational number. Compare with the statement of the Bertrand theorem, § 29.03.08.

29.06.12
Precession of the Perihelion of a Near Circular Stable Orbit in a Inverse Square Force Field

This § is referenced at pages:
[1585, 1585, 1628]

Consider the potential energy:

$$\Phi[r] = -\frac{\gamma}{r} - \frac{\epsilon}{r^\beta} \equiv \Phi_0[r] + \Phi_1[r] \qquad \text{with} \qquad r \geq 0 \quad , \qquad \gamma > 0 \quad , \qquad \beta > 0 \quad ,$$

→ 1584

$$f[r] = -\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\gamma}{r^2} - \frac{\beta\epsilon}{r^{1+\beta}} \quad ,$$

where the $\Phi_1[r]$ term is much smaller than the $\Phi_0[r]$ term at all the points in the orbit:

$$\frac{\epsilon}{r^\beta} \ll \frac{\gamma}{r} \qquad \text{for any } r \text{ along the orbit} \qquad \Longrightarrow \epsilon \ll \gamma r^{\beta-1} \quad . \tag{29.06.08}$$

→ 1584

The potential is therefore broadly attractive, as it is dominated by the Kepler attractive term.

Note that with the above convention the quantity ϵ (the perturbation parameter) is not dimensionless. To avoid complicating the notation one might keep in mind the it might be linked to a truly dimensionless quantity, $\bar{\epsilon}$, by:

$$\epsilon = \bar{\epsilon} r_0^\beta \quad ,$$

where r_0 is any suitable distance, such as the radius of the unperturbed orbit.

Determine the frequency of the small radial oscillations around a circular orbit and, based on it, determine the precession of the perihelion of the orbit due to the perturbation $\Phi_1[r]$, as the pure $\Phi_0[r]$ term will always give closed orbits, that is no precession of the perihelion.

$$\Phi'[r] = \frac{\gamma + \beta\epsilon r^{1-\beta}}{r^2} \quad ,$$

$$\Phi''[r] = -\frac{2\gamma}{r^3} - \frac{\epsilon\beta\,(1+\beta)}{r^{2+\beta}} \quad .$$

Let $r = r_0$ be the radius of the circular orbit (for the given energy and|or angular momentum). Equation (29.03.15) implies that the exact revolution frequency for circular orbits in the perturbed potential is:

$$\boxed{\omega_0^2 = \frac{1}{m}\left(\frac{\gamma}{r_0^3} + \frac{\beta\epsilon}{r_0^{\beta+2}}\right) = \frac{1}{mr_0^2}\left(\frac{\gamma}{r_0} + \frac{\beta\epsilon}{r_0^\beta}\right)} \quad .$$

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This revolution frequency differs by only a small amount (first-order in the perturbation parameter ϵ) from the revolution frequency of the unperturbed force:

$$\boxed{\omega_0^2\Big|_{\text{non-pert}} = \frac{\gamma}{mr_0^3}} \quad ,$$

thanks to the condition (29.06.08).

If the real orbit is only slightly different from a circular orbit in the exact perturbed potential the precession of perihelion can be calculated by means of the frequency of the radial oscillation around the equilibrium position. Equation (29.06.06) then implies:

$$\boxed{\omega_{\text{R}}^2 = \frac{\Phi''[r_0]}{m} + 3\omega_0^2 = \frac{1}{m}\left(\frac{\gamma}{r_0^3} + \frac{\epsilon\beta\,(2-\beta)}{r_0^{\beta+2}}\right) = \frac{1}{mr_0^2}\left(\frac{\gamma}{r_0} + \frac{\epsilon\beta\,(2-\beta)}{r_0^{\beta}}\right)} \quad .$$

This radial oscillation frequency, also, differs by only a small amount from the radial oscillation frequency of the unperturbed force thanks to the condition (29.06.08).

The approximate perihelion precession, $\Delta\varphi_0$, can be calculated by:

$$\boxed{\Delta\varphi_0 + 2\pi \simeq \omega_0 T_{\text{R}} = 2\pi\frac{\omega_0}{\omega_{\text{R}}} = 2\pi\left(1 + \frac{\beta\epsilon\,(\beta-1)}{2\gamma}r_0^{1-\beta} + \frac{\beta^2\epsilon^2\,(\beta-1)\,(3\beta-7)}{8\gamma^2}r_0^{2-2\beta} + \mathcal{O}\,(\epsilon)^3\right)} \quad .$$

The above expression assumes that the revolution angular velocity can be treated as a constant, ω_0, or, better to say, that the average revolution angular velocity equals ω_0, the revolution angular velocity of the circular orbit. This seems to be a reasonable assumption because the real orbit is, in any case, a slightly perturbed orbit with respect to the circular orbit.

To first-order the azimuthal frequency, $\dot{\varphi}$, (revolution angular velocity) is affected (changed) with respect to the circular orbit one, ω_0, because the angular momentum does not change and therefore, as the distance from the center in a non-circular orbit does actually change, the azimuthal frequency must change. In fact:

$$\ell \equiv mr^2\dot{\varphi} = \ell_0 = mr_0^2\omega_0 \implies \dot{\varphi} = \frac{\ell}{mr^2} \simeq \frac{\ell}{mr_0^2}\left(1 - 2\frac{\Delta r}{r_0}\right) = \omega_0\left(1 - 2\frac{\Delta r}{r_0}\right) \quad .$$

The latter shows that the value of $\dot{\varphi}$, when averaged along a full orbit, gives ω_0. In the small-oscillations approximations used here the potential energy near the minimum is supposed to be a quadratic function (which implies, in particular, that it is symmetrical) and the radial motion is then an harmonic motion. Therefore, in this approximation, the averaging can be done both in time and in the radial position.

In the significant case of $\beta = 2$ it reduces, to first-order in ϵ, to:

$$\boxed{\Delta\varphi_0 = \frac{2\pi\epsilon}{\gamma r_0}} \quad .$$

The result gives the precession of the perihelion due to the presence of the perturbation term, because the pure inverse square law gives no precession and $\omega_{\text{R}}^2 = \omega_0^2$. The additional term implies: $\omega_{\text{R}}^2 \neq \omega_0^2$.

29.06.13

Inverse Cube Law Perturbation to the Inverse Square Law

Consider the force law:

$$f[r] = -\frac{\gamma}{r^2} - \frac{2\epsilon}{r^3} \qquad \text{with } \frac{2|\epsilon|}{r^3} \ll \frac{|\gamma|}{r^2} \text{ for all } r \text{ in the interesting range} \quad , \tag{29.06.09}$$

→ 1584 1585

where an inverse cube law is a perturbation to the inverse square law. Note that the force law corresponds to the potential (29.06.12) with $\beta = 2$, which explains the presence of the factor 2 in the expression (29.06.09).

Determine the equation of the orbit.

The Binet equation gives:

$$u''[\varphi] + \left(1 - \frac{2m\epsilon}{\ell^2}\right)u[\varphi] = \frac{m\gamma}{\ell^2} \quad ,$$

IN PREPARATION

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whose general solution is:

$$u[\varphi] = B\cos\left[\alpha\left(\varphi-\varphi_0\right)\right] + \frac{m\gamma}{\ell^2}\left(\frac{1}{1-\dfrac{2m\epsilon}{\ell^2}}\right) \qquad \alpha \equiv \sqrt{1-\frac{2m\epsilon}{\ell^2}} \simeq 1-\frac{m\epsilon}{\ell^2} \quad .$$

The general solution shows that the angular distance (azimuthal distance) between any two successive maxima of u (that is perihelia) or between any two successive minima of u (that is aphelia), that is the period of the function, is:

$$\Delta\varphi = \frac{2\pi}{\alpha} \simeq 2\pi\left(1+\frac{m\epsilon}{\ell^2}\right) \quad .$$

The orbit, as long as the inverse cube law is only a small perturbation to the inverse square law, can be thought as an ellipse (for a bound system) whose perihelion is rotating at every orbit by

$$\Delta\varphi_0 = \frac{2m\pi\epsilon}{\ell^2} \quad . \tag{29.06.10}$$

→ 1585

This expression is generally correct, not only for almost circular orbits, as in § 29.06.12. When can then apply the result (29.06.10) to an almost circular orbit, in the force law (29.06.09):

$$\ell_0 = mr_0^2\omega_0 \quad ,$$
$$m\omega_0^2 r_0 = \frac{\gamma}{r_0^2} + \frac{2\epsilon}{r_0^3} \quad ,$$
$$2\epsilon \ll \gamma r_0 \quad ,$$

to obtain the same result as in § 29.06.12:

$$\Delta\varphi_0 = \frac{2m\pi\epsilon}{\ell^2} = \frac{2m\pi\epsilon}{m^2 r_0^4\omega_0^2} = \frac{2m^2\pi\epsilon r_0}{m^2 r_0^4\left(\dfrac{\gamma}{r_0^2}+\dfrac{2\epsilon}{r_0^3}\right)} = \frac{2\pi\epsilon}{(r_0\gamma+2\epsilon)} \simeq \frac{2\pi\epsilon}{r_0\gamma} \quad .$$

29.06.14
On Bertrand Theorem

It is possible to find closed particular orbits for a general potential law by finding a particular closed orbit for a generic central force field.

In fact, any attractive force field admits circular orbits as a solution.

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29.06.15
Total Mechanical Energy

- Verify using equation (29.03.09), by explicit calculation, the conservation of mechanical energy. Read § 29.03.04.
- Verify using equation (29.03.05), by explicit calculation, the conservation of mechanical energy.

29.06.16
Speed Along a Keplerian Orbit

Consider a mass particle of mass m orbiting around an object of mass M. Show that the velocity along the orbit is as follows.

Elliptic orbit:

$$v^2 = \frac{|\gamma|}{m}\left(\frac{2}{r} - \frac{1}{a}\right) \quad . \tag{29.06.11}$$

→ 1589

Note that γ is necessarily positive.

Parabolic orbit:

$$v^2 = \frac{|\gamma|}{m}\left(\frac{2}{r}\right) \quad . \tag{29.06.12}$$

Note that γ is necessarily positive.

Hyperbolic orbit:

$$v^2 = \frac{|\gamma|}{m}\left(\frac{2}{r} + \frac{1}{a}\right) \quad . \tag{29.06.13}$$

Use the mechanical energy conservation and the relations (29.04.21), (29.04.22), (2).

29.06.17
Third Kepler Law

Derive the third Kepler law.

From the expression of the areolar velocity, $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\ell}{2m}$, and the are of the ellipse, $A = \pi ab$, one finds:

$$T = \frac{2m\pi a^2}{\ell}\sqrt{1-e^2} = 2\pi\sqrt{\frac{ma^3}{\gamma}} \quad .$$

29.06.18
Third Kepler Law and the Measurement of the Masses of a Binary Stellar System

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Show how to use the Third Kepler law for the measurement of the total mass of a binary stellar system.

From the orbital period the sum of the masses is determined. From the relative distance with respect to common Center-Of-Mass the ration of the masses is determined.

This works in principle, in practice the geometry of the system with respect to the line of sight needs to be accounted for.

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29.06.19
Satellite Paradox

©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|6.2||
©|Symon, , ..., ..., ...Ed., ...|§ 4.5||

This § is referenced at pages:
[Never referenced.]

Un satellite di massa m è in orbita circolare attorno alla Terra ad una distanza $r = r_0$ dal centro della Terra. Sul satellite agisce una forza di attrito resistente, dovuta alla resistenza degli strati alti dell'atmosfera, di valore piccolo rispetto alla forza gravitazionale e avente direzione opposta alla sua velocità.

Supponendo che l'orbita sia sempre circolare con raggio lentamente variabile discutere l'evoluzione del moto del satellite in termini della sua energia meccanica e del suo momento angolare.

WORKED SOLUTION

A causa della presenza della forza di attrito resistente $\mathbf{f}$, che ha momento non nullo rispetto al centro della Terra, il momento angolare del satellite non si conserva. Tuttavia l'equazione del moto radiale, per le ipotesi fatte di orbita circolare e forza di attrito $\mathbf{f}$ antiparallela alla velocità $\mathbf{v}$ del satellite, resta la stessa che si ha nel caso in cui $\mathbf{f}$ non è presente

$$\frac{\mathcal{G}Mm}{r^2} = m\frac{v^2}{r} = m\omega^2 r \tag{29.06.14}$$

→ 1587 1588

L'energia meccanica del satellite non è conservata a causa della presenza della forza dissipativa $\mathbf{f}$ e si scrive, in virtù dell'equazione (29.06.14),

$$E = \mathcal{K} + \Phi = \frac{1}{2}mv^2 - \frac{\mathcal{G}Mm}{r} = -\mathcal{K} = \frac{\Phi}{2} < 0 \tag{29.06.15}$$

→ 1587

ed è negativa essendo il satellite gravitazionalmente legato. Denotando con L il lavoro fatto dalla resistenza del mezzo, non conservativa, il bilancio energetico diventa

$$\Delta E = L < 0 \tag{29.06.16}$$

In generale L può includere il lavoro di altre forze, conservative e non. Dalla relazione (29.06.15) si ha

$$\mathrm{d}L = \mathrm{d}E = \frac{1}{2}\,\mathrm{d}\Phi = \frac{\mathcal{G}Mm}{2r^2}\,\mathrm{d}r < 0 \implies \mathrm{d}r < 0 \quad ,$$

$$\mathrm{d}L = \mathrm{d}E = -\,\mathrm{d}\mathcal{K} = -\frac{m}{2}\,\mathrm{d}v^2 < 0 \implies \mathrm{d}v^2 > 0 \quad .$$

da cui si deduce che, essendo $\Delta E < 0$, il raggio dell'orbita diminuisce mentre il modulo della velocità del satellite aumenta. Si noti che il legame inverso tra r e v può essere dedotto già dall'equazione (29.06.14) che da sola non permette però di capire se r aumenti o diminuisca (e di conseguenza l'andamento di v). Le considerazioni energetiche, partendo dal fatto che $\Delta E = L < 0$, permettono di discriminare tra i due casi. In generale quindi se l'energia meccanica totale diminuisce anche il raggio dell'orbita diminuisce mentre se l'energia meccanica totale aumenta (cioè in presenza di una forza non resistente ma che compie lavoro positivo) il raggio dell'orbita aumenta. Le considerazioni energetiche fatte non dipendo dal tipo di forza tangenziale che agisce sul satellite ma solo dal segno del lavoro che questa compie. L'energia dissipata per attrito e quella che va in aumento di energia cinetica sono entrambe ottenute a spese dell'energia potenziale gravitazionale. Infatti si può scrivere dall'equazione (29.06.15),

$$-\,\mathrm{d}\Phi = -2\,\mathrm{d}E = -\,\mathrm{d}L + \,\mathrm{d}\mathcal{K}$$

da cui risulta come l'energia potenziale gravitazionale persa, $-\,\mathrm{d}\Phi > 0$, si ripartisce in parti uguali in aumento di energia cinetica, $\mathrm{d}\mathcal{K} > 0$ ed energia che il satellite cede all'esterno tramite la forza

dissipativa di attrito $-\,\mathrm{d}L > 0$ (L è il lavoro che l'ambiente esterno compie sul satellite, negativo, e $-L$ è il lavoro che il satellite compie sull'ambiente esterno, positivo).
Il momento angolare del satellite vale

$$l = mrv = m\sqrt{\mathcal{G}Mr} = \frac{\mathcal{G}Mm}{v} \quad ,$$

e quindi diminuisce nel caso in cui il raggio dell'orbita diminuisce (in quanto in tal caso il momento della forza resistente è opposto al momento angolare) e aumenta nel caso in cui il raggio dell'orbita aumenta (in quanto in tal caso il momento della forza resistente è concorde al momento angolare).
La legge di variazione del momento angolare fornisce l'equazione differenziale per r

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{2f[v]}{m\sqrt{\mathcal{G}M}}r^{3/2} = -k\frac{2}{m\sqrt{\mathcal{G}M}}r^{3/2}v \quad , \tag{29.06.17}$$

→ 1588

dove, in generale, la forza resistente $\mathbf{f} = -k\mathbf{v}$ dipende dal modulo della velocità che va espressa nell'equazione (29.06.17) in funzione di r, $v = v[r]$, tramite l'equazione (29.06.14).
Ne segue che la proiezione tangenziale dell'equazione del moto risulta

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = f[v] \quad .$$

Un caso di forza concorde con la velocità del satellite è nel problema § 31.04.05.

29.06.20
Supernovae in Sistemi Planetari O Sistemi Stellari Binari

This § is referenced at pages:
[1330]

©|Am.J.Phys, , ..., ..., ...Ed., ...|48 (3), 1980, 226||
©|B.W.Carroll & D.A.Ostlie, *An Introduction To Modern Astrophysics*, ..., ..., 1stEd., ...|§ 17.5||
©|Lim Yung-Kuo, *Problems And Solutions On Mechanics*, 1994, World Scientific, ...Ed., ...|1119||

Una stella a simmetria sferica espelle una parte ΔM della sua massa totale M sotto forma di un sottile guscio sferico a simmetria sferica che si allontana all'infinito in un tempo breve. Un pianeta di massa m orbita inizialmente attorno alla stella su un'orbita circolare a distnza $r = r_0$ dal centro della stella. Si supponga di poter trascurare l'effetto sul pianeta dell'onda d'urto causata dal passaggio del guscio di materia e di poter considerare tale passaggio istantaneo.

1. Determinare l'effetto dell'espulsione della massa sull'orbita del pianeta.
2. Discutere la dinamica del pianeta in coordinate polari.

WORKED SOLUTION

Ammettendo per il problema in questione che il passaggio del guscio sferico alla distanza del pianeta possa essere schematizzato come una variazione, con discontinuità di prima specie, della forza che agisce sul pianeta si ha quindi che la velocità è a priori una funzione non derivabile (in senso stretto) all'istante del passaggio. Si osservi che essendo il pianeta soggetto ad una forza dipendente dal tempo la sua energia non è conservata.
Fatte queste premesse si ha allora che l'energia meccanica totale dopo che la massa ΔM si è allontanata all'infinito è

$$E' = \frac{1}{2}mv_0^2 - \frac{\mathcal{G}\,(M - \Delta M)\,m}{r_0}$$

in cui r_0 e v_0 sono le stesse che si avevano prima del passaggio del guscio sferico. L'energia

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meccanica totale è dunque variata e denotando con α la frazione $\Delta M/M$ di massa espulsa si ha

$$E' = E_0 + \frac{\alpha \mathcal{G} M m}{r_0} > E_0$$

Il pianeta guadagna dunque energia a spesa dell'energia meccanica della stella. Essendo l'orbita iniziale circolare si ha poi

$$E_0 = \frac{\Phi_0}{2} = \mathcal{K}_0$$

da cui

$$E' = \frac{\Phi_0}{2} + \frac{\alpha \mathcal{G} M m}{r_0} = \frac{\mathcal{G} M m}{r_0}\left[\alpha - \frac{1}{2}\right]$$

Da questa relazione segue subito che

$$E' > 0 \qquad \Leftrightarrow \qquad \alpha > \frac{1}{2}$$

che è la condizione per cui il pianeta viene espulso dall'orbita. Supponendo $0 < 2\alpha < 1$ in modo che il pianeta resti legato il semiasse maggiore a della nuova orbita ellittica si determina usando la relazione (29.06.11) tra energia meccanica e semiasse per un'orbita Kepleriana

$$E' = -\frac{\mathcal{G} M m\,(1-\alpha)}{2a} = \frac{\mathcal{G} M m}{r_0}\left[\alpha - \frac{1}{2}\right] \qquad \text{per } 0 < 2\alpha < 1$$

Si osservi che l'orbita diventa ellittica perché l'energia del pianeta aumenta ma il momento angolare resta costante. Si ricava allora

$$a = \left[\frac{\alpha - 1}{2\alpha - 1}\right] r_0 > r_0 \qquad \text{per } 0 < 2\alpha < 1$$

e il semiasse maggiore dell'orbita aumenta sempre essendo $E' > E_0$.
In coordinate polari l'equazione del moto radiale del pianeta si scrive

$$\mu\left(\ddot{r} - r\dot{\varphi}^2\right) = -\frac{\mathcal{G} M[t] m}{r^2}$$

in cui la massa della stella è variabile in funzione del tempo. Introducendo la conservazione del momento angolare, giustificata dal fatto che si suppone sempre simmetria sferica della distribuzione di massa della stella e del guscio espulso e quindi il pianeta soggetto ad un campo di forza centrale, si ha

$$\mu\left(\ddot{r} - \frac{l^2}{\mu^2 r^3}\right) = -\frac{\mathcal{G} M[t] m}{r^2}$$

L'energia allora non è conservata essendo la massa della stella dipendente dal tempo e variabile con discontinuità di prima specie all'istante in cui il guscio oltrepassa il pianeta. Si ha allora che $r[t]$ è una funzione continua ma $\ddot{r}$ è discontinua quindi il pianeta acquista un'accelerazione radiale che muta l'orbita da circolare ($\ddot{r} = 0$) in ellittica. Dalla conservazione del momento angolare del pianeta, $l = mr^2\dot{\varphi} = mrv_\varphi$, e dalla continuità di $\dot{r}$ si ha pure la continuità delle componenti tangenziale e radiale della velocità.

29.06.21
Gravitational Quadrupole Moment of the Sun

The gravitational **Quadrupole** moment of the Sun affects the orbit of planets, producing precession. Its exact value is important, for instance, in order to compare the measured precession with the expectation from the theory of general relativity.

29.06.22
Orbits Around a Non-Spherical Cylindrically Symmetric Planet

Consider a point-mass satellite orbiting around a non-spherical cylindrically symmetric planet. Whenever the force is not a central force, the direction of the force on the satellite is not always passing for a fixed point, as for a central force.

On the equatorial plane, the overall force is central but not purely $1/r^2$, in general. Based on the gravitational **Multipole** development, the monopole field, $1/r^2$, can be modified by other power laws.

Outside the equatorial plane, the overall force is not central and not purely $1/r^2$, in general. In order to conserve the total angular momentum of the planet-satellite, assumed to be isolated, the point mass must exert a torque on the planet to compensate the torque the planet exerts on the satellite.

29.06.23
Potential Energy for a Generic Power-Law Force and Long-Range Interactions

This § is referenced at pages:
[2570]

Consider a large homogeneous distribution of identical point particles, whose properties are invariant under rotation, interacting via a potential energy given, at large distances, by:

$$\Phi[r] = -\frac{k}{r^\alpha} \qquad \alpha > 0 \qquad \text{for large } r \quad .$$

Consider a generic particle and calculate its potential energy due to its interaction with all the other particles in a spherical region of radius R. We are only interested in the long-range behavior of the interaction and therefore we set a cut-off at a small distance, $r = e$, to avoid problems with the convergence of the integral at $r = 0$.

Show that for $0 < \alpha < 3$ the contribution to the potential energy of the particle is increasing while R. Therefore, in this case, the contribution of distant particles to the potential energy is dominating the contribution of nearby particles. This interaction is called a long-range interaction.

DRAFT - In Progress

29.07 Exercises Problems and Physical Applications

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...||*A lot of problems on ...*

29-001 Ellipse With Vector Methods

©|M.R.Spiegel, *Theory And Problems Of Theoretical Mechanics*, 1967, McGraw-Hill, ...Ed., ...|5.36||

29-002 Expression of the Areolar Velocity

Demonstrate equation (29.02.03).

WORKED SOLUTION

In fact the area covered by the position vector in the (small) time interval Δt is given by:

$$\Delta\mathbf{S}[t,\Delta t] \simeq \frac{1}{2}|\,\mathbf{r}[t]\wedge\mathbf{r}[t+\Delta t]| = \frac{1}{2}|\,\mathbf{r}[t]\wedge(\mathbf{r}[t]+\Delta\mathbf{r}[t,\Delta t])| = \frac{1}{2}|\,\mathbf{r}[t]\wedge\Delta\mathbf{r}[t,\Delta t]| \quad ,$$

$$\Longrightarrow \lim_{\Delta t\longrightarrow 0}\frac{\Delta A}{\Delta t}[t,\Delta t] = \frac{1}{2}|\,\mathbf{x}\wedge\mathbf{v}| \quad .$$

Using the expression of $\mathbf{r}$ and $\mathbf{v}$ in polar coordinates the result that

$$\boxed{\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\varphi} \equiv \frac{c}{2}}$$

can be obtained.

29-003 Zero Angular Momentum Orbit in a Central Force Field

This § is referenced at pages:
[1567, 1592]

Show analytically that a zero angular momentum orbit in a Central Force Field is a rectilinear motion.

WORKED SOLUTION

A first solution can be obtained in the following way.
First of all:

$$\boldsymbol{\ell} \equiv \mathbf{r}\wedge\dot{\mathbf{r}} = 0 \longrightarrow \widehat{\mathbf{T}} \parallel \mathbf{r} \quad .$$

Moreover, due the central character of the force, equation (23.03.09) implies

$$\mathbf{r}\wedge\ddot{\mathbf{r}} = 0 \longrightarrow \mathbf{a} = \left(\frac{\mathrm{d}v}{\mathrm{d}t}\widehat{\mathbf{T}} + \frac{v^2}{\rho}\widehat{\mathbf{N}}\right) \parallel \mathbf{r} \longrightarrow \frac{v^2}{\rho}\mathbf{N}\wedge\mathbf{r} = 0 \quad .$$

As $\mathbf{N}\wedge\mathbf{r} \neq 0$, because $\mathbf{N}\wedge\mathbf{T} \neq 0$ by construction and $\widehat{\mathbf{T}} \parallel \mathbf{r}$, the latter equations imply $\rho^{-1} = 0$, that is the trajectory is a straight line.
Moreover, the straight line passes by the pole, because $\mathbf{r} \parallel \mathbf{v} \parallel \mathbf{a}$. In fact $\boldsymbol{\ell} = 0$ implies $\dot{\varphi} = 0$. Therefore $\varphi = \varphi_0$ is a solution and, because of the unicity of the solution, it is the solution.
A second, more formal, method to obtain the same solution is the following.

Consider any vector $\mathbf{u}$:

$$u^2 = \mathbf{u}\cdot\mathbf{u} \implies u\dot{u} = \mathbf{u}\cdot\dot{\mathbf{u}} \quad ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{u}}{u}\right) = \frac{\dot{\mathbf{u}}u - \dot{u}\mathbf{u}}{u^2} = \frac{(\mathbf{u}\cdot\mathbf{u})\,\dot{\mathbf{u}} - (\mathbf{u}\cdot\dot{\mathbf{u}})\,\mathbf{u}}{u^3} = \frac{(\mathbf{u}\wedge\dot{\mathbf{u}})\wedge\mathbf{u}}{u^3} \quad .$$

Read also § 29-005.
Applying the relation to $\mathbf{u} = \mathbf{r}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = \frac{(\mathbf{r}\wedge\dot{\mathbf{r}})\wedge\mathbf{r}}{r^3} = \frac{\mathbf{c}\wedge\mathbf{r}}{r^3} \quad .$$

If $\mathbf{c} = \mathbf{0}$ then the vector $\mathbf{r}/r$ is constant and the motion is thus a straight line through the origin.

29-004 Non Central Force Fields

1. Find an example of a conservative non central force field.
2. Find an example of a conservative non radial force field.
3. Find an example of a non conservative radial force field.

29-005 An Identity

This § is referenced at pages:
[1591]

Demonstrate that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{e}}_r = \frac{1}{r}(\widehat{\mathbf{e}}_r\wedge(\dot{\mathbf{r}}\wedge\widehat{\mathbf{e}}_r)) \quad . \tag{29.07.01}$$

Read also § 29-003.

29-006 Inverse Cube Force Law

Determine the equation of the orbit in a inverse cube force law.
Use Binet equation, which is easily solved in this case (in fact, also the centrifugal-barrier term is an inverse cube force law).

29-007 Energia Potenziale Gravitazionale Di Una Sfera Omogenea

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Data una distribuzione sferica e omogenea di massa, avente massa totale M e raggio R, si calcoli l'energia potenziale gravitazionale.

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WORKED SOLUTION

$$M[r] = \int_0^r \rho[r]\, \mathrm{d}V = \rho \int_0^r \mathrm{d}V = \frac{4\pi}{3}\rho r^3 \qquad M = M[R] \quad ,$$

$$\mathrm{d}m = 4\pi\rho r^2\, \mathrm{d}r \quad ,$$

$$\boxed{\mathrm{d}\Phi = -\frac{\mathcal{G}M[r]}{r}\, \mathrm{d}m} \quad ,$$

$$\mathrm{d}\Phi = -\frac{4\pi\mathcal{G}}{3}\rho r^2\, \mathrm{d}m = -\frac{(4\pi)^2\,\mathcal{G}}{3}\rho^2 r^4\, \mathrm{d}r \quad ,$$

$$\Phi = -\frac{(4\pi)^2\,\mathcal{G}}{3}\rho^2 \int_{r=0}^{R} r^4\, \mathrm{d}r = -\frac{3}{5}\frac{\mathcal{G}M^2}{R} \quad .$$

Result:

$$\Phi = -\frac{3}{5}\frac{\mathcal{G}M^2}{R} \quad .$$

29-008 Effetto Sull'orbita Terrestre Dell'accumulo Di Polvere Interplanetaria

La Terra, nella sua orbita attorno al Sole, raccoglie $m = 1.0{\cdot}10^4$ ton di polvere interplanetaria all'anno. Determinare l'effetto del cambio di massa della Terra sull'orbita. Si considerino due modelli del fenomeno:

- la polvere interplanetaria viene catturata dalla Terra con un processo a distribuzione totalmente isotropa;
- la polvere interplanetaria è stazionaria rispetto al Sole e viene catturata dalla Terra che attraversa questa distribuzione stazionaria.

WORKED SOLUTION

Momento angolare si conserva di certo, sistema isolato. Ma è una assunzione: se la polvere fa drag... non si conserva... e diventa il problema del satellite.
Energia meccanica potrebbe variare ... conversione in energia interna con la caduta della polvere...

29-009 Moto in Campo Di Forza Centrale

Un punto materiale di massa m in un campo di forza centrale è soggetto ad un potenziale

$$V[r] = -\frac{\gamma}{r^\alpha} \qquad \text{con } \alpha = 1/2 \text{ e } \gamma > 0 \quad .$$

1. Determinare il periodo orbitale delle orbite circolari del punto materiale.

   WORKED SOLUTION

   $$\omega^2 = \frac{\alpha\gamma}{m}\frac{1}{r^{\alpha+2}} \longrightarrow \frac{\gamma}{2m}\frac{1}{r^{5/2}} \qquad T = \frac{\pi\sqrt{8m}}{\sqrt{\gamma}} r^{5/4} \quad .$$

2. Da quali parametri dipende il periodo orbitale delle orbite circolari?
3. Disegnare e discutere il grafico del potenziale efficace.
4. Per ogni fissato valore del momento angolare orbitale determinare l'energia meccanica minima possibile.

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WORKED SOLUTION

$$E_{\text{MIN}} = \frac{\gamma\,(\alpha-2)}{2}\left(\frac{m\alpha\gamma}{l^2}\right)^{\frac{\alpha}{2-\alpha}} \longrightarrow -\frac{3\gamma}{4\,(2)^{1/3}}\left(\frac{m\alpha\gamma}{l^2}\right)^{1/3} \quad .$$

5. Sono stabili le orbite circolari in tale potenziale?

29-010 Total Energy for an Elliptic Orbit in an Inverse-Square Field

Demonstrate the relation (29.04.21) for the total energy for an elliptic orbit, starting from the equation (29.04.03), applied to the perihelion and aphelion, and using equation (29.04.01.03).

WORKED SOLUTION

As the orbit is elliptic one necessarily has: $\gamma > 0$. The perihelion and aphelion are expressed by (when $B > 0$, that is perihelion at $\varphi = 0$):

$$\frac{1}{r_P} = +B + \frac{m\gamma}{\ell^2} \quad ,$$
$$\frac{1}{r_A} = -B + \frac{m\gamma}{\ell^2} \quad .$$

Therefore:

$$2a = r_P + r_A = \frac{\frac{m\gamma}{\ell^2}}{\frac{m^2\gamma^2}{\ell^4} - B^2} \Longrightarrow E = -\frac{\gamma}{2a} < 0 \quad .$$

29-011 A Circular Orbit

A particle of mass m is moving on a circular orbit of radius $R = 2b$ passing through the center of force of a power-law attractive central force with unknown exponent:

$$\Phi[r] = -\frac{\gamma}{r^\alpha} \qquad \gamma > 0 \quad .$$

- Determine the exponent for which such an orbit exists.
- Find the energy and angular momentum of this orbit.
- Determine the period of the orbit as a function of the radius of the orbit, γ and m.

29-012 A Circular Orbit in the Yukawa Potential

A particle of mass m is moving in the Yukawa potential:

$$\Phi[r] = -\frac{\gamma}{r}\exp\left[r/r_0\right] \qquad \gamma > 0 \qquad r_0 > 0 \quad .$$

- Show that circular orbits exist only if the angular momentum does not exceed a certain value, ℓ_{M}, and determine such a value.
- Show that for any value $\ell < \ell_{\text{M}}$, there exist two circular orbits, at different radii, r_1 and r_2, and that one of them is a stable orbit and the other one is an unstable one. Determine r_1 and r_2.
- Plot $r_1[\ell]$ and $r_2[\ell]$.

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WORKED SOLUTION

$$\frac{\mathrm{d}\Phi_{\mathrm{E}}[r]}{\mathrm{d}r} \propto m\gamma r \exp[r/r_0]\,(r+r_0) - r_0\ell^2$$

The first derivative might have either zero or two zeros. In the first case the effective potential energy is always decreasing while in the second case it has one relative maximum and one relative minimum.

The positive zero of the first derivative (flex point) is at:

$$r = r_0\frac{1+\sqrt{5}}{2} \quad ,$$

from which the value of ℓ_{M} follows.

$$\frac{l^2}{2\,m\,r^2} - \frac{\gamma}{\exp\frac{r}{\rho}r} \quad ,$$

$$\left(-\left(\frac{l^2}{m\,r^3}\right) + \frac{\gamma}{\exp\frac{r}{\rho}r^2} + \frac{\gamma}{\exp\frac{r}{\rho}r\,\rho}\right)\left(\frac{l^2}{2\,m\,r^2} - \frac{\gamma}{\exp\frac{r}{\rho}r}\right) \quad ,$$

$$\left(\frac{3\,l^2}{m\,r^4} - \frac{2\,\gamma}{\exp\frac{r}{\rho}r^3} - \frac{\gamma}{\exp\frac{r}{\rho}r\,\rho^2} - \frac{2\,\gamma}{\exp\frac{r}{\rho}r^2\,\rho}\right)\left(\frac{l^2}{2\,m\,r^2} - \frac{\gamma}{\exp\frac{r}{\rho}r}\right) + \quad ,$$

$$\left(-\left(\frac{l^2}{m\,r^3}\right) + \frac{\gamma}{\exp\frac{r}{\rho}r^2} + \frac{\gamma}{\exp\frac{r}{\rho}r\,\rho}\right)^2\left(\frac{l^2}{2\,m\,r^2} - \frac{\gamma}{\exp\frac{r}{\rho}r}\right) \quad .$$

29-013 Orbita in Campo Centrale

Si supponga di sapere che il moto di una particella in un campo di forza centrale ha un'orbita data, in coordinate polari, da:

$$r[\varphi] = r_0 \exp -\alpha\varphi \quad ,$$

con r_0 ed α costanti positive e $-\infty < \varphi < +\infty$.

Note that the symmetry with respect to the apsidal vectors cannot be satisfied because apsidal vectors do not exist for this orbit.

1. Determinare la legge della forza a cui è soggetta la particella e descrivere qualitativamente il moto.
2. Ricavare $r[t]$ e $\varphi[t]$.

29-014 Another Orbita in Campo Centrale

A point mass follows a circular orbit passing by the force center. Determine the force law.

WORKED SOLUTION

Inverse fifth power.

RUGPHYS

29-015 Still Another Orbita in Campo Centrale

A point mass follows a lemniscate orbit: $r^2 = a^2 \cos[2\varphi]$ Determine the force law.

WORKED SOLUTION

Inverse seventh power.

29-016 Still Another Another Orbita in Campo Centrale

Prove that the orbits $r = r_0 \exp -\varphi$ and $r = r_0/\varphi$ are both possible for the case of an inverse cube field of force. Explain physically how this is possible.

29-017 Sonda Spaziale in Interazione Con Giove

Il pianeta Giove ha un periodo di rivoluzione attorno al Sole $T = 11.87$ anni, raggio $r_G = 0.7{\cdot}10^5$ km e massa $M_G = 1.9{\cdot}10^{27}$ kg. A fini del presente problema l'orbita si può supporre circolare (eccentricità pari a 0.05) e la velocità orbitale, $\mathbf{V}$, costante. Una sonda spaziale si trova inizialmente a distanza da Giove pari a $r \simeq 50 r_G$ e si avvicina a Giove inseguendolo con velocità, $\mathbf{v} = \alpha \mathbf{V}$ ($\alpha = 2$), nel sistema di riferimento solidale con il Sole, parallela a quella di Giove, concorde in verso. Il parametro di impatto della sonda con Giove è b. La massa del Sole vale $M_S = 2.0{\cdot}10^{30}$ kg e la distanza di Giove dal Sole vale: 778330000 km.

1. Dimostrare che in tali condizioni l'effetto dell'attrazione gravitazionale del Sole sulla sonda può essere trascurato rispetto all'effetto dell'attrazione gravitazionale di Giove.

   WORKED SOLUTION

   $$\frac{F_S}{F_G} \approx \frac{M_S}{M_G}\left(\frac{r}{a}\right)^2 = 0.02 \quad .$$

2. Quale deve essere il parametro di impatto, b, per ottenere un angolo di deviazione della sonda pari a $\Delta\varphi = \pi/2$ nel sistema di riferimento solidale con il Sole?
3. Quanto vale la variazione di energia cinetica nel sistema di riferimento solidale con il Sole e nel sistema di riferimento solidale con Giove?
4. Quale è la velocità, nel sistema di riferimento solidale con Giove, della sonda al perielio?
5. Tenendo conto del raggio finito di Giove quale è, idealmente, la massima deviazione della traiettoria che si può ottenere, con la sonda, cioè, che viene rispedita quasi all'indietro?

29-018 Escape Speed From the Earth and From Solar System

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Determine the minimum velocity necessary to a spaceship starting from the Earth to escape: the Earth; the Solar System.

DRAFT in progress

29-019 Quadrupole Moments: Spherical Versus Cartesian Coordinates Coordinate System

This § is referenced at pages:
[1670]

Find the relation between Q and J_2.

29-020 Quadrupole Moment and Laplace equation

Show directly that the expression (29.05.05) for the **Quadrupole** potential requires a traceless $Q^{\cdot\cdot}_{jk}$ in order to satisfy the **Laplace equation** for the gravitational potential.

29-021 Quadrupole Moment for Spherically Symmetrical Distributions

Show that for any spherically symmetric distribution the **Quadrupole** moment is identically zero.

29-022 Quadrupole Moment for Methane Molecule

Show that the electric **Quadrupole** moment of a methane molecule is identically zero.

29-023 Constancy of the Laplace-Runge-Lenz Vector for the Kepler Problem

Show, by direct computation, that the Laplace-Runge-Lenz vector is a constant of motion for the Kepler problem.

DRAFT-IN-PROGRESS

CHAPTER 30

Introduction to Many-Particles Systems

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Collision (also know as scattering processes) is the term used to refer to very fast interactions between two or more moving bodies that form, usually, an isolated system. During collisions very large Force|Torque|Interaction take place, usually of short range. A lot of information can be obtained about the results of collisions even without a precise knowledge of the interaction by using general principles, such as the exchange of energy, linear momentum, angular momentum.

Only collisions among point masses will be considered.

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30.01 Introduction

At the microscopic level all interactions are conservative: therefore assuming conservation of the total energy is not a real limitation.

It is assumed that the interaction potential energy tends to zero at infinity.

The only real limitation of the classical treatment is the non-relativistic assumption, that is low velocities. This excludes from the treatment mass-less particles, (for instance photons), that are always relativistic particles. However, in **Galilei-Newton Mechanics** , massless particles cannot exist; if they existed, thanks to the **Cardinal Equations** , they are not affected by external forces nor torques.

The collision processes are supposed to take place starting from some time ($t \to -\infty$) when the particles are very distant between each other, and ending at a time ($t \to +\infty$) when they are again very distant between each other, after the interaction. In other words, one studies the initial and final state and forgets about any intermediate state.

30.01.01 Conservation Laws

In **Galilei-Newton Mechanics** , **Galilei-Invarian*** and linear momentum conservation implies mass conservation, read § 24.10.04.

In many cases the kinematic conservation laws, that is the laws of conservation of linear momentum, angular momentum and energy, alone, can be used to obtain useful results concerning the properties of various processes, including possibly showing that some specific process is forbidden, without information about the details of the interaction. These properties are independent of the particular type of interaction between the particles involved.

Also, the laws of conservation of electric charge and other charges, such as baryonic and leptonic numbers, whenever applicable, can be used to constrain the collision processes.

30.01.02 Elastic and Non Elastic Collisions

An elastic collision between two generic isolated systems is a collision such that the total kinetic energy remains constant. This definition is independent from Reference Frame. In fact, **Koënig theorem** for kinetic energy implies that the variation of the kinetic energy during the collision is expressed in terms of the velocities relative to the Center-Of-Mass of the system, as the kinetic energy of the motion of the Center-Of-Mass remains constant for the conservation of linear momentum. The kinetic energy relative to the Center-Of-Mass, being expressed in terms of relative velocities, is independent of the choice of Reference Frame.

As the collision processes are supposed to take place starting from some time ($t \to -\infty$) when particles are very distant and ending at a time ($t \to +\infty$) when they are very distant again, the potential energy of interaction is negligible both at the beginning and at the end of the process. Moreover, the potential energy from external fields is often supposed to be negligible.

Under these conditions the scattering process is elastic if and only if the identity (and masses) of the particles do not change, because this condition is necessary and sufficient to ensure the conservation of kinetic energy.

A inelastic collision between particles is characterized by the change in kinetic energy of the system, quantified by the so-called Q-value, Q:

$$\Delta\mathcal{K} = \mathcal{K}_{\mathrm{F}} - \mathcal{K}_{\mathrm{I}} \equiv -Q \qquad \begin{cases} \Delta\mathcal{K} > 0 \text{exo-energetic reaction} \\ \Delta\mathcal{K} < 0 \text{endo-energetic reaction} \end{cases} ,$$

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It possibly includes the change in number and | or mass of the particles, as classically the constraint on masses is that the sum of the masses before the collision is equal to the sum of the masses after the collision, read § 24.10.04.

Types of inelastic collisions:

- $Q = 0$ elastic collision;
- $Q > 0$ inelastic (endothermal) collision: kinetic energy is converted into internal energy of the collision partners (excitation of the collision partners);
- $Q < 0$ inelastic (exothermal) collision: internal energy is converted into kinetic translational energy (de-excitation of the collision partners).

The Q-value is Galilei invariant (read § 30-001).

30.01.03
Impact Parameter

This § is referenced at pages:
[Never referenced.]

The impact parameter, for a generic collision with two point particles in the initial state, is defined as follows.

Assume that, at the beginning, the particles are far-away and their interaction is negligible, and are far-away from any other source of interaction, and move with constant velocity in some Inertial Reference Frame .

The impact parameter is defined as the geometrical distance between the two straight lines giving the trajectories of the particles before the interaction. This is, by definition, the minimum distance of the two straight lines. It is calculated as discussed in section § 15-015.

In **Galilei-Newton Mechanics** , as particles must have positive mass, one can always choose the Rest Frame of one particle. In this Reference Frame, usually defined as the Reference Frame of the **target** particle, the (scalar) impact parameter is defined as:

$$b \equiv \left| \frac{\mathbf{r} \wedge \mathbf{v}_\infty}{v_\infty} \right| \quad , \tag{30.01.01}$$

→ 1563

in terms of the velocity of the projectile particle at large distance, $\mathbf{v}_\infty$: it is the distance between the target particle and the line of flight, at large distances, of the projectile particle.

Note that it is not in general possible to find a Inertial Reference Frame such that the impact parameter is zero.

One distinguishes two-body collisions by their kinematics.

- Central/straight-line collisions: the impact parameter is zero; the colliding point-masses move along their connecting lines both before and after the collision; one coordinate is sufficient to describe the collision.
- Non-central collisions: the impact parameter is greater than zero; in the Rest Frame of any one of the two particles the position of the particles and the linear trajectory of the other define a plane, the plane of collision; for a given incident energy the impact parameter determines the orbital angular momentum of the relative motion of the two point-masses.

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30.02 Fixed-Target - Relation Between Laboratory and Center-Of-Mass System

Fixed target of mass M; projectile of mass m.

The kinematics of the elastic non-relativistic collision between two pint-particles is better discussed in the Center-Of-Mass Reference Frame.

La relazione tra angolo di scattering nel Reference Frame del laboratorio e nel Reference Frame del centro di massa è data da

$$\tan[\theta] = \frac{\sin[\theta^\star]}{\cos[\theta^\star] + (m/M)} \qquad 0 \leq \theta^\star \leq \pi \qquad 0 \leq \theta \leq \pi \quad , \tag{30.02.01}$$

completamente definita dal rapporto delle masse.

However, to make easier the choice of the quadrant of the angle in the Laboratory frame it is better to separate the sinus and cosinus:

$$\sin[\theta] = \sin[\theta^\star] \qquad \cos[\theta] = \cos[\theta^\star] + (m/M) \quad , \tag{30.02.02}$$

The angle in the Lab versus angle in the Center-Of-Mass for different values of the projectile to target masses ratio, ρ, is shown in figure 30.11.

Nel Reference Frame del Center-Of-Mass l'angolo non è vincolato (dai principi di conservazione), ma potrebbe esserlo dalla dinamica.

Se $m \gg M$ allora $|\tan[\theta]| \ll 1$: cioè se la massa del proiettile è grande rispetto a quella del bersaglio il proiettile viene poco deviato.

Se $m \ll M$ allora $\tan[\theta] \simeq \tan[\theta^\star]$: se la massa del proiettile è piccola rispetto a quella del bersaglio l'angolo di deflessione nel Reference Frame del Laboratorio diventa uguale a quello nel Reference Frame del centro di massa.

Esperimento di Rutherford: l'osservazione di eventi con scattering a grande angolo implica che ci siano dei bersagli di massa molto maggiore di quella dell'elettrone.

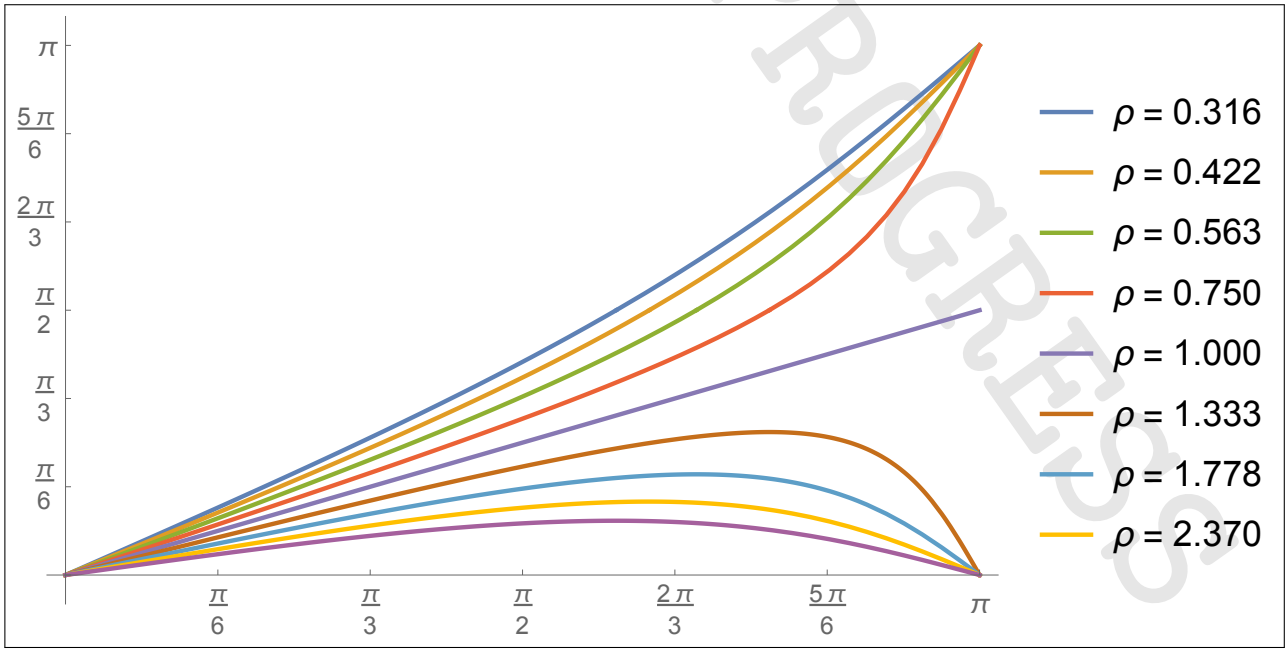

Figure 30.11: fixed target: angle in the Lab versus angle in the Center-Of-Mass for different values of the projectile to target masses ratio, ρ.

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30.03 Two-Body Elastic Collisions

This § is referenced at pages:
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Read § ?? for the relativistic treatment.

30.03.01 Classical Kinematics of Two-Particles Collisions

Consider a two-to-two particles elastic collision, between the two particles, A_1 and A_2:

$$A_1[m_1, \mathbf{p}_1] + A_2[m_2, \mathbf{p}_2] \longrightarrow A_1[m_1, \mathbf{p}'_1] + A_2[m_2, \mathbf{p}'_2] \quad ,$$

in the Laboratory frame.

As the collision is elastic, the identity and mass of each particle is unchanged by the collision.

Let θ_k be the scattering angle of the two particles, $(k = 1, 2)$, in the laboratory frame:

$$\cos[\theta_k] = \frac{\mathbf{p}'_k \cdot \mathbf{p}_k}{|\mathbf{p}'_k||\mathbf{p}_k|} \quad .$$

The relative position of the two particles is, by definition:

$$\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1 \quad .$$

The relative velocity of the two particles is, by definition:

$$\mathbf{v} \equiv \mathbf{v}_2 - \mathbf{v}_1 \quad .$$

30.03.01.01 SdRCM

The [glo:CMRF]Center-of-Momenta Reference Frame is defined as the non inertial (in general) Reference Frame frame whose origin is fixed at the Center-Of-Mass of the system and which is translating with respect to the inertial Laboratory frame.

30.03.01.02 CM and Positions

The position of the [glo:CMRF]Center-of-Momenta Reference Frame, as seen by the Laboratory frame, is:

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} = \mu\frac{\mathbf{r}_1}{m_2} + \mu\frac{\mathbf{r}_2}{m_1} \qquad M \equiv m_1 + m_2 \qquad \mu \equiv \frac{m_1 m_2}{M} \quad .$$

The relation between Center-Of-Mass ($\mathbf{r}^\star_k$) and Laboratory frame ($\mathbf{r}_k$) positions is:

$$\mathbf{r}_k \equiv \mathbf{r}^\star_k + \mathbf{R} \Longrightarrow \mathbf{r}^\star_k = \mathbf{r}_k - \mathbf{R} = m_j\left(\frac{\mathbf{r}_k - \mathbf{r}_j}{M}\right) \begin{cases} \text{if } k = 1 \text{ then } j = 2\text{, no sum on the indexes} & , \quad , \\ \text{if } k = 2 \text{ then } j = 1\text{, no sum on the indexes} & , \end{cases} \quad .$$

Equivalentely:

$$\mathbf{r}_k \equiv \mathbf{r}^\star_k + \mathbf{R} \Longrightarrow m_k\mathbf{r}^\star_k = m_k\mathbf{r}_k - m_k\mathbf{R} = \mu\left((\mathbf{r}_k - \mathbf{r}_j)\right) \begin{cases} \text{if } k = 1 \text{ then } j = 2\text{, no sum on the indexes} & , \quad , \\ \text{if } k = 2 \text{ then } j = 1\text{, no sum on the indexes} & , \end{cases} \quad .$$

The following relation apply, by definition of Center-Of-Mass:

$$m_1\mathbf{r}_1{}^\star + m_2\mathbf{r}_2{}^\star = 0 \quad .$$

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30.03.01.03 CM and Velocities/momenta

The velocity of the [glo:CMRF]Center-of-Momenta Reference Frame, as seen by the Laboratory frame, is:

$$\mathbf{V} \equiv \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1+m_2} \equiv \frac{\mathbf{P}}{M} \quad .$$

The relation between Center-Of-Mass ($\mathbf{v}_k^\star$) and Laboratory frame ($\mathbf{v}_k$) velocities is:

$$\mathbf{v}_k \equiv \mathbf{v}_k^\star + \mathbf{V} \implies \mathbf{v}_k^\star = \mathbf{v}_k - \mathbf{V} = m_j\left(\frac{\mathbf{v}_k - \mathbf{v}_j}{M}\right) \begin{cases} \text{if } k=1 \text{ then } j=2\text{, no sum on the indexes} & , \quad , \\ \text{if } k=2 \text{ then } j=1\text{, no sum on the indexes} & , \end{cases} .$$

Equivalently:

$$\mathbf{p}_k \equiv m_k\mathbf{v}_k^\star + m_k\mathbf{V} \implies \mathbf{p}_k^\star \equiv m_k\mathbf{v}_k - m_k\mathbf{V} = \mu\left(\mathbf{v}_k - \mathbf{v}_j\right) \begin{cases} \text{if } k=1 \text{ then } j=2\text{, no sum on the indexes} & , \quad , \\ \text{if } k=2 \text{ then } j=1\text{, no sum on the indexes} & , \end{cases} .$$

30.03.01.04 Expression of the Dynamical Quantities With Respect to the Center-Of-Mass

This § is referenced at pages:
[1605]

The following relation apply, by definition of Center-Of-Mass:

$$\mathbf{p}_1^\star + \mathbf{p}_2^\star = 0 \quad .$$

The linear momentum relative to the Center-Of-Mass, expressed in terms of the reduced mass of the pair of particles as follows:

$$\mathbf{p}_k^\star \equiv m_k\mathbf{v}_k^\star = \frac{m_1m_2}{m_1+m_2}\left(\mathbf{v}_k - \mathbf{v}_j\right) \equiv \mu\left(\mathbf{v}_k - \mathbf{v}_j\right) \begin{cases} \text{if } k=1 \text{ then } j=2\text{, no sum on the indexes} & , \quad , \\ \text{if } k=2 \text{ then } j=1 & , \end{cases} .$$

The linear momentum relative to the Center-Of-Mass, can be also expressed in terms of the relative velocity of the two particles, which is invariant under transformation between different inertial frames, as:

$$\begin{aligned} \mathbf{v} &\equiv \mathbf{v}_2 - \mathbf{v}_1 \qquad \text{relative velocity of particle 2 with respect to particle 1} \quad , \\ \mathbf{p}_1^\star &= -\mu\mathbf{v} \quad , \\ \mathbf{p}_2^\star &= +\mu\mathbf{v} \quad , \\ \mathbf{p}^\star &\equiv +\mu\mathbf{v} = +\mu\left(\mathbf{v}_2 - \mathbf{v}_1\right) \quad . \end{aligned}$$

The kinetic energy can be decomposed by means of the **Koënig theorem** : equation (24.11.05).

$$\mathcal{K} = \mathcal{K}_{\mathrm{CM}} + \mathcal{K}^\star = \frac{1}{2}MV^2 + \frac{1}{2}\left(m_1\mathbf{v}_1^{\star 2} + m_2\mathbf{v}_2^{\star 2}\right) = \frac{1}{2}MV^2 + \frac{1}{2}\mu v^2 = \frac{P^2}{2M} + \frac{p^{\star 2}}{2\mu} \quad .$$

The orbital angular momentum can be decomposed by means of the **Koënig theorem** : equation (24.11.02).

In the case of a two-particle system the angular relative to the Center-Of-Mass can be expressed (equation (24.11.02)), as:

$$\mathbf{L}^\star \equiv \mathbf{r}_1^\star \wedge \mathbf{p}_1^\star + \mathbf{r}_2^\star \wedge \mathbf{p}_2^\star = \mathbf{r}_1 \wedge \mathbf{p}_1^\star + \mathbf{r}_2 \wedge \mathbf{p}_2^\star = \mathbf{r} \wedge \mathbf{p}^\star = \mu\,\mathbf{r} \wedge \mathbf{v} \quad . \tag{30.03.01}$$

→ 1605

Conservazione dell'energia significa conservazione dell'energia cinetica in un processo di scattering tale che l'energia potenziale è nulla prima e dopo la interazione, e, dalla espressione dell'energia cinetica scritta sopra, questo significa conservazione dell'energia cinetica relativa e cioè del modulo dell'impulso nel Reference Frame del Center-Of-Mass (la quantità di moto si conserva). Therefore elastic scattering means:

$$\left|\mathbf{p}_k^{\star\prime}\right| = \left|\mathbf{p}_k^\star\right| \quad .$$

La massima energia che può essere persa dal sistema è data dal termine di energia cinetica relativa: $p^{\star 2}/(2\mu)$. Read § 30.04.

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30.03.01.05 Urto Elastico Contro Bersaglio Fermo: Sistema Del Laboratorio

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The kinematics of the elastic non-relativistic collision between a particle and a fixed target will be discussed in the laboratory Reference Frame. It is not a great kinematic limitation as for any collision one can stay in the Reference Frame of one of the two particles which is always possible because, in classical physics, all particles have a non-zero mass. The requirement of elastic collision is a limitation for macroscopic bodies but it is not a limitation at the microscopic level, where energy is always conserved.

Note that a different, and simplified, notation is used with respect to the general notation of section § 30.03.01.04. In that section the two particles are considered at the same level and therefore the notation $\{...\}_1$ and $\{...\}_2$ is used. On the other hand in this section there is a clear asymmetry between the two particles, as one of them is initially at rest: therefore the quantities related to the particle at rest will be denoted with upper-case letters while quantities related to the particle in motion (the projectile) will be denoted with lower-case letters.

The choice of a fixed target is not a real limitation, because one can always use as a Reference Frame the Reference Frame where the target is at rest.

Particella di massa m, velocità $\mathbf{v}$ e impulso $\mathbf{p}$ che incide su un bersaglio di massa M e impulso $\mathbf{P} = 0$ (bersaglio fermo). Gli impulsi e velocità dopo l'urto sono rispettivamente $\mathbf{p}'$, $\mathbf{P}'$ e $\mathbf{v}'$, $\mathbf{V}'$.

L'angolo di deflessione della prima particella è θ, l'angolo con cui parte la seconda particella, rispetto alla direzione originale dell'impulso $\mathbf{p}$ è α. That is:

$$\cos[\theta] \equiv \frac{\mathbf{p}'\cdot\mathbf{p}}{|\mathbf{p}'||\mathbf{p}|} \qquad \cos[\alpha] \equiv \frac{\mathbf{P}'\cdot\mathbf{p}}{|\mathbf{P}'||\mathbf{p}|} \quad .$$

Note that the impact parameter need not be zero.

If the orbital angular momentum in conserved (as it is the case, for instance, for isolated systems made of particles without intrinsic spin angular momentum), the motion is a plane motion, both before and after the collision, thanks to the expression (30.03.01) and the plane of motion does not change.

The plane defined by the linear momentum $\mathbf{p}$ of the incoming particle and the position of the fixed target defines the plane of motion before the interaction. The plane defined by the momenta, $\mathbf{p}'$ and $\mathbf{P}'$, of the outgoing particles defines the plane of motion after the interaction. The two planes need not to be coincident, in general, but they are coincident if the orbital angular momentum is conserved.

It will be assumed that the kinetic energy is just translational kinetic energy, as it is normally appropriate for elementary particles, while it is not for macroscopic bodies.

Applicando la conservazione dell'impulso e dell'energia cinetica si deduce

$$k \equiv M/m > 0 \quad ,$$

$$p = p'\cos[\theta] + P'\cos[\alpha] \qquad (30.03.02) \quad \rightarrow 1605$$

$$p'\sin[\theta] = P'\sin[\alpha] \qquad (30.03.03) \quad \rightarrow 1605$$

$$\frac{p^2}{2m} = \frac{p'^2}{2m} + \frac{P'^2}{2M} \qquad p^2 = p'^2 + \frac{P'^2}{k} \quad . \quad \rightarrow 1605$$

Ci sono tre equazioni e quattro incognite in quanto la dinamica non è del tutto determinata dai soli principi di conservazione.

Note that both the orientation of the final plane and the angles θ and α might be influenced by a possible intrinsic spin of the two particles and by the dynamics of the interaction. Moreover, the conservation of angular momentum, and|or other conservation laws, might add additional constraints. However, the three constraints given the linear momentum and energy conservation, equations (30.03.02), (30.03.03), (30.03.01.05), must always be obeyed. In other worlds: no matter possible additional constraints, imposed by either the properties of the particles or the dynamics of the interactions, the three constraints (30.03.02), (30.03.03), (30.03.01.05) are always valid. Once one of the four variables, (p', P', θ, α) is known the other three are determined. The determination of at least one of the four variables, (p', P', θ, α), as well as the determination of the final plane of motion, require additional information.

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Si possono ricavare p', P' e α in funzione di θ. Eliminando α dalle prime due equazioni si esplicita p' che poi si sostituisce nella terza equazione da cui si ricava p' in funzione di θ e p. Sostituendo si ricava poi P' in funzione di θ e p. Si trova

$$k \equiv M/m > 0 \qquad p > 0 \qquad p' \geq 0 \qquad P' \geq 0 \qquad p' \leq p \qquad P' \leq p\sqrt{k} \quad ,$$
$$0 \leq \theta \leq \pi \qquad 0 \leq \alpha \leq \pi \quad ,$$
$$\frac{p'}{p} = \frac{\cos\theta \pm \sqrt{\cos^2\theta + k^2 - 1}}{1+k} = \frac{\cos\theta \pm \sqrt{k^2 - \sin^2\theta}}{1+k} \quad ,$$
$$\frac{P'}{p} = \sqrt{k\left(1 - \frac{p'^2}{p^2}\right)} = \frac{\sqrt{2k}}{k+1}\sqrt{1 + k - \cos^2\theta \mp \cos\theta\sqrt{\cos^2\theta + k^2 - 1}} = \frac{\sqrt{2k}}{k+1}\sqrt{k + \sin^2\theta \mp \cos\theta\sqrt{k^2 - \sin^2\theta}} \quad ,$$
$$\cos[\alpha] = \frac{p - p'\cos[\theta]}{P'} \quad .$$

Note that the above equations are always valid non matter the final plane of motion is different from the initial plane of motion.

Le caratteristiche fondamentali, indipendenti dalle specificche proprietà delle particelle e dalla dinamica seguono, solo solggette alla limitazione di urto elastico non-relativistico su bersagio fermo con energia cinetica solo traslazionale:

- la soluzione mostra che P' è sempre automaticamente tale che $P' \leq p\sqrt{k}$; per P' occorre quindi solo accertarsi che risulti non negativo, ma per questo è sufficiente che sia soddisfatta la condizione $p' \leq p$; è quindi sufficiente verificare le condizioni su p' ($0 \leq p' \leq$) perchè quelle su P' siano automaticamente soddisfatte;
- per $k > 1$ ($M > m$) la sola soluzione accettabile è quella a segno positivo, perché l'altra fornirebbe un valore negativo di p', mentre si può verificare anche che la soluzione a segno positivo fornisce una soluzione tale che $0 \leq p' \leq p$;
- per $0 \leq k < 1$ ($M < m$) entrambe le soluzioni sono accettabili, e si hanno quindi due possibili valori di p', P' e α per ogni valore di θ;
- per $0 \leq k < 1$ ($M < m$) si ha necessariamente $|\sin[\theta]| \leq k < 1$: il proiettile, pesante, ha una deviazione massima possibile e risulta tanto più indirizzato in avanti quanto più la sua massa è grande rispetto a qualla del bersaglio;
- in entrambi i casi limite, $k \longrightarrow +\infty$ ($M \gg m$) e $k \longrightarrow 0$ ($M \ll m$), si ha $p' \longrightarrow p$ e $P' \longrightarrow 0$, cioè in entrambi i casi il bersaglio non acquista energia cinetica e il proiettile prosegue senza perdita di energia;
- nel caso limite $k \longrightarrow 0$ ($M \ll m$), si ha $|\sin[\theta]| \leq k \longrightarrow 0$, cioè il proiettile pesante prosegue diritto e senza perdita di energia;

The following relations are used to build a geometrical interpretation of the scattering in the Laboratory frame, the so-called scattering diagram.

Note that the relative velocity between the projectile and the target is:

$$\mathbf{v}_{\mathrm{R}} \equiv \mathbf{v} - \mathbf{V} = \mathbf{v} \qquad \text{because:} \qquad \mathbf{V} = \mathbf{0} \quad .$$
$$|\mathbf{p}^\star| = |\mathbf{P}^\star| = |\mathbf{p}^{\star\prime}| = |\mathbf{P}^{\star\prime}| = \mu|\mathbf{v}_{\mathrm{R}}|\mu|\mathbf{v}| \quad ,$$
$$\mathbf{p}' \equiv m\mathbf{v}' = m\left(\mathbf{v}^{\star\prime} + \mathbf{V}_{\mathrm{CM}}\right) = \mathbf{p}^{\star\prime} + m\frac{m\mathbf{v}}{m+M} = \mathbf{p}^{\star\prime} + \frac{m}{M}\mu\mathbf{v} \quad ,$$
$$\mathbf{P}' = -\mathbf{p}^{\star\prime} + \mu\mathbf{v} \quad .$$

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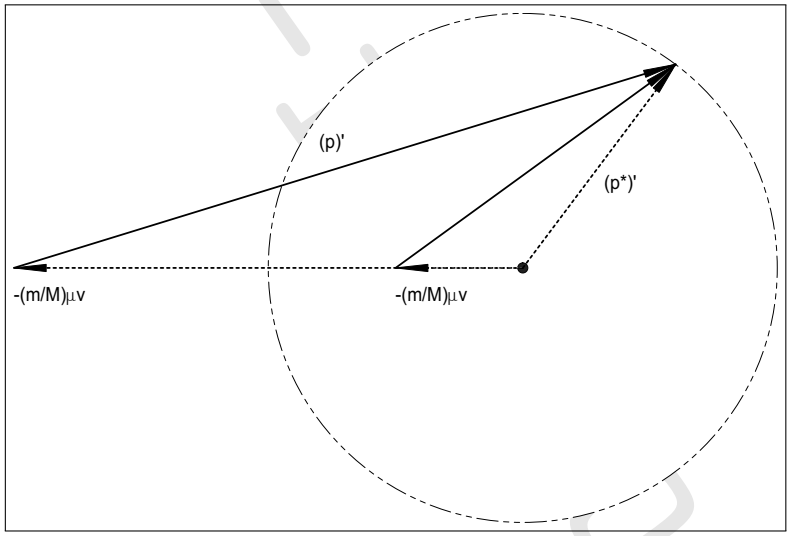

Figure 30.12: FIGURE

30.04 Two-Body Inelastic Collisions

This § is referenced at pages:
[1604]

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30.04.01 Types of Inelastic Two-Body Collisions

Types of inelastic two-body collisions include, as extreme cases:

- Capture Reaction: the two particles move as a whole after the collision.
- Particle Decay: a single particle decays into two different particles.

30.04.02 Examples of Inelastic Two-Body Collisions

A collisions between two particles such that one of the two particles breaks into two parts is certainly endo-energetic, because part of the kinetic energy must be used to break the particle.

An exo-thermal chemical reaction between two molecules is exo-energetic.

30.04.03 Coefficient of Restitution in Two-Body Collisions

The coefficient of restitution is defined as:

$$\epsilon \equiv \sqrt{\frac{\mathcal{K}^\star_{\mathrm{F}}}{\mathcal{K}^\star_{\mathrm{I}}}} = \frac{|\mathbf{v}'_2 - \mathbf{v}'_1|}{|\mathbf{v}_2 - \mathbf{v}_1|} \,.$$

Note that sometimes other similar definitions are used.

For $\epsilon = 0$ the loss of kinetic energy is maximal.

For $\epsilon = 1$ the collision is elastic.

Note that some of the total kinetic energy can never disappear, as dictated by **Koënig theorem** , because the kinetic energy of the Center-Of-Mass motion must remain constant for an isolated system whose linear momentum is conserved. This is the reason why the coefficient of restitution is defined in terms of the kinetic energy relative to the CM.

30.05 N-Body Collision Processes

The two-body problem can be, in general, reduced to two one-body problem and thus generally solved.

The N-body problem cannot be, in general, reduced to N one-body problems and no general solution is known.

The Center-Of-Mass position is defined as:

$$\mathbf{R} \equiv \frac{\sum_{k=1}^{N} m_k \mathbf{r}_k}{\sum_{k=1}^{N} m_k} \equiv \frac{\sum_{k=1}^{N} m_k \mathbf{r}_k}{M} \quad .$$

The Center-Of-Mass velocity is defined as:

$$\mathbf{V} \equiv \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{\sum_{k=1}^{N} m_k \mathbf{v}_k}{\sum_{k=1}^{N} m_k} \equiv \frac{\sum_{k=1}^{N} m_k \mathbf{v}_k}{M} \quad .$$

The Center-Of-Mass-coordinates are defined as:

$$\mathbf{r}_k^\star = \mathbf{r}_k - \mathbf{R} \qquad \text{with} \qquad \sum_{k=1}^{N} m_k \mathbf{r}_k^\star = \mathbf{0} \quad .$$

The Center-Of-Mass-velocities are defined as:

$$\mathbf{v}_k^\star \equiv \frac{\mathrm{d}\mathbf{r}_k^\star}{\mathrm{d}t} = \mathbf{v}_k - \mathbf{V} \qquad \text{with} \qquad \sum_{k=1}^{N} m_k \mathbf{v}_k^\star = \mathbf{0} \quad .$$

The linear momentum is given by:

$$\mathbf{P} = \sum_{k=1}^{N} m_k \mathbf{v}_k = M\mathbf{V} \quad .$$

The angular momentum with respect to a pole O is given by the first **Koënig theorem** (read § 24.11.02):

$$\mathbf{L}_0 = M\,(\mathbf{R} - \mathbf{x}_0) \wedge \mathbf{V} + \sum_{k=1}^{N} m_k\, \mathbf{r}_k^\star \wedge \mathbf{v}_k^\star \quad . \tag{30.05.01}$$

The kinetic energy is given by the second **Koënig theorem** (read § 24.11.03):

$$\mathcal{K} = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{k=1}^{N} m_k \mathbf{v}_k^{\star 2} \quad . \tag{30.05.02}$$

DRAFT - IN PROGRESS

30.06 Cross-Section

This § is referenced at pages:
[1103, 2078, 2328, 2726]

The only known way to investigate matter at a microscopic level is to used known microscopic projectiles to bombard the target to study and try to deduce properties of the target from the study of its interaction properties with the know projectiles. This method of investigation is usually employed to study particles, atoms, molecules but also bulk matter and surface properties. The projectiles are different types of particles, most often photons. The properties of the projectiles must be chosen so that their interaction with the target allows to draw interesting conclusions.

This explains the importance of the study of the interaction of a beam of projectiles with some target and the introduction of the concept of cross-section.

Si abbia un processo in cui un certo numero di particelle **bersaglio** sono bombardate da un fascio di particelle **proiettile** e si osservi il risultato di tale processo.

La sezione d'urto, σ, del generico processo si definisce, in termini del numero di eventi che interessano per unità di tempo, $\frac{\mathrm{d}N_S}{\mathrm{d}t}$, come:

$$\boxed{\frac{\mathrm{d}N_S}{\mathrm{d}t} \equiv \mathcal{N}\sigma\mathcal{F} \qquad \mathcal{F} \equiv \frac{d^2 N_0}{\mathrm{d}A\ \mathrm{d}t}} \quad , \tag{30.06.01}$$

→ 1610

dove $\mathcal{N}$ è il numero di bersagli ed $\mathcal{F}$ è il flusso delle particelle proiettile, cioé il numero di particelle incidenti per unità di superficie per unità di tempo.

Con tale definizione ci si libera dagli effetti non dinamici, intensità del fascio e numero di centri diffusori, ottenendo una grandezza fisica, la sezione d'urto σ, che dipende solo dalle proprietà dell'interazione tra una particella proiettile ed una particella bersaglio.

Note that the flux (or intensity) of incident particles, $\mathcal{F}$, is actually a current density of particles:

$$\boxed{\mathbf{j}_P = \eta_{\mathcal{N}}\mathbf{v} \equiv \rho_{\mathcal{N}}\mathbf{v}} \quad , \tag{30.06.02}$$

that is a number of particles per unit time per unit area perpendicular to the direction of motion (compare with the definition of current density of charge, the usual current density of ElectroMagnetism, read section ADD REFERENCE). In a similar way one can calculate the current of particles that is the total number of particles crossing any surface Σ:

$$\boxed{\frac{\mathrm{d}N}{\mathrm{d}t} = \iint_{\Sigma} \mathbf{j}_P \cdot \mathrm{d}\mathbf{S}} \quad .$$

The cross-section has dimension of an area. A typical unit of measure used and appropriate for microscopic physics is the barn:

$$\boxed{\mathrm{barn} = 10^{-28}\ \mathrm{m}^2} \quad . \tag{30.06.03}$$

Nei calcoli espliciti può essere necessario considerare i due casi distinti di un fascio più grande del bersaglio oppure di un fascio più piccolo del bersaglio. Nel seguito si assumerà sempre implicitamente che la sezione A di cui si parla è la sezione del bersaglio, nel primo caso, e la sezione del fascio, nel secondo caso.

Con riferimento alla definizione (30.06.01) si assume quanto segue.

- The spatial distribution of both the projectiles inside the incoming beam and the particles inside the target are random and homogeneous.
- Il numero di bersagli è così piccolo che la probabilità che un proiettile abbia più di una interazione, nell'attraversare il bersaglio, è trascurabile: assenza di scattering multiplo. Questo significa che, per ogni proiettile, il numero di bersagli potenziali deve essere sufficientemente piccolo.
- La lunghezza d'onda di De Broglie dei proiettili è piccola rispetto alle distanze tra i bersagli, in

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modo che gli effetti di interferenza dovuti allo scattering con diversi bersagli siano trascurabili. In altre parole i diversi bersagli devono agire come centri di scattering indipendenti e devono mancare effetti di interferenza/diffrazione.
This means that wave-properties must be negligible and that it should be possible to treat the projectiles as particles/corpuscles. The appropriate factor to discriminate between particle-like and wave-like behavior is, in general,

$$\frac{\lambda}{D} \qquad D\text{: characteristics dimensions involved in the processes} \quad .$$

- Finally the involved interaction forces should be short-range forces. Long range forces might require a special, more complex, treatment. However, short-range forces are rather common, with the exception of gravitational forces. Electric forces among charges, in fact, are in principle long-range forces as well but electric field are most often screened in conductive materials.

La sezione d'urto differenziale in angolo solido, $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$, del generico processo si definisce dal numero di particelle diffuse ad una certa direzione definita dagli angoli polare ed azimuthale θ e ϕ, $\frac{\mathrm{d}\mathsf{N}_\mathsf{S}}{\mathrm{d}t}[\theta,\phi]$:

$$\frac{\mathrm{d}\mathsf{N}_\mathsf{S}}{\mathrm{d}t}[\theta,\theta+\mathrm{d}\theta;\phi,\phi+\mathrm{d}\phi] \equiv \frac{d^2\mathsf{N}_\mathsf{S}}{\mathrm{d}\Omega\,\mathrm{d}t}[\theta;\phi]\,\mathrm{d}\Omega \equiv \mathcal{N}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}[\theta,\phi]\,\mathrm{d}\Omega\mathcal{F} \quad ,$$

$$\Longrightarrow \quad \boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}[\theta,\phi] \equiv \frac{1}{\mathcal{N}\mathcal{F}}\frac{d^2\mathsf{N}_\mathsf{S}}{\mathrm{d}\Omega\,\mathrm{d}t}[\theta;\phi]} \quad ,$$

$$\boxed{\sigma = \iint \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}[\theta,\phi]\,\mathrm{d}\Omega} \quad .$$

It depends, a priori, on the polar and azimuthal angles, θ and ϕ. In case the system is invariant by rotation around the axis of the incident beam the differential cross-section in angle does not depend on the azimuthal angle ϕ.

In a simpler way:

$$\boxed{\mathrm{d}\sigma \equiv \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}[\theta,\phi]\,\mathrm{d}\Omega} \quad .$$

In generale la sezione d'urto, $\sigma = \sigma[\varepsilon,\alpha]$, dipende dall'energia e da altri parametri, α.

Note that the number of targets, $\mathcal{N}$, is strictly liked to the cross section used. As a typical example if one is studying photon-matter interactions and uses a photon-electron cross-section the number of targets, $\mathcal{N}$, is the number of electrons; if, on the other hand, one uses the photon-atom cross-section the number of targets, $\mathcal{N}$, is the number of atoms.

30.06.01
Cammino Libero Medio E Sezione D'urto

This § is referenced at pages:
[2077, 2726, 2727]

Incident beam of projectiles of sectional area $\mathrm{d}\mathbf{S}$ crossing a thin sheet of targets of thickness $\mathrm{d}z$. The area of the target is assumed to be very much larger than the sectional area $\mathrm{d}\mathbf{S}$ of the beam.

First introduce the volume, $\delta V \equiv \mathrm{d}\mathbf{S}\,\mathrm{d}z$, of the region of intersection between the beam of particles and the target is introduced. This is assumed to be the region where interactions happen and the number of involved targets is thus the number of targets inside this volume.

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One can write:

$$\frac{\mathrm{d}\mathsf{N}_\mathsf{S}}{\mathrm{d}t} = \sigma \eta_\mathbb{N}\, \delta V\, \mathcal{F} \quad ,$$

$$\frac{\mathrm{d}\mathsf{N}_\mathsf{S}}{\mathrm{d}t} = \sigma\, \delta\mathcal{N}\, \mathcal{F} = \frac{\mathrm{d}\mathsf{N}_0}{\mathrm{d}t}\frac{\delta\mathcal{N}}{\delta V}\sigma\, \mathrm{d}z \quad ,$$

$$\frac{\mathrm{d}\mathsf{N}_\mathsf{S}}{\mathrm{d}t} = \sigma\, \delta\mathcal{N}\, \mathcal{F} = \sigma\frac{\delta\mathcal{N}}{\mathrm{d}\mathbf{S}\, \mathrm{d}z}\, \mathrm{d}\mathbf{S}\, \mathrm{d}z \frac{d^2\mathsf{N}_0}{\mathrm{d}A\, \mathrm{d}t} = \frac{\mathrm{d}\mathsf{N}_0}{\mathrm{d}t}\frac{\delta\mathcal{N}}{\delta V}\sigma\, \mathrm{d}z \quad ,$$

$$\frac{\left(\frac{\mathrm{d}\mathsf{N}_\mathsf{S}}{\mathrm{d}t}\right)}{\left(\frac{\mathrm{d}\mathsf{N}_0}{\mathrm{d}t}\right)} = \frac{\mathrm{d}\mathsf{N}_\mathsf{S}}{\mathrm{d}\mathsf{N}_0} = \frac{\mathcal{N}}{V}\sigma\, \mathrm{d}z \quad ,$$

$$N[z] \Longrightarrow \frac{\mathrm{d}N[z]}{N[z]} = -\frac{\mathcal{N}}{V}\sigma\, \mathrm{d}z \equiv -\frac{\mathrm{d}z}{\lambda} \quad ,$$

$$\frac{1}{\lambda} \equiv \frac{\mathcal{N}\sigma}{V} \quad ,$$

where

Dato un processo con una certa sezione d'urto σ il libero cammino medio per l'interazione corrispondente al processo vale dunque

$$\boxed{\frac{1}{\lambda[\varepsilon,\alpha]} \equiv \frac{\mathcal{N}\sigma[\varepsilon,\alpha]}{V} \equiv \eta_\mathbb{N}\sigma[\varepsilon,\alpha]} \tag{30.06.04}$$

→ 2078

dove $\eta_\mathbb{N} \equiv (\mathcal{N}/V)$ è il numero di bersagli per unità di volume. In generale, così come $\sigma = \sigma[\varepsilon,\alpha]$ dipende dall'energia e da altri parametri α, anche il cammino libero medio, $\lambda = \lambda[\varepsilon,\alpha]$, ne dipende.

Il numero medio di interazioni attese in un percorso $[z, z+\ \delta z\ ]$ è dunque:

$$\mathsf{N}_\mathsf{S}[z, z+\ \delta z\ ] = \frac{\delta z}{\lambda}\mathsf{N}_0[z] \quad .$$

Il numero di interazioni attese in un percorso $[z, z+\ \delta z\ ]$ ha dunque una distribuzione di Poisson con valore medio:

$$\boxed{\langle \mathsf{N}_\mathsf{S}[z, z+\ \delta z\ ]\rangle = \frac{\delta z}{\lambda}\mathsf{N}_0[z]} \quad .$$

In generale un fascio di particelle incidenti su un bersaglio viene attenuato dalle interazioni: le particelle possono essere assorbite o semplicemente diffuse in una direzione diversa. Il numero di particelle eliminate dal fascio, nell'intervallo infinitesimo $\mathrm{d}z$, è proporzionale al numero di particelle nel fascio e all'intervallo $\mathrm{d}z$:

$$-\frac{\mathrm{d}\mathcal{F}}{\mathcal{F}} = \frac{\mathrm{d}z}{\lambda} \tag{30.06.05}$$

da cui segue

$$\boxed{\mathcal{F}[z] = \mathcal{F}_0 \exp\left[-z/\lambda\right]} \quad .$$

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30.06.02
Differential Cross-Section and Impact Parameter

Assume that the system is invariant by rotation around the axis of the incident beam: the differential cross-section in angle does not depend on the azimuthal angle ϕ.

There exist a well defined relation between the impact parameter and the scattering angle θ:

$$\boxed{b = b[\theta]} \quad ,$$

determined by the interaction.

The number of particles scattered in the angular interval of polar angle $[\theta, \theta + \mathrm{d}\theta]$ and azimuthal angle $[\phi, \phi + \mathrm{d}\phi]$ is expressed in terms of the impact parameter, $b[\theta]$, as it is given by the particles passing through an infinitesimal circular corona of radius $b[\theta]$:

$$\mathrm{d}\mathsf{N}_\mathsf{S}[\theta, \theta + \mathrm{d}\theta; \phi, \phi + \mathrm{d}\phi] = \mathcal{N}\mathcal{F} b \, \mathrm{d}b \, \mathrm{d}\phi \, \mathrm{d}t = b\mathcal{N}\mathcal{F}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}t \quad .$$

Finally:

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}[\theta] = \frac{1}{\mathcal{N}\mathcal{F}}\frac{d^2\mathsf{N}_\mathsf{S}}{\mathrm{d}\Omega \, \mathrm{d}t}[\theta] = \frac{1}{\mathcal{N}\mathcal{F}}\frac{\left(b\mathcal{N}\mathcal{F}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}t\right)}{\mathrm{d}\Omega \, \mathrm{d}t} = \frac{b\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}t}{\sin[\theta] \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}t} = \frac{b}{\sin[\theta]}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|} \quad .$$

30.06.02.01 Angular Differential Cross-Section for a Rigid Sphere

Consider the elastic scattering of a mass point particle from a rigid sphere of radius R.

One finds, from geometrical considerations:

$$b = R\sin[\theta_0] \quad ,$$
$$2\theta_0 + \theta = \pi \quad ,$$
$$b[\theta] = R\sin\left[\frac{\pi - \theta}{2}\right] = R\cos\left[\frac{\theta}{2}\right] \quad .$$

It follows:

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}[\theta] = \frac{R^2}{4} \qquad \sigma = \iint \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}[\theta] \, \mathrm{d}\Omega = \pi R^2} \quad .$$

Therefore the total cross-section for a rigid sphere is just the geometrical area of the **shadow** presented by the sphere to the incident beam.

30.06.02.02 Angular Differential Cross-Section for Kepler/Rutherford Scattering

The relation between scattering angle and impact parameter for scattering from an inverse-square force law (Kepler/Rutherford scattering) is given by equation (29.04.27):

$$\boxed{b = \frac{\gamma}{2E}\cot[\theta/2] \qquad \left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| = \frac{\gamma}{4E}\frac{1}{\sin^2[\theta/2]}} \quad .$$

It follows:

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}[\theta] = \frac{\frac{\gamma}{2E}\cot[\theta/2]}{\sin[\theta]}\frac{\gamma}{4E}\frac{1}{\sin^2[\theta/2]} = \frac{\gamma^2}{16E^2}\frac{1}{\sin^4[\theta/2]}} \quad .$$

The divergence at $\theta = 0$ is not a problem. It just means that, if there is a beam of infinite extension and therefore an infinite number of incident particles, there are infinite scattered particles. Moreover, for any real beam of finite extension the impact parameter cannot be integrated to infinity, and therefore the scattering angle is not integrated down to zero.

For any beam of finite extension the impact parameter is to be integrated to less than infinity, and therefore the scattering angle should not be integrated down to zero !!!!!!

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30.07 Virial Theorem and Some Applications

This § is referenced at pages:
[1310, 2726]

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30.07.01
Jacobi Identity

Consider an arbitrary system made of N point particles, that is made of systems with no, or negligible, internal degrees of freedom. The particles can be complex systems themselves as long as the internal degrees of freedom can be ignored and only the translational degrees of freedom are relevant. In particular, the kinetic energy must be the translational kinetic energy only. More complex systems may led to different formulations. Note, however, that for small pieces of matter, assumed at fixed density, the rotational kinetic energy, goes to zero as the fifth power of size, due to the inertia moments, while the translational kinetic energy goes to zero as the third power of the size, due to mass.

Assume, therefore, that the kinetic energy of any particle is just given by the translational kinetic energy:

$$\mathtt{k} \equiv \frac{1}{2} m v^2 \quad .$$

If any particle has additional significant contributions to the kinetic energy the following derivations must be modified accordingly.

Starting from the total translational kinetic energy of the system of N point particles one can re-write it as follows:

$$\mathcal{K} \equiv \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{N} \mathbf{p_i}\cdot\mathbf{v_i} = \frac{1}{2}\sum_{i=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p_i}\cdot\mathbf{r_i}) - \frac{1}{2}\sum_{i=1}^{N}(\mathbf{r_i}\cdot\dot{\mathbf{p}}_\mathbf{i}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N}(\mathbf{p_i}\cdot\mathbf{r_i}) - \frac{1}{2}\sum_{i=1}^{N}(\mathbf{r_i}\cdot\mathbf{F_i}) = \quad ,$$

$$\frac{1}{4}\frac{d^2}{\mathrm{d}t^2}\sum_{i=1}^{N} m_i(\mathbf{r_i}\cdot\mathbf{r_i}) + \Omega = \quad ,$$

$$\frac{1}{4}\ddot{I}_0 + \Omega \quad ,$$

where the **virial**, Ω, has been defined as

$$\boxed{\Omega \equiv -\frac{1}{2}\sum_{k=1}^{N}(\mathbf{r_k}\cdot\mathbf{F_k})} \quad ,$$

and I_0, the polar moment of inertia with respect to the origin, have been introduced.

The result is the so called Jacobi identity:

$$\boxed{\mathcal{K} = \frac{1}{4}\ddot{I}_0 + \Omega} \quad . \tag{30.07.01}$$

→ 1616

For continuous systems, with only surface forces normal to the surface (normal stresses), the virial is:

$$\Omega = -\frac{1}{2}\iint_{\Sigma} p[\mathbf{x}]\,\mathbf{r}\cdot\,\mathrm{d}\mathbf{A} \qquad \text{provided } p[\mathbf{x}]\,\mathrm{d}\mathbf{S} \text{ has the same oriented direction as the force} \quad .$$

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For continuous systems, with volume forces the virial is:

$$\Omega = -\frac{1}{2}\iiint_V \mathbf{r}\cdot\mathbf{f}_V[\mathbf{x}]\ \mathrm{d}V \quad .$$

30.07.02
Virial Theorem

The virial theorem derives from Jacobi identity when, as sufficient conditions:

- either the motion of the system is finite (that is both coordinates and velocities are limited functions);
- or the system is in a macroscopically steady state.

The virial theorem provides the mean value in time of the kinetic energy.

For any given sufficiently well-behaved time-dependent function, $\xi[t]$, defined for $0 \le t \le \infty$, the average value in time is defined as:

$$\boxed{\langle \xi \rangle \equiv \lim_{T\longrightarrow\infty} \frac{1}{T}\int_0^T \xi[t]\ \mathrm{d}t}$$

In general, the coordinates, velocities and accelerations of the point particles building up the system are time-dependent. However, we may content ourselves with average values of macroscopic quantities when the number of particles is very large.

- Assume that the system is contained in a finite volume, that is the coordinates are limited functions at any time. Assume that also velocities are limited functions of time. The latter assumption is normally verified due to energy conservation, unless there is the possibility that the potential energy goes to $-\infty$ for some actual configuration. In any case relativity implies that the speeds cannot exceed the speed of light, and, if speeds increase close to the speed of light the current classical framework is not applicable anymore.
  It follows:
  $$\langle \ddot{I}_0 \rangle = \lim_{T\longrightarrow\infty} \frac{1}{T}\int_0^T \ddot{I}_0\ \mathrm{d}t = \lim_{T\longrightarrow\infty} \frac{1}{T}\Big(\dot{I}_0[T] - \dot{I}_0[0]\Big) = 0 \quad ,$$
  as the quantity $\Big(\dot{I}_0[T] - \dot{I}_0[0]\Big)$ is limited at any time. In fact,
  $$\dot{I}_0 \equiv 2\sum_{i=1}^{N} m_i\,(\mathbf{r_i}\cdot\mathbf{v_i}) \quad ,$$
  and, by hypothesis, position and velocities are limited functions.
  The virial theorem follows:
  $$\boxed{\langle \mathcal{K} \rangle = \langle \Omega \rangle} \quad . \qquad (30.07.02)$$
- Another sufficient condition for the validity of the virial theorem (30.07.02) is that the configuration of the system is steady from a thermodynamical, that is macroscopic, point of view. In this case all macroscopic quantities are constant in time, including I_0, and therefore the result
  $$\langle \ddot{I}_0 \rangle = 0 \quad ,$$
  follows again.
  The virial theorem follows:
  $$\boxed{\langle \mathcal{K} \rangle = \langle \Omega \rangle} \quad .$$

→ 1615

The quantity $\langle \Omega \rangle$ is called **virial** as well. Note, however, that sometimes the virial is defined without the factor $-1/2$; this is the original definition from Clausius.

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The virial can be written splitting the contribution from external and internal forces:

$$\boxed{\langle\Omega\rangle = -\frac{1}{2}\langle\sum_{i=1}^{N}\left(\mathbf{r_i}\cdot\mathbf{F}_\mathbf{i}^{I}\right)\rangle - \frac{1}{2}\langle\sum_{i=1}^{N}\left(\mathbf{r_i}\cdot\mathbf{F}_\mathbf{i}^{E}\right)\rangle \equiv \langle\Omega^{\mathrm{I}}\rangle + \langle\Omega^{\mathrm{E}}\rangle} \quad ,$$

where F_i^I and F_i^E are, respectively, the resultant on particle i of all the internal and external forces.

Note that the formulation of the virial theorem is based on time averages of infinite length. If the virial theorem is applied to systems (with finite motion) which are either expanding or contracting, then one must be careful in the calculation of time averages and compare the finite length of time of the averaging with the typical time-scales of the system changes.

The time average is often replaced, in systems made of a large number of constituents, by the ensemble average over all the constituents of the system.

30.07.03
Virial for a Homogeneous Function Potential Energy

Assume that the interaction potential energy among the particles only depends on the position of all the N particles,

$$\Phi = \Phi[\mathbf{r}_i] \equiv \Phi[\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N] \quad ,$$

and that it is an homogeneous function of degree n (read § 14.04.03).

It follows:

$$\Omega = -\frac{1}{2}\sum_{i=1}^{N}\left(\mathbf{r_i}\cdot\mathbf{F}_\mathbf{i}\right) = +\frac{1}{2}\sum_{i=1}^{N}\left(\mathbf{r_i}\cdot\frac{\partial\Phi}{\partial\mathbf{r}_i}\right) = \frac{n}{2}\Phi[\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N] \quad .$$

Note that the sum over all particles is actually the gradient of the potential energy function, which is a function of all the $\mathbf{r}_i$.

Jacobi identity (30.07.01) then implies:

$$\boxed{\frac{1}{2}\ddot{I}_0 = (n+2)\,\mathcal{K} - nE} \quad .$$

30.07.03.01 Newton and Coulomb Interactions

In the most relevant case of either newtonian or coulombian forces:

$$\Phi = -\frac{\mathcal{G}}{2}\sum_{a=1,b=1,a\neq b}^{a=N,b=N}\frac{m_a m_b}{|\mathbf{r}_a - \mathbf{r}_b|} \qquad n=-1 \implies \Omega = -\frac{1}{2}\Phi \qquad \frac{1}{2}\ddot{I}_0 = \mathcal{K} + E \qquad \langle\mathcal{K}\rangle = \langle\Omega\rangle = -\frac{1}{2}\langle\Phi\rangle \quad .$$

(30.07.03) → 1618

Note that it is a polar moment of inertia and not an axial moment of inertia.

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30.07.03.02 Harmonic Interactions

In the case of harmonic interactions one finds:

$$n = +2 \implies \Omega = \Phi \implies \langle \mathcal{K} \rangle = \langle \Phi \rangle \quad .$$

30.07.04 Some Applications of the Jacobi Identity and the Virial Theorem

30.07.04.01 Dark Matter in a Cluster of Galaxies

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The following method can be used to roughly estimate the total mass M of a cluster of galaxies with linear dimension of the order of R whose galaxies have an average squared velocity $\langle v^2 \rangle$.

The total kinetic and total potential energy is:

$$\Phi \approx -\mathcal{G}\frac{M^2}{R} \qquad \mathcal{K} = \frac{1}{2}M\langle v^2 \rangle \quad .$$

The virial theorem can be applied by assuming that the time average can be replaced by the average on the large number of elements making up the cluster of galaxies. If that is the case one can write, for the cluster of galaxies in a steady configuration, neglecting factors of the order of unity,

$$\langle \mathcal{K} \rangle = -\frac{\langle \Phi \rangle}{2} \implies M = \frac{R\langle v^2 \rangle}{\mathcal{G}} \quad .$$

The two quantities R and $\langle v^2 \rangle$ can both be estimated and therefore the total mass of the cluster can be estimated.

On the other hand the total mass of the cluster can be also estimated from the well known mass-luminosity relations for stars and by applying the necessary corrections for absorption and possibly other effects.

The mass estimated from the luminosity turns out to be invariably lower than the mass estimated from gravitational effect via the virial theorem. This points to the existence of dark matter in the universe.

The real estimation can be much more refined and needs to take into account geometrical and other effects.

30.07.04.02 Equation of State of a Ideal Molecular Gas

This § is referenced at pages:
[2754]

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A ideal gas is a model for a system for which the constituting particles are assumed to be point particles and the particle-particle interactions are negligible while the interactions with the walls are always present and are responsible for thermalization the gas; that is, particle-wall interactions are not elastic on a event-by-event basis, but only on average, and the wall, on average, does not change energy but just catalyses the thermalization of the gas.

An equivalent, more realistic, model of ideal gas is a model for a system for which the constituting particles are assumed to be almost point particles and the particle-particle interactions are elastic contact collisions, while the interactions with the walls are always present and elastic interactions.

Consider a ideal gas inside a closed fixed container at rest and assume the pressure is uniform.

The virial of internal forces is null. In fact at any contact interaction between any pair of point particles the two particles have the same positions, the forces are equal and opposite and the total virial of the pair is null.

We are only interested in the mean values and we can therefore calculate the virial of the external forces, given by the walls of the container, by introducing the pressure that is exactly the physical quantity describing the effect of collisions averaged in time, the impulse transferred per unit time per unit surface.

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One finds, passing to the integral on the surface of the container,

$$\langle \Omega^{\mathrm{E}} \rangle = -\frac{1}{2}\langle \sum_{i=1}^{N} \left(\mathbf{r}_{\mathrm{i}} \cdot \mathbf{F}_{\mathrm{i}}^{E} \right) \rangle \implies \langle \Omega^{\mathrm{E}} \rangle = +\frac{1}{2} p \oiint \mathbf{r} \cdot \mathrm{d}\mathbf{S} \quad ,$$

as the pressure is uniform and the unit vector is oriented towards the exterior of the surface. Note that one can change the sum over the particles to the integral on the surface of the container thanks to the large number of particles. However, the time-average implied by the virial theorem is required to use the concept of pressure, that is the time-averaged force per unit area.

By applying the **divergence theorem** :

$$\boxed{\langle \Omega^{\mathrm{E}} \rangle = \frac{3}{2} pV \implies \langle \mathcal{K} \rangle = \frac{3}{2} pV} \quad .$$

From the kinetic theory interpretation of the kinetic energy one finds the equation of state of ideal gases:

$$\boxed{pV = \frac{2}{3}\langle \mathcal{K} \rangle = N k_{\mathrm{B}} T} \quad .$$

The virial theorem is the main tool in classical kinetic theory to handle non-ideal gases.

30.07.04.03 Virial for a Set of Charged Particles in a Magnetic Field

Consider a set of charged particles in a uniform and constant magnetic field $\mathbf{B}$.

If all the particles have the same charge-to-mass ratio, $g \equiv q/m$ the virial is:

$$\Omega = -\frac{g}{2} \mathbf{L} \cdot \mathbf{B} \quad ,$$

as a function of the total orbital angular momentum $\mathbf{L}$.

30.07.04.04 Virial Theorem in Presence of Forces Depending Linearly on the Velocity

Show that the virial of viscous forces, proportional to the velocity of each particle, is zero.

30.07.04.05 A Proto-Star Collapse

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©|D.V.Schroeder, *Thermal Physics*, 1999, Addison-Wesley, ...Ed., ...|problem 1.55||

Consider a system made of particles subject to gravitational forces only in quasi-static equilibrium. Assume the internal energy is just given by translational kinetic energy and gravitational potential energy.

According to equation (30.07.03) if the system loses mechanical energy, (that is, by hypotesis, total energy), for instance by radiation (as a in collapsing proto-star) or by friction (as in the satellite paradox), $\Delta E < 0$ implies that its kinetic energy increases by the same amount, $\Delta \mathcal{K} = -\Delta E > 0$. These energies are supplied by the potential energy, which changes by, $\Delta \mathcal{U} = 2\Delta E < 0$. This normally implies that the system is shrinking in size.

Consider the protostar as composed of a ideal gas. The temperature is normally so high that molecules do not exist and atoms are largely ionized. The kinetic energy of the plasma is then:

$$\mathcal{K} = \frac{3}{2} N k_{\mathrm{B}} T \quad ,$$

with N total number of particles, that is assume that the total internal energy is just given by the translational kinetic energy only. In a high-temperature plasma all molecules and atoms are ionized so this is consistent. Therefore:

$$\Delta E = -\Delta \mathcal{K} = -\frac{3}{2} N k_{\mathrm{B}} \Delta T < 0 \quad .$$

Therefore while the system shinks and loses total energy it increases its temperature.

However, in real cases, also conservation of angular momentum plays a role

DRAFT - EM Physics

30.08 Exercises Problems and Physical Applications

30-001 Galilei-Invarian* of the Q-Value

This § is referenced at pages:
[1601]

Show that the Q-value is Galilei invariant.

30-002 Elastic Scattering of One Particle Onto a Stationary Particle of the Same Mass: Angles

Show that in the elastic scattering of one particle onto a stationary particle of the same mass the angle between the two particles in the final state is always $\pi/2$.

30-003 Elastic Scattering of One Particle Onto a Stationary Particle of the Same Mass: Momenta

Show that in the elastic scattering of one particle onto a stationary particle of the same mass the momenta of the two particles in the final state, $\mathbf{p}_1$ and $\mathbf{p}_1$, can be written, in terms of the unit vector in the direction of $\mathbf{p}'_1$, $\widehat{\mathbf{n}}$, and the module of the linear momentum of the particles in the Center-Of-Mass system, $p^\star$, as:

$$\mathbf{p}'_1 = +p^\star\widehat{\mathbf{n}} + \frac{m}{M}\mu\mathbf{v} \qquad \mathbf{P}'_1 = -p^\star\widehat{\mathbf{n}} + \mu\mathbf{v} \quad .$$

30-004 Energy Transfer to a Fixed Target in an Elastic Process

Show that:

$$\mathcal{K}'_M = \mathcal{K}_m \frac{2\mu}{M}(1-\cos[\theta^\star]) \quad .$$

30-005 Elastic Rutherford Scattering of One Particle Onto a Stationary Particle of the Same Mass

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Show that the energy loss of the incident particle is given by:

$$Q = \frac{4\mu}{m+M}\sin^2\left[\frac{\theta^\star}{2}\right]\mathcal{K} \quad ,$$

in terms of the kinetic energy of the incident particle, $\mathcal{K}$, and the scattering angle in the Center-Of-Mass system, $\theta^\star/2$.

Show that, for a given incident particle mass and kinetic energy, the maximum average energy loss is obtained when the fixed target is chosen so that $M = m$, assuming that the scattering is isotropic in the CM system.

WORKED SOLUTION

The function of M

$$\langle Q\rangle = \langle Q\rangle[M] = \frac{2\mu}{m+M}\mathcal{K} \quad ,$$

has maximum at $M = m$.

UNDER PEER REVIEW - IN PROGRESS

CHAPTER 31

Introduction to Celestial Mechanics and Astrodynamics

31.01 Perturbed Kepler Orbits 1622
31.02 Examples and Physical Applications 1627
31.03 Numerical Solution of Problems in Celestial Mechanics 1630
31.04 Exercises Problems and Physical Applications 1631

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DRAFT-IN-PROGRESS

31.01 Perturbed Kepler Orbits

This § is referenced at pages:
[1640, 1640]

©|J.A. Burns|Am.J.Phys, , ..., ..., ...Ed., ... 44 (1976) 944|*Elementary derivation of the perturbation equations to Kepler ...*
©|J.A. Burns|Am.J.Phys, , ..., ..., ...Ed., ... 45 (1977) 1230|*Errata Corrige to Am.J.Phys, , ..., ..., ...Ed., ... 44 (1976) 944|*
©|Boccaletti-Pucacco, Theory of orbits|||

©|T.Padmanabhan, *Theoretical Astrophysics, Vol. 1,2,3*, 2000/2001/2002, CUP, ...Ed., ...|2.8.10.2||
©|H.Pollard, *Mathematical Introduction To Celestial Mechanics*, 1966, Prentice-Hall, ...Ed., ...|1, 4.2 (with equation); 4.5 ...

We want to determine the effect of a small arbitrary perturbing force, $\mathbf{f}_{\text{P}}[\mathbf{r}]$, on the unperturbed conic orbit of a point particle in a pure and fixed inverse-square force field.

Assume that the interaction on a point particle of mass m is fully described by the total force, $\mathbf{f}_{\text{T}}$,:

$$\boxed{\mathbf{f}_{\text{T}} = -\frac{\gamma}{r^3}\mathbf{r} + \mathbf{f}_{\text{P}}[\mathbf{r}] \qquad \text{with } |\mathbf{f}_{\text{P}}| \ll \frac{|\gamma|}{r^2} \text{ at any point on the orbit}} \quad , \tag{31.01.01}$$

→ 1622

with no other external interaction, no intrinsic torques, for instance. Note that the perturbing force, $\mathbf{f}_{\text{P}}[\mathbf{r}]$, is not necessarily a central force.

In the presence of the small perturbation the orbit can be considered as a slowly changing conic. The case of the ellipse, that is a bound system, will be mostly considered, but the approach would be that same for parabolic and hyperbolic orbits, when using the appropriate equations. The osculating orbit is defined as the conic orbit which would be followed by the point particle if the perturbation would suddenly vanish.

The problem necessarily becomes a three-dimensional problem. Choose a Coordinate System in a Inertial Reference Frame such that its origin lies in the center of the inverse square force, at $\mathbf{r} = \mathbf{0}$, according to (31.01.01).

Define the line of nodes as the line defined by the crossing of the xy plane with the instantaneous orbit plane, that is the plane defined, at any instant, by the osculating ellipse (conic). Define the ascending node as the point where the osculating ellipse (conic) crosses the xy plane going in the direction of increasing z.

An unperturbed elliptical orbit will be only considered from now on. Therefore, in the current case,

$$\boxed{\gamma > 0} \quad .$$

See the figures 31.11, 31.12 for a graph and definition of variables. Note that with respect to the figures some notation in the text differ:

- we call φ the anomaly (called θ in the figure), because we must reserve θ for the polar angle in later applications;
- we call ν the eccentric anomaly (called ϵ in the figure);
- we call ℓ the orbital angular momentum of the mass point (called $\mathbf{H}$ in the figure).

Note finally that m and M in the figure refer to the mass of the two bodies while we will not use M as it is buried in our definition of γ. On the other hand we will later use M for the mean anomaly, as it is usually done in celestial mechanics.

See the text for comments on the notation

See the text for comments on the notation

The reference orbit equation (ellipse) in the plane of motion is:

$$\boxed{r[\varphi] = \frac{p}{1 + e\cos[\varphi - \omega]} \equiv \frac{p}{1 + e\cos[f]}} \quad , \tag{31.01.02}$$

→ 1623 1624

WORK IN PROGRESS

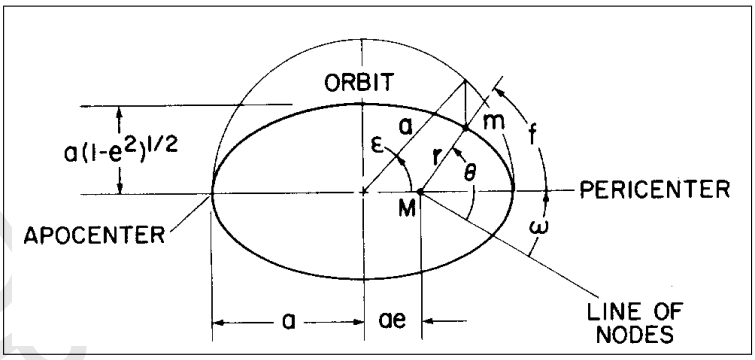

Figure 31.11: FIGURE

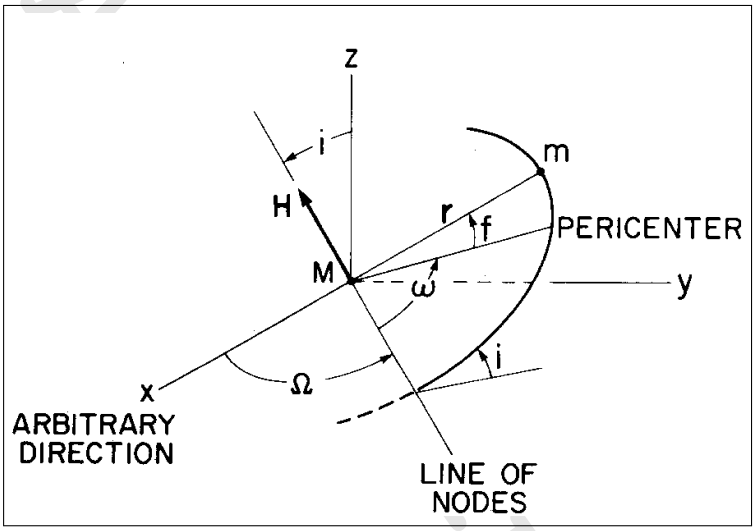

Figure 31.12: FIGURE

where φ is measured from the ascending node, such that it is increasing with time, and ω is the azimuth of the pericenter, measured in the plane of the osculating ellipse starting from the ascending node. The argument of the cosine term is called the true anomaly:

$$\boxed{f \equiv \varphi - \omega} \quad ,$$

and it represents the azimuth of the point particle from the pericenter.

One can show that an equivalent equation can be written in terms of the eccentric anomaly, ν:

$$\boxed{r[\varphi] = a\,(1 - e\cos[\nu])} \quad , \tag{31.01.03}$$

→ 1623 1640

with the following relation between the true and eccentric anomaly:

$$\boxed{\cos[\nu] = \frac{e + \cos[f]}{1 + e\cos[f]}} \quad , \tag{31.01.04}$$

→ 1640

which can be straightforwardly derived by comparing equations (31.01.02), (31.01.03).

See the references for the geometrical interpretation of the eccentric anomaly.

The eccentric anomaly allows one to solve the time dependence of the Kepler problem, via the mean anomaly M, defined in terms of the average angular velocity, which lacks a simple geometrical interpretation.

The time dependence is solved by the Kepler equations, which read:

$$M[t] \equiv \frac{2\pi}{T}(t - \tau) = e\,\mathrm{sih}\,[\nu[t]] - \nu[t] \qquad \text{for } e \geq 1 \tag{31.01.05}$$

$$M[t] \equiv \frac{2\pi}{T}(t - \tau) = \nu[t] - e\sin[\nu[t]] \qquad \text{for } 0 \leq e \leq 1 \quad , \tag{31.01.06}$$

→ 1641

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where T is the orbital period. Kepler equations allows to determine the time-dependence of the motion, as their solution is unique. Kepler equations are transcendental equations and therefore have no closed form solutions. Solutions via numerical methods is However, trivial and approximate methods have been developed to solve then in an approximate way.

The following expressions, which can be derived in a straightforward way by taking the time derivative of equation (31.01.02) and using the definition of the orbital angular momentum, will be necessary:

$$\boxed{\dot{r} = \frac{e\ell}{mp}\sin[f]} \qquad (31.01.07) \quad \rightarrow 1640$$

$$\boxed{r\dot{\varphi} = \frac{\ell}{mp}(1 + e\cos[f])} \quad . \qquad \rightarrow 1640$$

A useful expression for the forthcoming derivations is the time derivative of the anomaly expressed in terms of the distance only:

$$\boxed{\dot{\varphi} \equiv \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\ell}{mr^2} = \frac{1}{r^2}\sqrt{\frac{|\gamma|a\left(1-e^2\right)}{m}}} \quad , \qquad (31.01.08) \quad \rightarrow 1627\ 1640$$

derived from equation (29.04.17).

We need now a set of equation relating the geometrical parameters of the ellipse in space to the constant of motions, in such a waxy that we can consider how the two sets are related for small changes.

Equation (29.04.21) implies ($\gamma > 0$, by hypothesis):

$$\boxed{E = -\frac{\gamma}{2a} \equiv \frac{1}{2}mv^2 - \frac{\gamma}{r} \equiv K + \Phi_g} \quad .$$

The eccentricity is given by equation (29.04.16):

$$\boxed{e = \sqrt{1 + \frac{2E\ell^2}{m\gamma^2}}} \quad .$$

Two equations are required to describe the orientation of the orbit in space, which can be derived from the geometry, one giving the inclination of the plane of the osculating ellipse with respect to the xy plane,

$$\boxed{\cos[i] = \frac{\ell_z}{\ell}} \quad , \qquad (31.01.09) \quad \rightarrow 1640$$

and another one giving the angle between the line of nodes and the x axis,

$$\boxed{\tan[\Omega] = -\frac{\ell_x}{\ell_y}} \quad . \qquad (31.01.10) \quad \rightarrow 1640$$

Note that the three angles i, Ω and ω actually resemble Euler angles.

The two quantities i and Ω fix the orientation of the orbit plane in space.

Consider a small perturbing force:

$$\boxed{\mathbf{f}_{\mathrm{P}} = R\mathbf{e}_{\mathrm{R}} + T\mathbf{e}_{\mathrm{T}} + N\mathbf{e}_{\mathrm{N}}} \quad ,$$

in terms of the radial unit vector, $\mathbf{e}_{\mathrm{R}}$, the transverse unit vector in the plane of the orbit, positive in the direction of motion of the particle, $\mathbf{e}_{\mathrm{T}}$, and the unit vector normal to the osculating orbit plane, $\mathbf{e}_{\mathrm{N}} = \mathbf{e}_{\mathrm{R}} \wedge \mathbf{e}_{\mathrm{T}}$, given, in terms of the reference Coordinate System, by:

$$\begin{aligned} \mathbf{e}_{\mathrm{R}} &= \sin[\theta]\cos[\phi]\,\hat{\mathbf{e}}_1 + \sin[\theta]\sin[\phi]\,\hat{\mathbf{e}}_2 + \cos[\theta]\,\hat{\mathbf{e}}_3 \quad , \\ \mathbf{e}_{\mathrm{T}} &= \ldots \quad , \\ \mathbf{e}_{\mathrm{N}} &= \ldots \end{aligned} \qquad (31.01.11)$$

We will write the mechanical energy balance in terms of the work done by the perturbing force, in order to allow for possibly non conservative perturbing forces. The mechanical energy is defined as the sum of the kinetic energy plus the gravitational potential energy.

DRAFT WORK IN PROGRESS

The power balance then, at any instant of time, reads:

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t}=\frac{\mathrm{d}K}{\mathrm{d}t}+\frac{\mathrm{d}\Phi_g}{\mathrm{d}t}=\frac{\mathrm{d}W_{\mathrm{P}}}{\mathrm{d}t}=\mathbf{v}\cdot\mathbf{f}_{\mathrm{P}}=R\dot{r}+Tr\dot{\varphi}} \quad . \tag{31.01.12}$$

→ 1625

The second **Cardinal Equation** applied to the point, with pole at the center of force, gives:

$$\boxed{\frac{\mathrm{d}\boldsymbol{\ell}}{\mathrm{d}t}=\mathbf{r}\wedge\mathbf{f}_{\mathrm{P}}=rT\mathbf{e}_{\mathrm{N}}-rN\mathbf{e}_{\mathrm{T}}} \quad . \tag{31.01.13}$$

→ 1625

Moreover

$$\frac{\mathrm{d}\ell^2}{\mathrm{d}t}=2\ell\frac{\mathrm{d}\ell}{\mathrm{d}t}=2\,\boldsymbol{\ell}\cdot\frac{\mathrm{d}\boldsymbol{\ell}}{\mathrm{d}t}=2\ell\,(rT\mathbf{e}_{\mathrm{N}}-rN\mathbf{e}_{\mathrm{T}})\cdot\mathbf{e}_{\mathrm{N}}=2\ell rT \quad ,$$

implies

$$\boxed{\frac{\mathrm{d}\ell}{\mathrm{d}t}=rT} \quad . \tag{31.01.14}$$

→ 1625 1627 1628

The three equations (31.01.12), (31.01.13), (31.01.14) provide the relation between the force and the change in the basic constants of motion of the problem.

31.01.01
Perturbed Kepler Orbits Equations: Lagrange Planetary Equations

This § is referenced at pages:
[1640]

By carrying on the calculations one finds the following results.

H.Pollard, *Mathematical Introduction To Celestial Mechanics*, 1966, Prentice-Hall, ...Ed., ...-

J. A. Burns - Am.J.Phys, , ..., ..., ...Ed., ... 44 (1976) 944

J. A. Burns - Am.J.Phys, , ..., ..., ...Ed., ... 45 (1977) 1230

$$\frac{\mathrm{d}a}{\mathrm{d}t}=\frac{2a^{3/2}}{\sqrt{m\gamma\,(1-e^2)}}\left(Re\sin[f]+T\,(1+e\cos[f])\right) \tag{31.01.15}$$

→ 1626 1627

$$\frac{\mathrm{d}e}{\mathrm{d}t}=\frac{\sqrt{a\,(1-e^2)}}{\sqrt{m\gamma}}\left(R\sin[f]+T\,(\cos[f]+\cos[\nu])\right) \tag{31.01.16}$$

→ 1628

$$\frac{\mathrm{d}i}{\mathrm{d}t}=\frac{\sqrt{a\,(1-e^2)}}{\sqrt{m\gamma}}\frac{N\cos[\varphi]}{1+e\cos[f]}=\frac{rN\cos[\varphi]}{\ell} \tag{31.01.17}$$

→ 1626 1628

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t}=\frac{\sqrt{a\,(1-e^2)}}{\sqrt{m\gamma}}\frac{N\sin[\varphi]}{\sin[i]\,(1+e\cos[f])}=\frac{rN\sin[\varphi]}{\ell\sin[i]} \tag{31.01.18}$$

→ 1626 1628

$$\frac{\mathrm{d}\omega}{\mathrm{d}t}=\frac{\sqrt{a\,(1-e^2)}}{e\sqrt{m\gamma}}\left(-R\cos[f]+T\sin[f]\,\frac{(2+e\cos[f])}{(1+e\cos[f])}\right)-\cos[i]\,\frac{\mathrm{d}\Omega}{\mathrm{d}t} \tag{31.01.19}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}t}=\left(\frac{3\,(\tau-t)\,\sqrt{a}}{\sqrt{m\gamma\,(1-e^2)}}e\sin[f]+\frac{a^2\,(1-e^2)}{m\gamma}\left(\frac{2}{1+e\cos[f]}-\frac{\cos[f]}{e}\right)\right)R+ \quad ,$$
$$\left(\frac{3\,(\tau-t)\,\sqrt{a}}{\sqrt{m\gamma\,(1-e^2)}}(1+e\cos[f])+\frac{a^2\,(1-e^2)}{m\gamma}\left(\frac{\sin[f]\,(2+e\cos[f])}{e\,(1+e\cos[f])}\right)\right)T \quad .$$

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One should note that equations (31.01.17), (31.01.18) can be derived without any reference to the Kepler natuare of the problem.

It should be noted that equation (31.02.00.02) contains explicitly the time, which causes practical problems with this equations since the right-hand term grows with time.

It is clear from the previous equations that:

- R affects: E;
- T affects: E and, in addition, both $\boldsymbol{\ell}$ and ℓ;
- N affects: $\boldsymbol{\ell}$ (but not ℓ).

As far as the orbit parameters are concerned, it is clear from the previous equations that:

- R affects: a, e, ω and τ;
- T affects: a, e, ω and τ;
- N affects: Ω, ω and i.

31.01.01.01 Secular Perturbations From Lagrange Planetary Equations

In many cases the disturbing forces are very small with respect to the main Kepler force and therefore also the changes of the orbital elements are very small during the change of the true anomaly by 2π, the periond of the particle orbit. In this case one need not calculate perturbation equations exactly but one can average during one period.

The use of equations from (31.01.15) to (31.02.00.02) to obtain the secular (that is non-periodic) perturbation effects must go as follows. The right-hand side of equations (31.01.15) to (31.02.00.02) must be time-averaged over one period of the unperturbed orbit in order to obtain the change in the parameters during one unperturbed orbit. Next the updated parameters are used to build the new orbit with the new parameters. This means, in particular, that during the integration the parameters a, e, i, Ω, ω and τ, appearing in the right-hand sides of equations (31.01.15) to (31.02.00.02), must be kept constant in the averaging. The procedure is, obviously, an approximate one, but this is the core of the perturbation method. In fact the exact procedure would require to evaluate the effects on the orbit by using the real orbit, instead of the unperturbed one. As the real orbit is unknown, and it is what we need to determine, it is only possible to evaluate the approximate effect on the unperturbed orbit.

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31.02 Examples and Physical Applications

The following general comments apply.

Whenever the perturbing force is given by external static gravitational fields the mechanical energy is conserved,

as the perturbing field is conservative. Therefore the major axis of the osculating ellipse does not change (secularly). In general the presence of dissipative effects (for instance drag or deformation of extended bodies) will make the mechanical energy not conserved.

When the initial orbit is a circumference the application might be tricky. In fact the eccentricity function (29.04.16) is not differentiable for a zero value of the eccentricity, due to the presence of the square-root function. Moreover, the present method is slightly perverse in the case the initial orbit is a circumference, because the method would amount to approximate a circumference with an ellipse at any time. This would imply that the particle is always at the periastron (or apoastron) of the osculating ellipse, because its velocity is purely transverse.

Note that averaging of equations from (31.01.15) to (31.02.00.02) must be done in time and not in anomaly. Therefore in the present context when one talks about average the meaning is that one is taking the average on an integer number of periods of the unperturbed orbit. The use of equation (31.01.08) is useful to change the variable time to anomaly and vice-versa.

One should make sure to avoid confusion between the transverse component of the force and the component of the force parallel to the velocity, as they are, in general, different. Moreover, one has to be very careful not to confuse the transverse and normal components with respect to the osculating orbit with the $\mathbf{e}_\phi$ and $\mathbf{e}_z$ components with respect to the axis of symmetry of the field, if any.

Finally one should keep in mind the meaning of the osculating orbit: it is the Keplerian orbit which would be followed by the mass m in case one might suddenly switch off the perturbation.

31.02.00.01 Non-Spherical Static Gravitational Field With Cylindrical Symmetry

Introduce a cylindrical Coordinate System, (r, ϕ and z) with the z axis along the axis of cylindrical symmetry.

Energy is conserved and therefore the semi-major axis is constant. Note that this implies that the averaging of equation (31.01.15) will give zero.

The component of the angular momentum along the axis of symmetry (the z axis), ℓ_z, is conserved because the field is axisymmetric and therefore there is no component of $\mathbf{f}_\mathrm{P}$ in the ϕ direction. However, in general, neither the angular momentum nor its module are conserved. In fact (see equation (31.01.14)) the tangential component of the force is not zero in general. This can be easily shown when one considers that the field has, in general, a non null component in the meridional planes (planes passing by the z axis). As the orbit is in general inclined this component has in general a non null projection along the transverse direction to the orbit.

It can be shown that the both the eccentricity and inclination stay constant on average (that is they have no secular changes). However, they both have short period periodic changes.

On the other hand both the periastron and the line of nodes will show secular changes.

Some more conclusions can be easily found in some special cases.

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Circular Orbits

Some more conclusions can be easily drawn when the orbit is circular. In this case the averaging can be done also in the anomaly as this is equivalent to averaging in time.

The absolute value of the angular momentum does not change on average as equation (31.01.14) shows when it is averaged on a period of a circular orbit, due to the fact that the transverse component of the prturbing force, T, is periodic in the anomaly with period π.

The normal component of the perturbing force is an odd function of φ, changing its sign at $\varphi = 0, \pm\pi$ and therefore Δi will be zero during one period, as equation (31.01.17) shows (no secular term), beacuse the absolute value of the angular momentum does not change, on average. On the other hand, for the same reason, the line of nodes will precess, as it follows from (31.01.18), when the normal component of the force is an odd function of φ, beacuse the absolute value of the angular momentum does not change, on average.

In the case of a circular orbit the longitude of the perihelion and the time of passage at perihelion have non meaning.

Elliptic Orbit in the Equatorial Plane

Some more conclusions can be easily drawn when the orbit is elliptic in the equatorial plane. The case of elliptic orbits in the equatorial plane gives a central force given by an inverse square plus an inverse quartic force.

The angular momentum and the z component of it are conserved. As both energy and angular momentum are conserved the eccentricity does not change.

The orbit plane stays the same, because there is no force normal to the orbital plane.

The periastron precesses, as explained in section § 29.06.12.

Elliptic Orbit above the POles

31.02.00.02 Atmospheric Drag

Assume an atmospheric drag force of type:

$$\mathbf{f}_{\text{D}} = -k[r]v^{\alpha}\mathbf{v} \qquad k[r] > 0 \quad ,$$

→ 1626 1627

where the dependence $k[r]$ comes from the changing of the density of the atmosphere. Note that a special case of drag due to ElectroMagnetic waves absorption and emission, the method discussed in present section is applicalbe to, is givne by the Poynting-Robertson effect (read § 61.09.12).

The mechanical energy is dissipated and therefore the size of the orbit shrinks.

The drag force has both a radial and a transverse component, in general. Both $\boldsymbol{\ell}$ and ℓ change, and in particular ℓ_z changes.

It can be shown, from equation (31.01.16), that the eccentricity decreases down to zero.

It can be shown that for large eccentricities the decrease of the aphelion distance is larger than the decrease of the perihelion distance.

In case of a circular orbit the orbit will stay circular because if any perturbation would make it rise above zero it will go back to zero again.

As the normal component is zero the inclination and lines of nodes do not change, that is the orbit plane is fixed in space.

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31.02.00.03 Tides

Assume that the rotational angular velocity of the primary (planet) and the orbital angular velocity of the secondary (satellite) are parallel (along the z axis), and that both have a common sense of rotation and consider the common case when the former is larger than the latter.

In this case the tidal deformation provides a positive tangential component of the force on the satellite plus a radial component but no normal component of the force exist.

Therefore both the mechanical energy of the satellite alone and the angular momentum component along z increase.

The plane of the orbit does not change (inclination and line of nodes).

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31.03 Numerical Solution of Problems in Celestial Mechanics

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1-9.6,1-9.7||

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31.04 Exercises Problems and Physical Applications

31.04.01 Gravitational Attraction Between Spherically Symmetric Finite-Size Bodies

We want to demonstrate that the gravitational attraction between two spherically symmetric finite-size bodies can be calculated by assuming the mass of the two bodies of finite size is concentrated at their centers. This is not evident a priori because, even if the field produced by one body is given by the field produced by a point mass at the center, this field is applied to a finite-size body and the total force is given by the sum of all the forces acting on the different infinitesimal masses making up the body. The forces on the different infinitesimal masses are, in general, different.

First consider an isolated system made of one spherically symmetric finite-size body and one external point mass. The force on the point mass is given by Newton law of gravitation for the two equivalent point masses, because the body is spherically symmetrical. Therefore, as the sum of internal forces is zero, also the force exerted by the mass point on the finite size body is given by Newton law for the two equivalent point masses. Therefore the total force of a point mass applied to a spherically symmetrical body can be calculate from Newton law of gravitation for the two equivalent point masses.

It then follows that also the gravitational attraction between two spherically symmetric finite-size bodies can be calculated from Newton law of gravitation for the two equivalent point masses because the field produced externally by each of them is the same as the field of a point charge.

31.04.02 Roche Limit

This § is referenced at pages:
[1634, 1634]

Consider a satellite of mass m and radius d revolving around a central planet of mass M on a circular orbit of radius R. Assume that the angular velocities of rotation and revolution of the satellite are parallel vectors. Assume that the satellite is co-rotating, that is its rotation period is equal to the period of revolution around the planet, just like the Moon around the Earth. Let the radial axis, r, be directed in the radial direction, starting from the center of the planet towards the center of the satellite. Assume that both the satellite and planet have a spherically symmetrical mass distribution.

Consider a small mass μ at the surface of the satellite, just along the line joining the center of the planet to the center of the satellite (radial axis r). Consider both cases when the small mass lies on the side of the planet, in one case, and when the small mass lies on the opposite side with respect to the planet, in the other case. A constraint force, $\mathbf{N}$, directed along the line joining the center of the planet to the center of the satellite, acts on the mass, a radial force, oriented towards the exterior of the satellite as long as the small mass is just lying on the land; this is just the normal force keeping the small mass in equilibrium with respect to the gravitational attraction of the satellite. In this case, for the convention chosen, when we consider the case of the small mass lying on the side of the planet, the projection of $\mathbf{N}$ along the radial axis originating from the satellite center will be negative. This force might even have a positive component, in the case a contribution of a cohesion force exists capable to keep the mass from escaping. In the other case of the small mass lying on the opposite side with respect to the planet, the forces will be inverted. With this convention we will denote by N the component of the force $\mathbf{N}$ radial with respect to the center of the satellite. This is positive if and only if the small mass is just lying on the satellite surface while it will be negative if there is a cohesion force keeping the small mass from escaping high on the satellite surface.

We want to study the equilibrium condition of the small mass and show under which conditions a cohesive force is required to keep the mass from escaping from the satellite. This problem is related to the breaking of a satellite due to the tidal force of a planet.

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Assume: $\mu \ll m \ll M$ and $d \ll R$. The simplifying hypothesis $m \ll M$ is done in order to simplify the dynamics of the problem, as it allows one to assume that the small satellite is just revolving around the center of the large planet.

Let ω be the angular velocity of the revolution of the satellite around the planet. The orbit of the satellite around the planet satisfies, according to the **Cardinal Equations** :

$$\omega^2 R = \frac{\mathcal{G}M}{R^2} \quad .$$

Study the problem from the non-Inertial Reference Frame co-rotating with the satellite. The projection of the first **Cardinal Equation** on the r axis, applied to the small mass μ at equilibrium, in the non-Inertial Reference Frame co-rotating with the satellite, reads:

$$-N + \frac{\mathcal{G}m\mu}{d^2} - \frac{\mathcal{G}M\mu}{(R-d)^2} + \mu\omega^2 (R-d) = 0 \qquad \text{if the mass lies on the side of the planet} \qquad (31.04.01)$$

→ 1632

$$+N - \frac{\mathcal{G}m\mu}{d^2} - \frac{\mathcal{G}M\mu}{(R+d)^2} + \mu\omega^2 (R+d) = 0 \qquad \text{if the mass lies on the opposite side with respect to the planet} \quad .$$

(31.04.02)

→ 1632

The unknown constrain force **N** reads:

$$N = +\frac{\mathcal{G}m\mu}{d^2} - \frac{\mathcal{G}M\mu}{(R-d)^2} + \mu\omega^2 (R-d) \qquad \text{if the mass lies on the side of the planet} \qquad (31.04.03)$$

$$N = +\frac{\mathcal{G}m\mu}{d^2} + \frac{\mathcal{G}M\mu}{(R+d)^2} - \mu\omega^2 (R+d) \qquad \text{if the mass lies on the opposite side with respect to the planet} \quad .$$

(31.04.04)

Note that:

- $N > 0$, force oriented towards the exterior of the satellite, implies that the surface of the satellite is just sustaining the weight of the mass;
- $N < 0$ force oriented towards the interior of the satellite, implies that the surface of the satellite must keep the mass from getting high over the satellite surface (a cohesion force is required).

One can develop equations (31.04.01), (31.04.02), to first-order in d/R, by exploiting the fact that $d \ll R$:

$$\begin{aligned} N &= +\frac{\mathcal{G}m\mu}{d^2} \pm \frac{\mathcal{G}M\mu}{(R \pm d)^2} \mp \mu\omega^2 (R \pm d) \quad , \\ N &\simeq +\frac{\mathcal{G}m\mu}{d^2} \pm \frac{\mathcal{G}M\mu}{R^2}\left(1 \mp 2\frac{d}{R}\right) \mp \mu\omega^2 \left(1 \pm 2\frac{d}{R}\right) = \quad , \\ &+ \frac{\mathcal{G}m\mu}{d^2} \pm \frac{\mathcal{G}M\mu}{R^2}\left(1 \mp 2\frac{d}{R}\right) \mp \frac{\mathcal{G}M\mu}{R^2}\left(1 \pm 2\frac{d}{R}\right) = \quad , \\ &+ \frac{\mathcal{G}m\mu}{d^2} + \frac{\mathcal{G}M\mu}{R^2}\left(-3\frac{d}{R}\right) \quad . \end{aligned} \qquad (31.04.05)$$

→ 1632

The first term in equation (31.04.05) is just the weight due to the satellite, and the second one is a correction factor, going in the opposite direction with respect to the weight, that is tending to lift the mass μ from the satellite surface. This is true in both locations of the test mass μ as the equations show that the different signs cancel.

Coherently with the chosen conventions N is dominated by a positive term, to counteract the gravitational attraction of the satellite. However, the second term is negative, decreasing the effective weight of the test mass. Under appropriate conditions, the corrective term can exceed the first term, showing that a cohesion force is required to keep the small mass from escaping from the gravitational attraction of the satellite.

The condition of weightlessness is $N < 0$:

$$N \leq 0 \implies \boxed{\frac{R}{d} \leq \left(\frac{3M}{m}\right)^{1/3} \text{ for zero cohesion force}} \quad . \qquad (31.04.06)$$

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The overall effect is to deform the satellite in the direction of the line joining the centers of the planet and satellite.

In a real case one should also account for:

- the rotation of the satellite, if not geo-synchronous;
- the real geometry of the orbits (elliptic orbits, non parallel angular velocities of rotation and revolution of the satellites);
- tidal deformations of the planet and satellite, and effect on the tidal bulge of the rotation of the planet;
- the real cohesion force.

Note finally that the analysis is a purely static one, assuming that the satellite is co-rotating.

The problem has applications in astrophysics (dynamics of binary stars) as well as in planetary mechanics (tidal disruption of a satellite).

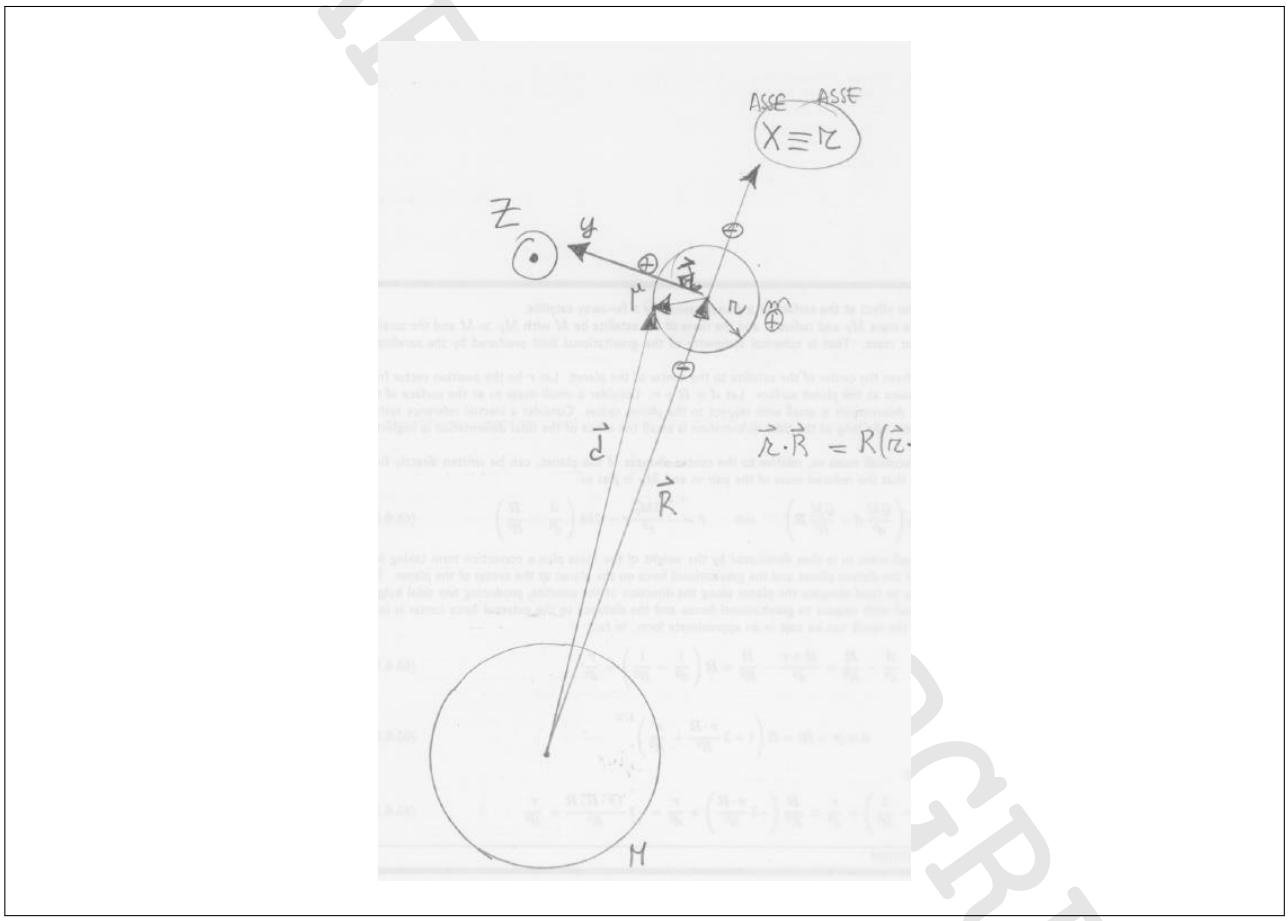

Figure 31.13: FIGURE

31.04.03
Roche Limit for Systems of Similar Size

This § is referenced at pages:
[Never referenced.]

Spherical planets of radii: r_0 and R_0.
Masses: m and M.
Circular orbits about the Center-Of-Mass.
Distances from the center of the planets to the Center-Of-Mass: r and R, with $mr = MR$.

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Distance betwen the centters of the two planets: $D \equiv r + R$.

$$\boxed{\omega^2 = \frac{\mathcal{G}}{D^2}\frac{M}{r} = \frac{\mathcal{G}}{D^2}\frac{m}{R} = \frac{\mathcal{G}}{D^3}(m+M)} \quad .$$

$$-N + \frac{\mathcal{G}m\mu}{r_0^2} - \frac{\mathcal{G}M\mu}{(D-r_0)^2} + \mu\omega^2(r-r_0) = 0 \qquad \text{if the mass lies on the side of the planet} \qquad (31.04.07)$$

$$+N - \frac{\mathcal{G}m\mu}{r_0^2} - \frac{\mathcal{G}M\mu}{(D-r_0)^2} + \mu\omega^2(r+r_0) = 0 \qquad \text{if the mass lies on the opposite side with respect to the planet} \quad . \qquad (31.04.08)$$

$$N = +\frac{\mathcal{G}m\mu}{r_0^2} - \frac{\mathcal{G}M\mu}{(D-r_0)^2} + \mu\omega^2(r-r_0) \qquad \text{if the mass lies on the side of the planet} \qquad (31.04.09)$$

$$N = +\frac{\mathcal{G}m\mu}{r_0^2} + \frac{\mathcal{G}M\mu}{(D-r_0)^2} - \mu\omega^2(r+r_0) \qquad \text{if the mass lies on the opposite side with respect to the planet} \quad . \qquad (31.04.10)$$

$$\begin{aligned} N = &+\frac{\mathcal{G}m\mu}{r_0^2} - \frac{\mathcal{G}M\mu}{(D-r_0)^2} + \mu\omega^2(r-r_0) \simeq \quad , \\ &+\frac{\mathcal{G}m\mu}{r_0^2} - \frac{\mathcal{G}M\mu}{D^2}\left(1+2\frac{r_0}{D}\right) + \frac{\mathcal{G}M\mu}{D^2}\left(1-\frac{r_0}{r}\right) = \quad , \\ &+\frac{\mathcal{G}m\mu}{r_0^2} - \frac{\mathcal{G}M\mu}{D^2}\left(\frac{r_0}{r}+2\frac{r_0}{D}\right) \end{aligned} \qquad (31.04.11)$$

$$\begin{aligned} N = &+\frac{\mathcal{G}m\mu}{r_0^2} + \frac{\mathcal{G}M\mu}{(D-r_0)^2} - \mu\omega^2(r+r_0) \simeq \quad , \\ &+\frac{\mathcal{G}m\mu}{r_0^2} + \frac{\mathcal{G}M\mu}{D^2}\left(1-2\frac{r_0}{D}\right) - \frac{\mathcal{G}M\mu}{D^2}\left(1+\frac{r_0}{r}\right) = \quad , \\ &+\frac{\mathcal{G}m\mu}{r_0^2} - \frac{\mathcal{G}M\mu}{D^2}\left(\frac{r_0}{r}+2\frac{r_0}{D}\right) \quad . \end{aligned} \qquad (31.04.12)$$

In the limit $m \ll M$ the latter results reduce to the results of problem § 31.04.02.

31.04.04

Tidal Potential

This problem represents the generalisation of problem § 31.04.02.

We want to determine the tidal effects at the surface of a first small spherically symmetrical body (conventionally called the satellite), with mass m and radius r, caused by a second far-away spherically symmetrical body (conventionally called the planet) with mass M; the distance between the centers the two bodies is D.

Assume that the satellite is spinning with angular velocity perpendicular to the plane of the two orbits.

Assume, for simplicity, $m \ll M$. Assume $r \ll D$.

Thanks to the hypotheses the planet can be considered just as a distant point mass. We are going to calculate the effects of the fact that the field produced by the planet is not uniform on the whole volume of the satellite, because the satellite has finite dimensions.

Assume that the planet is at rest with respect to some Inertial Reference Frame , with a Coordinate System centered at the center of the planet. Let $\mathbf{D}$ be the position vector from the center of the planet to the center of the satellite. Let $\mathbf{r}$ be the position vector from the center of the satellite. Let $\mathbf{d} \equiv \mathbf{D} + \mathbf{r}$ be the location of a small mass μ ($\mu \ll m$ and $\mu \ll M$) with respect to the center of the planet. Even if it is not strictly necessary we might assume, for concreteness, that the mass lies at the surface of the satellite. Let us assume that the tidal deformation is very small with respect to the radius of the satellite, that is consider the satellite spherical even if it is subject to the tidal force we are going to calculate.

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The two bodies are rotating around the common Center-Of-Mass with an angular angular frequency given by:

$$\frac{Mm}{m+M}\omega^2 D = \frac{\mathcal{G}Mm}{D^2} \Longrightarrow \omega^2 = \frac{\mathcal{G}\,(m+M)}{D^3} \simeq \frac{\mathcal{G}M}{D^3} \quad .$$

The equation of motion of the small mass μ with respect to the center of mass of the satellite can be written directly from equation (24.12.01), taking into account that the reduced mass of the pair satellite and small mass is just μ:

$$\mu\ddot{\mathbf{r}} = -\frac{\mathcal{G}\mu m}{r^3}\mathbf{r} + \mathbf{N} + \mu\left(-\frac{\mathcal{G}M}{d^3}\mathbf{d} + \frac{\mathcal{G}M}{D^3}\mathbf{D}\right) \Longrightarrow \ddot{\mathbf{r}} = -\frac{\mathcal{G}m}{r^3}\mathbf{r} + \frac{\mathbf{N}}{\mu} + \mathcal{G}M\left(-\frac{\mathbf{d}}{d^3} + \frac{\mathbf{D}}{D^3}\right) \quad .$$

The force $\mathbf{N}$ is the force constraining the small mass at the surface of the satellite, a constraint force to be determined from the equation of motion. At equilibrium $\ddot{\mathbf{r}} = 0$.

Note that the above equation is the equation for the relative coordinate $\mathbf{r}$ of the small mass with respect to the center of the satellite, and not the equation of motion of the small mass itself. This is indeed the correct expression to use, as we are intereted in the motion of the small mass with respect to the satellite. Moreover, we want to understand the change in the total force acting on the small mass when the third body (the planet) is present, with respect to the satellite-generated weight of the small mass alone.

The resulting force on the small mass μ is thus dominated by its weight and by the constraint force $\mathbf{N}$, plus a correction term taking into account the gravitational effects of the distant planet and the overall gravitational force of the planet on the satellite. The corrective term tends to elongate the satellite along the direction of the line connecting the centers of the planet and satellite, producing two tidal bulges.

The equation is the equation for the relative coordinate $\mathbf{r}$ of the small mass with respect to the center of the satellite. It shows that the forces on the small mass are, firstly, the weight of the mass and the constraint force $\mathbf{N}$. However, the presence of a third body manifests itself via two new additional terms, as it is shown by the fact that the two additional terms are proportional to the mass M of the third body (the planet). The two additional terms are the direct force on the small mass by the third body (the planet), $-\mathcal{G}M\frac{\mathbf{d}}{d^3}$, plus the direct force on the satellite by the planet with reversed sign, $\mathcal{G}M\frac{\mathbf{D}}{D^3}$, affecting the motion of the Center-Of-Mass of the satellite itself. As the force applied by the small mass on the satellite is negligible with respect to the latter force applied by the planet on the satellite, the latter term is just the acceleration of the satellite with respect to same Inertial Reference Frame . It is recognised that the acceleration of the satellite, when carried on the left side and summed to the relative acceleration of the small mass with respect to the Center-Of-Mass of the satellite, gives the absolute acceleration of the small mass with respect to same Inertial Reference Frame , thus expressing the first **Cardinal Equation** applied to the small mass. In case the mass of the planet is much large than the mass of the satellite and the planet is fixed with respect to some Reference Frame the aforementioned acceleration can be taken as the acceleration with respect to the center of the planet. In the additional simplifying assumption of a circular orbit the centripetal acceleration can be expressed as $-\omega^2\mathbf{D} = \mathcal{G}M\frac{\mathbf{D}}{D^3}$.

As long as tidal forces are small with respect to gravitational forces and the distance to the external force center, the planet, is large with respect to the satellite radius the result can be cast in an approximate form. In fact

$$\frac{\mathbf{d}}{d^3} - \frac{\mathbf{D}}{D^3} = \frac{\mathbf{D}+\mathbf{r}}{d^3} - \frac{\mathbf{D}}{D^3} = \mathbf{D}\left(\frac{1}{d^3} - \frac{1}{D^3}\right) + \frac{\mathbf{r}}{d^3} \quad ,$$

and

$$d \equiv |\mathbf{r}+\mathbf{D}| = D\left(1 + 2\frac{\mathbf{r}\cdot\mathbf{D}}{D^2} + \frac{r^2}{D^2}\right)^{1/2} = D\left(1 + \frac{\mathbf{r}\cdot\mathbf{D}}{D^2}\right) + \mathcal{O}\left(\frac{r}{D}\right)^2 \simeq D\left(1 + \frac{\mathbf{r}\cdot\mathbf{D}}{D^2}\right) \quad .$$

The binomial approximation gives, to first-order in $\frac{r}{D}$:

$$\mathbf{D}\left(\frac{1}{d^3} - \frac{1}{D^3}\right) + \frac{\mathbf{r}}{d^3} \simeq \frac{\mathbf{D}}{D^3}\left(-3\frac{\mathbf{r}\cdot\mathbf{D}}{D^2}\right) + \frac{\mathbf{r}}{d^3} = -3\frac{(\mathbf{r}\cdot\mathbf{D})\,\mathbf{D}}{D^5} + \frac{\mathbf{r}}{D^3} \quad .$$

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Note that the term $\frac{\mathbf{r}}{d^3}$ is already a first-order term in $\frac{r}{D}$ and therefore only the zero order approximation for d is used in developing it.

Note that the result has the functional expression of a **Dipole** field calculated at the position **D** produced by a Dipole moment proportional to **r**.

It follows:

$$\ddot{\mathbf{r}} \simeq -\frac{\mathcal{G}m}{r^3}\mathbf{r} + \frac{\mathbf{N}}{\mu} - \mathcal{G}M\left(\frac{\mathbf{r}}{D^3} - 3\frac{(\mathbf{r}\cdot\mathbf{D})\,\mathbf{D}}{D^5}\right) \quad .$$

Note that the tidal force changes the value of the constraint force **N** required to keep the body in equilibrium.

The potential at the surface of the satellite is thus:

$$\Phi[\mathbf{r}] = -\frac{\mathcal{G}m}{r} + \frac{\mathcal{G}M}{2D^3}\left(r^2 - 3\frac{(\mathbf{r}\cdot\mathbf{D})^2}{D^2}\right) \quad .$$

Note that this potential assumes that the satellite is of spherical shape, that is its deformation due to the tidal forces is negligible. As a successive approximation one should calcolate the deformation of the satellite due to the tidal force and after that one should calcolate the change of the potential produced by the satellite due to the deformation. Also note that the expression for the potential is not limited to the surface of the satellite.

As a fluid body will tend to reach an equilibrium situation such that its free surface is an equipotential surface a fluid satellite will tend to change its shape accordingly.

Choose the z axis perpendicularly to the plane where the orbits of the planet and satellite lies, that is along the rotation axis of the planet. Choose the x axis pointing to the center of the planet. Introducing a spherical Coordinate System the potential can be re-written as:

$$\Phi[\mathbf{r}] = -\frac{\mathcal{G}m}{r} + \frac{\mathcal{G}M}{2r}\frac{r^3}{D^3}\left(1 - 3\frac{(\mathbf{r}\cdot\mathbf{D})^2}{r^2D^2}\right) = -\frac{\mathcal{G}m}{r} - \frac{\mathcal{G}M}{r}\left(\frac{r}{D}\right)^3\left(\frac{3\sin^2[\theta]\cos^2[\phi] - 1}{2}\right) \quad .$$

The equilibrium shape of ideal planet covered by an ocean would be given by a constant potential.

Note, finally, that the tidal force is given by the gradient of the gravitational force exerted by the planet, and therefore it goes as an inverse-cube force. Therefore it rises at short distances more than the gravity force. Tides generated by different bodies must thus be compared by comparing the gradient of the force, not the force itself.

31.04.05
Tidal Evolution of the Earth-Moon System

This § is referenced at pages:
[1587]

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The Earth-Moon system can be considered, to a first approximation, as subject to a small overall torque due to external bodies, at least on average. Thus the total angular momentum is nearly constant. Note that the same argument does not apply equally well to the overall force due to external bodies, because that is driven by the Sun gravitational force and therefore the linear momentum is not conserved, not even in an approximate way.

To study the evolution of the system one can use a number of simplifying assumptions, useful to simplify the calculations while giving, at the same time, a clear identification of the physical mechanisms coming into play.

- Describe the situation from a Reference Frame tied to the Earth center, which is assumed to be approximately inertial.
- The Moon orbits around the Earth center on a circular orbit of radius r in the ecliptic plane.

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- The circular orbit of the Moon is slowly changing its radius, on a very long time scale compared to the orbital period.
- The spin angular momentum of the Earth, $\mathbf{S} = I\boldsymbol{\omega}$, is assumed to be parallel to the orbital angular momentum of the Moon around the Earth, $\mathbf{L}$. Moreover, one should keep in mind that the angular velocity of the Moon rotation is equal to the angular velocity of its revolution around the Earth. These assumptions are justified by the fact that the Earth spin precesses about the normal to the ecliptic plane with a period of about $2.5{\cdot}10^5$ years while the plane of the Moon orbit precesses similarly with a period of about 19 years. Therefore, on the long-time average, both $\mathbf{S}$ and $\mathbf{L}$ are perpendicular to the ecliptic plane.
- The total angular momentum, $\mathbf{J}$ is constant (we are neglecting the Solar tidal drag) and the spin angular momentum of the Moon is negligible with respect to $\mathbf{L}$ and $\mathbf{S}$. Thus: $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

We want to determine the evolution of the Earth-Moon system due to the effect of the tidal drag.

Let the Earth mass be M, its momentum of inertia with respect to its rotation axis be I, the Moon mass be m and the velocity of the Moon Center-Of-Mass be $\mathbf{v}$. The following numerical data are useful.

Earth Radius: $R = 6.3781{\cdot}10^6$ m.
Earth Mass: $M = 5.974{\cdot}10^{24}$ kg.
Moon Mass: $m = 7.353{\cdot}10^{22}$ kg.
Earth-Moon distance: $r = 3.841{\cdot}10^8$ m.
Earth rotation sidereal period: $T_{\mathtt{E}} = 86164$ s.
Moon revolution and rotation sidereal period: $T_{\mathtt{M}} = 2.361{\cdot}10^6$ s.

Moment of inertia of the Earth with respect to its rotation axis: $I = 0.3335MR^2 \neq (2/5)\,MR^2$. Note the significant difference with respect to the calculated value for a homogeneous spherical mass distribution: it is smaller than expected because the Earth density increases going towards the center of the Earth.

Let us use an Inertial Reference Frame with the Earth at rest and let z be the axis perpendicular to the Moon orbit. Let $\omega \equiv \omega_z$ be the component of the spin angular velocity of the Earth on the z axis (which might have any sign). Let $\Omega \equiv \Omega_z$ be the component of the orbital angular velocity of the Moon on the z axis (which might have any sign). Let $L \equiv L_z$ be the component of the orbital angular momentum of the Moon on the z axis (which might have any sign).

The total mechanical energy of the Earth-Moon system relative to the Center-Of-Mass of the Earth is:

$$E = \frac{1}{2}mv^2 - \frac{\gamma}{r} + \frac{1}{2}I\omega^2 \qquad \gamma \equiv \mathcal{G}Mm \quad ,$$

neglecting the spin energy of the Moon. Note that thanks to the hypothesis that the Moon orbits around the Earth center on a circular orbit this is actually the mechanical energy in the [glo:CMRF]Center-of-Momenta Reference Frame of the Earth-Moon system, following **Koënig theorem** for kinetic energy. Moreover, the kinetic energy of the motion of the Center-Of-Mass of the Earth-Moon system is driven by the external forces as it just depends on the total linear momentum of the Earth-Moon system, which is determined by the first **Cardinal Equation** .

From the first **Cardinal Equation** applied to the Moon one finds:

$$-m\frac{v^2}{r} \equiv -m\Omega^2 r = -\frac{\gamma}{r^2} \equiv -\frac{\gamma}{r^2} \quad , \tag{31.04.13}$$

→ 1637 1638

in terms of the angular velocity of the revolution of the Moon around the Earth, Ω.

It follows that:

$$\left\{L \equiv mrv \neq \text{constant} \qquad \text{and} \qquad mrv^2 = \gamma = \text{constant}\right\} \Longrightarrow \left\{v = \frac{\gamma}{L} \qquad r = \frac{L^2}{m\gamma}\right\} . \tag{31.04.14}$$

→ 1637 1638

As a consequence of (31.04.13), (31.04.14) we can express both Ω and ω as a function of either r alone or as a function of L alone:

$$\Omega = \frac{2\pi}{T_{\mathtt{M}}} = 2.66170{\cdot}10^{-6}\ \text{rad/sec} \quad ,$$

$$\Omega[r] = \sqrt{\frac{\gamma}{mr^3}} = \qquad \Omega[L] = \frac{m\gamma^2}{L^3} \quad ,$$

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and:

$$\omega = \frac{2\pi}{T_{\mathrm{E}}} = 7.29212{\cdot}10^{-5}\ \mathrm{rad/sec} \quad ,$$
$$\omega[L] = \frac{J-L}{I} \qquad \omega[r] = \frac{J-\sqrt{m\gamma r}}{I} \quad ,$$

→ 1638 1639

all these equations being ready for later use.

Moreover, as a consequence of (31.04.13), (31.04.14), the mechanical energy can be rewritten, by eliminating r, v and ω in favour of the angular momenta, as:

$$E[L] = -K + \frac{S^2}{2I} = +\frac{\Phi}{2} + \frac{S^2}{2I} = -\frac{1}{2}mv^2 + \frac{S^2}{2I} = -\frac{\gamma}{2r} + \frac{S^2}{2I} = -\frac{m\gamma^2}{2L^2} + \frac{S^2}{2} = -\frac{m\gamma^2}{2L^2} + \frac{(J-L)^2}{2I} \quad . \tag{31.04.15}$$

→ 1638

The expression (31.04.15) allows one to rewrite the mechanical energy as a function of either the Moon distance or the Moon velocity, by using equation (31.04.14).

If tidal friction is present the mechanical energy is not constant and it is dissipated into internal energy of the system. Moreover, only **J** is constant, not **L** nor **S** separately. The system will continue to dissipate the mechanical energy until this is compatible with the conservation of angular momentum. The ultimate state will feature the lowest possible mechanical energy compatible with the constraints (**J** constant). Therefore we look for the minimum of the energy:

$$\frac{\mathrm{d}E[L]}{\mathrm{d}L} = \frac{m\gamma^2}{L^3} - \frac{J-L}{I} = \Omega - \omega \quad ,$$
$$\frac{d^2E[L]}{\mathrm{d}L^2} = -\frac{3m\gamma^2}{L^4} + \frac{1}{I} \quad .$$

The extrema of the mechanical energy at fixed total angular momentum are thus characterized by co-rotation of the Moon: $\Omega = \omega$. The result is actually a very general theorem, stating that at thermal equilibrium a system must rotate as a rigid-body.

Using the dependence of the angular velocities on the distance, equations (31.04.05), (31.04.05), one can solve the equation for the distance of minimum energy.

The study of the function (31.04.15), defined for $L > 0$, shows the following. The limits at the boundary of the domain are:

$$\lim_{L\longrightarrow 0^+} E[L] = -\infty \qquad \lim_{L\longrightarrow +\infty} E[L] = +\infty \quad .$$

The limits at the boundary of the domain of the first derivative are:

$$\lim_{L\longrightarrow 0^+} \frac{\mathrm{d}E[L]}{\mathrm{d}L} = +\infty \qquad \lim_{L\longrightarrow +\infty} \frac{\mathrm{d}E[L]}{\mathrm{d}L} = +\infty \quad .$$

However, the second derivative has one and only one zero, as it is apparent from its expression, at $L_f = \left(3m\gamma^2 I\right)^{1/4}$ (flex point). It should be checked, by using the actual values of the physical parameters, that the minimum of the first derivative, $\frac{\mathrm{d}E}{\mathrm{d}L}[L = L_f]$, is negative in such a way that the function is not monotonically increasing but has one relative maximum at small L and one relative minimum at large L.

The plot of the total mechanical energy of the Earth-Moon system versus the Moon orbital angular momentum is shown in figure 31.14. Note that as $r \propto L^2$, as from equation (31.04.14), the qualitative behavior of $E[r]$ is similar to the qualitative behavior of $E[L]$.

One finds the following numerical values:

$$L_{\mathtt{today}} = 2.887{\cdot}10^{34}\ \mathrm{kg\cdot m^2/s} \quad ,$$
$$L_{\mathtt{MAX}} = 0.438{\cdot}10^{34}\ \mathrm{kg\cdot m^2/s} \quad ,$$
$$L_{\mathtt{MIN}} = 3.472{\cdot}10^{34}\ \mathrm{kg\cdot m^2/s} \quad ,$$
$$E[L_{\mathtt{today}}] = 3.931{\cdot}10^{29}\ \mathrm{J} \quad ,$$
$$E[L_{\mathtt{MIN}}] = -2.615{\cdot}10^{28}\ \mathrm{J} \quad .$$

In all the cases when $\Omega > \omega$ the Moon is orbiting faster than the Earth is rotating, therefore the tidal bulge is delayed with respect to the Moon position and it tends to decrease the orbital angular mo-

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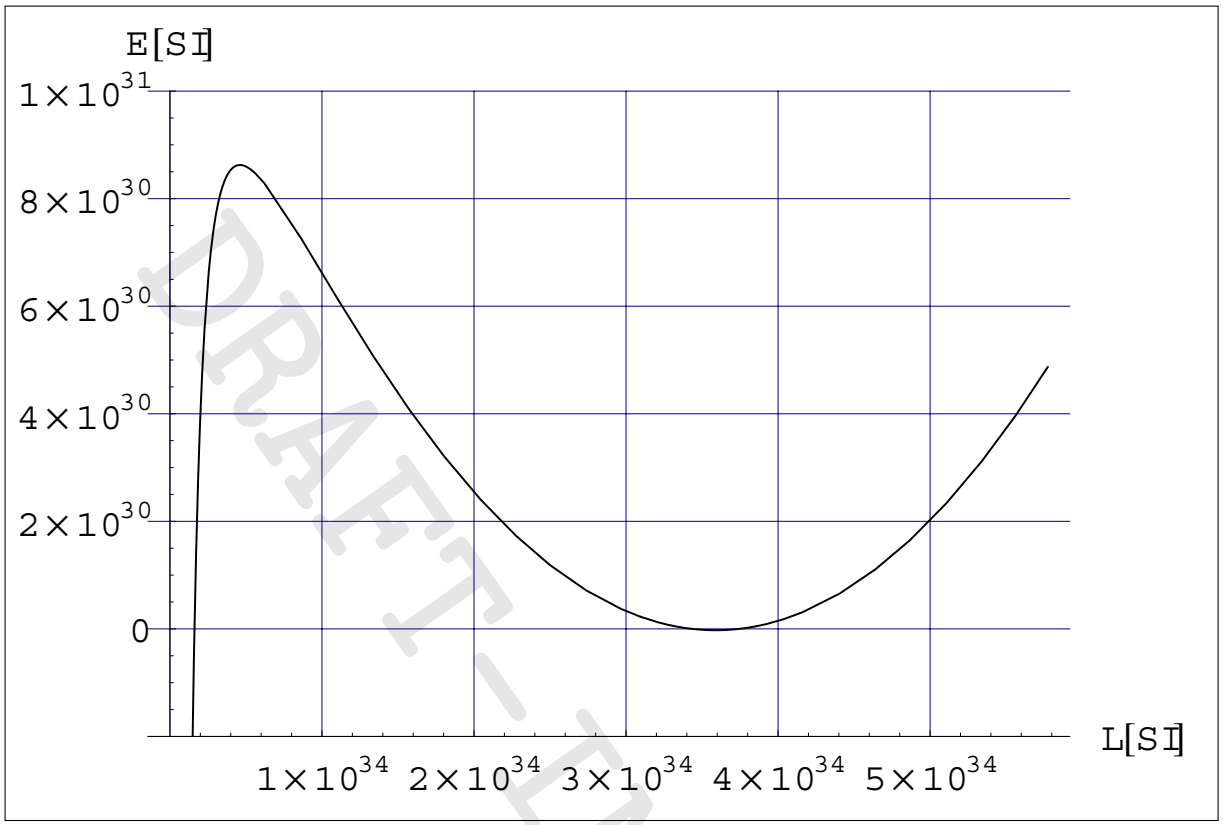

Figure 31.14: Total mechanical energy of the Earth-Moon system versus the Moon orbital angular momentum.

mentum of the Moon:

$$\Omega > \omega \implies \frac{\mathrm{d}L}{\mathrm{d}t} < 0 \implies \frac{\mathrm{d}r}{\mathrm{d}t} < 0 \quad .$$

In all the cases when $\Omega < \omega$ the Moon is orbiting slower than the Earth is rotating, therefore the tidal bulge is advanced with respect to the Moon position and it tends to increase the orbital angular momentum of the Moon:

$$\Omega < \omega \implies \frac{\mathrm{d}L}{\mathrm{d}t} > 0 \implies \frac{\mathrm{d}r}{\mathrm{d}t} > 0 \quad .$$

The extremum of the mechanical energy the at the smallest L is a relative maximum, such that the bulk of mechanical energy is in the spin of the Earth, because J is conserved. The extremum of mechanical energy at the largest L is a relative minimum, such that the bulk of mechanical energy is in the orbital angular momentum of the Moon, because J is conserved.

At the extrema, as they are characterized by co-rotation, both the Moon tidal drag on the Earth and the Earth tidal drag on the Moon will vanish. From then on the solar tidal drag (which is neglected in our model) will drive the system.

The minimum is obviously a stable configuration while the maximum is not.

From the current measured value of the lengthening of the Earth day length, $\dot{\omega} = -0.85 \cdot 10^{-21}\ \mathrm{rad/s^2}$, we can derive the value of the rate at which the Moon is receding from the Earth. In fact by taking the time deirvative of equation (31.04.05) and dividing the resulting equiation by r one finds:

$$\dot{S} = I\dot{\omega} = -\sqrt{\frac{m\gamma}{r}}\frac{\dot{r}}{2} \quad ,$$

$$\frac{\dot{r}}{r} = -2\frac{I\dot{\omega}}{\sqrt{m\gamma r}} = -2\frac{I\dot{\omega}}{L} = -2\frac{S\dot{\omega}}{L\omega} \quad .$$

The numerical result gives:

$$\dot{r} \simeq 0.058\ \mathrm{m/year} \quad .$$

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31-001 Perturbed Orbit Equation in Terms of the Intrinsic Unit Vectors

Express the perturbed orbit equation of § 31.01.01 in terms of the intrinsic unit vectors: **tangent**, **normal** and **binormal**. Note that the **normal** unit vector does not correspond to the $\mathbf{e}_N$ unit vector, which is normal to the orbit plane, of § 31.01. On the other hand the **binormal** unit vector corresponds to the $\mathbf{e}_N$ unit vector of § 31.01.

31-002 Equation Derivation

Derive equation (31.01.03) and equation (31.01.04).

31-003 Equation Derivation

Derive equation (31.01.07) and equation (31.01).

31-004 Equation Derivation

Derive equation (31.01.08).

31-005 Equation Derivation

Derive equation (31.01.09) and equation (31.01.10).

31-006 Magnitude of the Quadrupole Moment

Consider a point at a certain distance R from the Center-Of-Mass of an arbitrary mass distribution of total mass M, assumed for simplicity with cylindrical symmetry. Determine the condition on Q ensuring that the **Quadrupole** potential is negligible with respect to the monopole potential at the point.

31-007 Satellite Around an Oblate Planet

Determine the period of rotation T of a satellite in a circular orbit of radius a on the equatorial plane of a planet of mass M, radius R and **Quadrupole** moment Q.

31-008 True and Eccentric Anomaly

Show that the following relation links the true and eccentric anomaly for hyperbolic motion:

$$\tan\left[\frac{f}{2}\right] = \sqrt{\frac{1+e}{1-e}}\tanh\left[\frac{\nu}{2}\right] \quad .$$

Show that the following relation links the true and eccentric anomaly for elliptic motion:

$$\tan\left[\frac{f}{2}\right] = \sqrt{\frac{1+e}{1-e}}\tan\left[\frac{\nu}{2}\right] \quad .$$

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31-009 Kepler Equation

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©|T.J. Osler|An unusual approach to Kepler first law|*Am.J.Phys, , ..., ..., ...Ed., ... 69, 1036 (2001).*|

Use the conservation of angular momentum, and the geometrical interpretation of the eccentric anomaly to show that Kepler equation (31.01.06) applies.

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CHAPTER 32

Introduction to the Mechanics of Continuous Media

32.01 Lagrangian and Eulerian Approaches 1644
32.02 Deformations of a Continuous Medium 1645
32.03 Dynamics of the Continuous Media. 1650
32.04 Tensor of Elasticity and Hooke Law for Homogeneous and Isotropic Materials 1658
32.05 Equation of Motion of an Homogeneous and Isotropic Elastic Body 1660
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32.08 Examples and Physical Applications 1662

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This § is referenced at pages:
[1487, 1658, 2034, 2035]

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32.01 Lagrangian and Eulerian Approaches

This § is referenced at pages:
[1517, 1645]

Lagrangian approach: describe all quantities as a function of the space coordinates at some reference configuration of the continuous medium, $\mathbf{x}$. That is: follow the motion of each macroscopically infinitesimal volume of the continuum. Consider any reference configuration of the continuous medium and label each macroscopically infinitesimal volume of the continuum with its position at the reference configuration. In case of a solid use, for instance, the position of any particle at equilibrium.

Eulerian approach: describe all quantities as a function of the space coordinates of the point where the quantity is to be evaluated: $\mathbf{X}$. That is: fix a point in space a look at the properties there. Consider each point in space and describe what happens at that point irrespectively of which macroscopically infinitesimal volume of the continuum is passing there. It is most suitable for fluids, for which a reference configuration cannot be usually introduced.

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32.02 Deformations of a Continuous Medium

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32.02.01 Geometry of Deformation

Consider a continuous medium, either solid or fluid, in a reference configuration, for instance, but not necessarily, when it is unstrained and at rest in some Inertial frame. Each point of the medium, when it is in the reference configuration, is labeled by its position vector $\mathbf{x}$.

Its position at time t, $\mathbf{X}[\mathbf{x}, t]$, is given by the function:

$$\boxed{\mathbf{X}[\mathbf{x}, t] \equiv \mathbf{x} + \boldsymbol{\xi}[\mathbf{x}, t] \qquad \mathbf{x} \text{ does not depend on time}} \quad . \tag{32.02.01}$$

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The displacement vector is thus:

$$\boxed{\boldsymbol{\xi}[\mathbf{x}, t] \equiv \mathbf{X}[\mathbf{x}, t] - \mathbf{x}} \quad .$$

Note that the dynamical variable is either $\boldsymbol{\xi}[\mathbf{x}, t]$, the displacement vector, or $\mathbf{X}[\mathbf{x}, t]$, the actual position, while $\mathbf{x}$ just labels any point on the continuous medium via its position in the reference configuration.

The Jacobian of the transformation, which is supposed to be locally invertible, is given by:

$$\boxed{\frac{\partial \xi_j}{\partial x_k} \neq 0} \quad .$$

Every function defined for the continuous medium can be expresses either in terms of the initial coordinates $\mathbf{x}$ (the Lagrangian formulation) or in terms of the instantaneous coordinates $\mathbf{X}$ (the Eulerian formulation). Read § 32.01.

32.02.02 Infinitesimal Deformations and the Strain Tensor

Consider either a solid or a fluid continuous medium undergoing infinitesimal deformations. Use as a reference configuration, a configuration where the solid/fluid is unstrained at rest in an Inertial Reference Frame .

Then consider a situation outside equilibrium, under a small strain, such that the linear approximation can be applied. If this is not the case the much more complex theory for finite deformations must be used. Small displacements are always assumed since now on and therefore the linear approximation is always used.

The displacement from the equilibrium position of a point, whose position at equilibrium is at $\mathbf{x}$, is given by the function $\boldsymbol{\xi}[\mathbf{x}]$ (compare with the elementary introduction in section § 27.02).

Refer to figure 32.11.

Consider, at equilibrium, two nearby points $\mathbf{x}$ and $\hat{\mathbf{x}}$, separated by a displacement $\Delta\mathbf{x}$. One must look for the transformed vector of $\Delta\mathbf{x}$, that is $\Delta\mathbf{X}$, in order to understand how locally the continuous medium is deformed. This can be found as follows (the time dependence is omitted for brevity):

$$\mathbf{x} \longrightarrow \mathbf{X}[\mathbf{x}] = \mathbf{x} + \boldsymbol{\xi}[\mathbf{x}] \qquad \mathbf{X}[\mathbf{x}] = \mathbf{x} + \boldsymbol{\xi}[\mathbf{x}] \longrightarrow \mathbf{X}[\mathbf{x}] - \mathbf{x} = \boldsymbol{\xi}[\mathbf{x}] \quad ,$$

$$\hat{\mathbf{x}} \equiv \mathbf{x} + \Delta\mathbf{x} \longrightarrow \begin{cases} \mathbf{X}[\hat{\mathbf{x}}] = (\mathbf{x} + \boldsymbol{\xi}[\mathbf{x}]) + \Delta(\mathbf{x} + \boldsymbol{\xi}[\mathbf{x}]) \\ \mathbf{X}[\hat{\mathbf{x}}] = (\mathbf{x} + \Delta\mathbf{x}) + \boldsymbol{\xi}[\mathbf{x} + \Delta\mathbf{x}] \end{cases} , \qquad \mathbf{X}[\hat{\mathbf{x}}] = \mathbf{x} + \Delta\mathbf{x} + \boldsymbol{\xi}[\mathbf{x}] + \Delta\boldsymbol{\xi}[\mathbf{x}] \longrightarrow \mathbf{X}[\hat{\mathbf{x}}] - \hat{\mathbf{x}} = \boldsymbol{\xi}[\mathbf{x}] + \Delta\boldsymbol{\xi}[\mathbf{x}]$$

with the last equation showing two different, but equivalent, ways to obtain the displacement for $\hat{\mathbf{x}}$.

The basic result for equation (32.02.02) is thus:

$$\boxed{\mathbf{X}[\mathbf{x}] = \mathbf{x} + \boldsymbol{\xi}[\mathbf{x}] \qquad \mathbf{X}[\hat{\mathbf{x}}] = \mathbf{x} + \Delta\mathbf{x} + \boldsymbol{\xi}[\mathbf{x}] + \Delta\boldsymbol{\xi}[\mathbf{x}]}$$

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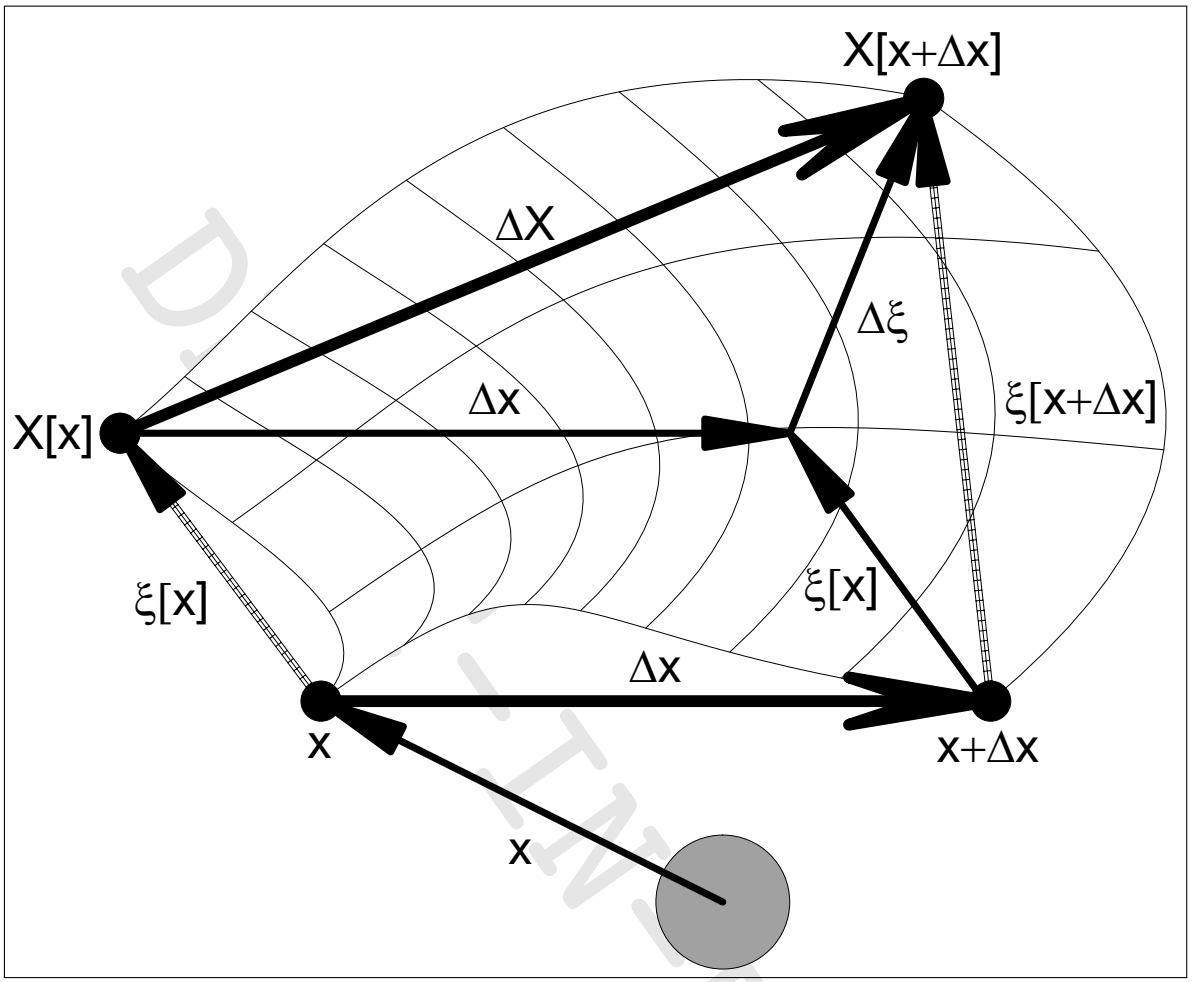

Figure 32.11: Sketchof the geometry of the displacement; the real displacements are in three dimensions

The displacement inside the medium is obtained by:

$$\Delta\mathbf{X} - \Delta\mathbf{x} = (\mathbf{X}[\hat{\mathbf{x}}] - \mathbf{X}[\mathbf{x}]) - (\hat{\mathbf{x}} - \mathbf{x}) = (\Delta\boldsymbol{\xi}[\mathbf{x}] + \Delta\mathbf{x}) - (\Delta\mathbf{x}) = \Delta\boldsymbol{\xi}[\mathbf{x}] \quad .$$

Alternatively:

$$\boxed{\Delta\mathbf{X} = \Delta\mathbf{x} + \Delta\boldsymbol{\xi}[\mathbf{x}]} \quad . \tag{32.02.02}$$

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Note that, as long as we treat with $\Delta\mathbf{x}$ and $\Delta\mathbf{X}$, we cannot describe any translation of the medium. The translation, on the other hand, is described by the function $\boldsymbol{\xi}[\mathbf{x}]$ itself.

Therefore:

- the translation of the medium at $\mathbf{x}$ is described by $\boldsymbol{\xi}[\mathbf{x}]$;
- the deformation at $\mathbf{x}$ of an infinitesimal segment of medium $\Delta\mathbf{x}$ is described by $\Delta\mathbf{x} + \Delta\boldsymbol{\xi}[\mathbf{x}]$.

The linearization, assuming small strains, gives:

$$\boxed{\mathrm{d}\xi_k[\mathbf{x}] = \mathrm{d}x_j \frac{\partial \xi_k}{\partial x_j}[\mathbf{x}] + \mathcal{O}\left(|\,\mathrm{d}\mathbf{x}|\right)^2 = \mathrm{d}x_j \left(\partial_j \xi_k[\mathbf{x}]\right) + \mathcal{O}\left(|\,\mathrm{d}\mathbf{x}|\right)^2 \equiv F^{\cdot\cdot}_{kj}[\mathbf{x}]\,\mathrm{d}x_j + \mathcal{O}\left(|\,\mathrm{d}\mathbf{x}|\right)^2} \quad , \tag{32.02.03}$$

which can be written without explicit tensor indexes as:

$$\boxed{\mathrm{d}\boldsymbol{\xi}[\mathbf{x}] = \mathrm{d}x_j \left(\frac{\partial \boldsymbol{\xi}[\mathbf{x}]}{\partial x_j}\right) + \mathcal{O}\left(|\,\mathrm{d}\mathbf{x}|\right)^2 = \left(\,\mathrm{d}\mathbf{x}\cdot\boldsymbol{\nabla}\right)\boldsymbol{\xi}[\mathbf{x}] + \mathcal{O}\left(|\,\mathrm{d}\mathbf{x}|\right)^2 \equiv \mathbb{F}[\mathbf{x}]\cdot\,\mathrm{d}\mathbf{x} + \mathcal{O}\left(|\,\mathrm{d}\mathbf{x}|\right)^2} \quad , \tag{32.02.04}$$

where the strain tensor $\mathbb{F}$ has been introduced:

$$\boxed{\mathbb{F} \longrightarrow F^{\cdot\cdot}_{kj}[\mathbf{x}] \equiv \frac{\partial \xi_k}{\partial x_j}[\mathbf{x}]} \quad . \tag{32.02.05}$$

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The order of the indexes in $\mathbb{F}$ is such that the matrix formed by $\mathbb{F}$ transforms the column vector $\mathrm{d}\mathbf{x}$ into $\mathrm{d}\boldsymbol{\xi}[\mathbf{x}]$ by normal matrix multiplication on the right of the matrix.

The strain tensor is defined as in (32.02.05) in orthonormal Cartesian Coordinates Coordinate System and it is defined in arbitrary coordinates starting from definition (32.02.05).

We have, in the linearised form, in terms of the matrix $\mathbb{F}$ with elements $F^{\cdot\cdot}_{kj}$:

$$\boxed{\mathrm{d}\mathbf{X} = \mathrm{d}\mathbf{x} + \mathrm{d}\boldsymbol{\xi}[\mathbf{x}] = \mathrm{d}\mathbf{x} + \mathbb{F}\ \mathrm{d}\mathbf{x} + \mathcal{O}\left(|\ \mathrm{d}\mathbf{x}|\right)^2 \simeq (\mathbb{I} + \mathbb{F})\ \mathrm{d}\mathbf{x}} \quad . \tag{32.02.06}$$

→ 1648

We will only be interested in the linear theory of elasticity, applicable to small deformations. The theory for finite deformations exists but it is obviously considerably more complex. Along this line we will always consider the linearised equations, neglecting terms of order higher than one.

32.02.03
Invariant Decomposition of the Strain Tensor

Equations (32.02.01), (32.02.02) show that the general infinitesimal deformation around any infinitesimal region around the point $\mathbf{x}$ is given by an overall translation by the amount $\boldsymbol{\xi}[\mathbf{x}]$ (described by equation (32.02.01)) plus a change in length and direction of every oriented line element originating at $\mathbf{x}$ (described by equation (32.02.02)).

In fact from equations (32.02.02), (32.02.02) we have:

$$\boxed{\mathbf{X}[\hat{\mathbf{x}}] = \mathbf{x} + \boldsymbol{\xi}[\mathbf{x}] + \Delta\mathbf{x} + \Delta\boldsymbol{\xi}[\mathbf{x}] = \mathbf{x} + \boldsymbol{\xi}[\mathbf{x}] + \Delta\mathbf{X}} \quad .$$

For an infinitesimal region:

$$\boxed{\mathbf{X}[\hat{\mathbf{x}}] = \mathbf{x} + \boldsymbol{\xi}[\mathbf{x}] + (\mathbb{I} + \mathbb{F})\ \mathrm{d}\mathbf{x}} \quad .$$

The general strain is thus defined, in orthonormal Cartesian Coordinates Coordinate System, in terms of the quantity

$$\boxed{F^{\cdot\cdot}_{kj}[\mathbf{x}] \equiv \frac{\partial \xi_k}{\partial x_j}[\mathbf{x}]} \quad , \tag{32.02.07}$$

→ 1647

the strain, a second-order tensor. That the strain is a Cartesian tensor is clear form its definition, equation (32.02.07), in terms of two vectors: ξ_k and x_j.

32.02.03.01 Trace of the Strain Tensor

The diagonal elements of the strain tensor represent the relative change of length in the corresponding direction. Its non diagonal elements, $F^{\cdot\cdot}_{kj}$, represent the shear deformations.

The trace of the strain tensor gives the relative volume dilatation of the element:

$$\frac{\Delta V}{V} = \mathrm{Tr}\,\mathbb{F} = F^{\cdot\cdot}_{jj} \quad .$$

In fact:

$$\frac{\Delta V}{V} = \frac{\Delta l_x}{l_x} + \frac{\Delta l_y}{l_y} + \frac{\Delta l_z}{l_z} \quad ,$$

becomes, in terms of the strain tensor,

$$\frac{\Delta V}{V} = \frac{\Delta \xi_x}{\Delta x} + \frac{\Delta \xi_y}{\Delta y} + \frac{\Delta \xi_z}{\Delta z} \qquad \lim_{\Delta x \Delta y \Delta z \longrightarrow 0} \frac{\Delta V}{V} = \frac{\partial \xi_x}{\partial x} + \frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z} \equiv \mathbf{div}\boldsymbol{\xi} \quad .$$

Alternatively one can note that:

$$\boxed{\delta V = \oiint_{\partial V} \boldsymbol{\xi} \cdot \mathrm{d}\mathbf{S} = \iiint_{V} \mathbf{div}\boldsymbol{\xi}\ \mathrm{d}V} \quad .$$

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32.02.03.02 Shear Tensor

The strain tensor is not symmetric but, as any tensor with two indexes, it can be always decomposed in its symmetric and anti-symmetric parts:

$$F_{kj}^{\cdot\cdot} = \frac{1}{2}\left(F_{kj}^{\cdot\cdot} + F_{jk}^{\cdot\cdot}\right) + \frac{1}{2}\left(F_{kj}^{\cdot\cdot} - F_{jk}^{\cdot\cdot}\right) \equiv F^{\mathtt{S}}{}_{kj}^{\cdot\cdot} + F^{\mathtt{A}}{}_{kj}^{\cdot\cdot} \qquad \Leftrightarrow \qquad \mathbb{F} \equiv \mathbb{F}^{\mathtt{S}} + \mathbb{F}^{\mathtt{A}} \quad . \tag{32.02.08}$$

The symmetric part can be additionally decomposed by extracting the traceless, $\mathbb{F}_0^{\mathtt{S}}$, part as:

$$\mathbb{F}^{\mathtt{S}} = \left(\mathbb{F}^{\mathtt{S}} - \frac{\operatorname{Tr} F}{3}\mathbb{I}\right) + \frac{\operatorname{Tr} F}{3}\mathbb{I} \equiv \mathbb{F}_0^{\mathtt{S}} + \frac{\operatorname{Tr} F}{3}\mathbb{I} \quad .$$

This shows that an arbitrary strain deformations, described by a symmetrical strain, can be expressed as the sum of a pure shear plus an isotropic dilatation/compression, because the term proportional to $\mathbb{I}$ is an isotropic tensor and it is multiplied by the trace of the strain tensor, that is the relative change of volume, having the same overall structure as an hydrostatic pressure stress tensor.

Principal Strains

As the pure shear tensor is symmetrical it can be diagonalized via an orthogonal matrix, that is a suitable rotation of the coordinates.

The eigenvalues are called the principal strains. Note that the pure shear tensor is not necessarily positive definite, and the eigenvalues may have any sign.

32.02.03.03 Anti-Symmetrical Part of the Strain Tensor

The anti-symmetric part can be written as a vector product, in terms of the vector $\boldsymbol{\Omega}$ as follows:

$$\mathrm{d}\boldsymbol{\xi}^{\mathtt{A}} = \mathbb{F}^{\mathtt{A}} \cdot \mathrm{d}\mathbf{x} = \boldsymbol{\Omega} \wedge \mathrm{d}\mathbf{x} \quad , \tag{32.02.09}$$

with

$$\Omega_q = +\frac{1}{2}\epsilon_{irq}^{\cdots} F^{\mathtt{A}}{}_{ri}^{\cdot\cdot} = -\frac{1}{2}\epsilon_{qri}^{\cdots} F^{\mathtt{A}}{}_{ri}^{\cdot\cdot} \qquad F^{\mathtt{A}}{}_{ri}^{\cdot\cdot} = \epsilon_{irq}^{\cdots}\Omega_q \quad . \tag{32.02.10}$$

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Therefore it describes a pure rotation around the direction of the vector $\boldsymbol{\Omega}$ by an amount described by the module of the vector $\boldsymbol{\Omega}$ itself.

Moreover, one has:

$$\Omega_q = \frac{1}{2}\epsilon_{irq}^{\cdots} F^{\mathtt{A}}{}_{ri}^{\cdot\cdot} = \frac{1}{2}\epsilon_{irq}^{\cdots}\left(F^{\mathtt{A}}{}_{ri}^{\cdot\cdot} + F^{\mathtt{S}}{}_{ri}^{\cdot\cdot}\right) = \frac{1}{2}\epsilon_{irq}^{\cdots} F_{ri}^{\cdot\cdot} = \frac{1}{2}\epsilon_{irq}^{\cdots}\frac{\partial \xi_r}{\partial x_i} = \frac{1}{2}\mathbf{e}_q \cdot \mathbf{rot}\boldsymbol{\xi} \quad . \tag{32.02.11}$$

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Therefore a deformation is without rotation if and only if the rotor of the displacement is zero.

One should be careful with the signs from equations (32.02.10), (32.02.11).

32.02.03.04 General Deformation

The general state of strain can thus be described as a translation, plus a rotation, plus a pure deformation (described by the symmetric traceless part of the tensor) plus a dilatation (described by the trace of the tensor).

Starting from equation (32.02.06) one can write:

$$\mathrm{d}\mathbf{X} = \mathrm{d}\mathbf{x} + \mathrm{d}\boldsymbol{\xi}[\mathbf{x}] = (\mathbb{I} + \mathbb{F})\ \mathrm{d}\mathbf{x} = \left(\mathbb{I} + \mathbb{F}^{\mathtt{S}} + \mathbb{F}^{\mathtt{A}}\right)\ \mathrm{d}\mathbf{x} = \left(\mathbb{I} + \mathbb{F}^{\mathtt{S}}\right)\ \mathrm{d}\mathbf{x} + \mathbb{F}^{\mathtt{A}}\ \mathrm{d}\mathbf{x} \quad .$$

Consider a infinitesimal vector $\mathrm{d}\mathbf{x}$, originating at $\mathbf{x}$. The head of this infinitesimal vector describes a sphere centered at $\mathbf{x}$.

After the most general deformation the center of this sphere is translated by $\boldsymbol{\xi}[\mathbf{x}]$, any infinitesimal vector $\mathrm{d}\mathbf{x}$ is mapped into a direction rotated as described by $\mathbb{F}^{\mathtt{A}}$ and the heads of the mapped infinitesimal vectors describe the surface of ellipsoid whose center is at $\mathbf{x} + \boldsymbol{\xi}[\mathbf{x}]$. The fact that the sphere is mapped into an ellipsoid comes from the fact that $\mathbb{I} + \mathbb{F}^{\mathtt{S}}$ is a real, symmetrical positive definite matrix, as the elements of the strain tensors are, by definition of small strains, small with respect to one. The matrix $\mathbb{F}^{\mathtt{S}}$ can then be diagonalized and its eigenvalues (principal strains) can have any sign but the sum with one is positive. The matrix $\mathbb{I} + \mathbb{F}^{\mathtt{S}}$ is thus positive definite.

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32.02.03.05 Types of Motion of Continuous Media

The decomposion of the strain tensor shows that the general motion of continuous media can be kinematicly descibed as follows.

Motions in which the trace of the strain tensor is null are solenoidal (zero-divergence) motions with no change in volume (iso-volumetric).

Motions in which the anti-symmetrical part of the strain tensor is null are irrotational (zero rotor) motions with no rotation at any point.

Motions in which the symmetrical part of the strain tensor is null are rigid motions.

32.02.04
Non-Diagonal Elements of the Strain Tensor

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Consider the plane finite deformation:

$$\boldsymbol{\xi}[\mathbf{x}] = \{y \tan[\theta], 0, 0\} \quad ,$$

shown in figure 32.12.

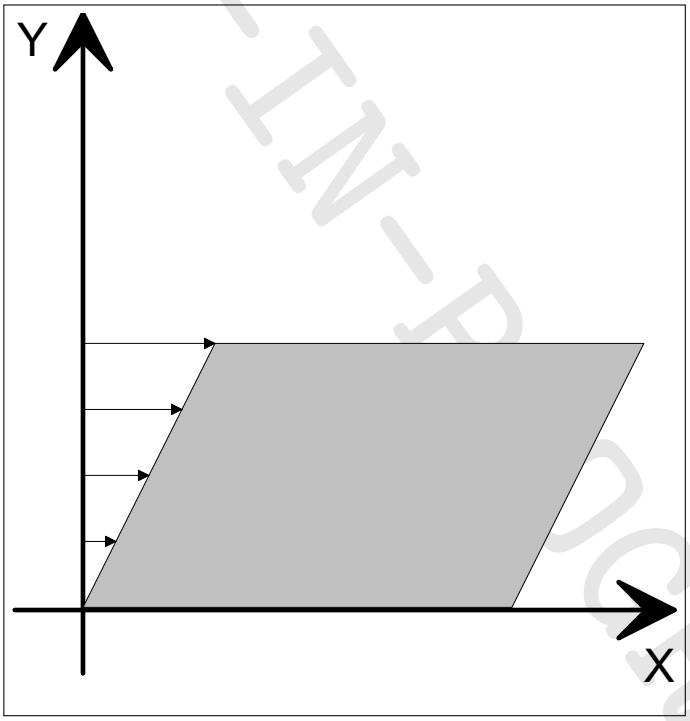

Figure 32.12: FIGURE

This corresponds to the deformation used in defining the shear modulus, G, see section § 27.02.02.

If we limit to infinitesimal deformations we can write the infinitesimal strain matrix, as a function of the infinitesimal angle $\mathrm{d}\theta$, as:

$$F^{\cdot}_{kj}[\mathbf{x}] \equiv \frac{\partial \xi_k}{\partial x_j}[\mathbf{x}] = \begin{pmatrix} 0 & \mathrm{d}\theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & +\mathrm{d}\theta/2 & 0 \\ +\mathrm{d}\theta/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & +\mathrm{d}\theta/2 & 0 \\ -\mathrm{d}\theta/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad .$$

The strain matrix is traceless and therefore the deformations preserves the volume. The strain is made of a pure strain (the symmetric traceless matrix) plus a rotation by an angle $-\theta/2$ (clockwise direction). In fact equation (32.02.10) gives

$$\boldsymbol{\Omega} = \{0, 0, -\mathrm{d}\theta/2\} \quad .$$

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If one considers a square figure in the xy plane the complete deformation can be decomposed into a pure deformation of the square, such that the square becomes a rhombus with its diagonals parallel to the diagonals of the original square followed by a clock-wise rotation by an angle $|\,\mathrm{d}\theta|/2$

The interpretation of the non-diagonal elements of the strain matrix coming from this particular case is valid in general: the non-diagonal elements (r, s) of the strain matrix represent one half of the decrease of the angle between the two initially orthogonal directions r and s.

32.02.05
Compatibility Conditions

32.02.06
Some Formal Details

32.03
Dynamics of the Continuous Media

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32.03.01
Volume and Surface Forces and Moments/Couples

This § is referenced at pages:
[1514]

Consider a continuous medium and an arbitrary ideal volume inside it, Ω, whose boundary is made by the closed surface, $\mathcal{A} \equiv \partial\Omega$, delimiting the system we are studying. Let $\hat{\mathbf{n}}$ denote the normal vector exiting the surface, a function of the point at the surface.

32.03.01.01 Volume (Body) Forces and Moments/Couples

Volume (body) forces and volume (body) moments/couples, which are forces and moments/torques acting at a distance on the volume, due either to external fields or to long range forces internal to the medium, need to be included as well.

In the following, except when specificed otherwise, the body moments/torques will be assumed to be zero, as it is done in the classical theory of elasticity.

Notable examples of body forces include: gravitational, centrifugal and Coriolis forces.

Notable examples of body moments include: Polarized material in an external electric field Magnetized material in an external magnetic field and material made of dipolar-molecules.

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32.03.01.02 Surface Forces and Moments/Couples

The classical force-stress and torque-stress tensors measure the internal forces and torques that parts of a medium exert on other parts (even though there may be zero net force and zero net moment at each point).

The description of the contact interaction between the material external to the surface and the material internal to the surface (which constitutes the system under study) as well as the description of the contact interactions between the system and external objects in contact with the system can be carried on as follows.

As the system is the part of material internal to the surface all the forces and torques are to be interpreted as the external forces and torques acting on the system.

Consider a small surface $\mathrm{d}\mathbf{S} \equiv \hat{\mathbf{n}}\,\mathrm{d}S$, oriented with its normal exiting the surface. Actually it might be noted that it is the unit vector $\hat{\mathbf{n}}$ which defines the internal and external parts to the system, in an unambiguous way, as the surface is closed.

The part of the material external to the surface applies to the internal part, that is the system, a set of contact forces and torques, whose extent depends on the area $\Delta\mathbf{S}$.

Consider the system of forces applied to the surface $\Delta\mathbf{S}$.

It is known, read § 15.04, that any system of applied forces is equivalent to a system composed of the total force (call it $\Delta\mathbf{F}$), applied at any point (pole) of the small area $\Delta\mathbf{S}$, plus a torque (call it $\Delta\mathbf{\Gamma}$), equal to the resultant momentum of the applied forces with respect to the same point (pole) of the small area $\Delta\mathbf{S}$. Note that, as we are dealing with torques, the choice of the pole is not relevant for them, that is their momentum is the same with respect to any pole. This decomposition is always implicitly assumed, except when specified otherwise.

For an alternative decomposition, read § 15.04, the concept of central axis can be used.

One should remember that the Force|Torque in the following discussion are the forces and torques applied to the system by the external world.

It is assumed that the following limits (which depend on $\hat{\mathbf{n}}$) exists as finite vectors:

$$\lim_{\Delta\mathbf{S}[\hat{\mathbf{n}}]\longrightarrow 0} \frac{\Delta\mathbf{F}}{\Delta\mathbf{S}[\hat{\mathbf{n}}]} = \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}A} \equiv \mathbf{S}[\hat{\mathbf{n}}] \qquad (32.03.01)$$

$$\lim_{\Delta\mathbf{S}[\hat{\mathbf{n}}]\longrightarrow 0} \frac{\Delta\mathbf{\Gamma}}{\Delta\mathbf{S}[\hat{\mathbf{n}}]} = \frac{\mathrm{d}\mathbf{\Gamma}}{\mathrm{d}A} \equiv \mathbf{M}[\hat{\mathbf{n}}] \quad .$$

→ 1651 1652

It is obvious, from its very definition, that the objects defined by equations (32.03.01), (32.03.02.01) are, for any fixed point at the surface (which determines in a unique way $\hat{\mathbf{n}}$ and $\mathrm{d}\mathbf{S}$), vectors. More generally: it is obvious, from its very definition, that the objects defined by equations (32.03.01), (32.03.02.01) are, for any fixed point and $\hat{\mathbf{n}}$ (which determines in a unique way $\mathrm{d}\mathbf{S}$), vectors. Later on one should ask what happens to the force when one changes $\hat{\mathbf{n}}$ at a fixed point.

The action of the two vectors, $\mathbf{S}[\hat{\mathbf{n}}]$, the force-stress, and $\mathbf{M}[\hat{\mathbf{n}}]$, the torque-stress, on the infinitesimal surface $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S$, is equipollent to the action of the set of external forces applied on $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S$.

Note that the previous assumptions are fundamental assumptions of the theory, or, in other words, they are constitutive equations for the material. Note in particular that in the theory of classical elastic materials one assumes that the limit in equation (32.03.02.01) is zero, that is absence of surface couples: $\mathbf{M}[\hat{\mathbf{n}}] = 0$.

The force-stress, $\mathbf{S}[\hat{\mathbf{n}}]$, and the torque-stress, $\mathbf{M}[\hat{\mathbf{n}}]$, describe the effect of contact interaction, acting across the surface $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S$.

In the following, except when specificed otherwise, the torque-stress will be assumed to be zero.

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32.03.01.03 Some Comments

Note that many authors define the interaction of the material with the external world in terms of the forces exerted by the material system on the external world, thus obtaining definitions of the quantities with the opposite sign. In fact the generalized Newton third law ensures that the two set of forces have equal and opposite total force and total torque. With the chosen convention the component of the force in the definition of equation (32.03.01) is positive whenever it is a traction force. It is, in fact, often called **traction**. With the chosen convention the component of the moment of the moments/torques in the definition of equation (32.03.02.01) is positive if it tends to rotate the material of the system in the counter-clockwise direction (right-hand rule) with respect to the local normal unit vector $\widehat{\mathbf{n}}$.

Cases when torque-stresses and body torques intervene include, for instance, Polarized material in an external electric field Magnetized material in an external magnetic field and material made of dipolar-molecules.

32.03.02
Forces: The Force-Stress Tensor and Volume (Body) Forces

This § is referenced at pages:
[Never referenced.]

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32.03.02.01 Force-Stress Tensor

From the force-stress, $\mathbf{S}[\widehat{\mathbf{n}}]$, one can introduce the force-stress tensor, usually just called the stress tensor, because in the theory of classical elastic materials the torque-stress tensor, $\mathbf{M}[\widehat{\mathbf{n}}]$, is zero.

The $S^{\cdot\cdot}_{jk}$ component of stress tensor is defined as the j component of the force across a unit area perpendicular to the k axis:

$$S^{\cdot\cdot}_{jk} \equiv \frac{\mathrm{d}F_j}{\mathrm{d}A_k} \quad , \qquad \rightarrow \text{1651 1652}$$

in terms of the j component of the force acting on the surface $\mathrm{d}\mathbf{S}_k$ perpendicular to the k axis.

Note the convention that the first index of $S^{\cdot\cdot}_{jk}$ gives the component of the force.

It can be shown that the force per unit area (the stress) acting on an arbitrary unit surface perpendicular to the unit vector $\widehat{\mathbf{n}}$ has a j component given by:

$$\frac{\mathrm{d}F_j}{\mathrm{d}A[\widehat{\mathbf{n}}]} \equiv \mathbf{e}_j \cdot \mathbf{S}[\widehat{\mathbf{n}}] = S^{\cdot\cdot}_{jk} n_k \quad . \qquad (32.03.02) \qquad \rightarrow \text{1652 1653}$$

Since both n_k and $\mathbf{e}_j \cdot \mathbf{S}[\widehat{\mathbf{n}}]$ are components of vectors, equation (32.03.02) demonstrates that $\mathbf{S}$ is actually a tensor, thanks to the **Quotient Law** § 20.03.07.

It was born as a native second rank tensor, in contrast to most of the other second-order tensor arising as coefficients in relating one vector to another.

R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...- 2.31

It follows that the force per unit area (the stress) along the direction of the unit vector $\mathbf{u}$ acting on a surface defined by the unit vector $\widehat{\mathbf{n}}$ is given by:

$$\mathbf{u} \cdot \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}A[\widehat{\mathbf{n}}]} \equiv S[\widehat{\mathbf{n}}, \mathbf{u}] = S^{\cdot\cdot}_{jk} n_k u_j \quad . \qquad (32.03.03)$$

DRAFT - In Progress

As the tensor $S^{\cdot\cdot}_{jk}$ is sufficient to calculate the force at any point in the direction perpendicular to any surface it completely describes the internal state of stress of the continuous medium.

The stress tensor, in general, depends on the point, and therefore it defines a tensor field (as the strain).

It can be shown that the stress tensor is actually symmetric in most common cases, but this is an assumption of the theory, valid as long as no body torques nor torque-stresses are present. That is the symmetry of the stress tensor is a constitutive equation of the material considered.

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As long as S is represented by a real symmetric matrix in Cartesian Coordinates it can be diagonalized to find the principal stress axes. For surfaces perpendicular to those three axes the stresses are particularly simple as they corresponds to push or pulls along these axes. There are no shear forces along those the surfaces perpendicular to the principal axes of inertia. For any stress one can choose the axes in such a way that shear components are zero.

In the case there is no isotropy the mean pressure is defined as the invariant:

$$\overline{p} \equiv -\frac{1}{3}\operatorname{Tr} S = -\frac{1}{3}(S_{11} + S_{22} + S_{33}) \quad .$$

Note the sign. Hydrostatic pressure gives a force that a piece of matter exerts on the surroundings, resisting compression, exiting from its boundary. Due to the convention used for the stress energy tensor this is a negative stress. The minus sign thus makes the average pressure positive.

Therefore in a similar way as the decomposition of the symmetrical part of the strain tensor into a traceless symmetrical tensor plus a tensor proportional to the unit tensor, also the stress tensor can be separated in a similar way.

Example: The Stress Tensor of a Fluid at Rest

The stress tensor of any fluid at rest:

$$S^{\cdot\cdot}_{kj} = -p\delta^{\cdot\cdot}_{kj} \quad .$$

Example: The Stress Tensor of an Ideal Fluid

The stress tensor of an ideal fluid, under any conditions, is:

$$S^{\cdot\cdot}_{kj} = -p\delta^{\cdot\cdot}_{kj} \quad .$$

32.03.02.02 Force-Stress Tensor and the Spatial Part of the Energy-LinMomentum Tensor

This § is referenced at pages:
[1929]

The force-stress tensor thus defined can be obviously introduced, by means of its definition (32.03.02) for any system, possibly by making use of the description of the force as change of linear momentum per unit time. When carrying on the relativistic generalisation (section § 60.03) the force-stress tensor becomes the spatial part of the Energy-LinMomentum four-tensor.

In order to show the relation between the stress tensor and the linear momentum flux one should note that if we refer to the linear momentum flux exiting the volume, as it is usually done when dealing with the Energy-LinMomentum tensor (read § 60.03), this represents the force per unit area applied by the system inside the volume to the external world. Therefore, based on our conventions, the stress tensor has the opposite sign with respect to the spatial components of the Energy-LinMomentum tensor. This can be also expressed by saying that the linear momentum flux described by the stress tensor is the flux of linear momentum entering the volume, that is given to the system inside the volume.

DRAFT - PROOFREAD

The force per unit area on a surface is equivalent to the flux of linear momentum, that is linear momentum crossing the surface per unit area and per unit time:

$$-S^{\cdot\cdot}_{jk} \equiv \frac{\mathrm{d}F_j[\text{IN} \longrightarrow \text{OUT}]}{\mathrm{d}A_k} = \frac{d^2 p_j[\text{IN} \longrightarrow \text{OUT}]}{\mathrm{d}A_k \, \mathrm{d}t} \equiv T^{\cdot\cdot}_{jk} \quad .$$

32.03.02.03 Principal Stresses

As long as the force-stress tensor is symmetrical it can be diagonalized via an orthogonal matrix, that is a suitable rotation of the coordinates.

The eigenvalues are called the principal stresses. Note that the force-stress tensor is not necessarily positive definite, and the eigenvalues may have any sign.

32-001 Von Mises Stress

Von Mises stress is often used as a measure of the total state of stress at a point, especially in cases related to the maximum allowables stress for a real material. It is defined as:

$$S_{\text{VM}} \equiv \sqrt{\frac{3}{2} S^{\cdot\cdot}_{jk} S^{\cdot\cdot}_{jk}} \quad . \tag{32.03.04}$$

Show that it can be written, in terms of the principal stresses, σ_1, σ_2 and σ_3, as:

$$S_{\text{VM}} = \sqrt{\frac{1}{2}\Big((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\Big)} \quad . \tag{32.03.05}$$

32.03.02.04 Examples of Stress Tensor: Fluids

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A fluid is a continuous system which does not keep any shear stress without starting to flow. That is a fluid at equilibrium with respect to any one Coordinate System has not shear stresses. In other words a fluid moving as a rigid-body has no shear stresses.

Ideal Fluid

An ideal fluid is totally isotropic, therefore its stress tensor must be proportional to the identity tensor, the only isotropic tensor with two indexes:

$$S^{\cdot\cdot}_{ij} = -p\delta^{\cdot\cdot}_{ij} \qquad \text{ideal fluid} \quad .$$

Note the minus sign coming from the convention that the stress tensor is the **traction** of the external world on the system.

Real Fluid

A real fluid is isotropic at equilibrium but it is not necessarily isotropic outisde equilibrium, due to the presence of viscosity which trasnfers shear stresses.

Therefore, for a real fluid at rest, one has:

$$S^{\cdot\cdot}_{ij} = -p\delta^{\cdot\cdot}_{ij} \qquad \text{real fluid at rest} \quad .$$

However, in general for a real fluid an additional non isotropic term, $\Pi^{\cdot\cdot}_{ij}$, arises:

$$S^{\cdot\cdot}_{ij} = -p\delta^{\cdot\cdot}_{ij} + \Pi^{\cdot\cdot}_{ij} \qquad \text{generic real fluid} \quad .$$

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Real Newtonian Fluids

A Newtonian fluid is a fluid such that the stress tensor for an infinitesimal parcel of fluid in the local Reference Frame of the parcel of fluid is:

$$S_{ij}^{\cdot\cdot} = -p\delta_{ij}^{\cdot\cdot} + \eta'\delta_{ij}^{\cdot\cdot}\mathbf{div}\,\mathbf{v} + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = -p\delta_{ij}^{\cdot\cdot} + \eta'\delta_{ij}^{\cdot\cdot}\mathbf{div}\,\mathbf{v} + 2\eta\partial_t F^{\mathrm{S}\,\cdot\cdot}_{ij} \qquad \text{real Newtonian fluid} \quad .$$

That is the shear stresses in a Newtonian fluid, by definition, are linear in the gradient of the velocity.

Read § 32.08.03.

32.03.02.05 Volume (Body) Forces

This § is referenced at pages:
[Never referenced.]

Volume forces are forces whose intensity is proportional to the volume Ω of the material included inside the closed surface $\mathcal{A} \equiv \partial\Omega$, that is the volume of the system:

$$\mathbf{F}_{\mathrm{V}} = \iiint_{\Omega} \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}V}\,\mathrm{d}V \equiv \iiint_{\Omega} \mathbf{f}\,\mathrm{d}V \quad .$$

32.03.02.06 First Cardinal Equation for Continuous Media

For any arbitrary part of the material medium, defined by a small volume Ω with a surface boundary $\partial\Omega$ and located at $\mathbf{r}$ we can write the first **Cardinal Equation** of motion:

$$\iiint_{\Omega} f_k\,\mathrm{d}V + \oiint_{\partial\Omega} S_{kj}^{\cdot\cdot}\,\mathrm{d}\mathbf{S}_j = \iiint_{\Omega} \partial_t p_k\,\mathrm{d}V = \frac{\mathrm{d}P_k}{\mathrm{d}t} \quad . \qquad (32.03.06)$$

→ 1662

It can be written in local form as:

$$f_k + \partial_j S_{kj}^{\cdot\cdot} = \partial_t p_k \quad . \qquad (32.03.07)$$

→ 1655 1657 1660

It can be written in local form also emphasising the linear momentum balance:

$$\partial_t p_k - \partial_j S_{kj}^{\cdot\cdot} = f_k \qquad \partial_t p_k + \partial_j T_{kj}^{\cdot\cdot} = f_k \quad . \qquad (32.03.08)$$

→ 1655

Equation (32.03.08) has the standard form of a balance (or continuity) equation.

Note that the derivation of equation (32.03.07) shows that surface forces can be expressed equivalently as volume forces, via the divergence theorem:

$$f_k^S \equiv \frac{\mathrm{d}F_k^S}{\mathrm{d}V} = \partial_j S_{kj}^{\cdot\cdot} \quad .$$

For material with symmetrical force-stress tensor the second **Cardinal Equation** does not provide additional informations besides those provided by the first **Cardinal Equation** . It is actually dependent of the first one.

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32-002 Density of Linear Momentum

Show that, in the Lagrangian approach:

$$\partial_t p_k = \rho_0 \frac{\partial^2 \xi}{\partial t^2} \quad .$$

32.03.03 Moments: The Torque-Stress Tensor and Volume (Body) Torques

This § is referenced at pages:
[Never referenced.]

Most of the common materials are adequately modeled without the use of couple-stresses and volume (body) torques. The classical theory of elastic bodies do not account for them.

However, there exists significant cases where couple-stresses and volume (body) torques do need to be included. In such cases it is mandatory to account for the second **Cardinal Equation** , which is independent of the first one in such cases.

32.03.03.01 Torque-Stress Tensor

The torque exerted by the external world onto a small surface

32.03.03.02 Volume (Body) Moments/couples

This § is referenced at pages:
[Never referenced.]

Volume moments/torques are moments/torques whose intensity is proportional to the volume Ω of the material included inside the closed surface $\mathcal{A} \equiv \partial\Omega$, that is the volume of the system:

$$\boxed{\mathbf{\Gamma}_{\mathrm{V}} = \iiint\limits_{\Omega} \frac{\mathrm{d}\mathbf{\Gamma}}{\mathrm{d}V}\, \mathrm{d}V \equiv \iiint\limits_{\Omega} \boldsymbol{\gamma}\, \mathrm{d}V} \quad .$$

32.03.03.03 Second Cardinal Equation for Continuous Media

For any arbitrary part of the material medium, defined by a small volume Ω with a surface boundary $\partial\Omega$ and located at $\mathbf{r}$ we can write the second **Cardinal Equation** of motion:

$$\boxed{\iiint\limits_{\Omega} \left(\gamma_k + \epsilon_{kab}^{\cdot\cdot\cdot} x_a f_b\right)\, \mathrm{d}V + \oiint\limits_{\partial\Omega} \left(M_{kj}^{\cdot\cdot} + \epsilon_{kab}^{\cdot\cdot\cdot} x_a S_{bj}^{\cdot\cdot}\right)\, \mathrm{d}\mathbf{S}_j = \iiint\limits_{\Omega} \partial_t j_k\, \mathrm{d}V = \frac{\mathrm{d}J_k}{\mathrm{d}t}} \quad . \tag{32.03.09}$$

It can be written in local form as:

$$\boxed{\gamma_k + \epsilon_{kab}^{\cdot\cdot\cdot} x_a f_b + \partial_j \left(M_{kj}^{\cdot\cdot} + \epsilon_{kab}^{\cdot\cdot\cdot} x_a S_{bj}^{\cdot\cdot}\right) = \partial_t j_k} \quad . \tag{32.03.10}$$

→ 1656

It can be written in local form also emphasizing the angular momentum balance:

$$\boxed{\partial_t j_k - \partial_j \left(M_{kj}^{\cdot\cdot} + \epsilon_{kab}^{\cdot\cdot\cdot} x_a S_{bj}^{\cdot\cdot}\right) = \gamma_k + \epsilon_{kab}^{\cdot\cdot\cdot} x_a f_b} \quad . \tag{32.03.11}$$

→ 1656

Equation (32.03.11) has the standard form of a balance (or continuity) equation.

For material with symmetrical force-stress tensor the second **Cardinal Equation** does not provide additional informations besides those provided by the first **Cardinal Equation** . It is actually dependent of the first one.

Note that the derivation of equation (32.03.10) shows that surface moments can be expressed equivalently as volume moments, via the divergence theorem:

$$\boxed{\gamma_k^S \equiv \frac{\mathrm{d}\Gamma_k^S}{\mathrm{d}V} = \partial_j M_{kj}^{\cdot\cdot}} \quad .$$

32-003 Acoustic Waves in Fluids/solids

Derive the equation for the acoustic waves in fluids/solids, as in section § 45.02.03, using the concept of stress tensor, instead of pressure, and assuming a linear relation (Hooke law) between the normal stress and the longitudinal strain.

Show that this is consistent with the local form of first **Cardinal Equation** for continuous media, equation (32.03.07).

WORKED SOLUTION

Equation (45.02.03) of section § 45.02.03, the first **Cardinal Equation** , can be re-written as:

$$\mathrm{d}F_z = -\left(S_{zz}[z] - S_{zz}[z+\Delta z_0]\right) \simeq \frac{\partial S_{zz}}{\partial z} A\Delta z_0 = \rho_0 A\Delta z_0 \frac{\partial^2 \xi_z}{\partial t^2} \quad .$$

It should be noted the negative sign, due to the different sign conventions between pressure and stress tensor.

The linear relation (Hooke law) implies:

$$S_{zz} = Y\frac{\partial \xi_z}{\partial z} \Longrightarrow \frac{\partial S_{zz}}{\partial z} = Y\frac{\partial^2 \xi_z}{\partial z^2} \quad ,$$

which, after substitution in the first **Cardinal Equation** , gives:

$$\frac{\partial S_{zz}}{\partial z} = \boxed{Y\frac{\partial^2 \xi_z}{\partial z^2} = \rho_0 \frac{\partial^2 \xi_z}{\partial t^2}} \quad .$$

The above expression

$$\frac{\partial S_{zz}}{\partial z} = \rho_0 \frac{\partial^2 \xi_z}{\partial t^2} \quad ,$$

is actually the local form of first **Cardinal Equation** for continuous media, equation (32.03.07):

$$f_k^V + \partial_j S_{kj}^{\cdot\cdot} = \partial_t p_k = \rho_0 \frac{\partial^2 \xi_k}{\partial t^2} \quad .$$

This shows, in this particular case, that the equation of motion of an elastic solid is a wave equation. Note that the signs are consistent with the expression of the stress tensor for an ideal fluid (and with the expression of the stress tensor for a real fluid at rest):

$$S_{kj}^{\cdot\cdot} = -p[\mathbf{x}]\delta_{kj}^{\cdot\cdot} \quad .$$

32.03.04
Statics of Continuous Media

Statics of continuous media is based on the elementary priciple that when the entire system is in equilibrium any portion of it must be in equilibrium as well.

32.04 Tensor of Elasticity and Hooke Law for Homogeneous and Isotropic Materials

This § is referenced at pages:
[1032, 1664]

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The case of materials with symmetrical stress tensor only will be considered. Read also § 32.

Consider homogeneous, isotropic and linear materials. The generalized Hooke law is expressed as:

$$\boxed{S^{\cdot\cdot}_{ij} = E^{\cdot\cdot\cdot\cdot}_{ijkl} F^{\cdot\cdot}_{kl}} \quad , \tag{32.04.01}$$

→ 1658

in terms of the fourth-rank tensor of elasticity $\mathbb{E}$.

Note that the genralized Hooke law (32.04.01) implies that the symmetrcal part of the strain tensor only intervene. The anti-symmetrical part of the strain tensor does not intervene in Hooke law, as it is appropriate because the anti-symmetrical part of the strain tensor describes a pure rotation and therefore no elastic effect is involved. In this section it is therefore implicitly assumed that the strain tensor is symmetrical.

A priori $\mathbb{E}$ has $3^4 = 81$ independent elements. However, at a closer inspection the number of independent components turn out to be much less than that. In fact both $\mathbb{S}$ and $\mathbb{F}$ are symmetric second rank tensors with six independent components and therefore there can be almost 36 independent components. There are actually much less. It can be shown that there are at most 21 independent components in the fourth-rank tensor of elasticity $\mathbb{E}$. In fact if one considers the six independent components of both $\mathbb{S}$ and $\mathbb{F}$ and the six times six matrix linking them. The latter is a symmetrical matrix, which has 21 independent components.

If the material is homogeneous the elasticity tensor does not depend on the position, that is its values are uniform.

If the material is isotropic many more additional constraints exist. In fact isotropy implies that the components of the elasticity tensor must be unchanged under any arbitrary rotation of the Coordinate System: the elasticity tensor of an isotropic medium must be and isotropic tensor, that is one whose components do not change under any arbitrary rotation of the Coordinate System. Therefore the stress tensor must be expressed in terms of the strain tensor via isotropic tensors only, that includes scalars as well as all the isotropic fourth-rank tensors with the correct symmetry properties. Read § 20.05.08.

The relation we are looking for is a linear one, as we are looking for a generalisation of Hooke law.

Therefore two possibilities arise:

$$S^{\cdot\cdot}_{ij} \propto F^{\cdot\cdot}_{ij} \qquad S^{\cdot\cdot}_{ij} \propto \delta^{\cdot\cdot}_{ij} F^{\cdot\cdot}_{kk} \quad .$$

Therefore the stress-strain relation for a isotropic (non necessarily homogeneous) medium can be written as:

$$\boxed{S^{\cdot\cdot}_{ij}[\mathbf{x}] = 2\mu[\mathbf{x}] F^{\cdot\cdot}_{ij} + \lambda[\mathbf{x}] \operatorname{Tr} F \delta^{\cdot\cdot}_{ij}} \quad ,$$

where the two constants are called Lamé constants.

It turns out that the Lamé constants relate to the constants Y, Σ, G and B in the following way:

$$\boxed{\mu = G \qquad \lambda = B - \frac{2}{3} G} \quad .$$

Finally the stress-strain relation for a homogeneous and isotropic medium, Hooke law, can be written as:

$$\boxed{S^{\cdot\cdot}_{ij} = 2\mu F^{\cdot\cdot}_{ij} + \lambda \operatorname{Tr} F \delta^{\cdot\cdot}_{ij} = 2G F^{\cdot\cdot}_{ij} + \left(B - \frac{2}{3} G \right) \operatorname{Tr} F \delta^{\cdot\cdot}_{ij}} \quad . \tag{32.04.02}$$

→ 1659 1660

In tensor terms one can write:

$$\mathbb{S} = 2\mu[\mathbf{x}]\mathbb{F} + \lambda[\mathbf{x}]\,\mathrm{Tr}\,F\mathbb{I} = 2\mu[\mathbf{x}]\mathbb{F}^{\mathrm{S}} + \lambda[\mathbf{x}]\,\mathrm{Tr}\,F\mathbb{I} = 2\mu[\mathbf{x}]\mathbb{F}_0^{\mathrm{S}} + \left(\lambda[\mathbf{x}] + \frac{2}{3}\mu[\mathbf{x}]\right)\mathrm{Tr}\,F\mathbb{I} = 2G[\mathbf{x}]\mathbb{F}_0^{\mathrm{S}} + B[\mathbf{x}]\,\mathrm{Tr}\,F\mathbb{I} \quad , \tag{32.04.03}$$

→ 1659

where the traceless part of the strain tensor has been made explicit.

The latest expressions can be also written as:

$$\mathbb{S} = 2G[\mathbf{x}]\mathbb{F}_0^{\mathrm{S}} + B[\mathbf{x}]\,\mathrm{Tr}\,F\mathbb{I} \quad , \tag{32.04.04}$$

→ 1659

which shows the decomposition into a pure shear part, given by $\mathbb{F}_0^{\mathrm{S}}$ with coefficient is $2G[\mathbf{x}]$, and a pure compression/dilatation part, given by $\mathrm{Tr}\,F$ with coefficient $B[\mathbf{x}]$. This is consistent with the definition of the coefficients G and B.

32-004 Elasticity Tensor

Find the expression of the elasticity tensor $\mathbb{E}$ for the Hooke law, equation (32.04.02).

32-005 Decomposition of Hooke Law

Equation (32.04.04) expresses each side of Hooke law as a symmetrical traceless term plus an isotropic term. Hooke law is now separated in an invariant manner into two equations, one involving the pure shear, the other one for pure hydrostatic compression/dilatation.

Show that this separation leads to one relation between Y, G and Σ and one relation between Y, B and Σ.

32-006 Hooke Law

Compare the elementary derivation of the effects of uniform stresses on a homogeneous and isotropic parallelepiped block in section § 27.07.06 with the general expression of Hooke law in equation (32.04.02).

32-007 Alternative Forms of the Hooke Law

Show that Hooke law (equation (32.04.03)), for homogeneous and isotropic materials, can be written in the alternative but equivalent form:

$$Y\mathbb{F} = (1+\Sigma)\,\mathbb{S} + 3\Sigma\langle p\rangle\mathbb{I} \quad , \tag{32.04.05}$$

→ 1659

in terms of the average pressure, $\langle p\rangle$.

Use the relations of § 27.02.04.

Note that the above expression (32.04.05) decomposes the strain tensor into a diagonal (that is isotropic) part plus a non diagonal part.

Show that equation (32.04.05) can be written as:

$$\mathbb{S} = \frac{Y}{1+\Sigma}\left(\mathbb{F} + \frac{\Sigma}{Y}\mathrm{Tr}\,S\mathbb{I}\right) \quad . \tag{32.04.06}$$

→ 1659

Show that equation (32.04.06) gives a relation among Y, Σ and G as well as a relation among Y, Σ and B. Show that these relations are the same relations of § 27.02.04.

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32.05 Equation of Motion of an Homogeneous and Isotropic Elastic Body

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Consider an homogeneous and isotropic elastic body whose constitutive relation is given by Hooke law, equation (32.04.02), of infinite extent (unbounded).

Determine the equation of motion.

WORKED SOLUTION

Equation (32.03.07) implies (Lagrangian approach), for a pure deformation, that is ignoring translations and rotations,

$$\partial_t p_k = \rho_0 \frac{\partial^2 \xi_k}{\partial t^2} = f_k + \frac{\partial S_{kj}^{\cdot\cdot}}{\partial x_j} = \quad ,$$

$$f_k + \frac{\partial}{\partial x_j}\left(2\mu F_{kj}^{\cdot\cdot} + \lambda F_{rr}^{\cdot\cdot}\delta_{kj}^{\cdot\cdot}\right) = f_k + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial \xi_k}{\partial x_j} + \frac{\partial \xi_j}{\partial x_k}\right) + \lambda \frac{\partial \xi_r}{\partial x_r}\delta_{kj}^{\cdot\cdot}\right) = \quad ,$$

$$\boxed{f_k + \mu \nabla^2 \xi_k + (\mu + \lambda)\frac{\partial}{\partial x_k}\mathbf{div}\boldsymbol{\xi} = \rho_0 \frac{\partial^2 \xi_k}{\partial t^2}} \quad ,$$

$$\boxed{f_k - \mu\, \mathbf{e}_k \cdot (\mathbf{rotrot}\boldsymbol{\xi}) + (2\mu + \lambda)\frac{\partial}{\partial x_k}\mathbf{div}\boldsymbol{\xi} = \rho_0 \frac{\partial^2 \xi_k}{\partial t^2}} \quad .$$

In vector form the two equivalent equation of motion can be written as:

$$\boxed{\mathbf{f} + \mu \boldsymbol{\nabla}^2 \boldsymbol{\xi} + (\mu + \lambda)\,\mathbf{graddiv}\boldsymbol{\xi} = \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}} \quad , \qquad (32.05.01)$$

→ 1661

which is equivalent to:

$$\boxed{\mathbf{f} + (2\mu + \lambda)\,\boldsymbol{\nabla}^2 \boldsymbol{\xi} + (\mu + \lambda)\,\mathbf{rotrot}\boldsymbol{\xi} = \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}} \quad . \qquad (32.05.02)$$

→ 1661 1662

Note that it is in fact often easier to write the equation of motion in terms of $\boldsymbol{\xi}$ than in terms of $\mathbb{F}$.

32.06 Waves in an Unbounded Homogeneous and Isotropic Elastic Body

This § is referenced at pages:
[2034]

Assume no volume forces: $\mathbf{f} = 0$.

32.06.01 Shear Waves (Solenoidal Waves)

Assume that the deformation in the medium is such that no change in volume takes place, that is $\mathbf{div}\boldsymbol{\xi} = 0$. These solenoidal waves can be described by a vector potential (in any two-connected domain) and their wave equation is easily deduced from equation (32.05.01).

$$\mu\boldsymbol{\nabla}^2\boldsymbol{\xi} = \rho_0\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} \Longrightarrow c_{\mathrm{T}}^2 = \frac{\mu}{\rho_0} = \frac{G}{\rho_0} \quad .$$

These waves are also called iso-voluminous waves.

32.06.01.01 Rotations Associated With an Arbitrary Deformation

Taking the rotor equation (32.05.01), for $\mathbf{f} = 0$, one obtains ($\mathbf{rotgrad}() = 0$):

$$\mu\boldsymbol{\nabla}^2\mathbf{rot}\boldsymbol{\xi} = \rho_0\frac{\partial^2}{\partial t^2}\mathbf{rot}\boldsymbol{\xi} \quad .$$

This is actually a wave equation for the rotor of the deformation, $\mathbf{rot}\boldsymbol{\xi}$, which propagates at the same speed as shear waves.

32.06.02 Dilatational Waves (Irrotational Waves)

Assume that the deformation in the medium is such that no rotation takes place, that is $\mathbf{rot}\boldsymbol{\xi} = 0$. These irrotational waves can be described by a scalar potential (in any one-connected domain) and their wave equation is easily deduced from equation (32.05.02).

$$(2\mu+\lambda)\,\boldsymbol{\nabla}^2\boldsymbol{\xi} = \rho_0\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} \Longrightarrow c_{\mathrm{L}}^2 = \frac{2\mu+\lambda}{\rho_0} = \frac{B+\frac{4}{3}G}{\rho_0} = \frac{Y\,(1-\Sigma)}{(1+\Sigma)\,(1-2\Sigma)} \quad .$$

This result shows that irrotational waves in an unbounded homogeneous and isotropic elastic body have a different velocity than purely longitudinal waves along a bar (section § 45.02.02). The case of section § 45.02.02 is recovered by setting $\Sigma = 0$, as it was implicit in the derivation in that section.

Even if these waves are called dilatational waves note that the deformation is not a pure dilatation as the shear module appears in addition to the bulk modulus.

These waves feature two independent states of Polarizarion.

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32.06.02.01 Dilatations Associated With an Arbitrary Deformation

Taking the divergence of equation (32.05.02), for $\mathbf{f} = 0$, one obtains ($\mathbf{divrot}() = 0$):

$$(2\mu + \lambda)\, \boldsymbol{\nabla}^2 \mathbf{div}\boldsymbol{\xi} = \rho_0 \frac{\partial^2}{\partial t^2} \mathbf{div}\boldsymbol{\xi} \quad .$$

This is actually a wave equation for the divergence of the deformation, $\mathbf{div}\boldsymbol{\xi}$, which propagates at the same speed as dilatational waves.

32.06.03
Properties of Solenoidal and Irrotational Waves

Helmholtz theorem, read § 16.07.05, states that any vector field can be decomposed uniquely (apart from additive constants) into the sum of a solenoidal and an irrotational vector fields. Moreover, if both rotor and divergence vanish in a certain region of space the vector field may be taken to be zero.

As a consequence of the above properties any deformation can be decomposed into into the sum of a solenoidal and an irrotational wave and the two waves propagate independently of one another with different speeds.

The different speeds have an impact on the kinematics, for instance via Snell law.

32.06.04
PW and Dispersion Relation

32.07 Examples and Physical Applications

32.08 Examples and Physical Applications

32.08.01
An Ideal Fluid at Equilibrium

Consider a system at rest. If the ellipsoid describing the force-stress tensor is a sphere there are only normal forces in any direction. This corresponds to hydrostatic pressure (either positive or negative). In this case, a totally isotropic system, the tensor is diagonal and all three components equal to the pressure p:

$$S^{\cdot\cdot}_{jk} = -p\delta^{\cdot\cdot}_{jk} \quad ,$$

in the Reference Frame where the system is at rest. The system is a ideal fluid.

Note that this shows that pressure, normally introduced as a scalar quantity, is actually a tensor.

Apply equation (32.03.06) to consider an equilibrium situation:

$$\iiint_\Omega f_k \, \mathrm{d}V - \delta^{\cdot\cdot}_{kj} \oiint_{\partial\Omega} p \, \mathrm{d}\mathbf{S}_j = 0 \longrightarrow \iiint_\Omega f_k \, \mathrm{d}V - \delta^{\cdot\cdot}_{kj} \iiint_\Omega \partial_j p \, \mathrm{d}V = 0 \longrightarrow \mathbf{f} = \mathbf{grad} p \quad .$$

The result shows that at equilibrium the volume forces must equal the gradient of the pressure (Archimedes' principle). Read also § 16.12.15.

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32.08.02
Pressure Waves in an Ideal Fluid

This § is referenced at pages:
[2027, 2029]

A fluid (read § 28.01) is defined as a system such that at equilibrium no shear stress exist. Outside equilibrium shear stress may exist, if viscosity is not negligible. An ideal fluid is such that viscosity is negligible.

Consider an ideal fluid at equilibrium. Let $p_0[\mathbf{x}]$ be the pressure field at equilibrium. Define the differential pressure, that is the variation of pressure at any point with respect to the equilibrium state. In terms of the absolute pressure, $p[\mathbf{x},t]$, we thus define:

$$p[\mathbf{x},t] \equiv P[\mathbf{x},t] - p_0[\mathbf{x}] \quad .$$

The three pressure fields are scalar fields. The displacement vector $\boldsymbol{\xi}[\mathbf{x},t]$ is a vector field. It is therefore easier to describe the situation using the pressure, a scalar field, than with the displacement, a vector field.

Consider a small parallelepiped with sides Δx, Δy and Δz. Assume that the sides are small, one will take the limit for the sides going to zero in the end. Assume that volume forces have a volume density $\mathbf{g}[\mathbf{x},t]$. Along the z axis we have:

$$\mathbb{d}F_z \simeq \left(P[x,y,z,t] - P[x,y,z+\Delta z,t]\right)\Delta x\Delta y + \rho g[x,y,\widetilde{z}]\Delta x\Delta y\Delta z + \mathcal{O}\left(\Delta z\right)^2 \qquad \text{with } z \leq \widetilde{z} \leq z+\Delta z \quad , \quad ,$$

$$\mathbb{d}F_z = \left(g - \frac{\partial P}{\partial z}\right)\Delta x\Delta y\Delta z = \rho_0 \frac{\partial^2 \xi_x}{\partial t^2} \quad .$$

The equation of motion follows, by putting together the three dimensions:

$$\boxed{\rho_0 \mathbf{g} - \mathbf{grad}\, P = \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}} \quad ,$$

where ρ_0 may or may not depend on the position.

At equilibrium:

$$\rho_0 \mathbf{g} - \mathbf{grad} p_0 = 0 \quad ,$$

and then

$$\boxed{-\mathbf{grad} p = \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}} \quad .$$

The latter is the equation to use in order to study wave propagation in a ideal fluid, as, in this case, only pressure differences with respect to the equilibrium configuration are relevant. With the differential pressure the (constant) gravity force disappears.

Assume that ρ_0 does not depend on the position. The equation of motion can be written entirely in terms of the pressure by taking the divergence of both sides:

$$-\nabla^2 p = \rho_0 \frac{\partial^2}{\partial t^2}(\mathbf{div}\,\boldsymbol{\xi}) = -\frac{\rho_0}{B}\frac{\partial^2 p}{\partial t^2} \quad ,$$

in terms of the bulk modulus and exploiting the fact that the divergence of $\boldsymbol{\xi}$ is the relative volume change.

The speed of sound follows:

$$\boxed{c^2 = \frac{B}{\rho_0}} \quad .$$

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32.08.03
Viscous Newtonian Fluids

This § is referenced at pages:
[1655, 2742]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|41.2||
©|T.Padmanabhan, *Theoretical Astrophysics, Vol. 1,2,3*, 2000/2001/2002, CUP, ...Ed., ...|Vol 1, chapt 8||

32.08.03.01 General Definition of Viscosity in the Local Rest Frame

A fluid is, by definition, a continuous medium which which does not keep any shear stress when it is at rest with respect to any Coordinate System. In the case shear stresses are present the fluid flows, even if it might flow with a very slow speed. This means, in particular, that a fluid moving as a rigid-body does not suffer shear stresses.

The stress tensor of a fluid at rest is thus symmetrical, due to the lack of shear stresses, but also isotropic, that is proportional to the identity matrix, thanks to Pascal Principle: The stress tensor of any fluid at rest:

$$\boxed{S^{\cdot\cdot}_{kj} = -p\delta^{\cdot\cdot}_{kj}} \quad .$$

In the case a real fluid is not at rest shear stresses appear, giving rise to an additional non isotropic term, the symmetrical rank-two tensor $\Pi^{\cdot\cdot}_{ij}$. Consider the comoving rest frame of the fluid, that is the local Rest Frame of the infinitesimal volume of fluid, moving at the velocity of the Center-Of-Mass of the infinitesimal volume of fluid; in this frame the bulk flow vanishes. In the comoving rest frame of the fluid the stress tensor is thus generalized to:

$$\boxed{S^{\cdot\cdot}_{ij} = -p\delta^{\cdot\cdot}_{ij} + \Pi^{\cdot\cdot}_{ij} \qquad \text{generic real fluid}} \quad .$$

Shear stresses are cause by viscous forces, giving rise to linear momentum transfer.

Note in the comoving Rest Frame convection terms are missing. Also note that in the comoving Rest Frame the gradient is meaningful: even if the velocity of the Center-Of-Mass is zero in the comoving Rest Frame the gradient can be non zero.

Viscous forces depend on the different velocities at neighboring points inside the fluid, that is they depend on the gradient of the velocity field. Newtonian fluids, by definitions, are fluids such that the viscosity tensor, $\Pi^{\cdot\cdot}_{ij}$, depends linearly on the gradient of the velocity. This is typically a good model for a real fluid only in the case the gradient of the velocity is small:

$$\eta\left|\frac{\partial v_j}{\partial x_k}\right| \ll p \qquad \eta'\left|\frac{\partial v_j}{\partial x_k}\right| \ll p \quad .$$

Note that as $\Pi^{\cdot\cdot}_{ij}$ contains the gradient of the velocity it is in general non null in the comoving rest frame (velocities with respect to the Center-Of-Mass).

The tensor $\dfrac{\partial v_j}{\partial x_k}$ is called the gradient velocity tensor or rate of strain tensor.

One should only consider the symmetrical part of the gradient velocity tensor because the anti-symmetrical part gives rise to a rigid rotation which produces no shear stresses.

Let

$$v^{\cdot\cdot}_{ij} \equiv \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) = \partial_t F^{\mathtt{s}\,\cdot\cdot}_{ij} \qquad \text{Lagrangian approach, x is a label not a variable} \quad .$$

In the comoving Rest Frame the fluid is isotropic. Therefore the same arguments used in section § 32.04 for isotropic elastic media lead to conclude that the only possible terms are a term proportional to the gradient velocity tensor plus a isotropic term proportional to the trace of the gradient velocity tensor.

WORK IN PROGRESS

Therefore, for a real Newtonian fluid in the comoving Rest Frame, one has:

$$\Pi^{\cdot\cdot}_{ij} = \eta' \delta^{\cdot\cdot}_{ij} \mathbf{div}\mathbf{v} + 2\eta v^{\cdot\cdot}_{ij} = \eta' \delta^{\cdot\cdot}_{ij} \mathbf{div}\mathbf{v} + \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = \eta' \delta^{\cdot\cdot}_{ij} \mathbf{div}\mathbf{v} + 2\eta \partial_t {F^{\mathrm{s}}}^{\cdot\cdot}_{ij} \qquad \text{real Newtonian fluid} \quad .$$

Here η and η' are known as first and second coefficient of viscosity.

The important case of incompressible fluids implies:

$$\eta' = 0 \qquad \text{for incompressible fluids} \quad .$$

Finally one has for a real Newtonian fluid in the comoving Rest Frame:

$$S^{\cdot\cdot}_{ij} = -p\delta^{\cdot\cdot}_{ij} + \eta' \delta^{\cdot\cdot}_{ij} \mathbf{div}\mathbf{v} + \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = -p\delta^{\cdot\cdot}_{ij} + \eta' \delta^{\cdot\cdot}_{ij} \mathbf{div}\mathbf{v} + 2\eta \partial_t {F^{\mathrm{s}}}^{\cdot\cdot}_{ij} \quad . \tag{32.08.01}$$

The force volume density is calculated from:

$$f_k = \frac{\partial}{\partial x_j} S^{\cdot\cdot}_{kj} \implies \mathbf{f} = \eta \boldsymbol{\nabla}^2 \mathbf{v} + (\eta + \eta') \, \mathbf{graddiv}\mathbf{v} \quad . \tag{32.08.02}$$

→ 1665

The use of equation (32.08.02) in the first **Cardinal Equation** gives rise to Navier-Stokes equations.

The force volume density for incompressible fluids is:

$$f_k = \frac{\partial}{\partial x_j} S^{\cdot\cdot}_{kj} \implies \mathbf{f} = \eta \boldsymbol{\nabla}^2 \mathbf{v} \quad . \tag{32.08.03}$$

Note that even if elasticity and viscosity are formally described in a very similar way they are actually very different phenomena.

32.08.04
Viscous Non-Newtonian Fluids

This § is referenced at pages:
[Never referenced.]

32.08.05
Order of Magnitude Estimations: Maximum Size of Non-Spherical Asteroids

This § is referenced at pages:
[1670]

32.08.06
Order of Magnitude Estimations: Maximum Height of Mountains

This § is referenced at pages:
[Never referenced.]

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32.08.07
Torsional Vibrations of a Cylindrical Round Homogeneous and Isotropic Elastic Bar

This § is referenced at pages:
[2033]

Consider a cylindrical, round, homogeneous and isotropic elastic bar of radius R, density ρ_0 and shear modulus G. Write the wave equation for torsional waves, neglecting the presence of volume forces, and show that the wave velocity is:

$$c_T^2 = \frac{G}{\rho_0} \tag{32.08.04}$$

WORKED SOLUTION

Assume that each section of the bar perpendicular to the bar axis will just rotate around the bar axis as a rigid sheet.
See problem § 27.07.12 for the derivation of the internal torques in a static situation. Let the z axis be coincident with the axis of the cylinder, let its origin located at one of the ends and let its orientation be such that the bar is defined by $0 \leq z \leq L$.
When a torsional wave is present the section of the bar at z, $\mathcal{S}[z]$, is rotated, with respect to the equilibrium position, by an angle $\phi[z]$, while the section at $z+\Delta z$, $\mathcal{S}[z+\Delta z]$, is rotated, with respect to the equilibrium position, by an angle $\phi[z+\Delta z]$.
Use the same notations as in problem § 27.07.12, with the time dependence always implicit, and let O be a point on the axis of the bar. The piece of the bar lying between $\mathcal{S}[z]$ and $\mathcal{S}[z+\Delta z]$ is subject to the two torques $\Gamma_O^-[z]$ and $\Gamma_O^+[z+\Delta z]$ on its two ends. Moreover, according to the generalized third Newton law we know that:

$$\Gamma_O^-[z+\Delta z] + \Gamma_O^+[z+\Delta z] = 0 \quad .$$

It follows that the second **Cardinal Equation** of dynamics, written in terms of the momentum of inertia of the piece of the bar, I, applied to the piece of bar, using the theorem of the average value for the integral, gives:

$$\Gamma_O^-[z] + \Gamma_O^+[z+\Delta z] = \Gamma_O^-[z] - \Gamma_O^-[z+\Delta z] = \frac{\mathrm{d}L_z}{\mathrm{d}t} \quad ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_z^{z+\Delta z} \frac{\rho_0 \pi R^4}{2}\omega[z']\,\mathrm{d}z' = \frac{\rho_0 \pi R^4}{2}\Delta z\,\partial_t\omega[z+\tilde{z}] \qquad \text{with} \qquad z \leq \tilde{z} \leq z+\Delta z \quad .$$

The expression of the angular momentum L_z of the piece of the bar has been obtained by integrating over the finite length of the piece of the bar, Δz, whose infinitesimal momentum of inertia is given by

$$\mathrm{d}I = \frac{\rho_0 \pi R^4}{2}\,\mathrm{d}z' \quad .$$

This is motivated by the fact that, by definition, the angular momentum is an additive quantity.
Dividing by Δz and taking the limit for $\Delta z \longrightarrow 0$ we get:

$$-\frac{\partial \Gamma_O^-[z]}{\partial z} = \frac{\rho_0 \pi R^4}{2}\partial_t\omega[z] \tag{32.08.05}$$

→ 1667

The expression for $\Gamma_O^-[z]$ is known from equation (27.07.12):

$$\Gamma_O^-[z] = \frac{\pi G}{2L}\left(r_2^4 - r_1^4\right)\phi \quad .$$

The latter equation must be adapted to the notation and setup of the present problem. Consider equation (27.07.12) applied to the current piece of bar lying between $\mathcal{S}[z]$ and $\mathcal{S}[z+\Delta z]$ and having length Δz (taking the place of the length L of the bar under uniform torsion). The torsion of the

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piece is measured by the angle:

$$\Delta\phi \equiv \phi[z+\Delta z] - \phi[z] \quad .$$

The linear momentum acting on the piece from the left of section $\mathcal{S}[z]$ and its limit for $\Delta z \longrightarrow 0$ are given by:

$$\Gamma^-_O[z;\Delta z] = -\frac{\pi G R^4}{2}\frac{\Delta\phi}{\Delta z} \qquad \Gamma^-_O[z] = \lim_{\Delta z \longrightarrow 0} \Gamma^-_O[z;\Delta z] = -\frac{\pi G R^4}{2}\frac{\partial\phi}{\partial z} \quad . \tag{32.08.06}$$

→ 1667

In fact in the formula (27.07.12) the torsion angle ϕ represents the overall torsion of the bar of length L under uniform torsion. In fact in the current framework the torsion angle of the piece of bar of length Δz is $\Delta\phi$. Note the minus sign, deriving from the fact that a positive torque $\Gamma^-_O[z]$ would produce a negative $\Delta\phi$ because it would tend to rotate the section $\mathcal{S}[z]$ in the positive angular direction more than it rotates the section $\mathcal{S}[z+\Delta z]$ thus producing a negative $\Delta\phi$.
The equation of motion follows from equations (32.08.05), (32.08.06):

$$\frac{\partial^2\phi}{\partial z^2} = \frac{\rho_0}{G}\frac{\partial^2\phi[z]}{\partial t^2} \qquad c_T^2 = \frac{G}{\rho_0} \quad .$$

Note that no approximation has been done within the elastic behavior.
Because of the relation (27.02.06) the speed of torsional waves is always less than the speed of longitudinal waves.

32.08.08
Hydrostatic Equilibrium of a Non-Rotating Gravitating Fluid Mass

This § is referenced at pages:
[1534]

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Read also § 28.07.14.

Consider a spherically symmetrical non rotating fluid mass (for instance a star) at equilibrium. Equilibrium is given by the balance between gravity and pressure. Let the mass density be $\rho[r]$ and and the radius R (that is $\rho[r] = 0$ for $r > R$). Assume also that $p[R] = 0$.

The equilibrium conditions (first **Cardinal Equation**) is:

$$f_k + \frac{\partial s_{kj}^{\cdot\cdot}}{\partial x_j} = 0 \quad .$$

In the present case the equilibrium condition becomes:

$$-\mathbf{grad}p[r] + \rho[r]g[r] = -\mathbf{grad}p[r] - \rho[r]\mathbf{grad}\phi[r] = 0 \quad ,$$

$$\mathbf{g}[r] = -\frac{\mathcal{G}M[r]}{r^2}\hat{\mathbf{e}}_r \quad ,$$

$$M[r] = \iiint_{V[r]} \rho[r]\, \mathbb{d}V = \int_{r=0}^{r} \rho[s] 4\pi s^2\, \mathbb{d}s \quad ,$$

$$\frac{\mathbb{d}M[r]}{\mathbb{d}r} = 4\pi\rho[r]r^2 \quad .$$

From the expression of the gradient in spherical coordinates, taking into account that there is only radial dependence, by assumption, one finds:

$$\frac{\mathbb{d}p[r]}{\mathbb{d}r} = -\rho[r]\frac{\mathcal{G}M[r]}{r^2} = -\frac{\mathbb{d}M[r]}{\mathbb{d}r}\frac{1}{4\pi r^2}\frac{\mathcal{G}M[r]}{r^2} < 0 \quad .$$

In general it is necessary to know a relation between pressure and density, that is the equation of state of the medium, to be able to solve the equation. The equation of state of the medium in general, will involve the temperature, too. Therefore the problem is a thermodynamical problem.

IN PROGRESS

However, regardless of the equation of state some general results can be derived.

One obvious conclusion is that the pressure is a decreasing function, that is the maximum pressure is a $r = 0$.

Another conclusion can be drawn by observing that the basic hydrostatic equilibrium equation can be re-written as:

$$\frac{\mathrm{d}p[r]}{\mathrm{d}r} = -\frac{\mathcal{G}}{4\pi}\left(\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{M^2[r]}{2r^4}\right) + 2\frac{M^2[r]}{r^5}\right) \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}r}\left(p[r] + \frac{\mathcal{G}}{8\pi}\frac{M^2[r]}{r^4}\right) = -\frac{\mathcal{G}}{2\pi}\frac{M^2[r]}{r^5} \leq 0 \quad .$$

The result implies that the function $p[r] + \dfrac{\mathcal{G}}{8\pi}\dfrac{M^2[r]}{r^4}$ is decreasing.

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(p[r] + \frac{\mathcal{G}}{8\pi}\frac{M^2[r]}{r^4}\right) \leq 0 \quad .$$

Moreover, the term added to pressure goes to zero for small r:

$$\frac{\mathcal{G}}{8\pi}\frac{M^2[r]}{r^4} \sim 0 \qquad \text{for } r \sim 0 \qquad \text{as } M^2[r] \sim r^6 \quad .$$

Therefore:

$$p[r=0] > p[r] + \frac{\mathcal{G}}{8\pi}\frac{M^2[r]}{r^4} > p[R] + \frac{\mathcal{G}}{8\pi}\frac{M^2[R]}{R^4} \Longrightarrow \boxed{p[r=0] > \frac{\mathcal{G}M^2[R]}{8\pi R^4} \equiv p_1 \qquad \text{exact}} \quad .$$

This limit is independent of the equation of state of the medium.

Note that the above equations, in fact, are a good approximation for solid planets, as well as for stars and liquid planets. In fact at the enormous stresses encountered in the interior of a solid planet, the strains are so large that plastic flow will occur, that is the limiting shear stresses are much smaller than the isotropic part of the stress tensor.

Note that this derivation of the static equilibrium solution does not give insight into the stability of the solution. In order to study oscillation of stars one should consider small motions around the stable equilibrium solution. In order to accomplish this task one must reintroduce the time derivative term in the first **Cardinal Equation** and then consider small time-dependent motion around the stable equilibrium solution.

32.08.08.01 Molecular Gas - the Polytropic Model for the Equation of State

One often used approximation is the polytropic one, describing a family of equations of state:

$$\boxed{p \propto \rho^{\frac{n+1}{n}}} \quad ,$$

where n is the polytropic index, defining the equation of state.

- The giant planets, Jupiter and Saturn mainly comprise a fluid made of Hydrogen and Helium which is well approximated by $n = 1$; the density of a small planets, like Mercury, is very roughly constant, $n = 0$.
  - $n = 0$: the density is constant; often used for small planets where the small gravity might allow to neglect the change in density due to the change in pressure.
  - $n = 1$: used for big planets.
- $n = (1.5 \div 3.0)$: used for stars.
  Low mass white dwarf stars are well approximated by $n = 1.5$; red giant stars are well approximated by $n = 3.0$.

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32-008 Solution for Uniform Density

Solve the equation of hydrostatic equilibrium for uniform density.

32.08.08.02 Photon Gas - Radiation Pressure

For a gas of thermal photons with vanishing chemical potential (that is an isotropic, black-body distribution) the radiation pressure is proportional to the fourth power of the absolute temperature. In fact, in terms of the radiation constant a, the energy per unit volume, $\rho \equiv u$ and pressure are given,

$$\rho \equiv u = aT^4 \qquad p = \rho/3 \qquad a \equiv \frac{8\pi^5}{15}\frac{k^4}{h^3c^3} = 7.56{\cdot}10^{-16}\ \mathrm{Jm^{-3}K^{-4}} \quad . \tag{32.08.07}$$

→ 1669

Equation (32.08.07) is in fact the equation of state for a gas of photons. When the Universe was younger than about 10^5 years, its energy density and pressure were predominantly due to thermal photons and neutrinos, contributed roughly the same amount, so its equation of state was given by equation (32.08.07) with a slight different coefficient.

In the case of stars a relevant role is played by radiation pressure: in addition to the pressure produced by matter there is the pressure produced by radiation (photons). Radiation pressure must be summed to the pressure produced by matter. In some cases it might exceed the pressure of matter.

32.08.08.03 Some Applications

Estimation of the Central Pressure

$$\frac{\mathrm{d}p[r]}{\mathrm{d}r} = -\rho[r]\frac{\mathcal{G}M[r]}{r^2} \quad ,$$

$$\frac{\Delta r}{\Delta R} \approx -\frac{p_0}{R} \, ,$$

$$\frac{\Delta r}{\Delta R} \approx -\langle\rho\rangle\frac{\mathcal{G}M}{R^2} \, ,$$

$$p_0 \approx \frac{\mathcal{G}M\langle\rho\rangle}{R} = \frac{3\mathcal{G}M^2}{4\pi R^4} \qquad \text{approximate} \quad .$$

$$p_0 \approx \frac{\mathcal{G}M\langle\rho\rangle}{R} = \frac{3\mathcal{G}M^2}{4\pi R^4} > p_1 \quad .$$

Estimations for the Sun

Assume for the Sun: $M = 2{\cdot}10^{30}$ kg; $R = 6.95{\cdot}10^5$ km; $\langle\rho\rangle = 1.42{\cdot}10^3$ kg/m^3.

$$p_0 \approx \frac{\mathcal{G}M\langle\rho\rangle}{R} = 2.7{\cdot}10^{14}\ \mathrm{Pa} \quad .$$

The inequality above gives: $p_1 > 4.5{\cdot}10^{13}$ Pa.

The accurate calculation gives: $p_0 = 2.5{\cdot}10^{14}$ Pa.

The surface acceleration of gravirty is:

$$g = \frac{\mathcal{G}M}{R^2} = 2.8{\cdot}10^2\ \mathrm{m/s^2} \quad .$$

The estimate of the central temperature of the Sun using the law of perfect gases gives: $T_0 = 10^7$ K.

The accurate calculation gives: $T_0 = 1.6{\cdot}10^7$ K.

PRELIMINARY - IN PROGRESS

Neutron Stars

Assume for a typical neutron star: $M = 2{\cdot}10^{30}$ kg; $R = 10$ km; $\langle\rho\rangle = 4.8{\cdot}10^{17}$ kg/m^3.

$$p_0 \approx \frac{\mathcal{G}M\langle\rho\rangle}{R} = 6.4{\cdot}10^{33}\ \text{Pa} \quad .$$

The inequality above gives: $p_1 > 1.1{\cdot}10^{33}$ Pa.

The surface acceleration of gravity is:

$$g = \frac{\mathcal{G}M}{R^2} = 1.3{\cdot}10^{12}\ \text{m/s}^2 \quad .$$

The gravitational potential energy (assuming uniform density) is:

$$\Phi_g \simeq -\frac{3}{5}\frac{\mathcal{G}M^2}{R} = -2{\cdot}10^{46}\ \text{J} \quad .$$

The number of nucleons is: $N = M/m_N = 1.2{\cdot}10^{57}$.

The average distance among neutrons is 1.5 fm.

Given the typical binding energy of nuclear matter, $U_b = 8$ MeV/nucleon, one can estimate the nuclear interaction energy: $U_n = NU_b = -1.5{\cdot}10^{45}$ J.

The gravitational potential energy increases with decreasing radius.

32.08.09

Hydrostatic Equilibrium of a Rotating Self-Gravitating Liquid Mass Body

Consider a mass of liquid of constant and uniform density ρ, mass M, self-gravitating and rotating with angular velocity Ω. Determine the shape of equilibrium in case of small Ω.

Neglect to a first approximation the effect on the gravitational field of the change of shape. This approximation can be justified in the case most of the mass is concentrated near the center of the liquid mass. As a second approximation include the effect of the change of the gravitational field by developing the gravitational potential to second-order.

WORKED SOLUTION

Let θ be the latitude and ϕ be the colatitude measured from the north-pole, such that $\theta + \phi = \pi/2$. Let the z axis be coincident with the rotation axis of the body, oriented as the angular velocity and with its origin at the center of the body. The body will assume a shape which is a revolution figure around the rotational axis. Assume that the body is rotating like a rigid-body, that is the steady state when internal viscous forces have dissipated the kinetic energy with respect to the Center-Of-Mass. Use a Reference Frame rotating with the body. At equilibrium the free surface of the body will be an equipotential surface, that is perpendicular to the volume forces, because a liquid at rest cannot resist shear stresses. The shape of the equipotential surfaces must be sought for, in particular the one corresponding to the surface of the body. This will be described by a function:

$$r[\theta] = a + \delta r\,[\theta] \quad , \tag{32.08.08}$$

→ 1671 1672 1673

where a is some reference, constant, radius, typically the equatorial radius or the polar radius. In any case $\delta r\,[\theta]$ will describe the deviation from a spherical shape.

The centrifugal potential, expressed in polar coordinates, is given by:

$$V_{\text{CEN}}[r,\theta] = -\frac{1}{2}\Omega^2 r^2\cos^2[\theta] = -\frac{1}{2}\Omega^2 r^2\sin^2[\phi] \quad .$$

Note that the requirement of small angular velocity implies that the centrifugal potential is small with respect to the gravitational potential. Take for instance a to be either the equatorial radius. The condition, in terms of the q parameter, the ration between the centrifugal and gravitational force at the equator, becomes:

$$\boxed{q \equiv \frac{\Omega^2 a^3}{2\mathcal{G}M} \ll 1} \quad \Longrightarrow \quad \frac{\Omega^2 a^3}{2\mathcal{G}M}\cos^2[\theta] \ll 1 \quad , \tag{32.08.09}$$

→ 1673

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where the condition $q \ll 1$ is a sufficient condition in order that the centrifugal potential is small with respect to the gravitational potential at any latitude. It is also obvious that is th condition is satisfied near the surface it will be valid at any interior point because the centrifugal to gravitational force ratio increases with the distance.

32.08.09.01 First Approximation: Neglect the Change of Gravitational Field Due to the Flattening

In this approximation one neglects the gravitational feedback due to the flattening of the body while rotating. The total (gravitational plus centrifugal) potential is thus:

$$V_{\text{TOT}}[r,\theta] = V_{\text{GRA}}[r,\theta] + V_{\text{CEN}}[r,\theta] = -\frac{\mathcal{G}M}{r} - \frac{1}{2}\Omega^2 r^2 \cos^2[\theta] \quad . \tag{32.08.10}$$

→ 1671

The expression (32.08.08) must be put into equation (32.08.10). All the constant terms must be set apart, inglobing them into a constant K, because in general we are looking for all the equipotential surfaces. In the non-constant terms, that is the terms containing either r or θ, the approximation

$$|\,\delta r\,[\theta]| \ll a$$

is to be done.
The result is:

$$V_{\text{TOT}}[r,\theta] = -\frac{\mathcal{G}M}{a} + \frac{\mathcal{G}M}{a^2}\,\delta r\,[\theta] - \frac{1}{2}\Omega^2\cos^2[\theta]\left(a^2 + 2a\,\delta r\,[\theta] + (\,\delta r\,[\theta])^2\right) \simeq \quad ,$$

$$K + \frac{\mathcal{G}M}{a^2}\,\delta r\,[\theta] - \frac{1}{2}\Omega^2 a^2\cos^2[\theta] \quad .$$

In order to have a constant potential one must then have:

$$\frac{\mathcal{G}M}{a^2}\,\delta r\,[\theta] - \frac{1}{2}\Omega^2 a^2\cos^2[\theta] = \text{constant} \qquad \Longrightarrow \qquad \boxed{\delta r\,[\theta] = K' + \frac{\Omega^2 a^4}{2\mathcal{G}M}\cos^2[\theta]}$$

The value of the constant K' will depend on the value chose for a in equation (32.08.08). If one chooses the equatorial radius for a one has:

$$\delta r\,[\theta = \pm\pi/2] = 0 \qquad \Longrightarrow \qquad K' = -\frac{\Omega^2 a^4}{2\mathcal{G}M} \qquad \Longrightarrow \qquad \boxed{\delta r\,[\theta] = \frac{\Omega^2 a^4}{2\mathcal{G}M}\left(\cos^2[\theta] - 1\right) = -\frac{\Omega^2 a^4}{2\mathcal{G}M}\sin^2[\theta]} \quad .$$

The flattening, f, and the q parameter are thus:

$$f \equiv \frac{r_{\text{EQ}} - r_{\text{PO}}}{r_{\text{EQ}}} \simeq \frac{\Omega^2 a^3}{2\mathcal{G}M} \equiv \frac{q}{2} \quad .$$

Note that a rough estimate of the maximum allowed angular velocity for the body, before braking apart due to the effect of the centrifugal forces, is:

$$q \approx 1 \qquad \Longrightarrow \qquad \Omega_{\text{MAX}} \approx \left(\frac{\mathcal{G}M}{a^3}\right)^{1/2} \approx a\,(\mathcal{G}\langle\rho\rangle)^{1/2} \quad . \tag{32.08.11}$$

→ 1671

In the case of Jupiter, for instance, the actual orbital period is $T_J = 9.9$ hours with a $T_{J|MIN} = 2.9$ hours.
Equation (32.08.11) giving a rough estimate of the angular velocity such the a rotating object without internal cohesion breaks apart can also roughly explain the breaking apart of a collapsing proto-star giving rise to a proto-planetary system.

32.08.09.02 Second Approximation: Account for the Change of Gravitational Field Due to the Flattening

As a second approximation one can try to account for or the change of gravitational field due to the flattening. Planets, but not always asteroids, compare with § 32.08.05, can be approximated as oblate spheroids, that is an ellipsoid with two identical axes, and with the equatorial radius greater than the polar radius the latter acting as the axis of symmetry of the system.

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The potential external to any body with an axis of symmetry can be developed in terms of terms of the spherical harmonics (see equation (29.05.11)):

$$V_{\text{GRA}}[r,\theta] = -\frac{\mathcal{G}M}{r}\left(1 - \sum_{k=2}^{\infty} J_k \frac{R^k}{r^k} P_k[\cos[\phi]]\right) \quad \text{with } R \text{ the equatorial radius} \quad . \tag{32.08.12}$$

→ 1672

Note that $J_1 = 0$ if, as usual, the origin is chosen at the Center-Of-Mass of the body. In order to make sure to stay outside the mass distribution one can use the equatorial radius for R and The potential thus obtained, when all terms are summed, is the exact potential externally to the mass distribution. If one wants the potential far and far away the potential can be calculated by truncating the series to a smaller and smaller number of terms. If one is close to the distribution all the terms, in principle, should be used. However, if one knows that the coefficients J_k are sufficiently small one can truncate the series accordingly and estimate the error. Note that the coefficients J_k can be calculated by the gravitational field far away.
Read also § 29-019.
The expression (32.08.12) can be used to calculate the gravitational potential at the surface of the body in order to study the equilibrium condition for the liquid at the surface.
The Legendre polynomials of second degree is:

$$P_2[s] = \frac{1}{2}\left(3s^2 - 1\right) \quad .$$

The centrifugal potential is given by:

$$V_{\text{CEN}}[r,\theta] = \frac{1}{3}\Omega^2 r^2 \left(P_2[\cos[\phi]] - 1\right) \quad .$$

The total potential, up to second-order in the development of the gravitational field, is:

$$V_{\text{TOT}}[r,\theta] = V_{\text{GRA}}[r,\theta] + V_{\text{CEN}}[r,\theta] = -\frac{\mathcal{G}M}{r} + \left(\frac{\mathcal{G}Ma^2}{r^3}J_2 + \frac{1}{3}\Omega^2 r^2\right) P_2[\cos[\phi]] - \frac{1}{3}\Omega^2 r^2 \quad . \tag{32.08.13}$$

→ 1672

One could immediately say that the term $\frac{1}{3}\Omega^2 r^2$, as well as its changes, is negligible with respect to the term $\frac{\mathcal{G}M}{r}$, as well as its changes. Both terms only depend on r only and have no angular dependence. The angular dependence in the centrifugal potential term is elsewhere. The centrifugal term could be then immediately dropped. However, we can keep it to show at the end that it has no impact.
The expression (32.08.08) must be put into equation (32.08.13). All the constant terms must be set apart, inglobing them into a constant K, because in general we are looking for all the equipotential surfaces. In the non-constant terms, that is the terms containing either r or θ, the approximation

$$|\,\delta r\,[\theta]| \ll a$$

is to be done.

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One then finds:

$$V_{\text{TOT}}[r,\theta] = -\frac{\mathcal{G}M}{a} + \frac{\mathcal{G}M}{a^2}\,\delta r\,[\theta] + \quad ,$$

$$\left(\frac{\mathcal{G}Ma^2}{a^3}J_2\left(1 - 3\frac{\delta r\,[\theta]}{a} + \mathcal{O}\left(\delta r\,[\theta]\right)^2\right) + \frac{1}{3}\Omega^2a^2\left(1 + 2\frac{\delta r\,[\theta]}{a} + \mathcal{O}\left(\delta r\,[\theta]\right)^2\right)\right)P_2[\cos[\phi]] - \quad ,$$

$$\frac{1}{3}\Omega^2a^2\left(1 + 2\frac{\delta r\,[\theta]}{a} + \mathcal{O}\left(\delta r\,[\theta]\right)^2\right) \simeq \quad ,$$

$$K + \frac{\mathcal{G}M}{a^2}\,\delta r\,[\theta] + \left(\frac{\mathcal{G}Ma^2}{a^3}J_2 + \frac{1}{3}\Omega^2a^2\right)P_2[\cos[\phi]] - \frac{2}{3}\Omega^2a\,\delta r\,[\theta] = \quad ,$$

$$K + \left(\frac{\mathcal{G}M}{a^2} - \frac{2}{3}\Omega^2a\right)\,\delta r\,[\theta] + \left(\frac{\mathcal{G}Ma^2}{a^3}J_2 + \frac{1}{3}\Omega^2a^2\right)P_2[\cos[\phi]] \simeq \quad ,$$

$$K + \frac{\mathcal{G}M}{a^2}\,\delta r\,[\theta] + \left(\frac{\mathcal{G}Ma^2}{a^3}J_2 + \frac{1}{3}\Omega^2a^2\right)P_2[\cos[\phi]] \quad ,$$

where the last passage follows from equation (32.08.09).
In order to have a constant potential one must then have:

$$\frac{\mathcal{G}M}{a^2}\,\delta r\,[\theta] + \left(\frac{\mathcal{G}Ma^2}{a^3}J_2 + \frac{1}{3}\Omega^2a^2\right)P_2[\cos[\phi]] = \text{constant} \quad ,$$

$$\Longrightarrow \quad \boxed{\delta r\,[\theta] = K' - \left(J_2 + \frac{\Omega^2a^3}{3\mathcal{G}M}\right)aP_2[\cos[\phi]]} \quad .$$

The value of the constant K' will depend on the value chose for a in equation (32.08.08). If one chooses the equatorial radius for a one has:

$$\delta r\,[\theta = \pm\pi/2] = 0 \qquad \Longrightarrow \qquad ,$$

$$K' = -\frac{a}{2}\left(J_2 + \frac{\Omega^2a^3}{3\mathcal{G}M}\right) \qquad \Longrightarrow \qquad ,$$

$$\boxed{\delta r\,[\theta] = -\frac{a}{2}\left(3J_2[\,\delta r\,[\theta]] + \frac{\Omega^2a^3}{2\mathcal{G}M}\right)\cos^2[\phi]} \quad .$$

The result is consistent with the result obtained with the previous approximation, equivalent to setting $J_2 = 0$.
Note, as emphasised in the relation (32.08.09.02) itself, the relation is actually an implicit equation for $\delta r\,[\theta]$ because J_2 is actually a function of $\delta r\,[\theta]$ itself.

→ 1673

32-009 Energy-LinMomentum Conservation of an Ideal Fluid

Verify by direct calculation from equation (**??**) that Energy-LinMomentum is conserved:

$$\partial_\beta T^{\alpha\beta}_{\cdot\cdot} = 0 \quad .$$

CHAPTER 33

Some Additional Topics in Mechanics

33.01 Constants of Motion and Integrals of Motion 1676
33.02 Additional Exercises Problems and Physical Applications. 1679

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33.01 Constants of Motion and Integrals of Motion

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Consider a generic system with n degrees of freedom, defined by generalized coordinates and momenta,

$$q_i \qquad p_i \qquad i = 1, \ldots n \quad .$$

Mathematically, the equations of motion constitute

- either a set of n second-order equations for n unknown functions $q_i[t]$;
- or a set of $2n$ first-order equations for $2n$ unknown functions $q_i[t]$ and $p_i[t]$.

The general solution contains in either case $2n$ arbitrary constants. To determine these constants and to define uniquely the motion of the system, it is necessary to know the initial conditions, which specify the state of the system at some given instant.

Constants of motion are functions of phase-space coordinates and time which are constant along orbits:

$$C[q_i[t], p_i[t], t] = \text{constant} \quad ,$$

where $q_i[t]$ and $p_i[t]$ are a solution to the equations of motion.

The function $C[q_i[t], p_i[t], t]$ must be constant along every trajectory, with a value which may depend on the trajectory. In a phase-space of $2n$ dimensions there are always $2n$ independent constants of motion. For example, the $2n$ initial conditions of a trajectory can serve as constants of motion. Once the phase-space coordinates at time t are given one can integrate the orbit backwards to $t = 0$ and read off the initial conditions.

Integrals of motion are functions of phase-space coordinates alone which are constant along orbits:

$$I[q_i[t], p_i[t]] = \text{constant} \quad ,$$

where $q_i[t]$ and $p_i[t]$ are a solution to the equations of motion.

An integral of motion can't depend on time. Thus all integrals of motions are constants of motion, but not all constants are integrals. Integrals of motions are important because they constrain the shapes of orbits. In fact, in a phase-space of $2n$ dimensions, the condition $I[q_i[t], p_i[t]] = \text{constant}$ defines a hyper-surface of $2n - 1$ dimensions. Thus if there are N independent integrals of motion then each orbit is confined to a sub-space of $2n - N$ dimensions.

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§6. Energy

DURING the motion of a mechanical system, the $2s$ quantities q_i and $\dot{q}_i$ ($i = 1, 2, ..., s$) which specify the state of the system vary with time. There exist, however, functions of these quantities whose values remain constant during the motion, and depend only on the initial conditions. Such functions are called *integrals of the motion*.

The number of independent integrals of the motion for a closed mechanical system with s degrees of freedom is $2s-1$. This is evident from the following simple arguments. The general solution of the equations of motion contains $2s$ arbitrary constants (see the discussion following equation (2.6)). Since the equations of motion for a closed system do not involve the time explicitly, the choice of the origin of time is entirely arbitrary, and one of the arbitrary constants in the solution of the cquations can always be taken as an additive constant t_0 in the time. Eliminating $t+t_0$ from the $2s$ functions $q_i = q_i(t+t_0, C_1, C_2, ..., C_{2s-1})$, $\dot{q}_i = \dot{q}_i(t+t_0, C_1, C_2, ..., C_{2s-1})$, we can express the $2s-1$ arbitrary constants $C_1, C_2, ..., C_{2s-1}$ as functions of q and $\dot{q}$, and these functions will be integrals of the motion.

Not all integrals of the motion, however, are of equal importance in mechanics. There are some whose constancy is of profound significance, deriving from the fundamental homogeneity and isotropy of space and time. The quantities represented by such integrals of the motion are said to be *conserved*, and have an important common property of being additive: their values for a system composed of several parts whose interaction is negligible are equal to the sums of their values for the individual parts.

It is to this additivity that the quantities concerned owc their especial importance in mechanics. Let us suppose, for example, that two bodies interact during a certain interval of time. Since each of the additive integrals of the whole system is, both before and after the interaction, equal to the sum of its values for the two bodies separately, the conservation laws for these quantities immediately make possible various conclusions regarding the state of the bodies after the interaction, if their states before the interaction are known.

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33.02 Additional Exercises Problems and Physical Applications

33-001 ...

Si risolva il seguente problema usando esclusivamente l'algebra vettoriale.
Sia data una piramide retta a base quadrata di lato l e altezza h e si consideri una delle sue facce laterali.

1. Si determini l'angolo θ formato, al vertice della piramide, tra i due spigoli della faccia della piramide.
2. Verificare che il risultato ottenuto è corretto nei casi limite:
   a) $h \to +\infty$;
   b) $h \to 0$;
   c) $l \to +\infty$;
   d) $l \to 0$.

33-002 ...

Siano dati i due vettori

$$\mathbf{a} = a_1\widehat{\mathbf{e}}_1 + a_2\widehat{\mathbf{e}}_2 \qquad \mathbf{b} = b_1\widehat{\mathbf{e}}_1 + b_3\widehat{\mathbf{e}}_3 \ ,$$

dove a_1, a_2, b_1 e b_3 sono costanti. Calcolare $\mathbf{a}\wedge\mathbf{b}$.

33-003 ...

Siano dati i due vettori

$$\mathbf{a} = a_1\widehat{\mathbf{e}}_1 + a_2\widehat{\mathbf{e}}_2 \qquad \mathbf{b} = x\widehat{\mathbf{e}}_1 + b_2\widehat{\mathbf{e}}_2 \quad .$$

Determinare il valore di x per cui $\mathbf{a}$ e $\mathbf{b}$ sono perpendicolari.

33-004 ...

Si consideri un moto lungo una linea retta (asse x) per cui l'accelerazione soddisfa alla:

$$a[v] = g - kv^2 \qquad v[t=0] = 0 \qquad x[t=0] = 0 \qquad ,$$

dove g e k sono costanti. Determinare $v = v[t]$ e $x = x[t]$.

33-005 ...

Si consideri un punto materiale la cui equazione del moto è data da:

$$x[t] = x_0\cos\omega t \qquad y[t] = y_0\sin\omega t \qquad z[t] = kt \qquad ,$$

con x_0, y_0, ω e k costanti opportune. Si calcoli l'accelerazione del punto materiale e si verifichi che è sempre perpendicolare all'asse z.

33-006 ...

Si consideri un punto materiale la cui equazione del moto piano è data da:

$$x[t] = x_0\cos\omega t \qquad y[t] = y_0\sin\omega t$$

con x_0, y_0 ed ω costanti opportune e positive. Si calcoli:

1. la componente tangenziale e quella normale dell'accelerazione;
2. la componente radiale e quella trasversale dell'accelerazione.

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33-007 ...

Un cannone spara lungo un pendio in salita. Se il cannone spara con alzo α e se il pendio forma un angolo β con l'orizzontale (con $\beta < \alpha$) si determini:

1. quanto deve valere il modulo v_0 della velocità iniziale del proiettile affinché questo colpisca un punto posto a distanza L lungo il pendio;
2. per una velocità iniziale fissata, v_0, determinare per quale valore dell'alzo α il proiettile cade alla massima distanza possibile dal cannone lungo il pendio.

33-008 ...

Un punto materiale si muove nel piano $x - y$ con equazione del moto:

$$x[t] = A \qquad y[t] = k\,(t - t_0)$$

con $A > 0$, $k > 0$ e t_0 costanti opportune.

- Determinare la velocità angolare e l'accelerazione angolare con cui il moto del punto materiale è visto da un osservatore posto nell'origine.
- Determinare la velocità radiale e quella tangenziale.

33-009 ...

Un'auto percorre una strada rettilinea in pianura. C'è del vento che soffia lungo la stessa direzione della strada e nello stesso verso del moto dell'auto. Per questo la pioggia che cade con velocità di modulo v_0 fa un certo angolo θ con la verticale. Se l'auto si muove con legge oraria

$$x[t] = At\,(1 - \exp(-t/\lambda))$$

con $A > 0$ e $\lambda > 0$ costanti opportune.

Determinare, in funzione del tempo, l'angolo con la verticale che fanno le tracce delle gocce di pioggia sui finestrini laterali dell'auto.

33-010 ...

Un aereo viaggia di moto rettilineo uniforme, velocità $\mathbf{v}_0$, ad una quota costante. Ad un certo istante passa esattamente sopra la verticale del centro di una giostra che ruota con velocità angolare Ω costante attorno ad un asse verticale.

1. Determinare l'accelerazione con cui il moto dell'aereo è visto da un osservatore che si trova al centro della giostra e ruota solidalmente con essa (componenti radiale e trasversale).

33-011 ...

Un blocchetto di massa m giace su un piano orizzontale. I coefficienti di attrito statico e cinetico tra il blocchetto e il piano sono μ_s e μ_c, con $\mu_s > \mu_c$. Una fune è collegata al blocchetto e viene tirata formando un angolo θ con l'orizzontale.

1. Determinare per quale angolo θ il modulo della forza necessario per mantenere il blocchetto in moto rettilineo uniforme è minimo.
2. Se $\theta = 0$, il blocchetto è inizialmente fermo e il modulo della forza applicata dipende dal tempo secondo la relazione $F(t) = kt^2$, con $k > 0$ costante, determinare il moto del blocchetto in funzione del tempo.

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33-012 ...

Due blocchetti di masse m_1 ed m_2 sono collegati da una fune ideale. Il blocchetto 1 è poi collegato ad una seconda fune ideale e, tramite questa fune, il tutto viene appeso ad un soffito.

1. Determinare le tensioni delle due funi
2. Se le due funi sono identiche e l'esperimento viene ripetuto con masse di valore progressivamente crescente quale delle due funi è piu probabile che si rompa per prima ? Perché' ?

33-013 ...

Una puleggia ideale ha un peso di massa m che pende ad un estremo della sua fune ideale. All'altro capo della fune c'è un uomo di massa M.

1. Se l'uomo si arrampica sulla fune con accelerazione costante A rispetto alla puleggia (verso l'alto) determinare il moto della massa m.
2. Se $M = m$ e l'uomo si arrampica sulla fune con velocità costante V determinare il moto della massa m.

WORKED SOLUTION

1. $a = (g(M - m) - aM)/m$.
2. $v = V$

33-014 ...

Una moneta è sistemata a $r = 13$ cm dall'asse del piatto di un giradischi. Si osserva che la moneta non si muove quando il giradischi ruota alla velocità angolare di 33.3 giri/s mentre scivola via quando il giradischi ruota alla velocità angolare di 45.0 giri/s.

1. Determinare entro quali limiti è compreso il coefficiente di attrito statico tra la moneta e il piatto del giradischi.
2. La moneta cade fuori dal piatto del giradischi e resta ferma a terra. Descrivere la dinamica della moneta rispetto ad un sistema di riferimento solidale con il piatto del giradischi.

33-015 ...

Si dimostri che il teorema lavoro-energia per un punto materiale è covariante, cioè che la legge assume la stessa forma per tutti i Sistemi di Riferimento Inerziali. Si ricordi che, in meccanica classica, le forze sono assolute, cioè sono le stesse in ogni Sistema di Riferimento Inerziale. Si usi la forma infinitesimale:

$$m\mathbf{v} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{F} \cdot \mathbf{v}$$

33-016 ...

Un campo di forze nel piano ha componenti:

$$F_x = y \qquad F_y = -x$$

1. Disegnare le linee di forza del campo.
2. Dimostare che il campo non è conservativo.

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CHAPTER 34

Other Topics

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Part D

Basics of ElectroMagnetism

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35 General References 1686

36 Basic Laws of ElectroMagnetism 1690

37 Electric|Magnetic Properties of Matter 1812

38 Electric Properties of Matter 1838

39 Magnetic Properties of Matter 1880

40 ElectroMagnetic Field - Energy Linear Momentum and Angular Momentum 1921

41 ElectroMagnetism - Miscellaneous Complements Applications and Phenomenology 1945

42 Other Topics 1983

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CHAPTER 35

General References

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Pagina Web del Corso

Informazioni sul corso, testi di studio, problemi ed esercizi, fonti di dati, materiale vario di studio e multimediale:
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Many classical texts are excellent texts, sometimes considered to be a *Bible* in their field, as they often are. However, physics progresses very very fast, so that excellent texts written many years ago remain a milestone, but may become old.

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Other references

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The SI of Units of Measure

Read § B.

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Some References for this section follow.

- M.Bertin & J.P.Faroux & J.Renault, *Milieux diélectriques et milieux aimantés*, 1984, Dunod, ...Ed., WEB - URL
- WEB - URL

Exercises and problems for this part

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CHAPTER 36

Basic Laws of ElectroMagnetism

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36.01 Introduction

this section deals with the basic laws of ElectroMagnetism. The laws discussed in this section are always valid, as they are expressed in terms of the basic ElectroMagnetic fields **E** and **B**.

The effect of the presence of matter will be considered explicitly in sections § 37, § 38, § 39 where, in addition to the Polarization, **P**, and Magnetization, **M**, fields, the auxiliary fields **D** and **H** are introduced as useful, but optional, tools to describe ElectroMagnetism in presence of matter.

36.01.01 ElectroMagnetism

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©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Introduction||

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|Introduction and Survey: I.0, I.1, I.2, I.3, I.6.||

- Once upon a time electricity and magnetism were two different subjects. Later it become clear that they are different aspects of the same thing: ElectroMagnetism. Later on it was found that optics was also one additional aspect of ElectroMagnetism. Today, Electricity, Magnetism and Optics are unified.
- As far as we know today universe is governed by four forces: gravity, ElectroMagnetism, whose effects are relevant at the macroscopic level, as well as the strong and the weak forces, whose effects are only important at the femto-meter level and below.
- ElectroMagnetism, together with gravity, governs all macroscopic phenomena (all physical and chemical properties of matter).
- Classical ElectroMagnetism survived both relativistic physics and quantum physics revolutions: it is compatible with relativity and it is ok with quantum physics down to pico-meters.
- Electric charges come in two species, distinguished by attraction/repulsion. In fundamental physics this duality becomes the duality between particle and anti-particles (that is matter and anti-matter). Other types of **charges** do exist as well, at the fundamental level: flavor charges in QCD, electroweak charges in the electro-weak model, for instance.
- Electric charge is conserved: the total electric charge in an isolated system, that is the algebraic sum of the positive and negative charge present at any time, never changes.
- Electric charge is Lorentz-invariant: observers in different inertial frames, measuring the charge of any system, obtain the same number. In other words the total electric charge of a system is a relativistically invariant number.
- Electric charge is quantised (as first shown by Millikan): particle and anti-particle have charges $\pm e$; electrons-protons have identically opposite charges; quarks and anti-quarks have $\pm 2/3e$ and $\pm 1/3e$ charges; leptons and anti-leptons have $\pm e$ charges. This is an experimental fact.

  see WEB - URL

- ElectroMagnetism is a field theory: charges and currents produces fields in space and other charges and currents feel the fields. Fields carry energy, linear momentum and angular momentum. The concept of field invites us to regard the fields as real dynamical entities in their own right.
- ElectroMagnetism is often described by different systems of units of measure. SI will be used (read § B).

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36.01.02
Phenomenology Examples and Applications

- Xerox
- SMES
- ElectroStatic induction.
- ElectroStatic separation of particles.
- The atmosphere.
- Hall probes.
- Millikan experiment.

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36.02 Names Notation Symbols and Conventions

As in any field of physics, names, notation, symbols and conventions change from author/teacher to author/teacher and there is no hope for universal agreement. Therefore one must always check names, notation, symbols and conventions at the beginning. Moreover, for this same reason, it is important to look at formulas as linking concepts, not symbols.

36.02.01 EM Fields

According to the modern point of view that the fields **E** and **B** are the fundamental ElectroMagnetic fields, the following names will be used in these notes:

- the electric field **E** is simply called **the electric field**;
- the magnetic field **B** is simply called **the magnetic field**;
- the field **D** is just called **the (auxiliary electric) field D** and it is often called, in the literature, the **electric excitation field**;
- the field **H** is just called **the (auxiliary magnetic) field H** and it is often called, in the literature, the **magnetic excitation field**.

Read § 38.05, § 39.05 for further details on the auxiliary fields.

Also note that **Maxwell Equations** written in terms of the fundamental fields **E** and **B** are often called **Maxwell Equations** in vacuum. Actually they are always valid, either in vacuum or in matter: they are just fundamental laws of physics: they are **the Maxwell equations**.

Finally, while the Electric|Magnetic field has a well-defined operational definition, from the Lorentz force, for instance, or from Coulomb|Biot-Savart laws.

36.02.01.01 EM Fields Versus Auxiliary Fields

While the electric field **E** has always been considered the fundamental field for electric phenomena, in contrast to **D**, the roles of **B** as the fundamental field for magnetic phenomena, has often been stolen by **H**.

Several facts support the idea of **B** begin the fundamental magnetic field, justifying naming it **the magnetic field**.

Read § 41.03.05.02.

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36.02.02
Real and Ideal Point Charges

As far as we know today there exist elementary particles that we can consider as point particles, in the sense that we are not able to measure any radius greater than zero with today instruments and technology.

In practice every charge distribution can be considered, to a first approximation, as a point charge as long as all the distances involved are much larger than the size of the particle. This statement is made quantitative via the **Multipole** expansion (read § 36.14).

36.02.03
Real and Ideal Dipoles

As far as **Dipoles** are concerned the following names will be used:

- real or physical **Dipole** will refer to the realistic object made of:
  - two point charges of equal and opposite charge, that is: the dimensions of the two charges are much smaller than their distance;
  - a closed plane thin current loop, that is: the dimensions of the section of the current carrying wire are much smaller than the dimensions of the current loop;
- pure or ideal **Dipole** will refer to the mathematical point-like abstraction:
  - electric: limiting case when the absolute value of the charge tends to infinity and distance tends to zero so that their product stays constant;
  - magnetic: limiting case when the absolute value of the current tends to infinity and the area of the plane circuit tends to zero in such a way that their product stays constant.

36.02.04
On the Meaning of Field

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.....

First we should say that the phrase **a real field** is not very meaningful. For one thing, you probably don't feel that the magnetic field is very real anyway, because even the whole idea of a field is a rather abstract thing. You cannot put out your hand and feel the magnetic field. Furthermore, the value of the magnetic field is not very definite; by choosing a suitable moving Coordinate System, for instance, you can make a magnetic field at a given point disappear.

What we mean here by a real field is this: a real field is a mathematical function we use for avoiding the idea of action at a distance. If we have a charged particle at the position P, it is affected by other charges located at some distance from P. One way to describe the interaction is to say that the other charges make some condition - whatever it may be - in the environment at P. If we know that condition, which we describe by giving the electric and magnetic fields, then we can determine completely the behavior of the particle - with no further reference to how those conditions came about.

In other words, if those other charges were altered in some way, but the conditions at P that are described by the Electric|Magnetic field at P remain the same, then the motion of the charge will also be the same. A real field is then a set of numbers we specify in such a way that what happens at a point depends only on the numbers at that point. We do not need to know any more about what's going on at other places.

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36.03 Kinematics of Charge Motion

This § is referenced at pages:
[Never referenced.]

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|6.1|*Excellent and detailed introduction and ...*

Charges and currents give rise to ElectroMagnetic fields and are affected by ElectroMagnetic fields.

Let us consider point charges of mass m_k, charge q_k and velocity $\mathbf{v}_k$ and describe the situation from a fixed Reference Frame.

36.03.01 Charge Densities

36.03.01.01 Volume Charge Density

Consider electric charges with a three-dimensional distribution.

The macroscopic volume charge density at $\mathbf{x}$, a point in the volume of interest, is defined by considering a macroscopically small but microscopically large volume around $\mathbf{x}$, having volume $\widetilde{\Delta V}$, as:

$$\rho \equiv \lim_{\widetilde{\Delta V}\to 0} \frac{1}{\widetilde{\Delta V}} \sum_{\text{charges in } \widetilde{\Delta V}} q \quad . \tag{36.03.01}$$

If the charges can be grouped in different subsets with the same charge, q_α, and number volume density, n_α, one can write:

$$\rho_\alpha = n_\alpha q_\alpha \qquad \rho = \sum_\alpha \rho_\alpha \quad . \tag{36.03.02}$$

Note that n_α in the relation above is a number of charges per unit volume.

36.03.01.02 Surface Charge Density

Consider electric charges with a bi-dimensional distribution, distributed on a regular surface. Read § 36.03.02.02 for a more detailed prescription of the meaning of surface density.

The macroscopic surface charge density at $\mathbf{x}$, a point on the surface, is defined by considering a macroscopically small but microscopically large surface around $\mathbf{x}$, having area $\widetilde{\Delta S}$, as:

$$\rho_{\mathrm{S}} \equiv \lim_{\widetilde{\Delta S}\to 0} \frac{1}{\widetilde{\Delta S}} \sum_{\text{charges in } \widetilde{\Delta S}} q \quad . \tag{36.03.03}$$

If the charges can be grouped in different subsets with the same charge, q_α, and number surface density, n_α, one can write:

$$\rho_{\mathrm{S}\alpha} = n_\alpha q_\alpha \qquad \rho_{\mathrm{S}} = \sum_\alpha \rho_{\mathrm{S}\alpha} \quad . \tag{36.03.04}$$

Note that n_α in the relation above is a number of charges per unit surface.

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36.03.01.03 Line Charge Density

Consider electric charges with a one-dimensional distribution, distributed on a regular curve. Read § 36.03.02.03 for a more detailed prescription of the meaning of line density.

The macroscopic surface charge density at $\mathbf{x}$, a point on the curve, is defined by considering a macroscopically small but microscopically large segment around $\mathbf{x}$, having length $\widetilde{\Delta L}$, as:

$$\rho_{\mathrm{L}} \equiv \lim_{\widetilde{\Delta L}\to 0} \frac{1}{\widetilde{\Delta L}} \sum_{\text{charges in } \widetilde{\Delta L}} q \quad . \tag{36.03.05}$$

If the charges can be grouped in different subsets with the same charge, q_α, and number line density, n_α, one can write:

$$\rho_{\mathrm{L}\alpha} = n_\alpha q_\alpha \qquad \rho_{\mathrm{L}} = \sum_\alpha \rho_{\mathrm{L}\alpha} \quad . \tag{36.03.06}$$

Note that n_α in the relation above is a number of charges per unit length.

36.03.01.04 Relationship Among the Different Charge Densities

Whenever applicable the following obvious relations apply, relating the different charge densities, whenever a volume can be approximated by either a surface or a line:

$$\mathrm{d}q \sim \rho_{\mathrm{L}}\, \mathrm{d}L \sim \rho_{\mathrm{S}}\, \mathrm{d}S \sim \rho\, \mathrm{d}V \quad .$$

The infinitesimal integration element is always, volume, surface or line.

36.03.02 Current Densities

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36.03.02.01 Volume Current Density

This § is referenced at pages:
[Never referenced.]

Consider a three-dimensional motion of charges. Consider, for simplicity, the motion of all identical charges, with charge q, moving with velocity $\mathbf{v}$. Consider an infinitesimal plane area, $\mathrm{d}\mathbf{S}$, with normal unit vector $\widehat{\mathbf{n}}$; let θ be the angle between $\widehat{\mathbf{n}}$ and $\mathbf{v}$; let $\mathrm{d}N/\mathrm{d}V$ the number of charges per unit volume. Calculate the charge passing across it in the time Δt:

$$\mathrm{d}I = q\left(\frac{\mathrm{d}N}{\mathrm{d}V}\right)\frac{(v\cos[\theta]\ \mathrm{d}S\Delta t)}{\Delta t} = \rho v\, \mathrm{d}S\cos[\theta] \implies \boxed{\mathrm{d}I = \rho v\cos[\theta]\ \mathrm{d}S \equiv (\mathbf{j}\cdot\widehat{\mathbf{n}})\ \mathrm{d}S \implies \mathbf{j} \equiv \rho\mathbf{v}} \quad . \tag{36.03.07}$$

→ 1702

Therefore the current is the flux of $\mathbf{j}$ across the surface:

$$I \equiv \iint_\Sigma (\mathbf{j}\cdot\widehat{\mathbf{n}})\ \mathrm{d}S \quad . \tag{36.03.08}$$

→ 1089

Unit of measure of $\mathbf{j}$: $\mathrm{A/m^2}$.

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36.03.02.02 Surface Current Density

This § is referenced at pages:
[1100, 1695]

Consider a two-dimensional motion of charges, that is a motion on a surface of maximum thickness h. The concept of surface current is useful any time one studies the effects of the current at a distance, r, very much larger than the sheet maximum thickness: $r \gg h$.

Consider, for simplicity, the motion of all identical charges, with charge q, moving with velocity $\mathbf{v}$ constrained to move on a regular surface. Any regular surface can be locally approximated by its tangent plane at any point. Therefore one can consider a plane motion.

Let $\widehat{\mathbf{n}}$ be the normal unit vector of the surface. Consider a line on the surface and a small line segment, $\mathrm{d}\mathbf{L}$, of the line. Let $\widehat{\mathbf{N}}$ be the unit vector on the surface perpendicular to $\mathrm{d}\mathbf{L}$:

$$\widehat{\mathbf{N}} \equiv \frac{\mathrm{d}\mathbf{L}\wedge\widehat{\mathbf{n}}}{\mathrm{d}L} \quad \Leftrightarrow \quad \left(\widehat{\mathbf{n}}\wedge\widehat{\mathbf{N}}\right)\,\mathrm{d}L = \mathrm{d}\mathbf{L} \quad .$$

Note that, as $\widehat{\mathbf{n}}$ is the normal unit vector of the surface, $\widehat{\mathbf{N}}$ is tangent to the surface, whatever $\mathrm{d}\mathbf{L}$ is.

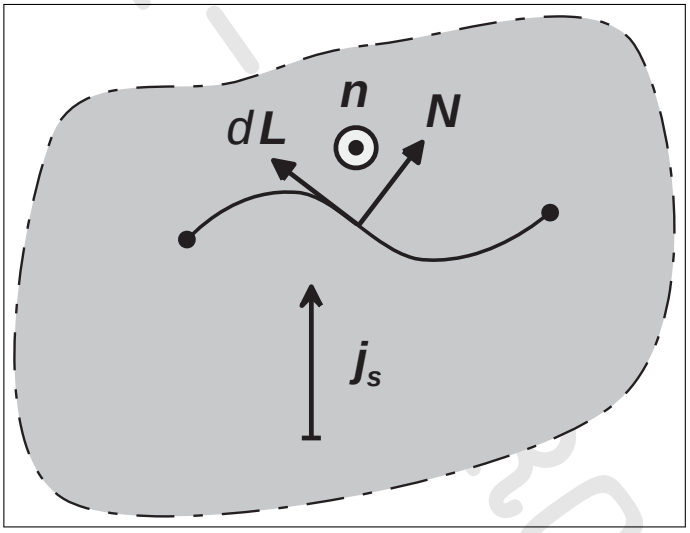

Figure 36.11: Surface current density kinematics

Let θ be the angle between $\widehat{\mathbf{N}}$ and $\mathbf{v}$; let $\mathrm{d}N/\mathrm{d}S$ the number of charges per unit area.

Calculate the current passing across the line in the time Δt:

$$\mathrm{d}I = q\left(\frac{\mathrm{d}N}{\mathrm{d}S}\right)\frac{(v\cos[\theta]\;\mathrm{d}L\Delta t)}{\Delta t} = \rho_{\mathrm{S}} v\;\mathrm{d}L\cos[\theta] \implies \boxed{\mathrm{d}I = \rho_{\mathrm{S}} v\cos[\theta]\;\mathrm{d}L \equiv \left(\mathbf{j}_{\mathrm{S}}\cdot\widehat{\mathbf{N}}\right)\,\mathrm{d}L \implies \mathbf{j}_{\mathrm{S}} \equiv \rho_{\mathrm{S}}\mathbf{v}} \quad .$$

(36.03.09) → 1703

It is understood that the velocity of the surface charges, $\mathbf{v}$, is such that the surface charges stay on the surface, that is $\mathbf{v}$ is tangent to the surface.

Therefore the current is the circulation of $\widehat{\mathbf{n}}\wedge\mathbf{j}_{\mathrm{S}}$ along the line:

$$\boxed{I \equiv \int_C \left(\mathbf{j}_{\mathrm{S}}\cdot\widehat{\mathbf{N}}\right)\,\mathrm{d}L = \int_C (\widehat{\mathbf{n}}\wedge\mathbf{j}_{\mathrm{S}})\cdot\,\mathrm{d}\mathbf{L}} \quad . \qquad \text{(36.03.10)} \quad \rightarrow 1090$$

Unit of measure of $\mathbf{j}_{\mathrm{S}}$: $\mathrm{A/m}$.

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36.03.02.03 Line Current Density

This § is referenced at pages:
[1696]

This is the common elementary definition of current, as flow of charges on a thin line.

Consider a tube of flux of a current of maximum section diameter d. The concept is useful any time one studies the effects of the current at a distance, r, very much larger than the wire maximum section diameter: $r \gg d$. If the tube is a thin one (section diameter d very much smaller than its length) it is possible to describe the current as a flow of charges along a line.

Consider, for simplicity, the motion of all identical charges, with charge q, moving with velocity $\mathbf{v}$ constrained to move on a regular curve. Any regular curve can be locally approximated by its tangent unit vector at any point. Therefore one can consider a rectilinear motion. Let $\mathbb{d}\mathbf{L}$ be the infinitesimal element of the curve.

One has:

$$\boxed{I = q\left(\frac{\mathbb{d}N}{\mathbb{d}L}\right)\frac{v\Delta t}{\Delta t} = \rho_{\mathrm{L}} v} \quad . \tag{36.03.11}$$

→ 1703

Note that the above definition can be given a vector character, even this is not usually done, as the direction of motion is dictated by the curve, as:

$$\boxed{\mathbf{I} = \rho_{\mathrm{L}}\mathbf{v} \equiv \mathbf{j}_{\mathrm{L}}} \quad .$$

Therefore, in complete analogy with the case of surface and volume densities, one might equivalently write:

$$\boxed{\mathbf{I}\,\mathbb{d}L \Leftrightarrow I\,\mathbb{d}\mathbf{L}} \quad .$$

It is understood that the velocity of the line charges, $\mathbf{v}$, is such that the line charges stay on the curve, that is $\mathbf{v}$ is tangent to the curve.

36.03.02.04 Relationship Among the Different Current Densities

Whenever applicable the following obvious relations apply, relating the different current densities, whenever a volume can be approximated by either a surface or a line:

$$\boxed{\mathbf{I}\,\mathbb{d}L \equiv I\,\mathbb{d}\mathbf{L} \equiv \mathbf{j}_{\mathrm{L}}\,\mathbb{d}L \sim \mathbf{j}_{\mathrm{S}}\,\mathbb{d}S \sim \mathbf{j}\,\mathbb{d}V} \quad .$$

The infinitesimal integration element is always: volume, surface or line.

Beware the nomenclature:

- volume current density describes a current in three-dimensions, that is a volume current, but is it is fact a current per unit area;
- surface current density describes a current in two-dimensions, that is a surface current, but is it is fact a current per unit length.

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Is Current a Vector?

So many people have written to me complaining about my treatment of current as a vector that I have decided to write out my explanation once and for all. Some of what follows may not be relevant to your personal concern, but I hope I have addressed the most common objections. Let me say at the outset that if it bothers you to think of current as a vector, by all means stick with the scalar notation! This is purely a matter of convenience, and most people think it is simpler to do it that way. But for consistency with more advanced work it is (in my opinion) preferable to switch over at some stage to the more general notation.

Let me begin with an analogy: A locomotive on a curving track. Suppose I want to tell someone how fast it is going. Should I specify its *velocity* (magnitude and direction), or just its *speed* (the magnitude)? Obviously it would be simplest to quote the *speed*—after all, the direction of the motion is dictated by the shape of the track, and it seems awkward and unnecessary to say "60 mph due east," and a moment later "60 mph at 37° west of north," and so on—why not just say the train is going 60 mph? In this context it is simpler to use the scalar quantity. But if you are a physics teacher, you know that in the next chapter you will be discussing projectile motion (say), in which the object in question moves in two dimensions, and its direction is *not* constrained by the shape of any track. In this context you might prefer to talk about the *velocity*. Is the rate of change of position a scalar, or a vector? I guess it depends on the context which is the more natural terminology.

In the case of current exactly the same issue arises: If you are talking about current constrained to flow along a wire (the analog to the track), then no doubt it is simpler to treat it as a scalar quantity. But when it comes to surface currents (charge flowing over a 2-dimensional surface) or volume currents (charge flowing through 3-dimensional space), then you *have* to keep track of the directions, and so we adopt the vector notation ($\mathbf{K}$ and $\mathbf{J}$, respectively). Looking back, one asks how come we didn't use a vector for the 1-dimensional case ($\mathbf{I}$). Perhaps, for consistency, we *should* have!

It is a peculiar fact that in almost all formulas (Biot-Savart, Lorentz, etc.) current appears multiplied by an element of displacement along the wire: $I\,d\mathbf{l}$. Because the current (I) and the displacement ($d\mathbf{l}$) are in the same direction, it doesn't matter whether you associate the vector sign with the one or the other—in elementary work everyone attaches it to $d\mathbf{l}$, keeping I a scalar. But again, when you come to surface and volume currents, where the analogous expression is $\mathbf{K}\,da$ and $\mathbf{J}\,d\tau$, you have no choice: here the direction is that of the current ($d\tau$ doesn't even *have* a direction). So if you write $\mathbf{I}\,dl$ for the one-dimensional current, everything is consistent, even though in that case you can do it either way.

Here's another consideration: If you have charges of one sign, with charge density ρ, flowing with velocity $\mathbf{v}$ (in 3-dimensions), this constitutes a volume current density

$$\mathbf{J} = \rho\mathbf{v}.$$

If you have a surface charge σ flowing over a surface with velocity $\mathbf{v}$, the surface current is

$$\mathbf{K} = \sigma\mathbf{v}.$$

What about a line charge λ, flowing down a wire at velocity $\mathbf{v}$? I would like to write the analogous expression

$$\mathbf{I} = \lambda\mathbf{v}.$$

But if you forbid me to treat current as a vector I have to adopt a different notation for the one-dimensional case. Doesn't that seem awkward?

OK: I have told you why I prefer (in appropriate cases) to represent current as a vector. Let me now address the standard objections to this:

- **In high school I was told that current is a scalar.** Sure—so was I. And that is entirely appropriate at that level, where surface and volume currents do not occur. It's not *wrong*, exactly, just limiting. At the college level it is time to liberate yourself from the more constraining notation.

- **Charge is a scalar, and time is a scalar, so $I = dq/dt$ must be a scalar.** To put it another way, the current in a wire is the amount of charge per unit time passing a particular point—there is no reference here to direction. Well, actually there *is*: if positive charge is passing to the right we say the current is positive (say), and if it is going to the left we say the current is negative. Of course, in a wire those are the only two possibilities, so a simple sign suffices to keep track of it. But in general we are talking about charge per unit time flowing in a particular direction—it's just that the direction (up to a sign) is dictated by the shape of the wire. Again, one could run the same argument with the locomotive on the track: distance (traveled) is a scalar, and time is a scalar, so velocity must be a scalar(?)! No: it is not distance we meant, but distance *in a particular direction*, which is to say *displacement*.

- **Currents do not add vectorially.** This is the most subtle objection, because it misconstrues the meaning of vector addition. Kirchhoff's law says that the sum of the currents *into* a junction is zero (counting currents flowing *out* as negative). They do *not* add vectorially—it doesn't matter what direction the wires meet at. Correct, but this is a *law of physics*, not a mathematical case of addition of vectors/scalars. If I say the sum of the momenta of some colliding particles is the same after the collision as before, that is (again) a *law of physics*, not a question of vector addition (though in this case the law says you should add them vectorially). Kirchhoff's law has nothing to do with the vectorial nature of current, since it refers to the *magnitudes*. The number of locomotives entering a junction is equal to the number leaving (in any given time interval)—scalar addition—but this tells me noting about the vectorial nature of velocity, only about the law of conservation of locomotives.

  In point of fact, currents *do* add vectorially. Here's an example. Suppose I have a fat pipe, and inside that pipe there are some wires, each carrying

a current. The wires do not run parallel to the axis of the pipe, but make various angles. When they reach the edge of the pipe, they bend back so as to make the same angle (like light rays reflecting off the surface of a light pipe). If you ask "What is the total current flowing down the pipe?" you would of course have to take into account the direction of each wire. If it happens to run parallel to the pipe, it counts at full value, but if it's going at 45° you'll need to throw in a factor of $1/\sqrt{2}$ (and if it is perpendicular to the pipe it contributes nothing). In other words, you must add the currents *vectorially*—the total current is the *vector* sum of the individual currents.

I hope this helps. But again: if you prefer to think of current as a scalar, that's fine. It won't hurt you; it just isn't very elegant.

David Griffiths

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36.03.03
Volume Surface and Line Charge|Current Densities

If a surface Charge|Current lies/flows in a thin layer of thickness h, with thickness very much smaller than the linear dimensions of the surface and any other significant dimension, one can write:

$$\boxed{\rho_S \sim h\rho \qquad \mathbf{j}_S \sim h\mathbf{j}} \quad . \tag{36.03.12}$$

→ 1793

The concept of surface Charge|Current is useful any time one studies the effects of the current at a distance, r, very much larger than the layer thickness, $r \gg h$, and any other significant dimension.

If a line Charge|Current lies/flows in a thin wire of cross-sectional area a, with transverse dimension ($\sim \sqrt{a}$) very much smaller than the dimensions of the wire and any other significant dimension, one can write:

$$\boxed{\rho_L \sim a\rho \qquad \mathbf{j}_L \sim a\mathbf{j}} \quad . \tag{36.03.13}$$

→ 1793

The concept of line Charge|Current is useful any time one studies the effects of the current at a distance, r, very much larger than the wire transverse dimension ($\sim \sqrt{a}$), $r \gg\sim \sqrt{a}$, and any other significant dimension.

36.03.04
Current Densities as Charge Transport

In the same way as mass transport is defined by the linear momentum, $\mathbf{p} = m\mathbf{v}$, charge transport can be described by introducing the charge transport quantity $q\mathbf{v}$.

36.03.04.01 Volume Current Density

This § is referenced at pages:
[1103]

One can define the volume density of the charge transport quantity $q\mathbf{v}$, that is the volume current (see (36.03.07)):

$$\mathbf{j} \equiv \lim_{\widetilde{\Delta V}\to 0} \frac{1}{\widetilde{\Delta V}} \sum_{\text{charges in } \widetilde{\Delta V}} q\mathbf{v} \quad .$$

If the charges can be grouped in different subsets with the same charge, $q[i]$, and number volume density, $n[i]$, one can first define the average mean velocity (drift velocity) of each kind of charges, $\mathbf{v}[i]$, and then write:

$$\boxed{\mathbf{j}[i] = n[i]q[i]\mathbf{v}[i] \equiv \rho[i]\mathbf{v}[i] \qquad \mathbf{j} = \sum[i]\mathbf{j}[i]} \quad . \tag{36.03.14}$$

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36.03.04.02 Surface Current Density

One can define the surface density of the charge transport quantity $q\mathbf{v}$, that is the surface current (see (36.03.09)):

$$\boxed{\mathbf{j}_{\mathtt{S}} \equiv \lim_{\widetilde{\Delta S}\to 0} \frac{1}{\widetilde{\Delta S}} \sum_{\text{charges in } \widetilde{\Delta \mathtt{S}}} q\mathbf{v}} \quad .$$

If the charges can be grouped in different subsets with the same charge, $q[i]$, and number surface density, $n[i]$, one can first define the average mean velocity (drift velocity) of each kind of charges, $\mathbf{v}[i]$, and then write:

$$\boxed{\mathbf{j}_{\mathtt{S}}[i] = n[i]q[i]\mathbf{v}[i] \equiv \rho_{\mathtt{S}}[i]\mathbf{v}[i] \qquad \mathbf{j}_{\mathtt{S}} = \sum[i]\mathbf{j}_{\mathtt{S}}[i]} \quad . \tag{36.03.15}$$

36.03.04.03 Line Current Density

This is the common elementary definition of current, as flow of charges on a thin line.

One can define the line density of the charge transport quantity $q\mathbf{v}$, that is the line current (see (36.03.11)):

$$\boxed{\mathbf{j}_{\mathtt{L}} \equiv \lim_{\widetilde{\Delta L}\to 0} \frac{1}{\widetilde{\Delta L}} \sum_{\text{charges in } \widetilde{\Delta \mathtt{L}}} q\mathbf{v}} \quad .$$

If the charges can be grouped in different subsets with the same charge, $q[i]$, and number surface density, $n[i]$, one can first define the average mean velocity (drift velocity) of each kind of charges, $\mathbf{v}[i]$, and then write:

$$\boxed{\mathbf{j}_{\mathtt{L}}[i] = n[i]q[i]\mathbf{v}[i] \equiv \rho_{\mathtt{L}}[i]\mathbf{v}[i] \qquad \mathbf{j}_{\mathtt{L}} = \sum[i]\mathbf{j}_{\mathtt{L}}[i]} \quad . \tag{36.03.16}$$

36.03.05
Some Comments About Charge and Current Densities

Note that a globally neutral medium, that is a medium with zero charge density, might be possibly obtained with two big charge densities of equal magnitude and opposite sign.

Note that while all the charges contribute to the charge densities, only the charges with non-zero velocity contribute to the current densities.

Also note that the condition $\mathbf{v}[i] = \mathbf{0}$ for a certain species of charges does not imply that all charges of that species have zero velocity, but it only implies that the velocity distribution has zero average value.

In a thermodynamics/statistical approach one can show that at equilibrium the velocity distribution is isotropic and therefore all partial currents vanish.

On the other hand the total current might be null while the different partial currents are not. In this case compensation among different partial currents occur. This is the case, for instance, when a neutral conductor translates with constant velocity: the total current is zero thanks to the cancellation of the currents due to each subset of charges.

Conducting media are often composed of fixed charges plus mobile charges, such that the total density is

$$\boxed{\rho = \rho_{\mathtt{F}} + \rho_{\mathtt{M}} \qquad \rho_{\mathtt{S}} = \rho_{\mathtt{SF}} + \rho_{\mathtt{SM}}}$$

Note that while **Maxwell Equations** refer to the total charge density, ρ, the equations which describe the charge movement only refer to the mobile charge density, $\rho_{\mathtt{M}}$. Inside neutral conducting media, either $\rho = 0$ or $\rho_{\mathtt{S}} = 0$, the density of mobile charges is often compensated by the density of fixed charges: either $\rho_{\mathtt{M}} = -\rho_{\mathtt{F}}$ or $\rho_{\mathtt{SM}} = -\rho_{\mathtt{SF}}$.

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36.03.06
Convection Currents

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|||

In case the material medium is moving the convection currents have to be included as well.

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36.04 Superposition Principle

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|||

Maxwell Equations in vacuum are linear in the Electric|Magnetic fields: the total field is the linear superposition of the fields produced by all the sources, charges and currents.

The principle of superposition is not obvious at all: if the ElectroMagnetic fields were proportional, for instance, to the square of the total charge (electric field), or to the square of the total current (magnetic field), then the principle of superposition would not hold at all.

The **superposition principle** is not a necessity, it is an experimental fact.

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36.05 Force|Torque

36.05.01 Force|Torque on Point Charges

This § is referenced at pages:
[1336, 1336, 1710, 2492]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 4.1.3, § 6.1.2||
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||

36.05.01.01 Force on Point Charges

In any Inertial Reference Frame , the force acting on an **electrically charged point test particle**, the **test charge** , characterized by an electric charge q and moving with velocity $\mathbf{v}$, is the Lorentz force:

$$\boxed{\mathbf{f} = q\,(\mathbf{E} + \mathbf{v}\wedge\mathbf{B})} \quad , \tag{36.05.01}$$

→ 1335

which actually defines $\mathbf{E}$ and $\mathbf{B}$, the external ElectroMagnetic fields, produced by anything else except the point charge itself.

Physically, **test charge** means that one can neglect the back-action of the test charge on the system of charges|currents that generates the external EM field.

For instance: put a small charge at the point at rest and measure the force to determine $\mathbf{E}$; send a small charge to pass a the point with a known velocity and several directions until the direction such that no force acts on the charge is found, to determine the direction of $\mathbf{B}$ and after do it in a different direction measuring the force to determine the module of $\mathbf{B}$.

36.05.01.02 Torque on Point Charges

It is obviously zero, because it is a spinless particle.

36.05.02 Force|Torque on Point Dipoles

This § is referenced at pages:
[1711, 1947]

36.05.02.01 Force on Point Dipoles

This § is referenced at pages:
[1710, 1711]

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|§ 10, § 11||

In case the external ElectroMagnetic fields are homogeneous the total force is zero.

In case the external ElectroMagnetic fields are not homogeneous a force also acts on the **Dipoles** . This is derived in section § 41.05.01.01 and § 41.05.02.01, and results in equations (41.05.01), (41.05.06). The force, in static conditions, when no other charge|current is present at the location of the ideal dipole, is as follows.

UNDER PEER REVIEW

Electric Dipole :

$$\mathbf{f}_e = (\mathbf{p}\cdot\boldsymbol{\nabla})\,\mathbf{E} \qquad \Longrightarrow \; \mathbf{f}_e\Big|_k = p_s^{\cdot}\partial_s^{\cdot}E_k^{\cdot} \quad . \tag{36.05.02}$$

→ 1707 1761 1763 1793

Magnetic Dipole :

$$\mathbf{f}_m = (\mathbf{m}\wedge\boldsymbol{\nabla})\wedge\mathbf{B} \qquad \Longrightarrow \; \mathbf{f}_m\Big|_k = \epsilon_{kpq}^{\cdot\cdot\cdot}\epsilon_{prs}^{\cdot\cdot\cdot}m_r^{\cdot}\partial_s^{\cdot}B_q^{\cdot} \quad . \tag{36.05.03}$$

→ 1707 1793

These expressions can be transformed in many ways, with some additional hypotheses.

In the electric case, in case of irrotational external electric fields and assuming fixed values (rigid dipole) of the **Electric Dipole** moment with respect to the nabla (partial derivative with respect to position) the expression in (36.05.02) is equivalent to:

$$\mathbf{f}_e = \boldsymbol{\nabla}\,(\mathbf{p}\cdot\mathbf{E}) \quad . \tag{36.05.04}$$

→ 1707 1761

Moreover, in the case the external electric fields are also divergence-less and assuming fixed values (rigid dipole) of the **Electric Dipole** moment with respect to the nabla (partial derivative with respect to position) expression (36.05.04) is equivalent to:

$$\mathbf{f}_e = (\mathbf{p}\wedge\boldsymbol{\nabla})\wedge\mathbf{E} \quad . \tag{36.05.05}$$

In the magnetic case, in case of divergence-less external magnetic fields (which is always the case) and assuming fixed values (rigid dipole) of the **Magnetic Dipole** moment with respect to the nabla (partial derivative with respect to position) the expression in (36.05.03) is equivalent to:

$$\mathbf{f}_m = \boldsymbol{\nabla}\,(\mathbf{m}\cdot\mathbf{B}) \quad . \tag{36.05.06}$$

→ 1707 1761

Moreover, in the case the external magnetic fields are also irrotational and assuming fixed values (rigid dipole) of the **Magnetic Dipole** moment with respect to the nabla (partial derivative with respect to position) expression (36.05.06) is equivalent to:

$$\mathbf{f}_m = (\mathbf{m}\cdot\boldsymbol{\nabla})\,\mathbf{B} \quad . \tag{36.05.07}$$

→ 1761 1785

A very common case when all the above formulas are equivalent (a sufficient condition, not a necessary one) is the case when the Charge|Current producing the external fields are outside the region of interest where the **Dipoles** are located, the dipole have fixed values (rigid dipoles) and the situation is a static one.

In all cases when the **Dipole** moment appears inside a nabla in the above expressions, it must be considered as a fixed **Dipole** moment, independent of the position and external fields, even in the case that it depends on the external (applied) field (Polarization|Magnetization) in order for the formulas to be valid.

Read § 41.05.01.01 and § 41.05.02.01.

See also § 41.05.01.02 and § 41.05.02.02.

It is worth recalling a few important facts. In the bare derivation of the force on a dipole, in both the Electric|Magnetic cases, when one moves the $\mathbf{p}|\mathbf{m}$ inside the nabla, it is done assuming $\mathbf{p}|\mathbf{m}$ do not depend on the coordinates.

In fact, the original expression for the force has in any case the $\mathbf{p}|\mathbf{m}$ outside the nabla,

$$\mathbf{p}\cdot\boldsymbol{\nabla} \qquad \mathbf{m}\wedge\boldsymbol{\nabla} \quad , \tag{36.05.08}$$

as the nabla comes from the Taylor series development of the external (applied) field. After that, you can move $\mathbf{p}|\mathbf{m}$ inside nabla, under the condition that it is fixed.

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D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ... at page 165.
... However, it is safer to stick with the dipole outside the nabla, because we will be applying the formula to materials in which the **Dipole** moment (per unit volume) is itself a function of position and the expression with the dipole inside the nabla would imply (incorrectly) that $\mathbf{p}|\mathbf{m}$ too is to be differentiated. ...

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36.05.02.02 Torque on Point Dipoles

This § is referenced at pages:
[1062, 1710, 1711]

In any Inertial Reference Frame a point particle with either **Electric Dipole** moment, $\mathbf{p}$, or **Magnetic Dipole** moment, $\mathbf{m}$, when placed in external ElectroMagnetic fields, $\mathbf{E}$ and $\mathbf{B}$, produced by anything else except the point **Dipoles** themselves, is given a pole-independent torque given by:

$$\boxed{\tilde{\gamma}_{\mathtt{e}}[\mathbf{r}] = \mathbf{p}\wedge\mathbf{E}[\mathbf{r}] \qquad \tilde{\gamma}_{\mathtt{m}}[\mathbf{r}] = \mathbf{m}\wedge\mathbf{B}[\mathbf{r}]} \quad , \tag{36.05.09}$$

The torque acting on a **Dipole** located at $\mathbf{r}$ with respect to any other pole, located at $\mathbf{r}_{\mathtt{P}}$, is obtained by adding to the term $\tilde{\gamma}_{\mathtt{e}}$ and|or $\tilde{\gamma}_{\mathtt{m}}$, the pole-independent torque, the term,

$$\boxed{(\mathbf{r}-\mathbf{r}_{\mathtt{P}})\wedge\mathbf{f}[\mathbf{r}]} \quad , \tag{36.05.10}$$

following the usual law of transformation of the moment when the pole changes:

$$\boxed{\gamma_{\mathtt{P}}[\mathbf{r}] = \tilde{\gamma} + (\mathbf{r}-\mathbf{r}_{\mathtt{P}})\wedge\mathbf{f}[\mathbf{r}]} \quad . \tag{36.05.11}$$

36.05.03 Some Comments on Modeling of Charge|Current Distributions

This § is referenced at pages:
[1922, 1926, 1928, 1930, 2235, 2492]

It might be useful to stress that one might consider point charged particles only, to model the basic components of matter, avoiding to introduce explicitly **Dipoles** and|or higher-order **Multipoles**, from the mathematical point of modeling. In fact any Charge|Current **Multipole** can be always described by a suitable combination of point charges and|or currents of point charges, possibly after a limiting mathematical process. Therefore, in principle, one does not need to introduce explicitly the expressions for **Dipoles** and|or higher-order **Multipoles**.

However, the explicit use of **Dipoles** and|or higher-order **Multipoles** is often a useful shortcut, allowing a much simpler description.

More importantly, today we know elementary (that is point) particles with an intrinsic multipole; at least, the electron, with its magnetic dipole.

Also note that when one considers a generic continuous distribution of Charge|Current, for calculations of integrated properties, such as Force|Torque, one can always model the Charge|Current density of point charges and|or currents of point charges with delta distributions and delta derivatives.

Therefore, modeling the system as a continuous distribution of Charge|Current, is capable to fully describe any **Multipole** distribution.

36.05.04 Force|Torque on Extended Distributions of Charge and Dipoles

This § is referenced at pages:
[1062, 1336, 1336]

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36.05.05
Forces and Torque on Continuous Distributions

The infinitesimal Lorentz Force|Torque acting on a generic infinitesimal element with Charge|Current density ρ / $\mathbf{j}$ and infinitesimal **Electric Dipole** | **Magnetic Dipole** moment $\mathbb{d}\mathbf{p}$ / $\mathbb{d}\mathbf{m}$, can be written as:

$$\boxed{\mathbb{d}\mathbf{f} = (\rho\mathbf{E} + \mathbf{j}\wedge\mathbf{B})\ \mathbb{d}V + (\ \mathbb{d}\mathbf{p}\cdot\boldsymbol{\nabla})\,\mathbf{E} + (\ \mathbb{d}\mathbf{m}\wedge\boldsymbol{\nabla})\wedge\mathbf{B}} \quad ,$$

$$\boxed{\mathbb{d}\boldsymbol{\gamma}_0 = (\mathbf{r} - \mathbf{r}_0)\wedge\mathbb{d}\mathbf{f} + \ \mathbb{d}\mathbf{p}\wedge\mathbf{E} + \ \mathbb{d}\mathbf{m}\wedge\mathbf{B}} \quad .$$

Note that in this case the Electric|Magnetic field to be used in the above formulas does not need subtraction of the Electric|Magnetic field produced by the infinitesimal element which is subject to the Force|Torque. In fact the infinitesimal element produces an infinitesimal field.

In fact the Electric|Magnetic field produced by the infinitesimal element is proportional to $\mathbb{d}V$ and its contribution goes like the square of the distance, which is internal to the infinitesimal volume element; therefore its contribution goes to zero.

Formally, from Coulomb and Biot-Savart laws, one has:

$$\boxed{\mathbb{d}\mathbf{E}[\ \mathbb{d}q;\widehat{\mathbf{e}}_r] \propto \frac{\mathbb{d}q}{r^2}\widehat{\mathbf{e}}_r = \frac{\rho\ \mathbb{d}V}{r^2}\widehat{\mathbf{e}}_r \xrightarrow[\mathbb{d}V\to 0]{} 0} \qquad \boxed{\mathbb{d}\mathbf{B}[i\ \mathbb{d}\mathbf{L};\widehat{\mathbf{e}}_r] \propto \frac{i}{r^2}\ \mathbb{d}\mathbf{L}\wedge\widehat{\mathbf{e}}_r = \frac{\mathbb{d}V}{r^2}\mathbf{j}\wedge\widehat{\mathbf{e}}_r \xrightarrow[\mathbb{d}V\to 0]{} 0} \quad .$$

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36.05.06
Forces and Torques on Real Fixed Electric Dipoles

Read § 41.05.01, § 41.05.01.01, § 41.05.01.02. Read also § 41.02.02.

Consider a static real **Electric Dipole** , a pair of charges $\pm|q|$ separated by $\Delta\mathbf{r}$; afterwards take the limiting case of an ideal **Electric Dipole** : $|\Delta\mathbf{r}| \equiv |\mathbf{r}_+ - \mathbf{r}_-| \to 0$ with $|q||\Delta\mathbf{r}| =$ constant.

It can be shown that the results are those of section § 36.05.01.

36.05.06.01 Total Force

This § is referenced at pages:
[1086]

Uniform and Constant External Field

In a uniform and constant external field the total force is clearly zero.

Generic External Field

In a generic external field, assume a positive charge at $\mathbf{r}_+$ and a negative charge at $\mathbf{r}_-$: the force is

$$\begin{aligned}\mathbf{F}[\mathbf{r}] &= |q|\,(-\mathbf{E}[\mathbf{r}_-] + \mathbf{E}[\mathbf{r}_+]) \quad ,\\ &= |q|\,(-\mathbf{E}[\mathbf{r}_-] + \mathbf{E}[\mathbf{r}_- + \Delta\mathbf{r}]) \quad ,\\ &\simeq |q|\,\Delta\mathbf{r}\cdot\boldsymbol{\nabla}\mathbf{E}[\mathbf{r}_-] \quad ,\\ \mathbf{F}[\mathbf{r}] &\longrightarrow \mathbf{p}\cdot\boldsymbol{\nabla}\mathbf{E}[\mathbf{r}] \quad .\end{aligned}$$

Read § 36.05.02.01, § 36.05.02.02.

36.05.06.02 Total Torque

This § is referenced at pages:
[1086]

Let a positive charge $+|q|$ at $\mathbf{r}_+$ and a negative charge $-|q|$ at $\mathbf{r}_-$. The total torque with respect to the origin of the Coordinate System is calculated as follows.

First method (full and direct calculation):

$$\begin{aligned}\boldsymbol{\Gamma}[\mathbf{r}] &= |q|\,(-\mathbf{r}_-\wedge\mathbf{E}[\mathbf{r}_-] + \mathbf{r}_+\wedge\mathbf{E}[\mathbf{r}_+]) \quad ,\\ &= |q|\,(-\mathbf{r}_-\wedge\mathbf{E}[\mathbf{r}_-] + (\mathbf{r}_- + \Delta\mathbf{r})\wedge\mathbf{E}[\mathbf{r}_- + \Delta\mathbf{r}]) \quad ,\\ &\simeq |q|\,(-\mathbf{r}_-\wedge\mathbf{E}[\mathbf{r}_-] + (\mathbf{r}_- + \Delta\mathbf{r})\wedge(\mathbf{E}[\mathbf{r}_-] + \Delta\mathbf{r}\cdot\boldsymbol{\nabla}\mathbf{E}[\mathbf{r}_-])) \quad ,\\ \boldsymbol{\Gamma}[\mathbf{r}] &\longrightarrow \mathbf{r}\wedge(\mathbf{p}\cdot\boldsymbol{\nabla}\mathbf{E}[\mathbf{r}]) + \mathbf{p}\wedge\mathbf{E}[\mathbf{r}] \quad .\end{aligned}$$

Second method (exploiting the results on the force):

$$\begin{aligned}\boldsymbol{\Gamma}[\mathbf{r}] &= \mathbf{r}_-\wedge\mathbf{F}_- + \mathbf{r}_+\wedge\mathbf{F}_+ \quad ,\\ &= \mathbf{r}_-\wedge\mathbf{F}_- + (\mathbf{r}_- + \Delta\mathbf{r})\wedge\mathbf{F}_+ \quad ,\\ &\simeq \mathbf{r}_-\wedge(\mathbf{F}_- + \mathbf{F}_+) + \Delta\mathbf{r}\wedge\mathbf{F}_+ \quad ,\\ \boldsymbol{\Gamma}[\mathbf{r}] &\longrightarrow \mathbf{r}\wedge\mathbf{F}[\mathbf{r}] + \mathbf{p}\wedge\mathbf{E}[\mathbf{r}] \quad .\end{aligned}$$

In the limiting point case the result is:

$$\boxed{\boldsymbol{\Gamma}[\mathbf{r}] = \mathbf{r}\wedge\mathbf{F}[\mathbf{r}] + \mathbf{p}\wedge\mathbf{E}[\mathbf{r}]} \quad .$$

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36.05.07
Forces and Torques on Real Fixed Magnetic Dipoles

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©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|§ 11.- ...
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||
©|D.J.Griffiths|Dipoles at rest|*Am. J. Phys. 1 November 1992; 60 (11): 979-987.* WEB - URL|

Read § 41.05.02, § 41.05.02.01, § 41.05.02.02.
Read also § 41.02.02.
Consider a static real **Magnetic Dipole** , that is a plane close current loop carrying a steady current I and area $\Delta\mathbf{S}$; afterwards take the limiting case of an ideal **Magnetic Dipole** : $|\Delta\mathbf{S}| \to 0$ with $|I||\Delta\mathbf{S}| =$ constant.
The derivations for the magnetic case are technically more cumbersome: see the references. It can be shown that the results are those of section § 36.05.02.

36.05.07.01 Total Force

Uniform and Constant External Field

In a uniform and constant external field the total force is zero:

$$\mathbf{F} = I \oint \mathrm{d}\mathbf{L} \wedge \mathbf{B} = I \left(\oint \mathrm{d}\mathbf{L} \right) \wedge \mathbf{B} = 0 \quad .$$

Generic External Field

A special instructive case is presented in section § 36.20.61.
In a generic external field, see the references. The result is:

$$\boxed{\mathbf{F}[\mathbf{r}] = (\mathbf{m} \wedge \boldsymbol{\nabla}) \wedge \mathbf{B}[\mathbf{r}]} \quad .$$

Read § 36.05.02.01, § 36.05.02.02.

36.05.07.02 Total Torque

In a generic external field, see the references.
In the limiting point case the result is:

$$\boxed{\boldsymbol{\Gamma}[\mathbf{r}] = \mathbf{r} \wedge \mathbf{F}[\mathbf{r}] + \mathbf{m} \wedge \mathbf{B}[\mathbf{r}]} \quad .$$

A wire current in a uniform magnetic field provides:

$$\boldsymbol{\gamma} = I \oint \mathbf{r} \wedge (\ \mathrm{d}\mathbf{L} \wedge \mathbf{B}) = I \oint ((\mathbf{r} \cdot \mathbf{B})\ \mathrm{d}\mathbf{L} - (\mathbf{r} \cdot \mathrm{d}\mathbf{L})\ \mathbf{B}) = I \oiint (\ \mathrm{d}\mathbf{S} \wedge (\mathbf{grad}\, \mathbf{r} \cdot \mathbf{B})) - 0 \quad , \qquad (36.05.12)$$

where for the last passage come from equation (16.07.07) (left term, with constant $\mathbf{B}$) and (16.07.08) (right term).

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36.05.07.03 Summary formulas

©|D.J.Griffiths|Dipoles at rest|*Am. J. Phys. 1 November 1992; 60 (11): 979-987.* WEB - URL|

Table I. Summary of results, for dipoles at rest. In the case of force, torque, energy, momentum, and angular momentum, the fields, charges, and potentials refer to the *external* quantities, evaluated at the location of the dipole. Formulas for energy momentum, and angular momentum assume *static* external fields. Torque and angular momentum are calculated with respect to the dipole.

Dipole species $\mathbf{p}_1,\mathbf{p}_2,\mathbf{m}_1,\mathbf{m}_2$	Potential V_e, V_m, $\mathbf{A}_e$, $\mathbf{A}_m$	Field $\mathbf{E},\mathbf{B}$	Force $\mathbf{F}$	Torque $\mathbf{N}$	Energy $\mathbf{W}$	Momentum $\mathbf{P}$	Angular momentum $\mathbf{L}$
Standard Electric $\mathbf{p}_1=\int \mathbf{r}\rho_e\, d^3r$	$V_e=\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_1\cdot\mathbf{r}}{r^3}$	$\mathbf{E}=\frac{1}{4\pi\epsilon_0 r^3}\left[3\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{p}_1)}{r^2}-\mathbf{p}_1\right]$ $-\frac{1}{3\epsilon_0}\mathbf{p}_1\delta^3(\mathbf{r})$	$(\mathbf{p}_1\cdot\nabla)\mathbf{E}$	$\mathbf{p}_1\times\mathbf{E}$	$\mathbf{p}_1\cdot(\nabla V_e)$	$(\mathbf{p}_1\cdot\nabla)\mathbf{A}_e$	$-\mathbf{p}_1 V_m+\mathbf{p}_1\cdot\mathbf{W}_m$ $+\mathbf{p}_1\times\mathbf{A}_e$
Nonstandard Magnetic $\mathbf{m}_1=\int \mathbf{r}\rho_m\, d^3r$	$V_m=\frac{\mu_0}{4\pi}\frac{\mathbf{m}_1\cdot\mathbf{r}}{r^3}$	$\mathbf{B}=\frac{\mu_0}{4\pi r^3}\left[3\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{m}_1)}{r^2}-\mathbf{m}_1\right]$ $-\frac{\mu_0}{3}\mathbf{m}_1\delta^3(\mathbf{r})$	$(\mathbf{m}_1\cdot\nabla)\mathbf{B}$	$\mathbf{m}_1\times\mathbf{B}$	$\mathbf{m}_1\cdot(\nabla V_m)$	$-\frac{1}{c^2}(\mathbf{m}_1\cdot\nabla)A_m$	$-\frac{1}{c^2}(-\mathbf{m}_1 V_e+\mathbf{m}_1\cdot\mathbf{W}_e$ $+\mathbf{m}_1\times\mathbf{A}_m)$
Nonstandard Electric $\mathbf{p}_2=-\frac{1}{2c^2}\int(\mathbf{r}\times\mathbf{J}_m)d^3r$	$\mathbf{A}_m=\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_2\times\mathbf{r}}{r^3}$	$\mathbf{E}=\frac{1}{4\pi\epsilon_0 r^3}\left[3\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{p}_2)}{r^2}-\mathbf{p}_2\right]$ $+\frac{2}{3\epsilon_0}\mathbf{p}_2\delta^3(\mathbf{r})$	$\nabla(\mathbf{p}_2\cdot\mathbf{E})$ $-\frac{1}{\epsilon_0}\rho_e\mathbf{p}_2$	$\mathbf{p}_2\times\mathbf{E}$	$\mathbf{p}_2\cdot(\nabla\times\mathbf{A}_m)$	$\nabla(\mathbf{p}_2\cdot\mathbf{A}_e)$ $-\mathbf{p}_2\times(\nabla V_m)$	$-\mathbf{p}_2 V_m+\mathbf{p}_2\cdot\mathbf{W}_m$ $+\mathbf{p}_2\times\mathbf{A}_e$
Standard Magnetic $\mathbf{m}_2=\frac{1}{2}\int(\mathbf{r}\times\mathbf{J}_e)\, d^3r$	$\mathbf{A}_e=\frac{\mu_0}{4\pi}\frac{\mathbf{m}_2\times\mathbf{r}}{r^3}$	$\mathbf{B}=\frac{\mu_0}{4\pi r^3}\left[3\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{m}_2)}{r^2}-\mathbf{m}_2\right]$ $+\frac{2\mu_0}{3}\mathbf{m}_2\delta^3(\mathbf{r})$	$\nabla(\mathbf{m}_2\cdot\mathbf{B})$ $-\mu_0\rho_m\mathbf{m}_2$	$\mathbf{m}_2\times\mathbf{B}$	$\mathbf{m}_2\cdot(\nabla\times\mathbf{A}_e)$	$-\frac{1}{c^2}[\nabla(\mathbf{m}_2\cdot\mathbf{A}_m)$ $-\mathbf{m}_2\times(\nabla V_e)]$	$-\frac{1}{c^2}(-\mathbf{m}_2 V_e+\mathbf{m}_2\cdot\mathbf{W}_e$ $+\mathbf{m}_2\times\mathbf{A}_m)$

Figure 36.12: Dipoles at rest (1). From the reference.

Table II. Summary of results for standard electrodynamcis (no magnetic monopoles), under the same conditions as Table I.

Dipole species $\mathbf{p},\mathbf{m}$	Potential $\mathbf{V},\mathbf{A}$	Field $\mathbf{E},\mathbf{B}$	Force $\mathbf{F}$	Torque $\mathbf{N}$	Energy $\mathbf{W}$	Momentum $\mathbf{P}$	Angular momentum $\mathbf{L}$
Electric $\mathbf{p}=\int \mathbf{r}\rho\, d^3r$	$V=\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}$	$\mathbf{E}=\frac{1}{4\pi\epsilon_0 r^3}\left[3\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{p})}{r^2}-\mathbf{p}\right]$ $-\frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r})$	$(\mathbf{p}\cdot\nabla)\mathbf{e}$	$\mathbf{p}\times\mathbf{E}$	$-\mathbf{p}\cdot\mathbf{E}$	$(\mathbf{p}\cdot\nabla)\mathbf{A}$	$\mathbf{p}\times\mathbf{A}$
Magnetic $\mathbf{m}=\frac{1}{2}\int(\mathbf{r}\times\mathbf{J})\mathrm{d}^3r$	$\mathbf{A}=\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}$	$\mathbf{B}=\frac{\mu_0}{4\pi r^3}\left[3\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{m})}{r^2}-\mathbf{m}\right]$ $+\frac{2\mu_0}{3}\mathbf{m}\delta^3(\mathbf{r})$	$\nabla(\mathbf{m}\cdot\mathbf{B})$	$\mathbf{m}\times\mathbf{B}$	$\mathbf{m}\cdot\mathbf{B}$	$-\frac{1}{c^2}(\mathbf{m}\times\mathbf{E})$	$-\frac{1}{c^2}(-\mathbf{m}V+\mathbf{m}\cdot\mathbf{W})$

Figure 36.13: Dipoles at rest (2). From the reference.

36.06 Coulomb|Biot-Savart Laws - ElectroStatics/MagnetoStatics

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|||

©|D.Griffiths, M.Heald|Time-dependent generalizations of the Biot-Savart and Coulomb laws|*Am.J.Phys*, , ..., ..., ...*Ed.*, ...

Stationary charges produce electric fields that are constant in time: ElectroStatics.
Steady currents produce magnetic fields that are constant in time: MagnetoStatics.
Therefore, ElectroStatics/MagnetoStatics:

$$\partial_t \rho = 0 \qquad \partial_t \mathbf{j} = 0 \qquad \text{all places and all times} \quad .$$

The Electric|Magnetic field at $\mathbf{x}$ produced by Static|Steady Charge|Current can be deduced via the Coulomb|Biot-Savart laws from the known static-charge/steady-current distributions at $\mathbf{y}$:

$$\mathbf{u} \equiv \mathbf{u}[\mathbf{x},\mathbf{y}] \equiv \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \quad ,$$

$$\mathbb{d}\mathbf{E}[\mathbf{x}] = \frac{1}{4\pi\varepsilon_0}\frac{\mathbb{d}q[\mathbf{y}]\mathbf{u}}{|\mathbf{x}-\mathbf{y}|^2} = \frac{1}{4\pi\varepsilon_0}\frac{\rho[\mathbf{y}]\mathbf{u}}{|\mathbf{x}-\mathbf{y}|^2}\,\mathbb{d}\mathbf{y} \quad , \qquad \rightarrow \text{1335 1756}$$

$$\mathbb{d}\mathbf{B}[\mathbf{x}] = \frac{\mu_0}{4\pi}\frac{i\,\mathbb{d}\mathbf{L}[\mathbf{y}]\wedge\mathbf{u}}{|\mathbf{x}-\mathbf{y}|^2} = \frac{\mu_0}{4\pi}\frac{\mathbf{j}[\mathbf{y}]\wedge\mathbf{u}}{|\mathbf{x}-\mathbf{y}|^2}\,\mathbb{d}\mathbf{y} \quad . \qquad \rightarrow \text{1088 1089 1335 1757}$$

Electric field is generated by superposition of the infinitesimal contributions, $\rho\ \mathbb{d}V$.
Magnetic field is generated by superposition of the infinitesimal contributions, $\mathbf{j}\ \mathbb{d}V$.
In ElectroStatics/MagnetoStatics the above relations are equivalent, respectively, to Gauss law and Ampère law.
It should be noted that both equations are inverse square laws. However, the vector character is very different.
Note that as a moving charge does not constitute a steady current MagnetoStatics cannot be built from the properties of a single charge in motion, while ElectroStatics can be built from the properties of a single stationary charge.
Stationary charges generate constant electric fields: ElectroStatics.
Steady currents generate constant magnetic fields: MagnetoStatics.

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36.07 Dipole Fields - ElectroStatics/MagnetoStatics

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|§ eq. 4.20, § eq. 5.64||

The electric field produced by a static electric dipole, read § 16.12.12, is:

$$\mathbf{E}[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\,(\mathbf{p}\cdot\mathbf{r})\,\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}\right) \quad . \tag{36.07.01}$$

→ 1129 1714 1759

It can be also derived from Coulomb law.

The magnetic field produced by a static magnetic dipole, read § 16.12.12, is:

$$\mathbf{B}[\mathbf{r}] = \frac{\mu_0}{4\pi}\left(\frac{3\,(\mathbf{m}\cdot\mathbf{r})\,\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right) \quad . \tag{36.07.02}$$

→ 1129 1714

It can be also derived from Biot-Savart law.

A subtle issue arise [1] with the point dipole field. On a closer inspection, it is in fact necessary to add a Dirac delta in equation (36.07.01), (36.07.02), with different signs and coefficients for the electric and magnetic cases. Outside the dipole this additional term is irrelevant.

The corrective term, deriving from § 36.20.03.01, modifies the expression so that the electric field produced by a static electric dipole, located at $\mathbf{r}_0$, equation (36.07.01), becomes:

$$\mathbf{E}[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\,(\mathbf{p}\cdot\mathbf{d})\,\mathbf{d}}{d^5} - \frac{\mathbf{p}}{d^3} - \frac{4\pi}{3}\mathbf{p}\delta^3[\mathbf{r}-\mathbf{r}_0]\right) \qquad \mathbf{d}\equiv\mathbf{r}-\mathbf{r}_0 \quad . \tag{36.07.03}$$

The corrective term, deriving from § 36.20.03.02, modifies the expression so that the magnetic field produced by a static magnetic dipole, located at $\mathbf{r}_0$, equation (36.07.02), becomes:

$$\mathbf{B}[\mathbf{r}] = \frac{\mu_0}{4\pi}\left(\frac{3\,(\mathbf{m}\cdot\mathbf{d})\,\mathbf{d}}{d^5} - \frac{\mathbf{m}}{d^3} + \frac{8\pi}{3}\mathbf{m}\delta^3[\mathbf{r}-\mathbf{r}_0]\right) \qquad \mathbf{d}\equiv\mathbf{r}-\mathbf{r}_0 \quad . \tag{36.07.04}$$

This corrective Dirac-delta term is sometimes called *contact term*.

[1] J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...

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36.08 Maxwell Equations and Charge Conservation in an Inertial Frame

Maxwell Equations unify Electric|Magnetic phenomena showing that electric fields can gives rise to magnetic fields and magnetic fields can gives rise to electric fields so that the two are strictly interconnected.

Moreover, electromagnetism predicts ElectroMagnetic waves, that is optics is an ElectroMagnetic phenomenon as well.

36.08.01 Maxwell Equations in Integral Form With Time-Independent Integration Domains

This § is referenced at pages:
[1717]

Read § 36.10 for the discussion of the case when integration domains are time-dependent.

The **Maxwell Equations** , written in **integral form for arbitrary volumes, surfaces and lines, but fixed in some Inertial Reference Frame** , read:

$$\oiint_{\partial\Omega} \mathbf{E}\cdot \mathbb{d}\mathbf{S} = \frac{1}{\varepsilon_0}\iiint_{\Omega} \rho\, \mathbb{d}V \quad , \tag{36.08.01}$$

→ 1724 2491 2512

$$\oiint_{\partial\Omega} \mathbf{B}\cdot \mathbb{d}\mathbf{S} = 0 \quad , \tag{36.08.02}$$

→ 1724 2512

$$\oint_{\partial\Sigma} \mathbf{E}\cdot \mathbb{d}\mathbf{L} = -\iint_{\Sigma} \partial_t\mathbf{B}\cdot \mathbb{d}\mathbf{S} = -\frac{\mathbb{d}}{\mathbb{d}t}\iint_{\Sigma} \mathbf{B}\cdot \mathbb{d}\mathbf{S} \quad , \tag{36.08.03}$$

→ 1715 1723 1724 2512

$$\oint_{\partial\Sigma} \mathbf{B}\cdot \mathbb{d}\mathbf{L} = \mu_0\iint_{\Sigma} \mathbf{j}\cdot \mathbb{d}\mathbf{S} + \mu_0\varepsilon_0\iint_{\Sigma} \partial_t\mathbf{E}\cdot \mathbb{d}\mathbf{S} = \mu_0\iint_{\Sigma} \mathbf{j}\cdot \mathbb{d}\mathbf{S} + \mu_0\varepsilon_0\frac{\mathbb{d}}{\mathbb{d}t}\iint_{\Sigma} \mathbf{E}\cdot \mathbb{d}\mathbf{S} \quad , \tag{36.08.04}$$

→ 1715 1722 1724 2491 2512

where Ω is an arbitrary fixed volume bounded by the closed surface $\partial\Omega$ and Σ is an arbitrary fixed surface bounded by the closed line $\partial\Sigma$.

The last equality of the last two equations applies **if and only the integration domains are time-independent**. As we are assuming here that all volumes, surfaces and lines are fixed in some Reference Frame, it does not make any difference whether the time derivatives are outside or inside the integral sign. However, the situation changes drastically if one accounts for moving volumes, surfaces and lines (see section § 36.10).

The **Maxwell Equations** must be complemented with the charge conservation law, which is, however, implied by the two **Maxwell Equations** containing Charge|Current:

$$\oiint_{\partial\Omega} \mathbf{j}\cdot \mathbb{d}\mathbf{S} + \frac{\mathbb{d}}{\mathbb{d}t}\iiint_{\Omega} \rho\, \mathbb{d}V_{\mathbf{x}} = 0 \quad . \tag{36.08.05}$$

→ 1715 1716

Note that the charge conservation law, described by the equation of continuity of charge (36.08.05) is a special case of the general balance equations, which includes the more general case of Sources|Sinks of the extensive physical quantity.

Note that Maxwell Equations contain two kinds of generators of ElectroMagnetic fields: charges and currents. However, they are related by the charge continuity equation and, therefore, they are not truly independent. This explains why it is often possible to solve problems by only considering either charge only or current only as the generators of ElectroMagnetic fields.

In the two circulation equations, (36.08.03), (36.08.04), it is implicit, for consistency reasons, that **the quantity on the right-hand side does not depend on the choice of the surface**, for any given closed line

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$\partial\Sigma$. This means that the integrand of surface-integrals at the right-hand side must be solenoidal, that is their integral over any closed surface is null. By using the charge conservation law (36.08.05) one can find:

$$0 = \oiint_{\partial\Omega} \partial_t \mathbf{B} \cdot \mathbb{d}\mathbf{S} \implies \oiint_{\partial\Omega} \mathbf{B} \cdot \mathbb{d}\mathbf{S} = \text{constant in time} \quad ,$$

$$0 = \oiint_{\partial\Omega} (\mathbf{j} + \varepsilon_0 \partial_t \mathbf{E}) \cdot \mathbb{d}\mathbf{S} \implies \oiint_{\partial\Omega} \mathbf{E} \cdot \mathbb{d}\mathbf{S} - \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho[\mathbf{x}] \, \mathbb{d}V_{\mathbf{x}} = \text{constant in time} \quad .$$

This means that if the charge conservation law is assumed the flux equations can be considered as just fixing the initial value at any time of the constant.

Finally note that **when the displacement current is neglected, for instance in steady conditions, the current density vector is solenoidal**:

$$0 = \oiint_{\partial\Omega} \mathbf{j} \cdot \mathbb{d}\mathbf{S} \quad .$$

Note that, **even if the magnetic field is such that its flux across any closed surface is zero, its field lines are not necessarily closed curves**: read § 16.12.19.

36.08.02
Maxwell Equations in Differential Form

This § is referenced at pages:
[1717, 2066]

Maxwell equations is a system of **coupled linear partial differential equations of first order**: **ElectroMagnetism is a linear theory**.

The **Maxwell Equations** , written in differential form and applicable in all the cases when the fields and Charge|Current producing the ElectroMagnetic fields are, from the mathematical point of view, sufficiently well-behaved functions, read:

$$\boxed{\mathbf{div}\mathbf{E} = \rho/\varepsilon_0} \quad , \tag{36.08.06}$$

→ 1736 2066

$$\boxed{\mathbf{div}\mathbf{B} = 0} \quad , \tag{36.08.07}$$

→ 1730 1736 2066 2491

$$\boxed{\mathbf{rot}\mathbf{E} = -\partial_t \mathbf{B}} \quad , \tag{36.08.08}$$

→ 1717 1726 1730 1736 2066 2491

$$\boxed{\mathbf{rot}\mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \partial_t \mathbf{E}} \quad .$$

The **Maxwell Equations** must be complemented with the charge conservation law, which is, however, implied by the two **Maxwell Equations** containing Charge|Current:

$$\boxed{\mathbf{div}\mathbf{j} + \partial_t \rho = 0} \quad . \tag{36.08.09}$$

→ 1716 1953

On the other hand, if the charge conservation law is assumed to hold, one can show that the divergence equations just fix the initial value in time, at any point, of the field divergence itself. In fact it is sufficient to take the divergence of both rotor equations and use equation (36.08.09) to obtain:

$$\partial_t \mathbf{div}\mathbf{B} = 0 \quad ,$$

$$\partial_t \left(\mathbf{div}\mathbf{E} - \frac{\rho}{\varepsilon_0} \right) = 0 \quad .$$

Basically, and very roughly, the applicability range of the **Maxwell Equations** written in differential form extends to all the cases and points in time and space when the written expressions are well defined. In all other cases the integral forms should be used.

Maxwell Equations in differential form show that:

- charge density gives the **sources/sinks** of the electric field: charge density is the same as electric field flux density while no Sources|Sinks exists for magnetic fields;
- current density gives the **vortexes** of the magnetic field: current density is the same as magnetic field circulation density while also changing Electric|Magnetic fields gives rise to magnetic/electric vortexes.

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36.08.03
Number of Independent Variables in Maxwell Equations

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The four vector **Maxwell Equations** are equivalent to eight scalar equations. However, the ElectroMagnetic fields, **E** and **B**, have only six independent components. This can be explained by the argument in sections § 36.08.01, § 36.08.02: when taking into account that charges and currents are constrained by the charge conservation equation it can be shown that the two divergence equations are basically initial condition on the values of the divergences (and therefore there are only $6 = 8 - 2$ equations). Note that the actual value of the constant can be assumed to be zero, by assuming as a starting time any time far in the past when charges and fields at the point were zero.

36.08.04
Integral Versus Differential Forms of Maxwell Equations

The integral form can be applied to any continuous field, as a sufficient, but by no means necessary, condition. The integral form is therefore of much broader applicability. The shortcoming of the integral form is the fact that one has one integral equation for each arbitrarily chose volume in space, while the differential form just provides one partial difference equation. Integral equations, in particular, are better suited to deal with discontinuities of the ElectroMagnetic fields.

36.08.05
Some Consequences of Maxwell Equations

As a consequence of Faraday-Neumann-Lenz equation, equation (36.08.08), the line-integral of the electric field between two points depends, in non-static cases, from the path. This is caused by the ElectroMagnetic induction. As a consequence of Faraday-Neumann-Lenz equation an EM-motive force (emf) originating from the electric field is present.

As a consequence of Maxwell-Ampère equation, equation (36.08.02), the line-integral of the magnetic field between two points depends, in non-static cases, from the path. This is caused by the magneto-electric induction, the dual of ElectroMagnetic induction. It also depends on the path in static cases when currents are present.

As discussed in section § 36.20.18 the magneto-electric induction is typically a much smaller effect than ElectroMagnetic induction, this is why it is often neglected, at least for slowly varying fields.

36.08.06
Structure of Maxwell Equations

Two different kinds of structures can be identified in **Maxwell Equations** .

- **Equations 2 and 3 are structural equations for the ElectroMagnetic fields as they only contain the fields** and not the Charge|Current producing the ElectroMagnetic fields. Equations 1 and 4 relate the ElectroMagnetic fields to the Charge|Current producing the ElectroMagnetic fields.
- **The rotor equations (3 and 4) link|mix Electric|Magnetic fields and include a time-dependence**. **The divergence equations (1 and 4) do not link|mix Electric|Magnetic fields and do not include any time-dependence** being identical in static as well as in dynamic conditions.

It is tempting to read the 3rd and 4th Maxwell equations, as: time-varying magnetic fields generate vortexes of the electric field and time-varying electric fields generate vortexes of the magnetic field; currents also generate vortexes of the magnetic field. However, in the relativistic view, Electric|Magnetic

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fields are parts of the same second-order field tensor, satisfying an inhomogeneous equation with the four-current as the generator, equation (61.05.04).

Therefore, **it seems to be more appropriate to state that the four-current generates the ElectroMagnetic fields (electric plus magnetic fields) at once**.

36.08.07
Maxwell Equations Unify Electric|Magnetic Phenomena

Maxwell Equations quantify to what extent Electric|Magnetic phenomena affect Magnetic|Electric phenomena, via the rotor equations.

36.08.08
On the Existence of Magnetic Monopoles

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|6.11||

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At present there is no experimental evidence.

The existence of magnetic Monopoles would explain electric charge quantisation.

Some condensed matter systems have a structure superficially similar to a real elementary magnetic monopole, a structure known as a flux tube. The ends of a flux tube form a Magnetic Dipole, but since they move independently, they can be treated for many purposes as independent magnetic monopole quasi-particles.

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36.09 Discontinuities of Fields and current at a fixed surface

This § is referenced at pages:
[1852, 1895]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|||
©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|||
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|Intro|*Deep discussion*|
©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|Appendix 6|*Deep discussion*|
©|R.B. McQuistan|Scalar and Vectar Fields: A Physical Interpretation - 11.8|*A mathematical demonstration*|
©|G.S.Smith, *An Introduction To Classical Electromagnetic Radiation*, 1997, CUP, ...Ed., ...|1.3||
©|V.Namias|Am.J.Phys, , ..., ..., ...Ed., ... 56, 898 (1988) - doi: 10.1119/1.15406|*Definitive!*|

Consider a smooth surface, separating two different media (media := just two arbitrary distinct portions of space), possibly with different physical properties, a point P at the surface where the normal is $\hat{\mathbf{n}}$, oriented from medium 1 towards medium 2, and define the discontinuity for any vector field $\mathbf{V}$ as:

$$\Delta\mathbf{V} \equiv \mathbf{V}_2 - \mathbf{V}_1 \quad .$$

The ($\perp$) and ($\parallel$) suffixes will refer, in our notation, to the plane tangent at the surface at P, not to the normal vector of the surface at P:

- $\mathbf{V}_\perp \equiv (\hat{\mathbf{n}}\cdot\mathbf{V})\,\hat{\mathbf{n}}$, the component orthogonal to the surface, that is the component having the same direction as the normal at the surface at P, **most often named the normal component**;
- $\mathbf{V}_\parallel \equiv \hat{\mathbf{n}}\wedge(\mathbf{V}\wedge\hat{\mathbf{n}})$, the component parallel to the surface, that is the component lying in the plane tangent at the surface at P, **most often named the tangential component**.

Note that the opposite conventions is often used in the literature! It is therefore important to clarify whether parallel/perpendicular refer to the surface or to the normal.

Surface charges and surface currents must be accounted for, that is a set of charges constrained to move on a smooth surface. Physically speaking, surface charges and surface currents are mathematical limiting cases for charges and currents limited to a very thin sheet whose thickness is much smaller than any other relevant dimension.

The discontinuities of the ElectroMagnetic fields, as derived from the integral forms of the **Maxwell Equations** , are:

$$\hat{\mathbf{n}}\cdot\Delta\mathbf{E}_\perp = \hat{\mathbf{n}}\cdot\Delta\mathbf{E} = \rho_S/\varepsilon_0 \quad , \tag{36.09.01}$$

→ 1723 1745

$$\hat{\mathbf{n}}\cdot\Delta\mathbf{B}_\perp = \hat{\mathbf{n}}\cdot\Delta\mathbf{B} = 0 \quad , \tag{36.09.02}$$

→ 1723 1745

$$\hat{\mathbf{n}}\wedge\Delta\mathbf{E}_\parallel = \hat{\mathbf{n}}\wedge\Delta\mathbf{E} = \mathbf{0} \quad , \tag{36.09.03}$$

→ 1723

$$\hat{\mathbf{n}}\wedge\Delta\mathbf{B}_\parallel = \hat{\mathbf{n}}\wedge\Delta\mathbf{B} = \mu_0\mathbf{j}_S \quad . \tag{36.09.04}$$

→ 1722 1723 2252

The above equations can be summarized in vector form as:

$$\Delta\mathbf{E} = \hat{\mathbf{n}}\rho_S/\varepsilon_0 \qquad \Delta\mathbf{B} = \mu_0\,\mathbf{j}_S\wedge\hat{\mathbf{n}} \quad .$$

Note that in general **the components of the magnetic field parallel to the surface on the two sides may be, in general, not parallel**.

Note that discontinuities in the fields might arise not only in presence of abrupt changes in the physical properties of the medium but **also in presence of moving Charge|Current**, starting/stopping to radiate at some instant.

Read § 36.16.03.

36.09.01
Discontinuity of Normal Components of fields at a fixed surface

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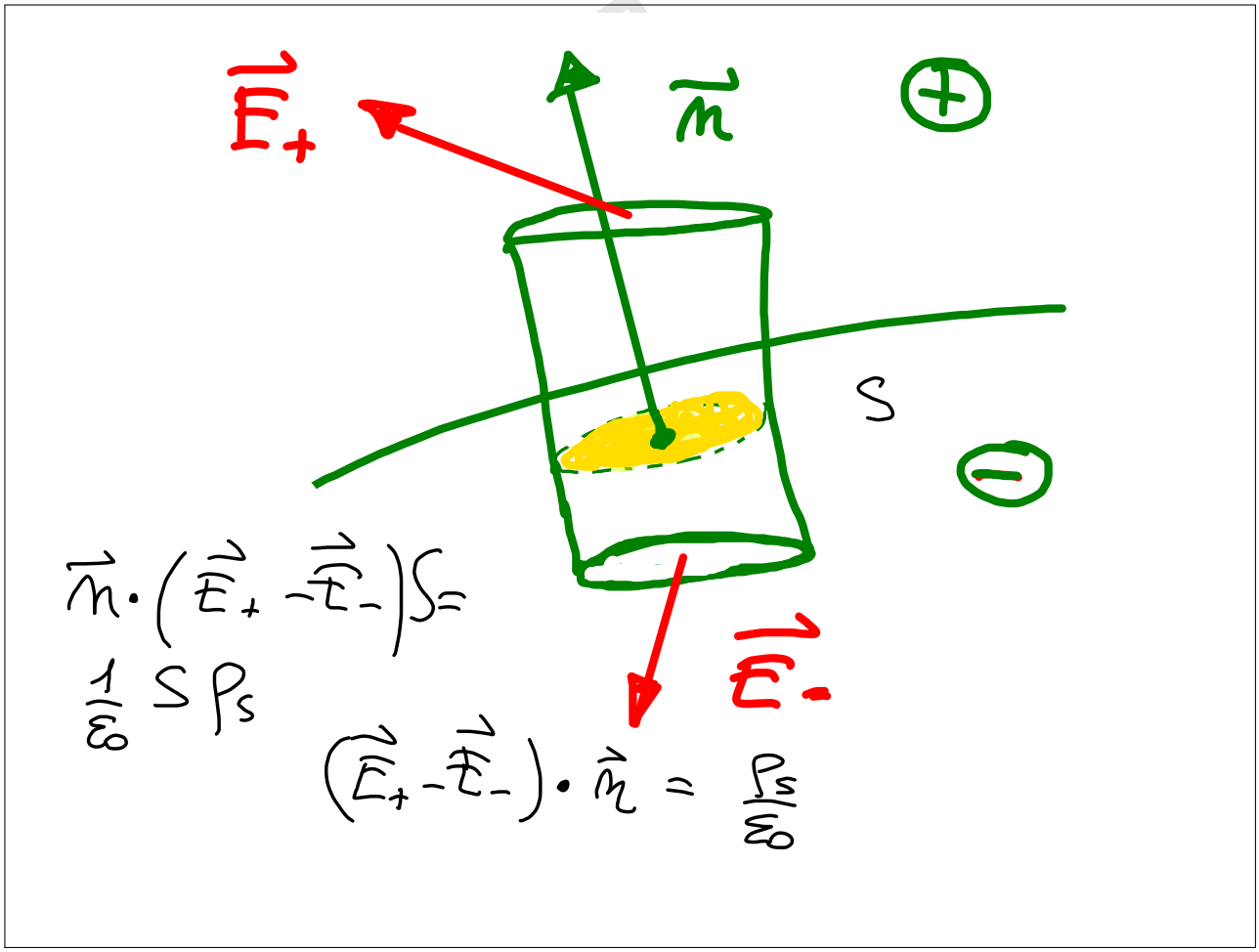
36.09.01.01 Normal Component of the Electric Field

This § is referenced at pages:
[Never referenced.]



36.09.01.02 Normal Component of the Magnetic Field

This § is referenced at pages:
[Never referenced.]

36.09.02
Discontinuity of Tangential Components of fields at a fixed surface

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Note that **the discontinuity relation for the tangential components do not imply at all that the tangential components on the two sides of the surface are parallel vectors: in general they are not collinear. On the other hand, the discontinuity of the normal components are necessarily parallel vectors**. The tangential components on the two sides of the surface can have an arbitrary direction in the tangent plane. In other words the two different tangential components of the fields in the two media are not necessarily in the same plane passing by the normal unit vector $\hat{\mathbf{n}}$.

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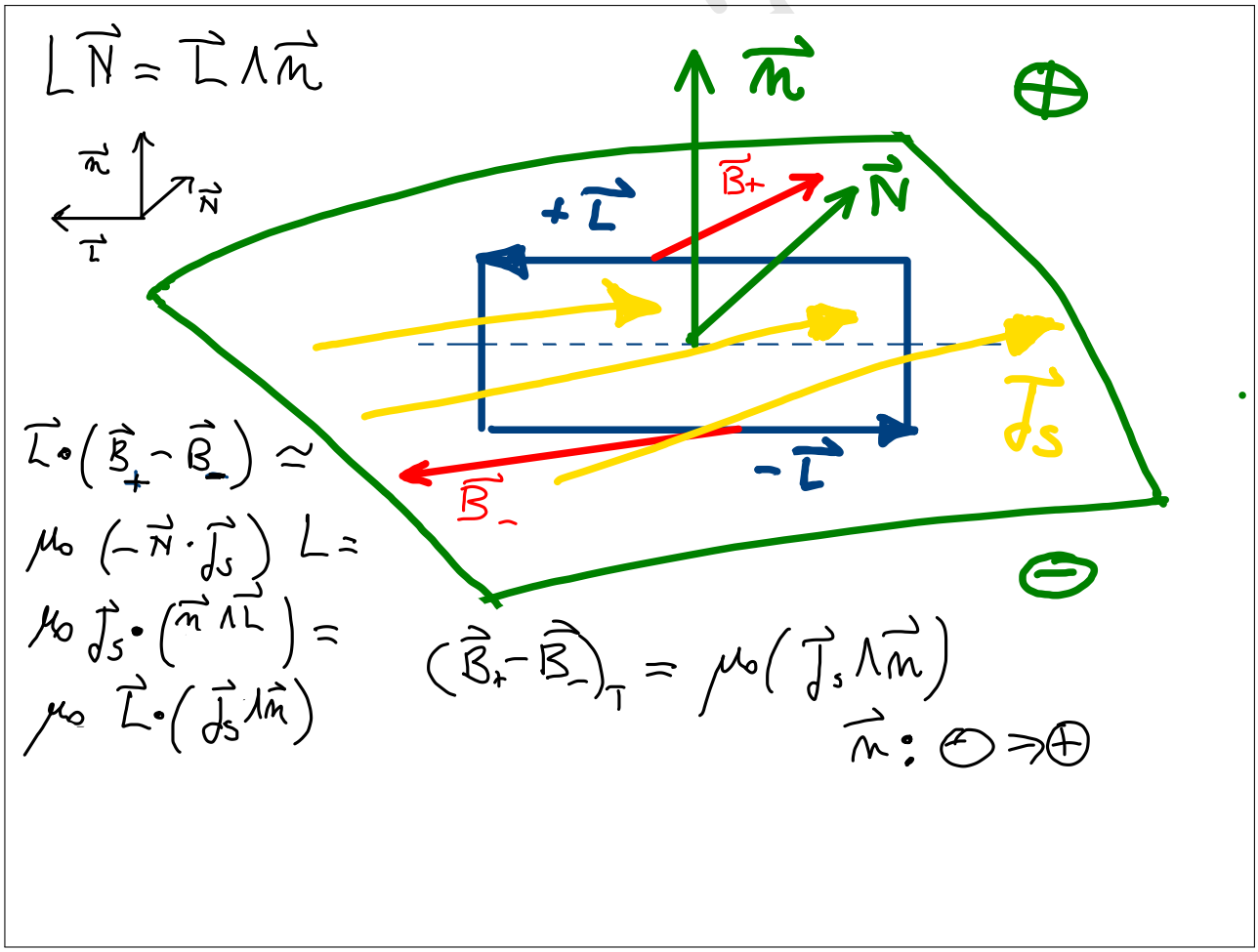
36.09.02.01 Tangential Component of the Magnetic Field



Let us demonstrate equation (36.09.04) starting from the integral form Maxwell equation (36.08.04).

Consider a smooth surface, Σ, separating two different media with different physical properties, a point P at the surface where the normal is $\hat{\mathbf{n}}$, oriented from medium 1 towards medium 2. Consider a small rectangle, S, whose frontier is the rectangular path, $\Gamma \equiv \partial S$, such that the normal vector $\hat{\mathbf{n}}$ is an axis of symmetry of the rectangle and the center of the rectangle is at P. The two sides parallel to $\hat{\mathbf{n}}$ are short and, moreover, the limit to zero will be taken at the end of the derivation. The other two sides, perpendicular to $\hat{\mathbf{n}}$, are short enough that neither the fields nor the Charge|Current, nor the properties of the surface change too much along it. Let $\mathbf{L}$ be the vector defining the oriented side perpendicular to $\hat{\mathbf{n}}$, the one of the two lying inside medium 1. Note that $\mathbf{L}$ will arbitrary but always parallel to the tangent plane.

The other side perpendicular to $\hat{\mathbf{n}}$ is then defined by $-\mathbf{L}$, in medium 2. Let $\widehat{\mathbf{N}}$ be the unit vector perpendicular to the rectangular loop, defined by

$$\boxed{L\widehat{\mathbf{N}} \equiv \mathbf{L}\wedge\hat{\mathbf{n}}} \quad .$$

The surface current concatenated with the path Γ is given by the line integral on the line determined by the intersection of the rectangle S and the surface Σ: $S \cap \Sigma$.

In the limit when the two sides parallel to $\hat{\mathbf{n}}$ tend to zero any contribution to the current concatenated to Γ originating from a volume current and any contribution of the time derivative of the flux of the electric field will tend to zero, because they are both proportional to the surface tending to zero. Any surface current, $\mathbf{j}_S$, will give a non zero contribution, in general, even in the limit when the surface tends to zero.

One can then write:

$$\oint_C \mathbf{B}\cdot \mathrm{d}\mathbf{L} \simeq \mathbf{L}\cdot(\mathbf{B}_1 - \mathbf{B}_2) = \mu_0 I =$$

$$-\mu_0 \int_{S\cap\Sigma} \left(\mathbf{j}_S\cdot\widehat{\mathbf{N}}\right) \mathrm{d}\mathbf{L} = -\frac{\mu_0}{L}\int (\mathbf{j}_S\cdot(\hat{\mathbf{n}}\wedge\mathbf{L})) \mathrm{d}\mathbf{L} \simeq$$

$$\mu_0\, \mathbf{j}_S\cdot(\mathbf{L}\wedge\hat{\mathbf{n}}) = \mu_0\, \mathbf{L}\cdot(\hat{\mathbf{n}}\wedge\mathbf{j}_S) \quad .$$

Finally, as the above relation is valid for any rectangular surface/path (S / Γ), one can conclude:

$$\boxed{\mathbf{L}\cdot(\mathbf{B}_2 - \mathbf{B}_1) = \mu_0\, \mathbf{L}\cdot(\mathbf{j}_S\wedge\hat{\mathbf{n}})} \quad .$$

However, **the relation does not constrain in any way the normal components**. In fact $\mathbf{L}$ is arbitrary but constrained to be in the tangential plane with $\hat{\mathbf{n}}\cdot\mathbf{L} = 0$. Therefore:

$$\mathbf{n}\cdot\mathbf{L} = 0 \implies \mathbf{L}\cdot(\mathbf{B}_2 - \mathbf{B}_1) = \mathbf{L}\cdot(\mathbf{B}_2 - \mathbf{B}_1)_{\parallel} + \mathbf{L}\cdot(\mathbf{B}_2 - \mathbf{B}_1)_{\perp} = \mathbf{L}\cdot(\mathbf{B}_2 - \mathbf{B}_1)_{\parallel} \quad .$$

On the other hand $\mathbf{j}_S\wedge\hat{\mathbf{n}}$ is always in the tangential plane.

One can then only draw conclusions about properties of $(\mathbf{B}_2 - \mathbf{B}_1)$ **in the tangential plane**. Thanks to the arbitrariness of $\mathbf{L}$ in the tangential plane one can then conclude:

$$\boxed{(\mathbf{B}_2 - \mathbf{B}_1)_{\parallel} \equiv \Delta\mathbf{B}_{\parallel} = \mu_0\, \mathbf{j}_S\wedge\hat{\mathbf{n}} \qquad \Longrightarrow \qquad \hat{\mathbf{n}}\wedge\Delta\mathbf{B}_{\parallel} = \mu_0\mathbf{j}_S} \quad , \qquad (36.09.05)$$

→ 1793

as, in fact, the component of $\mathbf{j}_S$ perpendicular to the normal vector is the same as $\mathbf{j}_S$, which is in the tangential plane.

For any material, the concept of **surface current is important because Magnetization of matter gives, in general, a surface Magnetization current**, see section § 39.05.02.

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36.09.02.02 Tangential Component of the Electric Field

In a similar way as the derivation in section § 36.09.02.01 one can demonstrate equation (36.09.03). One starts from the integral form **Maxwell Equation** (36.08.03) and, taking into account that the flux of the magnetic field tends to zero in the limit, one obtains:

$$\boxed{(\mathbf{E}_2 - \mathbf{E}_1)_{\parallel} \equiv \Delta\mathbf{E}_{\parallel} = \mathbf{0} \qquad \Longrightarrow \qquad \widehat{\mathbf{n}}\wedge\Delta\mathbf{E}_{\parallel} = \mathbf{0}} \quad . \tag{36.09.06}$$

→ 1793

36.09.02.03 Equivalent Formulas for Tangential Components

Two different equivalents forms of the equations for the discontinuity of the tangential components are possible:

$$\boxed{\widehat{\mathbf{n}}\wedge\Delta\mathbf{E} = \mathbf{0}} \quad \Leftrightarrow \quad \widehat{\mathbf{n}}\wedge\big(\widehat{\mathbf{n}}\,(\,\widehat{\mathbf{n}}\cdot\Delta\mathbf{E}) - \boldsymbol{\Delta}\mathbf{E}_{\parallel}\big) = \mathbf{0} \quad \Leftrightarrow \quad \boxed{\boldsymbol{\Delta}\mathbf{E}_{\parallel} = \mathbf{0}} \quad , \tag{36.09.07}$$

$$\boxed{\widehat{\mathbf{n}}\wedge\Delta\mathbf{B} = \mu_0\mathbf{j}_S} \quad \Leftrightarrow \quad \widehat{\mathbf{n}}\wedge\big(\widehat{\mathbf{n}}\,(\,\widehat{\mathbf{n}}\cdot\Delta\mathbf{B}) - \boldsymbol{\Delta}\mathbf{B}_{\parallel}\big) = \mu_0\,\widehat{\mathbf{n}}\wedge\mathbf{j}_S \quad \Leftrightarrow \quad \boxed{\boldsymbol{\Delta}\mathbf{B} = \mu_0\,\mathbf{j}_S\wedge\widehat{\mathbf{n}}} \quad . \tag{36.09.08}$$

36.09.03 Discontinuities of the Current Density at a fixed surface

The discontinuity of the current density can be derived from the charge continuity equation in a similar manner to equations (36.09.01), (36.09.02), because they are all based on a flux/divergence equation. One finds:

$$\boxed{\widehat{\mathbf{n}}\cdot\Delta\mathbf{j} = \widehat{\mathbf{n}}\cdot\Delta\mathbf{j}_{\perp} = -\mathbf{div}_S\mathbf{j}_S - \partial_t\rho_S} \quad , \tag{36.09.09}$$

→ 1723

in terms of the surface charge, ρ_S, surface current, $\mathbf{j}_S$ and the surface divergence operator, $\mathbf{div}_S\cdot$, (read § 16.06.02) ■

The physical meaning of the charge discontinuity equation is physically clear: any charge which is not found after crossing the surface must be found either as charge accumulated on the surface or flow as surface current density as leakage current.

In the often encountered case when at the interface there is no surface leakage current and there is a constant surface charge (as it is the case in a steady state) then the normal component of the current density is conserved:

$$\boxed{I_1 = I_2 \implies \widehat{\mathbf{n}}\cdot\Delta\mathbf{j} = \widehat{\mathbf{n}}\cdot\Delta\mathbf{j}_{\perp}} \qquad \text{sufficient condition: steady state and no surface current} \quad . \tag{36.09.10}$$

Note that, when the charge continuity equation is assumed, only three Maxwell equations are truly independent; similarly only three of the five boundary conditions, equations (36.09.01), (36.09.02), (36.09.03), (36.09.04), (36.09.09), are truly independent.

For moving surfaces, read § 36.16.

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36.10 Moving Lines Surfaces and Volumes in an Inertial Frame

This § is referenced at pages:
[1715, 1715]

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©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|16, 17||
©|S.Bobbio & E.Gatti, *Elettromagnetismo E Ottica*, 1991, Bollati-Boringhieri, ...Ed., ...|||
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|6.1||
©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|14||

Equations (36.08.01), (36.08.02) are always valid in any case, even if the integration domain moves.

Equations (36.08.03), (36.08.04) are valid in any case, even if the integration domain moves as long as the form with time derivative inside the integral sign is used. Taking the time derivative sign outside the integral is possible as long as the integration domain is fixed. If it is not fixed, it is clear that the derivative outside the integral sign gives a result which depends on the way the integration domain is changing in time.

In some applications it is useful to use moving integration boundaries. It is therefore necessary to understand how to treat those cases.

It is important to note that studying this problem, in particular the Faraday-Neumann-Lenz law, A.Einstein started to develop the theory of relativity.

36.10.01
Faraday-Neumann-Lenz Law

The most general form of the law of ElectroMagnetic Induction (Faraday-Neumann-Lenz) must take into account that, **in any fixed reference frame, there are two different and distinct physical effects capable to give rise to ElectroMagnetic Induction**:

- **the change, at any point, of the magnetic field**;
- **the movement of a real current-carrying circuit** (that is a **kinematic constraint** to the motion of the charges) inside a magnetic field.

The first effect is totally independent of any real current carrying circuit and it exists in empty space too.

The second effect is closely related to the presence of real current-carrying circuits. **A real current carrying circuit is actually modeled as a set of kinematic constraints to the motion of the charges, a kinematic constraint in the sense of mechanics**.

If the kinematic constraint is one-dimensional, that is a curve, as in the case of a thin wire, the description of ElectroMagnetic induction is relatively easy, as it will be shown below.

If the kinematic constraint is either two-dimensional or three-dimensional, that is either a surface or a volume current, as in the case of either surface or volume conductors, the description is typically much more complex, because the direction of the velocities of the charges are not uniquely determined by the kinematic constraint. In this case, using the law of ElectroMagnetic Induction (Faraday-Neumann-Lenz) in the form of the flux rule is hardly applicable.

© - QUOTE

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We know of no other place in physics where such a simple and accurate general principle requires

WEB-PROOFS

for its real understanding an analysis in terms of two different phenomena. Usually such a beautiful generalization is found to stem from a single deep underlying principle. Nevertheless, in this case there does not appear any such profound implication. We have to understand the rule as the combined effects of two quite separate phenomena. Italics in the original.

In order to avoid mistakes **both two different effects giving rise to EM induction phenomena must always be accounted for** and the physical origin of the EM induction phenomenon must be clearly identified.

A **clear understanding** of the way one passes from/to the **Maxwell Equations** in differential to/from **the flux-rule** is essential to avoid mistakes when applying the flux-rule to peculiar situations.

36.10.02
EM-Motive Force

This § is referenced at pages:
[1726]

The **EM-motive** force is typically introduced **for forces of non-ElectroMagnetic origin as the work done on the unit charge at any fixed time on a closed fixed circuit**:

$$\mathcal{E} \equiv \frac{\mathrm{d}W}{\mathrm{d}q} \equiv \frac{1}{q}\oint_C \mathbf{f}\cdot \mathrm{d}\mathbf{L} \qquad \text{at any fixed time on a closed fixed circuit} \quad . \tag{36.10.01}$$

→ 1725

The term **EM-motive force** is meant to describe the effect of external forces on electric charges (from which the term **EM-motive** originates) but it is by no means a force.

The physical entities responsible for the force f can be any one of many different things: a chemical force, as in a battery; mechanical pressure converted into an electric impulse, as in a piezoelectric crystal; a temperature gradient, as in a thermocouple; light, as in a photo-electric cell; and many others.

Whatever the mechanism, its net effect is quantified by the line integral of **f** around the circuit, that is, **force per unit charge integrated on a closed fixed path at fixed time**

In presence of ElectroMagnetic forces the definition must be generalized to account for and include the Lorentz force. One can therefore write the EM-motive force on a closed loop circuit, C, at any fixed time, for purely ElectroMagnetic forces, starting from its definition, as:

$$\mathcal{E} \equiv \frac{\mathrm{d}W}{\mathrm{d}q} = \oint_C (\mathbf{E} + \mathbf{v}\wedge\mathbf{B})\cdot \mathrm{d}\mathbf{L} \qquad \text{at any fixed time} \quad . \tag{36.10.02}$$

→ 1725 1726

The definition (36.10.02) **generalizes the concept of EM-motive force to the case when the origin of the force is of ElectroMagnetic origin**. The expression (36.10.02) is actually the **force per unit charge integrated on a closed fixed path at fixed time**. Therefore it can be identified with the EM-motive force, with forces of purely ElectroMagnetic origin. The EM-motive force of ElectroMagnetic origin (36.10.02) is thus an extension of the concept of EM-motive force.

In the case of the ElectroStatic field the integral along a closed line is zero.

In the above expression (36.10.02) the velocity **v** is the velocity of the charge, which, in general is the sum of the velocity of the current carrying circuit and the drift velocity of the charge, that is the velocity of the charge with respect to the circuit.

One should be careful that, whenever the real circuit is moving/changing, the definition of **around the closed circuit** requires some care [1]. In fact one should be careful to use the expression (36.10.02) **integrating over the closed loop at a fixed time**. In this sense it might be misleading to think, as suggested by equation (36.10.01), of work done on a specific charge, if the circuit is moving. The correct interpretation, as from equation (36.10.02), is the **circulation of the force vector over the closed path at fixed time per unit charge**. Interpreting the integral as the real work done on a specific charge can give two kinds

[1] D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...

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of troubles: if the circuit moves it might be unclear what a closed loop is; if the charges moves slowly with respect to the changes of the ElectroMagnetic fields then the ElectroMagnetic fields might change before the tour of the loop ends.

36.10.03
3rd Maxwell Equation in Differential/Flux-Rule Form

The discussion in this section shows an example where the mathematical modeling needs to be carried on having clearly in mind the physical processes.

In order to be consistent with the following mathematical passages, involving line and surface-integrals, it must be possible to **treat the real current-carrying circuit as an ideal line circuit, that is the kinematic constraint is a one-dimensional constraint**.

In this case it can be shown that it makes no difference to use, in the above expression (36.10.02), either the real velocity of the charge, $\mathbf{v}$, or the velocity of the circuit, $\mathbf{u}$, at the point where the charge is located. In fact the drift velocity, $\mathbf{V}_D$ is parallel to the line element, for a line circuit and therefore, from the law of composition of the velocities:

$$\mathbf{v} = \mathbf{u} + \mathbf{V}_D \implies (\mathbf{v} - \mathbf{u}) = \mathbf{V}_D \parallel \mathbb{d}\mathbf{L} \implies ((\mathbf{v} - \mathbf{u}) \wedge \mathbf{B}) \cdot \mathbb{d}\mathbf{L} = 0 \quad . \tag{36.10.03}$$

→ 1726 1727

The above result means that for a line circuit there is no difference in considering either $\mathbf{v}$ or $\mathbf{u}$ in the expression of the EM-motive force of ElectroMagnetic origin.

Let us start from the 3rd **Maxwell Equation** in differential form (36.08.08) and integrate across any surface, Σ, possibly a moving one. Let the boundary of the surface, $\partial\Sigma$, coincide with a real line current carrying-circuit, acting as a kinematic constraint to the motion of charges. The circuit shall be a fixed constitution circuit, to avoid a discontinuous evolution of the circuit.

For the general case of moving integration domains, Σ and $\partial\Sigma$, one finds, with $\mathbf{u}$ the velocity of any point of $\partial\Sigma$, with a purely mathematical derivation:

$$\oint_{\partial\Sigma} \mathbf{E} \cdot \mathbb{d}\mathbf{L} = \iint_{\Sigma} \mathbf{rot}\mathbf{E} \cdot \mathbb{d}\mathbf{S} = -\iint_{\Sigma} \partial_t \mathbf{B} \cdot \mathbb{d}\mathbf{S} = -\frac{\mathbb{d}}{\mathbb{d}t} \iint_{\Sigma} \mathbf{B} \cdot \mathbb{d}\mathbf{S} - \oint_{\partial\Sigma} (\mathbf{u} \wedge \mathbf{B}) \cdot \mathbb{d}\mathbf{L} \quad ,$$

$$\implies \boxed{\oint_{\partial\Sigma} (\mathbf{E} + \mathbf{u} \wedge \mathbf{B}) \cdot \mathbb{d}\mathbf{L} = -\frac{\mathbb{d}}{\mathbb{d}t} \iint_{\Sigma} \mathbf{B} \cdot \mathbb{d}\mathbf{S}} \quad . \tag{36.10.04}$$

→ 1726

The above relation (36.10.04) is just the **result of mathematical derivations**: **care is required in implementing them into physical laws**.

Following the definition of EM-motive force in section § 36.10.02 and relations (36.10.03) one can write, introducing the shortcut $\Gamma \equiv \partial\Sigma$:

$$\boxed{\mathcal{E}_\Gamma \equiv \mathcal{E}_\Gamma[\text{changing magnetic field}] + \mathcal{E}_\Gamma[\text{motional}] = -\frac{\mathbb{d}\Phi_\Sigma}{\mathbb{d}t}} \quad , \tag{36.10.05}$$

→ 1726

where Φ_Γ is the flux of magnetic field concatenated with the line Γ. Equation (36.10.05) is the **flux-rule** form of the Faraday-Neumann-Lenz law. Note that all quantities are measured in the same Inertial Reference Frame and not in the reference frame at rest (locally) with the moving integration domain.

Alternatively, starting from equation (36.10.04), one can write the total EM-motive force as:

$$\boxed{\mathcal{E}_{\partial\Sigma} = -\iint_{\Sigma[t]} \partial_t \mathbf{B} \cdot \mathbb{d}\mathbf{S} + \oint_{\partial\Sigma[t]} (\mathbf{u} \wedge \mathbf{B}) \cdot \mathbb{d}\mathbf{L}} \quad , \tag{36.10.06}$$

→ 1726 1727 1769

where $\Sigma[t]$ is a generic surface, possibly function of time, and $\partial\Sigma[t]$ is its boundary, which must coincide with the real current-carrying circuit, defined by the lines of current, acting as a one-dimensional kinematic constraint.

In case the circuit is a fixed-constitution one-dimensional circuit (a current line) changing in a sufficiently continuous way, the expression (36.10.06) is totally equivalent to the flux rule:

$$\boxed{\mathcal{E}_{\partial\Sigma} = -\frac{\mathbb{d}\Phi_\Sigma}{\mathbb{d}t}} \quad . \tag{36.10.07}$$

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The flux-rule law applies if and only if the mathematical integration line, $\Gamma \equiv \partial\Sigma$, whose velocity is $\mathbf{u}$, **is at any instant coincident with the real line current carrying wire and the evolution of the circuit can be modeled as a sufficiently continuous evolution**. In fact in this case the velocity $\mathbf{w}$ in the line integral

$$\oint_{\partial\Sigma[t]} (\mathbf{w}\wedge\mathbf{B})\cdot \mathbb{d}\mathbf{L}$$

can be set, indifferently, to either $\mathbf{w} = \mathbf{v}$, as required by the definition of the EM-motive force, or $\mathbf{w} = \mathbf{u}$, as required by the mathematical theorem.

The mathematical model [1] **used for the demonstration may fail when the conductor is not a line conductor, because it might be not possible to replace the velocity of the charge with the velocity of the circuit based on** (36.10.03).

The mathematical model used for the demonstration may also fail if the real circuit changes in a way that cannot be modeled as a smooth evolution, for instance when the circuit is not a fixed constitution circuit, because the mathematical theorem may not apply.

In fact, the troubles with the well-known flux rule are that, in many examples, either the current spreads out over a whole two-dimensional surface or three-dimensional volume and in in many such cases it is not even clear what the flux through the circuit would mean, or the line circuit varies in a non smooth-enough way.

In the most general case the use of the form (36.10.06) ensures the correct treatment.

Actually the presence of the two terms is related to the invariance of the ElectroMagnetic theory by **Lorentz Transformation** , because under **Lorentz Transformation** Electric|Magnetic fields can mix. Read § 61.

36.10.04

4th Maxwell Equation in Differential/Flux-Rule Form

The same considerations apply to (36.08.02), but in this case they are not very useful on the practical side.

UNDER-LINE-IN-PROGRESS

[1] This is in fact a nice non trivial example of the difference between physics and mathematics...

36.11 Modeling of Conductors and Insulators

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DRAFT PRELIMINARY

36.11.01 Conductivity and Resistivity

The **conductivity** , σ, is defined as the linear coefficient relating the current density with the applied force per unit charge, $\mathbf{f}/q$.

In the simple case of an isotropic material the **conductivity** is a scalar (possibly depending on the position) as the force and the current density must be parallel.

$$\boxed{\mathbf{j} \equiv \sigma \mathbf{f}/q} \quad . \tag{36.11.01}$$

In the simple case of an homogeneous material the **conductivity** is independent on the position (possibly depending on the direction of the fields)

$$\boxed{j_k \equiv \sigma_{ki}^{\cdot\cdot} f_i/q} \quad . \tag{36.11.02}$$

In the general case of a non-homogeneous and non-isotropic material the **conductivity** is a second-order tensor, relating the force and current density vectors:

$$\boxed{j_k \equiv \sigma_{ki}^{\cdot\cdot}[\mathbf{x}] f_i/q} \quad . \tag{36.11.03}$$

→ 1728

In reality the perfect insulator/conductor commonly used model is just an abstraction to simplify the modeling of very good insulators/conductors. The intermediate cases, as usual, are more difficult to deal with.

For steady currents and uniform **conductivity** in a Ohmic conductor the charge volume density inside the conductor is zero and any unbalanced charge must reside on the surface. This property reminds the case of stationary charges, which can only exist at the surface of a conductor.

36.11.02 Conductors

Note that the relation (36.11.03) is not Ohm law which is the following statement: **a Ohmic conductor is defined as a conductor whose conductivity , σ, is independent of the applied force per unit charge**, $\mathbf{f}/q$:

$$\boxed{\mathbf{j} = \sigma\,(\mathbf{f}/q) \qquad \text{with a fixed } \textbf{conductivity } \sigma\text{, independent of } \mathbf{f}} \quad . \tag{36.11.04}$$

A **perfect conductor** is defined as a Ohmic conductor having zero resistivity, that is: $\sigma \to \infty$. The model of a perfect conductor is used, for instance, in magneto-hydrodynamics, the study of perfectly conductive fluids, and in circuit theory when the connections between different components is modeled by a wire with no resistivity (but in real circuits the potential drop along wires can be significant, requiring wires with a large enough cross-section, also to reduce the power dissipation).

In case only ElectroMagnetic forces are acting on the charges the relation for Ohmic conductors becomes:

$$\boxed{\mathbf{j} = \sigma\,(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}) \qquad \text{with a fixed } \textbf{conductivity } \sigma} \quad . \tag{36.11.05}$$

→ 1728

Note that in many practical cases in (36.11.05) the magnetic term is negligible with respect to the electric term because velocity and|or magnetic field are small. In fact, for general ElectroMagnetic fields (that is ElectroMagnetic waves) typically: $E \approx cB$.

It follows that, in order to have a finite current, **the electric field inside an perfect conductor must be zero**. The 3rd **Maxwell Equation** then implies that **the magnetic field is independent on time**.

Therefore if the magnetic field is zero inside a conductor at a certain time it will remain zero forever. In other words the magnetic field does not penetrate inside a perfect conductor.

Moreover, it can be shown that Ohmic conductors might carry a surface currents if and only if they are perfect conductors, read § 36.20.22.

Summarizing, a perfect Ohmic conductor has the following properties:

perfect Ohmic conductor ($\sigma \to \infty$): $\{ \quad \mathbf{E} = \mathbf{0} \quad \mathbf{B} = \text{constant} \quad \mathbf{j}_s \,/\, \rho_s \text{ might be non zero} \quad \}$.

(36.11.06) → 1729

Note that while the electric field is zero in a generic conductor at equilibrium it is always zero inside a perfect conductor.

In case additional forces of non ElectroMagnetic origin exist, described by a force field per unit charge, $\mathbf{f}/q$, the constitutive relation for a Ohmic conductor becomes:

$$\mathbf{j} = \sigma\,(\mathbf{E} + \mathbf{v}\wedge\mathbf{B} + \mathbf{f}/q) \quad . \tag{36.11.07}$$

→ 1729

Forces of non ElectroMagnetic origin might be, for instance, inertial forces, like in the so-called Nichols disk.

It must be emphasized that the equation (36.11.07) is not a fundamental law of nature but is a constitutive relation, which defines a class of material media having the property that the current density is a linear function of the Electric|Magnetic fields as well as of any other EM-motive field, $\mathbf{f}/q$.

36.11.02.01 Perfect Conductors and Super-Conductors

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 7.42, 7.43||

A **perfect conductor** is characterized by the constitutive relation (36.11.06).

In a perfect conductor, the **conductivity** is infinite and any net charge resides at the surface, just as it does for any non-perfect conductor in static conditions.

A loop made of a perfect-conductor wire has constant concatenated magnetic field flux. In fact the circulation of the electric field is zero, because the electric field is zero, so that the concatenated magnetic field flux does not change.

A superconductor has the additional property that the constant magnetic field inside is zero, that is they are also **perfect diamagnets**. This is the **flux exclusion** effect, which is known as the Meissner effect.

The current in a superconductor is confined to the surface. In fact, from the 4th **Maxwell Equation** , zero electric field and zero magnetic field imply zero current density.

36.11.03 Insulators

36.11.03.01 Perfect Insulators

A perfect insulator is defined as a Ohmic conductor having zero **conductivity** , that is: $\sigma = 0$.

They will be called dielectrics.

Pure vacuum is a perfect dielectric.

DRAFT - IN PROGRESS

36.12 ElectroMagnetic Potentials

This § is referenced at pages:
[2489]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.15.4, 2.15.5||

In this section the short-hand notation:

$$\frac{1}{c^2} \equiv \varepsilon_0 \mu_0 \quad ,$$

will be often used, where c is the speed of light in vacuum. The relation will be derived later, when discussing ElectroMagnetic waves § 46.

Maxwell Equations consist of a set of four coupled first-order partial differential equations relating the Electric|Magnetic fields to their Sources|Sinks, charges and currents. **Maxwell Equations** can be solved directly only in simple situations. In general it is useful to introduce the ElectroMagnetic potentials, in order to obtain two uncoupled second-order partial differential equations. The 2nd and 3rd **Maxwell Equations** are identically satisfied thanks to the definition of the potentials themselves.

From the practical point of view, in ElectroStatics, **the use of the scalar potential allows a considerable simplification of the calculations** with respect to the use of the electric field (a scalar field against a vector field). On the other hand, in MagnetoStatics, **the use of the vector magnetic potential does not allow, usually, simplifications of the calculations** with respect to the use of the magnetic field (a vector field against a vector field), but it allows considerable advances in the development of the ElectroMagnetism. Moreover, since magnetic forces do no work, **the vector potential does not admit a simple physical interpretation as the scalar electric potential in terms of potential energy per unit charge**.

From the structure of the ElectroMagnetic fields, that is Maxwell equations (36.08.07), (36.08.08), the Electric|Magnetic fields can be expressed as a function of the potentials of the ElectroMagnetic fields, provided the conditions described in section § 16.09.03 on the domain of definition of the ElectroMagnetic fields are met:

$$\mathbf{div}\,\mathbf{B} = 0 \implies \mathbf{B} = +\mathbf{rot}\,\mathbf{A} \tag{36.12.01}$$

$$\mathbf{rot}\,\mathbf{E} + \partial_t \mathbf{B} = 0 \implies \mathbf{E} = -\mathbf{grad}\,\Phi - \partial_t \mathbf{A} \quad .$$

→ 1731 1735 1743
→ 1731 1735 1743

In order to use the **ElectroMagnetic potentials** a sufficient condition on the domain is that it is both 1-connected and 2-connected, which will be implicitly assumed in the rest of the section.

36.12.01 Gauge Invariance and Choice of a Gauge

This § is referenced at pages:
[1734]

©|J.D.Jackson|From Lorenz to Coulomb and other explicit gauge transformations|*Am.J.Phys*, , ..., ..., ...Ed., ... *70, 917* ...

©|J.A.Heras|How the potentials in different gauges yield the same retarded electric and magnetic fields|*Am.J.Phys*, , ..., ..., ...

The **ElectroMagnetic potentials** are not uniquely defined. In fact there exist infinitely many possible choices of the potentials giving rise to the same Electric|Magnetic fields.

In classical ElectroMagnetism only the ElectroMagnetic fields have a well-defined physical meaning, as they are ultimately defined from the Lorentz force law, while the **ElectroMagnetic potentials** do not, as

they are not uniquely defined. The fact that the **ElectroMagnetic potentials** are not uniquely defined does not pose any problem to the physics. On the other hand the freedom to choose a set of **ElectroMagnetic potentials** can lead to a considerable simplification of some equations.

The Electric|Magnetic fields do not change under the gauge transformations:

$$\Phi \longrightarrow \Phi' = \Phi - \partial_t \Lambda \tag{36.12.02}$$

→ 1731 1735 2489

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \mathbf{grad}\,\Lambda \quad , \tag{36.12.03}$$

→ 1731 1735 2489

that is: $\mathbf{E}$ and $\mathbf{B}$ calculated from equations (36.12.01), (36.12) are the same when using either the pair of **ElectroMagnetic potentials** Φ, $\mathbf{A}$ or the pair of **ElectroMagnetic potentials** Φ', $\mathbf{A}'$.

The freedom in the choice of the potentials which is provided by gauge invariance (equations (36.12.02), (36.12.03)) is used to simplify the developments.

Actually, under some circumstances quantum physics shows that the **ElectroMagnetic potentials** themselves have a physical significance (**Aharonov-Bohm effect** [1]) affecting the dynamics, in regions of space where Electric|Magnetic fields are absent [2].

The **ElectroMagnetic potentials** are discussed in the relativistic context in section § 61.04.

36.12.01.01 Lorenz Gauge

This § is referenced at pages:
[Never referenced.]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 10.1.3||

©|J.D.Jackson|From Lorenz to Coulomb and other explicit gauge transformations|*Am.J.Phys*, , ..., ..., ...*Ed.*, ... *70, 917* ...

©|J.A.Heras|How the potentials in different gauges yield the same retarded electric and magnetic fields|*Am.J.Phys*, , ..., ..., ...

A most used gauge choice in relativistic physics is the Lorenz gauge [3], which **has the virtue of providing a description with space and time treated on the same footing** (a relativistically covariant description):

$$\frac{1}{c^2}\partial_t \Phi + \mathbf{div}\,\mathbf{A} = 0 \quad . \tag{36.12.04}$$

→ 1731 2220

Given any set of **ElectroMagnetic potentials** one can determine a set of **ElectroMagnetic potentials** satisfying the **Lorenz Gauge** by replacing the gauge transformations (36.12.02), (36.12.03) into (36.12.04) and imposing the Lorenz condition on the new **ElectroMagnetic potentials** to find an equation for Λ:

$$\frac{1}{c^2}\partial_t \Phi' + \mathbf{div}\,\mathbf{A}' = 0 \implies \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = -\left(\frac{1}{c^2}\partial_t \Phi + \mathbf{div}\,\mathbf{A}\right) \quad .$$

Note that the condition does not specify uniquely the potentials. In fact any other set of **ElectroMagnetic potentials** derived from a set of **ElectroMagnetic potentials** satisfying the **Lorenz Gauge** , via a function Λ such that

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} \quad , \tag{36.12.05}$$

still satisfies the **Lorenz Gauge** condition (36.12.04). All these potentials are are said to belong to the **Lorenz Gauge** .

The **Lorenz Gauge** is particularly suitable for providing a relativistically covariant condition. Actually it is the only one possibility linear in the EM potentials.

The **Lorenz Gauge has the virtue of allowing the separation of the scalar electric potential and vector magnetic potential in the wave equations** derived by **Maxwell Equations** (see section § 49.01).

[1] R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ... 2.15.4 and 2.15.5
[2] The discussion about **Aharonov-Bohm effect** is still a lively argument among physicists, with an infinite scientific literature.
[3] It was proposed by the Danish physicist L.V.Lorenz, although it is often erroneously attributed to the Dutch physicist H.A.Lorentz.

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36.12.01.02 Coulomb Gauge

This § is referenced at pages:
[Never referenced.]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 10.1.3||
©|J.D.Jackson|From Lorenz to Coulomb and other explicit gauge transformations|*Am.J.Phys*, , ..., ..., ...*Ed.*, ... *70, 917* ...
©|J.A.Heras|How the potentials in different gauges yield the same retarded electric and magnetic fields|*Am.J.Phys*, , ..., ..., ...

In the static case the choice which is most often used is the so-called Coulomb gauge, which gives rise to **Poisson equation** for the potentials:

$$\boxed{\mathbf{div}\,\mathbf{A} = 0} \quad . \tag{36.12.06}$$

Note that the **Lorenz Gauge** is equivalent to the Coulomb gauge in stationary conditions.

From any given gauge one can deduce:

$$\mathbf{div}\,\mathbf{A}' = \mathbf{div}\,\mathbf{A} + \nabla^2\Lambda \quad ,$$

so that the Λ that solves the equation

$$\nabla^2\Lambda = -\mathbf{div}\,\mathbf{A} \quad ,$$

gives **ElectroMagnetic potentials** in the Coulomb gauge from any potential $\mathbf{A}$.

36.12.01.03 Temporal Gauge

This § is referenced at pages:
[Never referenced.]

©|J.D.Jackson|From Lorenz to Coulomb and other explicit gauge transformations|*Am.J.Phys*, , ..., ..., ...*Ed.*, ... *70, 917* ...
©|J.A.Heras|How the potentials in different gauges yield the same retarded electric and magnetic fields|*Am.J.Phys*, , ..., ..., ...

The temporal gauge:

$$\boxed{\Phi = 0} \quad , \tag{36.12.07}$$

is often used when describing the radiation of ElectroMagnetic waves. It has the virtue of describing both the $\mathbf{E}$ and $\mathbf{B}$ field by means of the **vector magnetic potential** $\mathbf{A}$ only. However, it might be rather impractical in static cases, as the **vector magnetic potential** for a static field would be time-dependent.

To keep the temporal gauge, for any gauge transformation:

$$\partial_t\Lambda = 0 \quad .$$

36.12.01.04 Kirchhoff Gauge

This § is referenced at pages:
[Never referenced.]

©|J.D.Jackson|From Lorenz to Coulomb and other explicit gauge transformations|*Am.J.Phys*, , ..., ..., ...*Ed.*, ... *70, 917* ...
©|J.A.Heras|How the potentials in different gauges yield the same retarded electric and magnetic fields|*Am.J.Phys*, , ..., ..., ...

DRAFT - UNDER REVIEW

36.12.01.05 Velocity Gauge

This § is referenced at pages:
[Never referenced.]

©|J.D.Jackson|From Lorenz to Coulomb and other explicit gauge transformations|*Am.J.Phys*, , ..., ..., ...*Ed.*, ... *70, 917* ...

©|J.A.Heras|How the potentials in different gauges yield the same retarded electric and magnetic fields|*Am.J.Phys*, , ..., ..., ...

36.12.02
Scalar Potential of the ElectroMagnetic Fields

©|R.H. Romer - Am.J.Phys, , ..., ..., ...Ed., ... 50, 1089 (1982)|||

©|F.Reif - Am.J.Phys, , ..., ..., ...Ed., ... 50, 1048 (1982)|||

The **scalar electric potential** , Φ, allows to express the line-integral (along any open curve) of the irrotational field $\mathbf{E} + \partial_t \mathbf{A}$, that is the circulation, in terms of the potential at the boundary of the curve:

$$\Phi[B] - \Phi[A] = -\int_A^B \mathbf{E} \cdot \mathbb{d}\mathbf{L} - \int_A^B \partial_t \mathbf{A} \cdot \mathbb{d}\mathbf{L} \quad , \tag{36.12.08}$$

thanks to the **gradient theorem** .

The difference of potential is the same for all curves having the same extremes.

It must be emphasized that, in non static cases, the potential difference is not the line-integral of the electric field.

Therefore, in the general case, one must be very careful not to confuse between:

- difference of potential: $\Phi[B] - \Phi[A]$, for the conservative field Φ;
- the line integral for the electric field: $\int_A^B \mathbf{E} \cdot \mathbb{d}\mathbf{L}$.

Obviously: $\Phi[P] - \Phi[P] = 0$.

Note that, as for the static case, even if the **scalar electric potential** is not uniquely determined, the difference of potential calculated by means of the difference of its values is always well-defined.

36.12.03
Vector Potential of the ElectroMagnetic Fields

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.15||

The **vector magnetic potential** , $\mathbf{A}$, allows to express the surface-integral (across any open surface) of the solenoidal field $\mathbf{B}$, that is the flux, in terms of the line-integral the boundary of the surface:

$$\Phi_\Sigma \equiv \iint_\Sigma \mathbf{B} \cdot \mathbb{d}\mathbf{S} = \oint_{\partial\Sigma} \mathbf{A} \cdot \mathbb{d}\mathbf{L} \quad .$$

thanks to the **rotor theorem** .

The flux of the magnetic field is the same for all surfaces having the same boundary.

Note that, as for the static case, even if the **vector magnetic potential** is not uniquely determined, its circulation around any given closed curve is the same for all possible equivalent **vector magnetic potentials** .

DRAFT - LAY-PROGRESS

36.12.04
Other (Less Useful) Potentials

Vector Electric Potential

In certain cases it is possible to introduce a vector electric potential. A necessary and sufficient condition is that the scalar charge density $\rho[\mathbf{r}]$ is zero inside the 2-connected space region considered. An alternative necessary and sufficient condition is that the flux of the electric field is zero for all possible closed surfaces in the region.

This is the analogous of the magnetic vector potential. However, it has little interest in practice, because the use of the scalar potential is normally easier. It might be somewhat useful to solve ElectroStatics problems by analogy with known solutions of MagnetoStatics problems, read § 41.03.07.

Scalar Magnetic Potential

©|J.Franklin, *Classical ElectroMagnetism*, 2017, 2ndEd, ...Ed., WEB - URL.|§ 7.10|*Different expressions and discussions.*|

In certain cases it is possible to introduce a scalar magnetic potential. A necessary and sufficient condition is that the vector current density $\mathbf{j}[\mathbf{r}]$ is zero inside the 1-connected space region considered. An alternative necessary and sufficient condition is that the circulation of the magnetic field is zero for all possible closed loops in the region.

This is the analogous of the **scalar electric potential** It has a wide interest for calculations in presence of matter and permanent magnets only, read § 39. It might be also useful to solve MagnetoStatics problems by analogy with known solutions of ElectroStatics problems, read § 41.03.07.

36.12.05
Introducing the ElectroMagnetic Potentials Into Maxwell Equations

The second and 3rd **Maxwell Equations** have been used to introduce the **ElectroMagnetic potentials** . One still need to exploit the first and 4th **Maxwell Equations** , those containing the generators of the ElectroMagnetic fields. The introduction of the **ElectroMagnetic potentials** into the first and fourth of **Maxwell Equations** gives:

$$\boxed{\nabla^2\Phi + \partial_t \mathbf{div}\,\mathbf{A} = -\rho/\varepsilon_0} \tag{36.12.09}$$

→ 1734 1735 2220

$$\boxed{\boldsymbol{\nabla}^2\mathbf{A} - \varepsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} - \mathbf{grad}(\mathbf{div}\,\mathbf{A} + \varepsilon_0\mu_0\partial_t\Phi) = -\mu_0\mathbf{j}} \; . \tag{36.12.10}$$

→ 1734 1735

The two equations (36.12.09), (36.12.10) are not decoupled. However, the gauge invariance freedom (read § 36.12.01) can be used to simplify the equations and decouple them, giving equations (49.01.01) and (49.01.02).

Note that **Maxwell Equations** contain two kind of generators of ElectroMagnetic fields: charges and currents. However, they are related by the charge continuity equation and, therefore, they are not truly independent. This explains why it is often possible to solve problems by only considering either charge only or current only as the generators of ElectroMagnetic fields. Actually, also points where $\partial_t\mathbf{E} \neq 0$ or $\partial_t\mathbf{B} \neq 0$ are vortexes, so they generate a rotor.

DRAFT - IN PROGRESS

36.12.06
ElectroMagnetic Potentials for Static ElectroMagnetic Fields

This § is referenced at pages:
[Never referenced.]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||
©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|||

The use of the **ElectroMagnetic potentials** definitions (equations (36.12.01), (36.12), as well as equations (36.12.02), (36.12.03)) in **Maxwell Equations** gives, in the static case and either the Coulomb or Lorenz gauges, identical in this case, the **Poisson equations** (read § 49.01 and the two equations (36.12.09), (36.12.10)):

$$\boxed{\nabla^2\Phi = -\rho/\varepsilon_0} \quad , \tag{36.12.11}$$

→ 1736 1743

$$\boxed{\boldsymbol{\nabla}^2\mathbf{A} = -\mu_0\mathbf{j}} \quad . \tag{36.12.12}$$

→ 1736 1743

In absence of charges the **Poisson equations** reduce to **Laplace equations** :

$$\boxed{\nabla^2\Phi = 0} \quad , \tag{36.12.13}$$

$$\boxed{\boldsymbol{\nabla}^2\mathbf{A} = 0} \quad . \tag{36.12.14}$$

Scalar/vector field satisfying **Laplace equation** are said to be harmonic functions.

Note that **Poisson equations** were derived from the first and fourth Maxwell equations, those involving the Charge|Current producing the ElectroMagnetic fields. In fact the second and third equations do not contain Charge|Current, and the information from those two equations were used to introduce the concept of **ElectroMagnetic potentials** .

Note also that the **Poisson equations** have accomplished the decoupling of the Electric|Magnetic parts of **Maxwell Equations** which can be treated in an independent way.

The equation for the **vector magnetic potential** has the additional difficulty that, due to the **Coulomb/Lorenz Gauge** condition, the three components of the **vector magnetic potential** are not independent.

36.12.07
Discontinuities of ElectroMagnetic Potentials

Read § 36.16.

Read § 36.16.03.

36.12.08
Lorentz Force Law in Terms of EM Potentials

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 10.1.4|*Illuminating!*|

$$\boxed{\frac{\mathrm{d}(\mathcal{K} + q\Phi)}{\mathrm{d}t} = \frac{\partial(q\Phi - q\,\mathbf{v}\cdot\mathbf{A})}{\partial t}}$$

$$\boxed{\frac{\mathrm{d}(\mathbf{p} + q\mathbf{A})}{\mathrm{d}t} = -\mathbf{grad}(q\Phi - q\,\mathbf{v}\cdot\mathbf{A}) \qquad \text{the derivative does not act on } \mathbf{v}} \quad .$$

PREFLIGHT-PROGRESS

36.13 Static Solution for Limited Distribution of the Sources|Sinks

In the static case, Electric|Magnetic fields and charge|current are decoupled in equations (36.08.06), (36.08.07), (36.08.08), (36.08.02).

Assume there are **limited distribution of the Sources|Sinks**:

- different from zero only in a limited region of space (functions with compact support);
- the module of the functions are limited functions (no infinity).

In the **Coulomb/Lorenz Gauge** the solutions to **Poisson equations** , equations (36.12.11), (36.12.12), subject to the boundary conditions of zero at infinity,

$$\lim_{|\mathbf{x}|\longrightarrow\infty} \Phi[\mathbf{x}] = 0 \qquad \lim_{|\mathbf{x}|\longrightarrow\infty} |\mathbf{A}[\mathbf{x}]| = 0 \quad , \tag{36.13.01}$$

→ 1736

in the case when ρ or $\mathbf{j}$ are **limited distribution of the Sources|Sinks**, are:

$$\Phi[\mathbf{x}] = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}V_{\mathbf{y}} \quad , \tag{36.13.02}$$

→ 1736 1743 2224

$$\mathbf{A}[\mathbf{x}] = \frac{\mu_0}{4\pi}\iiint \frac{\mathbf{j}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}V_{\mathbf{y}} \quad . \tag{36.13.03}$$

→ 1736 1743 1776 1780 2224

The **scalar electric potential** and **vector magnetic potential** , even in the **Coulomb/Lorenz Gauge** , are defined to within a constant term as the gradient, divergence and rotor of a constant are null. This arbitrariness is removed by the condition (36.13.01) of zero at infinity.

The **scalar electric potential** and **vector magnetic potential** , from equations (36.13.02) and (36.13.03), are clearly continuous for any well-behaved charge/current density.

The above expression (36.13.03) for the **vector magnetic potential** must have zero divergence (Lorenz/Coulomb gauge in steady conditions), as it can be shown by direct calculation of the divergence and use of formula (16.12.01) for a steady current distribution, which is divergence-less.

It is worth noting that the expression for the **vector magnetic potential** implies that any infinitesimal current element gives a contribution to the **vector magnetic potential** parallel to the current element itself.

In case the Charge|Current densities can be modeled as surface or line distributions, in addition to volume distributions, the above relations, (36.13.02) and (36.13.03), can be written as follows.

From (z: depth, w width and ℓ: length):

$$\rho\,\mathrm{d}V \longrightarrow \rho\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}L = (\rho\,\mathrm{d}z)\;\mathrm{d}w\,\mathrm{d}L = \rho_{\mathrm{S}}\,\mathrm{d}w\,\mathrm{d}L = (\rho_{\mathrm{S}}\,\mathrm{d}w)\;\mathrm{d}L = \rho_{\mathrm{S}}\,\mathrm{d}S = \rho_{\mathrm{L}}\,\mathrm{d}L \quad ,$$

one finds:

$$\Phi[\mathbf{x}] = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}V_{\mathbf{y}} + \frac{1}{4\pi\varepsilon_0}\iint \frac{\rho_{\mathrm{S}}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{S}_y + \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_{\mathrm{L}}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}L_y \quad . \tag{36.13.04}$$

→ 1743

From (z: depth, w width and ℓ: length):

$$\mathbf{j}\,\mathrm{d}V \longrightarrow \mathbf{j}\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}L = (\mathbf{j}\,\mathrm{d}z)\;\mathrm{d}w\,\mathrm{d}L = \mathbf{j}_{\mathrm{S}}\,\mathrm{d}w\,\mathrm{d}L = (\mathbf{j}_{\mathrm{S}}\,\mathrm{d}w)\;\mathrm{d}L = \mathbf{j}_{\mathrm{S}}\,\mathrm{d}S = I\,\mathbf{dL} \equiv \mathbf{j}_{\mathrm{L}}\,\mathrm{d}L$$

$$\text{with} \qquad I\,\mathbf{dL} \equiv \mathbf{j}_{\mathrm{L}}\,\mathrm{d}L \quad , \tag{36.13.05}$$

one finds:

$$\mathbf{A}[\mathbf{x}] = \frac{\mu_0}{4\pi}\iiint \frac{\mathbf{j}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}V_{\mathbf{y}} + \frac{\mu_0}{4\pi}\iint \frac{\mathbf{j}_{\mathrm{S}}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}S_y + \frac{\mu_0}{4\pi}\int \frac{\mathbf{j}_{\mathrm{L}}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}L_y \quad . \tag{36.13.06}$$

→ 1743 1805

Note that in the above equations, the surfaces and curves may be open; clearly, charge conservation must be obeyed.

Moreover, the Charge|Current densities may be not given in advance, but can depend on the Electric|Magnetic field present in the neighborhood, which in turn, depend on the Charge|Current densities themselves.

DRAFT - IN PROGRESS

36.14 Multipole Expansion of Static ElectroMagnetic Potentials

This § is referenced at pages:
[1127, 1127, 1569, 1569, 1693, 1737, 1737, 1828]

Multipole Expansion of Static ***ElectroMagnetic potentials*** *at Large Distance for Spatially-Limited Sources | Sinks to find approximate expressions of potentials and fields*

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|3.4||

©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|||

©|M.Castellanos *et al.*|Magnetostatic multipoles in Cartesian coordinates|*Am.J.Phys, , ..., ..., ...Ed., ... 46, 1116 (1978); ...*

©|F.Wegner, *Classical Electrodynamics*, 2003, ..., ...Ed., WEB - URL.||*Detailed magnetic calculations*|

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|4.1-4.2, 16.3||

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|1.7||

See the analogous discussion for the gravitational **Multipole** development in section § 29.05. Read also section, § 36.14, § 36.14, § 36.14.06, § 36.14.07, § 29.05.04.

The following general results can be derived for static ElectroMagnetic fields when the stationary charges and steady currents are different from zero in a spatially-limited region of space only.

Assume that all the stationary charges and steady currents are inside a sphere of radius R, centered on the origin of a suitable Coordinate System, and observe the **ElectroMagnetic potentials** /fields from a point $\mathbf{r}$ at a very large distance with respect to R: $|\mathbf{r}| \gg R$.

Note that the non-static and/or higher than second order development includes in general other multipole types, namely the so-called polar/axial toroidal moments [1]

Read § 29.05.01 for the analogous development of the gravitational potential, which is identical to the one of the ElectroStatic potential and very similar, apart from the vector current density instead of a scalar function.

See [2], for more details.

1 ©|The High-Order Toroidal Moments and Anapole States...|Laser Photonics Rev. 2019, 13, 1800266; WEB - URL ...

2 ©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|§ 5.6||

DRAFT IN PROGRESS

36.14.01
Cartesian Multipoles

The following notation will be used: $r \equiv |\mathbf{r}|$ and $x \equiv |\mathbf{x}|$.

Let denote generically with $\underline{s}[\mathbf{x}]$, either the mass volume density (scalar), the charge volume density (scalar) or the current volume density (vector); let $\underline{\Phi}[\mathbf{r}]$ be the correspondent scalar or component of the vector potential.

$$\underline{s}[\mathbf{x}] = 0 \qquad |\mathbf{x}| \geq r_0 \quad ,$$
$$\lim_{r \longrightarrow \infty} \Phi[\mathbf{r}] = 0 \quad , \tag{36.14.01}$$

$$(1+w)^{-1/2} = 1 - \frac{1}{2}w + \frac{3}{8}w^2 + \mathcal{O}\left(w\right)^3 \qquad \text{convergent for } w < 1 \quad , \tag{36.14.02}$$

$$\Phi[\mathbf{r}] = K \iiint \frac{\underline{s}[\mathbf{x}]}{|\mathbf{r}-\mathbf{x}|}\, \mathrm{d}V_{\mathbf{x}} = K \iiint \frac{\underline{s}[\mathbf{x}]}{\sqrt{r^2 + x^2 - 2\,\mathbf{r}\cdot\mathbf{x}}}\, \mathrm{d}V_{\mathbf{x}} = \quad ,$$

$$K \iiint \frac{\underline{s}[\mathbf{x}]}{r} \left(1 - 2\frac{\mathbf{r}\cdot\mathbf{x}}{r^2} + \frac{x^2}{r^2}\right)^{-1/2} \mathrm{d}\mathbf{x} = \quad ,$$

$$K \iiint \frac{\underline{s}[\mathbf{x}]}{r} \left(1 + \frac{\mathbf{r}\cdot\mathbf{x}}{r^2} + \frac{3}{2}\left(\frac{\mathbf{r}\cdot\mathbf{x}}{r^2}\right)^2 - \frac{x^2}{2r^2} + \mathcal{O}\left(\frac{x}{r}\right)^3\right) \mathrm{d}V_{\mathbf{x}} \quad ,$$

$$\Phi[\mathbf{r}] = \Phi_0[\mathbf{r}] + \Phi_1[\mathbf{r}] + \Phi_2[\mathbf{r}] + \mathcal{O}\left(\frac{r_0}{r}\right)^3 \quad . \tag{36.14.03}$$

The problem of the convergence of the series expansion is not discussed here.

36.14.02
Monopole Terms

Electric monopole term:

$$\boxed{\Phi[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} + \mathcal{O}\left(\frac{1}{r}\right)^2 \qquad \lim_{|\mathbf{r}|\to\infty} |\Phi|[\mathbf{r}] = 0} \quad . \tag{36.14.04}$$

Magnetic Monopole term:

$$\boxed{\mathbf{A}[\mathbf{r}] = \mathbf{0} + \mathcal{O}\left(\frac{1}{r}\right)^2 \qquad \lim_{|\mathbf{r}|\to\infty} |\mathbf{A}|[\mathbf{r}] = 0} \quad . \tag{36.14.05}$$

In the magnetic case the monopole terms is always zero, as there is no magnetic monopole. Mathematically, the volume integral of a space-limited stationary current distribution is zero, read § 36.20.48.

DRAFT-IN-PROGRESS

36.14.03
Electric|Magnetic Dipole terms

Read § 36.14.04, § 36.14.05 for discussion on quadruple moments.

36.14.03.01 Electric Dipole term

The dipole term is straightforward from equation 36.14.01 for the electric case.

Define the **Electric Dipole** moment:

$$\boxed{\mathbf{p} \equiv \iiint \rho \mathbf{x} \, \mathrm{d}V \longrightarrow \quad \text{for point charges} \quad = \sum_k q_k \mathbf{r}_k} \quad . \tag{36.14.06}$$

→ 1087 1781 1842

Read § 36.20.52: the **Electric Dipole** moment is independent of the origin of the Coordinate System as long as the total charge is zero.

The development, to first-order in $|\mathbf{r}|/R$, of the **scalar electric potential** at large distance, $|\mathbf{r}| \gg R$, for the spatially-limited stationary charge density gives:

$$\boxed{\Phi[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \mathcal{O}\left(\frac{1}{r}\right)^3 \qquad \lim_{|\mathbf{r}|\to\infty} |\Phi|[\mathbf{r}] = 0} \quad . \tag{36.14.07}$$

This development has the monopole and **Dipole** terms, the higher-order terms have been omitted.

Read § 36-018 for an example of electric field at large distance.

An **order-of-magnitude estimate** for the **Electric Dipole** moment is: $|\mathbf{p}| \approx QD$.

36.14.03.02 Scalar Potential of Two Equal Opposite Point Charges at Large Distances

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|||

Consider two equal and opposite charges, $\pm|q|$, located at $Q_\pm = \{0, 0, \pm d/2\}$ in a orthonormal Cartesian Coordinates Coordinate System. Consider a point $P = \{x, y, z\}$ located at distance r for the origin with a polar angle θ with respect to the z axis ($\cos[\theta] = z/r$). Let $r_\pm$ the distances between P and the two charges. At large distances with respect to the distance between the two charges one can find:

$$r_\pm \gg d \implies \Phi = \frac{1}{4\pi\varepsilon_0}|q|\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \simeq \frac{1}{4\pi\varepsilon_0}|q|\frac{d\cos[\theta]}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \qquad \text{with } \mathbf{p} = q\,(\mathbf{r}_+ - \mathbf{r}_-) = q\mathbf{d} \quad .$$

36.14.03.03 Magnetic Dipole term

The magnetic dipole term from equation 36.14.01 requires some rewriting.

$$\begin{aligned}
&\underline{s}[\mathbf{x}] \longrightarrow \mathbf{j}[\mathbf{x}] \quad , \\
&\frac{K}{r^3}\iiint \mathbf{j}[\mathbf{x}]\,(\mathbf{r}\cdot\mathbf{x})\;\mathrm{d}V_\mathbf{x} = \quad , \\
\longrightarrow (1) \qquad &\text{line circuit} \qquad = \frac{IK}{r^3}\oint (\mathbf{r}\cdot\mathbf{x})\;\mathrm{d}\mathbf{L} = \frac{K}{r^3}\mathbf{r}\wedge I\iint \mathrm{d}\mathbf{S} \quad , \\
\longrightarrow (2) \qquad &= \text{volume} \qquad \frac{K}{r^3}\mathbf{r}\wedge\iiint\left(\frac{\mathbf{x}\wedge\mathbf{j}}{2}\right)\mathrm{d}V_\mathbf{x} \quad ,
\end{aligned} \tag{36.14.08}$$

where the passage (1) comes from equation (16.07.08) and (16.13.06) the passage (2) comes from the properties of a stationary current [1].

Read § 36.14.03.04 a different derivation for a plane loop.

Define the **Magnetic Dipole** moment:

$$\boxed{\mathbf{m} \equiv \frac{1}{2}\iiint (\mathbf{x}\wedge\mathbf{j})\;\mathrm{d}V \longrightarrow \quad \text{for wire current loops} \quad = \sum_k I_k \mathbf{S}_k} \quad . \tag{36.14.09}$$

→ 1087 1780 1781 1884

[1] See J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...

UNDER IN PROGRESS

Read § 36.20.52: the **Magnetic Dipole** moment is independent of the origin of the Coordinate System as long as the current distribution has zero integral which is always the case for limited and steady current distribution, read § 16.12.07 and equation (16.12.03).

The development, to first-order in $|\mathbf{r}|/R$, of the **vector magnetic potential** at large distance, $|\mathbf{r}| \gg R$, for spatially-limited steady current density:

$$\mathbf{A}[\mathbf{r}] = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\wedge\mathbf{r}}{r^3} + \mathcal{O}\left(\frac{1}{r}\right)^3 \qquad \lim_{|\mathbf{r}|\to\infty} |\mathbf{A}|[\mathbf{r}] = 0 \quad . \tag{36.14.10}$$

The higher-order terms have been omitted.

Read § 36-025 for an example of magnetic field at large distance.

An **order-of-magnitude estimate** for the **Magnetic Dipole** moment is: $|\mathbf{m}| \approx IA$.

36.14.03.04 Vector Potential of a Plane Stationary Current Loop at Large Distances

This § is referenced at pages:
[1739]

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Consider a plane current loop, made of a thin wire, described by the area vector $\mathbf{S}$ and diameter of the order of $d \approx \sqrt{S}$. At large distances with respect to the linear dimensions of the plane circuit it is possible to show that:

$$r \gg d \implies \mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\wedge\mathbf{r}}{r^3} \qquad \text{with } \mathbf{m} = I\mathbf{S} \quad .$$

Consider the plane loop in the xy plane, with origin inside the loop (for simplicity of the geometry only, read § 36.20.52) and look for the potential at $P = \{0, y > 0, z > 0\}$, at a **distance large with respect to the linear dimensions of the plane circuit**. The constant current is $I > 0$ running counterclockwise, positive according to the usual conventions.

Note that, ***large distance*, implies that all directions from any current loop element to the observation point are treated as parallel, in the three dimensions.**

The line integral of the current can be split in its x and y components. As there is no z component in the integral, we know that $\hat{\mathbf{e}}_3 \cdot \mathbf{A} = 0$.

The x component is equal to the integral along the full range in x of the pairs of elements at $\{x, y_+\}$ and $\{x, y_-\}$, bringing to the result two contributions of opposite sign whose difference is

$$\propto \Delta r/r^2 \simeq \sin\theta\,(|y_+| + |y_-|)\,/r^2 \quad , \tag{36.14.11}$$

because in this approximation the two distances are measured from parallel lines in a plane at the fixed x, $\sin\theta$ is fixed and r is the distance of P from the loop. Finally, the resulting integral is:

$$\frac{\sin\theta}{r^2}\int_{x_{min}}^{x_{max}} (|y_+| + |y_-|)\ \mathrm{d}x = \frac{\sin\theta}{r^2}[\text{loop surface area}] \equiv \frac{A\sin\theta}{r^2} \tag{36.14.12}$$

and negative.

The y component is the integral along the full range in y of the pairs of elements at $\{x_+, y\}$ and $\{x_-, y\}$, which have opposite sign and same distance because in this approximation the two distances are measured from parallel lines in the plane containing $\{x_+, y\}$, $\{x_-, y\}$ and P.

Therefore:

$$\mathbf{A}[0, y, z] = \frac{\mu_0}{4\pi}\left\{-\frac{IA\sin\theta}{r^2}, 0, 0\right\} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\wedge\mathbf{r}}{r^3} \qquad \text{in the chosen Coordinate System} \quad .$$

As the vector equation is valid in a specific Coordinate System, we know it is a valid equation in general, thanks to the **Principle of Covariance** , read § 20.03.06:

$$\mathbf{A}[0, y, z] = \frac{\mu_0}{4\pi}\left\{-\frac{IA\sin\theta}{r^2}, 0, 0\right\} \longrightarrow \mathbf{A}[\mathbf{r}] = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\wedge\mathbf{r}}{r^3} \quad .$$

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Read also § 36.20.58 for a special case.

36.14.04
Electric Quadrupole Tensor

This § is referenced at pages:
[1203, 1738]

©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|§ 4||

The **traceless symmetric Cartesian electric Quadrupole tensor** is:

$$Q^{\mathtt{E}\,\cdot\cdot}_{rs} = \iiint \rho[\mathbf{x}]\left(3x_r x_s - x^2\delta^{\cdot\cdot}_{rs}\right)\,\mathrm{d}V_{\mathbf{x}} \qquad \text{with} \qquad \begin{cases} Q^{\mathtt{E}\,\cdot\cdot}_{rs} = Q^{\mathtt{E}\,\cdot\cdot}_{sr} & , \\ \sum_k Q^{\mathtt{E}\,\cdot\cdot}_{kk} = 0 \end{cases} \quad .$$

→ 1087

An **order-of-magnitude estimate** for the electric **Quadrupole tensor** , modeled as two equal and opposite real electric dipoles separated by a distance L, is: $\approx (QD)\,L$.

36.14.05
Magnetic Quadrupole Tensor

This § is referenced at pages:
[1203, 1738]

©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|§ 11||

The **traceless symmetric Cartesian magnetic Quadrupole tensor** is:

$$Q^{\mathtt{M}\,\cdot\cdot}_{rs} = \iiint (x_r g_s[\mathbf{x}] + x_s g_r[\mathbf{x}])\,\mathrm{d}V_{\mathbf{x}} \qquad \mathbf{g}[\mathbf{x}] \equiv \mathbf{x}\wedge\mathbf{j}[\mathbf{x}] \qquad \text{with} \qquad \begin{cases} Q^{\mathtt{M}\,\cdot\cdot}_{rs} = Q^{\mathtt{M}\,\cdot\cdot}_{sr} & , \\ \sum_k Q^{\mathtt{M}\,\cdot\cdot}_{kk} = 0 \end{cases} \quad .$$

→ 1087

An **order-of-magnitude estimate** for the magnetic **Quadrupole tensor** , modeled as two equal and opposite real magnetic dipoles separated by a distance L, is: $\approx (IA)\,L$.

BRAIN - IN - PROGRESS

36.14.06
Multipole Development of Energy of a Charge Distribution in External ElectroStatic Field

This § is referenced at pages:
[1127, 1569, 1737]

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36.14.07
Multipole Development of Energy of a Current Distribution in External MagnetoStatic Field

This § is referenced at pages:
[1127, 1569, 1737]

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36.15 Summary of ElectroStatics and MagnetoStatics

- Stationary charges give constant electric fields: ElectroStatics.
- Steady currents give constant magnetic fields: MagnetoStatics.

The relations (36.13.04), (36.13.06) allows to calculate the electric and magnetic fields from equations (36.12.01), (36.12) taken in static conditions:

$$\boxed{\mathbf{div}\mathbf{B}=0 \implies \mathbf{B}=+\mathbf{rot}\mathbf{A}} \quad , \qquad \boxed{\mathbf{rot}\mathbf{E}=0 \implies \mathbf{E}=-\mathbf{grad}\Phi} \quad . \tag{36.15.01}$$

Gauge invariance is obviously effective.

36.15.01 Equations of ElectroStatics

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|1.6.2, 2.3.5||

$\rho \to \mathbf{E}$	Coulomb law
$\mathbf{E} \to \rho$	1st **Maxwell Equation**
$\Phi \to \mathbf{E}$	$\mathbf{E}=-\mathbf{grad}\Phi$
$\mathbf{E} \to \Phi$	$\Phi[\mathbf{x}]-\Phi[\mathbf{x}_0]=-\int_{\mathbf{x}_0}^{\mathbf{x}}\mathbf{E}\cdot \mathbb{d}\mathbf{L}$
$\rho \to \Phi$	equation (36.13.02)
$\Phi \to \rho$	equation (36.12.11)

.

36.15.02 Equations of MagnetoStatics

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|1.6.2, 5.4.2||

$\mathbf{j} \to \mathbf{B}$	Biot-Savart law
$\mathbf{B} \to \mathbf{j}$	4th **Maxwell Equation**
$\mathbf{A} \to \mathbf{B}$	$\mathbf{B}=\mathbf{rot}\mathbf{A}$
$\mathbf{B} \to \mathbf{A}$	(see text)
$\mathbf{j} \to \mathbf{A}$	equation (36.13.03)
$\mathbf{A} \to \mathbf{j}$	equation (36.12.12)

.

The passage from the field to the potential is less trivial in the magnetic case than in electric case: see [1], and § 16.09.03.03.

[1] ©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 1.6.2, 5.4.2; problem 5.30, 5.50, 5.51||

DRAFT - IN PROGRESS

36.16 Discontinuities at a surface

This § is referenced at pages:
[1723, 1735]

©|V.Namias|Am.J.Phys, , ..., ..., ...Ed., ... 56, 898 (1988) - doi: 10.1119/1.15406|*Definitive!*|

The discontinuities for the **ElectroMagnetic potentials** ($c = 1$) read:

$$\boxed{\hat{\mathbf{n}} \wedge \Delta \mathbf{A} = \mathbf{0} \qquad \hat{\mathbf{n}} \Delta \Phi - (\mathbf{v} \cdot \hat{\mathbf{n}}) \, \Delta \mathbf{A} = \mathbf{0}} \quad . \tag{36.16.01}$$

→ 1744

In the **Lorenz Gauge** one finds:

$$\boxed{\hat{\mathbf{n}} \wedge \Delta \mathbf{A} - (\mathbf{v} \cdot \hat{\mathbf{n}}) \, \Delta \Phi = 0} \quad . \tag{36.16.02}$$

→ 1744

The condition deriving from the **Lorenz Gauge** (36.16.02) when combined with the general condition for the potentials (36.16.01) gives:

$$\boxed{\Delta \Phi = 0 \qquad \hat{\mathbf{n}} \wedge \Delta \mathbf{A} = \mathbf{0}} \quad . \tag{36.16.03}$$

Read § 36.16.03.

36.16.01 Singularities of fields and potentials at fixed points lines surfaces

This § is referenced at pages:
[Never referenced.]

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|§ 1||

	$\Phi/\|\mathbf{A}\|$ static **Potential**	**E/B** static **Field**	
volume Charge|Current	$\sim r^2$	$\sim r$	
surface Charge|Current	$\sim r$	constant	$\perp$ to the surface
line Charge|Current	$\sim \log r$	$\sim 1/r$	$\perp$ to the line
point charge/(...)	$\sim 1/r$	$\sim 1/r^2$	
point Dipole	$\sim 1/r^2$	$\sim 1/r^3$	
point quadrupole	$\sim 1/r^3$	$\sim 1/r^4$	
point 2^n-pole	$\sim 1/r^{n+1}$	$\sim 1/r^{n+2}$	

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36.16.02
Discontinuities of the Fields and Potentials at Moving Surfaces

This § is referenced at pages:
[2233]

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|Introduction||

©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|Appendix 6|*Moving surfaces.*|

©|V.Namias|Am.J.Phys, , ..., ..., ...Ed., ... 56, 898 (1988) - doi: 10.1119/1.15406|*Definitive!*|

The equation for discontinuities deriving from the divergence equations apply equally to moving surface, while the ones from the rotor equations needs to be modified, as it is clear from the fact that the latter equations involve time derivatives.

Read § 36.16.03.

36.16.02.01 Normal Components

The equations for the discontinuity of the normal components, equations (36.09.01), (36.09.02), are valid for surfaces in motion too, provided the surface charge density for the electric case is the one seen in the laboratory system.

36.16.02.02 Tangential Components

The equations for the discontinuity of the tangential components, must be modified, in the case of moving surfaces, and assume a more complex form depending on the velocity, $\mathbf{v}$, of the moving surface.

The discontinuity for the tangential component of the electric fields becomes:

$$\boxed{\widehat{\mathbf{n}}\wedge\Delta\mathbf{E} - (\mathbf{v}\cdot\widehat{\mathbf{n}})\,\Delta\mathbf{B} = \mathbf{0}} \quad . \tag{36.16.04}$$

The discontinuity for the tangential component of the magnetic fields becomes:

$$\boxed{\widehat{\mathbf{n}}\wedge\Delta\mathbf{B} + \varepsilon_0\mu_0\,(\mathbf{v}\cdot\widehat{\mathbf{n}})\,\Delta\mathbf{E} = \mu_0\mathbf{j}_S} \quad . \tag{36.16.05}$$

These two equations imply the necessary condition:

$$\boxed{\widehat{\mathbf{n}}\cdot\mathbf{j}_S - (\mathbf{v}\cdot\widehat{\mathbf{n}})\,\rho_S = 0} \quad .$$

36.16.03
Summary of EM discontinuities

This § is referenced at pages:
[1719, 1735, 1744, 1745]

$$c = 1 \quad , \tag{36.16.06}$$

$$\widehat{\mathbf{n}}\wedge\Delta\mathbf{E} - (\mathbf{v}\cdot\widehat{\mathbf{n}})\,\Delta\mathbf{B} = 0 \quad , \tag{36.16.07}$$

$$\widehat{\mathbf{n}}\wedge\Delta\mathbf{H} - (\mathbf{v}\cdot\widehat{\mathbf{n}})\,\Delta\mathbf{D} = \mathbf{j}_S \quad , \tag{36.16.08}$$

$$(\mathbf{v}\cdot\mathbf{j}_S) - (\mathbf{v}\cdot\widehat{\mathbf{n}})\,\rho_S = 0 \quad , \tag{36.16.09}$$

$$\widehat{\mathbf{n}}\Delta\Phi - (\mathbf{v}\cdot\widehat{\mathbf{n}})\,\Delta\mathbf{A} = 0 \qquad \Leftrightarrow \qquad \Delta\Phi - (\mathbf{v}\cdot\widehat{\mathbf{n}})\,\widehat{\mathbf{n}}\cdot\Delta\mathbf{A} = 0 \quad , \tag{36.16.10}$$

$$\widehat{\mathbf{n}}\cdot\Delta\mathbf{A} - (\mathbf{v}\cdot\widehat{\mathbf{n}})\,\Delta\Phi = 0 \qquad \text{from the Lorenz gauge condition} \quad , \tag{36.16.11}$$

$$\Delta\Phi = 0 \qquad \widehat{\mathbf{n}}\cdot\Delta\mathbf{A} = 0 \qquad \text{from the last two equations} \quad . \tag{36.16.12}$$

In the Lorentz gauge both **ElectroMagnetic potentials** are continuous also for moving surfaces.

For charge/current discontinuities at moving boundaries see the references.

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36.17 Capacitance and Inductance

36.17.01 Capacitance

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|||

Read § 36.20.30, § 36.20.31, § 36.20.32.

36.17.02 Inductance

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|||

©|WEB - URL ||*Excellent discussion of common mistakes found in some books.*|

Read § 36.20.33, § 36.20.34, § 36.20.35, § 36.20.36.

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36.18 Theorems of Existence and Uniqueness

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|1.9||

©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|||

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36.19 Quasi Static Fields

36.20 Examples and Physical Applications

36.20.01

☒ Some Units of Measure in ElectroMagnetic (SI)

All units are in the SI system of units of measure (read § B).

$$[\varepsilon_0] = \mathrm{F/m} \qquad [\mu_0] = \mathrm{H/m} \quad ,$$

$$[\Phi] = (\mathrm{N \cdot m/C}) \qquad [\mathbf{A}] = (\mathrm{N \cdot s/C}) \quad ,$$

$$[\Phi] = (\mathrm{m/s})\,[\mathbf{A}] \quad .$$

36.20.01.01 Tables form NIST

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Table 4. The 22 SI units with special names and symbols

Derived quantity	Special name of unit	Unit expressed in terms of base units [a]	Unit expressed in terms of other SI units
plane angle	radian [b]	rad = m/m	
solid angle	steradian [c]	sr = m^2/m^2	
frequency	hertz [d]	Hz = s^{-1}	
force	newton	N = kg m s^{-2}	
pressure, stress	pascal	Pa = kg m^{-1} s^{-2}	
energy, work, amount of heat	joule	J = kg m^2 s^{-2}	N m
power, radiant flux	watt	W = kg m^2 s^{-3}	J/s
electric charge	coulomb	C = A s	
electric potential difference [e]	volt	V = kg m^2 s^{-3} A^{-1}	W/A
capacitance	farad	F = kg^{-1} m^{-2} s^4 A^2	C/V
electric resistance	ohm	Ω = kg m^2 s^{-3} A^{-2}	V/A
electric conductance	siemens	S = kg^{-1} m^{-2} s^3 A^2	A/V
magnetic flux	weber	Wb = kg m^2 s^{-2} A^{-1}	V s
magnetic flux density	tesla	T = kg s^{-2} A^{-1}	Wb/m^2
inductance	henry	H = kg m^2 s^{-2} A^{-2}	Wb/A
Celsius temperature	degree Celsius [f]	°C = K	
luminous flux	lumen	lm = cd sr [g]	cd sr
illuminance	lux	lx = cd sr m^{-2}	lm/m^2
activity referred to a radionuclide [d,h]	becquerel	Bq = s^{-1}	
absorbed dose, kerma	gray	Gy = m^2 s^{-2}	J/kg
dose equivalent	sievert [i]	Sv = m^2 s^{-2}	J/kg
catalytic activity	katal	kat = mol s^{-1}	

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(*a*) The order of symbols for base units in this Table is different from that in the 8th edition following a decision by the CCU at its 21st meeting (2013) to return to the original order in Resolution 12 of the 11th CGPM (1960) in which newton was written kg m s^{-2}, the joule as kg m^2 s^{-2} and J s as kg m^{-2} s^{-1}. The intention was to reflect the underlying physics of the corresponding quantity equations although for some more complex derived units this may not be possible.

(*b*) The radian is the coherent unit for plane angle. One radian is the angle subtended at the center of a circle by an arc that is equal in length to the radius. It is also the unit for phase angle. For periodic phenomena, the phase angle increases by 2π rad in one period. The radian was formerly an SI supplementary unit, but this category was abolished in 1995.

(*c*) The steradian is the coherent unit for solid angle. One steradian is the solid angle subtended at the center of a sphere by an area of the surface that is equal to the squared radius. Like the radian, the steradian was formerly an SI supplementary unit.

(*d*) The hertz shall only be used for periodic phenomena and the becquerel shall only be used for stochastic processes in activity referred to a radionuclide.

(*e*) Electric potential difference is also called “voltage” in the United States and in many countries, as well as “electric tension” or simply “tension” in some countries.

(*f*) The degree Celsius is used to express Celsius temperatures. The numerical value of a temperature difference or temperature interval is the same when expressed in either degrees Celsius or in kelvin.

(*g*) In photometry the name steradian and the symbol sr are usually retained in expressions for units

(*h*) Activity referred to a radionuclide is sometimes incorrectly called radioactivity.

(*i*) See CIPM Recommendation 2 on the use of the sievert (PV, 2002, **70**, 205).

The seven base units and 22 units with special names and symbols may be used in combination to express the units of other derived quantities. Since the number of quantities is without limit, it is not possible to provide a complete list of derived quantities and derived units. Table 5 lists some examples of derived quantities and the corresponding coherent derived units expressed in terms of base units. In addition, Table 6 lists examples of coherent derived units whose names and symbols also include derived units. The complete set of SI units includes both the coherent set and the multiples and sub-multiples formed by using the SI prefixes.

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Table 5. Examples of coherent derived units in the SI expressed in terms of base units

Derived quantity	Typical symbol of quantity	Derived unit expressed in terms of base units
area	A	m^2
volume	V	m^3
speed, velocity	v	$m\ s^{-1}$
acceleration	a	$m\ s^{-2}$
wavenumber	σ	m^{-1}
density, mass density	ρ	$kg\ m^{-3}$
surface density	ρ_A	$kg\ m^{-2}$
specific volume	v	$m^3\ kg^{-1}$
current density	j	$A\ m^{-2}$
magnetic field strength	H	$A\ m^{-1}$
amount of substance concentration	c	$mol\ m^{-3}$
mass concentration	ρ, γ	$kg\ m^{-3}$
luminance	L_v	$cd\ m^{-2}$

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Table 6. Examples of SI coherent derived units whose names and symbols include SI coherent derived units with special names and symbols

Derived quantity	Name of coherent derived unit	Symbol	Derived unit expressed in terms of base units
dynamic viscosity	pascal second	Pa s	$kg\ m^{-1}\ s^{-1}$
moment of force	newton meter	N m	$kg\ m^{2}\ s^{-2}$
surface tension	newton per meter	$N\ m^{-1}$	$kg\ s^{-2}$
angular velocity, angular frequency	radian per second	$rad\ s^{-1}$	s^{-1}
angular acceleration	radian per second squared	rad/s^{2}	s^{-2}
heat flux density, irradiance	watt per square meter	W/m^{2}	$kg\ s^{-3}$
heat capacity, entropy	joule per kelvin	$J\ K^{-1}$	$kg\ m^{2}\ s^{-2}\ K^{-1}$
specific heat capacity, specific entropy	joule per kilogram kelvin	$J\ K^{-1}\ kg^{-1}$	$m^{2}\ s^{-2}\ K^{-1}$
specific energy	joule per kilogram	$J\ kg^{-1}$	$m^{2}\ s^{-2}$
thermal conductivity	watt per meter kelvin	$W\ m^{-1}\ K^{-1}$	$kg\ m\ s^{-3}\ K^{-1}$
energy density	joule per cubic meter	$J\ m^{-3}$	$kg\ m^{-1}\ s^{-2}$
electric field strength	volt per meter	$V\ m^{-1}$	$kg\ m\ s^{-3}\ A^{-1}$
electric charge density	coulomb per cubic meter	$C\ m^{-3}$	$A\ s\ m^{-3}$
surface charge density	coulomb per square meter	$C\ m^{-2}$	$A\ s\ m^{-2}$
electric flux density, electric displacement	coulomb per square meter	$C\ m^{-2}$	$A\ s\ m^{-2}$
permittivity	farad per meter	$F\ m^{-1}$	$kg^{-1}\ m^{-3}\ s^{4}\ A^{2}$
permeability	henry per meter	$H\ m^{-1}$	$kg\ m\ s^{-2}\ A^{-2}$
molar energy	joule per mole	$J\ mol^{-1}$	$kg\ m^{2}\ s^{-2}\ mol^{-1}$
molar entropy, molar heat capacity	joule per mole kelvin	$J\ K^{-1}\ mol^{-1}$	$kg\ m^{2}\ s^{-2}\ mol^{-1}\ K^{-1}$
exposure (x- and γ-rays)	coulomb per kilogram	$C\ kg^{-1}$	$A\ s\ kg^{-1}$
absorbed dose rate	gray per second	$Gy\ s^{-1}$	$m^{2}\ s^{-3}$
radiant intensity	watt per steradian	$W\ sr^{-1}$	$kg\ m^{2}\ s^{-3}$
radiance	watt per square meter steradian	$W\ sr^{-1}\ m^{-2}$	$kg\ s^{-3}$
catalytic activity concentration	katal per cubic meter	$kat\ m^{-3}$	$mol\ s^{-1}\ m^{-3}$

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36.20.02
Hypothetical Magnetic Charges and Currents

Consider an hypothetical world where Magnetic Charges and Currents exist.

This introduction will remove the asymmetry in **Maxwell Equations** : Electric Charges generate the **E** field while Electric Currents generate the **B** field; Magnetic Charges generate the **B** field while Magnetic Currents generate the **E** field.

1. Rewrite **Maxwell Equations** to account for Magnetic Charges and Currents by exploiting the parallel between Electric Charges, Currents and Electric Fields versus Magnetic Charges, Currents and Magnetic Fields.

36.20.03
Average EM Fields

36.20.03.01 Average Electric Fields

This § is referenced at pages:
[1714]

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|4.1||

It can be shown that, given a stationary charge distribution, the average value of the electric field over a spherical volume of radius R, centered at $\mathbf{r}_0$, is:

$$\iiint_{r\leq R} \mathbf{E}[\mathbf{x}]\, \mathrm{d}V_{\mathbf{x}} = -\frac{1}{3\varepsilon_0}\mathbf{p} \qquad \text{if the sphere encloses all the charges} \quad ,$$

$$\iiint_{r\leq R} \mathbf{E}[\mathbf{x}]\, \mathrm{d}V_{\mathbf{x}} = \frac{4\pi R^3}{3}\mathbf{E}[\mathbf{r}_0] \qquad \text{if the sphere is external to all the charges} \quad .$$

36.20.03.02 Average Magnetic Fields

This § is referenced at pages:
[1714, 1806, 1806]

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|5.6||

It can be shown that, given a steady current distribution, the average value of the magnetic field over a spherical volume of radius R, centered at $\mathbf{r}_0$, is:

$$\iiint_{r\leq R} \mathbf{B}[\mathbf{x}]\, \mathrm{d}V_{\mathbf{x}} = \frac{2\mu_0}{3}\mathbf{m} \qquad \text{if the sphere encloses all the currents} \quad ,$$

$$\iiint_{r\leq R} \mathbf{B}[\mathbf{x}]\, \mathrm{d}V_{\mathbf{x}} = \frac{4\pi R^3}{3}\mathbf{B}[\mathbf{r}_0] \qquad \text{if the sphere is external to all the currents} \quad .$$

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36.20.04
Static Charge Density at the Surface of a Conductor

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|problem 1.11||

Use Gauss law to prove that at the surface of a curved charged conductor, the normal derivative of the electric field, under static conditions, is given by:

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \quad .$$

where R_1 and R_2 are the principal radii of curvature of the surface.

36.20.05
☒ Infinite Uniformly Charged Plane

This § is referenced at pages:
[1787]

Let ρ_S be the uniform surface charge density and $\hat{\mathbf{n}}$ the unit vector normal to the plane, with an arbitrary orientation, from semi-space (1) to semi-space (2). It can be shown that electric fields on the two sides are given by:

$$\mathbf{E}_2 = +\frac{\rho_S}{2\varepsilon_0}\hat{\mathbf{n}} \qquad \mathbf{E}_1 = -\frac{\rho_S}{2\varepsilon_0}\hat{\mathbf{n}} \quad .$$

The law for the discontinuity of the normal components of $\mathbf{E}$ apply.

By symmetry and charges location the electric field is normal to the plane. The orientation of the electric field must be opposite on the two sides; in fact when rotating the plane by π around any axis in the plane the orientation of the electric field after rotation must be the same as before, as the charge density is unchanged.

The direction of the field can be also determined by the Coulomb law (36.06).

Gauss law immediately implies that the field does not depend on the distance from the plane.

Do the calculation by direct integration from Coulomb law and **superposition principle** .

36.20.06
☒ Infinite Uniform Current Plane

This § is referenced at pages:
[1787, 2253, 2253]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|E5.8||

Let $\mathbf{j}_S$ be the uniform surface current density and $\hat{\mathbf{n}}$ the unit vector normal to the plane, with an arbitrary orientation, from semi-space (1) to semi-space (2). It can be shown that magnetic fields on the two sides are given by:

$$\mathbf{B}_2 = +\frac{\mu_0}{2}\mathbf{j}_S \wedge \hat{\mathbf{n}} \qquad \mathbf{B}_1 = -\frac{\mu_0}{2}\mathbf{j}_S \wedge \hat{\mathbf{n}} \quad .$$

The law for the discontinuity of the tangential components of $\mathbf{B}$ apply.

The field cannot be perpendicular to the plane. In fact, a perpendicular field with the same orientation on the two sides is ruled out by the argument that when rotating the plane by π around any axis in the plane parallel to $\mathbf{j}_S$ the orientation of the magnetic field after rotation must be the same as before, as the current density is unchanged. Moreover, a perpendicular field with opposite orientations on the two sides is ruled out by the argument that it violates Gauss law for magnetism for any surface crossing the plane.

It can be shown that, by symmetry and currents location, the magnetic field is parallel to the plane and perpendicular to $\mathbf{j}_S$, with opposite orientation on the two sides of the plane.

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The direction of the field can be also determined by the Biot-Savart law (36.06). In fact every strip of current of infinite length and infinitesimal width parallel to $\mathbf{j}_S$ gives rise to a magnetic field perpendicular to $\mathbf{j}_S$. Moreover, for every point there are infinite pairs of strips at the same distance and opposite sides whose contributions perpendicular to the plane cancel.

Ampère law immediately implies that the field does not depend on the distance from the plane.

Do the calculation by direct integration from Biot-Savart law and **superposition principle** .

36.20.07

☒ Plane double layer of opposite charges

Consider two parallel planes, charged with uniform surface charges of opposite sign, $\pm|\rho_S|$, and separated by a distance d.

1. Determine the scalar electric potential as a function of the distance from the middle of the two planes.

36.20.08

☒ ElectroStatic Pressure

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|||

An application of the **superposition principle** .

Free charges at equilibrium locate at the surface of a conductor. Therefore an ElectroStatic pressure acts on free charges at the surface. As the macroscopic field external to the conductor is normally much smaller that the microscopic fields preventing the escape of the free charges this ElectroStatic pressure is actually transmitted to the material medium which is the support of the free charges.

It can be calculated by application of the **superposition principle** , by considering the force on an infinitesimal piece of surface.

Consider the plane surface with a surface charge density ρ_S and let the unit vector from side 1 to side 2 be $\hat{\mathbf{n}}$ and the electric fields on the two sides be $\mathbf{E}_1$ and $\mathbf{E}_2$. Assume $(\mathbf{E}_1 \parallel \mathbf{E}_2) \parallel \hat{\mathbf{n}}$. In fact in order to calculate the pressure, that is the force normal to the surface, only the components of $\mathbf{E}$ perpendicular to the surface contribute, thanks to the functional form of the Lorentz force.

In the most general case, when the direction of the fields are arbitrary, it is necessary to use the **Maxwell stress tensor** to determine all the forces [1].

Consider a small piece of surface and the field felt by its charges. Take a piece of surface of size D, much smaller that the absolute values of the two radii of curvature of the surface at the point:

$$D \ll |\rho_1| \qquad \text{and} \qquad D \ll |\rho_2| \quad ,$$

so that it can be considered plane.

The field felt by the small piece of surface $\mathrm{d}\mathbf{S}$ is given by the field generated by all other charges and it is given by:

$$\mathbf{E} = \frac{\mathbf{E}_1 + \mathbf{E}_2}{2} \parallel \hat{\mathbf{n}} \quad ,$$

as it is ready shown by using the **superposition principle** .

First note that $\mathbf{E}$ is continuous inside the hole left when ideally removing $\mathrm{d}\mathbf{S}$.

[1] W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...

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$$\mathbf{E}_{\text{hole}} + \mathbf{E}_{\text{by } \mathbb{d}\mathbf{S}\text{into2}} = \mathbf{E}_{\text{hole}} + \frac{\rho_S}{2\varepsilon_0} = \mathbf{E}_2 \quad ,$$

$$\mathbf{E}_{\text{hole}} + \mathbf{E}_{\text{by } \mathbb{d}\mathbf{S}\text{into1}} = \mathbf{E}_{\text{hole}} - \frac{\rho_S}{2\varepsilon_0} = \mathbf{E}_1 \quad ,$$

$$\Longrightarrow \mathbf{E}_{\text{hole}} = \frac{\mathbf{E}_2 + \mathbf{E}_1}{2} \quad ,$$

$$\Longrightarrow \mathbf{E}_{\text{by } \mathbb{d}\mathbf{S}\text{into2}} - \mathbf{E}_{\text{by } \mathbb{d}\mathbf{S}\text{into1}} = \mathbf{E}_2 - \mathbf{E}_1 \quad .$$

The force acting on the small piece of surface $\mathbb{d}S$ is thus:

$$\mathbb{d}\mathbf{f} = \rho_S \frac{\mathbf{E}_1 + \mathbf{E}_2}{2} \mathbb{d}S = \varepsilon_0 \left(E_2 - E_1\right) \frac{E_1 + E_2}{2} \hat{\mathbf{n}} \, \mathbb{d}S = \varepsilon_0 \frac{\left(E_2^2 - E_1^2\right)}{2} \hat{\mathbf{n}} \, \mathbb{d}S \quad .$$

The ElectroStatic pressure is thus:

$$\boxed{p_E = \varepsilon_0 \frac{\left(E_2^2 - E_1^2\right)}{2} \qquad \text{force directed from side 1 to side 2}} \quad . \tag{36.20.01}$$

→ 1759

It is thus given by the difference of the energy densities on the two sides.

Note the inverted sign with respect to equation (36.20.02). Note also that the force is directed along the normal to the surface, as in equation equation (36.20.02), which is parallel to the electric field, at variance with respect to equation (36.20.02).

It is important to stress that, in order to calculate the force acting on a small piece of surface charge, the ElectroStatic field produced by all the other charges must be used, excluding the contribution to the field by the small piece of surface charge itself.

36.20.09

☒ MagnetoStatic Pressure

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|||

An application of the **superposition principle** .

It can be calculated by application of the **superposition principle** , by considering the force acting on a strip of current of infinite length and infinitesimal width parallel to $\mathbf{j}_S$.

Consider a plane surface with a surface current density $\mathbf{j}_S$ and let the unit vector from side 1 to side 2 be $\hat{\mathbf{n}}$ and the magnetic fields on the two sides be $\mathbf{B}_1$ and $\mathbf{B}_2$.

Take a piece of surface of size D, much smaller that the absolute values of the two radii of curvature of the surface at the point:

$$D \ll |\rho_1| \qquad \text{and} \qquad D \ll |\rho_2| \quad ,$$

so that it can be considered plane.

Assume $(\mathbf{B}_1 \parallel \mathbf{B}_2) \perp \hat{\mathbf{n}}$ and $(\mathbf{B}_1 \parallel \mathbf{B}_2) \perp \mathbf{j}_S$.

In fact in order to calculate the pressure, that is the force normal to the surface, only the components of $\mathbf{B}$ perpendicular to $\mathbf{j}_S$ give a non-zero contribution and only the components parallel to the surface contribute to the pressure, thanks to the functional form of the Lorentz force.

In the most general case, when the direction of the fields are arbitrary, it is necessary to use the **Maxwell stress tensor** to determine all the forces [1].

Consider a small strip of surface, along the surface current, and the field felt by its currents. The field felt by the small strip of surface is given by the field generated by all other currents and it is given by:

$$\mathbf{B} = \frac{\mathbf{B}_1 + \mathbf{B}_2}{2} \perp \hat{\mathbf{n}} \quad ,$$

as it is readily shown by using the **superposition principle** .

[1] W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...

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Similarly to the case of ElectroStatic pressure one can determine the MagnetoStatic pressure:

$$p_B = \frac{(B_1^2 - B_2^2)}{2\mu_0} \qquad \text{force directed from side 1 to side 2} \quad . \tag{36.20.02}$$

→ 1758

It is thus given by the difference of the energy densities on the two sides.

Note the inverted sign with respect to equation (36.20.01). Note also that the force is directed along the normal to the surface, as in equation equation (36.20.01), which is perpendicular to the magnetic field, at variance with respect to equation (36.20.01).

It is important to stress that, in order to calculate the force acting on a small piece of surface current, the MagnetoStatics field produced by all the other currents must be used, excluding the contribution to the field by the small piece of surface current itself.

DRAFT

36.20.10

☒ Dipole Interactions: Internal Forces Between a Pair of Rigid Dipoles

This § is referenced at pages:
[1309, 1309, 1760, 1761]

Consider two ideal **Electric Dipoles** , $\mathbf{p}$ and $\mathbf{p}_0$, and find the internal forces of the system.

WORKED SOLUTION

The following properties of **Dipole** fields and interactions are exemplified for electric fields and Dipoles but are obviously equally valid for magnetic fields and **Dipoles** .

The electric potential produced at $\mathbf{r}$ by an **Electric Dipole** $\mathbf{p}_0$ located at the origin is, from equation (16.12.05):

$$\Phi[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\left(\frac{\mathbf{p}_0\cdot\mathbf{r}}{r^3}\right) \quad .$$

From this expression the electric field can be easily obtained by differentiation:

$$\mathbf{E}[\mathbf{r}] = -\mathbf{grad}\Phi[\mathbf{r}] \implies E_k[\mathbf{r}] = -\frac{\partial}{\partial x^k}\Phi[\mathbf{r}] = -\frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial x^k}\left(\frac{p_j x_j}{r^3}\right) \quad .$$

The result is (equation (36.07.01)):

$$\mathbf{E}[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\left(3\frac{(\mathbf{p}_0\cdot\mathbf{r})}{r^5}\mathbf{r} - \frac{\mathbf{p}_0}{r^3}\right) \quad . \tag{36.20.03}$$

Locate a second **Dipole** $\mathbf{p}$ at $\mathbf{r}$.

The electric field generated by the second **Dipole** at the origin (the location of the first **Dipole**) can be obtained by changing $\mathbf{r} \longrightarrow \mathbf{r}' = -\mathbf{r}$, (which does not make any change) and using the **Dipole** $\mathbf{p}$ as a Sources | Sinks:

$$\mathbf{E}_0[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\left(3\frac{(\mathbf{p}\cdot\mathbf{r})}{r^5}\mathbf{r} - \frac{\mathbf{p}}{r^3}\right) \quad . \tag{36.20.04}$$

The interaction energy can be calculated from equation (41.05.04) as:

$$\Phi = \Phi[r,\theta] = -\,\mathbf{p}\cdot\mathbf{E} \quad . \tag{36.20.05}$$

→ 1760

The force on the second **Dipole** due to the field of the first **Dipole** is given by:

$$\mathbf{f} = -\mathbf{grad}\Phi = \mathbf{grad}[\,\mathbf{p}\cdot\mathbf{E}] \implies f_k = \frac{\partial}{\partial x^k}(p_j E_j) \quad .$$

One finds:

$$\mathbf{f}[\mathbf{r}] = 3\frac{1}{4\pi\varepsilon_0}\left(\frac{\mathbf{p}_0\,(\mathbf{p}\cdot\mathbf{r}) + \mathbf{p}\,(\mathbf{p}_0\cdot\mathbf{r})}{r^5} - 5\frac{\mathbf{r}\,(\mathbf{p}\cdot\mathbf{r})\,(\mathbf{p}_0\cdot\mathbf{r})}{r^7} + \frac{\mathbf{r}\,(\mathbf{p}\cdot\mathbf{p}_0)}{r^5}\right) \tag{36.20.06}$$

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This force is clearly non-central. The two last terms are purely radial, while the first one has, in general, both a radial and a transverse part.
The force is symmetrical over the two **Dipoles** .
The force on the first **Dipole** due to the field of the second **Dipole** can be obtained by changing $\mathbf{r} \longrightarrow \mathbf{r}' = -\mathbf{r}$, causing the change of sign of the force (because $\mathbf{r}$ appears an odd number of times at the numerator):

$$\mathbf{f}_0[\mathbf{r}'] = -\mathbf{f}[\mathbf{r}] \quad .$$

36.20.11
☒ Dipole Interactions: Internal Torques Between a Pair of Rigid Dipoles

This § is referenced at pages:
[1309, 1309, 1761]

Consider two ideal **Electric Dipoles** , $\mathbf{p}$ and $\mathbf{p}_0$, and find the internal forces of the system.
Show that the force between the **Dipoles** is not radial, that is it is not necessarily directed along the line joining the centers of the two **Dipoles** . How does this compare with the expression of Newton third law?
Show that the internal torques sum to zero and that this is valid for any choice of the pole.

WORKED SOLUTION

Refer to problem § 36.20.10 for notations ad setup.
The torque acting on the second **Dipole** with respect to the pole $\mathbf{r}$ can be readily obtained from equation (36.20.05) as follows.
The interaction energy at a fixed point only depends on the angle between the **Dipole** moment and the electric field at the point, θ. In terms of θ one can write:

$$\delta\Phi \ = +Ep\sin[\theta]\ \delta\theta \quad ,$$

showing that the module of the torque is:

$$|\boldsymbol{\gamma}| = Ep\sin[\theta] \quad .$$

The direction is determined by noting that in order to have a torque tending to align the **Dipole** with the field, is it is required by minimizing the potential energy, the torque must be perpendicular to both $\mathbf{p}$ and $\mathbf{E}$, as it is required for a torque tending to align by rotation $\mathbf{p}$ towards $\mathbf{E}$. The orientation of the torque is determined by checking the right-hand rule to find that the correct sign is given by the expression:

$$\boldsymbol{\gamma} = \mathbf{p}\wedge\mathbf{E} \quad .$$

In an equivalent formal way the torque acting on the second **Dipole** with respect to the pole $\mathbf{r}$ can be readily obtained from equation (36.20.05) by varying $\mathbf{p}$ while keeping fixed both $\mathbf{r}$ and $\mathbf{p}_0$:

$$\delta\Phi \ = -\mathbf{E}\cdot\delta\mathbf{p} \ = -\mathbf{E}\cdot(\ \delta\boldsymbol{\theta}\wedge\mathbf{p}) = -\ \delta\boldsymbol{\theta}\cdot(\mathbf{p}\wedge\mathbf{E}) \Longrightarrow \boldsymbol{\gamma} = \mathbf{p}\wedge\mathbf{E} \quad .$$

The torque acting on the first **Dipole** with respect to the origin as a pole $\mathbf{r}$ can be readily obtained from the potential energy of the first **Dipole** in the field of the second Dipole by varying $\mathbf{p}_0$ while keeping fixed both $\mathbf{r}$ and $\mathbf{p}$:

$$\delta\Phi_0 \ = -\ \delta\boldsymbol{\theta}\cdot(\mathbf{p}_0\wedge\mathbf{E}_0) \Longrightarrow \boldsymbol{\gamma}_0 = \mathbf{p}_0\wedge\mathbf{E}_0 \quad .$$

The torque acting on the second **Dipole** with respect to the pole at the origin due to the force $\mathbf{f}$ can be readily obtained from:

$$\boldsymbol{\gamma}_L = \mathbf{r}\wedge\mathbf{f} = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\,\mathbf{r}\wedge(\mathbf{p}\,(\mathbf{p}_0\cdot\mathbf{r}) + \mathbf{p}_0\,(\mathbf{p}\cdot\mathbf{r}))}{r^5}\right) \quad ,$$

showing that the force is not central.
However, the internal torques with respect to the origin sum to zero:

$$\boldsymbol{\gamma}_L + \boldsymbol{\gamma}_0 + \boldsymbol{\gamma} = 0 \quad .$$

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36.20.12
☒ Dipole Interactions: Interaction Energy Between a Pair of Rigid Dipoles

This § is referenced at pages:
[1079, 1079, 1793]

1. Show that the interaction energy between a pair of rigid dipoles is:

$$\mathcal{E}[\mathbf{r},\mathbf{d}_1,\mathbf{d}_2] \propto \frac{1}{r^3}\left(\mathbf{d}_1\cdot\mathbf{d}_2 - \frac{3\,(\mathbf{d}_1\cdot\mathbf{r})\,(\mathbf{d}_2\cdot\mathbf{r})}{r^2}\right) \qquad \mathbf{r}\equiv(\mathbf{r}_2-\mathbf{r}_1) \quad . \tag{36.20.07}$$

2. Find at least one configuration corresponding to positive interaction energy.
3. Find at least one configuration corresponding to negative interaction energy.
4. Assume a fixed $\mathbf{r}$: determine the average interaction energy considering, at the fixed $\mathbf{r}$ all possible angular positions of the two dipoles.

Read also § 15-025.
Read also § 36.20.10, § 36.20.11.

36.20.13
Magnetic Scalar potential for a Point Magnetic Dipole

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 5.51||

In static conditions, in absence of current densities, Maxwell equations show that the magnetic file is irrotational. Under these conditions one can introduce a scalar magnetic potential, in parallel to what is done in electrostatics.

1. Construct a scalar magnetic potential for a Point Magnetic Dipole.
2. Construct a scalar magnetic potential for a spinning spherical shell uniformly charged.
3. Construct a scalar magnetic potential for the interior of a solid spinning sphere uniformly charged.

36.20.14
Which Expression of the Force on a Point Magnetic Dipole

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|E6.5||

A uniform current density, $\mathbf{j} = j\widehat{\mathbf{e}}_3$ fills a slab in the yz plane, from $x = -a$ to $x = +a$, perpendicular to the x axis.

- A **Magnetic Dipole** $\mathbf{m} = \mathbf{m}_0\widehat{\mathbf{e}}_1$ is situated at the origin. Find the force on the **Dipole** .
- A **Magnetic Dipole** $\mathbf{m} = \mathbf{m}_0\widehat{\mathbf{e}}_2$ is situated at the origin. Find the force on the **Dipole** .

In the ElectroStatic case the expressions (36.05.02), (36.05.04) are always equivalent.
However, this is not always the case for the magnetic analogs (36.05.06), (36.05.07) as this example shows.

1. Explain why equations (36.05.06), (36.05.07) are not equivalent in this example.

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36.20.15
Equilibrium Position of Two Dipoles

This § is referenced at pages:
[Never referenced.]

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|P 1.11||

Consider two **Dipoles** , $\mathbf{p}_1$ and $\mathbf{p}_2$, located at $\mathbf{r}_1$ and $\mathbf{r}_2$ such that the two positions can change but the distance cannot change. Let θ_1 and θ_2 be the angles made by the two **Dipoles** with the vector joining the position of the two **Dipoles** : $\mathbf{R} \equiv \mathbf{r}_2 - \mathbf{r}_1$.

1. Show that, if the distance $|\mathbf{R}|$ is kept constant, and the angle θ_1 is kept fixed as well then the equilibrium position is given by: $\tan[\theta_2] = -2\tan[\theta_1]$.

36.20.16
Dipolo in Campo Elettrico Uniforme

This § is referenced at pages:
[1763]

Dimostrare che ponendo un dipolo elettrico in un campo elettrico uniforme $\mathbf{E}_0$, con il momento di dipolo parallelo al campo:

1. esiste una superficie equipotenziale centrata sul dipolo che è una sfera;
2. determinarne il raggio.

WORKED SOLUTION

Il potenziale è la somma del potenziale prodotto dal dipolo più il potenziale prodotto dal campo elettrico uniforme ($\mathbf{E}_0 = E_0\widehat{\mathbf{e}}_z$). Introduciamo un asse z parallelo al campo elettrico $\mathbf{E}_0$ ed al momento di diplo elettrico. Allora:

$$\Phi[\mathbf{r}] = \Phi_0 - E_0 z + \frac{1}{4\pi\varepsilon_0}\frac{p\cos[\theta]}{r^2} = \Phi_0 - E_0 r\cos[\theta] + \frac{1}{4\pi\varepsilon_0}\frac{p\cos[\theta]}{r^2} = \Phi_0 - \cos[\theta]\left(E_0 r - \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2}\right) \quad .$$

Affinché ci sia una superficie equipotenziale che sia una sfera deve essere:

$$E_0 r - \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2} = 0 \qquad \Longrightarrow \qquad r = \left(\frac{1}{4\pi\varepsilon_0}\frac{p}{E_0}\right)^{1/3} \quad . \tag{36.20.08}$$

→ 1762

Tale sfera è a potenziale uniforme di valore uguale al valore del potenziale del campo $\mathbf{E}_0$ nel punto in cui si pone il dipolo.
Alla superficie della sfera il campo elettrico è radiale, essendo una superficie equipotenziale. Usando l'espressione del campo elettrico del dipolo si ha:

$$\mathbf{E} = \left(\frac{1}{4\pi\varepsilon_0}\frac{2p\cos[\theta]}{r^3} + E_0\cos[\theta]\right)\frac{\mathbf{r}}{r} = 3E_0\cos[\theta]\frac{\mathbf{r}}{r} \quad .$$

Tale risultato è utilizzabile per calcolare il campo elettrico di una sfera conduttrice in un campo elettrico uniforme. Si introduce al centro della sfera un dipolo immagine di valore dato dalla relazione (36.20.08):

$$p = 4\pi\varepsilon_0 r^3 E_0 \quad .$$

è importante notare che il campo elettrico indotto è proporzionale al campo elettrico esterno (inducente).
Il campo elettrico esterno alla sfera può essere determinato come sovrapposzione del campo elettrico $\mathbf{E}_0$ e del campo elettrico prodotto dal dipolo immagine. Il potenziale assunto dalla sfera è pari al

potenziale del campo elettrico $\mathbf{E}_0$ nel punto in cui viene posto il centro della sfera e la densità di carica superficiale indotta, che perturba il campo elettrico $\mathbf{E}_0$, vale:

$$\sigma = 3\varepsilon_0 E_0 \cos[\theta] \quad .$$

Poichè il campo elettrico entro la sfera conduttrice deve essere nullo tale densità di carica superficiale deve produrre all'interno della sfera un campo elettrico uguale ed opposto al campo esterno $\mathbf{E}_0$. Si può dimostrare che il momento di dipolo elettrico della distribuzione di carica superficiale vale esattamente $p\widehat{\mathbf{e}}_z$:

$$\mathbf{p} = \int \rho_S \mathbf{r}\, \mathrm{d}\mathbf{S} = -3\varepsilon_0 E_0 r^3 \int \cos^2[\theta]\ \mathrm{d}V\ \mathrm{d}\cos[\theta] = 4\pi\varepsilon_0 r^3 E_0 \quad .$$

36.20.17

☒ Forza Su Una Sfera Conduttrice Nel Campo Di Una Carica Puntiforme

Una piccola sfera conduttrice di raggio R è immersa nel campo elettrico prodotto da una carica puntiforme q, posta a distanza $r \gg R$. Determinare la forza che agisce sulla sfera.

WORKED SOLUTION

Il campo elettrico prodotto dalla carica puntiforme non è uniforme però, data al condizione $r \gg R$ lo si può considerare, in prima approssimazione, uniforme nella regione occupata dalla sfera.
Il momento di dipolo indotto sulla sfera conduttrice, si veda il problema § 36.20.16, vale:

$$p_i = 4\pi\varepsilon_0 R^3 E_0 \quad .$$

Considerando un Coordinate System sferiche centrato sulla carica puntiforme tale momento di dipolo indotto si scrive:

$$\mathbf{p}_i = 4\pi\varepsilon_0 R^3 \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)\widehat{\mathbf{e}}_r = \frac{pR^3}{r^2}\widehat{\mathbf{e}}_r \quad .$$

La forza che ne risulta è allora data dalla (36.05.02).
La forza è sempre attrattiva.
Notare che, se il campo elettrico fosse esattamente uniforme, la forza risultante sarebbe esattamente nulla.

36.20.18

☒ Magnitude of the Time-Varying Contributions in Maxwell Equations

This § is referenced at pages:
[1717]

The circulation of $\mathbf{E}$ is only given by the changes in the magnetic field with time, at variance with the circulation of $\mathbf{B}$, which depends not only on the changes in the electric field with time but also on the conduction current.

36.20.18.01 Estimate of the Displacement Current Density Term

Consider typical everyday values and a Sinusoidal|Cosinusoidal time-dependence of all the relevant quantities.

Note that in case the harmonic frequency is much larger than the values assumed below ($\nu \approx 10^2$ Hz), the conclusions may change drastically.

First Method

- Electric field:
  1. $E \approx 1.2{\cdot}10^2$ V/m (typical electric field at the Earth surface, directed downward);
  2. $E \approx 5.4{\cdot}10^{-3}$ V/m (electric field inside a copper wire with 1 mm radius carrying a current of 1 A);
- $\nu \approx 10^2$ Hz (frequency of household electric power);
- $I \approx 14$ A (typical maximum current of household electric power, from the difference of potential, $\Delta\Phi = 230$ V, and typical maximum available power, $P = 3.3$ kW);
- $j \approx I/S \approx 7{\cdot}10^6$ A/m^2 (for a typical wire cross-sectional area $S \approx 2$ mm^2);
- $L \approx 10$ m (order of magnitude of typical physical household lengths).

The displacement current term is thus $\varepsilon_0\mu_0\partial_t E \approx 1.3{\cdot}10^{-13}$ (SI units), for the larger value of the electric field above.

The conduction current term is thus $\mu_0 j \approx 9$ (SI units).

The order of magnitude of **B** resulting from the displacement current is: $B \approx L\varepsilon_0\mu_0\partial_t E \approx 1.3{\cdot}10^{-12}$ T, quite smaller than typical everyday values.

Therefore the rotor of **B** is typically affected much more by the conduction current than by the displacement current. The latter is often negligible with respect to the former.

The displacement current is obviously important whenever the conduction current is zero and|or the frequency is large.

Second Method

Assume a Ohmic conductor. In this case both the conduction current and the displacement current are proportional to the electric field (the displacement current to the frequency of the electric field also).

$$\left(\mu_0\sigma + \frac{\omega}{c^2}\right) E \approx \left(10^{-6}\sigma + 10^{-17}\omega\right) E \quad .$$

Also for a good insulator, such as pure water with $\sigma \approx 4{\cdot}10^{-6}$ (SI units), the conduction current is much larger than the displacement current for frequencies up to the kHz.

36.20.18.02 Estimate of the Faraday-Neuman-Lenz Term

Consider typical everyday values and a Sinusoidal|Cosinusoidal time-dependence of all the relevant quantities.

- Magnetic field:
  1. $B[t] \approx 0.5{\cdot}10^{-4}$ T (the magnetic field at the Earth surface);
  2. $B[t] \approx 2{\cdot}10^{-5}$ T (the magnetic field near a typical wire of current carrying a current of 1 A at a distance of 1 cm);
- $\nu \approx 10^2$ Hz (frequency of household electric power).
- $L \approx 10$ m (order of magnitude of typical physical lengths).

One finds, for the value of the time variation of the magnetic field near a wire $\nu B \approx 2{\cdot}10^{-3}$ (SI units), for the smaller value of the magnetic field above.

One finds then, for the induced electric field: $E \sim L\nu B \approx 2{\cdot}10^{-2}$ V/m.

The order of magnitude of the resulting E is thus: $\approx 10^{-2}$ V/m, not far from typical everyday values and close to the electric field inside a typical wire of current.

Note that in case the harmonic frequency is much larger than the assumed $\nu \approx 10^2$ Hz, the conclusions may change drastically.

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36.20.19
Some Conductibilities

Unit of measure of the resistance: Ohm (Ω).
Unit of measure of conductance: Siemens (S).
Relation: $\Omega = \mathrm{S}^{-1}$.
See tables in appendix: 7.13, 7.14, 7.15, 7.17, 9.16 and 9.17.

36.20.20
☒ A Junction Between Two Different Conductors

Consider two cylindrical conductors with the same section shape and dimension of area S, length L_1 and L_2, made of different materials with **conductivity** σ_1 and σ_2. The two conductors are coupled, on their bases, to made a conducting junction. A potential difference $\Delta\Phi \equiv \Phi_1 - \Phi_2 > 0$ is applied at the two extremes. The current then flows from the conductor 1 towards the conductor 2.

1. Determine the current at the discontinuity in steady conditions and no leakage currents at the surface.
2. Determine the electric fields at the discontinuity in steady conditions and no leakage currents at the surface.
3. Determine the charges at the discontinuity in steady conditions and no leakage currents at the surface.

WORKED SOLUTION

The normal component of $\mathbf{j}$ must be continuous, in steady conditions, as charges cannot escape from the interface between the two conductors. Therefore the normal component of the electric field must be discontinuous and a surface charge accumulates at the interface.

$$R \equiv R_1 + R_2 = \frac{1}{S}\left(\frac{L_1}{\sigma_1} + \frac{L_2}{\sigma_2}\right) \quad ,$$

$$I = \frac{\Delta\Phi}{R} \Longrightarrow |\mathbf{j}| = \frac{\Delta\Phi}{\left(\frac{L_1}{\sigma_1} + \frac{L_2}{\sigma_2}\right)} = \frac{\Delta\Phi\sigma_1\sigma_2}{\sigma_1 L_2 + \sigma_2 L_1} = |\mathbf{j}_1| = |\mathbf{j}_2| \quad ,$$

$$\mathbf{j} = \sigma\mathbf{E} \Longrightarrow \quad ,$$

$$E_1 = \frac{\Delta\Phi\sigma_2}{\sigma_1 L_2 + \sigma_2 L_1} \quad ,$$

$$E_2 = \frac{\Delta\Phi\sigma_1}{\sigma_1 L_2 + \sigma_2 L_1} \quad ,$$

$$\rho_S = \varepsilon_0 (E_2 - E_1) = \frac{\Delta\Phi\varepsilon_0 (\sigma_1 - \sigma_2)}{\sigma_1 L_2 + \sigma_2 L_1} = \varepsilon_0 \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right) |\mathbf{j}| \quad .$$

Note that the expression $\rho_S = \varepsilon_0 (E_2 - E_1)$, with our conventions, gives the correct sign of the charge. If $E_1 > E_2$ the flux across a small close surface crossing the interface is negative and the surface charge density is therefore negative. If $E_1 < E_2$ the flux across a small close surface crossing the interface is positive and the surface charge density is therefore positive.

Numerical Example

Copper and Silver, with $L_1 = L_2 = 0.1$ m and $\Delta\Phi = 100$ V.
One finds: $\rho_S = 10^{-8}$ C/m^2 $\simeq 7{\cdot}10^{11}$ electrons/m^2.

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36.20.21
☒ Relaxation Time Inside a Ohmic Homogeneous and Isotropic Conductor

This § is referenced at pages:
[1978, 2338]

©|J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|17.1.2||
©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|7.4||

Read also § 41-010.
Within a region of non-vanishing **conductivity** there can be no permanent distribution of free charges.

1. It is known that charge density is zero at interior points of conductors in equilibrium. How is this situation reached when there is a charge at some interior point ?
2. Suppose that at a certain time a free charge density ρ_0 exists inside a small region of a homogeneous and isotropic medium having **conductivity** σ. Determine the time development of the free charge density.

WORKED SOLUTION

Inside a homogeneous, isotropic and linear medium having **conductivity** σ one has: $\mathbf{j} = \sigma\mathbf{E}$, whenever only the effects of electric fields are relevant. From the charge continuity equation and from Maxwell first equation one finds:

$$\rho[\mathbf{x}, t] = \rho_0[\mathbf{x}] \exp\left[-\frac{\sigma}{\varepsilon_0} t\right] \quad ,$$

implying a time constant, τ, (relaxation time):

$$\boxed{\tau = \frac{\varepsilon_0}{\sigma}} \quad .$$

Therefore any initial free charge distribution inside a conductor decays exponentially with time at any point in a manner wholly independent of the applied field. If the charge is initially zero it remains zero forever.
This shows that within any region of non zero **conductivity** there can be no permanent distribution of free charge. If free charges are initially concentrated inside some small region their density starts to diminish exponentially but the charge cannot reappear at any other point within the conductor. Outside the region the charge density must vanish identically, as it was zero at the beginning. Therefore, outside the initial region, the flow is divergence-less. The flow will be arrested at the surface and this surface charge will start to appear at the exact instant that the interior charge begins to fade away, for the total charge is constant.
In all but the poorest conductors the relaxation time is exceedingly short:

- Copper: $\sigma^{-1} = 1.7{\cdot}10^{-8}\ \Omega \cdot \mathrm{m}$; it follows: $\tau \approx 1{\cdot}10^{-20}$ s.
- Quartz: $\sigma^{-1} = 1{\cdot}10^{16}\ \Omega \cdot \mathrm{m}$; it follows: $\tau \approx 1{\cdot}10^{5}$ s.

This fact justifies the use of the quasi-static approximation (zero charge and zero electric field inside any conductor) in many cases, not only is striclty static situations.

36.20.22
Surface Current in Ohmic Conductors

This § is referenced at pages:
[1728]

Ohmic conductors might carry surface currents if and only if they are perfect conductors.
It is an essential assumption that it is a Ohmic conductor.
In fact, consider a thin surface layer where the current flows, of thickness h:

$$\mathbf{j}_S = \mathbf{j}h = h\sigma\mathbf{E} \quad .$$

The above relations shows that in order to have a finite surface current, $0 < |\mathbf{j}_S| < \infty$, in the limit $h \to 0$, with a non infinite electric field the **conductivity** must tend to infinity.

Also note that in a perfect conductor $\mathbf{E}$ is zero inside, therefore the tangential component, which is continuous and the electric field is tangential in stationary conditions, is zero just outside, as well as inside. The fact that $\mathbf{E}$ is zero in a perfect conductor makes still stronger the previous conclusion that **conductivity** must tend to infinity.

36.20.23
☒ A Moving Bar in a Constant and Uniform Magnetic Field (1)

This § is referenced at pages:
[1769, 1769, 2394, 2395]

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|36.7||

Compare with § 36.20.24.
Consider a constant and uniform magnetic field, directed along the direction of the z axis.
A closed rectangular circuit is located in the xy plane. It is made of a U-shaped fixed part, having the short side with length a along the y direction and the two long sides along the x direction, plus a bar oriented along the y direction which can move along the x direction while still keeping the circuit electrically closed.
The bar moves in the positive x direction with speed $\mathbf{V}$ thanks to the intervention of a suitable external force. Let us determine the induced emf and describe the situation both from the point of view of the fixed observer and the observer moving with the bar.

Description of the Observer Seeing the Moving Bar

The observer at rest with the U-shaped part of the circuit is called, in short, the stationary observer.
The Lorentz force on the single free charge, with drift velocity $\mathbf{v}_D$, inside the circuit is:

$$\mathbf{F} = q\,(\mathbf{V} + \mathbf{v}_D) \wedge \mathbf{B} \quad ,$$

with

$$\mathbf{F}_{\parallel} = q\,\mathbf{V} \wedge \mathbf{B} \parallel (-\mathbf{e}_2) \qquad \mathbf{F}_{\perp} = q\,\mathbf{v}_D \wedge \mathbf{B} \parallel (-\widehat{\mathbf{e}}_1) \parallel (-\mathbf{V}) \quad .$$

The Lorentz force catalyzes the process putting in motion the free charges. Once the free charges are moving along $(-\mathbf{e}_2)$ they become subject to a component of the Lorentz force directed along $(-\mathbf{V})$.
In a steady situation, when charges move at constant speed, that is the current is uniform after any transient has disappeared, the two forces $\mathbf{F}_{\parallel}$ and $\mathbf{F}_{\perp}$ are counterbalanced by two equal and opposite forces, as in this situation the total force on any charge must be zero:

$$\mathbf{F}_{\parallel} + \mathbf{f} = q\,\mathbf{V} \wedge \mathbf{B} + \mathbf{f} = 0 \qquad \mathbf{f}\text{: friction encountered by the free moving charge} \quad ,$$

$$\mathbf{F}_{\perp} + \mathbf{N} = q\,\mathbf{v}_D \wedge \mathbf{B} + \mathbf{N} = 0 \qquad \mathbf{N}\text{: force exerted by the wire on the free charge} \quad .$$

Note that both forces $\mathbf{f}$ and $\mathbf{N}$ are forces provided by the wire and acting on the free charges and that $\mathbf{N}$ can arise from the internal electric field generated by the unbalance of charges due to force $\mathbf{F}_{\perp} \parallel (-\mathbf{V})$ (Hall effect). The force $\mathbf{N}$ prevents the free charges from exiting the wire. In the usual case of a crystalline

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wire the external agent move the wire structure while the free electrons are moved backwards and the unbalance creates the Hall field which in turn keeps the electrons from leaving the wire.

Note that the Lorentz force never does any work. The positive work on the free charges is done by force $\mathbf{N}$ ($\pi[\mathbf{N}] > 0$), in fact, while the negative work on the free charges is done by force $\mathbf{f}$ ($\pi[\mathbf{f}] < 0$). The total power on the (point) free charges is zero in steady conditions (no change of kinetic energy):

$$\pi[\mathbf{f}] \equiv \mathbf{f}\cdot(\mathbf{V}+\mathbf{v}_{\text{D}}) = \mathbf{f}\cdot\mathbf{v}_{\text{D}} < 0 \quad ,$$
$$\pi[\mathbf{N}] \equiv \mathbf{N}\cdot(\mathbf{V}+\mathbf{v}_{\text{D}}) = \mathbf{N}\cdot\mathbf{V} > 0 \quad ,$$
$$\pi = (\mathbf{f}+\mathbf{N})\cdot(\mathbf{V}+\mathbf{v}_{\text{D}}) = \mathbf{f}\cdot\mathbf{v}_{\text{D}} + \mathbf{N}\cdot\mathbf{V} = \pi[\mathbf{f}] + \pi[\mathbf{N}] = -q\,\mathbf{v}_{\text{D}}\wedge(\mathbf{V}\wedge\mathbf{B}) - q\,\mathbf{V}\wedge(\mathbf{v}_{\text{D}}\wedge\mathbf{B}) = 0 \quad ,$$
$$\pi[\mathbf{F}] = 0 \quad .$$

Note that the force $\mathbf{N}$ is, **from the mechanical point of view, a constraint force which can do work because the constraint is moving**.

The energy dissipated (by Joule effect or other means) by the circuit, in the end, comes from the work done on the moving bar by the external agent which keeps the bar moving at constant speed. **The role of the Lorentz force is essential in converting this work into kinetic energy of the free charges which is finally dissipated by the circuit**. This is similar to the role of static friction in a rolling cylinder.

Description of the Observer Moving With the Moving Bar

At startup, the observer moving with the moving bar (in short: the moving observer) does not see any force along $\mathbf{e}_2$ in the bar, as the charge is not moving, and thus she cannot explain the start of the current in the conductor via the magnetic part of the Lorentz force. After the current has started, she only sees a straight standard Hall effect, but she cannot explain the startup of the current when the bar is set into motion. In steady conditions, she needs to invoke and electric field to explain the charge moving at constant speed along the wire against the friction and producing the work which is dissipated by Joule effect. Moreover, she needs to invoke an electric field to startup and maintain the current in the circuit, as the relativity principle impose that the source of the emf must be located in the bar, the same as for the stationary observer.

So one might think that the observer moving with the bar also sees an electric field, as he is moving with respect to a pure magnetic field. Now, there is a big trouble, because she knows that any electric field must be produced either by charges or by a time changing magnetic field, and she sees none of them.

Moreover, she seems to have another problem, because if we assume a constant and uniform electric field, as it sounds reasonable given that we started from a constant and uniform magnetic field, this gives rise to an opposite effect on the far side of the U-shaped part of the circuit, if that is immersed into the same magnetic field too. In fact, one needs to remember that, when considering currents, one must always consider the full, closed circuit, that is the return path of currents.

The problem on the other side, however, is not there when one realizes that the far side of the U-shaped part of the circuit is in motion, with respect to the moving observer, with velocity $(-\mathbf{V})$, in a magnetic field. In fact, if we assume that the magnetic field seen by the moving observer is the same as the one seen by the stationary observer, an approximation which turns out to be correct only for non relativistic velocities $\mathbf{V}$, we can conclude that in the far side of the U-shaped part of the circuit the force due to the electric field exactly balances the part of force due to the magnetic field of the Lorentz force, that is the total Lorentz force on any charge is zero.

In fact the relativistic transformations of the ElectroMagnetic fields, equations (61.03.01), (61.03.02), (61.03.03), (61.03.04), give, in the non-relativistic limit with $\gamma \simeq 1$:

$$\boxed{\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} = 0} \quad ,$$
$$\boxed{\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} = 0} \quad ,$$
$$\boxed{\mathbf{E}'_{\perp} = \gamma\left(\mathbf{E} + c^{+1}\,\boldsymbol{\beta}\wedge\mathbf{B}\right)_{\perp}} = \gamma\left(c^{+1}\,\boldsymbol{\beta}\wedge\mathbf{B}\right)_{\perp} = \gamma c\,\boldsymbol{\beta}\wedge\mathbf{B} \simeq \mathbf{V}\wedge\mathbf{B} \quad ,$$
$$\boxed{\mathbf{B}'_{\perp} = \gamma\left(\mathbf{B} - c^{-1}\,\boldsymbol{\beta}\wedge\mathbf{E}\right)_{\perp}} = \gamma\mathbf{B} \simeq \mathbf{B} \quad .$$

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For this observer the forces on the far side of the circuit, taking into account relativistic effects, read:

$$q\gamma\,\mathbf{v}\wedge\mathbf{B} + q\,(-\mathbf{v})\wedge(\gamma\mathbf{B}) = 0 \quad .$$

Note that the **Galilei Transformation** , equations (58.04.01) provide the same result.

36.20.24
☒ A Moving Bar in a Constant and Uniform Magnetic Field (2)

This § is referenced at pages:
[1767, 1808, 2394, 2395]

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|36.7||

Compare with § 36.20.23.

An even simpler case than section § 36.20.23 is when the rectangular circuit is a rigid one and it has only one of the short sides inside a magnetic field. The circuit is removed with constant speed. In this case all happens in the short side of the circuit, as it is inside the magnetic field: one observer only sees the magnetic force while the other one only sees the electric force.

Note that the **Galilei Transformation** , equations (58.04.01) provide the same result.

36.20.25
ElectroMagnetic Induction Law: Some Critical Cases

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||

In order to respect the mathematical derivation of the equivalence between the integral and differential formulations one needs to identify a line circuit, actually traveled by the real charges, corresponding to the mathematical circuit, so that **v** can be replaced by **u** in the integral of the emf.

Sliding Contact on a Solenoid

©|J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|14.5.2||

A simple example shows how much care should be used when applying the Faraday-Neumann-Lenz to real circuits which are not simple line-circuits of invariable constitution.

A solenoid, made out of an helical winding, where a constant current I passes has two contacts: the first one is fixed at one the two extremes, while the second one can slide along the helical windings. The magnetic field flux is changing but it is clear that there cannot be any induced EM-motive force because there is no change of magnetic field anywhere nor any wire moving inside a magnetic field. Note also that the magnetic field flux is changing as a step function.

Homopolar Generator or Unipolar Generator or Disk Dynamo or Faraday Disc

©|J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|14.5.2||

What is the circuit? You cannot define a line such that current passes there, as the circuit is not a line circuit. The usual contours used to use the cut-flux concept makes no sense: there is no current flow along the circumference which is winding into itself.

If one defines a circuit with a fixed line joining the center and the edge of the Faraday disk there is no change of flux inside, while one knows that there is an indiced emf!

The correct treatment can be found by using equation (36.10.06).

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36.20.26
Hall Effect and Hall Effect Sensor

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 5.39||

A Hall effect sensor is a transducer that varies its output voltage according to the magnetic field where it is immersed.

36.20.27
ElectroMagnetic Induction Law Its Applications and Consequences

Metal Detectors

A metal detector is based on the principles of ElectroMagnetic induction. Metal detectors contain inductors used to produce magnetic fields. The magnetic fields, when interacting with a high-conductivity object, induce currents which produce a magnetic field which is finally detected by the instrument.

Focault Currents

EM Braking

With respect to standard methods it has the advantage that it does not produce dust: useful in metro systems!

Induction Oven/stoves/cooktops

Electric Guitar

©|Am.J.Phys, , ..., ..., ...Ed., ..., Vol. 77, No. 2, pp. 144, February 2009|||

Ground Fault Interrupter

How can one be sure that the forward and backward currents are perfectly in-phase in order to almost cancel the total current inside the GFI at any time?

Anti-Theft Systmes

36.20.28
Il Generatore Omopolare (Disco Di Faraday O Ruota Di Barlow)

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|36.36||

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|P14.9-P14.10||

Un disco circolare di rame, con raggio R e spesso s, ha **conductivity** $\sigma = 5.9{\cdot}10^7\ \Omega^{-1}\cdot\mathrm{m}^{-1}$. Il disco ruota attorno al suo asse con velocità angolare costante $\omega = 5\ \mathrm{turn/s}$ in un campo magnetico esterno, B, costante ed omogeneo, perpendicolare al piano del disco. Si applicano due contatti elettrici striscianti tra l'asse del disco, punto C, e un punto nel bordo, punto B. Tali contatti chiudono elettricamente il circuito attraverso il disco su una impedenza esterna Z.

Determinare la f.e.m. indotta e determinare la coppia che deve applicare chi fa girare il disco a velocità angolare costante quando la corrente nel circuito esterno vale i.

Si assuma, verificando l'assunzione, che la velocità di deriva degli elettroni nel rame sia piccola rispetto alla velocità di trascinamento del disco, eccetto per i punti molto vicino all'asse.

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36.20.29
Ideal Plane Capacitor

This § is referenced at pages:
[1783, 1785]

An ideal plane capacitor is made of two identical parallel plane conducting plates of infinite extension charged with charges of equal magnitude and opposite sign ($Q_+ \equiv +|Q|$ and $Q_- \equiv -|Q|$). Let the z axis be perpendicular to the plates.

1. Show that the electric field is along z and uniform inside the capacitor.

Read also § 36.20.63.

WORKED SOLUTION

The system is invariant by translation along the x and y axes. Introduce a orthonormal Cartesian Coordinates Coordinate System. The electric field must be along z and it cannot depend on neither x nor y:

$$\mathbf{E}[\mathbf{r}] = \widehat{\mathbf{e}}_3 E_z[z] \quad .$$

In between the armatures the electric field is divergence-less, and therefore, from the expression of the divergence in orthonormal orthonormal Cartesian Coordinates Coordinate System:

$$\frac{\partial E_z}{\partial z} = 0 \quad ,$$

that is the electric field is uniform inside the capacitor.

36.20.30
Capacitance Coefficients via the Scalar Electric Potential

This § is referenced at pages:
[1746]

©|I.Smolić and B.Klajn|Capacitance Matrix Revisited|*WEB - URL*|

The **superposition principle** of ElectroMagnetic fields in linear systems is assumed to hold, as it is appropriate for linear systems.

Note that in presence of non-linear effects, for instance of materials with non-linear response, the proportionality between potential difference and charge is lost as the **superposition principle** does not apply.

Assume zero potential at infinity.

Consider a set of conductors with a common reference potential (typically zero at infinity).

36.20.30.01 Self-Capacitance

The self-capacitance of a conductor is defined in terms of the potential of the electric field as:

$$Q \equiv C\Phi \propto \Phi \quad .$$

The charge is proportional to the potential, as it can be shown from **Poisson equation** .

36.20.30.02 Mutual-Capacitance

Consider two conductors.

Let us use the notation $\{\}_{xy}$ to denote that x is the source of an effect seen by y: $(\{\}_{xy} \equiv \{\}_{x\to y})$.

Their mutual-capacitance is defined in terms of the charge induced on one conductor for a given potential of the other:

$$Q_{21} \equiv C_{21}\Phi_2 \qquad Q_{12} \equiv C_{12}\Phi_1 \quad .$$

In the case of an arbitrary number of conductors it can be shown in general that the **capacitance** and mutual-capacitance coefficients are subject to the following relations:

$$\boxed{C_{kk} > 0 \qquad C_{kj}^{\cdot\cdot} = C_{jk}^{\cdot\cdot} < 0 \qquad C_{kk} > -\sum_{j\neq k} C_{jk}^{\cdot\cdot}} \quad .$$

36.20.31 Capacitance of a Pair of Conductors

This § is referenced at pages:
[1746]

Consider two conductors, 1 and 2. The charge on one conductor is given by the sum of the charge produced by the potential of the conductor itself plus the charge produced by the potential of the other conductor:

$$Q_1 = Q_{11} + Q_{21} = C_1\Phi_1 + C_{21}\Phi_2 \quad ,$$
$$Q_2 = Q_{12} + Q_{22} = C_{12}\Phi_1 + C_2\Phi_2 \quad .$$

The above relation is justified by the **superposition principle** of ElectroMagnetic fields in linear systems.

The matrix thus defined is called the capacitance matrix; the concept can be easily extended to an arbitrary number of conductors.

The **capacitance** coefficients only depends on the geometry of the system and on the properties of the (linear) media present.

Note, in particular, that, in linear systems:

$$\boxed{Q_{11} \propto \Phi_1 \qquad \text{for } \Phi_2 = 0 \qquad Q_{21} \propto \Phi_2 \qquad \text{for } \Phi_1 = 0} \quad ,$$
$$\boxed{Q_{12} \propto \Phi_1 \qquad \text{for } \Phi_2 = 0 \qquad Q_{22} \propto \Phi_2 \qquad \text{for } \Phi_1 = 0} \quad .$$

It can be shown that:

$$\boxed{C_{11} \geq 0 \qquad C_{22} \geq 0 \qquad C \equiv C_{21} = C_{12} \leq 0} \quad .$$

36.20.31.01 Complete Charge Induction and the Capacitor

A capacitor is a system of two conductors having complete charge induction: $Q_1 + Q_2 = 0$

The **capacitance** of a capacitor can be calculated in terms of C_1, C_2 and $C \equiv C_{12} = C_{21}$, for $Q_1 = +Q > 0$ and $Q_2 = -Q < 0$ as:

$$\boxed{Q = \left(\frac{C^2 - C_1C_2}{C_1 + C_2 + 2C}\right)(\Phi_2 - \Phi_1)} \quad .$$

In fact inversion of the capacitance matrix leads to:

$$V_1 = \frac{CQ_2 - Q_1C_2}{C^2 - C_1C_2} \quad ,$$
$$V_2 = \frac{CQ_1 - Q_2C_1}{C^2 - C_1C_2} \quad .$$

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36.20.32
Capacitance and Energy Balance With Self/Mutual Capacitance

This § is referenced at pages:
[1746]

36.20.33
Inductance Coefficients via the Magnetic Vector Potential

This § is referenced at pages:
[1746]

The **superposition principle** of ElectroMagnetic fields in linear systems is assumed to hold, as it is appropriate for linear systems.

Note that in presence of non-linear effects, for instance of materials with non-linear response, the proportionality between concatenated flux and current is lost as the **superposition principle** dose not apply.

Consider a set of thin closed current loops.

36.20.33.01 Self-Inductance

The self-inductance of a thin closed current loop is defined in terms of the concatenated flux of the magnetic field as:

$$\Phi \equiv LI = \oint_{\partial\Sigma} \mathbf{A} \cdot \mathrm{d}\mathbf{L} \propto I \quad .$$

The above relation clearly shows that the self-inductance does not depend on the current as the **vector magnetic potential** produced by a thin current wire is proportional to the current as it is clear from equation (**??**).

36.20.33.02 Mutual-Inductance

Consider two thin closed circuit loops carrying constant currents.

Let us use the notation $\{\}_{xy}$ to denote that x is the source of an effect seen by y: ($\{\}_{xy} \equiv \{\}_{x\to y}$).

Their mutual-inductance is defined in terms of the flux of the magnetic field of one circuit concatenated to the other one:

$$\Phi_{21} \equiv M_{21} I_2 \qquad \Phi_{12} \equiv M_{12} I_1 \quad .$$

The above relation clearly shows that the mutual-inductance does not depend on the current as the **vector magnetic potential** produced by a thin current wire is proportional to the current as it is clear from equation (**??**).

36.20.34
Inductance of a Pair of Circuits

This § is referenced at pages:
[1746]

Consider two thin closed circuit loops, 1 and 2. The flux of the magnetic field concatenated to one circuit is given by the sum of the flux given by the magnetic field produced by circuit itself plus the flux given by the magnetic field produced by the other circuit:

$$\Phi_1 = \Phi_{11} + \Phi_{21} = L_1 I_1 + M_{21} I_2 \quad ,$$
$$\Phi_2 = \Phi_{12} + \Phi_{22} = M_{12} I_1 + L_2 I_2 \quad .$$

The above relation is justified by the **superposition principle** of ElectroMagnetic fields in linear systems.

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The matrix thus defined is called the inductance matrix; the concept can be easily extended to an arbitrary number of circuits.

The inductance coefficients only depends on the geometry of the system and on the properties of the (linear) media present.

Note, in particular, that, for linear systems:

$$\boxed{\Phi_{11} \propto I_1 \quad \text{for } I_2 = 0 \qquad \Phi_{21} \propto I_2 \quad \text{for } I_1 = 0} \quad ,$$

$$\boxed{\Phi_{12} \propto I_1 \quad \text{for } I_2 = 0 \qquad \Phi_{22} \propto I_2 \quad \text{for } I_1 = 0} \quad .$$

36.20.35
Inductance and Energy Balance With Self/Mutual Inductance

This § is referenced at pages:
[1746]

Consider two closed fixed current circuits, with self-inductances L_1 and L_2 and mutual inductance M.

We want to determine the energy stored in the circuit by the generators when the currents I_1 and I_2 are switched on.

The result will show that the mutual inductance satisfies the relation:

$$\boxed{|M| \leq \sqrt{L_1 L_2}} \quad .$$

Let W be the work done by the two constant voltage generators (which are assumed to be ideal voltage generators).

Let us consider the energy balance between time $t = 0$, when $I_1 = I_2 = 0$, and the generic time $\bar{t}$.

Choose the conventional positive flow of the current as the one the voltage generators will push the current.

$$\mathcal{E}_1 - L_1 \frac{\mathrm{d}I_1}{\mathrm{d}t} - M \frac{\mathrm{d}I_2}{\mathrm{d}t} - R_1 I_1 = 0 \quad ,$$

$$\mathcal{E}_2 - L_2 \frac{\mathrm{d}I_2}{\mathrm{d}t} - M \frac{\mathrm{d}I_1}{\mathrm{d}t} - R_2 I_2 = 0 \quad ,$$

$$\mathrm{d}W = (\mathcal{E}_1 I_1 + \mathcal{E}_2 I_2)\ \mathrm{d}t \quad ,$$

$$W = \int_0^{\bar{t}} \mathrm{d}W = M I_1 I_2 + \frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} + \int_0^{\bar{t}} \left(R_1 I_1^2 + R_2 I_2^2\right)\ \mathrm{d}t \quad ,$$

[Work done by the generators] = [Energy stored in the magnetic field] + [Energy dissipated by Joule effect] .

As the generators will do positive work in order to let the current flow and build the magnetic fields the energy stored in the magnetic field must be positive:

$$M I_1 I_2 + \frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} \geq 0 \implies M^2 < L_1 L_2 \quad .$$

36.20.36
Mutual Inductance Coefficient of Two Circuits

This § is referenced at pages:
[1746, 1784]

Consider two thin closed circuit loops, Γ_1 and Γ_2, carrying currents I_1 and I_2. The mutual inductance coefficient of circuit Γ_1 to circuit Γ_2 is defined in terms of the flux of the magnetic field produced by the circuit Γ_1 and concatenated to circuit Γ_2, Φ_{12}, as:

$$\boxed{M_{12} \equiv \frac{\Phi_{12}}{I_1} = \frac{1}{I_1} \oint_{\Gamma_2} \mathbf{A}_1[\mathbf{y}_2] \cdot \mathrm{d}\mathbf{L}_2[\mathbf{y}_2] = \frac{\mu_0}{4\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{\mathrm{d}\mathbf{L}_1[\mathbf{y}_1] \cdot \mathrm{d}\mathbf{L}_2[\mathbf{y}_2]}{|\mathbf{y}_1 - \mathbf{y}_2|}} \qquad (36.20.09)$$

→ 1775

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Equation (36.20.09), called **Neumann formula** , clearly show the symmetry of the mutual inductance coefficient:

$$\boxed{M_{21} = M_{12}} \quad .$$

The mutual inductance only depends on the flux lines concatenated with both circuits.

36.20.37
Self-Inductance of a Loop

©|WGreinerEtAl ClassicalElectrodynamics|Example 14.3||

36.20.38
Mutual Inductance of a Pair of Loops

©|WGreinerEtAl ClassicalElectrodynamics|Example 14.2||

36.20.39
☒ Scalar Potential on the Axis of a Circular Charged Ring

This § is referenced at pages:
[1779]

Calculate the **scalar electric potential** on the axis of a thin plane circular ring of radius R carrying a uniform linear charge density λ.

From the knowledge of the **scalar electric potential** on the axis what can one say about the electric field?

WORKED SOLUTION

The electric field on the axis is directed along the axis by symmetry.

$$\mathrm{d}\Phi[0,0,z] = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,\mathrm{d}L}{\sqrt{R^2+z^2}} \quad ,$$

$$\Phi[0,0,z] = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}} \simeq \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}\left(1 - \frac{z^2}{2R^2} + \frac{3z^4}{8R^4} + \mathcal{O}\left(\frac{1}{R}\right)^6\right) \quad ,$$

$$\Longrightarrow E_z = -\frac{\partial\Phi}{\partial z} = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(R^2+z^2)^{3/2}} \qquad \text{the change of sign at } z=0 \text{ is correct;} \quad ,$$

$$|z| \gg R \Longrightarrow E_z \simeq \frac{1}{4\pi\varepsilon_0}\frac{Qz}{|z|^3} \quad .$$

It is very important to note that it is not possible, in general, to derive the electric field from the knowledge of the potential in some part of space only, as one must be to take the gradient, for which the knowledge of potential at one point only is not enough. However, in this problem we know the **scalar electric potential** at all the points of the z axis, so that the directional derivative along the z axis, that is the electric field component along the z axis, can be calculated. Moreover, thanks to symmetry considerations, we know that, in this problem, the electric field is directed along the z axis, at any point of the z axis.

Note that for motion along the z axis $z=0$ is a stable equilibrium position for point particles of opposite sign with respect to the charge of the ring. Our calculation is not enough to say anything about motions in the xy plane.

The development of the potential in series written above shows that at large distance the field

resembles the field of a point charge, that the **Dipole** term is missing and the **Quadrupole** term is not zero.
Study the limits: $|z| \ll R$ and $R \to +\infty$.

36.20.40
☒ Vector Potential of a Straight Infinite Wire With Circular Section

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|§ 12, pag. 218||

Calculate the **vector magnetic potential** produced by an infinite straight circular current wire having radius R and carrying a constant current I with uniform current distribution.
Note that one cannot use equation (36.13.03) since the current itself extends to infinity.

WORKED SOLUTION

Show first that the **vector magnetic potential** is zero on the axis, by symmetry arguments and|or direct integration. The use the known magnetic field to calculate the **vector magnetic potential** .

36.20.41
☒ Vector Potential of an Infinite Straight Circular Solenoid

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|E5.12||

Calculate the **vector magnetic potential** produced by an infinite straight circular solenoid having radius R and carrying a constant current I with n turns per unit length.
Note that one cannot use equation (36.13.03) since the current itself extends to infinity.

36.20.42
☒ Vector Potential on the Axis of a Circular Current Ring

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|11.4.4||

Calculate the **vector magnetic potential** on the axis of a thin plane circular ring of radius R carrying a steady current I.
From the knowledge of the **vector magnetic potential** on the axis what can one say about the magnetic field?

WORKED SOLUTION

At any point the axis of a circular current ring the **vector magnetic potential** is proportional to $\oint \mathbf{dL}$ that is zero for a circular current ring:

$$\mathbf{A} = \frac{\mu_0}{4\pi} I \oint \frac{\mathbf{dL}}{r} \quad .$$

In fact, the distance factor is constant and can be taken out the integral.
Nothing can be said about the magnetic field from the knowledge of the vector potential on a line as the rotor cannot be calculated from the knowledge of the values on a line only.
In this problem one needs to know the partial derivatives with respect to the x and y axes. In fact one should calculate the infinitesimal circulation per unit area in a plane perpendicular to the z axis, but the **vector magnetic potential** is only known on the z axis.

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36.20.43
☒ ElectroStatic Quadrupole Field (1)

This § is referenced at pages:
[1779]

Let $\Phi[x, y, z] = -xy$. Draw equipotential surface in 2D and field lines.

WORKED SOLUTION

Equipotential lines and field lines are drawn in figures 36.14 and 36.15.

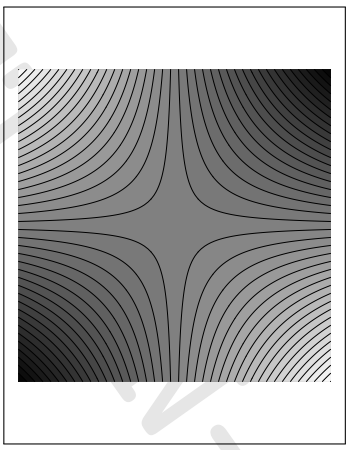

Figure 36.14: Electrostatics Quadrupole potential

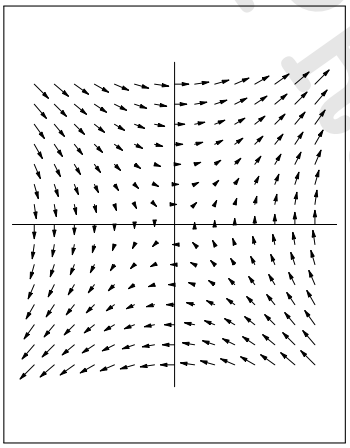

Figure 36.15: Electrostatics Quadrupole field

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36.20.44
☒ ElectroStatic Quadrupole Field (2)

This § is referenced at pages:
[1779]

Consider four point charges with the same magnitude, two positive and two negative ones. They are located at the vertexes of a square of side d, with the two couples of the same sign at opposite vertexes.

Let a Coordinate System centered at the center of the square with the x axis passing by the two positive charges and the y axis passing by the two negative charges.

1. Develop the potential in Taylor series around the center of the square and show that the center of the square is not a point of stable equilibrium.
2. Show that the electric potential is of the type:
$$\Phi[x,y,z] = \frac{1}{4\pi\varepsilon_0}\frac{6\sqrt{2}q}{d^3}\left(x^2-y^2\right) \quad .$$
Let $\Phi[x,y,z] = x^2 - y^2$.
3. Draw equipotential surface in 2D and field lines.

WORKED SOLUTION

Equipotential lines and field lines are drawn in figures 36.16 and 36.17.

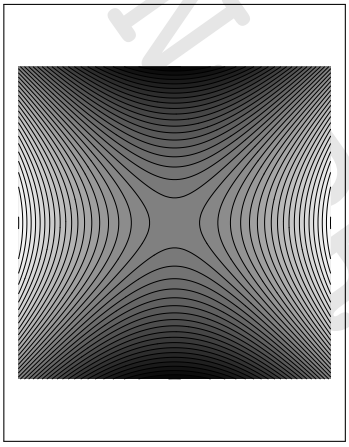

Figure 36.16: Electrostatics Quadrupole potential (another)

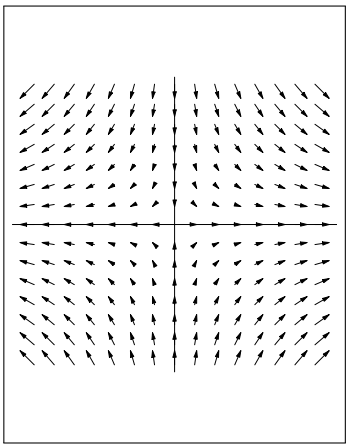

Figure 36.17: Electrostatics Quadrupole field (another)

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36.20.45

☒ ElectroStatic Field Produced by Four Charges

1. Show that the ElectroStatic fields in section § 36.20.43 and in section § 36.20.44 (called ElectroStatic **Quadrupole** fields) are the fields near the center of an arrangement of four identical point charges, two positive and two negative, at the edges of a square, as in figure 36.18.

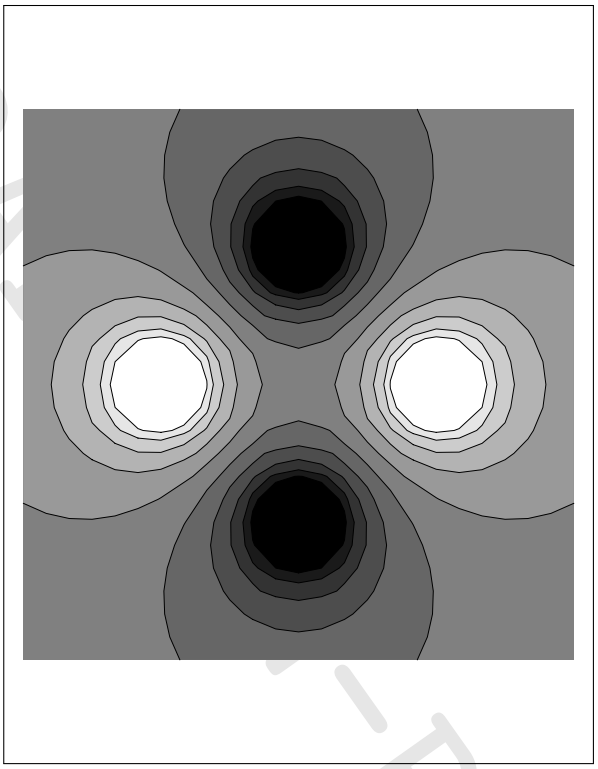

Figure 36.18: FIGURE

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36.20.46

☒ Motion On The Axis Of A circular uniformly charged Ring

This § is referenced at pages:
[Never referenced.]

Consider the situation of § 36.20.39.
A point charge is constrained to move on the axis of the ring, assumed to be of positive charge.

1. Study the equilibrium positions of the point charge, for both positive and negative charge.
2. Study the stability of equilibrium.
3. For a point charge starting at infinity with variable kinetic energy, determine the motion for different values of the initial kinetic energy and for both positive and negative charge.

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36.20.47
☒ EM Fields at Large Distances From Capacitors and Inductors

This § is referenced at pages:
[1782, 1782]

The electric field at a large distance from any capacitor is an **Electric Dipole** field, as it is the case for the electric field at a large distance from any limited time-independent charge distribution.

The magnetic field at a large distance from any solenoid is a **Magnetic Dipole** field, as it is the case for the electric field at a large distance from any limited time-independent current distribution.

36.20.48
☒ Magnetic Monopole Term

This § is referenced at pages:
[1738]

In equation (36.13.03) the first term in the multipole development is:

$$\iiint \mathbf{j}[\mathbf{r}]\, \mathrm{d}V \quad . \tag{36.20.10}$$

Show that is it always zero.

36.20.49
☒ Magnetic Dipole Moment of a Thin Closed Plane Current Loop (1)

This § is referenced at pages:
[1781]

Read § 16-009.

Show that the definition (36.14.09) reduces to the usual definition of **Magnetic Dipole** moment for a plane thin current loop in terms of the current, I, and the vector $\mathbf{S}$ describing the loop:

$$\boxed{\mathbf{m} = I\mathbf{S} \qquad \text{for a plane thin current loop}} \quad .$$

In fact for a thin plane current loop:

$$\boxed{\mathbf{m} = \frac{1}{2}\iiint (\mathbf{x}\wedge\mathbf{j}[\mathbf{x}])\ \mathrm{d}V_{\mathbf{x}} = \frac{I}{2}\oint \mathbf{x}\wedge \mathrm{d}\mathbf{L}[\mathbf{x}]} \quad .$$

In fact the integral can be evaluated on the plane surface having the plane current loop as a boundary, exploiting the geometrical interpretation of the vector product of two vectors, by taking an origin in the plane of the circuit. The integrand is the area of the triangle defined by $\mathbf{x}$ and $\mathrm{d}\mathbf{L}$. The integration over the closed circuit gives its surface.

The formal and more general derivation can be found in section § 36.20.51.

Read also § 36.20.50.

If the origin of the Coordinate System is outside the plane of the loop the result does not change as it is obvious from the geometrical interpretation and fact that

$$\oint \mathrm{d}\mathbf{L} = \mathbf{0} \quad .$$

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36.20.50
☒ Magnetic Dipole Moment of a Thin Closed Plane Current Loop (2)

This § is referenced at pages:
[1780]

Read § 16-009.
Read also sections § 36.20.49, § 36.20.51.

- The calculation of the **Magnetic Dipole** moment of a current loop leads to the integral
$$\oint \mathbf{r} \wedge \mathbb{d}\mathbf{r} \quad .$$
Show that the scalar magnitude of the integral equals twice the area of the enclosed surface.
Use one of the forms of equation (16.07.10) or directly equations (16.13.05) to apply the result
$$\frac{1}{2}\oint_{\partial\Sigma} \mathbf{r} \wedge \mathbb{d}\mathbf{r} = \iint_{\Sigma} \mathbb{d}\mathbf{S} \qquad \text{with } \partial\Sigma = \Gamma \quad ,$$
to the Magnetic Dipole moment of a thin plane current loop, where the right-hand integral is independent of the surface Σ.
- Evaluate
$$\oint_{C} \mathbf{r} \wedge \mathbb{d}\mathbf{r} \quad ,$$
on a plane curve using elementary geometrical methods.

36.20.51
☒ Magnetic Dipole Moment of a Thin Closed Arbitrary Non-Plane Current Loop

This § is referenced at pages:
[1780, 1781]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Example 5.13||

Read § 16-009.
Show, using theorem (16.07.09), that the definition (36.14.09) can be re-written, for an arbitrary thin current loop carrying a current I and described by the curve Γ, as:
$$\mathbf{m} = I \iint_{\Sigma} \mathbb{d}\mathbf{S} \qquad \text{with } \Sigma \text{ any surface such that } \partial\Sigma = \Gamma \quad .$$
In fact if the current loop is not plane the component of the **Magnetic Dipole** moment along any axis is proportional, via the current, to the projected area of the circuit on the plane perpendicular to the axis.
The above integral is obviously independent of the choice of the surface.

36.20.52
☒ Dipole Moments: Independence on the Origin

This § is referenced at pages:
[1739, 1739, 1740]

The definition of **Electric Dipole** | **Magnetic Dipole** moments, equation (36.14.06) and (36.14.09), do not give any prescription on the choice of the origin of the Coordinate System.
Show that the Dipole Moment is indeed independent of this choice:
1. for any ElectroStatic charge distribution with with compact support and zero total charge;
2. for any MagnetoStatic steady current distribution with compact support.

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WORKED SOLUTION

Consider the most general change of Coordinate System (roto-translation, Poincarè group):

$$\mathbf{x} \longrightarrow \mathbf{x}' \equiv \mathbb{R}\mathbf{x} + \mathbf{a} \quad .$$

- $$\mathbf{p} \longrightarrow \mathbf{p}' = \mathbf{p}' + Q\mathbf{a} \quad . \tag{36.20.11}$$
- Calculate $\mathbf{m}$ using this new Coordinate System, first assuming that $\mathbb{R} = \mathbb{I}$ (that is no rotation, translation only):

  $$\mathbf{m}' = (1/2) \iiint \mathbf{x}' \wedge \mathbf{j}'[\mathbf{x}'] \, \mathrm{d}V'_{\mathbf{x}} = (1/2) \iiint (\mathbf{x} + \mathbf{a}) \wedge \mathbf{j}[\mathbf{x}] \, \mathrm{d}V_{\mathbf{x}} = (1/2) \iiint \mathbf{x} \wedge \mathbf{j}[\mathbf{x}] \, \mathrm{d}V_{\mathbf{x}} = \mathbf{m} \quad , \tag{36.20.12}$$

  where the last steps follows from the fact that the volume integral of a steady current (that is a current satisfying $\mathbf{div}\mathbf{j} = 0$) which has a compact support is always zero, see (16.12.02).

The effect of a rotation of the Coordinate System is not a problem, as the **Dipole** moments are defined by a vector equation and are therefore independent of any rotation of the Coordinate System.

36.20.53
Electric Field of a Real Capacitor at Large Distances

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|9.6||

Read § 36.20.47.

Consider a parallel plate capacitor with capacitance $C = 1$ pF charged to $\Delta\Phi = 1$ kV. The plates are spaced at $d = 1$ mm apart. Consider the electric field outside the capacitor, which is usually ignored.

1. How can one estimate the electric field at a large distance from the capacitor?
2. Can $R = 3$ m be considered a large enough distance with respect to the answer to the previous question?
3. Estimate the electric field at the distance R, both in the plane of the capacitor and in the direction perpendicular to that plane.
4. Does the shape of the plates of the capacitor matters to answer the previous questions?

WORK IN PROGRESS

36.20.54
Magnetic Field of a Real Inductor at Large Distances

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|9.6||

Read § 36.20.47.

1. Determine the magnetic field of a real inductor at large distances.
2. Does the shape of the inductor matters to answer the previous questions?

36.20.55
ElectroStatic Multipole Momenta for Molecules/Atoms/Nuclei

Some examples.

- SO_4^{--}: at a distant point behaves like a point charge.
- HCl: at a distant point behaves like **Electric Dipole** moment.
- N_2: at a distant point behaves like no **Electric Dipole** moment but electric **Quadrupole** moment (two identical and opposite **Electric Dipole** moments).
- CH_4: at a distant point behaves like no **Electric Dipole** moment, no electric **Quadrupole** moment (as it is basically as a spherically symmetric charge distribution) but, possibly, but possibly an electric octupole moment. Another possible octupole: identical point charges, with alternate signs at the vertexes of a cube.
- Each of the above molecules, also has higher electric multipole moments. If the distance to the field point is not sufficiently large, then higher-order moments can contribute as well to the electric field.
- The phenomena of circular birefringence (aka optical activity) are caused by an electric quadrupole moment of the molecules. As such it cannot be described in terms of the usual Polarization/Magnetization only. Read § 46.14.02.02.
- Many nuclei possesses an electric quadrupole moment.

36.20.56
☒ Field on the Axis of an Ideal Plane Circular Capacitor

This § is referenced at pages:
[Never referenced.]

An ideal plane circular capacitor is made of two identical parallel plane conducting plates of radius R with the same axis and charged with charges of equal magnitude and opposite sign ($Q_+ \equiv +|Q|$ and $Q_- \equiv -|Q|$). Let the z axis be perpendicular to the plates and lying on the axis of the two planes.

1. Calculate the electric field along the z axis.
2. Show that the result reduces to the result of problem § 36.20.29 in the case of infinite radius.

36.20.57
☒ Magnetic field in a Ideal Circular Infinite Hollow Current Wire

This § is referenced at pages:
[Never referenced.]

An ideal hollow current wire is made of an infinitely long circular cylindrical conducting sheet where a uniform current exist along the axis of the wire. Assume that the thickness, a, of the sheet is negligible with respect to the radius, R, so that the current can be approximated with a surface current:

$$\mathbf{j}_S = |\mathbf{j}_S|\widehat{\mathbf{e}}_3 \qquad \text{for } R - \frac{a}{2} \leq r \leq R + \frac{a}{2}\text{; zero otherwise} \quad .$$

Let the z axis be the axis of the hollow wire.

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1. Show that the magnetic field is tangential outside the wire and zero inside it.

36.20.58
☒ Magnetic field in the plane of a plane stationary current loop

This § is referenced at pages:
[1740]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Problem 5.51||

36.20.59
Magnetic Field by a Steady Circular Current Loop

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|§ 5.5|*Solved problem*|

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Problem 5.48||

36.20.60
Forces between two stationary thin current loops with stationary currents

This § is referenced at pages:
[Never referenced.]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Problema 5.50||

Using similar calculations as in section § 36.20.36 show that the forces between the two stationary thin current loops with stationary currents are equal and opposite:

$$-\mathbf{f}_{2\to 1} = \mathbf{f}_{1\to 2} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{\Gamma_1}\oint_{\Gamma_2} \mathbb{d}\mathbf{L}_1[\mathbf{y}_1]\cdot \mathbb{d}\mathbf{L}_2[\mathbf{y}_2]\frac{\mathbf{y}_2-\mathbf{y}_1}{|\mathbf{y}_1-\mathbf{y}_2|^3} \quad . \tag{36.20.13}$$

The **Action-Reaction Principle** for the forces is thus obeyed.

36.20.61
☒ Force on a Small Circular Current Loop on the Axis of a Real Circular Solenoid

This § is referenced at pages:
[1711]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|||

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|11.4||

Assume that both the circular solenoid and the circular current loop (radius R) are coaxial with the z axis, with the solenoid centered at the origin, the currents having the same orientations, such that the magnetic field produced by the solenoid is parallel to the z axis and the circular current loop at positive z.

The current in the circular current loop is kept constant in some way.

$$\mathbb{d}\mathbf{f} = \mathbf{j}\wedge\mathbf{B}\ \mathbb{d}V = I\ \mathbb{d}\mathbf{L}\wedge\mathbf{B} \quad .$$

The radial components of the force sum to zero (the circular current loop tends to explode).

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Relation between $B_r[r,z]$ and $B_z[r,z]$ from the second **Maxwell Equation** : apply to a small cylinder coaxial with the z axis, with radius R and height Δz:

$$\pi R^2 \left(B_z[r,z+\Delta z] - B_z[r,z]\right) + 2\pi R \Delta z B_r[r,z] = 0 \implies R\frac{\partial B_z}{\partial z}[r,z] = -2B_r[r,z] \quad .$$

The z component of the force is (as $B_r > 0$) is attractive:

$$F_z = -2\pi R I B_r = \pi R^2 I \frac{\partial B_z}{\partial z} = \mathbf{m}_z \frac{\partial B_z}{\partial z} \quad .$$

The orientation of the current in the circular current loop with respect to the current in the solenoid determines whether the force is attractive or repulsive; it actually depends on the sign of the product IB_r.

Compare with equation (36.05.07), with $\mathbf{m} = \mathbf{m}_z \widehat{\mathbf{e}}_3$:

$$\mathbf{f}_m = \left(\mathbf{m}_z \frac{\partial}{\partial z}\right)\mathbf{B} \qquad \text{at } r = 0 \quad . \tag{36.20.14}$$

The magnetic field components perpendicular to the z axis are zero, by symmetry, on the axis; therefore their derivative along z is zero and the force is along z.

36.20.62
Magnetic Field on the Axis of a Plane Circular Loop

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|§ 35-2||

1. Calculate the magnetic field on the axis of a plane circular loop of radius R carrying a steady current I.

36.20.63
Ideal circular Solenoid

This § is referenced at pages:
[1771, 1787]

©|J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|11.6.2||

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|||

An ideal circular solenoid is made of an infinitely long circular conducting cylinder where a uniform current exist having no components along the axis of the solenoid. The axis of the solenoid is coincident with the z axis and a cylindrical coordinate system is used.

Assume that the thickness, t, of the sheet is negligible with respect to the radius, R, so that the current can be approximated with a surface current:

$$\mathbf{j}_S = \mathbf{j}_S = |\mathbf{j}_S|\widehat{\mathbf{e}}_\theta \qquad \text{for } r = R\text{; zero otherwise} \quad .$$

Show that the magnetic field is along z and uniform inside the solenoid.

WORKED SOLUTION

Introduce a cylindrical Coordinate System, with the z axis along the axis of the solenoid. The system is invariant by translations along the z axis and by rotations around the z axis. Therefore the components of the magnetic field cannot depend on z nor ϕ: any component of the field can only depend on r.

A radial component is forbidden. In fact if the current is reversed the magnetic field will reverse its direction. But reversing the current amount to rotate by π the solenoid around any axis perpendicular to the axis of the solenoid and and passing by it, which would not reverse the direction of the magnetic field. Note that a similar argument does not hold for a finite length solenoid, as its radial component is not, and must not be, independent of z.

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Alternatively, if a radial component would exist, it would be independent of both z and ϕ, so that the flux across any closed circular cylinder coaxial with the z axis would be non-zero, this violating **Maxwell Equations** . Note that a divergence-less radial field with the given constraints, must be of the form: $B_r = \alpha/r$, which is singular on the z axis, so that it is not a valid field for this problem.
A tangential component is forbidden as well. In fact it would be independent of ϕ so that Ampère law would be violated around any circular path concentric with the solenoid axis and perpendicular to it.
Therefore the magnetic field must be along z and it cannot depend on neither z nor ϕ:

$$\mathbf{B}[\mathbf{r}] = \widehat{\mathbf{e}}_3 B_z[r] \quad .$$

Read also § 36.20.29.
Ampère law then implies that the magnetic field is uniform both inside and outside the solenoid. Moreover, the discontinuity law applies, and it provides the value of the uniform field.
Alternatively, in between the solenoid as well as outside it the magnetic field is rotor-less, and therefore, from the expression of the rotor in cylindrical coordinates:

$$\frac{\partial B_z}{\partial r} = 0 \quad ,$$

that is the magnetic field is uniform inside and outside the solenoid.

36.20.64
A Circular Infinite Solenoid With Helicoidal Windings

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|§ 35-6||

Read also § 36.20.65 and § 39-010.

Consider a cylindrical circular solenoid of radius R and length L, with $L \gg R$ and helicoidal windings spaced by a distance p, $p \ll R$, which is made by a thin wire with $n \equiv 1/p$ windings per unit length. Let the z axis be coincident with the axis of the solenoid and the origin at the center of the solenoid.

1. Determine the ratio between the internal purely longitudinal magnetic field, $\mathbf{B}_{\mathrm{I}}$, and the external purely azimuthal magnetic field, $\mathbf{B}_{\mathrm{E}}$.
2. Calculate the magnetic field at very large distances, $|\mathbf{r}| \gg L$.

Note that the external field can be reduced by adding a second layer of windings with a backward stepping with respect to the first one.

WORKED SOLUTION

$$B_{\mathrm{E}}[\mathbf{r}] = \frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}} \qquad \text{for } \sqrt{x^2+y^2} > R \text{ and } |z| \ll L/2 \quad .$$

$$\frac{B_{\mathrm{E}}}{B_{\mathrm{I}}} = \frac{1}{2\pi n\sqrt{x^2+y^2}} \quad .$$

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36.20.65
☒ Field on the Axis of an Ideal Circular Finite Solenoid

This § is referenced at pages:
[1786, 1918]

An ideal circular solenoid is made of a circular cylindrical conducting plate of length L and radius R where a uniform surface current $\mathbf{j}_S$ exist having no components along the axis of the solenoid. Let the z axis be the axis of the solenoid.

1. Calculate the magnetic field along the z axis.
2. Show that the result reduces to the result of problem § 36.20.63 in the case of infinite length.

WORKED SOLUTION

Measuring the z coordinate with respect to the center of the solenoid and letting the angles ϕ_+ and ϕ_- be the angles with respect to the axis over which the edges of the solenoid are seen from the point where one wants to calculate the field one finds:

$$B_z[x=0,y=0,z]=\frac{\mu_0|\mathbf{j}_S|}{2}\left(\frac{L+2z}{\sqrt{(L+2z)^2+4R^2}}+\frac{L-2z}{\sqrt{(L-2z)^2+4R^2}}\right)=\frac{\mu_0|\mathbf{j}_S|}{2}(\cos[\phi_+]+\cos[\phi_-])\quad .$$

36.20.66
Field Near the End of a Solenoid

©|Lim Yung-Kuo, *Problems And Solutions On Electromagnetism*, 1993, World Scientific, ...Ed., ...|2008||

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|3rd ed., problems 5.4 and 5.5||

Consider a semi-infinite circular solenoid of radius R, n turns per unit length and carrying a current I.

1. Find an expression for the radial component of the magnetic field, $B[r]$, near the axis at the end of the solenoid, for $r \ll R$.

WORKED SOLUTION

Apply Gauss law for magnetism to a small cylinder coaxial with the axis of the cylinder.

36.20.67
☒ EM Potentials for a Infinite Uniform Charge|Current Sheet

1. Find the **scalar electric potential** for problem § 36.20.05.
2. Find the **vector magnetic potential** for problem § 36.20.06.

IN PROGRESS

36.20.68
☒ EM Potentials for Uniform and Static ElectroMagnetic Fields

This § is referenced at pages:
[1865]

1. Show that the **scalar electric potential** describing a uniform and constant electric field, $\mathbf{E}$, is:

$$\Phi = -\mathbf{E}\cdot\mathbf{x} + k \qquad \text{with } k \text{ arbitrary constant} \quad .$$

2. The term $\mathbf{E}\cdot\mathbf{x}$ of the potential sets an origin of the coordinates at $\mathbf{x} = \mathbf{0}$, which might seem to be incompatible with the fact the electric field is uniform in all space with no privileged points. Show that the arbitrariness of the origin is kept by the presence of the term k in the potential.

WORKED SOLUTION

Coordinate translation:

$$\mathbf{y} = \mathbf{x} + \mathbf{a} \quad .$$

It follows:

$$\begin{aligned}\Phi &= -\mathbf{E}\cdot\mathbf{x} + k = -\mathbf{E}\cdot(\mathbf{x}+\mathbf{a}) + \mathbf{E}\cdot\mathbf{a} + k \quad , \\ &= -\mathbf{E}\cdot\mathbf{y} + \mathbf{E}\cdot\mathbf{a} + k = \Phi' + k' \quad .\end{aligned}$$

Show that the **vector magnetic potential** describing a uniform and constant magnetic field, $\mathbf{B}$, is:

$$\mathbf{A} = \frac{1}{2}\mathbf{B}\wedge\mathbf{x} + \mathbf{grad}\Lambda[\mathbf{x},t] \qquad \text{with } \Lambda[\mathbf{x},t] \text{ arbitrary function} \quad .$$

The term $\frac{1}{2}\mathbf{B}\wedge\mathbf{x}$ of the potential sets an origin of the coordinates at $\mathbf{x} = \mathbf{0}$, which might seem to be incompatible with the fact that the magnetic field is uniform in all space with no privileged points.

Show that the arbitrariness of the origin is kept by the presence of the term $\mathbf{grad}\Lambda[\mathbf{x},t]$ in the potential.

WORKED SOLUTION

Coordinate translation:

$$\mathbf{y} = \mathbf{x} + \mathbf{a} \quad .$$

It follows:

$$\begin{aligned}\mathbf{A} &= \frac{1}{2}\mathbf{B}\wedge\mathbf{x} + \mathbf{grad}\Lambda[\mathbf{x},t] = \frac{1}{2}\mathbf{B}\wedge(\mathbf{x}+\mathbf{a}) - \frac{1}{2}\mathbf{B}\wedge\mathbf{a} + \mathbf{grad}\Lambda[\mathbf{x},t] \quad , \\ &= \frac{1}{2}\mathbf{B}\wedge(\mathbf{x}+\mathbf{a}) - \frac{1}{2}\mathbf{grad}(\,\mathbf{x}\cdot(\,\mathbf{B}\wedge\mathbf{a})) + \mathbf{grad}\Lambda[\mathbf{x},t] = \mathbf{A}' + \mathbf{grad}\big(\Lambda'[\mathbf{x},t]\big) \quad .\end{aligned}$$

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36.20.69

☒ Vector Potential for Uniform and Static Magnetic Field

1. Show that the **vector magnetic potential** of a homogeneous field, $\mathbf{B} = B_0\widehat{\mathbf{e}}_3$, in a rectangular system of coordinates, can be expressed as:
$$\mathbf{A} = -\widehat{\mathbf{e}}_1 C_1 B_0 y + +\widehat{\mathbf{e}}_2 C_2 B_0 x \qquad C_1 + C_2 = 1 \qquad C_1 \text{ and } C_2 \text{ constants} \quad .$$
2. Show that the **vector magnetic potential** of a homogeneous field, $\mathbf{B} = B_0\widehat{\mathbf{e}}_3$, in a cylindrical system of coordinates, can be expressed as:
$$\mathbf{A} = \frac{B_0 r}{2}\widehat{\mathbf{e}}_\theta \quad .$$
3. Show that the **vector magnetic potential** of a homogeneous field, $\mathbf{B} = B_0\widehat{\mathbf{e}}_3$, in a spherical system of coordinates, can be expressed as:
$$\mathbf{A} = \frac{B_0 r \sin\theta}{2}\widehat{\mathbf{e}}_\psi \quad .$$
4. In all cases, how is it possible that a uniform field (invariant under translations) is described by a potential whose module depends on the distance from a arbitrary origin of the Coordinate System?

36.20.70

☒ ElectroStatic Energy of a Thin Circular Ring

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|||

Consider a thin circular ring of radius R with a total charge Q uniformly distributed along it. Model the system as a finite number, n, of point charges uniformly distributed on a circle of radius R and consider the appropriate limiting process.

1. Calculate the ElectroStatic energy.

Read § 36-017.

36.20.71

☒ Energia Elettrostatica Di Una Sfera Carica Con Densità Di Carica Omogenea

1. Determine the ElectroStatic potential energy of a spherical charge distribution of radius R and uniform charge density, with a total charge Q.

WORKED SOLUTION

$$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{3Q^2}{5R}$$

36.20.72

☒ Energia Di Due Distribuzioni Di Carica

Si supponga di avere due distribuzioni di carica distinte, con densità di carica di volume: $\rho_1[\mathbf{x}]$ e $\rho_2[\mathbf{x}]$.

1. Determinare l'energia elettrostatica della configurazione, in funzione dei campi elettrici $\mathbf{E}_1$ e $\mathbf{E}_2$ prodotti dalle due singole distribuzioni di carica.

WORKED SOLUTION

Data una qualunque distribuzione di carica di volume, $\rho[\mathbf{x}]$, limitata si ha:
$$\Phi_e = \frac{1}{2}\iiint \rho[\mathbf{x}]\Phi[\mathbf{x}]\,\mathrm{d}V_{\mathbf{x}} = \Phi_e = \iiint u[\mathbf{x}]\,\mathrm{d}V_{\mathbf{x}} \qquad u[\mathbf{x}] = \frac{1}{2}\varepsilon_0|\mathbf{E}[\mathbf{x}]|^2 \quad .$$
Siano $\mathbf{E}_1[\mathbf{x}]$ e $\mathbf{E}_2[\mathbf{x}]$ i campi elettrici prodotti dalle due distribuzioni di carica $\rho_1[\mathbf{x}]$ e $\rho_2[\mathbf{x}]$ da sole.

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Per il principio di sovrapposizione

$$\mathbf{E}[\mathbf{x}] = \mathbf{E}_1[\mathbf{x}] + \mathbf{E}_2[\mathbf{x}] \quad ,$$

ne allora

$$\Phi_e = \frac{1}{2}\varepsilon_0 \int \left(|\mathbf{E}_1|^2 + |\mathbf{E}_2|^2 + 2\,\mathbf{E}_1 \cdot \mathbf{E}_2 \right) \mathrm{d}\mathbf{x}$$

Se ne conclude che l'energia complessiva del sistema è data dalla somma delle due auto-energie più l'energia di interazione.
Come corollario si può dire che, tutte le volte che in un cambio di configurazione dei campi elettrici una certa distribuzione di carica non cambia il suo termine di auto-energia può essere trascurato perchè immutato.
This is true even if the self-energy turns out to be infinite, that is for point charges.

36.20.73

☒ Volume Integral of Current Density

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 5.7||

For any limited Charge|Current distribution:

1. show that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = \iiint \mathbf{j}\, \mathrm{d}V} \quad ,$$

where $\mathbf{p}$ is the total electric dipole moment.

36.20.74

☒ Differences of Potential and What Voltmeters Do Measure

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 7.50||

©|WEB - URL |||

©|R.H. Romer - Am.J.Phys, , ..., ..., ...Ed., ... 50, 1089 (1982)|||

©|F.Reif, Am.J.Phys, , ..., ..., ...Ed., ... 50 (11) 1048, 1982|||

Consider a cylindrical circular solenoid of radius R and length L, with $L \gg R$ and treat it as an ideal solenoid. Assume the current is changing with time in some way, so that the flux of the magnetic field concatenated with the circuit is changing with time. Consider a plane circular closed circuit consisting of two resistors, R_1 and R_2, located in a plane perpendicular to the axis of the solenoid and concentric to it. Two voltmeters are connected to the two points, A and B, where the two resistors are connected. The closed circuit voltmeter-resistor do not encircle the solenoid at all.

Note that any real voltmeter always measures, between A and B: $\int_A^B E \cdot \mathrm{d}\mathbf{L}$. In fact a typical voltmeter is a high-resistance instrument which measures the current passing inside it. If the voltemter is connected to two points in such a away that the closed loop does not have any change of the magnetic filed flux inside it then the voltage betwen the two points is the same as the one measured by the voltmeter, neglecting the perturbation induced by the insertion of the voltmeter.

The voltmeter, assumed to be a Ohmic instrument, measures the line integral of the electric field. If it is crossed by a measured current I on a wire of sectional area A and its internal resistence is R one has:

$$\mathrm{d}\Phi = I\, \mathrm{d}R = I\frac{\mathrm{d}\mathbf{L}}{\sigma A} = \frac{j\, \mathrm{d}\mathbf{L}}{\sigma} = E\, \mathrm{d}\mathbf{L} \quad .$$

1. What do the two voltmeters measure?

Do not confuse differences between the **scalar electric potential** at different points with line integrals of the electric field, as these are two different concepts in non-static conditions.

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Figura da inserire

36.20.75
Electricity in the Atmosphere

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||

36.20.76
Magnetic Field of the Earth

©|WEB - URL |||

The magnetic field of the Earth (read § 39.08.02) is modeled, near the earth surface, as a **Magnetic Dipole** field, as we are reasonably far away from the Charge|Current producing the ElectroMagnetic fields, which are typically located close the center of the Earth. The **Magnetic Dipole** field of the Earth is closely described by a **Magnetic Dipole** located close to the center of the Earth inclined with respect to the Earth rotation axis. Uranus has a magnetic field modeled by a **Magnetic Dipole** located far away from the center of Uranus and inclined by $59\ ^\circ$ with respect to the rotation axis.

36.20.77
Applications to Biological Systems

- ECG.
- EEG.
- Defibrillator.

36.20.78
Particle Accelerators

A particle accelerator is a device that uses ElectroMagnetic fields to accelerate charged particles to high speeds, to contain them in well-defined beams and to direct them wherever it is required.

They are used for fundamental physics studies, for studies in physics of matter and for medical therapy.

- CRT
- Betatron
- Cyclotron
- Synchrotron
- ...

36.20.79
Topology of Magnetic Field Lines

©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|§ 10.6|*The Topology of Magnetic Field Lines*|

Read also § 16.12.19.

UNDER-IN-PROGRESS

36.20.80
Newton Demo That a Isotropic Charge Shell Gives Zero Field at Internal Points

36.20.81
Magnetic Circuits

36.20.82
Rasetti Experiment

This § is referenced at pages:
[1954]

©|F.Rasetti|Phys. Rev. 66 (1944) 1|*Deflection of muons in Magnetized iron*|

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36.21 Exercises Problems and Physical Applications

36-001 ✓ Exercises and Problems in ElectroMagnetism From D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...

Problems from D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ..., § 2.
Problems from D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ..., § 5.
Problems from D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ..., § 7.

36-002 ✓ Exercises and Problems in ElectroMagnetism From I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...

3.001, 3.002, 3.003, 3.005, 3.012, 3.013, 3.014, 3.016, 3.017, 3.018, 3.024, 3.025, 3.026, 3.028, 3.029, 3.034, 3.036, 3.037, 3.044, 3.045, 3.046, 3.047, 3.048, 3.049, 3.050, 3.051, 3.053.

36-003 ✓ Line Surface Volume - Charges Currents

Demonstrate equations (36.03.12) and (36.03.13).

36-004 ✓ Tangential components of discontinuities of the fields

Demonstrate the implications in equation (36.09.05) and (36.09.06).

36-005 ✓ Rotational Equilibrium of Two Dipoles

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|37.3||

Consider two identical **Magnetic Dipoles** , away from any other external magnetic field. The first **Dipole** is kept fixed at the origin of a orthonormal Cartesian Coordinates Coordinate System with its **Dipole** moment directed along the z axis. The second **Dipole** is kept at some distance r from the first **Dipole** , free to rotate.

1. Determine the equilibrium position, with respect to rotation, when the second **Dipole** is located on the z axis. Is the equilibrium stable or unstable?
2. Determine the equilibrium position, with respect to rotation, when the second **Dipole** is located on the $x - y$ plane. Is the equilibrium stable or unstable?
3. Write the expression of the potential energy of interaction of the two **Dipoles** .

Read § 36.20.12.

36-006 Force on a fixed dipole in terms of the shear tensor of the field

This § is referenced at pages:
[1965, 1967]

Show that the force on a fixed dipole (equation (36.05.02) and equation (36.05.03)) can be written in terms of the symmetrical shear tensor of the external field, $\mathbb{S}$, outside charges/currents and in static conditions, as:

$$f_k = p_i \mathbb{S}[\mathbf{E}]_{ik} \qquad f_k = m_i \mathbb{S}[\mathbf{B}]_{ik} \quad . \tag{36.21.01}$$

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36-007 ✓ Super-Conducting Magnet of the CMS Experiment at CERN

It is the world largest superconducting solenoid magnet, as of year 2006 [1] Main parameters of the CMS magnet: size: $13\ \mathrm{m} \times 5.9\ \mathrm{m}$ (length times diameter); current: $I = 19.5\ \mathrm{kA}$; magnetic field: $4\ \mathrm{T}$; stored energy: $E = 2.7\ \mathrm{GJ}$; mass: $M = 220\ \mathrm{ton}$.

The solenoid tends to explode due to the radial magnetostatic pressure.

1. Calculate the radial magnetostatic pressure.
2. Compare with a typical pressure of automotive tires, relative to atmospheric pressure, which is about two atm.

36-008 ✓ A Superconducting Loop

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 5.21||

Show that the magnetic flux through a perfectly conducting loop is constant.

WORKED SOLUTION

Apply Faraday-Neumann-Lenz (integral form) to the loop: as $\mathbf{E}$ is always zero the magnetic flux must be constant.

36-009 A Discharging Capacitor

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|3.348||

A plane capacitor is made of two circular flat disks and a homogeneous weakly conducting medium inside them. The capacitor is charged and the battery is removed afterwards.

1. Show that, as long as edge effects can be neglected, the magnetic field inside the capacitor is zero.

36-010 Resistance and Discharge Time on a Cylindrical Capacitor

1. Determine the time constant for the discharge of a real cylindrical capacitor as a function of the geometrical parameters and properties of the material between the armatures.

36-011 ✓ Force on a Current Carrying Conductor of Arbitrary Shape

Consider a piece of current wire of arbitrary shape, with starting point (S) and end point (E), carrying a steady current I.

1. Write the expression of the total force on it when it is immersed in a constant and uniform magnetic field.
2. What is the result for a closed loop?
3. Every piece of the wire is immersed in the magnetic field produced by the other pieces of the wire and therefore also feels a force due to the other pieces of wire. What is the effect of these forces?

WORKED SOLUTION

Let $\mathbf{L}$ be the vector from the starting point (S) to the end point (E) of the wire

$$\mathbf{F} = I\,\mathbf{L}\wedge\mathbf{B} \quad .$$

[1] See, for instance WEB - URL and WEB - URL

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36-012 ✓ ElectroStatic Quadrupole Field (1)

Consider the ElectroStatic field of § 16.12.04.06:

$$\mathbf{E}[\mathbf{r}] = k\,(y\widehat{\mathbf{e}}_1 + x\widehat{\mathbf{e}}_2) \quad .$$

1. Calculate the electric potential.
2. Calculate the field lines.
3. Rewrite the potential in cylindrical coordinates and show that it satisfies **Poisson equation** .

WORKED SOLUTION

As the field is both irrotational and solenoidal it can be an ElectroStatic field outside the charges producing the electric field.

1. The potential difference does not depend on the path and therefore one can choose a path that makes the integral easy. Let us calculate the potential at $P = \{x, y, 0\}$ using as a reference point the origin of the Coordinate System: O. Use the intermediate point $A = \{x, 0, 0\}$.

$$\Phi(P) - \Phi(O) = -\int_O^P \mathbf{E}\cdot \mathbb{d}\boldsymbol{\ell} = \quad ,$$

$$\Longrightarrow \quad ,$$

$$-\left(\int_O^A + \int_A^P\right) \mathbf{E}\cdot \mathbb{d}\boldsymbol{\ell} = -\int_O^A \mathbf{E}\cdot \mathbb{d}\boldsymbol{\ell} - \int_A^P \mathbf{E}\cdot \mathbb{d}\boldsymbol{\ell} = -\int_0^x ky\ \mathbb{d}x - \int_0^y kx\ \mathbb{d}y = 0 - \int_0^y kx\ \mathbb{d}y = -kxy \quad .$$

Equipotential surfaces are the equilateral hyperbolas:

$$xy = \text{constant} \quad .$$

A suitable set of electrodes held at the appropriate potential can be used to build such a field.

2.

$$\frac{\mathbb{d}\mathbf{r}}{\mathbb{d}s} = \mathbf{E} \qquad \text{with the field lines parameterised by } \mathbf{r} = \mathbf{r}[s] \quad ,$$

$$\begin{cases} \dfrac{\mathbb{d}x[s]}{\mathbb{d}s} = ky \\ \dfrac{\mathbb{d}y[s]}{\mathbb{d}s} = kx \end{cases} \quad , \qquad \Longrightarrow \frac{\mathbb{d}x}{y} = \frac{\mathbb{d}y}{x} \Longrightarrow x^2 - y^2 = \text{constant} \equiv c \quad .$$

The above solution of the system of differential equations gave the field lines in implicit form. The explicit parameterised form of the field lines can be found to be:

$$x[s] = A\exp[+ks] + B\exp[-ks] \quad ,$$
$$y[s] = A\exp[+ks] - B\exp[-ks] \quad ,$$

where the integration constants A and B are linked to the integration constant c of the implicit form by:

$$c = 4AB \quad .$$

3.

$$\Phi[r, \phi] = -kr^2 \sin[\phi]\cos[\phi] \quad .$$

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36-013 ✓ ElectroStatic Quadrupole Field (2)

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||
©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|3.10||

Si consideri un potenziale elettrico dato da $\Phi[x,y,z] = h\left(x^2 - y^2\right)$.

- Verificare che si tratta di una funzione armonica.
- Esprimere il potenziale in coordinate polari.
- Determinare il campo elettrico e le linee di campo.
- Discutere la forza che tale campo esercita su una carica puntiforme.

WORKED SOLUTION

- Si verifica che il potenziale soddisfa l'equazione di Laplace:
$$\nabla^2\Phi = 0 \quad .$$
- Si ha:
$$\Phi[r,\theta] \propto r^2 \cos[2\theta] \quad .$$
- Campo elettrico:
$$E_x = -2hx \quad ,$$
$$E_y = +2hy \quad .$$
Modulo del campo elettrico: $E = 2hr$.
Le linee equipotenziali sono iperboli equilatere con $h\left(x^2 - y^2\right) = costante$ con gli assi paralleli agli assi coordinati.
Le linee di campo si trovano dalla (16.12.12):
$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \mathbf{E} \quad ,$$
da cui si ottiene: $x[s]y[s] =$ constant. Le linee di campo sono dunque iperboli equilatere con gli assi coordinati come asintoti.
- The origin of the axes is an unstable equilibrium position. In fact the force, for small displacements, acting on a positive test charge is a force tending to restore the equilibrium position in case the displacement is along the x axis and a tending to send the test particle further-away in case of a e displacement is along the y axis.
The module of the force is increasing linearly with distance.
It is not a force tending to restore the equilibrium position in all cases, but it only restores along some direction.
However, a force restoring the equilibrium for small displacement along any direction would require a non-zero divergence and this implies that some of the charges producing the field must exist at the origin. In case a focusing system is required these charges might be a problem for the dynamics of the charges to be focused.
It can be shown that a suitable linear sequence of **Quadrupole** fields as the one just described, each one rotated by $\pi/2$ with respect to the neighboring ones, actually has an overall focusing effect [a].

[a] R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...

36-014 ✓ A Simple Quadrupole Charge Distribution

Consider two identical real **Electric Dipoles** made of point charges $\pm q$ at a distance d. The four charges are all located along the z axis, with the two negative charges at the origin and the two positive charges at $z = \pm d$.

1. Calculate the electric field at large distances along the z axis, and show that is goes like z^{-4}.

WORKED SOLUTION

$$E_z = \frac{3qd^2}{2\pi\varepsilon_0 z^4} \quad .$$

36-015 ✓ MagnetoStatic Quadrupole Field

Study the focusing effect of a MagnetoStatic **Quadrupole** field, whose field lines are the same as the field lines of an ElectroStatic **Quadrupole** field.

Note that the focusing/defocusing effect is rotated by an angle $\pi/4$ with respect to the case of the ElectroStatic **Quadrupole** lens and it is directed along the mid-plane between the poles.

36-016 ✓ A Magnetic Sextupole Lens

©|Lim Yung-Kuo, *Problems And Solutions On Electromagnetism*, 1993, World Scientific, ...Ed., ...|2093||

A **Magnetic Dipole** , $\mathbf{m}$, is placed in a magnetic field given by:

$$\mathbf{B} = \alpha\left(x^2 - y^2\right)\widehat{\mathbf{e}}_1 - 2\alpha xy\widehat{\mathbf{e}}_2 \quad ,$$

where α is a constant.

1. Determine the Force|Torque on the **Dipole** as a function of the position.
2. Determine the force on a point particle with velocity $\mathbf{v} = v_z\widehat{\mathbf{e}}_3$ as a function of its xy coordinates.

36-017 ✓ Point Charges Along a Circumference

This § is referenced at pages:
[1789]

1. Three identical point charges of charge q are uniformly distributed along a circumference of radius R, forming the vertexes of a regular triangle. One of them is removed: what is the electric field at the center of the circumference?
2. Redo the same problem above for n charges.
3. A number $2n$ point charges, n of charge $+q$ and n of charge $-q$, are uniformly distributed along a circumference of radius R, forming the vertexes of a regular polygon, in such a way that each charge is alternating with charges of opposite sign. One of them is removed: what is the electric field at the center of the circumference?

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36-018 ✓ A Uniformly Charged Circular Ring/disk

This § is referenced at pages:
[1739]

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|11.4.4||

Consider a uniformly charged plane thin circular ring of internal radius R_1 and external radius R_2, carrying a total charge Q. Let the z axis be perpendicular to the plane with origin at the center of the ring/disk.

- Calculate the electric field on the axis, at z, both in a direct way and using the electric potential (if that is possible). Interpret the results in terms of the **superposition principle** .
- Calculate the electric potential on the axis, at z. From the knowledge of the electric potential on the axis what can one say about the electric field?

Consider and study, for the above questions, the following limiting cases, making sure that the above results reduce to what one expects in these cases:

- $R_1 = 0$;
- $R_1 = 0$ and $R_2 \to \infty$;
- $R_2 \to \infty$;
- $|z| \gg R_2$;
- $R_2 - R_1 \ll R_1$ (a thin ring);
- $|z| \ll R_2$ with $R_1 = 0$;
- $|z| \ll R_1$ with $R_1 > 0$;

Does the mathematical limit for $\{z, R_1\} \to \{0, 0\}$ exist? Does it make sense from the physical point of view?

36-019 ✓ A Uniformly Charged Circular Ring/Disk/Sphere

1. Consider a uniformly charged plane thin circular **ring** of radius R carrying a total charge Q. Let z be the axis of the ring, with origin at the center of the ring.
   a) Calculate the potential and electric field on the axis by direct calculation.
   b) Calculate, if possible, the electric field from the potential.

WORKED SOLUTION

$$\Phi[z] = \frac{1}{4\pi\varepsilon_0}\int \frac{đq}{d} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}} \quad .$$

2. Consider a uniformly charged plane thin circular **disk** of radius R carrying a total charge Q. Let z be the axis of the disk, with origin at the center of the ring.
   a) Calculate the potential and electric field on the axis by direct calculation and using the results of the previous point, that is considering the disk made of infinitesimal rings carrying an infinitesimal charge $đq$.
   b) Calculate, if possible, the electric field from the potential.
   c) Check the signs of potential and electric field and the validity of the discontinuity relations.

WORKED SOLUTION

$$\Phi[z] = \frac{1}{4\pi\varepsilon_0}\int_{r=0}^{R} \frac{2\pi\rho_S r\, đr}{\sqrt{r^2+z^2}} = \frac{\rho_S}{2\varepsilon_0}\left(\sqrt{R^2+z^2} - |z|\right) \quad .$$

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$$E_z[z] = -\frac{\partial \Phi}{\partial z} = \frac{\rho_S}{2\varepsilon_0}\left(\frac{z}{\sqrt{R^2+z^2}} - \frac{z}{|z|}\right) \quad .$$

d) Consider the two limiting cases of very large and very small distance with respect to the radius and check that the results reduce to what one expects from known results.

WORKED SOLUTION

$$\begin{cases} \lim_{z\to z^+} E_z[z] = +\dfrac{\rho_S}{2\varepsilon_0} \\ \lim_{z\to z^-} E_z[z] = -\dfrac{\rho_S}{2\varepsilon_0} \end{cases} \quad , \qquad \text{as for an infinite plane} \quad .$$

$$\lim_{|z|\to\infty} E_z[z] = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|z|} \quad .$$

e) Calculate the potential at one point at the edge of the disk.

WORKED SOLUTION

Use a Coordinate System in polar coordinate centered at the point on the edge.

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int_{\theta=-\pi/2}^{\theta=+\pi/2}\int_{r=0}^{2R\cos[\theta]} \frac{\rho_S r \, \mathrm{d}r \, \mathrm{d}\theta}{r} = \frac{\rho_S R}{\pi\varepsilon_0} \quad .$$

f) Calculate the potential on the axis by direct integration in two dimensions.

3. Consider a uniformly charged **sphere** of radius R carrying a total charge Q. Let z be the axis of the sphere, with origin at the center of the ring.
   a) Calculate the potential and electric field on the axis by direct calculation.
   b) Calculate the potential and electric field on the axis using the results of the previous point, that is considering the sphere made of infinitesimal disks carrying an infinitesimal charge $\mathrm{d}q$.

36-020 ✓ A Uniformly Charged Fixed Spherical Shell

This § is referenced at pages:
[Never referenced.]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 2.7||

Consider a uniformly charged thin spherical shell of radius R carrying a total charge Q. Let z be the axis, with origin at the center of the sphere.

1. Calculate the potential at any point, internal and external to the shell, by direct integration.

WORKED SOLUTION

Use a system of spherical coordinates centered at the center of the sphere and with its z axis passing by the point P where the potential must be calculated. Then integrate in rings perpendicular to z axis, so that all the points of the ring have the same distance from the point P.

2. Compare the results with the known result deriving from Gauss law.
3. Use the above result to calculate the potential of a sphere with uniform charge density.

Read § 36-037.

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36-021 ✓ Electric Potential on the Axis of a Thin Circular Right Cylinder

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|||

1. Calculate the electric potential on the axis of a thin circular right cylinder of radius R and length L with uniform charge density and total charge Q.

36-022 ✓ Electric Potential on the Axis of a Thick Circular Right Cylinder

1. Calculate the electric potential on the axis of a thick circular right cylinder of internal radius R_1, external radius R_2 and thickness D with a uniform charge density and total charge Q.

36-023 ✓ A Sheet of Charge

A plane sheet of charge, with thickness d much smaller than its dimensions, carries a uniform charge density per unit volume, ρ. Let z be the axis perpendicular to the plane, with origin at the center of the sheet of charge, so that the charge density is uniform for $|z| \leq d/2$ and zero otherwise.

1. Determine the **scalar electric potential** produced by the sheet of charge as a function of z taking zero potential at $z = 0$.

WORKED SOLUTION

Solve **Poisson equation** and apply the continuity of the **scalar electric potential** and its derivative. Fix the arbitrary constant remaining

36-024 ✓ Three Wires

©|Lim Yung-Kuo, *Problems And Solutions On Electromagnetism*, 1993, World Scientific, ...Ed., ...|2006||

Consider three straight long parallel wires lying on the same plane and equally spaced by a distance a. Let the radius of the wires be much smaller than the distance between wires. Each wire carries the same current I in the same direction.

1. Calculate the points were the magnetic field is zero.
2. Suppose that the two external wires are kept fixed while the middle wire is moved rigidly by a small amount, $x \ll a$, in the same plane of the other two wires and keeping it parallel to the other two wires. Determine the subsequent motion of the middle wire after it is left.
3. Suppose that the two external wires are kept fixed while the middle wire is moved rigidly by a small amount, $y \ll a$, in the direction perpendicular to the plane of the other two wires and keeping it parallel to the other two wires. Determine the subsequent motion of the middle wire after it is left.

WORKED SOLUTION

1. $x = \pm a/\sqrt{3}$.
2. Both harmonic motion with $\omega_0^2 = \dfrac{\mu_0 I^2}{m\pi a^2}$, with m mass of the wire.

DRAFT - In Progress

36-025 ✓ An Infinite Strip of Current

This § is referenced at pages:
[1740]

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|35.2||

Consider a plane rectilinear thin infinite strip of current of width D and carrying a steady current I.

1. Calculate the magnetic field at a point on a line perpendicular to the strip and passing by the axis of the strip at distance H for the strip.
2. Show that in the limit $D \to \infty$ the result reduces to the field of an infinite sheet of current.
3. Show that in the limit $D \to 0$ the result reduces to the field of an infinite rectilinear current wire.
4. Show that in the limit $H \to 0$ the result reduces to the field of an infinite sheet of current.

WORKED SOLUTION

The magnetic field $\mathbf{B}$ lies parallel to the plane of the strip, perpendicularly to the long direction of the strip. Its modulus is:

$$B = \frac{\mu_0}{\pi} |\mathbf{j}_S| \tan\left[\frac{D}{2H}\right] \quad .$$

Its orientation is easily given by the Biot-Savart law.

36-026 A Finite Length and Thick Solenoid

Consider a realistic model of a solenoid whose thickness, $a = 0.5$ cm, is not negligible with respect to its internal radius $R_1 = 3$ cm. The external radius is $R_2 \equiv R_1 + a$. Model the current density as a uniform volume current density with zero component along the axis:

$$\mathbf{j}_S = |\mathbf{j}_S| \widehat{\mathbf{e}}_\theta \qquad \text{for } R_1 \leq r \leq R_2; \text{ zero otherwise} \quad .$$

The total current is $I = 10$ A. The solenoid has finite length $L = 10$ cm.

1. Calculate the field at any point on the axis.
2. Show that for $a \ll R_1$ the expressions of the field on the axis reduce to the one for a thin solenoid.
3. Calculate the field as a function of the distance from the axis in the case $L \gg R_2$.
4. Calculate the field at $r = 100$ m on the axis of the solenoid and in the plane perpendicular to the axis ans passing by the solenoid.

36-027 ✓ A Plane Square Loop

Consider a square plane loop of current of side L carrying a constant current I.

1. Calculate the magnetic field along its axis.

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36-028 ✓ A Plane Polygonal Loop

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|P 11.3||

Consider a plane loop of current carrying a constant current I and having the shape of a regular polygon with n sides inscribed in a circle of radius R.

1. Calculate the magnetic field at the center.
2. Calculate the magnetic field along its axis.
3. Find, for both previus questions, the limiting case for $n \to \infty$ at fixed R.

36-029 ✓ Ampère Law and Plane Loops

1. Consider a circular plane loop of current of radius R carrying a constant current I. Calculate the circulation of the magnetic field along a circuit going along the axis of the loop closing the integration path with a semi-circle of infinite radius. Be careful to evaluate the value of the circulation along the semi-circle of infinite radius. Verify that Ampère law is satisfied.
2. Consider a square plane loop of current of side L carrying a constant current I. Calculate the circulation of the magnetic field along a circuit going along the axis of the loop closing the integration path with a semi-circle of infinite radius. Be careful to evaluate the value of the circulation along the semi-circle of infinite radius. Verify that Ampère law is satisfied.

36-030 ✓ A Non-Plane Current Loop

A circular loop of wire carrying a steady current $I = 1$ A has radius $r = 1$ cm lies in the xy plane with its center at the origin. It is then bent so that half loop lies in the yz plane and the other half-loop lies in the xy plane.

1. What is the **Magnetic Dipole** moment?
2. What is the magnetic field at $r_1 = \{D, 0, 0\}$, $r_2 = \{0, D, 0\}$ and $r_3 = \{0, 0, D\}$, with $D = 15$ m ?

36-031 ✓ E.Purcell Loop

This § is referenced at pages:
[Never referenced.]

©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|problem 11.11||

Consider the figure 36.19, geometrically a cube of sides $2a$ centered at the origin and with sides parallel to the axes. Imagine the thick black line is a closed current loop carrying a fixed current I.

1. Determine the magnetic dipole moment.
2. Determine the magnetic quadruple moment.

36-032 ✓ Helmholtz Coils

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|35.10 35.26 35.27||

Helmholtz coils are a practical tool to obtain a uniform magnetic field.

Consider two identical plane circular loops, of radius R, located coaxially and separated by a variable distance D and carrying a constant current I. Let the z axis be the axis of the two loops and let the origin be located at the middle point.

1. Determine the magnetic field at the middle point ($\mathbf{B}_0$) as a function of D and compare it with the magnetic fields ($\mathbf{B}_1$) at the center of the two loops.
2. Study the family of functions $B_z[z, D]$ for different values of the parameter D.

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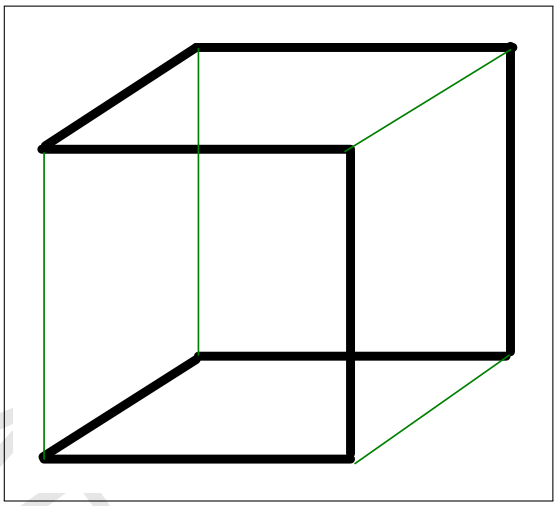

Figure 36.19: E.Purcell Loop

3. Determine the distance D such that the relative difference between B_1 and B_0 is minimal and show that, placing the two loops at the distance D, the magnetic field near the middle point is uniform up to third-order in the distance from the middle point along the axis of the two loops.

WORKED SOLUTION

$$B_z = \frac{\mu_0 I R^2}{2}\left(\frac{1}{\left(R^2+(z+D/2)^2\right)^{3/2}}+\frac{1}{\left(R^2+(z-D/2)^2\right)^{3/2}}\right) \quad .$$

36-033 ✓ A Rotating uniformly charged Bar

A long and thin bar, made of an insulating material, is uniformly charged. Its length is L, its mass is M, and the total charge is Q, uniformly distributed along its length. The bar rotates with angular velocity ω around an axis orthogonal to the bar and passing by its center.

Calculate the **Magnetic Dipole** moment of the bar and the **gyromagnetic ratio** .

WORKED SOLUTION

$$\mathbf{m} = \frac{QL^2}{24}\boldsymbol{\omega} \qquad g = 1 \quad .$$

36-034 ✓ A Rotating Uniformly Charged Cylindrical Shell

©|Lim Yung-Kuo, *Problems And Solutions On Electromagnetism*, 1993, World Scientific, ...Ed., ...|2019||

A cylindrical shell of electric charge has length L and radius R, with $L \gg R$. The surface charge density is ρ_S and the cylinder rotates with an angular velocity ω, which increases with time as $\omega = kt$ with k a positive constant.

1. Determine the magnetic field inside the cylinder.
2. Determine the electric field inside the cylinder.
3. Determine the total electric field energy and the total magnetic field energy inside the cylinder.

DRAFT PROOF Edits

36-035 ✓ A Rotating Uniformly Charged circular Ring

This § is referenced at pages:
[1806]

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|37.9||

Consider a thin ring of radius R with a uniformly distributed charge Q on it. The ring rotates around its axis with constant angular velocity ω.

1. Show that the **Magnetic Dipole** moment is:

$$\mathbf{m} = \frac{QR^2}{2}\boldsymbol{\omega} \quad .$$

2. Find the relation of the **Magnetic Dipole** moment to the angular moment.

36-036 ✓ A Rotating Uniformly Charged Circular Disk

This § is referenced at pages:
[1806]

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|35.32||

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|P 11.10||

A thin circular disk of radius R carries a uniformly distributed charge Q. The disk rotates with a constant angular velocity ω around its axis.

1. Calculate the magnetic field at the center of the disk.
2. Calculate the **Magnetic Dipole** moment.

WORKED SOLUTION

$$\mathbf{B} = \frac{\mu_0 Q}{2\pi R}\boldsymbol{\omega} \quad .$$

$$\mathbf{m} = \frac{QR^2}{4}\boldsymbol{\omega} \quad .$$

36-037 ✓ A Rotating Uniformly Charged Spherical Shell

This § is referenced at pages:
[1799, 1806, 1806, 1908]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Example 5.11|*Explicit calculation*|

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|P 11.9||

A spherical shell of radius R carries a uniformly distributed charge Q. The sphere rotates with a constant angular velocity ω around one of its diameters.

1. Calculate the **vector magnetic potential** at any point in space.
2. Calculate the **Magnetic Dipole** moment.
3. Calculate the magnetic field at any point in space.

WORKED SOLUTION

The integration is not trivial but it can be carried on. A suitable choice of the Coordinate System make the integration a lot easier.
Choose an Coordinate System centered at the center of the sphere with its z axis passing by the point P where the **vector magnetic potential** is going to be calculated and let $\boldsymbol{\omega}$ lie in the zx plane and making an angle ψ with the z axis. Use spherical coordinates, with the vector position $\mathbf{x}$ defining the integration variable on the spherical shell. This Coordinate System is useful because it allows to integrate on one variable at constant distance from the infinitesimal element and the point P, making the integration in one of the variables trivial.
Therefore, using equations (36.13.06) in the surface current expression, one finds:

$$\boldsymbol{\omega} = \omega\left(\sin\left[\psi\right]\widehat{\mathbf{e}}_1 + \cos\left[\psi\right]\widehat{\mathbf{e}}_3\right) \quad ,$$

$$\mathbf{x} = R\left(\sin\left[\theta\right]\cos\left[\phi\right]\widehat{\mathbf{e}}_1 + \sin\left[\theta\right]\sin\left[\phi\right]\widehat{\mathbf{e}}_2 + \cos\left[\theta\right]\widehat{\mathbf{e}}_3\right) \quad ,$$

$$\mathbf{j}_S = \rho_S\left(\boldsymbol{\omega}\wedge\mathbf{x}\right) \quad ,$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\iint \frac{\rho_S\left(\boldsymbol{\omega}\wedge\mathbf{x}\right)R^2\sin\left[\theta\right]}{\sqrt{R^2+r^2-2rR\cos\left[\theta\right]}}\,\mathrm{d}\theta\,\mathrm{d}\phi \quad ,$$

all terms in $\boldsymbol{\omega}\wedge\mathbf{r}$ but $-\widehat{\mathbf{e}}_2\sin\left[\psi\right]\cos\left[\theta\right]$ are proportional to either $\sin\left[\phi\right]$ or $\cos\left[\phi\right]$,

the integral in $\mathrm{d}\phi$ is zero: ,

$$\mathbf{A} = 2\pi\frac{\mu_0}{4\pi}\rho_S\omega R^3\widehat{\mathbf{e}}_2\sin\left[\psi\right]\int_{+1}^{-1}\frac{u\,\mathrm{d}u}{\sqrt{R^2+r^2-2rRu}} = \frac{\mu_0\rho_S\omega R^3\sin\left[\psi\right]\widehat{\mathbf{e}}_2}{2}\begin{cases}-\dfrac{2r}{3R^2} & \text{for } 0<r<R \quad ,\\ -\dfrac{2R}{3r^2} & \text{for } 0<R<r\end{cases} \quad ,$$

the above result can be written in vector form as ,

$$\mathbf{A}[\mathbf{r}] = \begin{cases}\dfrac{\mu_0\rho_S R^4}{3}\dfrac{\boldsymbol{\omega}\wedge\mathbf{r}}{r^3} & \text{for } 0<R<r \quad ,\\ \dfrac{\mu_0\rho_S R}{3}\boldsymbol{\omega}\wedge\mathbf{r} & \text{for } 0<r<R\end{cases} \quad .$$

As the result has been written in vector form and it is correct for the special choice of the Coordinate System it is always valid.
The **Magnetic Dipole** moment, integrating the infinitesimal rings perpendicular to the z axis, is:

- using the concept of total current:

$$\mathrm{d}\mathbf{m}_z = \mathrm{d}IA = \nu\rho_S\left(2\pi R\sin\left[\theta\right]\right)R\,\mathrm{d}\theta\left(\pi R^2\sin^2\left[\theta\right]\right) \quad ,$$

using the concept of surface current:

$$\mathrm{d}\mathbf{m}_z = |\mathbf{j}_S|R\,\mathrm{d}\theta A = \rho_S\omega R\sin\left[\theta\right]R\,\mathrm{d}\theta\left(\pi R^2\sin^2\left[\theta\right]\right) \quad ,$$

Integration gives:

$$\mathbf{m} = \frac{QR^2}{3}\boldsymbol{\omega} \quad .$$

The magnetic field inside the sphere is uniform:

$$\mathbf{B} = \frac{2}{3}\mu_0\rho_S R\omega \qquad r<R\text{, uniform inside the sphere} \quad .$$

The magnetic field outside the sphere is a pure **Magnetic Dipole** moment field.

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36-038 ✓ A Rotating Uniformly Charged Sphere

This § is referenced at pages:
[Never referenced.]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Problem 5.58|*Explicit calculation*|

A uniformly charged solid sphere of radius R carries a total charge Q. It is rotating with angular velocity ω about one diameter, the z axis.

1. What is the **Magnetic Dipole** moment of the sphere?
2. Find the average magnetic field within the sphere (use the results in section § 36.20.03.02).
3. Find the approximate **vector magnetic potential** at a point at distance $r \gg R$ from the center of the sphere.
4. Find the exact potential at a point at a point at distance $r \gg R$ from the center of the sphere, outside the sphere and check that it is consistent with the previuos question.
5. Find the magnetic field at a point inside the sphere, and check that it is consistent with § 36.20.03.02.

WORKED SOLUTION

Use the result of problem § 36-037: $\mathbf{m} = \dfrac{QR^2}{3}\boldsymbol{\omega}$ and let $\rho_S \to \rho\, dr$:

$$\mathbf{m} = \frac{QR^2}{5}\boldsymbol{\omega} \quad .$$

$$\mathbf{B}_{ave} = \frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3} \quad .$$

The field is a point **Dipole** field outside the sphere.

36-039 ✓ A Comparison of some Rotating Uniformly Charged Objects

Compare the **Magnetic Dipole** momenta of problems § 36-035, § 36-036 and § 36-037 and explain why the numerical factors of one with respect to the other are as they are for the same total charge.

WORKED SOLUTION

As far as the production of magnetic moment is concerned:

- Ring: it is the most efficient: the total charge is all at the maximum distance;
- Disk: it is less efficient than the ring;
- Spherical shell: it is less efficient than the ring but more efficient than the disk as the projection of the charge on the equatorial plane is not uniform but more densely packed near the edge.

DRAFT PRO

36-040 ☒ Explicit Construction of the Vector Potential

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 5.31||

Read also section § 16.09.03.03 and equation (16.09.02).

36-041 ✓ Static Fields in the Temporal Gauge

1. Consider two constant and homogeneous fields, **E** and **B**, and determine the **ElectroMagnetic potentials** that, in the temporal gauge ($\Phi = 0$), give rise to them.

36-042 ✓ ElectroMagnetic Potentials

This § is referenced at pages:
[2511]

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|P16.3||

Si consideri il campo elettromagnetico descritto dal potenziale:

$$\Phi_1 = 0 \qquad \mathbf{A}_1 = (xB_0 - tE_0)\,\widehat{\mathbf{e}}_2 \quad .$$

1. Calcolare i campi elettrico e magnetico.

WORKED SOLUTION

$$\mathbf{E} = E_0\widehat{\mathbf{e}}_2 \qquad \mathbf{B} = B_0\widehat{\mathbf{e}}_3 \quad .$$

2. Calcolare la circuitazione del campo elettrico attorno ad una qualunque linea chiusa.
3. Calcolare l'integrale di linea del campo elettrico tra i due punti generici $\{x_0, y_0, z_0\}$ e $\{x_1, y_1, z_1\}$.

WORKED SOLUTION

Il campo elettrico in questo problema è conservativo.

$$\int_O^P \mathbf{E}\cdot \mathbb{d}\mathbf{r} = E_0\,(y_1 - y_0) \quad .$$

4. Si consideri il potenziale:

$$\Phi_2 = -E_0 y \qquad \mathbf{A}_2 = \frac{1}{2}\mathbf{B}\wedge\mathbf{r} \qquad \text{con } \mathbf{B} = B_0\widehat{\mathbf{e}}_3 \quad .$$

Dimostrare che descrive lo stesso campo elettromagnetico del potenziale (1). Determinare la funzione che descrive la trasformazione di gauge che fa passare dal primo four-potenziale al secondo.

WORKED SOLUTION

$$\Lambda[\mathbf{x}, t] = -B_0\frac{xy}{2} + E_0 yt + \text{constant} \quad .$$

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36-043 ✓ A homopolar generator

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|3.290||

A Rotating Metal Disk as a homopolar generator.
A metal disc of radius $a = 25\ \mathrm{cm}$ rotates with a constant angular velocity $\omega = 130\ \mathrm{rad/s}$ about its axis. Find the potential difference between the center and the rim of the disc if:

1. the external magnetic field is absent;
2. the external uniform magnetic field is $B = 5.0\ \mathrm{mT}$ is directed perpendicular to the disc.

36-044 ✓ Energy Balance of a Bar Moving in a Magnetic Field

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|§ 36-4||

Consider the situation described in section § 36.20.24.

1. Write the energy balance, that is who does work and where the work done ends-up.

36-045 Freno Elettromagnetico E Correnti Di Focault

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|36.39||

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Un freno elettromagnetico a correnti di Focault è formato da un disco circolare di raggio R di materiale conduttore spesso s e con **conductivity** σ che ruota attorno al suo asse con velocità angolare ω. Un campo magnetico esterno costante ed omogeneo B perpendicolare al piano del disco è applicato su una piccola porzione del disco di dimensioni sia radiale che trasversale pari ad $a \ll R$ che dista $r \ll R$ dal centro del disco.

1. Determinare l'espressione approssimata del momento frenante sul disco e della potenza dissipata per effetto Joule.

WORKED SOLUTION

Si consideri l'areola alla stregua di un conduttore in moto in un campo magnetico esterno disposta radialmente al disco. Il conduttore è lungo a (misurato radialmente) e ha sezione as, nel piano trasverasale.
Il campo magnetico mette in moto le cariche libere e la corrente che ne risulta fa si che il conduttore venga frenato. La forza elettromotrice indotta vale allora:

$$|\mathcal{E}| \simeq vBa \quad ,$$

e può essere considerata localizzata nella areola dove c'è il campo magnetico. La corrente indotta assume un valore che dipende dalla resistenza del circuito. La corrente indotta si chiuderà nel disco in qualche modo. Di certo però la resistenza complessiva offerta dal circuito sarà poco superiore a quella della sola areola, in quanto il resto del conduttore è molto più grande. Allora, essendo,

$$R \simeq \frac{a}{\sigma a s} \quad ,$$

valore approssimato per difetto, si ha:

$$I \simeq \frac{vBa}{R} = \frac{vBa}{\frac{a}{\sigma a s}} = \sigma v B a s \quad .$$

La forza frenante sulla areola vale:

$$F \simeq \sigma v B^2 a^2 s \quad ,$$

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e il momento che ne risulta vale:

$$\Gamma \simeq \sigma\omega r^2 B^2 a^2 s \quad ,$$

Notare che la schematizzazione adottata per **B** è molto cruda in quanto il campo magnetico non può annularsi bruscamente al di fuori della areola entro cui è supposto essere costante.
La potenza dissipata per effetto Joule vale allora:

$$P_J \simeq RI^2 = \sigma\omega^2 r^2 B^2 a^2 s \quad .$$

Discussione della legge di induzione nella forma completa...

36-046 ✓ Forza Tra Le Armature Di Un Condensatore Piano

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|31.40||

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||

Si determini la forza di attrazione tra le due armature di un condensatore piano con due armature di area A a distanza x, cariche con carica Q.

Do the calculation by considering:

1. the electric field and surface charges;
2. the method of virtual works, that is use the change of the energy of the system for an infinitesimal change of the distance between the armatures, x, at constant charge;
3. the method of virtual works, that is use the change of the energy of the system for an infinitesimal change of the distance between the armatures, x, at constant voltage;
4. the concept of ElectroStatic pressure.

36-047 ✓ Generation of the Earth Magnetic Field

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|37.27 37.28||

The Earth magnetic field can be described as a **Magnetic Dipole** field generated by a **Dipole** moment $m = 8.0{\cdot}10^{22}$ J/T located - approximately - at the center of the Earth and aligned - approximately - with the Earth rotation axis.

1. What current, flowing inside a hypothetical current loop running all around the equator, would be able to produce, at large distances, such a field?
2. Could one use such an apparatus to cancel the magnetic field at large distances?
3. Could one use such an apparatus to cancel the magnetic field at the Earth surface?
4. If the field would be produced by a iron sphere uniformly Magnetized, at saturation, what radius would be necessary for such a sphere? The **Magnetic Dipole** moment of a iron atom is $m = 2.1{\cdot}10^{-23}$ J/T.

36-048 ✓ Earth Magnetic Field

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|37.29||

The Earth magnetic field can be described as a **Magnetic Dipole** field generated by a **Dipole** moment $m = 8.0{\cdot}10^{22}$ J/T located - approximately - at the center of the Earth and aligned - approximately - with the Earth rotation axis. The real axis is not exactly aligned with the Earth rotation axis, however. The latitude as measured from the equator of the equatorial plane of the magnetic field is called the magnetic latitude, L_m.

1. Show that the intensity of the field is given by:

$$B = \frac{\mu_0}{4\pi}\frac{m}{r^3}\sqrt{1 + 3\left(\sin\left[L_m\right]\right)^2} \quad .$$

2. Show that the inclination, ϕ, of the field related to the magnetic latitude by:

$$\tan\left[\phi\right] = 2\tan\left[L_m\right] \quad .$$

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36-049 Earth Magnetic Field MagnetoStatic Pressure

Consider the Earth Magnetic Field.

1. Determine the magnetic pressure due to the Earth magnetic field at the magnetic poles.
2. Compare the value with the Earth atmospheric pressure
3. Assume that the **Magnetic Dipole** moment is proportional to the angular velocity of the Earth: how much faster would the angular velocity need to be for the magnetic pressure to be comparable to the atmospheric pressure?

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CHAPTER 37

Electric|Magnetic Properties of Matter

37.01 Introduction 1813
37.02 A Bird's-eye View of Microscopic Properties of Matter and Radiation 1816
37.03 Macroscopic Description of ElectroMagnetic Fields in Presence of Matter 1828
37.04 Examples and Physical Applications 1833
37.05 Exercises Problems and Physical Applications 1836

DRAFT-IN-PROGRESS

This § is referenced at pages:
[1691]

Pre-requisites: review the ***Multipole*** *expansion.*

©|W.Demtröder|Atoms, Molecules and Photons; § 1 and § 11|*Excellent! With Measuring Techniques*|
©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|6|*Good text.*|
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|1.4||
©|J.Franklin, *Classical ElectroMagnetism*, 2017, 2ndEd, ...Ed., WEB - URL.|||
©|F.N.H.Robinson, *Macroscopic ElectroMagnetism*, ..., ..., ...Ed., .||*Excellent! Definitive*|
©|Haken-Wolf||*Simple Introduction*|
©|JMRaimond||*Material media and linear response*|

DRAFT - IN-PROGRESS

37.01 Introduction

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|6.0||

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Everyday macroscopic world is dominated by the ElectroMagnetic interactions. In fact, gravitational interactions are so weak that, apart from weight, have little effect on everyday phenomena. On the other hand, weak and strong interactions are only relevant at microscopic level, because they are short-range interactions, with ranges of the order of $1\ \mathrm{fm}$ or less.

Matter is almost everywhere: therefore it is important to understand how ElectroMagnetic fields behave in presence of matter and the other way round.

The interaction between ElectroMagnetic fields (including ElectroMagnetic waves) and matter explains many common phenomena, such as:

- Electric phenomena:
  - the attraction of small pieces of paper by an electrised object;
  - the attraction of a Polarizable material inside a charged capacitor;
  - the increase of the **capacitance** of a capacitor when inserting a material medium between its faces;
  - the electric screening effect of a Polarizable material.
- Magnetic phenomena:
  - the strong attraction of iron materials from a magnet;
  - the weak attraction of some materials (such a aluminum) from a magnet;
  - the weak repulsion of some materials (such as copper and silver) from a magnet;
  - the increase of the inductance of a solenoid when a iron material is inserted inside it;
  - the magnetic screening effect of certain materials.
- EM waves phenomena:
  - why the sky is blue and the Sun is red at sunset;
  - why gold is yellowish;
  - how **Polaroid** glasses work;
  - why X-rays are refracted in a opposite way with respect to light rays;
  - why at a certain point of the reenter maneuver of a spaceship the radio communication is not working;
  - why silver coatings give good mirrors.

The ElectroMagnetic fields interacts with the constituents of matter, including atomic nuclei, electrons, atoms and molecules, as well as elementary particles. **The correct and full explanation of these interactions must use quantum-relativistic physics** but most phenomena can be approximately and easily described by means of classical physics on the macroscopic scale.

The interaction of charged (point) particles with matter is classically described by the Lorentz force:

$$\boxed{\mathbf{F} = q\,(\mathbf{E} + \mathbf{v}\wedge\mathbf{B})} \quad .$$

Particles, atoms and molecules might also possess an Electric|Magnetic permanent **Dipole** moment and also higher order Electric|Magnetic **multipole** moments. In the former case the interaction must include the effect of the interaction of the **Dipole** moment with the field (Read § 38, § 39), the interaction energy being:

$$\boxed{\Phi_e = -\,\mathbf{p}\cdot\mathbf{E} \qquad \Phi_m = -\,\mathbf{m}\cdot\mathbf{B} \qquad \text{ifoif } \mathbf{p}/\mathbf{m} \text{ are permanent/fixed/rigid Dipoles}} \quad ,$$

in the case the **Electric Dipole** | **Magnetic Dipole** moment are considered to be permanent (that is fixed) and not dependent of the external (applied) fields. Read § 41.02.01 for linear Polarizable|Magnetizable Dipoles. The involved interaction energies turn out to be small in most cases.

Even if no permanent Dipole moment is present, in absence of external fields, the presence of external fields may induce one; the system is Polarizable|Magnetizable. Note that this may be also the case for subatomic particles. Matter made of constituents having **permanent dipoles** is normally called **(di)polar matter**, for both the electric and the magnetic case.

The interactions with matter of electric with respect to magnetic fields play very different roles. The reason is simple and fundamental. Atoms and molecules are made of electricly charged particles which move with velocities generally small compared with the speed of light. A magnetic field exerts no force at all on a stationary electric charge while the force on a moving charged particle is proportional to the particle speed and therefore small for non-relativistic velocities. For instance, modules of electric and magnetic fields of ElectroMagnetic waves are linked by the relation $E = cB$, so that **the ratio of electric to magnetic forces is of the order of** v/c (see section § 46). Electric forces overwhelmingly dominate the atomic scene, not magnetic forces. Magnetism appears, in our world at least, to be a relativistic effect. The story would be different if matter were made of magnetically charged particles.

Two different extreme models are usually considered to simplify the modeling of phenomena, in the so-called classical theory of ElectroMagnetism in matter:

- the case of a **perfectly insulating material**, where all the electrons are bound near a certain region (atom, molecule, ...) and kept there by an attractive force; generally speaking in a perfectly insulating material there are no charges free to move at all; a perfectly insulating material is called a **dielectric material** or simply **a dielectric**;
- the case of a **perfectly conducting material**, where there exist either electrons or ions (and|or any other charged particles) totally free to move; generally speaking in a perfectly conducting material there are a number of charges completely free to move, such as in a plasma; a perfectly conducting material is often just called a **conducting material** or simply **a conductor**.

Obviously the distinction between perfect insulators/conductors is not a sharp one and most material cannot be considered neither perfect insulators nor perfect conductors.

Two approaches are possible.

- In the **microscopic approach** the interaction is described at a microscopic level.
- In the **macroscopic approach** one does not attempt to describe every single particle, but the bulk properties of matter are considered on a large scale, expressing the properties of the medium in presence of the ElectroMagnetic fields; **electric susceptibility**, **magnetic susceptibility** and **conductivity** are the three basic properties. The advantage of the latter approach is that the properties used to describe the interaction between ElectroMagnetic fields and matter can be easily measured and can be incorporated in the basic laws of ElectroMagnetism. Note that those macroscopic properties, by their definition, are meaningful in every context, also beyond classical physics.

A decent microscopic discussion requires the use of modern quantum-relativistic physics as well as statistical physics.

37.01.01
Inadequacy of Classical Physics to a Microscopic Description

Classical physics in unable to give a full and coherent description of microscopic ElectroMagnetic phenomena, even if in most cases it gives models and predictions very close to the reality.

- One first example is the picture of a neutral atom as one positive point charge (the nucleus) with the electrons orbiting around the nucleus which would give rise, classically, to a time-changing **Electric Dipole** which would radiate energy thus making the atom unstable.
- One second example is that classical physics does not give any magnetic phenomenon in matter, that is to say if classical physics is followed coherently until the end it can be shown that no magnetic phenomenon exists [a].

[a] R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...

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R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...
...it is not possible to understand the magnetic effects of materials in any honest way from the point of view of classical physics. Such magnetic effects are a completely quantum-mechanical phenomenon. It is, however, possible to make some phony classical arguments and to get some idea of what is going on.

37.01.02
Nomenclature and Conventions

- The term **dielectric material** will be used to identify a perfectly insulating material.
- The effects of external electric fields on matter are called **(induced) Polarization**.
- The effects of external magnetic fields on matter are called **(induced) Magnetization**.
- The term **Polarizable material** will be used to identify a material which Polarizes, that is it acquires a **Electric Dipole** moment, if and only if it is under the effect of external electric fields.
- The term **Magnetizable material** will be used to identify a material which Magnetizes, that is it acquires a **Magnetic Dipole** moment, if and only if it is under the effect of external magnetic fields.

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37.02 A Bird's-eye View of Microscopic Properties of Matter and Radiation

This § is referenced at pages:
[2081, 2083]

©|Perspectives on high-energy-density physics|Physics of Plasmas 16, 055501 (2009- ...

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.1; 1.2|*Elementary overview*|

©|dePodesta|||

The components of the microscopic world necessary to describe ElectroMagnetism in matter are: molecules; atoms (and their constituents: protons, electrons and neutrons); EM radiation (photons); ionized Atoms|Molecules, producing positively or negatively charged atomic/molecular ions and free electrons.

The behavior of such components may be substantially different from that of ordinary macroscopic objects and must be described by **Quantum Relativistic Physics**. However, many phenomena can be usefully described, at least from a qualitative point of view, by semi-classical models.

It is useful to introduce, without any attempt of derivation nor completeness, some ultra-elementary concepts about the quantum-relativistic treatment of the particle-like and wave-like properties of elementary particles, even if this goes beyond the realm of Classical Physics.

In what follows it will be always assumed that one deals with the microscopic ElectroMagnetic fields and their microscopic description, that is no macroscopic description of matter is included. In other words ElectroMagnetic fields and particles are considered in vacuum.

37.02.01
Dynamical Quantities: Mass Energy Linear|Angular Momentum in Relativistic Physics

Relativistic Physics links space and time and is relevant only for speeds close to the speed of light.

Energy and Linear Momentum

Energy and linear momentum are given by:

$$\epsilon = m\gamma c^2 \qquad \mathbf{p} = m\gamma \mathbf{v} \qquad \text{with} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad .$$

The relation between energy and linear momentum is given by:

$$\epsilon = \sqrt{p^2c^2 + m^2c^4} \quad .$$

Moreover, the speed is given by:

$$\mathbf{v} = c^2 \frac{\mathbf{p}}{\epsilon} \quad .$$

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Angular Momentum

37.02.02

Dynamical Quantities: Mass Energy Linear|Angular Momentum in Quantum Physics

Quantum Physics introduces a wave-like character for any particle as well as a particle-like character for any wave.

Energy and Linear Momentum

The link between energy and frequency of the particle/wave is:

$$\epsilon = h\nu = \hbar\omega \qquad \text{with the Planck constant, a fundamental constant of Nature,} \qquad \hbar \equiv \frac{h}{2\pi} = 1.055 \cdot 10^{-34}\ \mathrm{J \cdot s} \quad .$$

Physical observables of a system may be **quantised**, in Quantum Physics, that is they cannot assume any value but they can only assume a well-defined set of discrete values related to the Planck constant, the fundamental constant of Quantum Physics.

The energy and linear momentum may be **quantised**.

Angular Momentum

The angular momentum is always **quantised**, in Quantum Physics, that is it cannot assume any value but it can only assume a well-defined set of discrete values related to the Planck constant, the fundamental constant of Quantum Physics.

The full description of angular momentum requires Quantum Physics.

Let us only consider a generic angular momentum, $\mathbf{j}$, and its projection along any arbitrary but fixed axis (let us say the z axis).

Any angular momentum, $\mathbf{j}$, in Quantum Physics can assume a squared module value given by:

$$|\mathbf{j}|^2 = j\,(j+1)\,\hbar^2 \qquad \text{with } j \text{ any non-negative integer or positive half-odd value} \quad .$$

For an angular momentum j the projection of the angular momentum along any fixed axis (assume the z axis) can assume the $2j+1$ values:

$$j_z = (-j, -j+1, \ldots, +j-1, +j)\,\hbar \quad .$$

Orbital Angular Momentum

As for classical physics one deals with orbital angular momentum:

$$\boldsymbol{\ell} \equiv \mathbf{r} \wedge \mathbf{p} \quad .$$

Any orbital angular momentum, $\boldsymbol{\ell}$, in Quantum Physics can assume a squared module value given by:

$$|\boldsymbol{\ell}|^2 = \ell\,(\ell+1)\,\hbar^2 \qquad \text{with } \ell \text{ any non negative integer} \quad .$$

The component along any axis (assume the z axis) can only assume integer values (in units of $\hbar$):

$$\ell_z = k\hbar \qquad \text{with } k \text{ any integer number with } |k| \leq \ell \quad .$$

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Spin Angular Momentum

Elementary particles also have an intrinsic angular momentum, the spin angular momentum, **s**, not connected to any kind of **rotation** of the particle around itself, but acting under all respects as an intrinsic angular momentum, that is an angular momentum with respect to the location of the particle (for point particles) or with respect to the Center-Of-Mass of the system (for extended, non point systems).

Most common elementary particles have either spin $1/2$ (matter particles) or spin 1 (mediators of interactions).

Most common particles, including protons, neutrons and electrons, have spin $1/2$, that is the component of the spin angular momentum along any axis (assume the z axis) can only assume the values:

$$\boxed{s_z = \pm\frac{\hbar}{2}} \quad .$$

Other particles, the most common being the photons, have spin 1, that is the component of the spin angular momentum along any axis (assume the z axis) can only assume the values:

$$\boxed{s_z = -\hbar, 0, +\hbar} \quad .$$

The photon is special as a spin 1 particle, because its mass is zero. In fact, the $s_z = 0$ value does not exist and therefore the component of the spin angular momentum along any axis can only assume the two values: $s_z = \pm\hbar$.

Total Angular Momentum

The total angular momentum of any system of particles is given by the sum of the orbital angular momenta, $\boldsymbol{\ell}$, plus the spin angular momenta, **s**:

$$\mathbf{J} = \sum_k (\boldsymbol{\ell}_k + \mathbf{s}_k) \quad .$$

37.02.03
Mole and Avogadro Number

The concept of mole and Avogadro number provide the link between the microscopic world and the macroscopic one.

The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of Carbon 12 (at rest and in their ground state); the symbol is mol. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles (or possibly any kind of objects).

A mole of objects contains a **number of Avogadro** objects.

The molar mass of any substance is the mass of a **number of Avogadro** objects (measured in $\mathrm{kg/mol}$ in the SI).

$$\boxed{N_A = 6.022{\cdot}10^{23}\ \mathrm{mol}^{-1}} \quad .$$

37.02.04
Matter - Atoms Molecules and Common Elementary Particles

All common matter is made of atoms.

Atoms may bond each other via **chemical bonds**. A chemical bond is a **purely quantum-mechanical phenomenon**.

Atoms may group in small molecules (a few atoms), in large molecules (more than hundreds atoms), or in crystalline solids, which are large regular structures made of, a priori, an infinite number of atoms or molecules. The terms amorphous solid, non-crystalline solid, disordered solid, glass or liquid have no precise structural meaning beyond the description that the structure is not crystalline on any significant scale. The separation of nearest neighbor atoms or molecules is approximately constant. Disordered crystalline alloys are such that different atoms randomly occupy the sites of a regular crystal lattice.

Atoms group by sharing some of their electrons.

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Atoms and molecules may appear in either the gaseous or liquid state.

Quite often Atoms|Molecules may be ionized, by loosing or capturing electrons and therefore ordinary matter may be also composed of positive ions and negative free electrons. A plasma is a gas made of positive ions and negative free electrons.

Every atom is made of a nucleus, made of protons and neutrons, plus electrons orbiting around the nucleus. The orbiting electrons are more appropriately thought as an **negative charge diffuse electron cloud**.

The number of protons in an atom is called the atomic number, Z. It is equal to the number of electrons, for a neutral atom. The total number of protons plus neutrons in a nucleus (collectively called nucleons) is called the mass number of the nucleus, A. It is $A \geq Z$. Chemical properties are only determined by the atomic number Z and there exist, for many elements, different isotopes, that is atoms with the same atomic number but different mass number.

Atoms of different elements form a logic structure described by D. I. Mendeleev, explained by the shell structure of the electron orbitals which, in turn, is explained by Quantum Relativistic Physics.

The atomic radius as a function of the atomic number for the elements is shown in figure 37.11, showing the behavior dictated by the electronic shell structure.

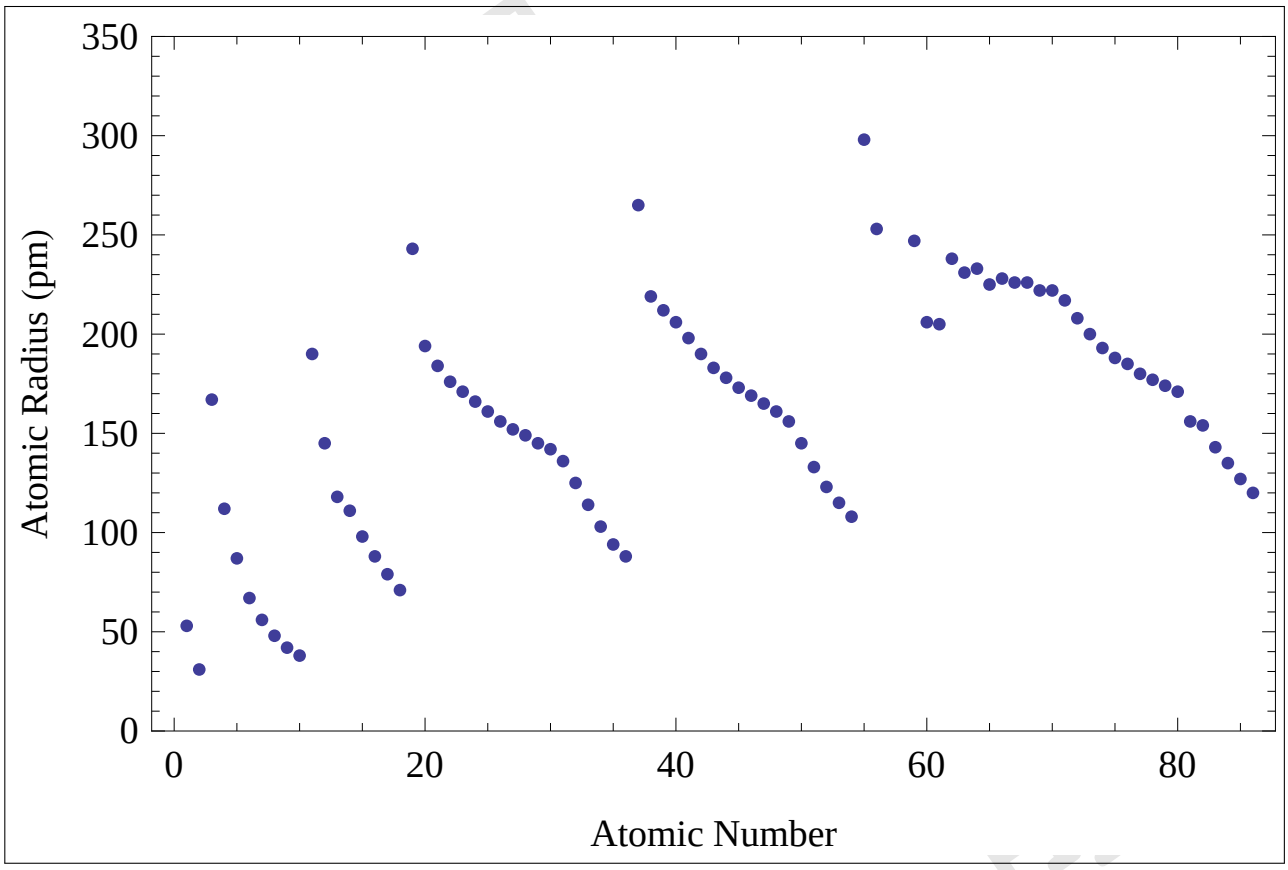

Figure 37.11: Atomicradius as a function of the atomic number for the elements.

The molar volume as a function of the atomic number for the elements is shown in figure 37.12, showing the behavior dictated by the electronic shell structure.

All the interaction between atoms and on scales larger that the atomic scale up to the everyday world are dominated by ElectroMagnetic interactions.

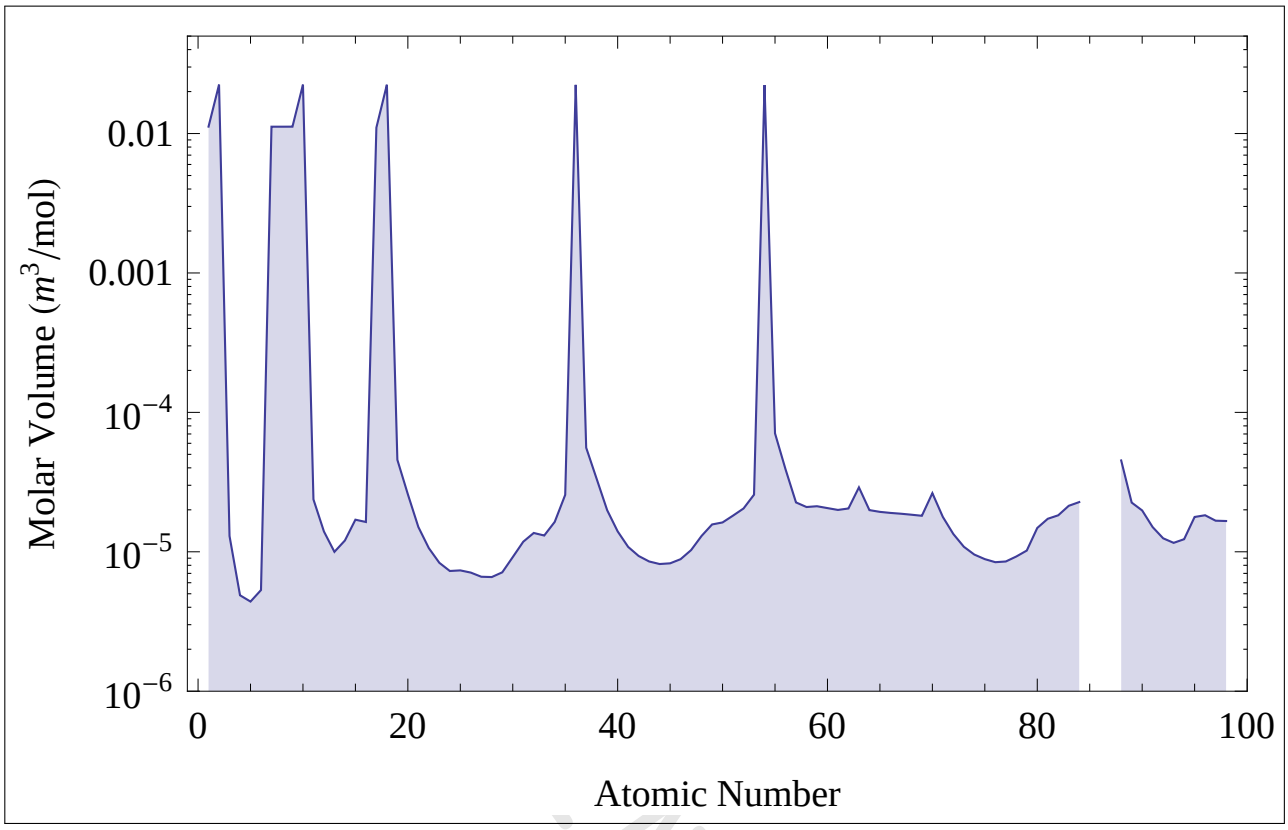

Figure 37.12: Molar volume as a function of the atomic number for the elements.

37.02.04.01 Mass Charge Size and Lifetime

It is: $m_p \simeq m_n \simeq 1836 \cdot m_e$ and $m_e = 9.11 \cdot 10^{-31}$ kg.

Given the above values of the masses it turns out that, to a first approximation, every atom has a mass roughly equal to $M \simeq A$ u, with u the atomic mass unit. It is defined as one twelfth of the rest mass of an unbound atom of Carbon-12 in its nuclear and electronic ground state: u $= 1.66 \cdot 10^{-27}$ kg.

It is: $-q_e = q_p = 1.6 \cdot 10^{-19}$ C and $q_n = 0$.

The typical size of an atomic nucleus is given by: $r = r_0 A^{1/3}$, with $r_0 = 1.25 \cdot 10^{-15}$ m.

All the neutral isolated atoms have a radius of the order of $R \approx (0.3 \div 3.0) \cdot 10^{-10}$ m.

Electrons, as far as one knows, are elementary structure-less point particles. Protons and neutrons have a non zero size of about 10^{-15} m and are made of quarks.

Electrons and protons, as far as one knows, are stable particles. Neutrons are stable when bound inside an atomic nucleus but they have a lifetime of about 15 min when they are free.

37.02.05
Atoms and Aggregates of Atoms

Atoms: perfectly well defined.

Atoms can aggregate in many ways into molecules via a chemical bond, a purely quantum-mechanical effect:

- single-atom molecules: noble gases;
- a-few-atoms molecules: water, carbon dioxide, etc.;
- many-atoms molecules: organic molecules, sugar, etc.;
- macro-molecules: millions of AMU (e.g. DNA);
- crystalline solids: lattice of positive and negative ions, atom identity lost;
- metallic solid: valence electrons shared by the positive ions of the lattice, atom identity lost.

In general, any recognizable aggregation among atoms will be called, in short, molecule.

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37.02.06
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37.02.06.01 The Periodic table of the Elements

This § is referenced at pages:
[Never referenced.]

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Table 4.1. Revised June 2019 by D.E. Groom (LBNL). The atomic number (top left) is the number of protons in the nucleus. The atomic masses (bottom) of stable elements are weighted by isotopic abundances in the Earth's surface. Atomic masses are relative to the mass of ^{12}C, defined to be exactly 12 unified atomic mass units (u) (1 u $\approx$ 1 g/mole). The exceptions are Th, Pa, and U, which have no stable isotopes but do have characteristic terrestrial compositions. Relative isotopic abundances often vary considerably, both in natural and commercial samples; this is reflected in the number of significant figures given for the mass. Masses may be found at `https://www.nist.gov/pml/atomic-weights-and-isotopic-compositions-relative-atomic-masses`. If there is no stable isotope, the atomic mass of the most stable isotope known as of June 2019 is given in parentheses.

IUPAC announced verification of the discoveries of elements 113, 115, 117, and 118 in December 2015. The names were approved November 2016. The 7th period of the periodic table is now complete.

PERIODIC TABLE OF THE ELEMENTS

1 IA												13 IIIA	14 IVA	15 VA	16 VIA	17 VIIA	18 VIIIA
1 H hydrogen 1.008	2 IIA																2 He helium 4.002602
3 Li lithium 6.94	4 Be beryllium 9.012182											5 B boron 10.81	6 C carbon 12.0107	7 N nitrogen 14.007	8 O oxygen 15.999	9 F fluorine 18.998403163	10 Ne neon 20.1797
11 Na sodium 22.98976928	12 Mg magnesium 24.305	3 IIIB	4 IVB	5 VB	6 VIB	7 VIIB	8	9 VIII	10	11 IB	12 IIB	13 Al aluminum 26.9815385	14 Si silicon 28.085	15 P phosphorus 30.973761998	16 S sulfur 32.06	17 Cl chlorine 35.45	18 Ar argon 39.948
19 K potassium 39.0983	20 Ca calcium 40.078	21 Sc scandium 44.955908	22 Ti titanium 47.867	23 V vanadium 50.9415	24 Cr chromium 51.9961	25 Mn manganese 54.938044	26 Fe iron 55.845	27 Co cobalt 58.933195	28 Ni nickel 58.6934	29 Cu copper 63.546	30 Zn zinc 65.38	31 Ga gallium 69.723	32 Ge germanium 72.630	33 As arsenic 74.921595	34 Se selenium 78.971	35 Br bromine 79.904	36 Kr krypton 83.798
37 Rb rubidium 85.4678	38 Sr strontium 87.62	39 Y yttrium 88.90584	40 Zr zirconium 91.224	41 Nb niobium 92.90637	42 Mo molybdenum 95.95	43 Tc technetium (97.907212)	44 Ru ruthenium 101.07	45 Rh rhodium 102.9055	46 Pd palladium 106.42	47 Ag silver 107.8682	48 Cd cadmium 112.414	49 In indium 114.818	50 Sn tin 118.710	51 Sb antimony 121.760	52 Te tellurium 127.60	53 I iodine 126.90447	54 Xe xenon 131.293
55 Cs caesium 132.90545196	56 Ba barium 137.327	57–71 LANTHA-NIDES	72 Hf hafnium 178.49	73 Ta tantalum 180.94788	74 W tungsten 183.84	75 Re rhenium 186.207	76 Os osmium 190.23	77 Ir iridium 192.217	78 Pt platinum 195.084	79 Au gold 196.966569	80 Hg mercury 200.592	81 Tl thallium 204.38	82 Pb lead 207.2	83 Bi bismuth 208.98040	84 Po polonium (208.98243)	85 At astatine (209.98715)	86 Rn radon (222.01758)
87 Fr francium (223.01974)	88 Ra radium (226.02541)	89–103 ACTINIDES	104 Rf rutherford. (267.12169)	105 Db dubnium (268.12567)	106 Sg seaborgium (269.12863)	107 Bh bohrium (270.13336)	108 Hs hassium (269.13375)	109 Mt meitnerium (278.1563)	110 Ds darmstadt. (281.1645)	111 Rg roentgen. (282.16912)	112 Cn copernicium (285.17712)	113 Nh nihonium (286.18221)	114 Fl flerovium (289.19042	115 Mc moscovium (290.19598)	116 Lv livermorium (293.20449	117 Ts tennessine (294.2105)	118 Og oganesson (294.21392)

Lanthanide series	57 La lanthanum 138.90547	58 Ce cerium 140.116	59 Pr praseodym. 140.90765	60 Nd neodymium 144.242	61 Pm promethium (144.91276)	62 Sm samarium 150.36	63 Eu europium 151.964	64 Gd gadolinium 157.25	65 Tb terbium 158.92535	66 Dy dysprosium 162.500	67 Ho holmium 164.93033	68 Er erbium 167.259	69 Tm thulium 168.93422	70 Yb ytterbium 173.054	71 Lu lutetium 174.9668
Actinide series	89 Ac actinium (227.02775)	90 Th thorium 232.0377	91 Pa protactinium 231.03588	92 U uranium 238.02891	93 Np neptunium (237.04817)	94 Pu plutonium (244.06420)	95 Am americium (243.06138)	96 Cm curium (247.07035)	97 Bk berkelium (247.07031)	98 Cf californium (251.07959)	99 Es einsteinium (252.08298)	100 Fm fermium (257.09511)	101 Md mendelevium (258.09844)	102 No nobelium (259.10103)	103 Lr lawrencium (262.10961)

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37.02.06.02 Atomic electronic structure of the elements

This § is referenced at pages:
[Never referenced.]

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5. Electronic Structure of the Elements

Table 5.1. Reviewed 2011 by J.E. Sansonetti (NIST). The electronic configurations and the ionization energies are from the NIST database, "Ground Levels and Ionization Energies for the Neutral Atoms," W.C. Martin, A. Musgrove, S. Kotochigova, and J.E. Sansonetti, `http://www.nist.gov/pml/data/ion_energy.cfm`. The electron configuration for, say, iron indicates an argon electronic core (see argon) plus six $3d$ electrons and two $4s$ electrons.

	Element		Electron configuration ($3d^5$ = five $3d$ electrons, *etc.*)		Ground state $^{2S+1}L_J$	Ionization energy (eV)
1	H	Hydrogen	$1s$		$^2S_{1/2}$	13.5984
2	He	Helium	$1s^2$		$^1S_0$	24.5874
3	Li	Lithium	(He) $2s$		$^2S_{1/2}$	5.3917
4	Be	Beryllium	(He) $2s^2$		$^1S_0$	9.3227
5	B	Boron	(He) $2s^2$ $2p$		$^2P_{1/2}$	8.2980
6	C	Carbon	(He) $2s^2$ $2p^2$		$^3P_0$	11.2603
7	N	Nitrogen	(He) $2s^2$ $2p^3$		$^4S_{3/2}$	14.5341
8	O	Oxygen	(He) $2s^2$ $2p^4$		$^3P_2$	13.6181
9	F	Fluorine	(He) $2s^2$ $2p^5$		$^2P_{3/2}$	17.4228
10	Ne	Neon	(He) $2s^2$ $2p^6$		$^1S_0$	21.5645
11	Na	Sodium	(Ne) $3s$		$^2S_{1/2}$	5.1391
12	Mg	Magnesium	(Ne) $3s^2$		$^1S_0$	7.6462
13	Al	Aluminum	(Ne) $3s^2$ $3p$		$^2P_{1/2}$	5.9858
14	Si	Silicon	(Ne) $3s^2$ $3p^2$		$^3P_0$	8.1517
15	P	Phosphorus	(Ne) $3s^2$ $3p^3$		$^4S_{3/2}$	10.4867
16	S	Sulfur	(Ne) $3s^2$ $3p^4$		$^3P_2$	10.3600
17	Cl	Chlorine	(Ne) $3s^2$ $3p^5$		$^2P_{3/2}$	12.9676
18	Ar	Argon	(Ne) $3s^2$ $3p^6$		$^1S_0$	15.7596
19	K	Potassium	(Ar) $4s$		$^2S_{1/2}$	4.3407
20	Ca	Calcium	(Ar) $4s^2$		$^1S_0$	6.1132
21	Sc	Scandium	(Ar) $3d$ $4s^2$	Transition elements	$^2D_{3/2}$	6.5615
22	Ti	Titanium	(Ar) $3d^2$ $4s^2$		$^3F_2$	6.8281
23	V	Vanadium	(Ar) $3d^3$ $4s^2$		$^4F_{3/2}$	6.7462
24	Cr	Chromium	(Ar) $3d^5$ $4s$		$^7S_3$	6.7665
25	Mn	Manganese	(Ar) $3d^5$ $4s^2$		$^6S_{5/2}$	7.4340
26	Fe	Iron	(Ar) $3d^6$ $4s^2$		$^5D_4$	7.9024
27	Co	Cobalt	(Ar) $3d^7$ $4s^2$		$^4F_{9/2}$	7.8810
28	Ni	Nickel	(Ar) $3d^8$ $4s^2$		$^3F_4$	7.6399
29	Cu	Copper	(Ar) $3d^{10}$ $4s$		$^2S_{1/2}$	7.7264
30	Zn	Zinc	(Ar) $3d^{10}$ $4s^2$		$^1S_0$	9.3942
31	Ga	Gallium	(Ar) $3d^{10}$ $4s^2$ $4p$		$^2P_{1/2}$	5.9993
32	Ge	Germanium	(Ar) $3d^{10}$ $4s^2$ $4p^2$		$^3P_0$	7.8994
33	As	Arsenic	(Ar) $3d^{10}$ $4s^2$ $4p^3$		$^4S_{3/2}$	9.7886
34	Se	Selenium	(Ar) $3d^{10}$ $4s^2$ $4p^4$		$^3P_2$	9.7524
35	Br	Bromine	(Ar) $3d^{10}$ $4s^2$ $4p^5$		$^2P_{3/2}$	11.8138
36	Kr	Krypton	(Ar) $3d^{10}$ $4s^2$ $4p^6$		$^1S_0$	13.9996
37	Rb	Rubidium	(Kr) $5s$		$^2S_{1/2}$	4.1771
38	Sr	Strontium	(Kr) $5s^2$		$^1S_0$	5.6949
39	Y	Yttrium	(Kr) $4d$ $5s^2$		$^2D_{3/2}$	6.2173
40	Zr	Zirconium	(Kr) $4d^2$ $5s^2$		$^3F_2$	6.8339
41	Nb	Niobium	(Kr) $4d^4$ $5s$		$^6D_{1/2}$	6.7589
42	Mo	Molybdenum	(Kr) $4d^5$ $5s$		$^7S_3$	7.0924
43	Tc	Technetium	(Kr) $4d^5$ $5s^2$		$^6S_{5/2}$	7.28
44	Ru	Ruthenium	(Kr) $4d^7$ $5s$		$^5F_5$	7.3605
45	Rh	Rhodium	(Kr) $4d^8$ $5s$		$^4F_{9/2}$	7.4589
46	Pd	Palladium	(Kr) $4d^{10}$		$^1S_0$	8.3369
47	Ag	Silver	(Kr) $4d^{10}$ $5s$		$^2S_{1/2}$	7.5762
48	Cd	Cadmium	(Kr) $4d^{10}$ $5s^2$		$^1S_0$	8.9938

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49	In	Indium	(Kr) $4d^{10}5s^2$	$5p$				$^2P_{1/2}$	5.7864
50	Sn	Tin	(Kr) $4d^{10}5s^2$	$5p^2$				$^3P_0$	7.3439
51	Sb	Antimony	(Kr) $4d^{10}5s^2$	$5p^3$				$^4S_{3/2}$	8.6084
52	Te	Tellurium	(Kr) $4d^{10}5s^2$	$5p^4$				$^3P_2$	9.0096
53	I	Iodine	(Kr) $4d^{10}5s^2$	$5p^5$				$^2P_{3/2}$	10.4513
54	Xe	Xenon	(Kr) $4d^{10}5s^2$	$5p^6$				$^1S_0$	12.1298
55	Cs	Cesium	(Xe)	$6s$				$^2S_{1/2}$	3.8939
56	Ba	Barium	(Xe)	$6s^2$				$^1S_0$	5.2117
57	La	Lanthanum	(Xe) $5d$	$6s^2$				$^2D_{3/2}$	5.5769
58	Ce	Cerium	(Xe) $4f$ $5d$	$6s^2$				$^1G_4$	5.5387
59	Pr	Praseodymium	(Xe) $4f^3$	$6s^2$				$^4I_{9/2}$	5.473
60	Nd	Neodymium	(Xe) $4f^4$	$6s^2$				$^5I_4$	5.5250
61	Pm	Promethium	(Xe) $4f^5$	$6s^2$				$^6H_{5/2}$	5.582
62	Sm	Samarium	(Xe) $4f^6$	$6s^2$				$^7F_0$	5.6437
63	Eu	Europium	(Xe) $4f^7$	$6s^2$				$^8S_{7/2}$	5.6704
64	Gd	Gadolinium	(Xe) $4f^7$ $5d$	$6s^2$				$^9D_2$	6.1498
65	Tb	Terbium	(Xe) $4f^9$	$6s^2$				$^6H_{15/2}$	5.8638
66	Dy	Dysprosium	(Xe) $4f^{10}$	$6s^2$				$^5I_8$	5.9389
67	Ho	Holmium	(Xe) $4f^{11}$	$6s^2$				$^4I_{15/2}$	6.0215
68	Er	Erbium	(Xe) $4f^{12}$	$6s^2$				$^3H_6$	6.1077
69	Tm	Thulium	(Xe) $4f^{13}$	$6s^2$				$^2F_{7/2}$	6.1843
70	Yb	Ytterbium	(Xe) $4f^{14}$	$6s^2$				$^1S_0$	6.2542
71	Lu	Lutetium	(Xe) $4f^{14}$ $5d$	$6s^2$				$^2D_{3/2}$	5.4259
72	Hf	Hafnium	(Xe) $4f^{14}5d^2$	$6s^2$				$^3F_2$	6.8251
73	Ta	Tantalum	(Xe) $4f^{14}5d^3$	$6s^2$				$^4F_{3/2}$	7.5496
74	W	Tungsten	(Xe) $4f^{14}5d^4$	$6s^2$				$^5D_0$	7.8640
75	Re	Rhenium	(Xe) $4f^{14}5d^5$	$6s^2$				$^6S_{5/2}$	7.8335
76	Os	Osmium	(Xe) $4f^{14}5d^6$	$6s^2$				$^5D_4$	8.4382
77	Ir	Iridium	(Xe) $4f^{14}5d^7$	$6s^2$				$^4F_{9/2}$	8.9670
78	Pt	Platinum	(Xe) $4f^{14}5d^9$	$6s$				$^3D_3$	8.9588
79	Au	Gold	(Xe) $4f^{14}5d^{10}$	$6s$				$^2S_{1/2}$	9.2255
80	Hg	Mercury	(Xe) $4f^{14}5d^{10}$	$6s^2$				$^1S_0$	10.4375
81	Tl	Thallium	(Xe) $4f^{14}5d^{10}6s^2$	$6p$				$^2P_{1/2}$	6.1082
82	Pb	Lead	(Xe) $4f^{14}5d^{10}6s^2$	$6p^2$				$^3P_0$	7.4167
83	Bi	Bismuth	(Xe) $4f^{14}5d^{10}6s^2$	$6p^3$				$^4S_{3/2}$	7.2855
84	Po	Polonium	(Xe) $4f^{14}5d^{10}6s^2$	$6p^4$				$^3P_2$	8.414
85	At	Astatine	(Xe) $4f^{14}5d^{10}6s^2$	$6p^5$				$^2P_{3/2}$	
86	Rn	Radon	(Xe) $4f^{14}5d^{10}6s^2$	$6p^6$				$^1S_0$	10.7485
87	Fr	Francium	(Rn)	$7s$				$^2S_{1/2}$	4.0727
88	Ra	Radium	(Rn)	$7s^2$				$^1S_0$	5.2784
89	Ac	Actinium	(Rn) $6d$	$7s^2$				$^2D_{3/2}$	5.3807
90	Th	Thorium	(Rn) $6d^2$	$7s^2$				$^3F_2$	6.3067
91	Pa	Protactinium	(Rn)$5f^2$ $6d$	$7s^2$				$^4K_{11/2}$	5.89
92	U	Uranium	(Rn)$5f^3$ $6d$	$7s^2$				$^5L_6$	6.1938
93	Np	Neptunium	(Rn)$5f^4$ $6d$	$7s^2$				$^6L_{11/2}$	6.2657
94	Pu	Plutonium	(Rn)$5f^6$	$7s^2$				$^7F_0$	6.0260
95	Am	Americium	(Rn)$5f^7$	$7s^2$				$^8S_{7/2}$	5.0738
96	Cm	Curium	(Rn)$5f^7$ $6d$	$7s^2$				$^9D_2$	6.0914
97	Bk	Berkelium	(Rn)$5f^9$	$7s^2$				$^6H_{15/2}$	6.1979
98	Cf	Californium	(Rn)$5f^{10}$	$7s^2$				$^5I_8$	6.2817
99	Es	Einsteinium	(Rn)$5f^{11}$	$7s^2$				$^4I_{15/2}$	6.3676
100	Fm	Fermium	(Rn)$5f^{12}$	$7s^2$				$^3H_6$	6.50
101	Md	Mendelevium	(Rn)$5f^{13}$	$7s^2$				$^2F_{7/2}$	6.58
102	No	Nobelium	(Rn)$5f^{14}$	$7s^2$				$^1S_0$	6.65
103	Lr	Lawrencium	(Rn)$5f^{14}$	$7s^2$	$7p$?			$^2P_{1/2}$?	4.9?
104	Rf	Rutherfordium	(Rn)$5f^{14}6d^2$	$7s^2$?				$^3F_2$?	6.0?

Lanthanides: La, Ce, Pr, Nd, Pm, Sm, Eu, Gd, Tb, Dy, Ho, Er, Tm, Yb, Lu

Transition elements: Hf, Ta, W, Re, Os, Ir, Pt, Au, Hg

Actinides: Ac, Th, Pa, U, Np, Pu, Am, Cm, Bk, Cf, Es, Fm, Md, No, Lr

* The usual *LS* coupling scheme does not apply for these three elements. See the introductory note to the NIST table from which this table is taken.

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37.02.07
Other Properties

37.02.07.01 Intrinsic Electric Dipole Moments of Common Elementary Particles

As far as one knows today the intrinsic **Electric Dipole** moments of common elementary particles are compatible with zero.

37.02.07.02 Magnetic Dipole Moments

Orbital Magnetic Dipole Moment

In general a orbital **Magnetic Dipole** moment is produced by orbital angular momentum.

The associated orbital **Magnetic Dipole** (read § 39.10.02) moment is:

$$\boldsymbol{\mu}_\ell = g_\ell \frac{q}{2m}\boldsymbol{\ell} = \frac{q}{2m}\boldsymbol{\ell} \qquad q \text{ with algebraic sign; } g_\ell = +1 \quad .$$

Intrinsic Magnetic Dipole Moments of Common Elementary Particles

Associated to the intrinsic angular momentum (spin) elementary particles may have an intrinsic magnetic **Dipole** moment, typically expressed as:

$$\boldsymbol{\mu}_s = g_s \frac{|q|}{2m}\mathbf{s} \quad ,$$

in terms of the absolute value of the electron charge, a mass (typically the mass of the particle or a strictly related mass) and the **gyromagnetic ratio** g_s, a specific property of any particle (a pure number).

Total Magnetic Dipole Moments

Associated to both the spin angular momentum and orbital angular momentum there is a total **Magnetic Dipole** moment, connected to the total angular momentum, **j**, of the particle:

$$\boldsymbol{\mu}_j = g_j \frac{|q|}{2m}\mathbf{j} \quad .$$

The factor g_j is called the **gyromagnetic ratio** : it is the ratio of the **Magnetic Dipole** moment to the total angular momentum of a system.

In general one defines the **gyromagnetic ratio** of a system in terms of its mass mass M and the absolute value of the elementary charge of electron/proton ($e \equiv e > 0$) as:

$$\boldsymbol{\mu}_j \equiv g_j \frac{e\hbar}{2M}(\mathbf{j}/\hbar) \quad . \tag{37.02.01}$$

→ 1826 1906

Note that there is sometimes confusion in the literature as the name **gyromagnetic ratio** is sometimes used as in the above equation (37.02.01), for g, while it is sometimes used as in equation (39.10.01), for γ.

The quantities for the electron are usually defined in terms of the Bohr-**magneton** . It is defined, in terms of the mass of the electron, as:

$$\mu_{\mathrm{B}} = \frac{e\hbar}{2m_e} = 0.9274{\cdot}10^{-23}\ \mathrm{A\cdot m^2} = 0.9274{\cdot}10^{-23}\ \mathrm{J/T} \quad . \tag{37.02.02}$$

→ 1827

The quantities for protons and neutrons (collectively called nucleons) are usually defined in terms of the nuclear-**magneton** . It is defined, in terms of the mass of the nucleon, as:

$$\mu_{\mathrm{N}} = \frac{e\hbar}{2m_N} = 0.5051{\cdot}10^{-26}\ \mathrm{A\cdot m^2} = 0.5051{\cdot}10^{-26}\ \mathrm{J/T} \quad , \tag{37.02.03}$$

where the nucleon mass is defined as the average value of the proton and neutron mass:

$$m_N \equiv \frac{m_p + m_n}{2} \quad .$$

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Common values are:

$$g_s = +5.586 \qquad \text{proton} \quad ,$$
$$g_s = -3.826 \qquad \text{neutron} \quad ,$$
$$g_s = -2.002 \qquad \text{electron} \quad ,$$
$$g_\ell = 1 \qquad \text{orbital angular momentum, exact} \quad .$$

Magnetic Dipole Moments of Atoms and Molecules

Some atoms and molecules have electron orbitals such that there is net un-balance of the negative electron currents which gives rise to an overall **Magnetic Dipole** moment.

The order of magnitude of these **Magnetic Dipole** moments is the Bohr-**magneton** (37.02.02).

37.02.07.03 Intrinsic Polarisability |Magnetizability of Common Elementary Particles

As far as one knows today the electron is a point particle, that is it has not been possible, yet, to measure any size of it with the current experimental instruments.

On the other hand one knows that protons and neutrons have a finite size of the order of 10^{-15} m. Therefore, in principle, they have an intrinsic **Polarisability** and and intrinsic Magnetizability, and both of them have been, in fact, measured.

37.02.08
Radiation: Particle-Like Properties of ElectroMagnetic Radiation (Photons)

In quantum-relativistic physics ElectroMagnetic radiation is described as particles called photons (γ) with zero mass, zero electric charge, zero **Electric Dipole** | **Magnetic Dipole** , and intrinsic spin angular momentum equal to one in $\hbar$ units:

$$\boxed{m_\gamma = 0 \qquad (s_\gamma)_z = \pm\hbar} \quad .$$

Photons, as far as one knows, are stable elementary structure-less point particles.

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37.03 Macroscopic Description of ElectroMagnetic Fields in Presence of Matter

This § is referenced at pages:
[1842, 1851, 1884, 1893]

Read also sections § 38.04, § 39.04.

All the properties of matter in presence of ElectroMagnetic fields can be described, to a first approximation, from a macroscopic point of view, via an **Electric Dipole** and|or a **Magnetic Dipole** moment acquired by the material medium.

In presence of ElectroMagnetic fields we can think that any volume element, macroscopically small but large on a microscopic scale, acquires an induced **electric Dipole** moment and|or a induced **Magnetic Dipole** moment. It will be then described by a **Dipole** moment per unit volume. The medium has thus become Polarized/Magnetized.

We will assume that this is sufficient to describe the macroscopic properties of matter in presence of ElectroMagnetic fields. Note that this kind of model is not always sufficient and a more complex description might be required, which takes into account additional higher-order ElectroMagnetic **Multipoles** , read § 36.14 and references therein..

Under certain circumstances, some materials may have a Polarization/Magnetization even in absence of external fields. In this case, the induced moments add, not necessarily linearly, to the pre-existing permanent moments.

It is usually assumed that the material medium has **no free charge and no free currents**. If this is not the case, the principle of superposition can be applied.

In any case:

$$\mathbf{p}_0 \to \mathbf{p}_0 + \Delta\mathbf{p} \quad ,$$
$$\mathbf{m}_0 \to \mathbf{m}_0 + \Delta\mathbf{m} \quad .$$

We limit to a macroscopic description only. In fact any serious microscopic description must be based on Relativistic Quantum Physics and on Statistical Physics.

37.03.01 Average Fields

This § is referenced at pages:
[2335]

©||Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...||

When ElectroMagnetic fields are considered in presence of matter one should consider that the ElectroMagnetic fields are rapidly changing at the atomic-molecular scale and that macroscopic pieces of matter contain a number of elementary unit of the order of the Avogadro number. A detailed knowledge is clearly not possible. In most cases it is not even necessary nor useful. One should consider the average value of the ElectroMagnetic fields on a scale that is large, microscopically, but macroscopically small. The macroscopic, space averaged ElectroMagnetic fields ($\mathbf{F} \equiv \{\mathbf{E}, \mathbf{B}\}$) are thus defined as the space average of the microscopic ElectroMagnetic fields, as long as the average is taken on a volume which is small on a macroscopic scale but large on the microscopic scale:

$$\langle \mathbf{F} \rangle[\mathbf{x}] \equiv \lim_{\tau \longrightarrow 0} \frac{1}{\tau} \iiint_{\Omega} \mathbf{F}[\mathbf{x}+\mathbf{y}] \, \mathrm{d}V_{\mathbf{y}} \qquad \tau := |\mathbf{y}| \leq R \qquad R \text{ microscopically large and macroscopically small} \quad .$$

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The volume τ is a suitable volume around the point $\mathbf{x}$ (that is around $\mathbf{y} = \mathbf{0}$) and the limiting process must be intended in the above-mentioned **macroscopic** sense: the volume τ must be large on a microscopic scale but small on the macroscopic scale, the scale of change of macroscopic properties.

In fact the volume must be small enough to be able to describe wave phenomena of wavelength λ, in case of changing ElectroMagnetic fields. Therefore if the atomic scale size is $\approx a_0$, the size of the averaging region, R, must be such that:

$$\boxed{a_0 \ll R \ll \lambda} \quad ,$$

clearly depending on the wavelength.

Clearly these assumptions prevent from dealing with very short wavelength waves, depending on the inter-molecular distances.

In a more more sophisticated approach, one can introduce a kernel smoothing function, $K[\mathbf{y}]$, to define the average field:

$$\boxed{\langle \mathbf{F} \rangle[\mathbf{x}] \equiv \lim_{\tau \longrightarrow 0} \frac{1}{\tau} \iiint_{\Omega} \mathbf{F}[\mathbf{x}+\mathbf{y}] K[\mathbf{y}] \, \mathrm{d}V_{\mathbf{y}} \qquad \tau := |\mathbf{y}| \leq R \qquad R \text{ microscopically/macroscopically large/small}} \quad .$$

Whenever the definition and ubiquitous use of macroscopic (averaged) ElectroMagnetic fields inside matter is clear, the averaging symbol $\langle\rangle$ is normally omitted.

For the equivalence between average field inside matter and the equivalent field generated by the equivalent Charge|Current distribution see: Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...

The averaging to define average fields inside matter is exactly the same procedure with exactly the same motivations as the averaging when defining the concept of density of a piece of real matter.

See figure 37.14 for the equivalence between microscopic and averaged (macroscopic) ElectroMagnetic fields.

Read § 41.03.01 for more discussion.

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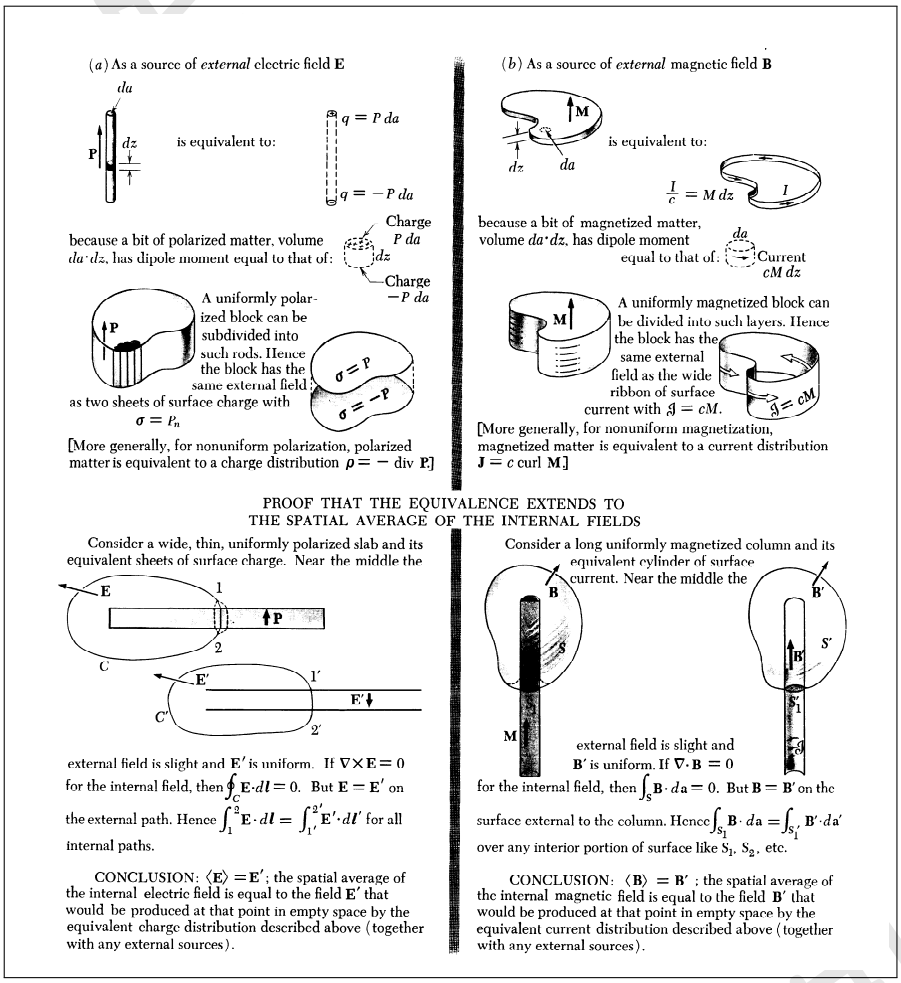

Figure 37.13: AverageElectroMagnetic fields (from E.Purcell)

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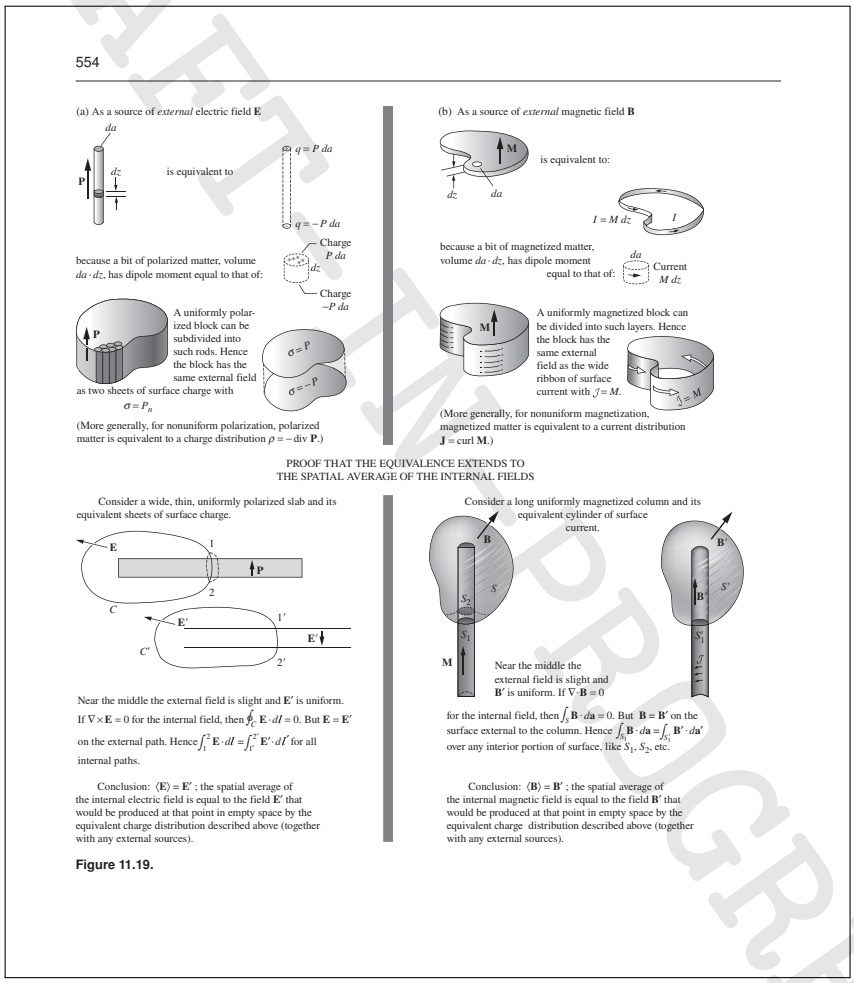

Figure 37.14: AverageElectroMagnetic fields (from E.Purcell)

37.03.02
Charges and Currents as Producers of ElectroMagnetic Fields

This § is referenced at pages:
[1951]

©|F.Wegner, *Classical Electrodynamics*, 2003, ..., ...Ed., WEB - URL.|14||
©|L.D.Landau *et al.*, *Electrodynamics of continuous media*, ..., ..., ...Ed., ...|§ 6, 29||

The density of the charges and currents can be separated into different contributions:

$$\rho = \rho^{\mathrm{F}} + \rho^{\mathrm{P}} \quad ,$$

$$\mathbf{j} = \mathbf{j}^{\mathrm{F}} + \mathbf{j}^{\mathrm{P}} + \mathbf{j}^{\mathrm{M}} \quad .$$

The distinction is between **freely accessible Charge|Current**, those **which can be to some extent controlled by the experimenter**, and **bound/inaccessible Charge|Current**, over which, typically, **the experimenter has no access**.

The free charge and current densities, ρ^{F} and $\mathbf{j}^{\mathrm{F}}$, are the freely moving contributions which can be controlled is some way, in principle. As their study is not fundamentally different from the case of absence of matter and the superposition principle applies, they will be normally assumed to be zero.

The Polarization charge and current densities, ρ^{P} and $\mathbf{j}^{\mathrm{P}}$, are the Polarization contributions coming from bound charges which can only move/deform/orient to a limited extent nearby some point inside the medium. They are charges within regions that are inaccessible to observation but which might give rise to net boundary or volume charge densities, due to imperfect charge cancellation on an atomic scale. These charges might include the equivalent charges from elementary **Electric Dipole** moments of particles/Atoms|Molecules.

The Magnetization current density, $\mathbf{j}^{\mathrm{M}}$, comes from the currents at the atomic/molecular level which can only move/deform/orient to a limited extent nearby some point inside the medium. They are steady currents that flow within regions that are inaccessible to observation but which might give rise to net boundary or volume currents, due to imperfect orbit cancellation on an atomic scale. These currents include the equivalent currents from elementary **Magnetic Dipole** moments of particles/Atoms|Molecules.

Note, however, that the distinction has some degree of arbitrariness, and it is partly conventional, see section § 41.03.01. **The most important issue, however, is not how Charge|Current are separated into free and bound/inaccessible parts, but to make sure that all Charge|Current are counted once and only once**.

No real Magnetization charge density exist, coherently with the fact that magnetic charges do not exist. However, under certain circumstances, it might be useful to introduce a formal Magnetization charge density, ρ^{M}, to solve some problems, read § 39.

The common characteristic of Polarization charges and Polarization|Magnetization current densities is that they cannot basically be controlled by external as they are strongly bound, but they are the consequence of the reaction of matter to external ElectroMagnetic fields.

The following fundamental assumption is done: it is assumed that **the different contributions can always be identified and that they never mix** each other. Therefore for these charge and current densities one has:

$$0 = \partial_t \rho^{\mathrm{F}} + \mathbf{div}\mathbf{j}^{\mathrm{F}} \quad ,$$

$$0 = \partial_t \rho^{\mathrm{P}} + \mathbf{div}\mathbf{j}^{\mathrm{P}} \quad ,$$

$$0 = \mathbf{div}\mathbf{j}^{\mathrm{M}} \qquad \text{because } \rho^{\mathrm{M}} = 0 \quad .$$

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37.04 Examples and Physical Applications

37.04.01

✓ Inter-atomic/Inter-Molecular Distances

Show that the relation giving the average distance between Atoms|Molecules is:

$$d = \left(\frac{\mathcal{M}_0}{\eta_{\mathtt{M}} \mathcal{N}_{\mathtt{A}}} \right)^{1/3} \quad ,$$

in terms of the molar mass, $\mathcal{M}_0$, and the mass density, $\eta_{\mathtt{M}}$, of the substance. Find the value of d for solid iron and liquid water.

The ratio between molar mass and mass density is called molar volume (volume per mole), v_m, show that:

$$\frac{\Delta V}{N} = \frac{\mathcal{M}_0}{\eta_{\mathtt{M}} \mathcal{N}_{\mathtt{A}}} \Longrightarrow v_m \equiv \frac{\Delta V}{N} \mathcal{N}_{\mathtt{A}} = \frac{\mathcal{M}_0}{\eta_{\mathtt{M}} \mathcal{N}_{\mathtt{A}}} \quad .$$

WORKED SOLUTION

$$d_{\text{solid Fe}} = 2.3 \cdot 10^{-10} \text{ m} \qquad d_{\text{liquid water}} = 3.1 \cdot 10^{-10} \text{ m} \quad .$$

37.04.02

✓ Molar Volume

This § is referenced at pages:
[1840, 1882]

How many iron atoms per unit volume are in a piece of solid iron?

How many molecules per unit volume are in a ideal gas at a temperature of zero degrees Celsius and one atmosphere pressure?

WORKED SOLUTION

Molar volume of solid iron: $7.11 \text{ cm}^3/\text{mole}$. Therefore the number of iron atoms per cubic meters is: $8.5 \cdot 10^{28} \text{ atoms/m}^3$.

Molar volume of ideal gas: RT/p. At $T = 273.15$ K and $p = 101.325$ kPa:

$$v_m = 22.414 \text{ l/mol} \Longrightarrow 44.6 \text{ mol/m}^3 \Longrightarrow 2.7 \cdot 10^{25} \text{ molecules/m}^3 \quad .$$

37.04.03

✓ Drift Velocity in Copper

Copper has: $Z = 29$, $\eta_{\mathtt{M}} = 8.96 \text{ kg/dm}^3$ and $\mathcal{M}_0 = 63.6 \text{ g/mol}$.

Calculate the molar volume of Copper:

$$v_m = \frac{\mathcal{M}_0}{\eta_{\mathtt{M}}} = 6 \text{ cm}^3/\text{mol} \quad .$$

Assume a simple model of copper such that one electron per atom is free to move and contributes to the current, that is the valence is $\mathcal{V} = 1$.

Take a current $I = 10$ A flowing in a copper wire of section $S = 1 \text{ mm}^2$: $j \approx 10^7 \text{ A/m}^2$.

$$\eta_{\mathtt{N}} = \frac{N}{\Delta V} = \frac{m}{\mathcal{M}_0} \frac{\mathcal{N}_{\mathtt{A}}}{V} \mathcal{V} = \eta_{\mathtt{M}} \frac{\mathcal{N}_{\mathtt{A}}}{\mathcal{M}_0} \mathcal{V} = 8.4 \cdot 10^{28} \text{ m}^{-3} \quad .$$

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The resulting drift velocity is:

$$v \equiv \frac{j}{e\eta_{\mathbb{N}}} = 1 \text{ mm/s} \quad .$$

Note that the drift velocity is very small but when, for instance, one closes/opens a switch in a circuit with a lamp the lamp switches on/off very quickly: why?

37.04.04

✓ Drift Velocity in Semi-Conductors

Consider a current density $j = 10^{-5}$ A/m^2 in Silicon wire (a semi-conductor) with of section $S = 1$ mm^2. In Silicon charge conduction is ensured by electron-hole pairs with a total number density $n = 2.4{\cdot}10^{16}$ m^{-3} One find typical values of $v \approx 3$ mm/s.

37.04.05

✓ Semi-Classical ElectroMagnetic Properties of an Hydrogen Atom

This § is referenced at pages:
[Never referenced.]

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|37.7||

Consider the simple classical model of an hydrogen atom as a heavy point charge, the proton, plus an electron orbiting around the proton in a circular orbit of radius $a_0 = 0.5{\cdot}10^{-10}$ m (the so-called Bohr radius determined by quantum physics).

Calculate the electric field produced by the proton at the position of the electron.

WORKED SOLUTION

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q_e}{a_0^2} = 6{\cdot}10^{11} \text{ V/m} \quad .$$

This is a very large field, with respect to everyday electric fields.

Calculate the current associated to the electron orbit.

WORKED SOLUTION

$$\frac{1}{4\pi\varepsilon_0}\frac{q_e^2}{a_0^2} = m_e\omega^2 a_0 \implies \nu = \frac{q_e}{2\pi a_0}\sqrt{\frac{1}{4\pi\varepsilon_0}\frac{1}{m_e a_0}} = 7{\cdot}10^{15} \text{ Hz} \implies I = q_e\nu = 1 \text{ mA} \quad .$$

Note that the electron revolution frequency is in the range of the order of magnitude of the frequency of visible light.

The velocity of the electron is:

$$v = 2\pi a_0\nu = 2{\cdot}10^6 \text{ m/s} = 0.007c \quad .$$

The non relativistic approximation is thus justified.

Calculate the magnetic field produced by the electron at the center of its orbit (that is where the proton is).

WORKED SOLUTION

$$B_0 = \frac{\mu_0 I}{2a_0} = 12 \text{ T} \quad .$$

This is a very large field, with respect to everyday magnetic fields.

Calculate the **Magnetic Dipole** moment due to the motion of the electron around its orbit.

WORKED SOLUTION

$$\mu_L = I\pi a_0^2 = 8{\cdot}10^{-24}\ \mathrm{A\cdot m^2} \approx \frac{q_e\hbar}{2m_e} \equiv \text{the Bohr-}\textbf{magneton}\ \equiv \mu_\mathrm{B}\quad .$$

In fact all the magnetic orbital moments of atoms have the same order of magnitude of the Bohr-**magneton** .

Calculate the interaction energy between the **Magnetic Dipole** moment of the proton and the magnetic field produced by the electron ($\mu_p = (m_e/m_N)\,\mu_B = 0.5{\cdot}10^{-26}\ \mathrm{A\cdot m^2}$):

WORKED SOLUTION

$$|\Phi| \simeq \mu_p B_0 = 6{\cdot}10^{-26}\ \mathrm{J}\quad .$$

Calculate the interaction energy between the **Magnetic Dipole** moment of one electron and an external magnetic field $B_0 = 1.0\ \mathrm{T}$:

WORKED SOLUTION

$$|\Phi| \simeq \mu_B B_0 = 0.9{\cdot}10^{-23}\ \mathrm{J}\quad .$$

The above energies must be compared with the much larger energies:

- the typical thermal energies at room temperature of $\approx k_\mathrm{B}T \simeq 0.025\ \mathrm{eV} \simeq 4{\cdot}10^{-21}$ J, that is the order of magnitude of the kinetic energies of the atom and its constituents;
- the typical electric interaction energies, for instance the one between the electron and the proton of the hydrogen atom: $4.6{\cdot}10^{-18}$ J.

37.04.06

✓ Equivalent Current Due to Magnetization

Consider a sphere of iron ($\eta_\mathrm{M} = 7.86{\cdot}10^3\ \mathrm{kg/m^3}$, $\mathcal{M}_0 = 56\ \mathrm{g/mol}$) with $R = 10\ \mathrm{cm}$ and assume that every atom contributes with one orbital electron to the **Magnetic Dipole** moment for one Bohr-**magneton** . Imagine all Bohr-**magnetons** in the sphere are aligned.

Find the current in a loop of radius R giving the same **Magnetic Dipole** moment.

WORKED SOLUTION

$I = 1.05{\cdot}10^5\ \mathrm{A}$

37.05 Exercises Problems and Physical Applications

37-001 D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...- Problem 5.56

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CHAPTER 38

Electric Properties of Matter

38.01 Phenomenology of the Electric Properties of Materials 1839
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This § is referenced at pages:
[1691, 1813]

Pre-requisites: review the ***Multipole*** *expansion.*

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.10; 2.11||

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|6|*Good text.*|

©|F.N.H.Robinson, *Macroscopic ElectroMagnetism*, ..., ..., ...Ed., .||*Excellent!*|

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|1.4||

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38.01 Phenomenology of the Electric Properties of Materials

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Experimentally one can study the effects of matter on electric fields (and vice-versa) by inserting a material inside a an empty capacitor. When any material is inserted inside an empty capacitor the **capacitance** always increases.

Similarly any piece of material is attracted inside an empty capacitor.

Under the influence of an external electric field the atoms and|or molecules of the medium react collectively: the medium becomes Polarized.

In fact, we know that matter is made of electrons, ions, atoms, molecules etc.. These are in turn composed of other particles, possibly composed particles. Instead of describing the material as a volume of structure-less but non-vacuum jelly, it is described as a collection of electrons|ions|atoms|molecules etc. inhabiting a vacuum.

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38.02 Polarization of Matter

This § is referenced at pages:
[1882]

The effects of external electric fields on matter can be described, to a first approximation, by means of a suitable distribution of **Electric Dipoles** . This hypothesis is normally assumed when studying the response of matter to electric fields. Note, however, that under certain circumstances the inclusion of higher-order multipoles fields might become necessary.

Matter generally reacts to an external electric field by Polarization. On the microscopic point of view this can be understood by either of two mechanisms:

- **deformation** of the microscopic charge distributions: Polarization by deformation of non-dipolar matter; it is oriented with the external field.
- **orientation** of pre-existing microscopic **Electric Dipoles** : Polarization by orientation of dipolar matter; in these cases a Polarization by deformation exists as well, but it is normally negligible with respect to the Polarization by orientation; it is oriented with the external field.

Induced Polarization is always directed with the external field, so it is often called **para-electricity (para-electric matter)**, similar to para-magnetism, as opposed to dia-magnetism; see section § 39.02.

Note that, while the entity of the deformation and|or orientation of the single atom/molecule may be very small, the large number of Atoms|Molecules in macroscopic matter can make the effect a very large one (read § 37.04.02). In fact the number of atoms per cubic meter for ordinary matter has values in the range $\left(10^{25} \div 10^{29}\right)\ \mathrm{atoms/m^3}$.

Polarization by deformation and orientation can be understood, from a qualitative point of view, without invoking quantum-relativistic physics, which is However, necessary for a quantitative understanding.

Note that materials exists having a **built-in Polarization** even in absence of external applied electric fields: they are **generically called here ferro-electric materials**.

Because, to a first approximation, the effect of external fields on matter may be described either as induction of Dipoles and|or orientation of existing Dipoles, the electric field produced by the Polarized material can be thus replaced by the electric field produced by a continuous **Dipole** distribution.

38.02.01 Atomic/Molecular Polarisability - Microscopic

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Typical values [1] for Atoms|Molecules: $\left(1 \div 10\right) \cdot 10^{-30}\ \mathrm{m^3}$.

1 ©||W.M.Haynes *et al.*, *CRC Handbook of Chemistry and Physics*, 2016, CRC press, 97thEd., WEB - URL.|*§ 10.188*|

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38.02.01.01 Polarisability of an Isotropic Molecule

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The **Polarisability** of an isotropic atom/molecule (or any other microscopic system) is defined as the average **Electric Dipole** moment, $\mathbf{p}$, induced by an external electric field $\mathbf{E}_0$:

$$\boxed{\alpha^{(e)} \equiv \frac{1}{\varepsilon_0}\left.\frac{\partial \mathbf{p}}{\partial E_0}\right|_{E_0=0}} \quad \text{for the case } \mathbf{p} \parallel \mathbf{E}_0 \text{ (isotropic atom/molecule)} \quad , \qquad \rightarrow \text{1861 1961 1962}$$

$$\boxed{\Longrightarrow \mathbf{p} \simeq \mathbf{p}_0 + \varepsilon_0 \alpha^{(e)} \mathbf{E}_0 + \mathcal{O}\left(E_0\right)^2} \quad \text{for a permanent } \textbf{Electric Dipole} \text{ moment: } \mathbf{p}_0 \quad .$$

The Polarisability is the second derivative of the interaction energy with respect to external electric field calculated at zero field:

$$\boxed{\alpha^{(e)} \equiv -{\varepsilon_0}^{-1}\left.\frac{\partial^2 \mathcal{U}_e}{\partial {E_0}^2}\right|_{E_0=0}} \quad ,$$

with no factor $1/2$, because it is a Polarizable systems with factor $1/2$ in the interaction energy.

The above discussion applies to a single atom/molecule: a permanent Dipole may be present, even in absence of an external field, and an induced moment is generally always present.

Whenever a large number of Atoms|Molecules is present, one should introduce the macroscopic averaging, in terms of the Polarization density, section § 38.02.02; in fact, $\mathbf{E}_0$ is actually the **local field** :

$$\boxed{\mathbf{E}_0 \longrightarrow \mathbf{E}_{\text{LOC}}} \quad .$$

Even if each atom/molecule possesses a permanent Dipole, a random arrangement may result in a zero total Dipole in absence of an external field.

The induced Polarization includes the effect of alignment of the permanent Dipoles, which is generally opposed by temperature. See section § 41.04.06.

The **Polarisability** is always a positive number:

$$\boxed{\alpha^{(e)} \geq 0} \quad .$$

One should assume that a linear relation applies as long as the external electric field is not too big. Note that for large external fields the relation may become non-linear, so that the **Polarisability** may loose some of its meaning and become field-dependent.

Note that in general whenever a permanent **Electric Dipole** moment, $\mathbf{p}_0$, is present, this is normally much larger than the induced **Electric Dipole** moment, at least for external fields not too large.

By definition the physical dimensions of the **Polarisability** is a volume. Note, however, that in the literature slightly different definitions may be found.

38.02.01.02 Polarisability of a Non-Isotropic Axially-Symmetric Molecule

If a molecule is not isotropic, but axially-symmetric, it is rather obvious that the **Polarisability** might depend of the direction of the applied field. Consider for instance a bi-atomic molecule: one expects that the induced **Dipole** moment is different for an external electric field applied along the axis of the molecule with respect to an external electric field applied perpendicularly to it. Therefore one can write for this kind of molecule (having one axis of symmetry):

$$\boxed{\mathbf{p} \simeq \varepsilon_0 \left({\alpha^{(e)}}_{\parallel}[E_0]\mathbf{E}_{\parallel} + {\alpha^{(e)}}_{\perp}[E_0]\mathbf{E}_{\perp} \right) \qquad \textbf{superposition principle}} \quad .$$

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38.02.01.03 Polarisability of a General Non-Isotropic Molecule

The general case is discussed in section § 38.06.03 in the completely analogous case of the Polarization vector.

38.02.01.04 All Materials Undergo Polarization by Deformation

This § is referenced at pages:
[1884]

Consider an atom/molecule: all charge densities associated with electrons and ions will be deformed by the application of an external field giving an induced **Electric Dipole** moment.

Assume that two charge distributions, with equal and opposite charges, are superimposed, and have the same centers of charge, like the electron and proton clouds in a neutral atom. The electric dipole moment is zero.

In presence of an external electric field, both charge distributions are affected and move their centers of charge in opposite directions. An electric dipole moment is induced, **parallel** to the external field.

Let us then assume a simple semi-classical model such that, in absence of external fields, the two opposite center of charges coincide and cancel, while in presence of external fields, the two center of charges do not coincide anymore anymore so that there is an induced electric dipole moment, parallel to the external field.

This will be modeled as two equal and opposite charges, separated by the relative positions of the two centers of charge.

Read § 39.02.01.04, for similarities and differences.

38.02.02 Definition of the Polarization

This § is referenced at pages:
[1841]

Remembering that the definition of electric dipole is, by definition, **additive**, see (36.14.06), the macroscopic Polarization field is defined as:

$$\boxed{\mathbf{P} \equiv \lim_{\Delta V \longrightarrow 0} \frac{\Delta \mathbf{p}}{\Delta V} \equiv \lim_{\Delta V \longrightarrow 0} \frac{\sum_k \mathbf{p}_k(\text{inside } \Delta V)}{\Delta V} \equiv \eta_{\mathrm{N}} \langle \mathbf{p} \rangle} \quad . \tag{38.02.01}$$

→ 1842

The limiting process in equation (38.02.01) must be interpreted in the sense of taking **macroscopic** volumes, containing a large enough number of microscopic systems, as discussed in section § 37.03.

The Polarization **P** turns out to be **the electric dipole moment per unit volume**.

38.02.02.01 A Polarized Medium: Some Simple Cases

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Note that it is difficult to imagine the electric field produced by a large number (of the order of Avogadro number) of **Electric Dipoles** .

However, it will be shown that the electric field is the one given by an equivalent charge distribution.

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A Large plane slab Uniformly Polarized perpendicular to is faces

This § is referenced at pages:
[1844]

From the elementary interpretation of Polarization as aligned **Electric Dipoles** one easily finds the following results.

Zero volume charge density at internal points.

Zero surface charge density on the lateral surface.

Surface charge density at the two bases: $\pm\mathbf{P}\cdot\mathbf{n}$.

Consider, in fact, a surface of area A at one of the two bases and calculate the charge per unit area:

$$\Delta Q = q\eta_{\mathbb{N}} A d = (q d \eta_{\mathbb{N}})\, A = PA \implies |\rho_{\mathbb{S}}| = P \equiv |\mathbf{P}| \quad .$$

Regardless the details of the **Electric Dipole** distribution the electric field produced is the same as the one produced by a surface charge on the bases.

A long circular cylinder Uniformly Polarized Perpendicular along its axis

A Cylinder Uniformly Polarized Along Its Axis

A Sphere Uniformly Polarized

A Spherical Shell With Radial Polarization

Assume a uniform radial distribution of the all elementary identical electric Dipoles in a sphere:

$$\begin{cases} \mathbf{P} = p\eta_{\mathbb{N}}\widehat{\mathbf{e}}_r \equiv P_r\widehat{\mathbf{e}}_r & R_1 < r < R_2 \quad , \\ \mathbf{P} = 0 \qquad \text{otherwise} \end{cases} \quad .$$

From equations (38.02.03), (38.02.04) one finds: $\rho^{\mathsf{P}} = -2P_r/r$ as well as $\rho^{\mathsf{P}}[R_1] = -P_r$ and $\rho^{\mathsf{P}}[R_2] = +P_r$.

Total charge inside any spherical shell with radii $R_1 \le r_1 < r_2 \le R_2$: $Q = -4\pi P_r \left(r_2^2 - r_1^2\right)$.

Total surface charges: $Q[R_2] = 4\pi P_r R_2^2$ and $Q[R_1] = -4\pi P_r R_1^2$.

Total Polarization charge is zero.

Note that every spherical shell has more electric charge, when increasing the distance from the center.

A Plane Capacitor

This § is referenced at pages:
[Never referenced.]

A capacitor completely filled with a material medium always increases its **capacitance** by a factor ϵ, with respect to its vacuum **capacitance** , where ϵ, the relative **Electric-Permittivity** under static conditions, is a property of the material and its state (for instance temperature).

Note that the definition of **capacitance** does not make any reference to the material medium between the conductors.

Consider in fact an isolated large plane capacitor filled with a material medium. **Electric Dipoles** either align with the electric field, if they already exist, or are created, aligned with the electric field on average. The matter close to the positive armature is then negatively charged and the matter close to the negative armature is positively charged. The charge on the armatures has not changed while the electric field between the armatures has decreased, so the difference of potential. The definition of **capacitance** then shows that **capacitance** increased.

In case the capacitor is connected to a battery the electric field between the armatures must not change and, due to the Polarization charges of opposite sign near the armatures, the free charge has to increase to keep the same field. The definition of **capacitance** then shows that **capacitance** increased, as well.

Electric Dipoles align with the electric field and the electric field is decreased. This is opposite with respect to the magnetic case due to the basic difference between **Electric Dipoles** and **Magnetic Dipoles** (read § 41.01).

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38.02.03
Polarization Charge Density

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©|S.Bobbio & E.Gatti, *Elettromagnetismo E Ottica*, 1991, Bollati-Boringhieri, ...Ed., ...|5||

G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...- for a simpler, simplified and more elementary derivation.

38.02.03.01 A Non-Polarized Medium

Consider a non Polarized insulating medium. The electric field effects due to positive and negative charges in atoms and|or molecules cancel each other:

- either because, for non-dipolar molecules, (that is molecules without a net **Electric Dipole** moment), the positive and negative charges of each molecule have the same center-of-charge (and any other higher-order **Electric Dipole** moment vanishes);
- or because, for dipolar-molecules, the random orientation of the **Dipoles** produces a net zero **Dipole** moment (which is contrasted by thermal motion, featuring kinetic energies of the order of $\approx k_B T$).

38.02.03.02 A Polarized Medium With No Free Charges (A Simple Heuristic Model)

This § is referenced at pages:
[Never referenced.]

A simple heuristic model is developed in this section. A better and more general, but still heuristic model, is developed in section § 38.02.03.03 An exact, but purely formal, derivation is carried on in section § 38.03.

From the simple examples one might guess that, in general, that the Polarization surface charge density is (read § 38.02.02.01):

$$\rho_S = \mathbf{P}\cdot\mathbf{n} \quad .$$

Consider a piece of matter divided into small parallelepiped blocks, with sides parallel to the axes, Δx, Δy and Δz long. Consider, for instance, P_z changing along z and two neighboring blocks centered at the same x and y and at z and $z+\Delta z$. The total charge at the common face is:

$$\Delta x \Delta y \left(P_z[z] - P_z[z+\Delta z]\right) \simeq \Delta x \Delta y \Delta z \left(-\frac{\partial P_z}{\partial z}\right) \quad .$$

Considering, similarly, the contributions of P_x changing along x and P_y changing along y one finds the total charge per unit volume:

$$\rho^P = -\mathbf{div}\mathbf{P} \quad .$$

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38.02.03.03 A Polarized Medium With No Free Charges (A Simple Heuristic Model) a More General Case

This § is referenced at pages:
[1844]

Consider any fixed volume Ω, inside a Polarized medium, and a small (infinitesimal) part of its boundary, the closed surface $\partial\Omega$; the infinitesimal part of the closed surface is defined by the vector $\mathbb{d}\mathbf{S}$.

Assume that there are no free charges in the medium and the medium is not Polarized in absence of the external (applied) field. In case there are free charges the **superposition principle** may be applied.

Assume a very simplified model where Polarization is built from a number of all identical real **Electric Dipoles** of value $\mathbf{p} = |q|\mathbf{d}$, all with the same charges, $\pm|q|$, and where $\mathbf{d}$ is the vector from the negative charge to the positive one; both of them, a priori, may depend on the position.

The Polarization may be due to any effect (deformation and|or orientation); for simplicity, the demonstration only considers deformation.

The **Electric Dipole** moments, a priori, might depend on the location: $\mathbf{p} = \mathbf{p}[\mathbf{x}]$.

The charge exiting outside the element of surface $\mathbb{d}\mathbf{S}$ in $\partial\Omega$, is calculated by counting the number of **Dipoles** cut by the surface element $\mathbb{d}\mathbf{S}$. Note that this might have both a contribution from **Dipoles** near the surface, internal to it, whose charge exits the volume when Polarized and from **Dipoles** near the surface, external to it, whose charge enters the volume when Polarized. In any case one can write (by implicitly considering space-averaged values):

$$\boxed{\mathbf{P} \parallel \mathbf{p} \qquad \delta Q_{\mathrm{P}}\,[\mathbf{x}] = |q|\eta_{\mathbb{N}}[\mathbf{x}] d\;\mathbb{d}S\cos[\theta] = \mathbf{P}[\mathbf{x}]\cdot\mathbb{d}\mathbf{S}} \quad .$$

Note: the above is meaningful for a closed surface only. In general: the total charge crossing the surface as a consequence of Polarization is thus given by the contribution of all **Dipoles** which are cut by the piece of surface. **Rule: look for real Electric Dipoles cut by the closed surface**.

Therefore the total charge exiting any closed surface inside the medium is the flux of the Polarization vector:

$$Q_{\mathrm{P}} = \oiint_{\partial\Omega} \mathbf{P}\cdot\mathbb{d}\mathbf{S} \quad . \tag{38.02.02}$$

→ 1845

As equation (38.02.02) is valid for any arbitrary surface inside the medium we always have that the charge exiting any closed surface is given by the flux of the Polarization vector $\mathbf{P}$. Therefore inside the surface there is an equal and opposite charge left:

$$-Q_{\mathrm{P}} = -\oiint_{\partial\Omega} \mathbf{P}\cdot\mathbb{d}\mathbf{S} = -\iiint_{\Omega} \mathbf{div}\mathbf{P}\;\mathbb{d}V \equiv \iiint_{\Omega} \rho^{\mathrm{P}}\;\mathbb{d}V \quad .$$

and, as the surface is arbitrary, one can conclude, by applying the **divergence theorem** , (Read also equation (38.03.03)):

$$\rho^{\mathrm{P}} \equiv -\mathbf{div}\mathbf{P}[\mathbf{x}] \quad .$$

Applying the result to the surface $\partial\Omega$ one finds the surface Polarization charge density (Read also equation (38.03.03)):

$$\rho^{\mathrm{P}}_{\mathrm{S}} \equiv +\widehat{\mathbf{n}}\cdot\mathbf{P}[\mathbf{x}] \quad .$$

All in all it was found, for a very special model of Polarization and setup that Polarization charge density is expressed by the relations:

$$\boxed{\rho^{\mathrm{P}} \equiv -\mathbf{div}\mathbf{P}[\mathbf{x}]} \quad , \tag{38.02.03}$$

→ 1843 1845 1846 1847

$$\boxed{\rho^{\mathrm{P}}_{\mathrm{S}} \equiv +\widehat{\mathbf{n}}\cdot\mathbf{P}[\mathbf{x}]} \quad . \tag{38.02.04}$$

→ 1843 1845 1846 1847

Equations (38.02.03), (38.02.04), derived in a static situation and in absence of free charges, are assumed to be valid in all cases.

Another derivation, a formal one, under the same hypothesis is summarised in section § 38.03.01.

It can be shown, in a more formal way, that the above results are indeed valid under any circumstance, with a mathematical derivation giving less room to physical intuition.

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Note that, in the derivation above, the Polarization can in fact change along the surface, without spoiling the validity of the derivation. However, when using the **divergence theorem** to derive equation (38.02.03) one must think of choosing a smaller and smaller (infinitesimal) surface in order to pick-up a point property inside the medium.

38.02.03.04 Equivalence of a Polarized Material to Coulombian Charge Densities

This § is referenced at pages:
[Never referenced.]

In general, a piece of material with a given Polarization produces the same electric fields as the volume charge density (38.02.03) plus the surface charge density (38.02.04).

See Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ..., for the equivalence of the equivalent field with the average field in matter.

38.02.04 Polarization Current Density

This § is referenced at pages:
[1846, 1951]

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In non-static situations a changing Polarization produces a current:

$$\boxed{\mathbf{j}^{\mathrm{P}} = \partial_t \mathbf{P}} \quad . \tag{38.02.05}$$

→ 1846

Assume a very simplified model where Polarization (by deformation) is built from a number of **Electric Dipoles** of value $\mathbf{p} = |q|\mathbf{d}$, all with the same charges, $\pm|q|$, and where $\mathbf{d}$ is the vector from the negative charge to the positive one, a priori depending on the position. The number density, $\eta_{\mathbb{N}}[\mathbf{x}]$, is assumed to be independent of time but, possibly, dependent of the position. In terms of the definition of current density, $\mathbf{j}^{\mathrm{P}}[\mathbf{x},t] \equiv |q|\eta_{\mathbb{N}}[\mathbf{x}]\mathbf{v}[\mathbf{x},t]$, one then has:

$$\mathbf{P}[\mathbf{x},t] = |q|\eta_{\mathbb{N}}[\mathbf{x}]\mathbf{d}[\mathbf{x},t] = \eta_{\mathbb{N}}[\mathbf{x}]\mathbf{p}[\mathbf{x},t] \quad ,$$
$$\mathbf{j}^{\mathrm{P}}[\mathbf{x},t] \equiv |q|\eta_{\mathbb{N}}[\mathbf{x}]\mathbf{v}[\mathbf{x},t] = \partial_t \mathbf{P}[\mathbf{x},t] \quad .$$

It can also be shown that the term $\mathbf{j}^{\mathrm{P}}$ is necessary in the **Maxwell Equation** in presence of matter in order to warrant the charge conservation. Read § 38.02.04.

More formally, based on conservation of the Polarization charge, and taking the time derivative of the Polarization charge density, the conservation equation provides immediately equation (38.02.05).

Read also § 41.03.

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38.03 Calculation of the Electric Field Produced by a Given Polarization

This § is referenced at pages:
[1844, 1889]

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Note that when dealing with systems with pre-assigned Polarization (ferro-electric materials, for instance), as opposed to the case of linear systems, for which the Polarization is related to the electric field, there is no a priori relation between electric field and Polarization. In particular, as opposed to isotropic materials, electric field and Polarization need not to be parallel.

38.03.01 Fields Produced by Polarized Matter: A Formal Treatment

This § is referenced at pages:
[1845]

The expressions for the volume and surface Polarization charge densities, equations (38.02.03), (38.02.04), can be also deduced, in a purely formal way, from the ElectroStatic field of a distribution of **Electric Dipoles** as follows.

The electric field produced by a given (pre-assigned) Polarization $\mathbf{P}[\mathbf{x}]$ can be calculated from the definition itself of the Polarization as a volume density of electric **Dipoles** .

The ElectroStatic potential, using equations (16.13.07), (16.13.01), (16.07.03), is given by:

$$\Phi[\mathbf{x}] = \frac{1}{4\pi\varepsilon_0}\iiint_\tau \frac{\mathbf{P}[\mathbf{y}]\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,\mathrm{d}V_\mathbf{y} = \quad , \tag{38.03.01}$$

→ 1847 1866

$$\frac{1}{4\pi\varepsilon_0}\iiint_\tau \mathbf{P}[\mathbf{y}]\cdot\mathbf{grad}_\mathbf{y}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}V_\mathbf{y} = \frac{1}{4\pi\varepsilon_0}\iiint_\tau \mathbf{div}\left(\frac{\mathbf{P}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}V_\mathbf{y} + \frac{1}{4\pi\varepsilon_0}\iiint_\tau \left(\frac{-1}{|\mathbf{x}-\mathbf{y}|}\right)\mathbf{div}\mathbf{P}[\mathbf{y}]\,\mathrm{d}V_\mathbf{y} = \quad ,$$

$$\frac{1}{4\pi\varepsilon_0}\oiint_{\partial\tau} \frac{\widehat{\mathbf{n}}\cdot\mathbf{P}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}S[\mathbf{y}] + \frac{1}{4\pi\varepsilon_0}\iiint_\tau \left(\frac{-\,\mathbf{div}\mathbf{P}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}V_\mathbf{y} \quad , \tag{38.03.02}$$

→ 1850

where the integral is extended to the region of space, τ, where the Polarizable material is present, that is where the Polarization is non zero.

Equation (38.03.01) shows that the electric field produced by any Polarization can be expressed in terms of an equivalent volume charge density, ρ^{P}, and an equivalent surface charge density, $\rho^{\mathrm{P}}_{\mathrm{S}}$, such that:

$$\rho^{\mathrm{P}} \equiv -\mathbf{div}\mathbf{P}[\mathbf{y}] \qquad \rho^{\mathrm{P}}_{\mathrm{S}} \equiv +\,\widehat{\mathbf{n}}\cdot\mathbf{P}[\mathbf{y}] \quad . \tag{38.03.03}$$

→ 1845 1847 1850

Note that the relations (38.03.03) imply that the material medium is globally neutral, as it follows from the **divergence theorem** .

Read also § 39.03.

GO TO AFTER included File

CHAPTER II

ELECTROSTATICS OF DIELECTRICS

§6. The electric field in dielectrics

WE SHALL now go on to consider a static electric field in another class of substances, namely dielectrics. The fundamental property of dielectrics is that a steady current cannot flow in them. Hence the static electric field need not be zero, as in conductors, and we have to derive the equations which describe this field. One equation is obtained by averaging equation (1.3), and is again

$$\mathbf{curl}\ \mathbf{E} = 0. \tag{6.1}$$

A second equation is obtained by averaging the equation div $\mathbf{e} = 4\pi\rho$:

$$\operatorname{div} \mathbf{E} = 4\pi\bar{\rho}. \tag{6.2}$$

Let us suppose that no charges are brought into the dielectric from outside, which is the most usual and important case. Then the total charge in the volume of the dielectric is zero; even if it is placed in an electric field we have $\int\bar{\rho}\,dV = 0$. This integral equation, which must be valid for a body of any shape, means that the average charge density can be written as the divergence of a certain vector, which is usually denoted by $-\mathbf{P}$:

$$\bar{\rho} = -\operatorname{div}\mathbf{P}, \tag{6.3}$$

while outside the body $\mathbf{P} = 0$. For, on integrating over the volume bounded by a surface which encloses the body but nowhere enters it, we find $\int\bar{\rho}\,dV = -\int \operatorname{div}\mathbf{P}\,dV = -\oint\mathbf{P}\cdot d\mathbf{f} = 0$. $\mathbf{P}$ is called the *dielectric polarization*, or simply the *polarization*, of the body. A dielectric in which $\mathbf{P}$ differs from zero is said to be *polarized*. The vector $\mathbf{P}$ determines not only the volume charge density (6.3), but also the density σ of the charges on the surface of the polarized dielectric. If we integrate formula (6.3) over an element of volume lying between two neighbouring unit areas, one on each side of the dielectric surface, we have, since $\mathbf{P} = 0$ on the outer area (cf. the derivation of formula (1.9)),

$$\sigma = P_n, \tag{6.4}$$

where P_n is the component of the vector $\mathbf{P}$ along the outward normal to the surface.

To see the physical significance of the quantity $\mathbf{P}$ itself, let us consider the total dipole moment of all the charges within the dielectric; unlike the total charge, the total dipole moment need not be zero. By definition, it is the integral $\int\mathbf{r}\bar{\rho}\,dV$. Substituting $\bar{\rho}$ from (6.3) and again integrating over a volume which includes the whole body we have

$$\int\mathbf{r}\bar{\rho}\,dV = -\int\mathbf{r}\operatorname{div}\mathbf{P}\,dV = -\oint\mathbf{r}(d\mathbf{f}\cdot\mathbf{P}) + \int(\mathbf{P}\cdot\mathbf{grad})\mathbf{r}\,dV.$$

The integral over the surface is zero, and in the second term we have $(\mathbf{P}\cdot\mathbf{grad})\mathbf{r} = \mathbf{P}$, so that

$$\int\mathbf{r}\bar{\rho}\,dV = \int\mathbf{P}\,dV. \tag{6.5}$$

Thus the polarization vector is the dipole moment (or *electric moment*) per unit volume of the dielectric.†

Substituting (6.3) in (6.2), we obtain the second equation of the electrostatic field in the form

$$\operatorname{div} \mathbf{D} = 0, \tag{6.6}$$

where we have introduced a quantity $\mathbf{D}$ defined by

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \tag{6.7}$$

called the *electric induction*. The equation (6.6) has been derived by averaging the density of charges in the dielectric. If, however, charges not belonging to the dielectric are brought in from outside (we shall call these *extraneous charges*), then their density must be added to the right-hand side of equation (6.6):

$$\operatorname{div} \mathbf{D} = 4\pi\rho_{\text{ex}}. \tag{6.8}$$

On the surface of separation between two different dielectrics, certain boundary conditions must be satisfied. One of these follows from the equation $\mathbf{curl}\ \mathbf{E} = 0$. If the surface of separation is uniform as regards physical properties,‡ this condition requires the continuity of the tangential component of the field:

$$\mathbf{E}_{t1} = \mathbf{E}_{t2}; \tag{6.9}$$

cf. the derivation of the condition (1.7). The second condition follows from the equation $\operatorname{div} \mathbf{D} = 0$, and requires the continuity of the normal component of the induction:

$$D_{n1} = D_{n2}. \tag{6.10}$$

For a discontinuity in the normal component $D_n = D_z$ would involve an infinity of the derivative $\partial D_z/\partial z$, and therefore of $\operatorname{div} \mathbf{D}$.

At a boundary between a dielectric and a conductor, $\mathbf{E}_t = 0$, and the condition on the normal component is obtained from (6.8):

$$\mathbf{E}_t = 0, \quad D_n = 4\pi\sigma_{\text{ex}}, \tag{6.11}$$

where σ_{ex} is the charge density on the surface of the conductor; cf. (1.8), (1.9).

§7. The permittivity

In order that equations (6.1) and (6.6) should form a complete set of equations determining the electrostatic field, they must be supplemented by a relation between the induction $\mathbf{D}$ and the field $\mathbf{E}$. In the great majority of cases this relation may be supposed linear. It corresponds to the first terms in an expansion of $\mathbf{D}$ in powers of $\mathbf{E}$, and its correctness is due to the smallness of the external electric fields in comparison with the internal molecular fields.

The linear relation between $\mathbf{D}$ and $\mathbf{E}$ is especially simple in the most important case, that

† It should be noticed that the relation (6.3) inside the dielectric and the condition $\mathbf{P} = 0$ outside do not in themselves determine $\mathbf{P}$ uniquely; inside the dielectric we could add to $\mathbf{P}$ any vector of the form $\mathbf{curl}\ \mathbf{f}$. The exact form of $\mathbf{P}$ can be completely determined only by establishing its connection with the dipole moment.

‡ That is, as regards composition of the adjoining media, temperature, etc. If the dielectric is a crystal, the surface must be a crystallographic plane.

GO TO BEFORE included File

38.03.01.01 An Important Warning

It should be stressed that, rigorously, the previous derivation only applies to the case of ideal **Dipoles** , so that the point where the field is calculated can be considered far enough that the field of a ideal (point) **Dipole** can be used. The problem of the previous derivation is the fact that the starting point of (39.03.01) is only valid for distant points, not certainly inside the Dipole distribution itself.

The analysis showing that the above results is, nevertheless, exact can be found in the literature [1].

© - QUOTE

All in all it is the result the same as integrating for ideal **Dipoles** . All revolves around the fact that the average field over any sphere due to the charge inside is the same as the field at the center of a uniformly Polarized sphere with the same total **Dipole** moment. That is no matter how the actual microscopic charge configuration is, one can always replace it by a smooth distribution of perfect **Dipoles** , if one wants the macroscopic (average) field.

38.03.02
Fields Produced by Polarized Matter: A Physical Approach

The result of equation (38.03.02) is what one obtains straightforwardly considering that a Polarized material has a volume and surface Polarization charge density, given by equations (38.03.03), which must be considered as normal charges. The result of equation (38.03.02) is then the expression of the potential from the given charge densities.

WORK IN PROGRESS

[1] See, for instance, D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ..., § 4.2.3 and 6.2.3

38.04 Electric Field Inside Polarized Matter

This § is referenced at pages:
[1828, 1893]

Read § 37.03, § 39.04.

The electric field inside matter is rapidly changing in time as well as on distances of the order of the inter-atomic distances. For instance the electric field near the surface of an hydrogen nucleus is about $10 \cdot 10^{21}$ V/m; the electric field in the inter-atomic region is basically zero, due to the neutrality of the atoms.

The macroscopic electric field is defined as the spatial average of the microscopic electric field, as long as the average is taken on a volume which is small on a macroscopic scale and large on the microscopic scale:

$$\langle \mathbf{E} \rangle [\mathbf{x}] \equiv \lim_{\tau \longrightarrow 0} \frac{1}{\tau} \iiint_{\Omega} \mathbf{E}[\mathbf{x}+\mathbf{y}] \, \mathrm{d}V_{\mathbf{y}} \qquad \tau := |\mathbf{y}| \leq R, \; R \text{ macroscopically small and microscopically large} \quad , \tag{38.04.01}$$

where the limiting process must be intended in the above-mentioned **macroscopic** sense.

The macroscopic electric field can be calculated from the Polarization.

38.04.01 Average Field Inside Matter

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||

For the equivalence between average field inside matter and the equivalent field generated by the equivalent Charge|Current distribution see figure 37.14, from:

Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...

PRE-IN-PROGRESS

38.05 Electric Fields and Maxwell Equations in Matter

This § is referenced at pages:
[1693, 1894]

38.05.01 Maxwell Equations in Matter: Electric Fields

Let us rewrite **Maxwell Equations** in terms of Polarization and Magnetization as well as Electric|Magnetic fields.

The total charge density is given by the sum of the density of **free** charges, ρ^{F}, plus the bound, Polarization charges, ρ^{P}. It follows, for the divergence equation:

$$\boxed{\varepsilon_0 \mathbf{div}\mathbf{E} = \left(\rho^{\mathrm{F}} + \rho^{\mathrm{P}}\right)} \tag{38.05.01}$$

$$\boxed{\mathbf{div}\mathbf{D} \equiv \rho^{\mathrm{F}}} \tag{38.05.02}$$

$$\boxed{\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P}} \quad .$$

→ 1852
→ 1954

It follows, for the rotor equation:

$$\boxed{\mathbf{rot}\mathbf{E} = -\partial_t \mathbf{B}} \tag{38.05.03}$$

$$\boxed{\mathbf{rot}\mathbf{D} = +\mathbf{rot}\mathbf{P} - \varepsilon_0 \partial_t \mathbf{B}} \quad .$$

→ 1852

Only the non-homogeneous equations change.

Note that, in general, the producers of vector $\mathbf{D}$ are not only the free charges, as it follows from equation (38.05.02), but also the Polarization, as it follows from equation (38.05.01). In fact according to Helmholtz theorem (section § 16.07.05) a vector field can be uniquely reconstructed from the knowledge of both its divergence and its rotor.

In fact the analogous of **Coulomb law** for $\mathbf{D}$ and free charges does not exist!

Note, however, that in special configurations endowed with some symmetry it might be possible to determine the $\mathbf{D}$ vector by using only one of the equations (38.05.02), (38.05.01). In particular the vector $\mathbf{D}$ is not conservative, even in stationary conditions, while $\mathbf{E}$ is. The utility of the $\mathbf{D}$ vector lies in the fact that, usually, the free charges are known while the Polarization charges are not.

38.05.02 Discontinuity of the Electric Fields

This § is referenced at pages:
[Never referenced.]

The discontinuity equations derived in section § 36.09 translate, in presence of matter, into:

$$\boxed{\widehat{\mathbf{n}} \cdot \Delta \mathbf{D}_{\perp} = \rho_{\mathrm{S}}^{\mathrm{F}}} \quad ,$$

$$\boxed{\widehat{\mathbf{n}} \wedge \Delta \mathbf{E}_{\parallel} = \mathbf{0}} \quad .$$

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38.06 Constitutive Relations for Electric Response

38.06.01 Homogeneity Isotropy and Time-(Non)Invariance in Electric Response

Read § 21.02 and § 21.04.

38.06.02 Constitutive Relation - Rigid (Frozen-In) Polarization

In case the material is such that, for instance,

$$\boxed{\mathbf{P}[\mathbf{E}=0] \equiv \mathbf{P}_{\mathrm{R}} \equiv \quad \text{fixed} \quad \neq 0} \quad ,$$

the material has a Polarization even in absence of external (applied) field (a rigid | frozen-in Polarization).

The case of so-called rigid | frozen-in Polarization concerns materials having a fixed Polarization, in particular a Polarization independent of any external electric field or, better, a non zero Polarization when the external (applied) field is zero.

Note, in particular, that the concept of susceptibility has no meaning when dealing with a rigid | frozen-in Polarization.

rigid | frozen-in Polarization implies anisotropy. However, the lack of isotropy, that is failure to obey rotational symmetry in space, is in this case just hidden (spontaneously broken symmetry), by the need to choose one specific direction in space, among infinite equivalent directions. Normally every substance, above its Curie temperature, loses its rigid | frozen-in Polarization to become manifestly isotropic.

38.06.03 Equation of State (Constitutive Relation) of Linear Electric Response

This § is referenced at pages:
[1842]

The introduction of the vector field $\mathbf{D}$ (or $\mathbf{P}$) amounts to introducing a new variable. It is thus necessary to introduce one more equation to solve the problem. The equation to add is not a law of Nature but it is a constitutive equation characterizing the properties of the piece of matter involved, that is an equation of state:

$$\boxed{\mathbf{P}=\mathbf{P}[\mathbf{E},\mathbf{B}] \qquad \text{or} \qquad \mathbf{D}=\mathbf{D}[\mathbf{E},\mathbf{B}]} \quad . \tag{38.06.01}$$

→ 1855

A linear material is defined by a linear equation in terms of the symmetrical **electric susceptibility** tensor, $\{\chi_{\mathrm{E}}\}_{kj}^{\cdot\cdot}[\mathbf{x}]$:

$$\boxed{P_k[\mathbf{x}] = \varepsilon_0 \{\chi_{\mathrm{E}}\}_{kj}^{\cdot\cdot} E_j[\mathbf{x}] \implies \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \{\chi_{\mathrm{E}}\}_{xx}^{\cdot\cdot} & \{\chi_{\mathrm{E}}\}_{xy}^{\cdot\cdot} & \{\chi_{\mathrm{E}}\}_{xz}^{\cdot\cdot} \\ \{\chi_{\mathrm{E}}\}_{yx}^{\cdot\cdot} & \{\chi_{\mathrm{E}}\}_{yy}^{\cdot\cdot} & \{\chi_{\mathrm{E}}\}_{yz}^{\cdot\cdot} \\ \{\chi_{\mathrm{E}}\}_{zx}^{\cdot\cdot} & \{\chi_{\mathrm{E}}\}_{zy}^{\cdot\cdot} & \{\chi_{\mathrm{E}}\}_{zz}^{\cdot\cdot} \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}} \quad ,$$

$$\boxed{\text{with } \{\chi_{\mathrm{E}}\} = \{\chi_{\mathrm{E}}\}[\mathbf{x}]} \quad ,$$

$$\boxed{\text{assuming } \textbf{local and instantaneous response}} \quad . \tag{38.06.02}$$

A linear and homogeneous material is such that the susceptibility does not depend on the position:

$$\boxed{P_k[\mathbf{x}] = \varepsilon_0 \{\chi_{\mathrm{E}}\}_{kj}^{\cdot\cdot} E_j[\mathbf{x}]} \quad .$$

An linear and isotropic material is such that the susceptibility does not depend on the direction, that is the **electric susceptibility** tensor is multiple of the identity:

$$\boxed{P_k[\mathbf{x}] = \varepsilon_0 \chi_{\mathrm{E}}[\mathbf{x}] \delta_{kj}^{\cdot\cdot} E_j[\mathbf{x}] = \varepsilon_0 \chi_{\mathrm{E}}[\mathbf{x}] E_k[\mathbf{x}]} \quad .$$

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In all the above cases one introduces the **Electric-Permittivity** , linking $\mathbf{D}$ and $\mathbf{E}$:

$$\boxed{\mathbf{D} \equiv \varepsilon_{\mathrm{R}} \varepsilon_0 \mathbf{E} \qquad \varepsilon_{\mathrm{R}} \equiv 1 + \chi_{\mathrm{E}} \qquad \text{in static conditions}} \quad .$$

In the isotropic case one can show that the **Electric-Permittivity** satisfies:

$$\boxed{\varepsilon_{\mathrm{R}} \equiv 1 + \chi_{\mathrm{E}} \geq 1} \quad .$$

Note that linearity is not a property of the medium itself but a property of the medium and the applied external (applied) fields: for small enough fields the response is linear, that is the linear approximation of the development in series of Taylor is applied, while for large enough fields the response will certainly cease to be linear.

38.06.03.01 Refraction of Electric Field Lines in LHI Electric Media (No Free Charges)

This § is referenced at pages:
[1897]

Read also § 39.06.03.01.

Consider two electric LHI media, such that:

$$\boxed{\mathbf{D} = \varepsilon_{\mathrm{R}} \varepsilon_0 \mathbf{E}} \quad .$$

Consider a smooth surface, separating the two different media with different physical properties, a point P at the surface where the normal is $\widehat{\mathbf{n}}$, oriented from medium (1) towards medium (2), and define $\Delta\mathbf{A} \equiv \mathbf{A}_2 - \mathbf{A}_1$.

Assume there is no free surface charge density at the surface in such a way that both the normal component of $\mathbf{D}$ and the tangential component of $\mathbf{E}$ are continuous.

Note that the two tangential components and the normal vector to the surface are co-planar vectors, thanks to the continuity of the tangential component of the $\mathbf{E}$ field.

The **incident|refraction angle**, α_e, (read § 15.05.07), is defined as the arc-tangent of the ratio of the modules of the tangential, V_t, and normal, V_n, components of the vector field:

$$\boxed{\tan[\alpha_e] = \frac{|V_t|}{|V_n|} \qquad 0 \leq \alpha_e \leq \pi/2} \quad .$$

As the angle is always $0 \leq \alpha_e \leq \pi/2$ and the therefore the modules of the components must be used.

The following relations apply:

$$\boxed{E_{t2} = E_{t1} \qquad \varepsilon_{\mathrm{R}2} E_{n2} = \varepsilon_{\mathrm{R}1} E_{n1}}$$

$$\boxed{D_{n2} = D_{n1} \qquad \frac{D_{t2}}{\varepsilon_{\mathrm{R}2}} = \frac{D_{t1}}{\varepsilon_{\mathrm{R}1}}} \quad .$$

If one defines the refraction angle, α, of the field lines, as the angle between the vector and the normal to the surface, one can write the above relations as:

$$\boxed{\frac{\tan[\alpha_1]}{\varepsilon_{\mathrm{R}1}} = \frac{\tan[\alpha_2]}{\varepsilon_{\mathrm{R}2}} \Longrightarrow \frac{\tan[\alpha]}{\varepsilon_{\mathrm{R}}} = \text{constant}} \quad ,$$

valid for both $\mathbf{E}$ and $\mathbf{D}$ field lines.

Remember that Field lines are, by definition, continuous: in fact, they are solutions of a differential equation.

Work in Progress

A High Electric-Permittivity Medium

Suppose that $\varepsilon_{R2} \gg \varepsilon_{R1}$: it follows that $\alpha_2 \gg \alpha_1$ so that the field lines are, in medium 1, almost perpendicular to the surface.

In fact in case medium (2) has a very high **Electric-Permittivity** , $\varepsilon_{R2} \gg \varepsilon_{R1}$, the field lines inside the high-**Electric-Permittivity** medium tend to run almost parallel to the surface, while outside the high-**Electric-Permittivity** medium the field lines tend to exit the medium almost perpendicularly to the surface.

As an example: medium 2 is a conductor, which means $\chi_E \to \infty$

38.06.04
Non-Linear Materials

This § is referenced at pages:
[Never referenced.]

In case the constitutive relation between the Polarization and electric field is not linear the concept of **electric susceptibility** is generalized as a differential concept, because one is interested in the description of the Polarization for small changes of the electric field near a prescribed value of the electric field.

Actually, in general, the relation is a non linear-one, except for very small external fields, and therefore one considers the expansion in Taylor series truncated to first-order.

Considering for simplicity isotropic materials, such that $\mathbf{P} \parallel \mathbf{E}$, the definition of susceptibility is thus generalized to:

$$\chi_E \equiv \frac{1}{\varepsilon_0}\frac{\partial P}{\partial E} \quad \text{or better, in case other parameters } X \text{ may affect the polarization:} \quad \chi_E \equiv \frac{1}{\varepsilon_0}\left.\frac{\partial P}{\partial E}\right|_X \quad , \tag{38.06.03}$$

as a function of the components of the Polarization and electric field along their common direction and keeping constant any other possibly relevant parameter, X (such as temperature).

Note that the linear response in normally limited to small fields, while for large enough fields the response shows non-linear terms; see, for a relevant and important example, § 41.04.06. In other words linearity is not a property of the material alone but a property of the material in a certain field.

The general response, equation (38.06.01), can be developed in Taylor series such that linear materials are described by the linear term of the development.

The linearity/non-linearity is in fact a property of the material plus the intensity of the field, not a property of the material alone as the linearity of the response depends on the intensity of the field.

DRAFT - IN PROGRESS

38.07 Forces and Torques on Matter in External (Applied) Electric Fields

Read § 41.02, § 41.05.

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38.08 Additional Topics in Electricity

G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...- ...

R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...- ...

Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...- ...

38.08.01
Break-Down Electric Field

Applying a large enough electric field to a piece of matter the material undergoes break-down, that is an electric discharge crosses the material typically destroying it.

38.08.02
Electric Discharge in Gases

©|Y.P. Raizer|Gas Discharge Physics||

Many different phenomena with many important examples and applications: lighting, gas discharge lamps WEB - URL such as fluorescent lamp or fluorescent tube (it is a gas-discharge lamp that uses electricity to excite mercury vapor; the excited mercury atoms produce short-wave ultraviolet light that then causes a phosphor to fluoresce, producing visible light).

DRAFT IN PROGRESS

38.09 Electric Properties in Dense LI non-dipolarMaterials

This § is referenced at pages:
[1905, 1905, 1905, 1905, 1961, 1977, 2325, 2325]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.10-2.11||
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||
©|C.Kittel, *Introduction To Solid State Physics*, 1996, J.Wiley & Sons, ...Ed., ...|||
©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...| esercizio 4.38 | *SOLUZIONE rigorosa* |

WORK IN PROGRESS

We want to find the relation between the microscopic Polarisability and the macroscopic susceptibility.

In fact there is in general a difference between the average field in matter and the local field acting on a certain molecule, at least **for dense media**. The average field in fact also includes the effect of the field produced by the molecule itself and its neighboring molecules.

The desired relation is **Clausius-Mossotti equation** , giving the relation between the susceptibility and **Polarisability** |Magnetizability based on the relation between the local field (acting on the single molecule) and the average macroscopic fields. It applies to **substances which are linear and isotropic on the macroscopic scale**. Therefore, even if it is not totally general, it is very good to understand the relation between **Polarisability** |Magnetizability, the local field and the (macroscopic) fields.

Proceed as in section § 39.09. Consider a linear isotropic non electric-dipolar dense medium. The local field can be calculated via a spherical cavity, as it looks reasonable for a isotropic material. The medium outside the sphere is treated as a continuous medium; the medium inside the sphere is treated as a set of Polarized molecules. Read § 38.10.07.02 and § 38.10.08 for the calculation of relevant quantities. Imagine a iper-simplified model such that the space allotted to each molecule is a sphere of radius R.

It is a good choice for gases and liquids. Actually for gases there is no significant difference between the local fields and the average field.

Under these assumptions one finds, for the local field on the test molecule, $\mathbf{E}_{\mathrm{L}}$, in term of the macroscopic $\mathbf{E}$ and $\mathbf{P}$:

$$\boxed{\mathbf{E}_{\mathrm{L}} = \mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}} \quad ,$$

$$\mathbf{P} = \eta_e \alpha^{(\mathrm{e})} \varepsilon_0 \mathbf{E}_{\mathrm{L}} \quad ,$$

$$\mathbf{P} = \frac{\eta_e \alpha^{(\mathrm{e})} \varepsilon_0}{1 - \eta_e \alpha^{(\mathrm{e})}/3} \mathbf{E} \quad .$$

Finally the **Clausius-Mossotti equation** is found:

$$\boxed{\chi_{\mathrm{E}} = \frac{3\eta_e \alpha^{(\mathrm{e})}}{3 - \eta_e \alpha^{(\mathrm{e})}} \qquad \frac{\eta_e \alpha^{(\mathrm{e})}}{3} = \frac{\chi_{\mathrm{E}}}{3 + \chi_{\mathrm{E}}} = \frac{\varepsilon_{\mathrm{R}} - 1}{\varepsilon_{\mathrm{R}} + 2}} \quad . \tag{38.09.01}$$

The **Polarisability** $\alpha^{(\mathrm{e})}$ has dimensions of a volume: $\left[\alpha^{(\mathrm{e})}\right] = L^3$.

Note that the above relations are somewhat different from equation (38.10.01): there the **Polarisability** is linked to the polarization of the sphere itself; here, the **Polarisability** is linked to the polarization of the embedding material.

The corrective factor at denominator increases the susceptibility. When $\eta_e \alpha^{(\mathrm{e})} \ll 1$ (for instance at very low density) one finds $\chi_{\mathrm{E}} \simeq \eta_e \alpha^{(\mathrm{e})}$.

If $\alpha^{(\mathrm{e})}$ increases then the **electric susceptibility** increases and $\mathbf{P}$ increases and $\mathbf{E}_{\mathrm{L}}$ increases and this makes $\mathbf{P}$ increase: it is positive feedback.

One cannot extrapolate too much because the linearity is then lost. However, if $\eta_e \alpha^{(e)}$ is too big the system can lock in a Polarized state, even in absence of external electric field, just due to fluctuations.

Consider, as an example, $BaTiO_3$. Below $T = 118$ °C it is a ferro-electric material. Going above $T = 118$ °C it becomes a normal electric material because the density decreases due to the increase of the temperature. It remains a normal electric material with large ε_R, due to the polar character of its structure. If the temperature is decreased below $T = 118\,°C$ the density increases and the internal **Electric Dipoles** locks in a Polarized state.

In general, compare with § 39.09: $\eta_m \neq \eta_e$.

If the molecules are dipolar-molecules the above derivation is not correct. The correct model was worked out in this case by Onsager.

Read also § 39.09.

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38.10 Examples and Physical Applications

38.10.01 Some Values of the Static Electric-Permittivity

WHAT	CONDITIONS	VALUE
Aria	Gas 0°C 1 Atm	1.00059
Acido Cloridrico	Gas 0°C 1 Atm	1.0046
Acqua	Gas 110°C 1 Atm	1.0126
Acqua	Liquido 20°C	80
Benzene	Liquido 20°C	2.28
Ammoniaca	Liquido -34°C	22
Olio Da Trasformatori	Liquido 20°C	2.24
Cloruro Di Sodio	Cristallo 20°C	6.12
Zolfo	Solido 20°C	4.0
Quarzo	Cristallo 20°C (1 Asse Ottico)	4.34
Quarzo	Cristallo 20°C (2 Asse Ottico)	4.27
Polietilene	Solido 20°C	2.3
Neoprene	Solido 20°C	4.1
Porcellana	Solido 20°C	6.0 ÷ 8.0
Cera Di Paraffina	Solido 20°C	2.1 ÷ 2.5
Vetro Pirex 7070	Solido 20°C	4.0

38.10.02 Electric Dipole Moment and Interactions of a HCl Molecule

This § is referenced at pages:
[1862]

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|9.4||

Consider a molecule of HCl. The distance between the two atoms is $d = 126$ pm. Assumes that it is a purely ionic molecule, that is the electron from the H atom is totally transferred to the Cl atom, as in a pure ionic bond. In this hypothesis one estimates its **Electric Dipole** moment as: $\widetilde{p} = ed = 2.01 \cdot 10^{-29}$ C · m. However, the real measured **Electric Dipole** moment is smaller than that, $p = 3.43 \cdot 10^{-30}$ C · m, so that $p = 0.17\widetilde{p}$.

Permanent **Electric Dipole** moments have values of the order of one debye $\equiv 1 \cdot 10^{-30}$ C · m.

Consider then the following model: a positive H ion plus a neutral Cl atom with one electron at some distance d' from the H positive ion and determine d'.

Also consider the following alternative model: a positive H ion plus a positive Cl ion with one pair of electrons at some distance d'' from the H positive ion and determine d''.

Calculate the interaction energy with a typical inter-atomic electric field of $E \approx 1.4 \cdot 10^{11}$ V/m produced by a charge e at a distance of one A.

DRAFT - UNDER PROGRESS

WORKED SOLUTION

First model: the average distance between the electron and the H nucleus is $d' = 0.17d$. In fact (the neutral Cl atom gives no contribution):

$$p = ed' = 0.17ed \implies d' = 0.17d \quad .$$

In fact: $d' = 0 \implies p = 0$ and $d' = d \implies p = ed = \widetilde{p}$.
Interaction energy: $|\Phi_e| = pE = 5{\cdot}10^{-19}$ J. It is large, comparable to interaction energies between point charges and much larger than magnetic interactions. However, $E \approx 10^{12}$ V/m requires distances of about 10^{-10} m, but this is the size of the HCl molecule. Moreover, the interaction gives rise to a strong attractive force between the point charge (nucleus or electron) leading to some kind of collapse of the system.

38.10.03
Polarization of a Hydrogen Atom According to J. J. Thomson Model

This § is referenced at pages:
[1863, 1977]

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|||

Consider a Hydrogen atom.

Model a neutral atom as a point positive charge (the nucleus) surrounded by a uniform cloud of negative charge with radius $a \approx 0.5{\cdot}10^{-10}$ m. This is J.J.Thomson model.

First consider the stability of the equilibrium of the system, under small perturbations.

Assume that the negative electron cloud does not suffer any deformation, which is certainly a good approximation for small perturbations. Imagine the proton is shifted with respect to the center of the negative electron cloud. The negative electron cloud applies a restoring force on the proton, which pushes it back to the center, so that the equilibrium is a stable one.

Secondly, consider the effects of an external static and uniform electric field.

Assume that the cloud is shifted by an amount b, without any deformation, when an external electric field, $\mathbf{E}_0$, is applied; assume the external field $\mathbf{E}_0$ is along z. The hypothesis of non deformation of the spherical shape of the electron cloud is justified by the result we are going to find, showing that the displacement is a tiny one.

One might expect that the relative shift is proportional to the ratio between the external electric field and the typical electric fields inside the atom. It can be roughly estimated as follows:

$$\frac{\Delta z}{a} \approx \frac{E_0}{E} \implies \Delta z \approx \frac{4\pi\varepsilon_0 a^3}{e} E_0 \qquad E \approx \frac{1}{4\pi\varepsilon_0}\frac{e}{a^2} \approx 1.4{\cdot}10^{11}\ \mathrm{V/m} \quad .$$

Note the huge value of the electric field inside the atom.

The result can be made more quantitative as follows. Let the external electric field be along the z axis and only consider modules along the z direction. According to this simple model the equilibrium condition for the nucleus implies that force acting on the nucleus by the cloud of negative charge, E_-, must be equal and opposite to the force on the nucleus by the external (applied) field, E_0:

$$qE_0 = qE[b] = q\frac{\rho b}{3\varepsilon_0} = \frac{q^2 b}{4\pi\varepsilon_0 a^3} \qquad E[b] = E_0 \text{ at equilibrium} \quad ,$$

and the induced **Dipole** moment

$$p = bq = 4\pi\varepsilon_0 a^3 E_0 \quad .$$

The **Polarisability** , $\alpha^{(e)}$, is defined, in terms of the external (applied) field, in equation (38.02.01.01).

Assume $a = 0.5{\cdot}10^{-10}$ m and $E_0 = 10^5$ V/m. The electric **Polarisability** is: $\alpha^{(e)} = 1.6{\cdot}10^{-30}$ m^3.

Note that the electric **Polarisability** has dimension of a volume and that the result is the correct order of magnitude of the volume of the hydrogen atom.

A.F.T. - LTEX-BOOKS

One finds: $b = 9 \cdot 10^{-18}$ m that is much less than the dimensions of the nucleus. Therefore the atom is very rigid with respect to typical external electric fields. This approximation of non-distorted electron cloud is justified by the result: $b \ll a_0$.

The resulting induced **Electric Dipole** moment is: $p \simeq 1.4 \cdot 10^{-36}$ C $\cdot$ m; this is much less that a typical permanent **Dipole** moment (see the HCl molecule § 38.10.02, for which $p \simeq 3.4 \cdot 10^{-30}$ C $\cdot$ m).

For the sake of comparison the measured **Polarisability** is: $\alpha^{(e)} = 0.7 \cdot 10^{-30}$ m^3.

The value of $3 \cdot 10^{-30}$ C $\cdot$ m is quite typical for **Electric Dipole** moments of molecules. sometimes the unit of measure called **debye**, is used: $3.3356 \cdot 10^{-30}$ C $\cdot$ m.

As the restoring electric field is of the order of $E \approx 10^{11}$ V/m one may expect a linear response up to external electric fields $E_0 \ll 10^{11}$ V/m.

38.10.04
Polarization of a Hydrogen Atom With Exponential Electron Charge Density

This § is referenced at pages:
[Never referenced.]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 4.2||

According to quantum physics, the electron in a hydrogen atom, in the fundamental state, has a charge density:

$$\rho[r] = \frac{e}{\pi a_0^3} \exp\left[-2r/a_0\right] \quad ,$$

where q is the charge of the electron and a_0 is the Bohr radius.

Find the atomic **Polarisability** of such an atom assuming that the electron charge distribution is not distorted by the external electric field and that the nucleus moves by an amount $b \ll a_0$. This approximation is justified by the results of the problem with uniform charge density as well as, afterwards, by the result found with this approximation.

WORKED SOLUTION

$$4\pi r^2 E[r] = \frac{1}{\varepsilon_0} \int_{s=0}^{r} \rho[s] 4\pi s^2 \, \mathrm{d}s = \frac{4\pi}{\varepsilon_0} \frac{e}{\pi a_0^3} \int_{s=0}^{r} s^2 \exp\left[-2s/a_0\right] \, \mathrm{d}s \qquad \text{Gauss law for a sphere of radius } r \quad ,$$

$$\int_{s=0}^{r} s^2 \exp\left[-2s/a_0\right] \, \mathrm{d}s = \frac{1}{4} a_0 \left(a_0^2 - \exp -\frac{2r}{a_0} \left(2a_0 r + a_0^2 + 2r^2 \right) \right) \simeq \frac{r^3}{3} - \frac{r^4}{2a_0} + \frac{2r^5}{5a_0^2} + O\left(r^6\right) \quad ,$$

$$E[r] = \frac{er}{3\varepsilon_0 \pi a_0^3} \quad ,$$

$$\alpha^{(e)} = 3\pi a_0^3 \quad .$$

Not so different from the model with uniform charge distribution.

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38.10.05
Polarization of a Hydrogen Atom With Electron Charge Density Proportional to the Distance

This § is referenced at pages:
[Never referenced.]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|||

Let us study a simple case of Non-Linear response.

The induced **Electric Dipole** moment of an atom/molecule is proportional to the external (applied) electric field. This is a first-order approximation, not a fundamental law of nature, and it is easy to invent exceptions examples.

Suppose, for example, that the charge density of the electron cloud is proportional to the distance from the center, up to a distance R and all the other hypotheses of example § 38.10.03:

$$\begin{cases} \rho[r] = \beta r & \text{for } r < R \\ \rho[r] = 0 & \text{for } r > R \end{cases} ,$$

- How much is β for a total charge q ?
- To what power of E is the induced **Electric Dipole** moment proportional in this case?
- Under what conditions the first-order approximation holds?

WORKED SOLUTION

$$\begin{cases} E[r] = \dfrac{\beta r^2}{4\varepsilon_0} & \text{for } r < R \\ E[r] = \dfrac{Q}{4\pi\varepsilon_0 r^2} & \text{for } r > R \end{cases} ,$$

One finds: $p \propto \sqrt{E_0}$

38.10.06
Van Der Waals Forces

This § is referenced at pages:
[1950]

©|M.dePodesta, *Understanding The Properties Of Matter*, 2002, CRC Press, 2ndEd., ...|6.2.2||

The name **Van Der Waals forces** refers to inter-molecular forces between point charges, permanent or induced **Dipoles** and|or higher-order ElectroMagnetic **Multipoles** . It is also sometimes used loosely as a synonym for the totality of inter-molecular forces.

Van Der Waals forces are relatively weak compared to normal chemical bonds.

They typically have a repulsive core, preventing the two systems to collapse.

They fall into three different categories.

- Keesom forces (orientation): forces between permanent charges, **Dipoles** and|or higher-order **Multipoles** .
- Debye forces (induction): forces between permanent charges, **Dipoles** and|or higher-order **Multipoles** and an induced **Dipole** and|or higher-order **Multipole** .
- London forces (dispersion): transient induced **Dipole** -**Dipole** or higher-order **Multipoles** interactions. These are sometimes called **Van Der Waals forces** themselves. The instantaneous fluctuations of **Electric Dipole** momenta of neutral molecules/atoms give rise to an **Electric Dipole** field which polarises nearby molecules/atoms in such a way that an attractive force with an interaction energy

DRAFT PROOF

going as the inverse sixth-power of the distance. The interaction is a quantum effect, in the sense that it disappears if the Planck constant is let to go to zero. The zero point energy of the system is lowered by the **Dipole** -Dipole coupling. The interaction does not depend for its existence on any overlap of the charge densities of the two atoms.

Point charge with **Dipole** interactions: attractive force with an interaction energy going as the inverse third-power of the distance.

Dipole with **Dipole** interactions: attractive force with an interaction energy going as inverse sixth-power of the distance.

In case of induced **Multipoles** the force is always attractive, as explained in section § 41.02.02.

38.10.07
Rigid (Frozen-In) Polarization

Note that for a uniform Polarization one simply has:

$$\mathbf{p} \equiv \iiint \mathbf{P}\, \mathrm{d}V_{\mathbf{y}} = \mathbf{P}\Delta V \qquad \text{for uniform } \mathbf{P} \quad .$$

38.10.07.01 A Uniformly Polarized Cylinder

By taking into account the Polarization charges it is clear that the electric field, **E**, produced by such a cylinder is exactly the same as the one produced by a circular capacitor with the same surface charge density. As the Polarization is known a priori the **D** field can be calculated from its definition.

The electric field obviously resembles the field of a **Dipole** , at large distances, as always. Near the cylinder one can think of the field as produced by an array of real **Electric Dipoles** (pairs of equal and opposite sign charges) put side by side.

38.10.07.02 A Uniformly Polarized Sphere

This § is referenced at pages:
[1858, 1866, 1907, 1908]

Compare with § 39.10.04.02

The field inside a Polarized sphere is uniform, while the external field is exactly a **Dipole** field.

The **E** field has Sources|Sinks at the surface of the sphere, therefore the density of field lines changes crossing the surface.

The **D** field has no Sources|Sinks at the surface of the sphere and its flux is conserved, so the density of flux lines is.

Direct Calculation

©|S.Bobbio & E.Gatti, *Elettromagnetismo E Ottica*, 1991, Bollati-Boringhieri, ...Ed., ...|5.3||

From the microscopic interpretation of the Polarization, if one assumes that matter is made of all identical systems, one can imagine that the uniformly Polarized sphere comes from the uniform displacement of a grid of positive/negative charges, such that relative position of the positive charges with respect to negative charges is $\boldsymbol{\delta}$.

The **scalar electric potential** is thus the potential of two uniformly charged spheres whose centers are displaced by a small amount $\boldsymbol{\delta}$, with $\delta \ll R$. Let $|q|$ be the absolute value of the charge of our model and $\eta_{\mathbb{N}}$ the number of charges per unit volume.

Direct calculation: external field.

The potential external to the sphere is the potential of an **Electric Dipole** , as the two uniformly charged spheres behave like point charges outside their radius:

$$\mathbf{p} = q\eta_{\mathbb{N}} \left(\frac{4}{3}\pi R^3\right) \boldsymbol{\delta} \equiv \mathbf{P} \left(\frac{4}{3}\pi R^3\right) \qquad \text{with} \qquad \mathbf{P} = q\eta_{\mathbb{N}}\boldsymbol{\delta} \equiv \eta_{\mathbb{N}}\mathbf{p} \quad ,$$

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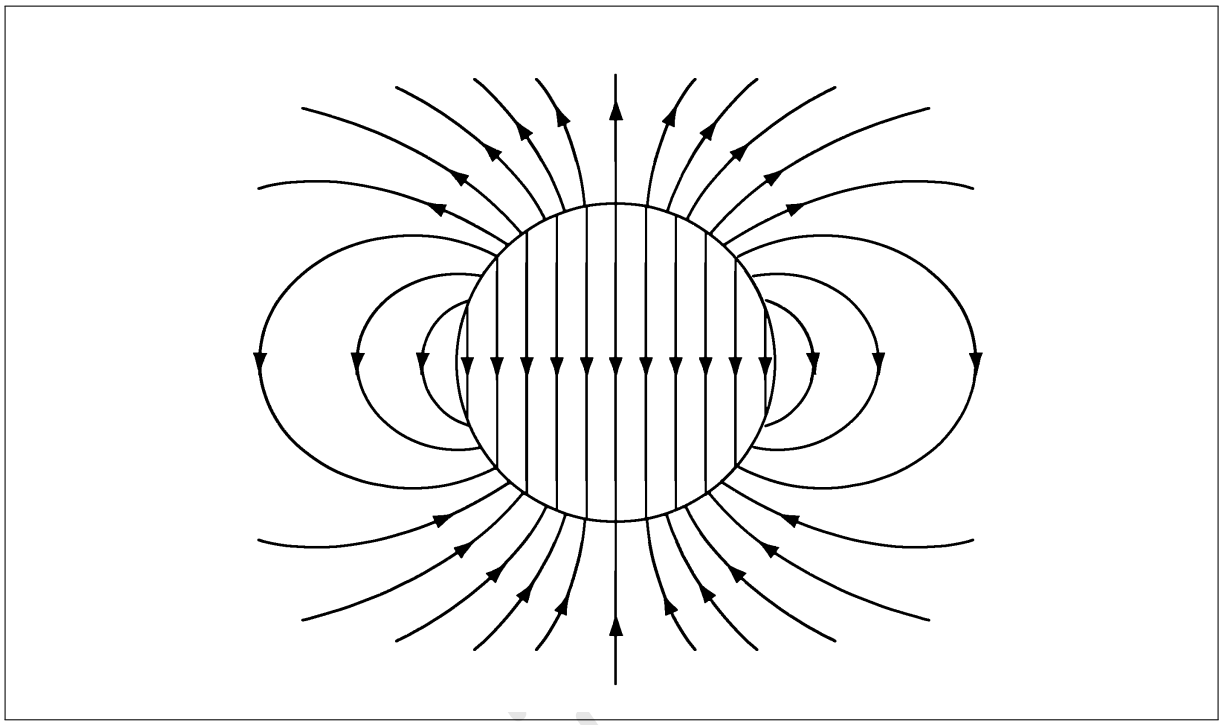

Figure 38.11: The**D** field has nowhere Sources|Sinks, but it may have vortexes. The **E** field has Sources|Sinks at the surface of the sphere, therefore the density of field lines changes crossing the surface.

giving a potential, outside the sphere,

$$\boxed{\Phi[\mathbf{x}] = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3} \qquad \text{for } |\mathbf{x}| > R \qquad \text{with:} \qquad \mathbf{p} = \mathbf{P}\left(\frac{4}{3}\pi R^3\right)} \quad ,$$

for a uniformly Polarized sphere of radius R and Polarization $\mathbf{P}$. Note that this is exactly the potential of an **Electric Dipole** at any point externally to the sphere, not only at large distances.

Direct calculation: internal field.

The potential internal to the sphere is the potential of two superimposed uniformly charged spheres, with charges of different sign, and with the center of the positive charge displaced with respect to the center of the negative charge by an amount $\boldsymbol{\delta}$. Let $\mathbf{r}_+$ and $\mathbf{r}_-$ be, respectively, the position of the centers of the two charge distributions and $\mathbf{r}$ the position of the point, internal to the sphere, where we want to calculate the electric field. Therefore we have:

$$-\boldsymbol{\delta} \equiv (\mathbf{r}_+ - \mathbf{r}_-) \quad ,$$

giving an electric field:

$$\mathbf{E}_{\text{I}} = \frac{|q|\eta_{\text{N}}}{3\varepsilon_0}(\mathbf{r}_+ - \mathbf{r}_-) = -\frac{|q|\eta_{\text{N}}}{3\varepsilon_0}\boldsymbol{\delta} = -\frac{\mathbf{P}}{3\varepsilon_0} \qquad \mathbf{D}_{\text{I}} \equiv (\varepsilon_0\mathbf{E}_{\text{I}} + \mathbf{P}) = +\frac{2}{3}\mathbf{P} \quad .$$

Therefore:

$$\boxed{\mathbf{E}_{\text{I}} = -\frac{\mathbf{P}}{3\varepsilon_0} \qquad \mathbf{D}_{\text{I}} \equiv (\varepsilon_0\mathbf{E}_{\text{I}} + \mathbf{P}) = +\frac{2}{3}\mathbf{P}} \quad .$$

Therefore the potential internal to the sphere (according to the results in section § 36.20.68) is:

$$\boxed{\Phi[\mathbf{r}] = \frac{\mathbf{P}\cdot\mathbf{r}}{3\varepsilon_0} \qquad \text{for } |\mathbf{r}| < R} \quad ,$$

Note that the internal electric field is such that:

$$\boxed{\mathbf{E}_{\text{I}}\cdot\mathbf{P} \leq 0} \quad ,$$

IN PROGRESS

and compare with the analogous magnetic situation in section § 39.10.04.02. The difference of the two cases is a reflection of the fundamental difference, at small distances, between the fields produced by a real **Electric Dipole** and a real **Magnetic Dipole** (read § 41.01).

A Purely Formal Calculation

Start from equation (38.03.01), with $\mathbf{P}$ uniform,

$$\Phi[\mathbf{x}] = \frac{1}{4\pi\varepsilon_0}\iiint_\tau \frac{\mathbf{P}[\mathbf{y}]\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,\mathrm{d}\mathbf{y} = \frac{1}{4\pi\varepsilon_0}\mathbf{P}\cdot\iiint_\tau \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,\mathrm{d}V_\mathbf{y} = \frac{1}{4\pi\varepsilon_0}\mathbf{P}\cdot\mathbf{I}[\mathbf{x}] \quad .$$

Read § 41-014 for the calculation of $\mathbf{I}[\mathbf{x}]$.

It follows:

$$\Phi[\mathbf{x}] = \frac{1}{4\pi\varepsilon_0}\mathbf{P}\cdot\mathbf{I}[\mathbf{x}] = \begin{cases} \dfrac{1}{3\varepsilon_0}\mathbf{P}\cdot\mathbf{x} & \text{for } |\mathbf{x}| \le R \quad , \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{\mathbf{P}\cdot\mathbf{x}}{x^3}\tau & \text{for } |\mathbf{x}| > R \end{cases} \qquad \text{with } \tau = \frac{4}{3}\pi R^3 \quad .$$

The **scalar electric potential** Φ is the same as a **Electric Dipole** , outside the sphere. It is the **scalar electric potential** of a uniform field, inside the sphere. The equipotential surfaces of Φ inside the sphere are planes perpendicular the axis of the sphere (the diameter of the sphere parallel to $\mathbf{P}$). Remember that the **scalar electric potential** , in static conditions, is defined to within an arbitrary constant.

38.10.08
A Sphere of LHI Material in a Uniform and Constant External Electric Field

This § is referenced at pages:
[1858, 1872, 1874, 1874, 1909, 1910, 1948]

©|J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|21.V.4||

Compare with § 39.10.05.

It can be shown that, for any LHI sphere or ellipsoid, the induced Polarization is uniform in an external uniform field.

For a sphere of LHI material with **electric susceptibility** χ_E in an external electric field $\mathbf{E}_0$ the following relations apply.

Be careful to distinguish between the external applied field $\mathbf{E}_0$ and the average field inside the sphere $\mathbf{E}$.

One first need to calculate the Polarization, so that the **Electric Dipole** moment is known and the results of § 38.10.07.02 can be used.

The internal field can be calculated as follows.

$$\boxed{\mathbf{P} \equiv \chi_\mathrm{E}\varepsilon_0\mathbf{E}} \quad ,$$

$$\boxed{\mathbf{E} = \mathbf{E}_0 - \frac{\mathbf{P}}{3\varepsilon_0}} \quad ,$$

$$\boxed{\mathbf{P} = \left(\frac{3\chi_\mathrm{E}}{3+\chi_\mathrm{E}}\right)\varepsilon_0\mathbf{E}_0} \quad ,$$

$$\boxed{\mathbf{E} = \frac{3}{3+\chi_\mathrm{E}}\mathbf{E}_0 \implies |\mathbf{E}| \le |\mathbf{E}_0| \qquad \mathbf{E}\cdot\mathbf{E}_0 \ge 0} \quad ,$$

$$\mathbf{D} = \frac{3\,(1+\chi_\mathrm{E})}{3+\chi_\mathrm{E}}\mathbf{D}_0 \implies |\mathbf{D}| \ge |\mathbf{D}_0| \qquad \mathbf{D}\cdot\mathbf{D}_0 \ge 0 \quad .$$

The meaning of **dePolarizing field** $\mathbf{E}$ is clear when noting that the internal $\mathbf{E}$ field is always smaller that the external (applied) field.

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Note that in any case $\mathbf{E}$ is parallel to $\mathbf{E}_0$ and $\mathbf{D}$ is parallel to $\mathbf{D}_0$ (the coefficients are always positive).
The case of a strictly non Polarizable object (or vacuum) is found for $\chi_{\mathrm{E}} = 0$.
The case of a perfect conductor can be recovered as $\chi_{\mathrm{E}} \to \infty$: $\mathbf{E} = 0$, $\mathbf{D} = 3\mathbf{D}_0$.
The Polarization is always parallel to the external (applied) field, as $\chi_{\mathrm{E}} > 0$.
The **Polarisability** of the sphere is:

$$\alpha^{(\mathrm{e})} = \frac{3\chi_{\mathrm{E}}}{3+\chi_{\mathrm{E}}}\Delta V = \frac{3\,(\varepsilon_{\mathrm{R}}-1)}{2+\varepsilon_{\mathrm{R}}}\Delta V \quad , \tag{38.10.01}$$

→ 1858

and it is not strictly proportional to the **electric susceptibility** χ_{E}.

The formal analogy, $\mathbf{E} \Leftrightarrow \mathbf{H}$, with the case of the sphere in an external magnetic field (section § 39.10.05) is obvious, the only basic difference being that the electric susceptibility is non negative while the **magnetic susceptibility**, χ^*_{M}, is constrained to be $\chi^*_{\mathrm{M}} \geq -1$.

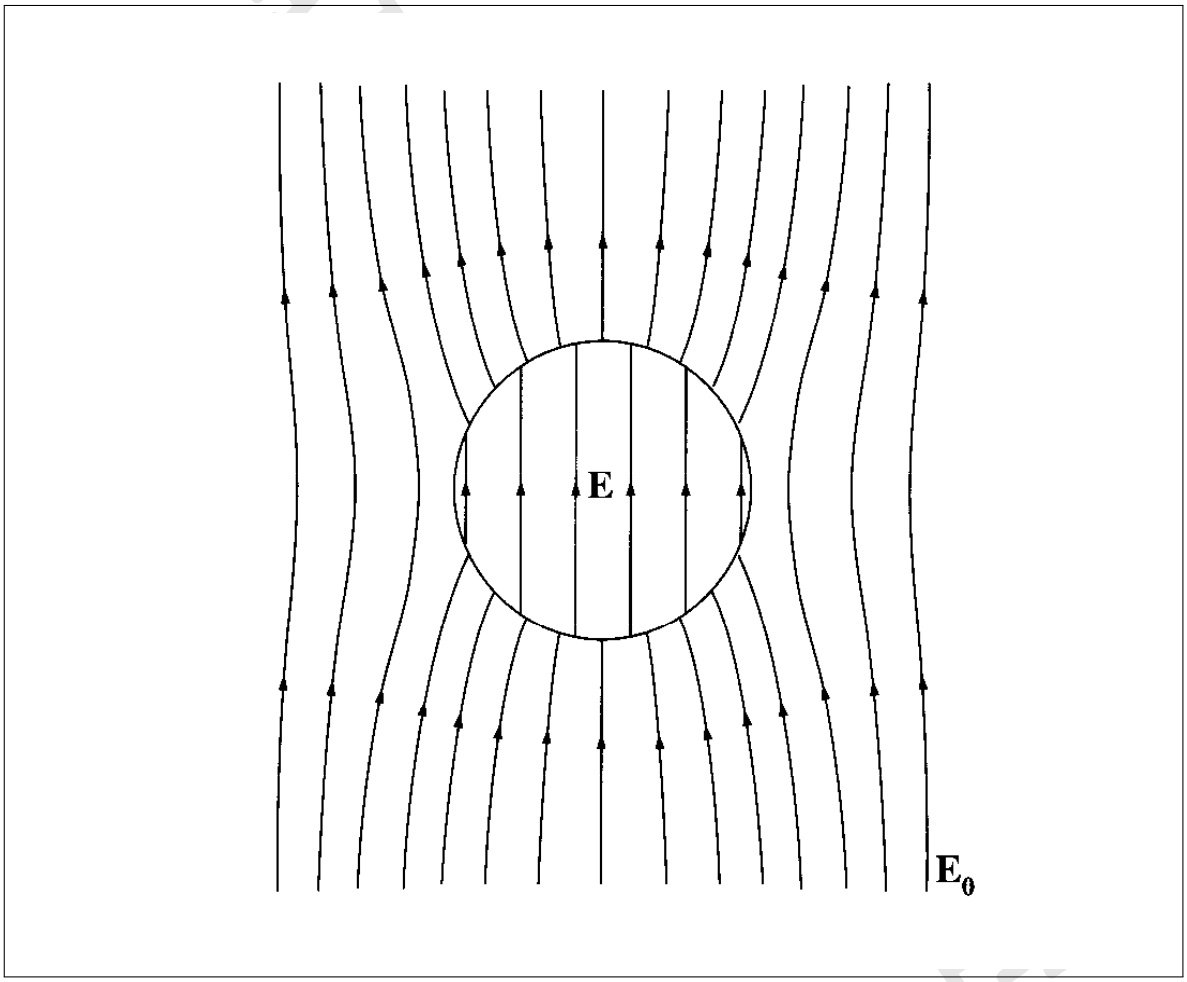

Figure 38.12: The**E** field has Sources | Sinks at the surface of the sphere, therefore the density of field lines changes crossing the surface (the flux is not conserved while crossing the Polarization surface charges).

38.10.09
Cavities in a Polarized Material

38.10.09.01 Cylindrical Cavity
Read § 41.03.08.

38.10.09.02 Spherical Cavity
Read § 41.03.08.

38.10.10
A Point Charge and Two Polarizable LHI Materials

This § is referenced at pages:
[1953]

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|4.4||

Assume that space is full with two Polarizable LHI materials separated by the plane $z = 0$. A point charge, q, is placed at $\mathbf{r}_0 = \{0, 0, d\}$. The material at $z > 0$ has **Electric-Permittivity** ε_{R1}. The material at $z < 0$ has **Electric-Permittivity** ε_{R2}.

The charge polarises both media and in particular a Polarization surface charge appears at the surface at $z = 0$. The point charge and the Polarization surface charge at the boundary exert a force one on the other. There is no Polarization volume charge density as there is no free charges in the material.

WORKED SOLUTION

One exploits the uniqueness of the solution: we use one good solution so that it must be the unique one.
Everywhere at $z \neq 0$: $\mathbf{rot}\,\mathbf{E} = 0$. We can therefore define a potential, continuous at $z = 0$.
Assume the following potential, with image charges q' at $-\mathbf{r}_0$ and q'') at $\mathbf{r}_0$:

$$z > 0 \qquad \Phi_1[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\frac{1}{\varepsilon_{R1}}\left(\frac{q}{|\mathbf{r}_0 - \mathbf{r}|} + \frac{q'}{|\mathbf{r}_0 + \mathbf{r}|}\right) \quad ,$$

$$z < 0 \qquad \Phi_2[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\frac{1}{\varepsilon_{R2}}\left(\frac{q''}{|\mathbf{r}_0 - \mathbf{r}|}\right) \quad .$$

The potential at $z > 0$ is the potential of a **Dipole** with, a priori, different charges (q' not yet known).
The potential at $z < 0$ is the potential of a point charge (q'' not yet known). A priori the sign of the two image charges q' at $-\mathbf{r}_0$ and q'' at $\mathbf{r}_0$, is not determined at all.
We have:

$$r_1 = \sqrt{x^2 + y^2 + (z - d)^2} \qquad r_2 = \sqrt{x^2 + y^2 + (z + d)^2} \quad .$$

Continuity of the potential at $z = 0$ implies:

$$\Phi_1[\{x, y, 0\}] = \Phi_2[\{x, y, 0\}] \Longrightarrow \frac{q}{\varepsilon_{R1}} + \frac{q'}{\varepsilon_{R1}} = \frac{q''}{\varepsilon_{R2}} \quad .$$

Continuity of the normal components of $\mathbf{D}$ at $z=0$ implies (no free charges):

$$D_{n1}[\{x,y,0\}] = D_{n2}[\{x,y,0\}] \quad ,$$
$$\varepsilon_{\mathrm{R}1} E_{n1}[\{x,y,0\}] = \varepsilon_{\mathrm{R}2} E_{n2}[\{x,y,0\}] \quad ,$$
$$\varepsilon_{\mathrm{R}1} \left.\frac{\partial \Phi_1}{\partial z}\right|_{z=0} = \varepsilon_{\mathrm{R}2} \left.\frac{\partial \Phi_2}{\partial z}\right|_{z=0} \quad ,$$
$$q = q' + q'' \quad .$$

Continuity of the tangential components of $\mathbf{E}$ at $z=0$ implies:

$$E_{t1}[\{x,y,0\}] = E_{t2}[\{x,y,0\}] \quad ,$$

but this condition is not used to solve the problem.
The results are:

$$q' = q\left(\frac{\varepsilon_{\mathrm{R}1}-\varepsilon_{\mathrm{R}2}}{\varepsilon_{\mathrm{R}1}+\varepsilon_{\mathrm{R}2}}\right) \qquad q'' = q\left(\frac{2\varepsilon_{\mathrm{R}2}}{\varepsilon_{\mathrm{R}1}+\varepsilon_{\mathrm{R}2}}\right) \quad .$$

The above results imply that the image charge q'' has the same sign as the real charge.
The surface at $z=0$ is Polarized with Polarization surface charge:

$$\rho_{\mathrm{S}}^{\mathrm{P}} = \widehat{\mathbf{n}}_{12}\cdot(\mathbf{P}_1-\mathbf{P}_2) \equiv \widehat{\mathbf{n}}_{1\to 2}\cdot(\mathbf{P}_1-\mathbf{P}_2) = P_{z2}-P_{z1} \quad ,$$

where one should be careful with $\widehat{\mathbf{n}}_{12}=\{0,0,-1\}$ to avoid making errors with the sign of the surface Polarization charge.
As the free charge density is zero one can calculate the Polarization charge density as the total charge density:

$$\rho_{\mathrm{S}}^{\mathrm{P}} = \varepsilon_0\left(E_{z2}-E_{z1}\right) = \frac{1}{4\pi\varepsilon_0}\frac{2\varepsilon_0 q d}{r^3}\frac{1}{\varepsilon_{\mathrm{R}1}}\left(\frac{\varepsilon_{\mathrm{R}1}-\varepsilon_{\mathrm{R}2}}{\varepsilon_{\mathrm{R}1}+\varepsilon_{\mathrm{R}2}}\right) = \frac{q}{2\pi}\frac{d}{\left(d^2+\rho^2\right)^{3/2}}\frac{1}{\varepsilon_{\mathrm{R}1}}\left(\frac{\varepsilon_{\mathrm{R}1}-\varepsilon_{\mathrm{R}2}}{\varepsilon_{\mathrm{R}1}+\varepsilon_{\mathrm{R}2}}\right) \equiv \frac{\alpha q d}{\left(d^2+\rho^2\right)^{3/2}} \quad ,$$

with $\rho \equiv \sqrt{x^2+y^2}$.
The total Polarization charge can be calculated by integration:

$$Q_P = \int_{\rho=0}^{+\infty} \rho_{\mathrm{S}}^{\mathrm{P}} 2\pi\rho \,\mathrm{d}\rho = \frac{q}{\varepsilon_{\mathrm{R}1}}\left(\frac{\varepsilon_{\mathrm{R}1}-\varepsilon_{\mathrm{R}2}}{\varepsilon_{\mathrm{R}1}+\varepsilon_{\mathrm{R}2}}\right) \quad .$$

The force between the real charges (the component perpendicular to the interface) is calculated from Coulomb law (in vacuum!). The z component of force on the point charge (with the correct sign) is given by:

$$\mathrm{d}F_\perp = \frac{1}{4\pi\varepsilon_0}\frac{q\,\mathrm{d}q_P}{r^2}\frac{d}{r} = \frac{1}{4\pi\varepsilon_0}\frac{2\pi q d\rho\rho_{\mathrm{S}}^{\mathrm{P}}\,\mathrm{d}\rho}{r^3} = \frac{\alpha q^2 d^2 \rho\,\mathrm{d}\rho}{2\varepsilon_0\left(d^2+\rho^2\right)^3} \quad ,$$

$$\Longrightarrow F_\perp = \frac{\alpha q^2}{8\varepsilon_0 d^2} = \frac{q^2}{16\pi\varepsilon_0 d^2}\frac{1}{\varepsilon_{\mathrm{R}1}}\left(\frac{\varepsilon_{\mathrm{R}1}-\varepsilon_{\mathrm{R}2}}{\varepsilon_{\mathrm{R}1}+\varepsilon_{\mathrm{R}2}}\right) = \frac{q^2}{4\pi\varepsilon_0\left(2d\right)^2}\frac{1}{\varepsilon_{\mathrm{R}1}}\left(\frac{\varepsilon_{\mathrm{R}1}-\varepsilon_{\mathrm{R}2}}{\varepsilon_{\mathrm{R}1}+\varepsilon_{\mathrm{R}2}}\right) \quad .$$

Limiting case: $\varepsilon_{\mathrm{R}2}=\varepsilon_{\mathrm{R}1} \Longrightarrow \rho_{\mathrm{S}}^{\mathrm{P}}=0$; ok as there is not interface in this case.
Limiting case: $\varepsilon_{\mathrm{R}2} \gg \varepsilon_{\mathrm{R}1}$; medium (2) becomes much more Polarized than medium (1) and therefore the induced density of **Dipoles** is much bigger in medium (2) than in medium (1) and therefore at the interface the density of **Dipoles** from medium (2) (a negative one if $q>0$) is bigger than the density of **Dipoles** from medium (1) (a positive one if $q>0$). Therefore $\rho_{\mathrm{S}}^{\mathrm{P}}<0$ if $q>0$.
Limiting case: $\varepsilon_{\mathrm{R}2} < \varepsilon_{\mathrm{R}1}$; medium (1) becomes more Polarized than medium (2) and therefore the induced density of **Dipoles** is bigger in medium (1) than in medium (1) and therefore at the interface the density of **Dipoles** has the same sign as the charge q. Therefore $\rho_{\mathrm{S}}^{\mathrm{P}}>0$ if $q>0$.
Limiting case: $\varepsilon_{\mathrm{R}1}=1$ and $\varepsilon_{\mathrm{R}2}\to\infty$; it is in fact a charge in vacuum in front of a conductor, with electric field lines perpendicular to the interface. In this case:

$$\rho_{\mathrm{S}}^{\mathrm{P}} = -\frac{q}{2\pi}\frac{d}{\left(d^2+\rho^2\right)^{3/2}} \qquad Q_P = -q \qquad F_\perp = -\frac{q^2}{4\pi\varepsilon_0\left(2d\right)^2} \quad .$$

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Limiting case: $d = 0$; there is no charge density at the interface because the Polarization is everywhere perpendicular to the interface.
The Polarization charge volume density inside the medium is given by the divergence of $\mathbf{P}$ that is proportional to the divergence of $\mathbf{E}$, that is zero away from the position of the real point charge.
It is very important to note that when one calculates the force one must use all and only the real charges: the free point charge and the induced Polarization charges. One must use the normal Coulomb law in vacuum and not the Coulomb law with ε_{R} ! The latter, in fact, only applies when one uses the free charges because then the ε_{R} factor keeps memory of the screening of the Polarization charges.

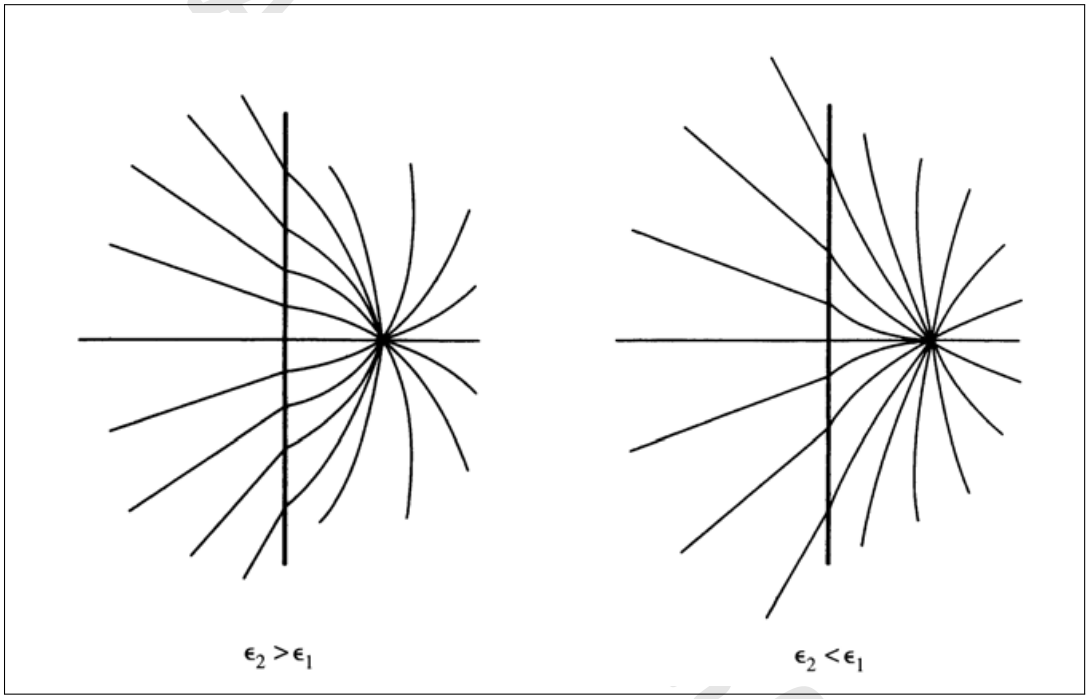

Figure 38.13: Fieldlines of $\mathbf{D}$ for a point charge embedded in a Polarizable material (with $\varepsilon_{\mathrm{R}} = \varepsilon_{\mathrm{R}1}$) near a semi-infinite slab of Polarizable material (with $\varepsilon_{\mathrm{R}} = \varepsilon_{\mathrm{R}2}$).

WORK IN PROGRESS

38.10.11
Force on a Point Charge Embedded in a Polarizable LHI Material

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38.11 Exercises Problems and Physical Applications

38-001 D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...

All problems from D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ..., § 4.

38-002 A Cubic Polarizable Material With rigid|frozen-in Polarization

A cube of Polarizable material of side d, with one vertex at the origin and sides parallel to the Cartesian planes, carries a rigid|frozen-in Polarization $\mathbf{P} = k\mathbf{r}$ where k is a constant.

1. Determine the dimensions of k.
2. Find all the Polarization charges.
3. Write the expression of the electric field at large distances.

38-003 A Spherical Shell With rigid|frozen-in Polarization

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Problem 4.15||

A spherical shell, with internal radius r_1 and external radius r_2, has a rigid|frozen-in Polarization $\mathbf{P}[\mathbf{r}] = k\hat{\mathbf{e}}_r/r$ where k is a suitable constant. There is no free charge anywhere. Determine the electric field everywhere

- either calculating directly the electric field from the Polarization charges;
- or calculating $\mathbf{D}$ first and $\mathbf{E}$ after.

Note that the medium is not a LHI medium.

38-004 A Polarizable LHI Sphere in an Applied Uniform and Constant Electric Field (1)

Read also § 38.10.08.

A Polarizable LHI sphere of radius R in put into an external applied uniform and constant electric field, $\mathbf{E}_0$.

Assume that the electric field produced by the Polarization charges of the sphere is $\mathbf{E}' \parallel \mathbf{E}_0$, uniform ad parallel to the applied electric field internally to the sphere, and a **Dipole** field, externally to the sphere.

Show that this defines a solution at all points in space satisfying the required conditions at the boundaries, including infinity (so this is the solution for the uniqueness of the solution).

Determine the value of the electric field produced by the Polarization charges by imposing the appropriate boundary conditions.

WORKED SOLUTION

Let $\mathbf{E}$ be the total electric field, superposition of the applied field and the field produced by the Polarization charges of the sphere.
The relation between **Electric Dipole** moment and Polarization is:

$$\mathbf{p} = \mathbf{P}\frac{4\pi R^3}{3} \quad .$$

Use spherical coordinates with the z axis along the direction of the applied electric field $\mathbf{E}_0$.
The external, $r \geq R$, potential is:

$$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} - zE_0 \quad .$$

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The external, $r \geq R$, field is:

$$\mathbf{E}[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\,(\mathbf{p}\cdot\mathbf{r})\,\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}\right) + E_0\widehat{\mathbf{e}}_3 \quad ,$$

$$\begin{cases} E_r = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2\mathbf{p}\cos[\theta]}{r^3} + E_0\cos[\theta] & \text{in fact: } \widehat{\mathbf{e}}_3\cdot\widehat{\mathbf{e}}_r = \cos[\theta] \quad , \\ E_\theta = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\mathbf{p}\sin[\theta]}{r^3} - E_0\sin[\theta] & \text{in fact: } \cos[\theta+\pi/2] = -\sin[\theta] \end{cases} .$$

Take, conventionally, the fields $\mathbf{P}$ and $\mathbf{E}'$ oriented as the z axis; if their components turns out to be negative its real orientation would be opposite.
The internal, $r \leq R$, potential is:

$$\Phi = -zE' \quad .$$

The internal, $r \leq R$, field is:

$$\mathbf{E}[\mathbf{r}] = E'\widehat{\mathbf{e}}_3 \quad ,$$

$$\begin{cases} E_r = E'\cos[\theta] \quad , \\ E_\theta = -E'\sin[\theta] \end{cases} .$$

The above electric fields are solutions to the **Maxwell Equations** both externally and internally to the sphere; in fact they are both divergence-less and rotor-less.
The unknown internal field $\mathbf{E}'$ can be determined imposing the boundary conditions.
The boundary conditions for the tangential component of $\mathbf{E}$ at $r = R$ (continuity) and the definition of **electric susceptibility** form a system of two equations with two unknowns, $\mathbf{E}'$ and $\mathbf{P}$ which can be solved:

$$\sin[\theta]\left(\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{R^3} - E_0\right) = -E'\sin[\theta] \implies E' = E_0 - \frac{P}{3\varepsilon_0} \quad ,$$

$$\mathbf{P} = \chi_{\mathrm{E}}\varepsilon_0\mathbf{E}' \quad .$$

Note that P, E' and E_0 are the components along the $\widehat{\mathbf{e}}_3$ direction, not modules. The results shows that the internal field is the sum of the uniform applied field and the field produced by the Polarization charges. Solving the above system for the two unknowns, $\mathbf{E}'$ and $\mathbf{P}$ one finds:

$$\mathbf{P} = \frac{3\chi_{\mathrm{E}}}{3+\chi_{\mathrm{E}}}\varepsilon_0\mathbf{E}_0 \quad ,$$

$$\mathbf{E}' = \frac{3}{3+\chi_{\mathrm{E}}}\mathbf{E}_0 \quad .$$

The discontinuity of the radial component gives the Polarization surface charge density equal, in this problem, to the total surface charge density as there is no free charge.

$$\Delta E_r = \left(\frac{1}{4\pi\varepsilon_0}\frac{2\mathbf{p}}{R^3} + E_0 - E'\right)\cos[\theta] = \Delta E_r = \frac{P}{\varepsilon_0}\cos[\theta] \quad ,$$

$$\implies \rho_{\mathrm{S}}^{\mathrm{P}} = P\cos[\theta] \quad .$$

The external field gives, for the refraction of the field lines:

$$\tan[\alpha_{\mathtt{ext}}] \equiv \left|\frac{E_\theta}{E_r}\right| \implies \quad ,$$

$$\tan[\alpha_{\mathtt{ext}}] = \left|\frac{\dfrac{1}{4\pi\varepsilon_0}\dfrac{\mathbf{p}}{R^3} - E_0}{\dfrac{1}{4\pi\varepsilon_0}\dfrac{2\mathbf{p}}{R^3} + E_0}\right|\tan[\theta] = \left|\frac{\dfrac{P}{3\varepsilon_0} - E_0}{\dfrac{2P}{3\varepsilon_0} + E_0}\right|\tan[\theta] = \frac{1}{\varepsilon_{\mathrm{R}}}\tan[\theta] > 0 \quad .$$

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The internal field gives, for the refraction of the field lines:

$$\tan\left[\alpha_{\mathtt{int}}\right] \equiv \left|\frac{E_\theta}{E_r}\right| \Longrightarrow \quad ,$$

$$\tan\left[\theta\right] > 0 \quad .$$

The result is consistent with the general law for refraction of the filed lines, as the **Electric-Permittivity** outside is one.

38-005 A Polarizable LHI Sphere in an Applied Uniform and Constant Electric Field (2)

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Problem 4.22||

Read § 38.10.08.

Calculate the internal field by successive approximations as follows. Start assuming the internal electric field is the applied field $\mathbf{E}_0$ and find the Polarization as in a LHI medium. Calculate the resulting internal electric field $\mathbf{E}_1$ and find the Polarization as in a LHI medium. Go on and sum the series of the resulting electric fields.

38-006 A Polarizable LHI Sphere in an Applied Uniform and Constant Electric Field (3)

Carry one the same problem as in section § 38.10.08 with a LHI medium having **Electric-Permittivity** different from the one of the LHI sphere. Show that the fields internal to the sphere are, in this case:

$$\mathbf{E} = \frac{3{\varepsilon_{\mathtt{R}}}^E}{2{\varepsilon_{\mathtt{R}}}^E + {\varepsilon_{\mathtt{R}}}^I}\mathbf{E}_0 \qquad \mathbf{P} = \left(\frac{3{\varepsilon_{\mathtt{R}}}^E\left({\varepsilon_{\mathtt{R}}}^I - 1\right)}{2{\varepsilon_{\mathtt{R}}}^E + {\varepsilon_{\mathtt{R}}}^I}\right)\varepsilon_0\mathbf{E}_0 \quad .$$

Show, in particular, that the electric field internal to the sphere is increased/decreased according to the relative values of the **Electric-Permittivity** : ${\varepsilon_{\mathtt{R}}}^E \gtreqless {\varepsilon_{\mathtt{R}}}^I$.

Study the qualitative behavior of the field lines in the two cases of **Electric-Permittivity** of the external medium larger or smaller than the one of the sphere.

WORKED SOLUTION

If the **Electric-Permittivity** of the external medium is larger than the one of the sphere the field lines tend to be repelled by the sphere, like for a diamagnetic sphere in an external magnetic field.

38-007 Capacitor With a Polarizable Material or a Conductor Inside

Take a plane ideal capacitor of area A and distance between the armatures equal to D. Insert a slab of Polarizable material with relative **Electric-Permittivity** ϵ, thickness $d \leq D$ parallel to the armatures, without touching the armatures . Find the **capacitance** .

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WORKED SOLUTION

Capacitor with the Polarizable material slab inserted:

$$\Delta V = \frac{\sigma}{\varepsilon_0}\left(\frac{d}{\epsilon} + (D-d)\right) \quad ,$$

$$C[\epsilon] = \frac{\varepsilon_0 A}{D + d\left(\frac{1}{\epsilon} - 1\right)} \quad ,$$

$$\lim_{\epsilon \longrightarrow \infty} C[\epsilon] = \frac{\varepsilon_0 A}{D-d} \quad .$$

Capacitor with the conducting slab inserted. It can be calculated as the series of two capacitors; only the sum of the two gaps, $D-d$, matters.

38-008 A Spherical Capacitor

A spherical capacitor is made of two surfaces of radii R_1 and R_2. It is filled with a LHI material with **Electric-Permittivity** $\varepsilon_{\mathrm{R}} = A\left(\frac{R_1}{r}\right)^2$.

1. Find the capacitance C of the capacitor.
2. If the charge on the capacitor is Q find the total energy stored in the capacitor.

38-009 An Ideal Plane Capacitor Filled With a Non Uniform Electric Material

This § is referenced at pages:
[1876]

Determine the **capacitance** of an ideal plane capacitor, with the armatures having area A and distance d, filled with an isotropic and non uniform electric material with relative **Electric-Permittivity** depending on the distance from the armatures only. Let z be the axis perpendicular to the armatures, with origin at the negatively charged armature and directed towards the positively charged one. The relative **Electric-Permittivity** is the function $\varepsilon_{\mathrm{R}}[z]$ for $0 \leq z \leq d$.

Calculate the Polarization charge (both volume and surface charges).

WORKED SOLUTION

$$D = \rho_{\mathrm{S}}^{\mathrm{F}}$$

$$C \equiv \frac{|Q|}{|\Delta V|} = \varepsilon_0 A \left(\int_{z=0}^{z=D} {\varepsilon_{\mathrm{R}}}^{-1}[z]\, \mathrm{d}z\right)^{-1}$$

Order of magnitude: for $A =$, $d =$ one has $C =$.
Alternative method: consider the Polarizable material medium as an infinite series of slices acting as capacitors in series.

38-010 An Ideal Plane Capacitor Filled With Two Uniform Electric Materials

Solve problem § 38-009 with two electric materials with different **Electric-Permittivity** whose interface is parallel to the armatures.

Calculate the Polarization charges.

38-011 A Spherical Insulating LHI Shell Surrounding a Point Charge

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|P-21.12||

A point charge Q is at the center, O, of a insulating spherical shell of radii R_1 and R_2, previously uncharged and made of a LHI material with **electric susceptibility** χ_E.

1. Calculate the electric and Polarization fields in all space.
2. Determine the total charge inside a sphere centered in O, as a function of the radius r.
3. Study the case when the **electric susceptibility** χ_E is very big, in the limit $\chi_E \longrightarrow \infty$.

WORKED SOLUTION

in the polarizable material shell: $\rho^F = \rho^F_S = 0$,

$$D[r] = \frac{1}{4\pi}\frac{Q}{r^2} \quad \text{everywhere} \ ,$$

$$E[r] = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2} & \text{for: } 0 < r < R_1 \text{ or } r > R_2 \ , \\ \dfrac{1}{4\pi\varepsilon_0\varepsilon_R}\dfrac{Q}{r^2} & \text{for } R_1 < r < R_2 \end{cases} \quad \text{as for the single point charge} \ ,$$

$$P[r] = \begin{cases} \dfrac{1}{4\pi}\dfrac{\chi_E}{1+\chi_E}\dfrac{Q}{r^2} & \text{for } R_1 < r < R_2 \ , \\ 0 \quad \text{otherwise} \end{cases} \ ,$$

$$\rho^P = -\mathbf{div}\mathbf{P} = 0 \quad \text{as the polarization is an inverse-square field} \ ,$$

$$Q_1 = -Q\frac{\chi_E}{1+\chi_E} \qquad Q_2 = +Q\frac{\chi_E}{1+\chi_E} = -Q_1 \quad \text{because the material medium is globally neutral} \ ,$$

if $R_2 \to \infty$ (observer immersed in the polarizable material) the effective charge seen is: $Q + Q_1 = \dfrac{Q}{1+\chi_E}$,

if $\chi_E \to \infty$ one finds the results for a conductor: ,

$$\chi_E \to \infty \implies Q_2 = -Q_1 = Q \ .$$

The Polarizable material screening effect is obtained when $R_2 \to \infty$, that is the observer is immersed in the Polarizable material and the effective charge seen is reduced with respect to the free point charge.

38-012 A Model for the Electric Polarization

This § is referenced at pages:
[1971]

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|P-21.10||
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||

Consider the following model for a Polarizable medium. The medium is made of small conducting spheres of radius r much smaller than the average distance between the spheres.

Calculate the **electric susceptibility** of the medium as a function of the radius r and the number density, $\eta_{\mathbb{N}}$, of the spheres. Assume that the medium is sufficiently dilute such that each sphere reacts to the external (applied) field only and it is not sensitive to the field produced by the other spheres.

38-013 Cavities Inside a Polarizable Material

A uniform ElectroStatic field $E_0 = 2{\cdot}10^3\ \mathrm{V/m}$ is present inside a linear, homogeneous isotropic material of **Electric-Permittivity** $\varepsilon_{\mathrm{R}} = 3$. A long and thin cavity inside the material makes an angle $\alpha = 30\ ^\circ$ with the direction of the electric field.

Calculate the value of the electric field inside the cavity.

WORKED SOLUTION

Module: $E = 3.46{\cdot}10^3\ \mathrm{V/m}$. Direction: at an angle $\beta = 60\ ^\circ$ with the direction of $\mathbf{E}_0$.

38-014 A Conducting Sphere Surrounded by a Polarizable Material

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|P-21.9||

A conducting sphere has a radius $R_1 = 1\ \mathrm{cm}$ and a charge $Q = 6{\cdot}10^{-8}\ \mathrm{C}$. It is surrounded by a non homogeneous Polarizable material sphere with internal radius $R_1 = 1$ and external radius $R_2 = 3\ \mathrm{cm}$. The **Electric-Permittivity** of the medium changes with the radial distance as $\varepsilon_{\mathrm{R}} = b/r^2$, with $b = 9{\cdot}10^{-4}\ \mathrm{m}^2$.

1. Calculate, in all the space, the electric field, the Polarization field, the Polarization charge volume density and the Polarization charge surface density.
2. Study the case $\varepsilon_{\mathrm{R}} \longrightarrow \infty$.

38-015 A Polarized Cylinder

©|Lim Yung-Kuo, *Problems And Solutions On Electromagnetism*, 1993, World Scientific, ...Ed., ...|2020, 2021||

A long solid cylinder of radius a is made of a material with a fixed Polarization, so that the Polarization is everywhere directed radially outward with a magnitude proportional to the distance from the axis: $\mathbf{P} = P_0 r \widehat{\mathbf{e}}_r$. The cylinder rotates with constant angular velocity ω around its axis, without any change in $\mathbf{P}$.

1. Determine the charge density inside the cylinder.
2. Determine the magnetic field on the axis, far away from the edges of the cylinder.
3. Determine the electric field at any distance r for the axis, far away from the edges of the cylinder.
4. Determine the magnetic field at any distance r for the axis, far away from the edges of the cylinder.
5. Determine the total ElectroMagnetic energy stored per unit length of the cylinder both before the cylinder starts spinning and while it is spinning. Where does the extra energy come from?

UNDER CONSTRUCTION

38-016 Another Polarized Cylinder

A very long cylinder, of radius R, has $\mathbf{P}$ Polarization uniform and perpendicular to the axis. For example, let the axis of the cylinder be along z, and the Polarization is $\mathbf{P} = \{0, P, 0\}$.

1. Find the electric field inside the cylinder.
2. Show that on the outside we have:

$$\mathbf{E}(\mathbf{r}) = \frac{R^2}{2\epsilon_0 r^2}(2(\mathbf{P}\cdot\hat{r})\hat{\mathbf{r}} - \mathbf{P}) \quad .$$

Here, r and $\hat{\mathbf{r}}$ are meant in cylindrical coordinates.

38-017 A Electric Dipole Inside an LHI Sphere

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|4.34||

A electric point **Dipole** is embedded at the center of a sphere of LHI material. Find the electric field inside and outside the sphere.

DRAFT-IN-PROGRESS

CHAPTER 39

Magnetic Properties of Matter

39.01 Phenomenology of the Magnetic Properties of Materials 1881
39.02 Magnetization of Matter 1882
39.03 Calculation of the Magnetic Field Produced by a Given Magnetization 1889
39.04 Magnetic Field Inside Magnetized Matter 1893
39.05 Magnetic Fields and Maxwell Equations in Matter 1894
39.06 Constitutive Relations for Magnetic Response 1896
39.07 Forces and Torques on Matter in External (Applied) Magnetic Fields 1900
39.08 Additional Topics in Magnetism 1901
39.09 Magnetic Properties in Dense LI Non-Magnetic Materials 1905
39.10 Examples and Physical Applications 1906
39.11 Exercises Problems and Physical Applications 1916

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This § is referenced at pages:
[1691, 1734, 1813, 1832]

Pre-requisites: review the ***Multipole*** *expansion.*

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.34; 2.35; 2.36||

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|6|*Good text.*|

©|F.N.H.Robinson, *Macroscopic ElectroMagnetism*, ..., ..., ...Ed., .||*Excellent!*|

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|1.4||

DRAFT-IN-PROGRESS

39.01 Phenomenology of the Magnetic Properties of Materials

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|§ 11||
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||

Experimentally one can study the effects of matter on magnetic fields (and vice-versa) by inserting a material inside a an empty inductor. When any material is inserted inside an empty inductor the inductance may change either slightly (either increasing or decreasing) or it may largely increase, depending on the material.

Similarly, some kind of materials are weakly attracted by a magnet, some kind of materials are strongly attracted by a magnet and some kind of materials are weakly repelled by a magnet.

Under the influence of an external magnetic field the atoms and|or molecules of the medium react collectively: the medium becomes Magnetized.

In fact, we know that matter is made of electrons, ions, atoms, molecules etc.. These are in turn composed of other particles, possibly composed particles. Instead of describing the material as a volume of structure-less but non-vacuum jelly, it is described as a collection of electrons|ions|atoms|molecules etc. inhabiting a vacuum.

DRAFT-IN-PROGRESS

39.02 Magnetization of Matter

This § is referenced at pages:
[1840]

The effects of external magnetic fields on matter can be described, to a first approximation, by means of a suitable distribution of **Magnetic Dipoles** . This hypothesis is normally assumed when studying the response of matter to magnetic fields. Note, however, that under certain circumstances the inclusion of higher-order multipoles fields might become necessary.

Matter generally reacts to an external magnetic field by Magnetization. On the microscopic point of view this can be understood by either of two mechanisms:

- **deformation** of the microscopic currents distributions: Magnetization by deformation of non-magnetic matter, **dia-magnetism**; it is opposite to the external field.
- **orientation** of pre-existing microscopic **Magnetic Dipoles** : Magnetization by orientation of magnetic matter; in these cases a Magnetization by deformation exists as well, but it is normally negligible with respect to the Magnetization by orientation; it is often called **para-magnetism (para-magnetic matter)**; it is oriented with the external field.

Induced Magnetization can be either parallel (para-magnetism) or anti-parallel (dia-magnetism) to the external field, at variance with **para-electricity**; see section § 38.02.

Note that, while the entity of the deformation and|or orientation of the single atom/molecule may be very small, the large number of Atoms|Molecules in macroscopic matter can make the effect a very large one (read § 37.04.02). In fact the number of atoms per cubic meter for ordinary matter has values in the range $\left(10^{25} \div 10^{29}\right)\ \mathrm{atoms/m^3}$.

Magnetization by deformation (diamagnetism) and orientation (para-magnetism) can be understood, from a qualitative point of view, without invoking quantum-relativistic physics, which is However, necessary for a quantitative understanding. On the other hand ferro-magnetism cannot be explained without quantum physics.

Note that materials exists having a **built-in Magnetization** even in absence of external applied magnetic fields: they are **generically called here ferro-magnetic materials**.

Because, to a first approximation, the effect of external fields on matter may be described either as induction of Dipoles and|or orientation of existing Dipoles, the magnetic field produced by the Magnetized material can be thus replaced by the magnetic field produced by a continuous **Dipole** distribution.

Note that an extremely important role is played in the Magnetization by orientation of magnetic matter by **the fact that physical systems, down in scale to elementary particles, may have an intrinsic Magnetic Dipole moment**.

WORK IN PROGRESS

39.02.01
Atomic/Molecular Magnetisability - Microscopic

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©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|||
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39.02.01.01 Magnetizability of an Isotropic Molecule

©|Raimond|excellent||

The magnetisability of an isotropic atom/molecule (or any other microscopic system) is defined as the average **Magnetic Dipole** moment, $\mathbf{m}$, induced by an external magnetic field $\mathbf{B}_0$:

$$\alpha^{(\mathtt{m})} \equiv \mu_0 \left.\frac{\partial \mathbf{m}}{\partial B_0}\right|_{B_0=0} \quad \text{for the case } \mathbf{m} \parallel \mathbf{B}_0 \text{ (isotropic atom/molecule)} \quad , \qquad \rightarrow \text{1961 1962}$$

$$\Longrightarrow \mathbf{m} \simeq \mathbf{m}_0 + \frac{1}{\mu_0}\alpha^{(\mathtt{m})}\mathbf{B}_0 + \mathcal{O}\left(B_0\right)^2 \quad \text{for a permanent } \textbf{Magnetic Dipole} \text{ moment: } \mathbf{m}_0 \quad .$$

The magnetisability is the second derivative of the interaction energy with respect to external magnetic field calculated at zero field:

$$\alpha^{(\mathtt{m})} \equiv -\mu_0 \left.\frac{\partial^2 \mathcal{U}_m}{\partial {B_0}^2}\right|_{B_0=0} \quad ,$$

with no factor $1/2$, because it is a Magnetizable systems with factor $1/2$ in the interaction energy.

The above discussion applies to a single atom/molecule: a permanent Dipole may be present, even in absence of an external field, and an induced moment is generally always present.

Whenever a large number of Atoms|Molecules is present, one should introduce the macroscopic averaging, in terms of the Magnetization density, section § 39.02.02; in fact, $\mathbf{B}_0$ is actually the **local field** :

$$\mathbf{B}_0 \longrightarrow \mathbf{B}_{\mathtt{LOC}} \quad .$$

Even if each atom/molecule possesses a permanent Dipole, a random arrangement may result in a zero total Dipole in absence of an external field.

The induced Magnetization includes the effect of alignment of the permanent Dipoles, which is generally opposed by temperature. See section § 41.04.06.

The magnetisability can be either positive or negative:

$$\alpha^{(\mathtt{m})} \gtreqless 0 \quad .$$

One should assume that a linear relation applies as long as the external magnetic field is not too big.

Note that for large external fields the relation may become non-linear, so that the magnetisability may loose some of its meaning and become field-dependent.

Note that in general whenever a permanent **Magnetic Dipole** moment, $\mathbf{m}_0$, is present, this is normally much larger than the induced **Magnetic Dipole** moment, at least for external fields not too large.

By definition the physical dimensions of the magnetisability is a volume. Note, however, that in the literature slightly different definitions may be found.

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39.02.01.02 Magnetisability of a Non-Isotropic Axially-Symmetric Molecule

If a molecule is not isotropic, but axially-symmetric, it is rather obvious that the magnetisability might depend of the direction of the applied field. Consider for instance a bi-atomic molecule: one expects that the induced **Dipole** moment is different for an external magnetic field applied along the axis of the molecule with respect to an external magnetic field applied perpendicularly to it.

Therefore one can write for this kind of molecule (having one axis of symmetry):

$$\mathbf{m} \simeq \frac{1}{\mu_0}\left(\alpha^{(\mathtt{m})}{}_{\parallel}[B_0]\mathbf{B}_{\parallel} + \alpha^{(\mathtt{m})}{}_{\perp}[B_0]\mathbf{B}_{\perp}\right) \qquad \textbf{superposition principle} \quad .$$

39.02.01.03 Magnetisability of a General Non-Isotropic Molecule

The general case is discussed in section § 39.06.03 in the completely analogous case of the Magnetization vector.

39.02.01.04 All Materials Undergo Magnetization by Deformation (Dia-Magnetism)

This § is referenced at pages:
[1842]

Consider an atom/molecule and assume that the overall **Magnetic Dipole** moment of the is zero.

If an external magnetic field is introduced the current densities will change in such a way to oppose to the change put in force by the external magnetic field. Therefore the current densities will change all generating a magnetic field opposing the change. Therefore the induced fields of the current densities will sum up each other. This is, qualitatively, diamagnetism, which is present in every atom/molecule.

Assume that two identical current distributions, with equal and opposite currents, are superimposed, like two electrons in an atom doing the same orbit with equal and opposite angular momenta. The magnetic dipole moment is zero.

In presence of an external magnetic field, both current distributions induce a magnetic field opposing the external one. A magnetic dipole moment is induced, opposite to the external field.

Let us then assume a simple semi-classical model such that, in absence of external fields, the opposite and identical currents cancel, while in presence of external fields the currents do not cancel anymore so that there is an induced magnetic dipole moment, **anti-parallel** to the external field.

This will be modeled as two non-equal and opposite total currents, on the same original space distributions.

Read § 38.02.01.04, for similarities and differences.

39.02.02 Definition of the Magnetization

This § is referenced at pages:
[1883]

Remembering that the definition of magnetic dipole is, by definition, **additive**, see (36.14.09), the macroscopic Magnetization field is defined as:

$$\mathbf{M} \equiv \lim_{\Delta V \longrightarrow 0} \frac{\Delta \mathbf{m}}{\Delta V} \equiv \lim_{\Delta V \longrightarrow 0} \frac{\sum_k \mathbf{m}_k(\text{inside } \Delta V)}{\Delta V} \equiv \eta_{\mathbb{N}}\langle \mathbf{m} \rangle \quad . \tag{39.02.01}$$

→ 1884

The limiting process in equation (39.02.01) must be interpreted in the sense of taking **macroscopic** volumes, containing a large enough number of microscopic systems, as discussed in section § 37.03.

The Magnetization M turns out to be **the magnetic dipole moment per unit volume**.

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39.02.02.01 A Magnetized Medium: Some Simple Cases

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Note that it is difficult to imagine the magnetic field produced by a large number (of the order of Avogadro number) of **Magnetic Dipoles** .

However, it will be shown that the magnetic field is the one given by an equivalent current distribution.

A Long Circular Cylinder Uniformly Magnetized Along Its Axis

From the elementary interpretation of Magnetization as aligned **Magnetic Dipoles** one easily finds the following results.

Zero volume current density at internal points.

Zero surface current density at the two bases.

Surface current density on the lateral surface: $\mathbf{M} \wedge \mathbf{n}$.

Consider, in fact, a line of length L on the lateral surface of the cylinder, parallel to its axis and calculate current concatenated per unit length:

$$\Delta I = I \eta_{\mathtt{N}} A L = (I A \eta_{\mathtt{N}})\, L = M L \implies |\mathbf{j}_{\mathtt{S}}| = M \equiv |\mathbf{M}| \quad .$$

Regardless the details of the **Magnetic Dipole** distribution the magnetic field produced is the same as the one produced by a surface current on the lateral surface.

A Large Thin Slab Uniformly Magnetized Perpendicular to Its Faces

A Cylinder Uniformly Magnetized Along Its Axis

A Sphere Uniformly Magnetized

A Spherical Shell With Radial Magnetization

Assume a uniform radial distribution of the all elementary identical magnetic Dipoles in a sphere:

$$\begin{cases} \mathbf{M} = m \eta_{\mathtt{N}} \widehat{\mathbf{e}}_r \equiv M_r \widehat{\mathbf{e}}_r & R_1 < r < R_2 \\ \mathbf{M} = 0 \qquad \text{otherwise} \end{cases} \quad , \qquad .$$

One finds $\mathbf{j}^{\mathtt{M}} = 0$ and $\mathbf{j}^{\mathtt{M}}_{\mathtt{S}} = 0$, from equations (39.02.03), (39.02.04).

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A Circular Cylindrical Solenoid

Consider a long circular cylindrical solenoid kept at constant current. When the solenoid is filled with a material the elementary **Magnetic Dipoles** may either align or anti-align with the magnetic field.

In the first case the Magnetization currents on the lateral face of the cylinder conspire with the free currents of the coil to increase the magnetic field inside the solenoid and thus the concatenated flux, that is the inductance. In the second case the reverse happens.

Magnetic **Dipoles** align with the magnetic field and the magnetic field is increased. This is opposite with respect to the electric case due to the basic difference between **Electric Dipole** | **Magnetic Dipole** (read § 41.01).

39.02.03 Magnetization Current Density

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©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|||

©|S.Bobbio & E.Gatti, *Elettromagnetismo E Ottica*, 1991, Bollati-Boringhieri, ...Ed., ...|8||

G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...- for a simpler, simplified and more elementary derivation.

39.02.03.01 A Non-Magnetized Medium

Consider a non Magnetized insulating medium. The magnetic field effects due the currents at the atomic and|or molecular level as well as the microscopic magnetic fields due to intrinsic **Dipole** moments of the particles cancel each other:

- either because, for non magnetic molecules, (that is molecules without a net **Magnetic Dipole** moment), the magnetic **Dipole** moment of the currents at the atomic and|or molecular level as well as the microscopic **Magnetic Dipole** moments of the particles sum to zero (and any other higher-order **Magnetic Dipole** moment vanishes);
- or because, for magnetic molecules, the random orientation of the **Dipoles** produces a net zero **Dipole** moment (which is contrasted by thermal motion, featuring kinetic energies of the order of $\approx k_B T$).

39.02.03.02 A Magnetized Medium With No Free Currents (A Simple Heuristic Model)

This § is referenced at pages:
[Never referenced.]

A simple heuristic model is developed in this section. A better and more general, but still heuristic model, is developed in section § 39.02.03.03 An exact, but purely formal, derivation is carried on in section § 39.03.

From the simple examples one might guess that, in general, the Magnetization surface current density is:

$$\mathbf{j}_S = \mathbf{M} \wedge \mathbf{n} \quad .$$

Consider the piece of matter divided into small parallelepiped blocks, with sides parallel to the axes, Δx, Δy and Δz long.

.....

See D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...

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39.02.03.03 A Magnetized Medium With No Free Currents (A Simple Heuristic Model) a More General Case

This § is referenced at pages:
[1886]

Consider any fixed open surface Σ, inside a Magnetized medium, and a small (infinitesimal) part of its boundary, the closed line $\partial\Sigma$; the infinitesimal part of the closed line is defined by the vector $\mathbb{d}\mathbf{L}$.

Assume that there are no free currents in the medium and the medium is not Magnetized in absence of the external (applied) field. In case there are free currents the **superposition principle** may be applied.

Assume a very simplified model where Magnetization is built from a number of all identical real **Magnetic Dipoles** of value $\mathbf{m} = I\mathbf{A}$, all with the same currents, I, and where $\mathbf{A}$ is the vector describing the area of the current loop; both of them, a priori, may depend on the position.

The Magnetization may be due to any effect (deformation and|or orientation); for simplicity, the demonstration only considers deformation.

The **Magnetic Dipole** moments, a priori, might depend on the location: $\mathbf{m} = \mathbf{m}[\mathbf{x}]$.

Assume that the distribution of microscopic currents is steady, in such a way that it is divergence-less:

$$\boxed{\mathbf{div}\,\mathbf{j}^{\mathrm{M}} = 0} \quad .$$

This means that the current concatenated to a surface only depends on the boundary of the surface.

The current concatenated with the surface Σ can be calculated from the elementary current loops which are crossed by the closed line $\partial\Sigma$, as any other current loop will give a zero net contribution. Any elementary circular current loop, of radius R, concatenates with the line if and only if its center is at a distance less than $R\cos[\theta]$ from the line.

If θ is the angle between the Magnetization vector, $\mathbf{M}$, and the infinitesimal line element, $\mathbb{d}\mathbf{L}$, it is, (by implicitly considering space-averaged values):

$$\boxed{\mathbf{M} \parallel \mathbf{m} \qquad \delta I_{\mathrm{M}}[\mathbf{x}] = I\eta_{\mathrm{N}}[\mathbf{x}]A\,\mathbb{d}L\cos[\theta] = \mathbf{M}[\mathbf{x}]\cdot\mathbb{d}\mathbf{L}} \quad .$$

→ 1887

Note: the above is meaningful for a closed curve only. In general: the total current crossing the surface as a consequence of Magnetization is thus given by the contribution of all **Dipoles** which are cut by the boundary of the surface. **Rule: look for real magnetic Dipoles concatenated by the closed curve.**

The current concatenated with the surface Σ is thus given by the line-integral over its boundary of the Magnetization:

$$\boxed{I_{\mathrm{M}} = \oint_{\partial\Sigma} \mathbf{M}\cdot\mathbb{d}\mathbf{L}} \quad , \qquad (39.02.02)$$

→ 1887

where the orientation of the surface is connected to the orientation of the line by the right-hand rule in such a way that a positive $\mathbf{M}\cdot\mathbb{d}\mathbf{L}$ gives rise to a positive current.

As equation (39.02.02) is valid for any arbitrary surface inside the medium we always have that the current crossing any open surface is given by the circulation of the Magnetization vector $\mathbf{M}$ along the boundary of the surface. One can replace the line-integral using the rotor theorem:

$$\boxed{I_{\mathrm{M}} = \oint_{\partial\Sigma} \mathbf{M}\cdot\mathbb{d}\mathbf{L} = \iint_{\Sigma} \mathbf{rot}\,\mathbf{M}\cdot\mathbb{d}\mathbf{S} \equiv \iint_{\Sigma} \mathbf{j}^{\mathrm{M}}\cdot\mathbb{d}\mathbf{S}} \quad ,$$

showing that the current can be expressed as the flux across the surface of the rotor of the Magnetization.

As the surface Σ is arbitrary the above result implies that the rotor of the Magnetization is equal to the volume current density:

$$\boxed{\mathbf{j}^{\mathrm{M}} \equiv +\mathbf{rot}\,\mathbf{M}[\mathbf{x}]} \quad .$$

Equation (39.02.03.03) can be used to derive the surface Magnetization current. Apply it to an open curve, Γ, at the surface of the material. Let the normal vector to the surface be $\widehat{\mathbf{n}}$. Define the unit vector $\widehat{\mathbf{N}}$ on the surface and orthogonal to $\mathbb{d}\mathbf{L}$ such that

$$\widehat{\mathbf{N}} \equiv \frac{\mathbb{d}\mathbf{L}\wedge\widehat{\mathbf{n}}}{\mathbb{d}L} \qquad \Leftrightarrow \qquad \mathbb{d}L\,(\widehat{\mathbf{n}}\wedge\widehat{\mathbf{n}}) = \mathbb{d}\mathbf{L} \quad .$$

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One can then write, with Γ being in general any open line on the surface:

$$I_{\mathrm{M}} = \int_C \mathbf{M}\cdot \mathbb{d}\mathbf{L} = \int_C \mathbf{M}\cdot(\,\widehat{\mathbf{n}}\wedge\widehat{\mathbf{n}}\,)\,\mathbb{d}\mathbf{L} = \int_C \widehat{\mathbf{n}}\cdot(\,\mathbf{M}\wedge\widehat{\mathbf{n}}\,)\,\mathbb{d}\mathbf{L} = \int_C \mathbf{j}_{\mathrm{S}}^{\mathrm{M}}\cdot\widehat{\mathbf{n}}\,\mathbb{d}\mathbf{L} \quad .$$

As the curve Γ is arbitrary on the surface the above result implies that the surface current density is thus:

$$\boxed{\mathbf{j}_{\mathrm{S}}^{\mathrm{M}} \equiv -\,\widehat{\mathbf{n}}\wedge\mathbf{M}[\mathbf{x}]} \quad .$$

It can be shown by direct inspection that the signs are correct.

All in all it was found, for a very special model of Magnetization and setup that Magnetization current density is expressed by the relations:

$$\boxed{\mathbf{j}^{\mathrm{M}} \equiv +\mathbf{rot}\mathbf{M}[\mathbf{x}]} \quad , \qquad (39.02.03) \quad \rightarrow 1885\ 1888\ 1889$$

$$\boxed{\mathbf{j}_{\mathrm{S}}^{\mathrm{M}} \equiv -\,\widehat{\mathbf{n}}\wedge\mathbf{M}[\mathbf{x}]} \quad . \qquad (39.02.04) \quad \rightarrow 1885\ 1888\ 1889$$

Equations (39.02.03), (39.02.04), derived in a static situation and in absence of free currents, are assumed to be valid in all cases.

Another derivation, a formal one, under the same hypotesis is summarized in section § 39.03.01.

It can be shown, in a more formal way, that the above results are indeed valid under any circumstance, with a mathematical derivation giving less room to physical intuition.

Note that, in the derviation above, the Magnetization can in fact change along the curve, without spoiling the validity of the derivation. However, when using thre **rotor theorem** to derive equation (39.02.03) one must think of choosing a smaller and smaller (infinitesimal) curve in order to pick-up a point property inside the medium.

39.02.03.04 Equivalence of Magnetized Material to Amperian Current Densities

This § is referenced at pages:
[Never referenced.]

In general, a piece of material with a given Magnetization produces the same magnetic fields as the volume current density (39.02.03) plus the surface current density (39.02.04). In case a Polarization current exists, it must be included as well to calculate the total magnetic field.

See Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ..., for the equivalence of the equivalent field with the average field in matter.

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39.03 Calculation of the Magnetic Field Produced by a Given Magnetization

This § is referenced at pages:
[1847, 1886]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|||

Note that when dealing with systems with pre-assigned Magnetization (ferro-magnetic materials, for instance), as opposed to the case of linear systems, for which the Magnetization is related to the magnetic field, there is no a priori relation between magnetic field and Magnetization. In particular, as opposed to isotropic materials, magnetic field and Magnetization need not to be parallel.

39.03.01 Fields Produced by Magnetized Matter: A Formal Treatment

This § is referenced at pages:
[1888]

The expressions for the volume and surface Magnetization current densities, equations (39.02.03), (39.02.04), can be also deduced, in a purely formal way, from the MagnetoStatic field of a distribution of **Magnetic Dipoles** as follows.

The magnetic field produced by a given (pre-assigned) Magnetization $\mathbf{M}[\mathbf{x}]$ can be calculated from the definition itself of the Magnetization as a volume density of **Magnetic Dipoles** .

The MagnetoStatic potential, using equations (16.13.08), (16.13.01), (16.07.04), is given by:

$$\boxed{\mathbf{A}[\mathbf{x}] = \frac{\mu_0}{4\pi} \iiint_\tau \frac{\mathbf{M}[\mathbf{y}] \wedge (\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} \, \mathrm{d}\mathbf{y}} = \quad , \tag{39.03.01}$$

→ 1850 1889 1892 1908

$$\frac{\mu_0}{4\pi} \iiint_\tau \mathbf{M}[\mathbf{y}] \wedge \mathbf{grad}_\mathbf{y} \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} \right) \mathrm{d}V_\mathbf{y} = \frac{\mu_0}{4\pi} \iiint_\tau \mathbf{rot} \left(\frac{-\mathbf{M}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|} \right) \mathrm{d}V_\mathbf{y} + \frac{\mu_0}{4\pi} \iiint_\tau \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} \right) \mathbf{rot}\mathbf{M}[\mathbf{y}] \, \mathrm{d}V_\mathbf{y} = \quad ,$$

$$\boxed{\frac{\mu_0}{4\pi} \oiint_{\partial\tau} \frac{-\widehat{\mathbf{n}} \wedge \mathbf{M}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|} \, \mathrm{d}S[\mathbf{y}] + \frac{\mu_0}{4\pi} \iiint_\tau \left(\frac{\mathbf{rot}\mathbf{M}[\mathbf{y}]}{|\mathbf{x}-\mathbf{y}|} \right) \mathrm{d}V_\mathbf{y}} \quad , \tag{39.03.02}$$

→ 1892

where the integral is extended to the region of space, τ, where the magnet is present, that is where the Magnetization is non zero.

Equation (39.03.01) shows that the magnetic field produced by any Magnetization can be expressed in terms of an equivalent volume current density, $\mathbf{j}^\mathrm{M}$, and an equivalent surface current density, $\mathbf{j}_\mathrm{S}^\mathrm{M}$, such that:

$$\boxed{\mathbf{j}^\mathrm{M} \equiv +\mathbf{rot}\mathbf{M}[\mathbf{y}] \qquad \mathbf{j}_\mathrm{S}^\mathrm{M} \equiv -\widehat{\mathbf{n}} \wedge \mathbf{M}[\mathbf{y}]} \quad . \tag{39.03.03}$$

→ 1892

Read also § 38.03.

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CHAPTER IV

STATIC MAGNETIC FIELD

§29. Static magnetic field

A STATIC magnetic field in matter satisfies two of Maxwell's equations, obtained by averaging the microscopic equations

$$\operatorname{div} \mathbf{h} = 0, \qquad \mathbf{curl}\, \mathbf{h} = \frac{1}{c}\frac{\partial \mathbf{e}}{\partial t} + \frac{4\pi}{c}\rho\mathbf{v}. \tag{29.1}$$

The mean magnetic field is usually called the *magnetic induction* and denoted by **B**:

$$\overline{\mathbf{h}} = \mathbf{B}. \tag{29.2}$$

Hence the result of averaging the first equation (29.1) is

$$\operatorname{div} \mathbf{B} = 0. \tag{29.3}$$

In the second equation, the time derivative gives zero on averaging, since the mean field is supposed constant, and so we have

$$\mathbf{curl}\, \mathbf{B} = (4\pi/c)\overline{\rho\mathbf{v}}. \tag{29.4}$$

The mean value of the microscopic current density is in general not zero in either conductors or dielectrics. The only difference between these two classes is that in dielectrics we always have

$$\int \overline{\rho\mathbf{v}} \cdot \mathbf{df} = 0, \tag{29.5}$$

where the integral is taken over the area of any cross-section of the body; in conductors, this integral need not be zero. Let us suppose to begin with that there is no net current in the body if it is a conductor, i.e. that (29.5) holds.

The vanishing of the integral in (29.5) for every cross-section of the body means that the vector $\overline{\rho\mathbf{v}}$ can be written as the curl of another vector, usually denoted by $c\mathbf{M}$:

$$\overline{\rho\mathbf{v}} = c\,\mathbf{curl}\, \mathbf{M}, \tag{29.6}$$

where **M** is zero outside the body; compare the similar discussion in §6. For, integrating over a surface bounded by a curve which encloses the body and nowhere enters it, we have $\int \overline{\rho\mathbf{v}} \cdot \mathbf{df} = c \int \mathbf{curl}\, \mathbf{M} \cdot \mathbf{df} = c \oint \mathbf{M} \cdot \mathbf{dl} = 0$. The vector **M** is called the *magnetization* of the body. Substituting it in (29.4), we find

$$\mathbf{curl}\, \mathbf{H} = 0, \tag{29.7}$$

where the vector **H** and the magnetic induction **B** are related by

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}, \tag{29.8}$$

which is analogous to the relation between the electric field **E** and induction **D**. Although **H** is, by analogy with **E**, usually called the *magnetic field*, it must be remembered that the true mean field is really **B** and not **H**.

To see the physical significance of the quantity **M**, let us consider the total magnetic moment due to all the charged particles moving in the body. By the definition of the magnetic moment (see *Fields*, §44), this is†

$$\int \mathbf{r} \times \overline{\rho \mathbf{v}}\, dV/2c = \tfrac{1}{2}\int \mathbf{r} \times \mathbf{curl}\, \mathbf{M}\, dV.$$

Since $\rho \mathbf{v} \equiv 0$ outside the body, the integral can be taken over any volume which includes the body. We transform the integral as follows:

$$\int \mathbf{r} \times \mathbf{curl}\, \mathbf{M}\, dV = -\oint \mathbf{r} \times (\mathbf{M} \times d\mathbf{f}) - \int (\mathbf{M} \times \mathbf{grad}) \times \mathbf{r}\, dV.$$

The integral over the surface outside the body is zero. In the second term we have $(\mathbf{M} \times \mathbf{grad}) \times \mathbf{r} = -\mathbf{M}\,\mathrm{div}\,\mathbf{r} + \mathbf{M} = -2\mathbf{M}$. Thus we obtain

$$\frac{1}{2c}\int \mathbf{r} \times \overline{\rho \mathbf{v}}\, dV = \int \mathbf{M}\, dV. \tag{29.9}$$

We see that the magnetization vector is the magnetic moment per unit volume.‡

The equations (29.3) and (29.7) must be supplemented by a relation between **H** and **B** in order to complete the system of equations. For example, in non-ferromagnetic bodies in fairly weak magnetic fields, **B** and **H** are linearly related. In isotropic bodies, this linear relation becomes a simple proportionality:

$$\mathbf{B} = \mu \mathbf{H}. \tag{29.10}$$

The coefficient μ is called the *magnetic permeability*. We also have $\mathbf{M} = \chi \mathbf{H}$, where the coefficient

$$\chi = (\mu - 1)/4\pi \tag{29.11}$$

is called the *magnetic susceptibility*.

Unlike the permittivity ε, which always exceeds unity, the magnetic permeability may be either greater or less than unity. (It is, however, always positive, as we shall prove in §31. The reason for the differing behaviour of μ and ε is discussed in §32.) The magnetic susceptibility χ may correspondingly be either positive or negative.

Another, quantitative, difference is that the magnetic susceptibility of the great majority of bodies is very small in comparison with the dielectric susceptibility. This difference arises because the magnetization of a (non-ferromagnetic) body is a relativistic effect, of order v^2/c^2, where v is the velocity of the electrons in the atoms.§

In anisotropic bodies (crystals), the simple proportionality (29.10) is replaced by the linear relations

$$B_i = \mu_{ik} H_k. \tag{29.12}$$

---

† For clarity, it should be emphasized that **r** here is a variable coordinate of integration, not the position vector of a microscopic particle; it therefore does not come under the averaging sign.

‡ The quantity **M** is completely determined only when this relation is established. The relation (29.6) inside the body, and $\mathbf{M} = 0$ outside it, do not uniquely define **M**: the gradient of any scalar could be added to **M** inside the body without affecting (29.6) (cf. the similar remark in the first footnote to §6).

§ The ratio v/c appears with **H** in the Hamiltonian of the interaction of the body with the magnetic field, and again in the magnetic moments of the atoms or molecules.

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39.03.01.01 An Important Warning

It should be stressed that, rigorously, the previous derivation only applies to the case of ideal **Dipoles** , so that the point where the field is calculated can be considered far enough that the field of a ideal (point) **Dipole** can be used. The problem of the previous derivation is the fact that the starting point of (39.03.01) is only valid for distant points, not certainly inside the Dipole distribution itself.

The analysis showing that the above results is, nevertheless, exact can be found in the literature [1].

© - QUOTE

All in all it is the result the same as integrating for ideal **Dipoles** . All revolves around the fact that the average field over any sphere due to the current inside is the same as the field at the center of a uniformly Magnetized sphere with the same total **Dipole** moment. That is no matter how the actual microscopic current configuration is, one can always replace it by a smooth distribution of perfect **Dipoles** , if one wants the macroscopic (average) field.

39.03.02
Fields Produced by Magnetized Matter: A Physical Approach

The result of equation (39.03.02) is what one obtains straightforwardly considering that a Magnetized material has a volume and surface Magnetization current density, given by equations (39.03.03), which must be considered as normal currents. The result of equation (39.03.02) is then the expression of the potential from the given current densities.

WORK-IN-PROGRESS

[1] See, for instance, D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ..., § 4.2.3 and 6.2.3

39.04 Magnetic Field Inside Magnetized Matter

This § is referenced at pages:
[1828, 1851]

Read § 37.03, § 38.04.

The magnetic field inside matter is rapidly changing in time as well as on distances of the order of the inter-atomic distances.

The macroscopic magnetic field is defined as the spatial average of the microscopic magnetic field, as long as the average is taken on a volume which is small on a macroscopic scale and large on the microscopic scale:

$$\langle \mathbf{B} \rangle[\mathbf{x}] \equiv \lim_{\tau \longrightarrow 0} \frac{1}{\tau} \iiint_{\Omega} \mathbf{B}[\mathbf{x}+\mathbf{y}] \, \mathrm{d}V_{\mathbf{y}} \qquad \tau := |\mathbf{y}| \leq R, \; R \text{ macroscopically small and microscopically large} \quad , \tag{39.04.01}$$

where the limiting process must be intended in the above-mentioned **macroscopic** sense.

The macroscopic magnetic field can be calculated from the Magnetization.

39.04.01 Average Field Inside Matter

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||

For the equivalence between average field inside matter and the equivalent field generated by the equivalent Charge|Current distribution see figure 37.14, from:

Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...

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39.05 Magnetic Fields and Maxwell Equations in Matter

This § is referenced at pages:
[1693]

39.05.01 Maxwell Equations in Matter: Magnetic Fields

Let us rewrite **Maxwell Equations** in terms of Polarization and Magnetization as well as Electric|Magnetic fields [1].

The total current density is given by the sum of the density of **free** currents, $\mathbf{j}^{\mathrm{F}}$, plus the Polarization currents, $\mathbf{j}^{\mathrm{P}}$, plus the bound, Magnetization currents, $\mathbf{j}^{\mathrm{M}}$:

$$\boxed{\mathbf{j} = \mathbf{j}^{\mathrm{F}} + \mathbf{j}^{\mathrm{P}} + \mathbf{j}^{\mathrm{M}}} \quad .$$

In case of macroscopic moving matter, also conventions currents should be added (for fluids), read § 21.03.03.03

It follows, for the rotor equation:

$$\boxed{\mathbf{rot}\mathbf{B} = \mu_0 \left(\mathbf{j}^{\mathrm{F}} + \mathbf{j}^{\mathrm{P}} + \mathbf{j}^{\mathrm{M}}\right) + \varepsilon_0 \mu_0 \partial_t \mathbf{E}} \tag{39.05.01}$$

$$\boxed{\mathbf{rot}\mathbf{H} \equiv \mathbf{j}^{\mathrm{F}} + \partial_t \mathbf{D}} \tag{39.05.02}$$

→ 1894

$$\boxed{\mathbf{H} \equiv \mathbf{B}/\mu_0 - \mathbf{M}} \quad .$$

→ 1954

It follows, for the divergence equation:

$$\boxed{\mathbf{div}\mathbf{B} = 0} \tag{39.05.03}$$

$$\boxed{\mathbf{div}\mathbf{H} = -\mathbf{div}\mathbf{M}} \quad .$$

→ 1894

Only the non-homogeneous equations change.

Note that, in general, the producers of vector $\mathbf{H}$ are not only the free currents, as it follows from equation (39.05.02), but also the Magnetization, as it follows from equation (39.05.01). In fact according to Helmholtz theorem (section § 16.07.05) a vector field can be uniquely reconstructed from the knowledge of both its divergence and its rotor.

In fact the analogous of **Biot-Savart law** for $\mathbf{H}$ and free currents does not exist!

Note, however, that in special configurations endowed with some symmetry it might be possible to determine the $\mathbf{H}$ vector by using only one of the equations (39.05.02), (39.05.01). In particular the vector $\mathbf{H}$ is not solenoidal, even if $\mathbf{B}$ is. The utility of the $\mathbf{H}$ vector lies in the fact that, usually, the free currents are known while the Polarization and Magnetization currents are not.

[1] Please note that most physicists call **H the magnetic field** and **B** the **magnetic induction field**. This nomenclature is an heritage of the old times when it was thought that **H** was the basic magnetic field. Nowadays we know that **B** is the basic magnetic field while **H** is just an auxiliary field defined as a suitable combination of the field **B** and the Magnetization field, **M**. In order to emphasize this facts we will use the nomenclature **the magnetic field** for **B** and we will avoid a specific name for **H** which will be just called **the auxiliary magnetic field H** . Read also § 38.05.

DRAFT - PROOFS

39.05.02
Discontinuity of the Magnetic Fields

This § is referenced at pages:
[1722]

The discontinuity equations derived in section § 36.09 translate, in presence of matter, into:

$$\boxed{\widehat{\mathbf{n}}\cdot\Delta\mathbf{B}_{\perp}=0} \quad ,$$

$$\boxed{\widehat{\mathbf{n}}\wedge\Delta\mathbf{H}_{\parallel}=\mathbf{j}_{S}^{F}} \quad .$$

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39.06 Constitutive Relations for Magnetic Response

39.06.01 Homogeneity Isotropy and Time-(Non)Invariance in Magnetic Response

Read § 21.02 and § 21.04.

39.06.02 Constitutive Relation - Rigid (Frozen-In) Magnetization

In case the material is such that, for instance,

$$\boxed{\mathbf{M}[\mathbf{B}=0] \equiv \mathbf{M}_{\mathrm{R}} \equiv \quad \text{fixed} \quad \neq 0} \quad ,$$

the material has a Magnetization even in absence of external (applied) field (a rigid|frozen-in Magnetization).

The case of so-called rigid|frozen-in Magnetization concerns materials having a fixed Magnetization, in particular a Magnetization independent of any external magnetic field or, better, a non zero Magnetization when the external (applied) field is zero.

Note, in particular, that the concept of susceptibility has no meaning when dealing with a rigid|frozen-in Magnetization.

rigid|frozen-in Magnetization implies anisotropy. However, the lack of isotropy, that is failure to obey rotational symmetry in space, is in this case just hidden (spontaneously broken symmetry), by the need to choose one specific direction in space, among infinite equivalent directions. Normally every substance, above its Curie temperature, loses its rigid|frozen-in Magnetization to become manifestly isotropic.

39.06.03 Equation of State (Constitutive Relation) of Linear Magnetic Response

This § is referenced at pages:
[1884]

The introduction of the vector field $\mathbf{H}$ (or $\mathbf{M}$) amounts to introducing a new variable. It is thus necessary to introduce one more equation to solve the problem. The equation to add is not a law of Nature but it is a constitutive equation characterizing the properties of the piece of matter involved, that is an equation of state:

$$\boxed{\mathbf{M} = \mathbf{M}[\mathbf{E},\mathbf{B}] \quad \text{or} \quad \mathbf{H} = \mathbf{H}[\mathbf{E},\mathbf{B}]} \quad . \tag{39.06.01}$$

→ 1899

A linear material is defined by a linear equation in terms of the symmetrical **magnetic susceptibility** tensor, $\{\boldsymbol{\chi}_{\mathrm{E}}\}^{\cdot\cdot}_{kj}[\mathbf{x}]$:

$$\boxed{M_k[\mathbf{x}] = \frac{1}{\mu_0}\{\boldsymbol{\chi}_{\mathrm{M}}\}^{\cdot\cdot}_{kj} B_j[\mathbf{x}] \Longrightarrow \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} = \frac{1}{\mu_0}\begin{pmatrix} \{\boldsymbol{\chi}_{\mathrm{M}}\}^{\cdot\cdot}_{xx} & \{\boldsymbol{\chi}_{\mathrm{M}}\}^{\cdot\cdot}_{xy} & \{\boldsymbol{\chi}_{\mathrm{M}}\}^{\cdot\cdot}_{xz} \\ \{\boldsymbol{\chi}_{\mathrm{M}}\}^{\cdot\cdot}_{yx} & \{\boldsymbol{\chi}_{\mathrm{M}}\}^{\cdot\cdot}_{yy} & \{\boldsymbol{\chi}_{\mathrm{M}}\}^{\cdot\cdot}_{yz} \\ \{\boldsymbol{\chi}_{\mathrm{M}}\}^{\cdot\cdot}_{zx} & \{\boldsymbol{\chi}_{\mathrm{M}}\}^{\cdot\cdot}_{zy} & \{\boldsymbol{\chi}_{\mathrm{M}}\}^{\cdot\cdot}_{zz} \end{pmatrix} \cdot \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}} \quad ,$$

$$\boxed{\text{with } \{\boldsymbol{\chi}_{\mathrm{M}}\} = \{\boldsymbol{\chi}_{\mathrm{M}}\}[\mathbf{x}]} \quad ,$$

$$\boxed{\text{assuming } \textbf{local and instantaneous response}} \quad . \tag{39.06.02}$$

A linear and homogeneous material is such that the susceptibility does not depend on the position:

$$\boxed{M_k[\mathbf{x}] = \frac{1}{\mu_0}\{\boldsymbol{\chi}_{\mathrm{M}}\}^{\cdot\cdot}_{kj} B_j[\mathbf{x}]} \quad .$$

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An linear and isotropic material is such that the susceptibility does not depend on the direction, that is the **magnetic susceptibility** tensor is multiple of the identity:

$$M_k[\mathbf{x}] = \frac{1}{\mu_0}\chi_{\mathtt{M}}[\mathbf{x}]\delta_{kj}^{:}B_j[\mathbf{x}] = \frac{1}{\mu_0}\chi_{\mathtt{M}}[\mathbf{x}]B_k[\mathbf{x}] \quad .$$

In all the above cases one introduces the **Magnetic-Permeability** , linking $\mathbf{H}$ and $\mathbf{B}$:

$$\mathbf{B} \equiv \mu_{\mathtt{R}}\mu_0\mathbf{H} \qquad \mu_{\mathtt{R}} \equiv 1 + \chi_{\mathtt{M}}^{*} \equiv \frac{1}{1-\chi_{\mathtt{M}}} \quad .$$

In the isotropic case one can show that the **Magnetic-Permeability** satisfies:

$$\mu_{\mathtt{R}} \equiv 1 + \chi_{\mathtt{M}}^{*} \equiv \frac{1}{1-\chi_{\mathtt{M}}} \geq 0 \qquad \text{in static conditions} \quad .$$

Note that linearity is not a property of the medium itself but a property of the medium and the applied external (applied) fields: for small enough fields the response is linear, that is the linear approximation of the development in series of Taylor is applied, while for large enough fields the response will certainly cease to be linear.

39.06.03.01 Refraction of Magnetic Field Lines in LHI Magnetic Media (No Free Currents)

This § is referenced at pages:
[1854]

Read also § 38.06.03.01.

Consider two magnetic LHI media, such that:

$$\mathbf{B} = \mu_{\mathtt{R}}\mu_0\mathbf{H} \quad .$$

Consider a smooth surface, separating the two different media with different physical properties, a point P at the surface where the normal is $\widehat{\mathbf{n}}$, oriented from medium (1) towards medium (2), and define $\Delta\mathbf{A} \equiv \mathbf{A}_2 - \mathbf{A}_1$.

Assume there is no free surface current density at the surface in such a way that both the normal component of $\mathbf{B}$ and the tangential component of $\mathbf{H}$ are continuous.

Note, that in this case, the two tangential components and the normal vector to the surface are co-planar vectors, thanks to the absence of free surface currents density at the surface which implies the continuity of the tangential component of the $\mathbf{H}$ field.

The **incident|refraction angle**, α_m, (read § 15.05.07), is defined as the arc-tangent of the ratio of the modules of the tangential, V_t, and normal, V_n, components of the vector field:

$$\tan[\alpha_m] = \frac{|V_t|}{|V_n|} \qquad 0 \leq \alpha_m \leq \pi/2 \quad .$$

As the angle is always $0 \leq \alpha_m \leq \pi/2$ and the therefore the modules of the components must be used.

The following relations apply:

$$B_{n2} = B_{n1} \qquad \frac{B_{t2}}{\mu_{\mathtt{R}2}} = \frac{B_{t1}}{\mu_{\mathtt{R}1}} \quad ,$$

$$H_{t2} = H_{t1} \qquad \mu_{\mathtt{R}2}H_{n2} = \mu_{\mathtt{R}1}H_{n1} \quad .$$

If one defines the refraction angle, α, of the field lines, as the angle between the vector and the normal to the surface, one can write the above relations as:

$$\frac{\tan[\alpha_1]}{\mu_{\mathtt{R}1}} = \frac{\tan[\alpha_2]}{\mu_{\mathtt{R}2}} \Longrightarrow \frac{\tan[\alpha]}{\mu_{\mathtt{R}}} = \text{constant} \quad ,$$

valid for both $\mathbf{B}$ and $\mathbf{H}$ field lines.

Remember that Field lines are, by definition, continuous: in fact, they are solutions of a differential equation.

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A High Magnetic-Permeability Medium

Suppose that $\mu_{R2} \gg \mu_{R1}$: it follows that $\alpha_2 \gg \alpha_1$ so that the field lines are, in medium 1, almost perpendicular to the surface.

In fact in case medium (2) has a very high **Magnetic-Permeability** , $\mu_{R2} \gg \mu_{R1}$, the field lines inside the high-**Magnetic-Permeability** medium tend to run almost parallel to the surface, while outside the high-**Magnetic-Permeability** medium the field lines tend to exit the medium almost perpendicularly to the surface.

As an example: medium 2 is a LHI medium with very high **Magnetic-Permeability** .

39.06.03.02 Old Historical Definition of the Magnetic Susceptibility

For an isotropic material the following relations apply.

The correct definition of magnetic susceptibility should be:

$$\boxed{\chi_M \equiv \mu_0 \frac{M}{B}} \quad . \tag{39.06.03}$$

However, historically, the following one has been adopted:

$$\boxed{\chi_M^* \equiv \frac{M}{H}} \quad . \tag{39.06.04}$$

The relation between the two susceptibilities is:

$$\boxed{\chi_M = \frac{\chi_M^*}{1+\chi_M^*} \qquad \chi_M^* = \frac{\chi_M}{1-\chi_M}} \quad . \tag{39.06.05}$$

→ 1955

The relation with the **Magnetic-Permeability** is:

$$\boxed{\mu_R = \frac{1}{1-\chi_M} = 1+\chi_M^*} \quad . \tag{39.06.06}$$

It is to be noted that, on practical grounds, the difference between χ_M and χ_M^* is negligible. In fact for all linear media (diamagnetic and paramagnetic media) the **magnetic susceptibility** is so small that the two values of χ_M and χ_M^* are very close. In fact:

$$\boxed{\frac{\chi_M}{\chi_M^*} = \frac{1}{1+\chi_M^*} = 1-\chi_M} \quad .$$

Note, in particular, that $\chi_M^* \geq -1$ and $\chi_M \leq +1$ always have the same sign (in their range of definition).

On the other hand for non-linear materials the definition of **magnetic susceptibility** itself must be generalized (read § 39.06.04).

39.06.04 Non-Linear Materials

This § is referenced at pages:
[1898]

In case the constitutive relation between the Magnetization and magnetic field is not linear the concept of **magnetic susceptibility** is generalized as a differential concept, because one is interested in the description of the Magnetization for small changes of the magnetic field near a prescribed value of the magnetic field.

Actually, in general, the relation is a non linear-one, except for very small external fields, and therefore one considers the expansion in Taylor series truncated to first-order.

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Considering for simplicity isotropic materials, such that $\mathbf{M} \parallel \mathbf{B}$, the definition of susceptibility is thus generalized to:

$$\chi_{\mathrm{M}} \equiv \mu_0 \frac{\partial M}{\partial B} \qquad \text{or better, in case other parameters } X \text{ may affect the magnetisazion:} \qquad \chi_{\mathrm{M}} \equiv \mu_0 \left.\frac{\partial M}{\partial B}\right|_X \quad , \tag{39.06.07}$$

as a function of the components of the Magnetization and magnetic field along their common direction and keeping constant any other possibly relevant parameter, X (such as temperature).

Note that the linear response in normally limited to small fields, while for large enough fields the response shows non-linear terms; see, for a relevant and important example, § 41.04.06. In other words linearity is not a property of the material alone but a property of the material in a certain field.

The general response, equation (39.06.01), can be developed in Taylor series such that linear materials are described by the linear term of the development.

The linearity/non-linearity is in fact a property of the material plus the intensity of the field, not a property of the material alone as the linearity of the response depends on the intensity of the field.

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39.07 Forces and Torques on Matter in External (Applied) Magnetic Fields

Read § 41.02, § 41.05.

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39.08 Additional Topics in Magnetism

G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...

R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...

Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...

39.08.01 Einsten-deHaas Effect

©|M.Bertin & J.P.Faroux & J.Renault, *Milieux diélectriques et milieux aimantés*, 1984, Dunod, ...Ed., WEB - URL|Pag. ...

Consider a cylindrical bar with circular section made of an LHI material with high **Magnetic-Permeability** (soft iron, for instance) suspended along its length, at the center of its basis, by means of wire with very small torsional constant.

Let us introduce a mechanical model which is capable to explicitly describe the contribution of the molecular angular momenta. The model is a rigid body with particles with spin uniformly distributed inside it.

In case the bar is rotating around its axis the angular momentum can be calculated, at the molecular level in a classical way, by means of the Koënig theorem, as the orbital angular momentum of the whole molecule plus the orbital angular momentum of the electrons orbiting around the nuclei, which we can consider the spin angular momentum of the molecule. The mass of the nuclei is assumed to be much larger than the mass of the electrons. The contributions of all the molecules must be summed.

If a magnetic field is switched on, parallel to the axis of the un-Magnetized cylinder, when the cylinder is standing still, the cylinder assumes a Magnetization parallel to the external (applied) field and therefore no torque is expected. Nevertheless the cylinder starts to rotate against the restoring torque by the wire.

As the initial angular momentum was zero, in fact, the alignment of the microscopic currents build an angular momentum which must be counter-balanced by the rotation of the cylinder. At the initial time, assuming an instantaneous switching on of the external magnetic field, when there is no effect from the torsion wire, the angular velocity is given by the total angular momentum balance:

$$0 = I\omega + MV/\gamma \qquad \mathbf{m} \equiv \gamma \mathbf{j} = g\frac{e}{2m}\mathbf{j} \quad ,$$

in terms of the moment of inertia, I, of the bar with respect to it axis, its Magnetization $\mathbf{M}$ and its volume.

In practice, in the real experiment, the external magnetic field is quickly switched up and down so that the cylinder starts an oscillatory motion and one looks for the resonance.

The measurements give a **gyromagnetic ratio** whose order of magnitude is close to γ_e but in general it is often close to twice its value. Therefore, the experiment suggests that Magnetization must have additional contributions in addition to the orbital currents.

The inverse effect exists: the Barnett effect is the Magnetization of an uncharged body when spun around an axis.

DRAFT - NOT FOR PUBLICATION

39.08.02
Earth Magnetic Field

This § is referenced at pages:
[1791]

D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...- 37.5: Il magnetismo dei pianeti

The details of the formation of the magnetic fields of the Earth, planets and stars are not well understood. Anyway, at large enough distance, we know that any magnetic field can be approximated as a diple field, regardsless of the (uknown) details of the generating current distributions.

Therefore the Earth magnetic field is approximated by a **Dipole** field. The charts of the magnetic field at the Earth surface are shown in figure 39.11 (intensity), figure 39.12 (inclination) and figure 39.13 (declination), all from : WEB - URL .

Some planets have magnetic fields with a magnetic **Quadrupole** component of the same order of magnitude as the **Dipole** moment.

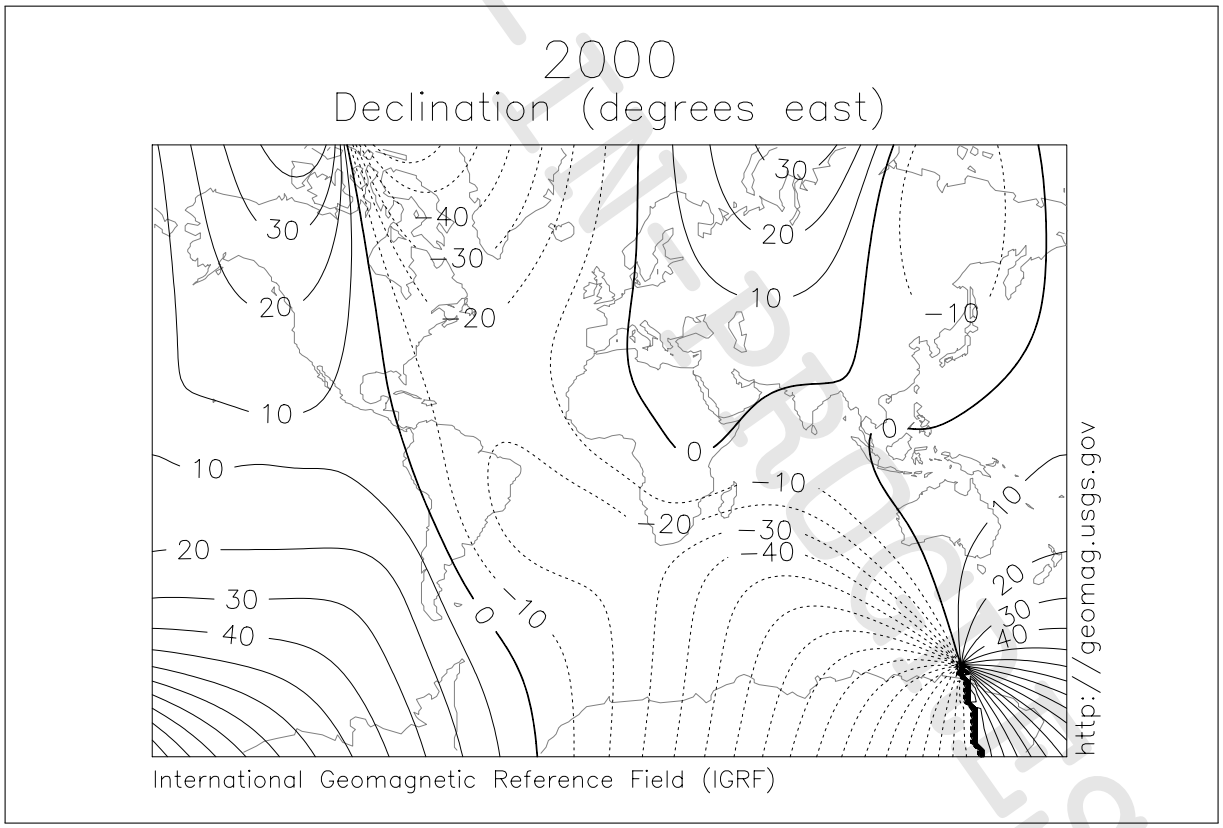

Figure 39.11: Earth Surface Magnetic Field Map (1)

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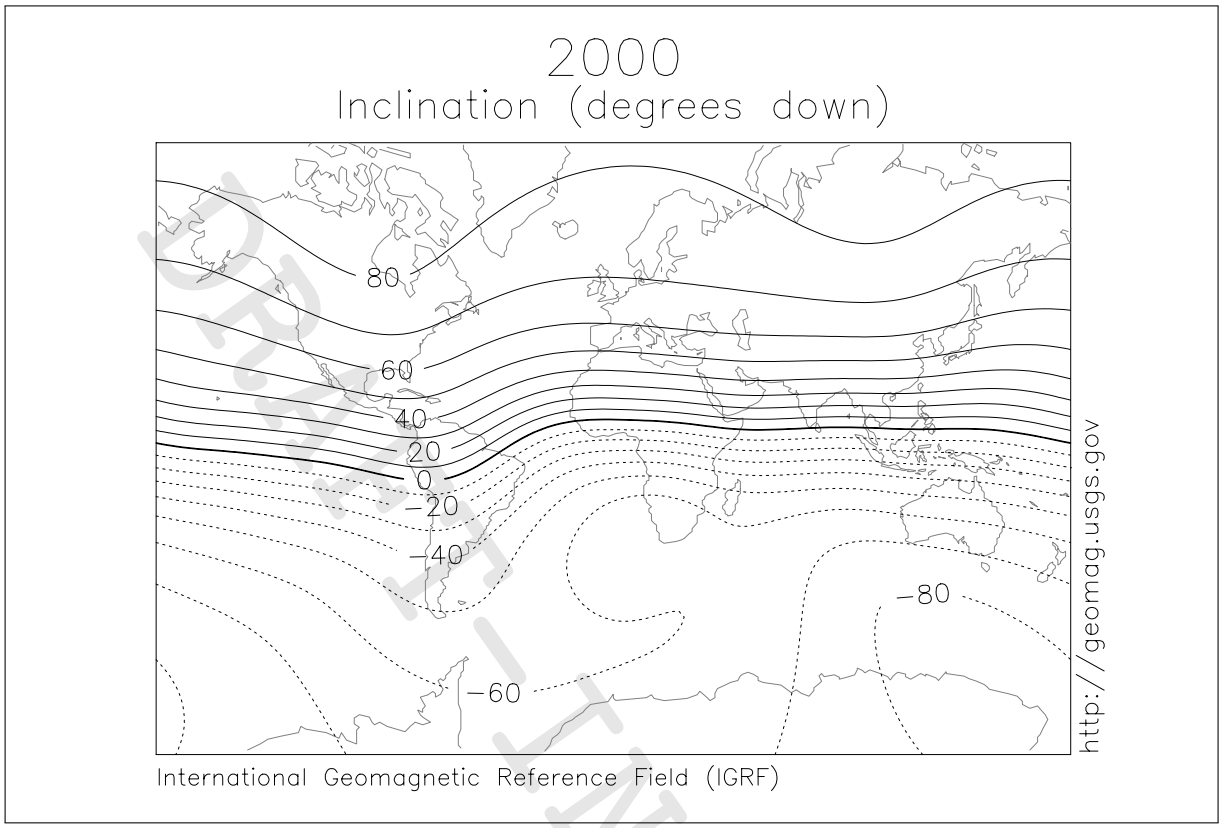

Figure 39.12: Earth Surface Magnetic Field Map (2)

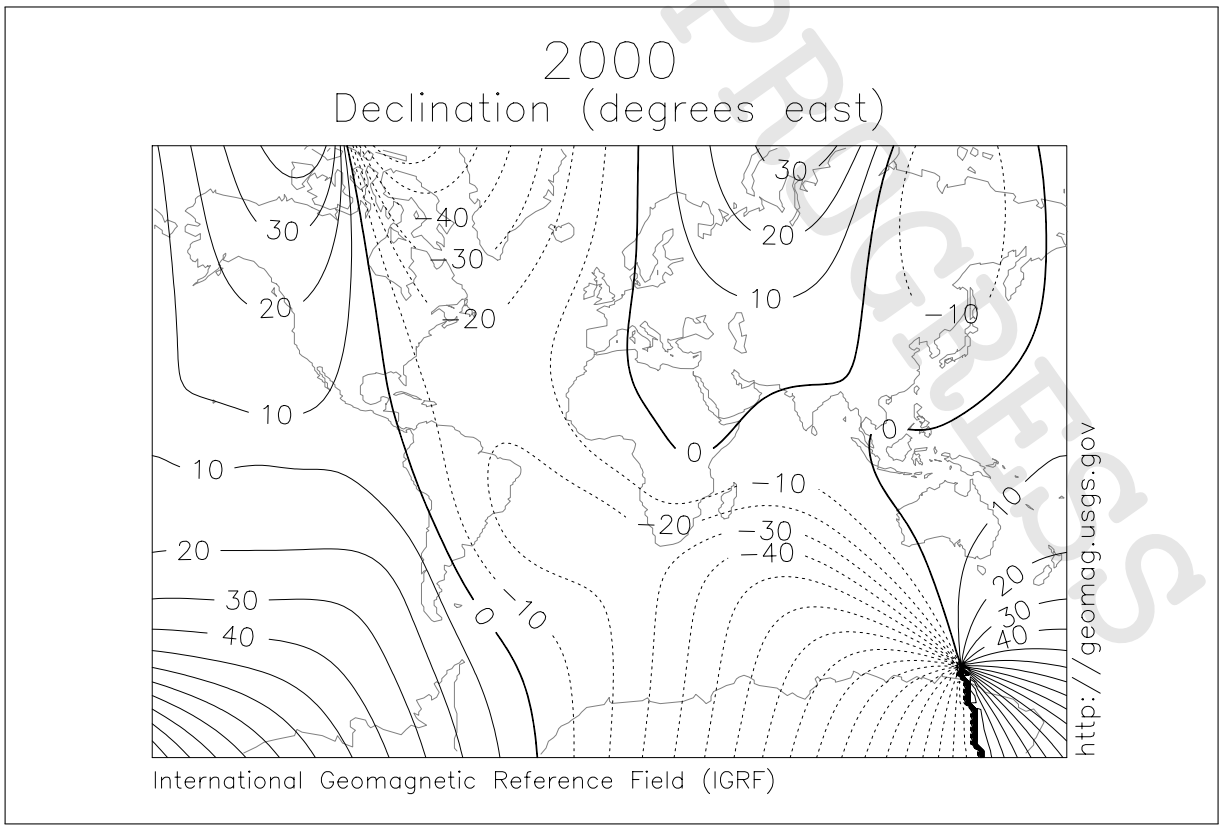

Figure 39.13: Earth Surface Magnetic Field Map (3)

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39.08.03
Magnetic Shielding

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|5.12|*Calculation for a spherical shell*|

There are many situations where a certain region of space must be shielded by external magnetic fields. This can be achieved by using a material with high **magnetic susceptibility** to surround the region. In fact a material with high **magnetic susceptibility** has the tendency to **attract the magnetic field lines**.

It can be shown that a spherical shell of material with high magnetic susceptibility when palced inside an external magnetic field has a residual internal magnetic field is inversely proportional to χ^*_{M}.

The most effective magnetic shielding material is called mumetal. It is an alloy of Copper, Chromium, Nickel and Iron. The maximum **magnetic susceptibility** of mumetal is about $\approx 10^5$ and, moreover, it has a low residual and coercitive field.

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39.09 Magnetic Properties in Dense LI Non-Magnetic Materials

This § is referenced at pages:
[1858, 1859, 1859, 1961, 1977]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.36-2.37||

©|C.Kittel, *Introduction To Solid State Physics*, 1996, J.Wiley & Sons, ...Ed., ...|||

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...| esercizio 4.38 | *SOLUZIONE rigorosa* |

We want to find the relation between the microscopic magnetisability and the macroscopic susceptibility.

In fact there is in general a difference between the average field in matter and the local field acting on a certain molecule, at least **for dense media**. The average field in fact also includes the effect of the field produced by the molecule itself and its neighboring molecules.

The desired relation is **Clausius-Mossotti equation** , giving the relation between the susceptibility and **Polarisability** |Magnetizability based on the relation between the local field (acting on the single molecule) and the average macroscopic fields. It applies to **substances which are linear and isotropic on the macroscopic scale**. Therefore, even if it is not totally general, it is very good to understand the relation between **Polarisability** |Magnetizability, the local field and the (macroscopic) fields.

Proceed as in section § 38.09. Consider a linear isotropic non magnetic-dipolar dense medium. The local field can be calculated via a spherical cavity, as it looks reasonable for a isotropic material. The medium outside the sphere is treated as a continuous medium; the medium inside the sphere is treated as a set of Magnetized molecules. Read § 39.10.04.02 and § 39.10.05 for the calculation of relevant quantities. Imagine a iper-simplified model such that the space allotted to each molecule is a sphere of radius R.

It is a good choice for gases and liquids. Actually for gases there is no significant difference between the local fields and the average field. The case of linear magnetic response is always such that the susceptibility is very small with respect to one. Therefore the linear Magnetization effects are so small that there would be no real need, in practice, to carry on the developments of section § 38.09.

One finds, as in section § 38.09, the magnetic **Clausius-Mossotti equation** :

$$\boxed{\chi_{\mathtt{M}} = \frac{3\eta_m\alpha^{(\mathtt{m})}}{3+2\eta_m\alpha^{(\mathtt{m})}} \neq \chi^*_{\mathtt{M}} \qquad \frac{\eta_m\alpha^{(\mathtt{m})}}{3} = \frac{\chi_{\mathtt{M}}}{3-2\chi_{\mathtt{M}}} = \frac{\mu_{\mathtt{R}}-1}{\mu_{\mathtt{R}}+2}} \quad . \tag{39.09.01}$$

The **Magnetisability** $\alpha^{(\mathtt{m})}$ has dimensions of a volume: $\left[\alpha^{(\mathtt{m})}\right] = L^3$.

Note that the above relations are somewhat different from equation (39.10.02): there the **Magnetisability** is linked to the magnetization of the sphere itself; here, the **Magnetisability** is linked to the magnetization of the embedding material.

The corrective factor at denominator decreases the susceptibility, if $\alpha^{(\mathtt{m})} > 0$ (paramagnetism) and increases the susceptibility, if $\alpha^{(\mathtt{m})} < 0$. When $\eta_m\alpha^{(\mathtt{m})} \ll 1$ (for instance at very low density) one finds $\chi_{\mathtt{M}} \simeq \eta_m\alpha^{(\mathtt{m})}$.

In general, compare with § 38.09: $\eta_m \neq \eta_e$.

If the molecules are magnetic molecules the above derivation is not correct.

The case of non-linear magnetic response requires a much more complex treatment.

DRAFT - IN PROGRESS

39.10 Examples and Physical Applications

39.10.01 Some Values of the Static Magnetic-Permeability

39.10.02 Gyromagnetic Ratio and the Bohr-Magneton

This § is referenced at pages:
[1826]

The **Magnetic Dipole** moment of a point charge orbiting around a point in a central force field, located at the origin of the Coordinate System and parameterized in terms of time, is given, by:

$$\mathbf{m} = \frac{I}{2}\oint \mathbf{r}\wedge \mathbb{d}\mathbf{r} = \frac{I}{2}\int_{t=0}^{T} \mathbf{r}[t]\wedge\mathbf{v}[t]\ \mathbb{d}t = \frac{I}{2}\frac{\boldsymbol{\ell}}{m}T = \frac{q}{2m}\boldsymbol{\ell} \equiv \gamma\boldsymbol{\ell} \quad ,$$

in terms of the current I, and the conserved orbital angular momentum; it is either parallel or anti-parallel to the angular momentum depending on the sign of the charge.

The value γ, characteristic of each particle/system, is sometimes called the **gyromagnetic ratio** . It has the sign of the electric charge:

$$\boxed{\mathbf{m} \equiv \gamma\boldsymbol{\ell}} \quad . \tag{39.10.01}$$

→ 1826 1906

Note that there is sometimes confusion in the literature as the name **gyromagnetic ratio** is sometimes used as in the above equation (39.10.01), for γ, while it is sometimes used as in equation (37.02.01), for g.

Due to the additivity both of angular momenta and of **Magnetic Dipole** momenta, the relation can be generalized to a generic set of charges:

$$\mathbf{m} = \sum_k \frac{q_k}{2m_k}\boldsymbol{\ell}_k \equiv \sum_k \gamma_k\boldsymbol{\ell}_k \quad ,$$

which can be further simplified in the case all the particles have the same **gyromagnetic ratio** , γ:

$$\mathbf{m} = \gamma\mathbf{L} \quad ,$$

In the case of the electron the **gyromagnetic ratio** is:

$$\gamma_e \equiv -\frac{e}{2m_e} \equiv -\frac{|q_e|}{2m_e} < 0 \quad .$$

39.10.03 Effect of a External Magnetic Field on Electron Orbits

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|6.1.3||

In a very rough semi-classical model, diamagnetism can be described as follows in a simplified qualitative way. In fact, the physics of the process is just **ElectroMagnetic induction**.

After introduction of the external magnetic field, **B**, (possibly perpendicular to the orbit), the centripetal force is given by both the electric force and the magnetic force, so that the speed of the electron changes, either increasing or decreasing, depending on the orientation of magnetic field and orbital plane. If the change of speed is small is small: $\Delta v = eRB/(2m_e)$.

The change of speed of the electron is not due to work done by the magnetic force but it is due to the work done by the induced electric field.

The change of speed of the electron causes a change of the **Magnetic Dipole** moment, always opposite to the change of magnetic field, as from ElectroMagnetic induction.

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Normally, the electron orbits are randomly oriented, and the orbital **Dipole** moments cancel out. But in the presence of a magnetic field, each atom picks up a little extra **Dipole** moment, and these increments are all anti-parallel to the external magnetic field. This is **diamagnetism**.

39.10.04
Rigid (Frozen-In) Magnetization

Note that for a uniform Magnetization one simply has:

$$\mathbf{m} \equiv \iiint \mathbf{M}\, \mathrm{d}V_{\mathbf{y}} = \mathbf{M}\Delta V \qquad \text{for uniform } \mathbf{M} \quad .$$

39.10.04.01 A Uniformly Magnetized Cylinder

By taking into account the Magnetization currents it is clear that the magnetic field, **B**, produced by such a cylinder is exactly the same as the one produced by a circular solenoid with the same surface current density. As the Magnetization is known a priori the **H** field can be calculated from its definition.

The magnetic field obviously resembles the field of a **Dipole** , at large distances, as always. Near the cylinder one can think of the field as produced by an array of real **Magnetic Dipoles** (plane circular current loops) aligned one after the other.

39.10.04.02 A Uniformly Magnetized Sphere

This § is referenced at pages:
[1864, 1865, 1905]

Compare with § 38.10.07.02

The field inside a Magnetized sphere is uniform, while the external field is exactly a **Dipole** field.

The **H** field has Sources|Sinks at the surface of the sphere, therefore the density of field lines changes crossing the surface.

The **B** field has no Sources|Sinks at the surface of the sphere and its flux is conserved, so the density of flux lines is.

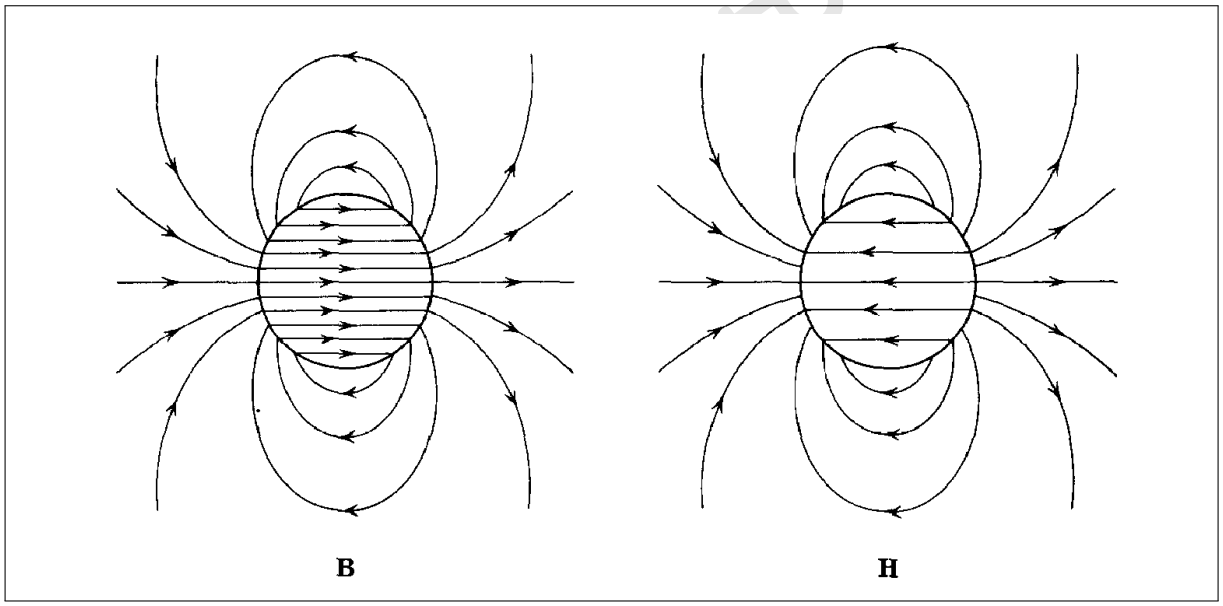

Figure 39.14: The**B** field has nowhere Sources|Sinks, but it may have vortexes. The **H** field has Sources|Sinks at the surface of the sphere, therefore the density of field lines changes crossing the surface.

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Direct Calculation

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Example 5.11||

Use the results of § 36-037 and integrate the Magnetization surface current density $\mathbf{M}\wedge\mathbf{n}$.

A Purely Formal Calculation

Start from equation (39.03.01), with $\mathbf{M}$ uniform,

$$\mathbf{A}[\mathbf{x}]=\frac{\mu_0}{4\pi}\iiint_\tau\frac{\mathbf{M}[\mathbf{y}]\wedge(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,\mathrm{d}\mathbf{y}=\frac{\mu_0}{4\pi}\mathbf{M}\wedge\iiint_\tau\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,\mathrm{d}V_\mathbf{y}=\frac{\mu_0}{4\pi}\mathbf{M}\wedge\mathbf{I}[\mathbf{x}]\quad .$$

Read § 41-014 for the calculation of $\mathbf{I}[\mathbf{x}]$.

It follows:

$$\mathbf{A}[\mathbf{x}]=\frac{\mu_0}{4\pi}\mathbf{M}\wedge\mathbf{I}[\mathbf{x}]=\begin{cases}\dfrac{\mu_0}{3}\mathbf{M}\wedge\mathbf{x} & \text{for } |\mathbf{x}|\le R\\ \dfrac{\mu_0}{4\pi}\dfrac{\mathbf{M}\wedge\mathbf{x}}{x^3}\tau & \text{for } |\mathbf{x}|>R\end{cases}\quad,\qquad \text{with } \tau=\frac{4}{3}\pi R^3\quad .$$

The **vector magnetic potential** $\mathbf{A}$ is the same as a **Magnetic Dipole** , outside the sphere. It is the **vector magnetic potential** of a uniform field, inside the sphere. The field lines of $\mathbf{A}$ inside the sphere are lines perpendicular to the direction of $\mathbf{M}$, running in circumferences centered on the axis of the sphere (the diameter of the sphere parallel to $\mathbf{M}$). Remember that the **vector magnetic potential** , in static conditions, is defined to within the gradient of an arbitrary function.

Analogy With the Electric Case

Use the definition of magnetic charges, generated by the field $\mathbf{H}$ and the formal analogy between $\mathbf{E}$ and $\mathbf{H}$. It follows that the equations for $\mathbf{H}$ are formally the same equations as the ones for the electric field produced by a uniformly Polarized sphere, to within, possibly, the dimensional constants ε_R and μ_R.

The external magnetic field is written from the analogy with the electric field of an electric Dipole taking again into account the correspondence described in section § 41.03.07 and the fact that $\mathbf{M}_\mathrm{E}=0$:

$$\mathbf{H}_\mathrm{E}=\frac{1}{4\pi}\left(3\frac{(\mathbf{m}\cdot\mathbf{r})}{r^5}\mathbf{r}-\frac{\mathbf{m}}{r^3}\right)\qquad\Longrightarrow\qquad\mathbf{B}_\mathrm{E}=\frac{\mu_0}{4\pi}\left(3\frac{(\mathbf{m}\cdot\mathbf{r})}{r^5}\mathbf{r}-\frac{\mathbf{m}}{r^3}\right)\quad .$$

The expression of the magnetic field, deduced by analogy, is the same as the expression of the magnetic field produced by a **Magnetic Dipole** moment: $\mathbf{m}=\Delta\tau\mathbf{M}$.

Use the correspondence described in section § 41.03.07 to find the internal magnetic fields:

$$\mathbf{H}_\mathrm{I}=-\frac{\mathbf{M}}{3}\qquad\mathbf{B}_\mathrm{I}\equiv\mu_0\left(\mathbf{H}_\mathrm{I}+\mathbf{M}\right)=+\frac{2\mu_0}{3}\mathbf{M}\quad .$$

Note that the internal magnetic field is such that:

$$\boxed{\mathbf{B}_\mathrm{I}\cdot\mathbf{M}\ge 0}\quad ,$$

and compare with the analogous electric situation in section § 38.10.07.02. The difference of the two cases is a reflection of the fundamentanl difference, at small distances, between the fields produced by a real **Electric Dipole** and a real **Magnetic Dipole** (read § 41.01).

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39.10.05
A Sphere of LHI Material in a Uniform and Constant External Magnetic Field

This § is referenced at pages:
[1866, 1867, 1905, 1948]

©|J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|22.V.4||

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Compare with § 38.10.08.

It can be shown that, for any LHI sphere or ellipsoid, the induced Magnetization is uniform in an external uniform field.

For a sphere of LHI material with **magnetic susceptibility** χ^*_{M} in an external magnetic field $\mathbf{B}_0$ the following relations apply.

Be careful to distinguish between the external applied field $\mathbf{B}_0$ and the average field inside the sphere $\mathbf{B}$.

The internal field can be calculated as follows.

$$\mathbf{M} \equiv \chi^*_{\mathrm{M}}\mathbf{H} \quad ,$$

$$\mathbf{H} = \mathbf{H}_0 - \frac{\mathbf{M}}{3} \quad ,$$

$$\mathbf{M} = \left(\frac{3\chi^*_{\mathrm{M}}}{3+\chi^*_{\mathrm{M}}}\right)\frac{\mathbf{B}_0}{\mu_0} \quad ,$$

$$\mathbf{B} = \frac{3\,(1+\chi^*_{\mathrm{M}})}{3+\chi^*_{\mathrm{M}}}\mathbf{B}_0 \implies |\mathbf{B}| \geq |\mathbf{B}_0| \quad \text{if } \chi^*_{\mathrm{M}} > 0 \quad , \quad |\mathbf{B}| \leq |\mathbf{B}_0| \quad \text{if } \chi^*_{\mathrm{M}} < 0 \quad \mathbf{B}\cdot\mathbf{B}_0 \geq 0 \quad ,$$

$$\mathbf{H} = \frac{3}{3+\chi^*_{\mathrm{M}}}\mathbf{H}_0 \implies |\mathbf{H}| \leq |\mathbf{H}_0| \quad \text{if } \chi^*_{\mathrm{M}} > 0 \quad , \quad |\mathbf{H}| \geq |\mathbf{H}_0| \quad \text{if } \chi^*_{\mathrm{M}} < 0 \quad \mathbf{H}\cdot\mathbf{H}_0 \geq 0 \quad .$$

Expressions in terms of the **magnetic susceptibility** follow.

$$\boxed{\mathbf{M} \equiv \chi_{\mathrm{M}}\mathbf{B}/\mu_0} \quad ,$$

$$\boxed{\mathbf{B} = \mathbf{B}_0 + \frac{2}{3}\mu_0\mathbf{M}} \quad ,$$

$$\boxed{\mathbf{M} = \frac{3\chi_{\mathrm{M}}}{3-2\chi_{\mathrm{M}}}\frac{\mathbf{B}_0}{\mu_0}} \quad ,$$

$$\boxed{\mathbf{B} = \frac{3}{3-2\chi_{\mathrm{M}}}\mathbf{B}_0 \leq \mathbf{B}_0} \quad ,$$

$$\boxed{\mathbf{H} = \frac{3-3\chi_{\mathrm{M}}}{3-2\chi_{\mathrm{M}}}\frac{\mathbf{B}_0}{\mu_0}} \quad .$$

The meaning of **deMagnetizing field H** is clear when noting that the internal $\mathbf{H}$ field is always smaller that the external (applied) field for $\chi^*_{\mathrm{M}} > 0$ (but not for $\chi^*_{\mathrm{M}} < 0$).

Note that in any case $\mathbf{H}$ is parallel to $\mathbf{H}_0$ and $\mathbf{B}$ is parallel to $\mathbf{B}_0$ (the coefficients are always positive).

The case of a strictly non Magnetizable object (such as vacuum) is found for $\chi^*_{\mathrm{M}} = \chi_{\mathrm{M}} = 0$.

A paramagnetic medium **attracts the field lines**: in fact $|\mathbf{B}| \geq |\mathbf{B}_0|$, if $\chi^*_{\mathrm{M}} > 0$.

A diamagnetic medium **expels the field lines**: in fact $|\mathbf{B}| \leq |\mathbf{B}_0|$, if $\chi^*_{\mathrm{M}} < 0$.

The case of a perfect paramagnet is found for $\chi^*_{\mathrm{M}} = +\infty$ ($\chi_{\mathrm{M}} = 1$): $\mathbf{H} = 0$, $\mathbf{B} = 3\mathbf{B}_0$.

The case of a perfect diamagnet is found for $\chi^*_{\mathrm{M}} = -1$ ($\chi_{\mathrm{M}} = -\infty$): $\mathbf{B} = 0$, $\mathbf{H} = -\mathbf{M} = {}^3\!/_2\mathbf{H}_0$.

The Magnetization is parallel to the external (applied) field for $\chi^*_{\mathrm{M}} > 0$.

The Magnetization is anti-parallel to the external (applied) field for $\chi^*_{\mathrm{M}} < 0$.

The **Magnetisability** of the sphere is:

$$\alpha^{(\mathtt{m})} = \frac{3\chi^*_{\mathtt{M}}}{3+\chi^*_{\mathtt{M}}}\Delta V = \frac{3\,(\mu_{\mathtt{R}}-1)}{2+\mu_{\mathtt{R}}}\Delta V \quad , \tag{39.10.02}$$

→ 1905

and it is not strictly proportional to the **magnetic susceptibility** $\chi^*_{\mathtt{M}}$.

Note that for most typical materials, whose **magnetic susceptibility** is very close to zero, the difference between the $\mathbf{B}$ and $\mathbf{H}$ fields is very small.

The formal analogy, $\mathbf{H} \Leftrightarrow \mathbf{E}$, with the case of the sphere in an external electric field (section § 38.10.08) is obvious, the only basic difference being that the electric susceptibility is non negative while the **magnetic susceptibility** ($\chi^*_{\mathtt{M}}$) is constrained to be greater or equal -1.

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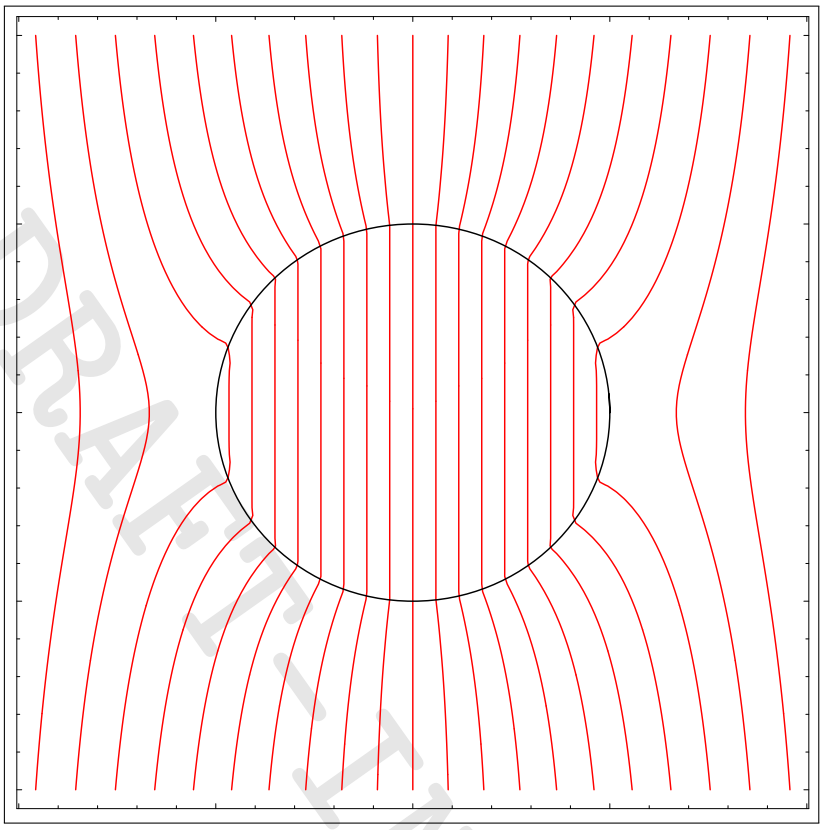

Figure 39.15: A LHI sphere with $\mu_{\mathrm{R}} = 100$ in an external uniform and constant field ($\mathbf{B}$ field lines)

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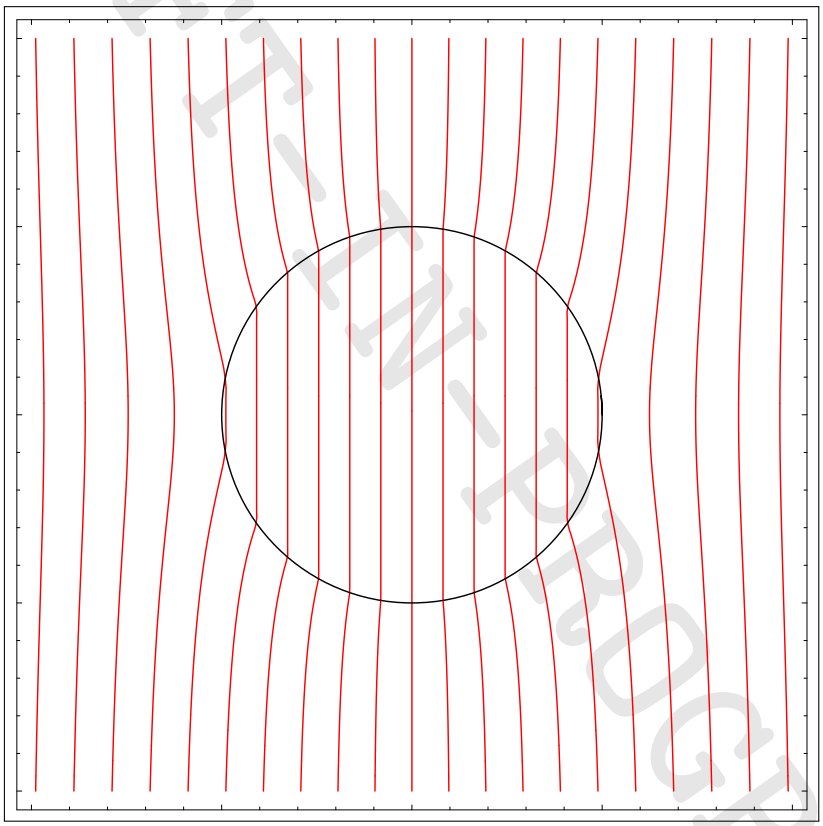

Figure 39.16: A LHI sphere with $\mu_{\mathrm{R}} = 2$ in an external uniform and constant field ($\mathbf{B}$ field lines)

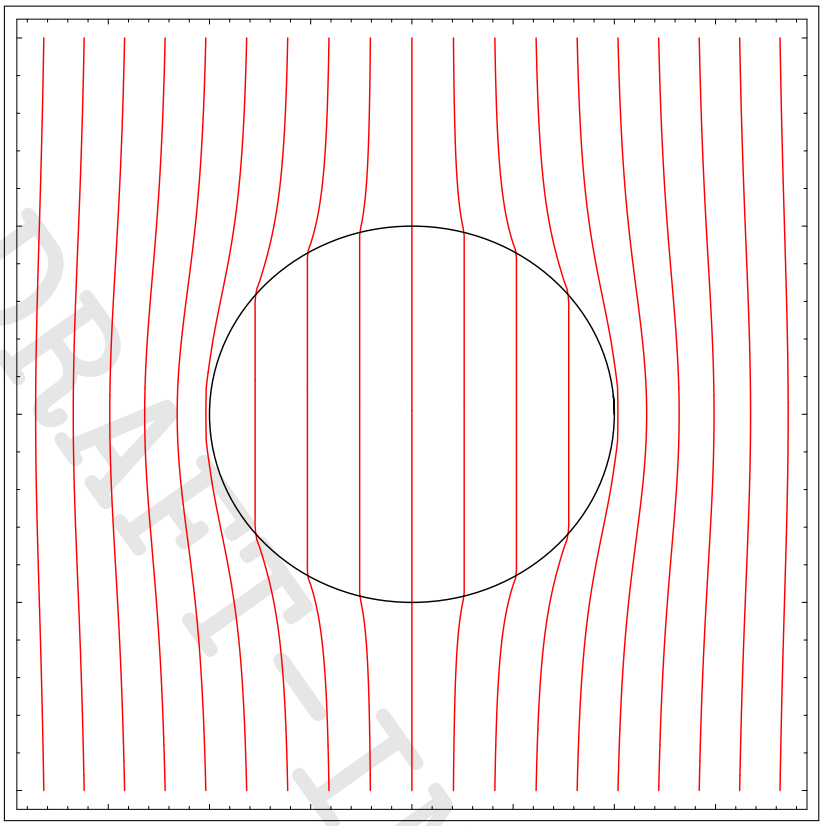

Figure 39.17: A LHI sphere with $\mu_{\mathrm{R}} = 0.5$ in an external uniform and constant field (**B** field lines)

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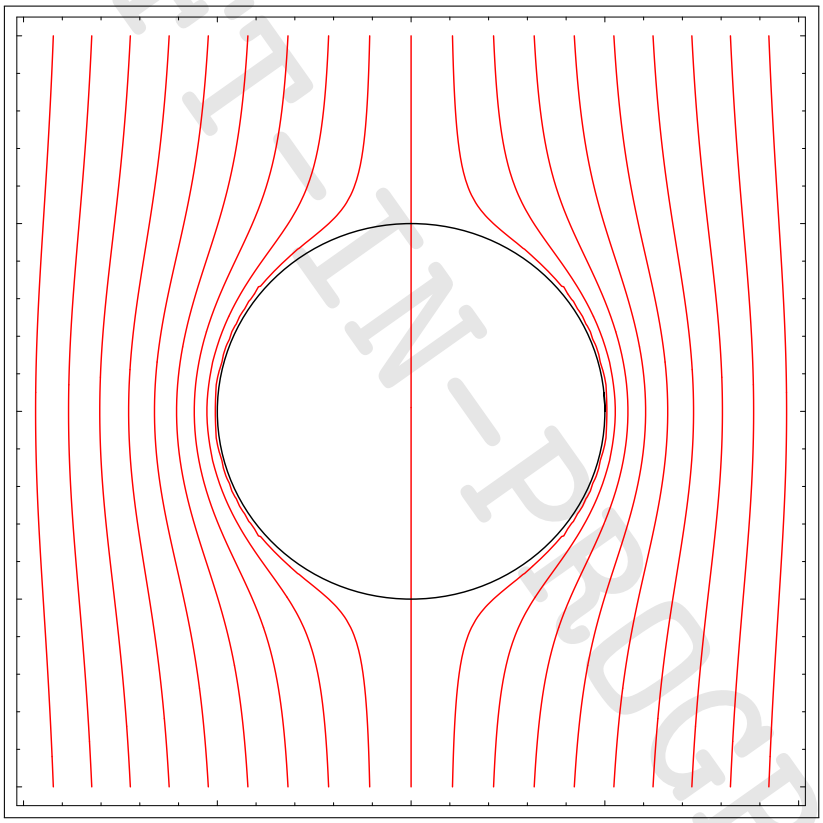

Figure 39.18: A LHI sphere with $\mu_{\mathrm{R}} = 0.001$ in an external uniform and constant field ($\mathbf{B}$ field lines)

39.10.06
Cavities in a Magnetized Material

39.10.06.01 Cylindrical Cavity
Read § 41.03.08.

39.10.06.02 Spherical Cavity
Read § 41.03.08.

39.10.07
Non-Circular Toroidal Coil

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Example 5.10||

A toroidal coil is made of a circular ring (donut) around which a long wire is tightly wrapped. The winding is uniform and tight enough so that each turn can be considered a closed loop and the current as a surface current. The cross-sectional shape of the coil is not important, the results do not change as long as the shape remains the same all around the ring, that is the toroidal coil is a solid of revolution. In that case it can be shown that the magnetic field of the toroidal coil is circumferential at all points, both inside and outside the toroidal coil.

WORKED SOLUTION

The magnetic field is:

$$\mathbf{B} = \frac{\mu_0 N I}{2\pi r}\widehat{\mathbf{e}}_\theta$$ inside the toroidal coil; the magnetic field is zero outside the toroidal coil .

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39.11 Exercises Problems and Physical Applications

39-001 D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...

All problems from D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ..., § 6.

39-002 A Cubic Magnetizable Material With rigid|frozen-in Magnetization

A cube of Magnetizable material of side d, with one vertex at the origin and sides parallel to the Cartesian planes, carries a rigid|frozen-in Magnetization $\mathbf{M} = k\mathbf{r}$ where k is a constant.

1. Determine the dimensions of k.
2. Find all the Magnetization currents.
3. Write the expression of the magnetic field at large distances.

39-003 A Magnetic Material on the Axis of a Solenoid: Force|Torque

Consider a small piece of LHI Magnetizable material on the axis of a circular solenoid. Where is the force on the piece maximum? What is the torque?

39-004 Magnetic Dipole Moment of Iron Atoms

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|37.12||

The **Magnetic Dipole** moment of an iron atom in a piece of iron is $\mathbf{m} = 2.22\mu_B$. Suppose that all the **Dipoles** are aligned in a bar $L = 4.86$ m long and with a sectional area $S = 1.31$ cm^2.

1. Calculate the **Magnetic Dipole** moment of the bar.
2. What is the torque required to keep the bar perpendicular to a uniform external magnetic field $B = 1.53$ T ?

39-005 A Solenoid With a Chrome Core

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|37.14 CHECK||

A solenoid has a number of windings per unit lengthh $n = 16$ w/cm and a current $I = 1.3$ A.

1. A chrome bar is inserted, filling completely the interior of the solenoid: what is the magnetic field increase?
2. What is the Magnetization of the bar?

39-006 Saturation Magnetization of Nickel

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|37.22||

The saturation Magnetization of nickel is $M = 511$ kA/m. Calculate the **Magnetic Dipole** moment of a nickel atom, knowing that Nickel has $Z = 28$, $\mathcal{M}_0 = 58.7$ g/mol and $\eta_{\mathtt{M}} = 8.9$ g/cm^3.

WORKED SOLUTION

$$m = \frac{M\mathcal{M}_0}{\rho\mathcal{N}_{\mathtt{A}}} = 0.6\mu_B \quad .$$

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39-007 magnetic susceptibility|permeability for Non-Linear Media

Show that for non linear (but isotropic for simplicity) media, using the differential definitions of magnetic susceptibility and **Magnetic-Permeability** , it is still valid that:

$$\mu_{\mathrm{R}} = \frac{\partial}{\partial H}\left(\frac{B}{\mu_0}\right) = 1 + \frac{\partial M}{\partial H} = 1 + \chi^*_{\mathrm{M}} \quad .$$

What are the limits at saturation?

WORKED SOLUTION

$$\chi^*_{\mathrm{M}} \longrightarrow 0 \qquad \mu_{\mathrm{R}} \longrightarrow 1 \quad ,$$
$$\mathbf{B} \simeq \mu_0 \mathbf{H} \qquad \text{when } |H| \gg |M| \text{ and|or } |B|/\mu_0 \gg |M| \quad .$$

39-008 Magnetization Curve of Iron

The measured first Magnetization curve of some piece of ferro-magnetic iron ($\eta_{\mathrm{M}} = 7.87\ \mathrm{kg/dm^3}$ and $\mathcal{M}_0 = 55.8\ \mathrm{g/mol}$) is described by the function:

$$M[H] = \alpha \arctan[\beta H] \quad ,$$

with $\alpha = 1.11{\cdot}10^6\ \mathrm{A/m}$ and $\beta^{-1} = 10\ \mathrm{A/m}$.

Determine the saturation Magnetization.

Determine the **magnetic susceptibility** at $H = H_0 = 0\ \mathrm{A/m}$ and $H = H_1 = 1500\ \mathrm{A/m}$.

Calculate the number of Bohr-**magnetons** per iron atom contributing to the saturation Magnetization.

WORKED SOLUTION

$$M_{\mathtt{sat}} \lim_{H \longrightarrow +\infty} M[H] = \frac{\pi\alpha}{2} \Longrightarrow M_{\mathtt{sat}} = 1.74{\cdot}10^6\ \mathrm{A/m} \quad .$$

$$\chi^*_{\mathrm{M}} = \frac{\alpha\beta}{1+\beta^2 H^2} \qquad \chi^*_{\mathrm{M}}[H = H_0] = 1.11{\cdot}10^5 \qquad \chi^*_{\mathrm{M}}[H = H_1] = 4.93$$

Average number of contributing electrons per iron atom (g-factor):

$$g = \frac{M_{\mathtt{sat}} \mathcal{M}_0}{\mu_{\mathrm{B}} N_A \eta_{\mathrm{M}}} = 2.22$$

39-009 Magnetic Field Inside a Solenoid

A long and thin solenoid is filled with a ferro-magnetic material. Inside the material there are two small cavities, coaxial with the solenoid. Inside the first cavity, shaped as a large and thin disk, one measures a B field $B_1 = 7.54{\cdot}10^{-2}\ \mathrm{T}$. In the second cavity, shaped as a long and thin cylinder, one measures a B field $B_2 = 1.26{\cdot}10^{-3}\ \mathrm{T}$.

Calculate the susceptibility and Magnetization of the ferromagnetic material.

WORKED SOLUTION

$\chi^*_{\mathrm{M}} = 59$; $M = 5.9{\cdot}10^4\ \mathrm{A/m}$.

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39-010 Force on a Magnetic Sample on the Axis of a Ideal Circular Finite Solenoid

This § is referenced at pages:
[1786]

Use the results of § 36.20.65 to determine the force acting on a small sample (volume τ) of a homogeneous, isotropic and linear magnetic material with susceptibility $\chi^*_{\mathtt{M}}$ located on the axis (z axis) of a circular finite-length solenoid (of radius r), as a function of the position.

Does the reasoning and results change if the material is ferromagnetic ?

Consider the sample located at the center of one of the basis of the solenoid and find the expression if the force in this case if $|\mathbf{j}_{\mathtt{S}}| = 10^3$ A/m, $r = 5$ cm and $\tau = 1.0$ cm^3.

WORKED SOLUTION

$$f[z] = \frac{\chi^*_{\mathtt{M}}}{\mu_0\,(1+\chi^*_{\mathtt{M}})}\tau B_z[z]\frac{\mathrm{d}B_z}{\mathrm{d}z}[z] \simeq \frac{\chi^*_{\mathtt{M}}}{\mu_0}\tau B_z[z]\frac{\mathrm{d}B_z}{\mathrm{d}z}[z] \quad .$$

39-011 An Ideal Circular Solenoid Filled With a Non Uniform Magnetic Material

This § is referenced at pages:
[1918]

Determine the inductance of an ideal circular solenoid, with number of windings per unit length n, radius R and length l, filled with an isotropic and non uniform magnetic material with relative **Magnetic-Permeability** depending on the distance from the axis of the solenoid only. Let r be the radial distance form the axis of the solenoid. The relative **Magnetic-Permeability** is the function $\mu_{\mathtt{R}}[r]$ for $0 \le r \le R$.

Calculate the Magnetization current (both volume and surface currents).

WORKED SOLUTION

$$H = |\mathbf{j}^{\mathtt{F}}_{\mathtt{S}}|$$

$$L \equiv \frac{|\Phi|}{|I|} = 2\pi\mu_0 n^2 l \int_{r=0}^{r=R} \mu_{\mathtt{R}}[r] r\, \mathrm{d}r$$

Order of magnitude: for $n = 10^3$ m^{-1}, $R = 3$ cm and $l = 10$ cm one has $L = 0.36$ mH.

Alternative method of solution.

39-012 An Ideal Circular Solenoid Filled With Two Uniform Magnetic Materials

Solve problem § 39-011 with two magnetic materials with different **Magnetic-Permeability** whose interface is at constant distance from the axis.

Calculate the Magnetization currents.

39-013 A Current Wire With Non Zero Magnetic-Permeability

©|Lim Yung-Kuo, *Problems And Solutions On Electromagnetism*, 1993, World Scientific, ...Ed., ...|2001||

A straight circular cylindrical wire has radius R and carries a current I. If the material of the wire has **Magnetic-Permeability** μ_{R} find the expression of the magnetic field and the **H** field inside and outside the wire. Assume that the **Magnetic-Permeability** outside the wire is 1.

39-014 Lamina in External Magnetic Field

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|P22-7||

A thin lamina with palne parallel faces is made of LHI magnetic material with **magnetic susceptibility** χ_{M} is immersed in a uniform externla magnetic field, $\mathbf{B} = B_x\widehat{\mathbf{e}}_1 + B_y\widehat{\mathbf{e}}_2$. The lamina lies in the yz plane. Calculate **B**, **M** and **H**, assumed to be uniform, internally to the lamina and the Magnetization currents.

DRAFT-IN-PROGRESS

CHAPTER 40

ElectroMagnetic Field - Energy Linear Momentum and Angular Momentum

40.01 ElectroMagnetic Field Linear Momentum Angular Momentum and Energy 1922
40.02 Examples and Physical Applications 1932
40.03 Exercises Problems and Physical Applications 1942

DRAFT-IN-PROGRESS

This § is referenced at pages:
[2462, 2463]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.27||
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|P6.10, P6.11||
©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|§ 6, § 10||
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|II 13.6-13.7, 26, 27, 29||
©|B.W.Carroll & D.A.Ostlie, *An Introduction To Modern Astrophysics*, ..., ..., 1stEd., ...|4.3||
©|S.Bobbio, *Electrodynamics of Materials*, 2000, Academic Press, 1stEd., ...||*Definitive*|
©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...||*Excellent and modern!*|
©|O.Emile, J.Emile|**WEB - URL** |*Excellent Review*|

WORK-IN-PROGRESS

40.01 ElectroMagnetic Field Linear Momentum Angular Momentum and Energy

This § is referenced at pages:
[1969, 2493, 2578]

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|10.5||
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 2, § 27.2||
©|K.T.McDonald|WEB - URL |*Extensive Recent Review - for instance see about Feynman, Stratton, Bossavit.*|
©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...||*Excellent and modern!*|

In this section we will always deal with **ElectroMagnetic relations in free space**, which are the basic law of nature and always valid, and never ever the description in presence of matter, nor the **D** and **H** fields, will be used. In the rest of this section we will consider ElectroMagnetic fields interacting with charges and currents. We will use interchangeably the words **particles** and **matter**.

EM fields store and transport energy, linear momentum and angular momentum.

Note that, in order to simplify the formulas, the result:

$$\boxed{\varepsilon_0 \mu_0 c^2 = 1} \quad ,$$

which turns out from the solution of **Maxwell Equations** for ElectroMagnetic waves, will be often used.

40.01.01 Local Balance of Conservation Laws

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.27.1||

Read § 21.03, § 60.03.01.

40.01.02 Energy of the ElectroMagnetic Fields

This § is referenced at pages:
[2493]

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||
©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|||
©|G.S.Smith, *An Introduction To Classical Electromagnetic Radiation*, 1997, CUP, ...Ed., ...|1.4||
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||
©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 8||
©|F.Wegner, *Classical Electrodynamics*, 2003, ..., ...Ed., WEB - URL.|15||

See the important comment in section § 36.05.03 about the modeling of matter with point charges, **Dipoles** and higher-order **Multipoles** .

WEB - PROOF

40.01.02.01 Introduction

- **Capacitors and inductors demonstrate that ElectroMagnetic fields can store energy**.
- **Heating by ElectroMagnetic radiation** (including light) shows that ElectroMagnetic radiation can transport energy.
- The ElectroMagnetic braking effect shows how kinetic energy can be converted into internal energy and transferred as heat via ElectroMagnetic fields (the magnetic field in this case).

40.01.02.02 Expression of the Energy of the ElectroMagnetic Fields

We know, from the energy equations of mechanics, that **energy changes are related to the total power applied by both internal and external interactions**.

One needs to start from the expression of the work per unit time (that is power) done by ElectroMagnetic fields on matter, in order to derive the expression of the energy balance, that is the first principle of thermodynamics:

$$\Pi \equiv \frac{\mathrm{d}\mathcal{W}_{\mathrm{EM}\longrightarrow\mathrm{MAT}}}{\mathrm{d}t} \quad .$$

The work per unit time (that is power) per unit volume done by the ElectroMagnetic fields on charges and currents (that is matter) is:

$$\frac{\mathrm{d}\Pi}{\mathrm{d}\Omega} = \rho\,\mathbf{v}\cdot(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}) = \mathbf{j}\cdot\mathbf{E} \quad .$$

By using the 4th **Maxwell Equation** to replace $\mathbf{j}$ and the identity:

$$\mathbf{div}(\mathbf{E}\wedge\mathbf{B}) = \mathbf{B}\cdot\mathbf{rot}\mathbf{E} - \mathbf{E}\cdot\mathbf{rot}\mathbf{B} \quad ,$$

one finds Poynting theorem which expresses the conservation of energy in local form at any point:

$$\frac{\mathrm{d}\Pi}{\mathrm{d}\Omega} = \mathbf{j}\cdot\mathbf{E} \equiv k^0 = -\partial_t u_{\mathrm{EM}} - \mathbf{div}\mathbf{S}_{\mathrm{EM}} \quad . \tag{40.01.01}$$

→ 1924

ElectroMagnetic field energy density (energy per unit volume) is:

$$u[\mathbf{x},t] = \frac{\varepsilon_0\mathbf{E}^2[\mathbf{x},t]}{2} + \frac{\mathbf{B}^2[\mathbf{x},t]}{2\mu_0} \quad . \tag{40.01.02}$$

→ 1923 2584

The expression for the energy density (40.01.02)**, which in elementary and static cases is shown to be the energy stored in ElectroMagnetic fields, is assumed to be a valid expression for the energy stored in ElectroMagnetic fields in all situations**, following Poynting theorem.

In static conditions only, it can be shown that an alternative form for the ElectroMagnetic field energy density (energy per unit volume) can be written in terms of the potentials as:

$$U_{\mathrm{EM}} = \frac{1}{2}\iiint \rho[\mathbf{r}]\Phi[\mathbf{r}]\,\mathrm{d}V + \frac{1}{2}\iiint \mathbf{j}[\mathbf{r}]\cdot\mathbf{A}[\mathbf{r}]\,\mathrm{d}V \qquad \textbf{static} \text{ conditions - } \textbf{total EM energy} \quad . \tag{40.01.03}$$

→ 1923

In presence of matter the above relations apply only in case of a linear relation between the charge and scalar electric potential (for the first term) and the current and the vector magnetic potential (for the second term). The above relation, equation (40.01.03)**, is the total energy required to build the full configuration; note, in fact, the factor** $1/2$.

DRAFT - IN PROGRESS

40.01.02.03 Poynting Vector and Its Non-Uniqueness

The ElectroMagnetic field energy flux (energy crossing a unit area perpendicular to the direction of energy flow per unit time) is given by Poynting vector:

$$\mathbf{S}_{\text{EM}}[\mathbf{x},t] = \frac{\mathbf{E}[\mathbf{x},t]\wedge\mathbf{B}[\mathbf{x},t]}{\mu_0} \quad . \tag{40.01.04}$$

→ 1924

The Poynting vector describes the energy flux: energy per unit time per unit area perpendicular to the direction of energy flow. This is evident from the form of equation (40.01.01), which is a **balance equation**, reducing to the equation for the conservation of ElectroMagnetic energy in case no work is exchanged with matter by ElectroMagnetic fields. In fact equation (40.01.01) has the form of a typical balance law, with a **non zero Sources|Sinks** term: $-\mathbf{j}\cdot\mathbf{E}$.

The time-averaged value of the Poynting vector thus describes the time-averaged energy flux: average energy per unit time per unit area perpendicular to the direction of energy flow); this is called the **intensity of ElectroMagnetic radiation**.

Note that only the entire surface-integral of the Poynting vector, on a closed surface, has a clear physical meaning. In fact one has **to be careful with the strict identification of the Poynting vector at a point with the local energy flux per unit time per unit area perpendicular to the direction of energy flow at that point**. In fact the Poynting vector is defined to within a divergence-less field, by Poynting theorem. No troubles arise when one uses the theorem (40.01.01) in integral form, applied to an arbitrary volume, without attempting to localize the energy flux in space. On the other hand the differential form is correct at any point. In fact **Poynting theorem defines, in a unambiguous way, only the divergence of $\mathbf{S}_{\text{EM}}$**, that is the infinitesimal flux per unit volume, not $\mathbf{S}_{\text{EM}}$ itself.

One needs to postulate that the Poynting vector, equation (40.01.04) represent the local energy flux.

40.01.02.04 Local Conservation of Energy

It must be emphasized that equation (40.01.01) implies that **energy conservation is a local energy conservation**, that is whenever there is a change of energy inside a certain volume the energy must be either given to (received from) matter or has to flow across the boundaries of the surface.

Equation (40.01.01) can be written [1], in integral form and for the fixed integration domain Ω (Poynting theorem for the energy), as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\Omega} u_{\text{EM}}\,\mathrm{d}V + \iiint_{\Omega}\mathbf{j}\cdot\mathbf{E}\,\mathrm{d}V = -\oiint_{\partial\Omega}\mathbf{S}_{\text{EM}}\cdot\mathbf{e}_k\,\mathrm{d}S_k = -\oiint_{\partial\Omega}\mathbf{S}_{\text{EM}}\cdot\,\mathrm{d}\mathbf{S} \quad , \tag{40.01.05}$$

→ 1924 1925

with

$$U_{\text{EM}}[t] \equiv \iiint_{\Omega} u_{\text{EM}}[\mathbf{x},t]\,\mathrm{d}V \quad .$$

Equation (40.01.05) represents the energy balance for the system made of the ElectroMagnetic fields inside volume Ω. It shows that, in non-stationary situations, ElectroMagnetic energy can be stocked in ElectroMagnetic fields and not only transformed into other forms of energy via its transformation into mechanical energy. In fact the immediate effect of the ElectroMagnetic fields on matter is to transform ElectroMagnetic energy into kinetic energy of the particles. This energy can then be transformed into other forms of energy.

The balance is only the balance for EM fields inside volume Ω: if other sources of energies or work are present as well as other parts, they should be taken into account in the balance of the overall system by writing the energy balance for any component.

[1] One has to be obviously careful not to mix the symbols for the Poynting vector, $\mathbf{S}_{\text{EM}}$, and the one of the integration surface, $\mathbf{S}$ in these relations

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40.01.02.05 Some Particular Cases

- **steady conditions**:

$$\partial_t u_{\text{EM}} = 0 \quad .$$

- **No work** by/to ElectroMagnetic fields:

$$\mathbf{j}\cdot\mathbf{E} = 0 \quad .$$

- **Localized fields**: there exists a large enough sphere of radius R enclosing volume Ω_0 such that outside it any field is zero:

$$\oiint_{\partial\Omega|\Omega_0\subset\Omega} \mathbf{S}_{\text{EM}}\cdot \mathbb{d}\mathbf{S} = 0 \quad .$$

40.01.02.06 Matter and ElectroMagnetic Fields Only

Note that:

$$\boxed{\iiint_{\Omega} \mathbf{j}\cdot\mathbf{E}\ \mathbb{d}V \equiv \frac{\mathbb{d}\mathcal{W}_{\text{EM}\longrightarrow\text{MAT}}}{\mathbb{d}t} \equiv \frac{\mathbb{d}U_{\text{MAT}}}{\mathbb{d}t}} \quad .$$

Therefore the integrated form of equation (40.01.05) can be also written as:

$$\frac{\mathbb{d}}{\mathbb{d}t}(U_{\text{EM}} + U_{\text{MAT}}) = -\oiint_{\partial\Omega} \mathbf{S}_{\text{EM}}\cdot\mathbf{e}_k\ \mathbb{d}S_k = -\oiint_{\partial\Omega} \mathbf{S}_{\text{EM}}\cdot\mathbb{d}\mathbf{S} \quad , \tag{40.01.06}$$

showing that the energy (of the ElectroMagnetic fields plus matter) inside the volume can change because energy flows into or out of the volume across the surface.

If the fields are localized inside a finite spatial region the volume can be taken as large as it is required to contain all the fields. This implies the conservation of the total energy:

$$\boxed{\frac{\mathbb{d}}{\mathbb{d}t}(U_{\text{EM}} + U_{\text{MAT}}) = 0 \qquad \text{(localized fields)}} \quad .$$

Note that in general, for radiation fields, the surface contribution cannot be neglected, even taking the surface at infinity, because for radiation fields the Electric|Magnetic fields decrease only as r^{-1} at infinity.

40.01.02.07 Ohmic Conductors

Note that in the case of a Ohmic conductor the power per unit volume transferred by the fields to matter is the positive quantity (Joule effect):

$$\boxed{\frac{\mathbb{d}\Pi}{\mathbb{d}\Omega} = \mathbf{j}\cdot\mathbf{E} = \frac{\mathbf{j}^2}{\sigma}} \quad .$$

This is the Joule effect. Read § 40.02.01 for the description of the energy transfers.

40.01.02.08 Energy Balance With External Generators (Batteries)

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Consider the case when a battery, providing a **linear EM-motive field $\mathbf{E}'$, of non ElectroMagnetic origin**, different from zero only inside the battery, is feeding energy to matter inside the batteries to recover energy lost by matter in other parts of the circuit. One has:

$$\boxed{\mathbf{j} = \sigma\left(\mathbf{E} + \mathbf{E}'\right) \implies \mathbf{j}\cdot\mathbf{E} = \frac{\mathbf{j}^2}{\sigma} - \mathbf{j}\cdot\mathbf{E}'} \qquad \mathbf{E}' \neq 0 \text{ only inside the battery} \quad .$$

The power per unit volume transferred by the ElectroMagnetic fields to matter is: $\mathbf{j}\cdot\mathbf{E}$.

The power per unit volume transferred by the EM-motive fields to matter is: $\mathbf{j}\cdot\mathbf{E}'$.

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The energy balance, for ElectroMagnetic fields plus conduction charges plus battery, becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\Omega} u_{\text{EM}}\,\mathrm{d}V + \oiint_{\partial\Omega}\mathbf{S}_{\text{EM}}\cdot\,\mathrm{d}\mathbf{S} + \iiint_{\Omega}\frac{\mathbf{j}^2}{\sigma}\,\mathrm{d}V = \iiint_{\Omega}\mathbf{j}\cdot\mathbf{E}'\,\mathrm{d}V \quad . \tag{40.01.07}$$

The above equation means that ElectroMagnetic energy is transferred out of the ElectroMagnetic fields via the Joule effect, taking out energy from the ElectroMagnetic fields, and that the linear EM-motive field feeds energy into the ElectroMagnetic fields.

40.01.03 Linear Momentum of the ElectroMagnetic Fields

This § is referenced at pages:
[2082, 2493]

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©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|||
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||
©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 8||

See the important comment in section § 36.05.03 about the modeling of matter with point charges, **Dipoles** and higher-order multipoles.

40.01.03.01 Introduction

ElectroMagnetic fields can carry linear momentum.

The following example motivates the above statement. Consider two identical point charges constrained to move at constant speed along the x and y axis, both of them moving in the negative direction and approaching the origin, that is still at $x > 0$ and $y > 0$. It can be shown that the electric field is radial and that the magnetic field is purely tangential to circumferences perpendicular and concentric to the velocity. The ElectroMagnetic force between the charges would tend to drive them off the axes but we can assume that the charges are mounted on tracks and forced to maintain the same direction and the same speed by some external agent. The electric force between them is repulsive, and the two forces are equal and opposite. The magnetic field produced by the charge on the x axis points in the negative direction of the z axis, on the y at $y > 0$, so that the force is in the positive direction of the x axis. The magnetic field produced by the charge on the y axis points in the positive direction of the z axis, on the x at $x > 0$, so that the force is in the positive direction of the y axis.

All, in all, the ElectroMagnetic forces on the two charges are equal but not opposite, in violation of Newton third law. As Newton third law is linked to linear momentum conservation this results challenges linear momentum conservation.

The problem is solved once one realizes that the ElectroMagnetic field carry a linear momentum.

Even perfectly static ElectroMagnetic fields can harbor linear momentum and angular momentum, as long as $\mathbf{E}\wedge\mathbf{B}$ is non-zero. It is only when these contributions from ElectroMagnetic fields are included that the classical conservation laws hold.

Linear momentum conservation for isolated systems is maintained if and only if the linear momentum carried by the ElectroMagnetic fields is accounted for, and experiments have shown that ElectroMagnetic waves can carry and transport linear momentum.

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40.01.03.02 Expression of the Linear Momentum of the ElectroMagnetic Fields

This § is referenced at pages:
[1928, 1929]

linear momentum is a vector, its volume density must be a vector. A local conservation law implies that one must take the divergence of the flux. But the flux must be a quantity capable to provide the flux of all the three components of the linear momentum, that is we need one flux for each component of the linear momentum. In other words the divergence of the linear momentum flux must be a vector, it cannot be a scalar. Therefore the flux of linear momentum must be a second-rank tensor.

Consider the linear momentum balance in vacuum in presence of a charge density ρ and current density $\mathbf{j}$. The ElectroMagnetic fields loose linear momentum which is transferred to the charges. We know, from the first **Cardinal Equation** of mechanics, that linear momentum changes are connected to the total external force. The linear momentum per unit volume per unit time given by the ElectroMagnetic fields to matter can be written, after using the **Maxwell Equations** , as:

$$\boxed{\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}\Omega} = (\rho\mathbf{E} + \mathbf{j}\wedge\mathbf{B}) \equiv \mathbf{k} = -\partial_t \mathbf{G}_{\mathrm{EM}} - \frac{\partial \Xi_{\mathrm{EM}}{}_{ik}^{\cdot\cdot}}{\partial x^k}\mathbf{e}_i} \quad , \tag{40.01.08}$$

→ 1927 1928 2083

in terms of the force $\mathbf{F}$ applied by the ElectroMagnetic fields to matter.

ElectroMagnetic field linear momentum density (linear momentum per unit volume) and linear momentum flux (linear momentum crossing a unit area perpendicular to the direction of linear momentum flow per unit time) are:

$$\boxed{\mathbf{G}_{\mathrm{EM}}[\mathbf{x},t] = \varepsilon_0\,\mathbf{E}[\mathbf{x},t]\wedge\mathbf{B}[\mathbf{x},t] = \frac{\mathbf{S}_{\mathrm{EM}}[\mathbf{x},t]}{c^2}} \quad . \tag{40.01.09}$$

$$\boxed{\Xi_{\mathrm{EM}}{}_{ij}^{\cdot\cdot}[\mathbf{x},t] = -\left(\varepsilon_0 E_i E_j + \frac{B_i B_j}{\mu_0} - \frac{1}{2}\delta_{ij}^{\cdot\cdot}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)\right) = -\left(\varepsilon_0 E_i E_j + \frac{B_i B_j}{\mu_0} - \delta_{ij}^{\cdot\cdot} u_{\mathrm{EM}}\right)} \quad .$$

The vector $\mathbf{G}_{\mathrm{EM}}$ is the volume density of the ElectroMagnetic linear momentum and it describes the linear momentum density (linear momentum per unit volume). ElectroMagnetic radiation (see section § 46) is given by the propagation of ElectroMagnetic fields. The EM fields in ElectroMagnetic radiation are time-varying. The time-averaged value of the $\mathbf{G}_{\mathrm{EM}}$ vector thus describes the linear momentum density (linear momentum per unit volume) of EM radiation.

The tensor $\Xi_{\mathrm{EM}}{}_{ij}^{\cdot\cdot}$ is a tensor describing the flux of linear momentum out of the volume defining the system, that is the linear momentum per unit area perpendicular to the direction of the flow per unit time. It is analogous of the Poynting vector for linear momentum and as the Poynting vector it conventionally describe a positive flux when the flux is out-going. The tensor $-\Xi_{\mathrm{EM}}{}_{ij}^{\cdot\cdot}$ (note the minus sign) is the ElectroMagnetic stress tensor (**Maxwell stress tensor**). The linear momentum flux per unit time out any surface Σ can thus be written as:

$$\frac{\mathrm{d}\mathbf{G}_{\mathrm{EM}}}{\mathrm{d}t} = \mathbf{e}_i \oiint_{\Sigma} \Xi_{\mathrm{EM}}{}_{ij}^{\cdot\cdot}\,\mathrm{d}S_j \quad .$$

Note that only the entire surface-integral of the ElectroMagnetic stress tensor has a clear physical meaning. One has to be careful with the identification of the ElectroMagnetic stress tensor with the local linear momentum flux per unit area per unit time. In fact the ElectroMagnetic stress tensor is defined to within a divergence-less field. No troubles arise when one uses the theorem (40.01.08) in integral form, applied to an arbitrary volume without attempting to localise the linear momentum flow in space. On the other hand the differential form is correct at any point, as the derivation provide the divergence of the linear momentum flow.

It must be emphasised that equation (40.01.08) implies that linear momentum conservation is a local linear momentum conservation, that is whenever there is a change of linear momentum inside a certain volume the linear momentum must be either given to (received from) matter or has to flow across the boundaries of the surface.

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Note also the equation (40.01.08) has the form of a typical balance law, with a non zero source term.
Equation (40.01.08) can be written, in integral form (Poynting theorem for the linear momentum), as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\Omega}\mathbf{G}_{\mathrm{EM}}[\mathbf{x}]\;\mathrm{d}V_{\mathbf{x}}+\iiint_{\Omega}\mathbf{k}[\mathbf{x}]\;\mathrm{d}V_{\mathbf{x}}=-\oiint_{\partial\Omega}\mathbf{e}_i\Xi_{\mathrm{EM}ij}^{\cdot\cdot}[\mathbf{x}]\;\mathrm{d}S_j \quad , \tag{40.01.10}$$

→ 1928

with

$$\mathbf{P}_{\mathrm{EM}}\equiv\iiint_{\Omega}\mathbf{G}_{\mathrm{EM}}\;\mathrm{d}V \quad .$$

Equation (40.01.10) represents the linear momentum balance for the system made of the ElectroMagnetic fields inside volume Ω.

The left side gives the rate of change of ElectroMagnetic linear momentum inside the volume Ω.

Note that:

$$\iiint_{\Omega}\left(\rho\mathbf{E}+\mathbf{j}\wedge\mathbf{B}\right)\;\mathrm{d}V\equiv\mathbf{F}_{\mathrm{EM}\longrightarrow\mathrm{MAT}}=\frac{\mathrm{d}\mathbf{P}_{\mathrm{MAT}}}{\mathrm{d}t} \quad .$$

Therefore the integral form can be also written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{P}_{\mathrm{EM}}+\mathbf{P}_{\mathrm{MAT}}\right)=-\oiint_{\partial\Omega}\mathbf{e}_i\Xi_{\mathrm{EM}ij}^{\cdot\cdot}\;\mathrm{d}\mathbf{S}_j \quad ,$$

showing that the linear momentum (of the ElectroMagnetic fields plus matter) inside the volume can change because linear momentum flows into or out of the volume across the surface.

If the fields are localized inside a finite spatial region the volume can be taken as large as it is required to contain all the fields. This implies the conservation of the total linear momentum:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{G}_{\mathrm{EM}}+\mathbf{P}_{\mathrm{MAT}}\right)=0 \qquad \text{(localized fields)} \quad .$$

Note that in general, for radiation fields, the surface contribution cannot be neglected, even taking the surface at infinity, because for radiation fields the Electric|Magnetic fields decrease only as r^{-1} at infinity. Compare the analogous behavior of the Poynting vector.

40.01.03.03 Examples of ElectroMagnetic Linear Momentum in Action

- Tail of dust of comets: it is produced by the Sun radiation pressure.
- Radiation pressure has a relevant role in the internal equilibrium of a star.
- Radiation pressure has a relevant role the dynamics of very small bodies in solar system, with effects such as the Poynting-Robertson effect, the Yarkovsky effect and the YORP effect).
- Radiation pressure is used in laser cooling (WEB - URL).

40.01.03.04 Maxwell Stress Tensor

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A slightly different approach, equivalent to the general one presented above when one limits to static situations, leads directly to **Maxwell stress tensor** as follows.

Maxwell stress tensor describes, in static situations, the forces acting on the matter enclosed inside a certain volume. When the situation is not static one needs to revert to the general expressions in section § 40.01.03.02.

See the very important comment in section § 36.05.03 about the modeling of matter with point charges, **Dipoles** and higher-order **Multipoles** .

The total force acting on the matter enclosed inside a certain volume Ω is given by:

$$\mathbf{F}=\iiint_{\Omega}\left(\rho\mathbf{E}+\mathbf{j}\wedge\mathbf{B}\right)\;\mathrm{d}V \quad .$$

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In static situations, using **Maxwell Equations** for static situations, one can rewrite it as:

$$\partial_t \{\ldots\} = 0 \implies \mathbf{F} = \mathbf{e}_i \iiint\limits_{\Omega} \partial_k \Xi_{\mathrm{EM}}{}_{ik}^{\cdot\cdot}\, \mathrm{d}V \quad ,$$

where **Maxwell stress tensor** is given by:

$$\boxed{\Xi_{\mathrm{EM}}{}_{ij}^{\cdot\cdot} \equiv \varepsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij}^{\cdot\cdot} E^2 \right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}^{\cdot\cdot} B^2 \right)} \quad . \tag{40.01.11}$$

Maxwell stress tensor gives the ElectroMagnetic stresses in a static situation.

The total ElectroMagnetic force acting on the matter enclosed inside a a volume Ω can be also expressed in terms of a surface integral as:

$$\boxed{\mathbf{F} = \mathbf{e}_i \iiint\limits_{\Omega} \partial_k \Xi_{\mathrm{EM}}{}_{ik}^{\cdot\cdot}\, \mathrm{d}V = \mathbf{e}_i \oiint\limits_{\partial\Omega} n_k \Xi_{\mathrm{EM}}{}_{ik}^{\cdot\cdot}\, \mathrm{d}\mathbf{S}} \quad .$$

It is rather amazing that the total ElectroMagnetic force acting on the matter enclosed inside the volume Ω can be expressed in terms of the ElectroMagnetic fields on the boundary of the volume only without any reference to the Charge|Current inside the volume. The reason why this makes sense is that the ElectroMagnetic fields on the boundary of the volume includes also the contribution generated by the Charge|Current inside the volume themselves.

Note that the resulting ElectroMagnetic pressure is not isotropic.

Note also that the result is unambiguous when the total ElectroMagnetic force acting on the matter enclosed inside the volume Ω is considered. If one tries to give a meaning to the **Maxwell stress tensor** limiting to a part of the surface only the objection might arise that **Maxwell stress tensor** can be changed by adding any divergence-less function to it. As for the case of Poynting vector only the integral on a closed surface is unambiguous.

Note that **Maxwell stress tensor** gives the ElectroMagnetic stresses in a static situation acting on the system, that is the flux of linear momentum entering the volume. Therefore it has the opposite sign with respect to the tensor $\Xi_{\mathrm{EM}}{}_{ij}^{\cdot\cdot}$ introduced in section § 40.01.03.02 (Read also § 32.03.02.02).

The stress tensor is conventionally defined to be positive for a **traction** (acting on the system) while it is negative for a **compression** (acting on the system).

Special case, when $\mathbf{B} = 0$:

$$\boxed{\mathbf{F}_{\mathrm{E}} = \varepsilon_0 \oiint\limits_{\partial\Omega} \left((\mathbf{n}\cdot\mathbf{E})\,\mathbf{E} - \frac{1}{2}\widehat{\mathbf{n}} E^2 \right) \mathrm{d}\mathbf{S}} \quad . \tag{40.01.12}$$

→ 1929

Special case, when $\mathbf{E} = 0$:

$$\boxed{\mathbf{F}_{\mathrm{B}} = \frac{1}{\mu_0} \oiint\limits_{\partial\Omega} \left((\mathbf{n}\cdot\mathbf{B})\,\mathbf{B} - \frac{1}{2}\widehat{\mathbf{n}} B^2 \right) \mathrm{d}\mathbf{S}} \quad . \tag{40.01.13}$$

→ 1929

Diagonalization of the two expressions (40.01.12), (40.01.13) provides:

$$\boxed{\Xi_{\mathrm{EM}}{}_{ik}^{\cdot\cdot}[\mathbf{E},\mathbf{0}] = \frac{\varepsilon_0}{2}\begin{pmatrix} +E^2 & 0 & 0 & , \\ 0 & -E^2 & 0 & , \\ 0 & 0 & -E^2 & , \end{pmatrix}} \qquad \boxed{\Xi_{\mathrm{EM}}{}_{ik}^{\cdot\cdot}[\mathbf{0},\mathbf{B}] = \frac{1}{2\mu_0}\begin{pmatrix} +B^2 & 0 & 0 & , \\ 0 & -B^2 & 0 & , \\ 0 & 0 & -B^2 & , \end{pmatrix}} \quad ,$$

where the positive eigenvalue corresponds to an eigenvector along the direction of the field.

This shows that a tension is transmitted along the direction of the field and a transverse pressure is transmitted perpendicularly to the direction of the field, in both cases.

Note that the diagonalized stress tensor can be considered as the superposition of one isotropic stress tensor, giving rise to a uniform pressure, plus a stress tensor giving rise to a tension along the field equal to twice the pressure.

The effect of **Maxwell stress tensor** can be understood as follows. Consider a small circular cylinder coaxial with the field at some point. The expressions (40.01.12), (40.01.13) show that the contribution to

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the surface integral at the two bases and at the lateral surface are (with $\mathbf{V}=\{\mathbf{E},\mathbf{B}\}$):

$$\text{lateral surface} \implies \mathbf{n}\cdot\mathbf{V}=0 \implies \mathbb{d}\mathbf{F}\propto -\hat{\mathbf{n}} \implies \text{compression on the lateral sides} \quad ,$$

$$\text{bases} \implies \mathbf{V}\,(\mathbf{n}\cdot\mathbf{V})=\hat{\mathbf{n}}V^2 \implies \mathbb{d}\mathbf{F}\propto \left(\hat{\mathbf{n}}V^2-\hat{\mathbf{n}}V^2/2\right)=\hat{\mathbf{n}}V^2/2 \implies \text{traction on both bases} \quad .$$

As a comment to the previous relations one should note that the term $\mathbf{V}\,(\mathbf{n}\cdot\mathbf{V})$ is invariant for changing $\mathbf{V}\rightarrow-\mathbf{V}$.

When summing all the contributions on the whole surface the balance of the different contribution will give the overall force.

Stresses on the Surface Charge at the Surface of a Closed Conductor

It always tends to expand the surface charge.

Two Point Charges With the Same Sign

Two Point Charges With Opposite Signs

Stresses in a Solenoid (1)

Stresses in a Solenoid (2)

Stress on the sides.

It tends to explode the solenoid.

40.01.04
Angular Momentum of the ElectroMagnetic Fields

This § is referenced at pages:
[2084, 2493]

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See the important comment in section § 36.05.03 about the modeling of matter with point charges, **Dipoles** and higher-order **Multipoles** .

40.01.04.01 Introduction

ElectroMagnetic fields can carry angular momentum.

Even perfectly static ElectroMagnetic fields can harbor linear momentum and angular momentum, as long as $\mathbf{E}\wedge\mathbf{B}$ is non-zero. It is only when these contributions from ElectroMagnetic fields are included that the classical conservation laws hold.

The Feynman Cylinder paradox is solved by taking into account the angular momentum of ElectroMagnetic fields.

Angular momentum conservation for isolated systems is maintained if and only if the angular momentum carried by the ElectroMagnetic fields is accounted for, and experiments have shown that ElectroMagnetic waves can carry and transport angular momentum.

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40.01.04.02 Expression of the Angular Momentum of the ElectroMagnetic Fields

This § is referenced at pages:
[Never referenced.]

$$\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}\Omega} = \mathbf{x}\wedge(\rho\mathbf{E} + \mathbf{j}\wedge\mathbf{B}) \quad . \tag{40.01.14}$$

The density of angular momentum and the angular momentum of the ElectroMagnetic fields are:

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}\Omega}[\mathbf{x},t] = \mathbf{x}\wedge\mathbf{g} \qquad \mathbf{J}[t] = \iiint_{\Omega} \mathbf{x}\wedge\mathbf{g}\ \mathrm{d}V_{\mathbf{x}} \quad . \tag{40.01.15}$$

→ 1931 2084

This angular momentum density equation (40.01.15) includes both spin and orbital angular momenta of the ElectroMagnetic fields. That is, whether the angular momentum is produced by circular Polarization (spin) or by vorticity (orbital), equation (40.01.15) yields the total angular momentum density at any point in space and any instant in time, regardless its appearance seemingly suggesting an orbital part only.

40.01.04.03 Examples of ElectroMagnetic Angular Momentum in Action

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- The Feynman cylinder paradox [a].
- A variation on the Feynman cylinder paradox.
  Consider a cylindrical capacitor immersed in a magnetic field parallel to the axis.
  First case. If the capacitor is discharged there is an average current in the radial direction which interacts with the magnetic field. This generates an overall torque on the capacitor which tends to put it into rotation: angular momentum of the ElectroMagnetic field is transferred into mechanical angular momentum.
  Second case. If the magnetic field is reduced to zero a circular electric field is induced, whose modulus depends on the distance form the axis. Therefore the torques acting on the two charged faces of the capacitor are not balanced and the capacitor acquires angular momentum: angular momentum of the ElectroMagnetic field is transferred into mechanical angular momentum.

[a] R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ..., 17.4

40.01.05 General Expressions in Presence of Matter

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The expressions in presence of matter, using the macroscopic description, leads to some complications.
The Abraham-Minkowski controversy, over the form of the ElectroMagnetic Energy-LinMomentum tensor in a dielectric medium, has been resolved by the recognition that division of the total Energy-LinMomentum tensor into ElectroMagnetic and matter parts is somewhat arbitrary.

DRAFT UNDER REVIEW

40.02 Examples and Physical Applications

40.02.01 Energy Balance for a Resistive Wire

This § is referenced at pages:
[1925]

Consider a circular cylindrical wire of radius R [1] whose axis is the z axis, where a constant and uniform current I flows. Consider a finite part of the wire, of length L, and apply the energy balance to it.

Consider the microscopic picture for a typical solid conductor, where free electrons move inside a fixed lattice of positive ions. The power per unit volume given by the ElectroMagnetic fields to free electrons first goes into kinetic energy of the free electrons. The kinetic energy gained by the free electrons is then given to the lattice of ions, as potential and kinetic energy of the ions in the lattice, thanks to collisions. Both the kinetic energy of the free electrons and the total energy of the lattice contribute to the internal energy of the resistive material. All in all the ElectroMagnetic field transfers some energy into internal energy of the material, which is eventually given to the surrounding ambient as heat.

We will first consider the integral balance for a whole piece of wire and later the differential balance at the internal points.

The equipotential surfaces and the electric field for the very idealized situation of a long rectilinear and cylindrical coaxial wire with circular cross-section are shown in figures 40.11 and 40.12. The external shell is kept at zero potential and it is assumed to have zero resistance, while the difference of potential is applied at the two bases of the wire. Note that in general at the surface of the wire there is both a normal component of the electric field and a surface charge.

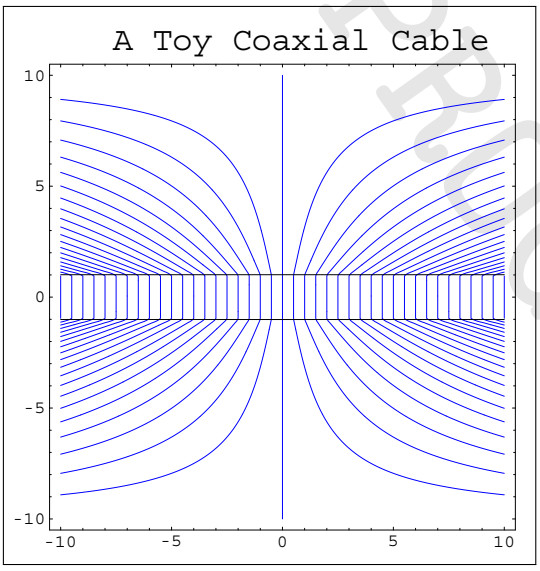

Figure 40.11: A toy coaxial cable (potential)

[1] Not to be confused with the resistance of the wire, which will not be used.

WORK IN PROGRESS

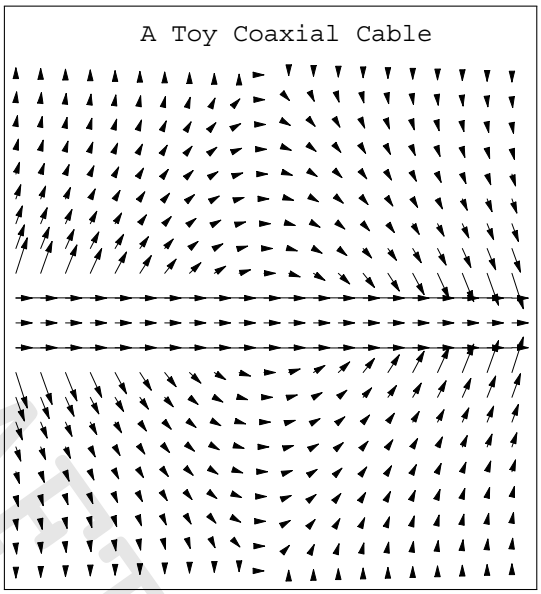

Figure 40.12: A toy coaxial cable (field)

40.02.01.01 Integral Balance

Under steady conditions the Poynting theorem shows that the energy given by the ElectroMagnetic fields into internal energy of the resistor (the Joule effect) can be considered to enter from the lateral perimeter of the wire, feeded by the ElectroMagnetic fields. However, it should be kept in mind that the Poynting vector is not unique and therefore the spatial localisation of the energy flux is not relevant, but only the divergence of the Poynting vector is as well as the integral on a closed surface are.

It is important to stress that the EM-motive field of the battery does not intervene here, because the battery is outside the volume where the energy balance is applied.

Consider, in fact, a circular cylinder surface, Σ, coaxial with the wire and with radius $r > R$, just slightly larger than R. Inside the wire the electric field is parallel to the axis of the wire, thanks the steady situation, and therefore, its tangential component, thanks to its continuity, has the same value just outside the wire. Note that, a priori, a normal component of the electric field is present at the surface of the wire, as well as a corresponding surface charge density. The normal component of the electric field, only present externally to the wire, is not relevant because it does not affect the energy balance as it provides a component of the Poynting vector parallel to the wire, whose flux across the cylindrical surface is zero. To avoid complications with this normal component at the two bases of the cylinder it is necessary to choose the surface Σ just slightly bigger that the wire in such a way that the part of the cylinder basis outside the wire has a negligible area, tending to zero, as the normal component has a non-zero contribution there. In fact, a priori the normal components of the electric field at the two bases are different because ρ_S may be different.

The radial component of the Poynting vector is directed towards the interior of the wire. It has the value, at $r = R$:

$$\mathbf{S}_{\mathrm{EM}} = \varepsilon_0 c^2\, \mathbf{E}\wedge\mathbf{B} \implies |S| = \varepsilon_0 c^2 \left(\frac{|j|}{\sigma}\right)\left(\frac{\mu_0 |I|}{2\pi R}\right) = \frac{j^2 R}{2\sigma} \qquad \text{and} \qquad \mathbf{S}_{\mathrm{EM}} = -\frac{j^2 R}{2\sigma}\widehat{\mathbf{e}}_r \quad .$$

The negative flux of the Poynting vector across the surface Σ is thus:

$$-\oiint \mathbf{S}_{\mathrm{EM}}\cdot \mathrm{d}\mathbf{S} = \frac{j^2 R}{2\sigma}(2\pi R L) = \frac{\pi j^2 R^2 L}{\sigma} = |I\Delta V| = \iiint \mathbf{j}\cdot\mathbf{E}\ \mathrm{d}V \quad .$$

The negative flux of the Poynting vector implies that ElectroMagnetic energy is entering the volume.

DRAFT - IN PROGRESS

40.02.01.02 Differential Balance at the Internal Points

The internal Electric|Magnetic fields and the Poynting vector are:

$$\mathbf{E} = E_z \widehat{\mathbf{e}}_3 \qquad \mathbf{B} = \frac{\mu_0 j_z r}{2} \widehat{\mathbf{e}}_\psi \qquad \mathbf{S}_{\text{EM}} = -\frac{E_z j_z r}{2} \widehat{\mathbf{e}}_r \quad .$$

It follows:

$$-\mathbf{div}\mathbf{S}_{\text{EM}} = -\frac{\partial\, \mathbf{S}_{\text{EM}} \cdot \widehat{\mathbf{e}}_r}{\partial r} - \frac{\mathbf{S}_{\text{EM}} \cdot \widehat{\mathbf{e}}_r}{r} = +E_z j_z = +\frac{j_z^2}{\sigma} > 0 \quad .$$

This shows clearly in a non-ambiguous way the meaning of Poynting theorem in differential form: at any point internal to the wire the ElectroMagnetic fields are feeding a power per unit volume to matter, energy which is then dissipated by the Joule effect.

40.02.01.03 Differential Balance at the External Points

The calculation of the Poynting vector outside the wire is difficult but Poynting theorem allows to state that, in steady conditions, $\mathbf{div}\mathbf{S}_{\text{EM}} = 0$ because $\mathbf{j} = 0$.

40.02.01.04 Energy Transfers

Inside the wire the electric field gives energy to the free charges (electrons), which becomes kinetic energy of the free charges. The free charges (electrons) collide with the fixed particles (positive ions) giving them mechanical energy. The energy of all the particles contributes to the internal energy. In steady conditions the work done by the electric field is thus totally transferred into internal energy, which is then possibly given to the external world as heat.

Poynting theorem, in steady conditions, reads:

$$\{\text{Work done by ElectroMagnetic fields on free electrons inside the wire}\} = \iiint \mathbf{j}\cdot\mathbf{E}\, \mathrm{d}V \quad ,$$

$$\{\text{EM Energy entering the system from outside}\} = \oint \mathbf{S}_{\text{EM}} \cdot \mathrm{d}\mathbf{S} \quad ,$$

$$\iiint \mathbf{j}\cdot\mathbf{E}\, \mathrm{d}V + \oint \mathbf{S}_{\text{EM}} \cdot \mathrm{d}\mathbf{S} = 0 \quad .$$

In fact in steady conditions the energy of the ElectroMagnetic fields inside the system does not change and therefore any energy must come from outside.

40.02.02 Hidden Linear Momentum and Angular Momentum

This § is referenced at pages:
[2472]

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Hidden momentum is perfectly ordinary relativistic mechanical momentum, as the example above indicates; it occurs in systems with internally moving parts, such as current-carrying loops, and it is "hidden" only in the sense that it not associated with motion of the objects a whole.

GO TO AFTER included File

Mansuripur's paradox

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A recent article claims that the Lorentz force law is incompatible with special relativity. We discuss the "paradox" on which this claim is based. The resolution depends on whether one assumes a "Gilbert" model for the magnetic dipole (separated monopoles) or the standard "Ampère" model (a current loop). The former case was treated in these pages many years ago; the latter, as several authors have noted, constitutes an interesting manifestation of "hidden momentum."

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I. INTRODUCTION

On May 7, 2012, a remarkable article appeared in *Physical Review Letters*.[1] The author, Masud Mansuripur, claimed to offer "incontrovertible theoretical evidence of the incompatibility of the Lorentz [force] law with the fundamental tenets of special relativity," and concluded that "the Lorentz law must be abandoned." The Lorentz law,

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] \tag{1}$$

tells us the force $\mathbf{F}$ on a charge q moving with velocity $\mathbf{v}$ through electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Together with Maxwell's equations, it is the foundation on which all of classical electrodynamics rests. If it is incorrect, 150 years of theoretical physics is in serious jeopardy.

Such a provocative proposal was bound to attract attention. *Science*[2] published a full-page commentary, and within days several rebuttals were posted.[3] Critics pointed out that since the Lorentz force law can be embedded in a manifestly covariant formulation of electrodynamics, it is guaranteed to be consistent with special relativity,[4] and and some of them identified the specific source of Mansuripur's error: neglect of "hidden momentum." Nearly a year later *Physical Review Letters* published four rebuttals,[5] and *Science* printed a follow-up article declaring the "purported relativity paradox resolved."[6]

Mansuripur's argument is based on a "paradox" that was explored in this journal by Victor Namias and others[7] many years ago: a magnetic dipole moving through an electric field can experience a torque, with no accompanying rotation. In Section II we introduce Mansuripur's version of the paradox, in simplified form, and explain Namias's resolution. The latter is based on a "Gilbert" model of the dipole (separated magnetic monopoles); it does not work for the (realistic) "Ampère" model (a current loop). For Amperian dipoles the resolution involves "hidden" momentum, so in Section III we discuss the physical nature of this often-misunderstood phenomenon. Mansuripur himself treated the dipole as the point limit of a magnetized object, so in Section IV we repeat the calculations in that context (for both models), and confirm our earlier results. In Section V we discuss the Einstein–Laub force law, which Mansuripur proposed as a replacement for the Lorentz law, and in Section VI we offer some comments and conclusions.

II. GILBERT DIPOLES: NAMIAS'S RESOLUTION

First the paradox: In $\mathcal{S}'$ (the "proper" frame) there is an ideal magnetic dipole $\mathbf{m} = m_0\,\hat{\mathbf{x}}$ at $(0, 0, d)$, and a point charge q at the origin, both at rest. The torque on $\mathbf{m}$ is (obviously) zero. Now examine the same configuration in $\mathcal{S}$ (the "lab" frame), with respect to which $\mathcal{S}'$ moves at constant speed v in the z direction (Fig. 1). In $\mathcal{S}$ the (moving) point charge generates electric and magnetic fields

$$\mathbf{E}(x, y, z, t) = \frac{q}{4\pi\epsilon_0}\frac{\gamma}{R^3}\left(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + (z - vt)\,\hat{\mathbf{z}}\right), \tag{2}$$

$$\mathbf{B}(x, y, z, t) = \frac{q}{4\pi\epsilon_0}\frac{v\gamma}{c^2 R^3}\left(-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}\right), \tag{3}$$

($\gamma \equiv 1/\sqrt{1 - (v/c)^2}$, $R \equiv \sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}$), and the (moving) magnetic dipole acquires an electric dipole moment[8]

$$\mathbf{p} = \frac{1}{c^2}(\mathbf{v} \times \mathbf{m}) = \frac{1}{c^2} v m_0\,\hat{\mathbf{y}}. \tag{4}$$

The torque on the dipole is

$$\mathbf{N} = (\mathbf{m} \times \mathbf{B}) + (\mathbf{p} \times \mathbf{E}) = \frac{q m_0}{4\pi\epsilon_0}\frac{v}{c^2 d^2}\,\hat{\mathbf{x}} \tag{5}$$

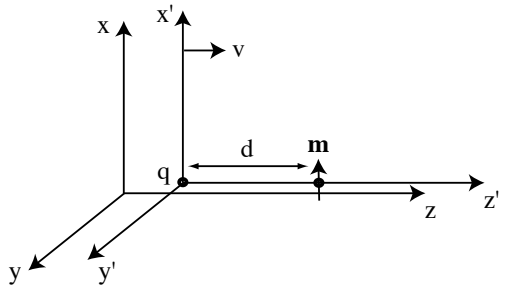

FIG. 1. Electric charge (q) and magnetic dipole ($\mathbf{m}$) in proper (primed) and lab (unprimed) frames.

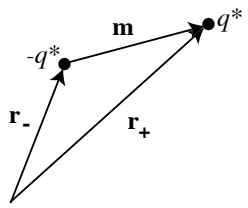

FIG. 2. "Gilbert" magnetic dipole.

(by Lorentz transformation, $d = \gamma(z - vt)$; the magnetic contribution is zero, because $\mathbf{B}$ vanishes on the z axis). The torque is zero in one inertial frame, but *non*-zero in the other! Mansuripur concludes that the Lorentz force law (on which Eq. 5 is predicated) is inconsistent with special relativity.

This "paradox" was resolved years ago by Victor Namias.[7] The standard torque formulas ($\mathbf{p} \times \mathbf{E}$ and $\mathbf{m} \times \mathbf{B}$) apply to dipoles *at rest*, but they do not hold, in general, for dipoles in motion. Suppose we model the magnetic dipole as separated monopoles (Fig. 2). The "Lorentz force law" for a magnetic monopole q^* reads[9]

$$\mathbf{F} = q^* \left[\mathbf{B} - (1/c^2)\mathbf{v} \times \mathbf{E}\right], \tag{6}$$

so the torque[10] on a moving dipole $\mathbf{m} = q^*(\mathbf{r}_+ - \mathbf{r}_-)$ is

$$\begin{aligned}\mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) \\ &= (\mathbf{m} \times \mathbf{B}) - \frac{1}{c^2}\mathbf{m} \times (\mathbf{v} \times \mathbf{E}).\end{aligned}$$

But $\mathbf{m} \times (\mathbf{v} \times \mathbf{E}) = \mathbf{v} \times (\mathbf{m} \times \mathbf{E}) + (\mathbf{m} \times \mathbf{v}) \times \mathbf{E}$, so

$$\begin{aligned}\mathbf{N} &= (\mathbf{m} \times \mathbf{B}) - \frac{1}{c^2}(\mathbf{m} \times \mathbf{v}) \times \mathbf{E} - \frac{1}{c^2}\mathbf{v} \times (\mathbf{m} \times \mathbf{E}) \\ &= (\mathbf{m} \times \mathbf{B}) + (\mathbf{p} \times \mathbf{E}) - \frac{1}{c^2}\mathbf{v} \times (\mathbf{m} \times \mathbf{E}).\end{aligned} \tag{7}$$

There is a third term, missing in Eq. 5, which (it is easy to check) exactly cancels the offending torque; the net torque is zero in both frames.

III. AMPÈRE DIPOLES: HIDDEN MOMENTUM

Namias believed that his formula (Eq. 7) applies just as well to an Ampère dipole as it does to a Gilbert dipole. He was mistaken. An Ampère dipole in an electric field carries "hidden" momentum,[11]

$$\mathbf{p}_h = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E}). \tag{8}$$

Because it is crucial in understanding the resolution to Mansuripur's paradox, we pause to review the derivation of this formula, in a simple model.

Imagine a rectangular loop of wire carrying a steady current. Picture the current as a stream of noninteracting positive charges that move freely within the wire.[12] When

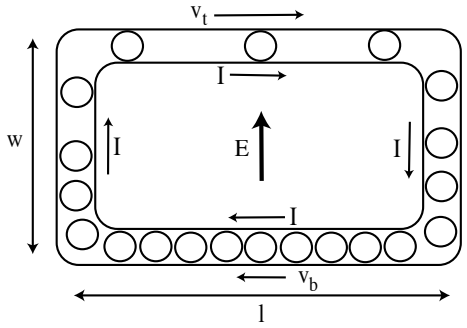

FIG. 3. Current loop in an external electric field.

a uniform electric field $\mathbf{E}$ is applied (Fig. 3), the charges accelerate up the left segment, and decelerate down the right one. *Question:* What is the total momentum of all the charges in the loop? The left and right segments cancel, so we need only consider the top and bottom. Say there are N_t charges in the top segment, going to the right at speed v_t, and N_b charges in the lower segment, going at (slower) speed v_b to the left. The *current* ($I = \lambda v$) is the same in all four segments (otherwise charge would be piling up somewhere). Thus

$$I = \frac{qN_t}{l}v_t = \frac{qN_b}{l}v_b, \quad \text{so} \quad N_t v_t = N_b v_b = \frac{Il}{q}, \tag{9}$$

where q is the charge of each particle, and l is the length of the rectangle. *Classically,* the momentum of a single particle is $p = mv$, where m is its mass, so the total momentum (to the right) is

$$p_{\text{classical}} = mN_t v_t - mN_b v_b = m\frac{Il}{q} - m\frac{Il}{q} = 0, \tag{10}$$

as one would certainly expect (after all, the loop as a whole is not moving). But *relativistically* the momentum of a particle is $p = \gamma mv$, and we get

$$p_{\text{relativistic}} = \gamma_t mN_t v_t - \gamma_b mN_b v_b = \frac{mIl}{q}\left(\gamma_t - \gamma_b\right), \tag{11}$$

which is *not* zero, because the particles in the upper segment are moving faster. In fact, the gain in energy (γmc^2), as a particle goes up the left side, is equal to the work done by the electric force, qEw, where w is the height of the rectangle, so

$$\gamma_t - \gamma_b = \frac{qEw}{mc^2}, \quad \text{and hence} \quad p_{\text{rel}} = \frac{IlEw}{c^2}. \tag{12}$$

Now Ilw is the magnetic dipole moment of the loop; as vectors, $\mathbf{m}$ points into the page and $\mathbf{p}$ is to the right, so

$$\mathbf{p}_{\text{rel}} = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E}). \tag{13}$$

This is the "hidden" momentum in Eq. 8.

The term "hidden momentum" was coined by Shockley;[11] it was an unfortunate choice. The phenomenon itself was first studied in the context of static electromagnetic systems with nonzero field momentum ($\mathbf{p}_{\text{field}} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d^3\mathbf{r}$). In such configurations the hidden momentum exactly cancels the field momentum ($\mathbf{p}_h = -\mathbf{p}_{\text{field}}$), leaving a total of zero, as required by the "center of energy theorem."[13] This has created the impression that hidden momentum is something artificial and ad hoc—invented simply to rescue an abstract theorem.[14] Nothing could be farther from the truth. Hidden momentum is perfectly ordinary relativistic mechanical momentum, as the example above indicates; it occurs in systems with internally moving parts, such as current-carrying loops, and it is "hidden" only in the sense that it not associated with motion of the object as a whole. A *Gilbert* dipole in an electric field, having no moving parts, harbors *no* hidden momentum (and the fields—with the crucial delta-function term in $\mathbf{B}$ included—carry no compensating momentum).[15]

Returning to the configuration in Fig. 1, the hidden momentum in $\mathcal{S}'$ is

$$\mathbf{p}_h = \frac{1}{c^2}\left[(m_0\,\hat{\mathbf{x}}) \times \left(\frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\,\hat{\mathbf{z}}\right)\right] = -\frac{qm_0}{4\pi\epsilon_0 c^2 d^2}\,\hat{\mathbf{y}}. \quad (14)$$

Because $\mathbf{p}_h$ is perpendicular to $\mathbf{v}$, and transverse components are unaffected by Lorentz transformations, this is also the hidden momentum in $\mathcal{S}$. It is constant (in time), so there is no associated force. But the hidden *angular* momentum,

$$\mathbf{L}_h = \mathbf{r} \times \mathbf{p}_h, \quad (15)$$

is *not* constant (in the lab frame), because $\mathbf{r}$ is changing. In fact,

$$\frac{d\mathbf{L}_h}{dt} = \mathbf{v} \times \mathbf{p}_h = \frac{qm_0}{4\pi\epsilon_0}\frac{v}{c^2 d^2}\,\hat{\mathbf{x}}. \quad (16)$$

This increase in angular momentum requires a torque,

$$\mathbf{N} = \frac{qm_0}{4\pi\epsilon_0}\frac{v}{c^2 d^2}\,\hat{\mathbf{x}}, \quad (17)$$

and this is precisely what we found in Eq. 5.

Recapitulating: In the Gilbert model there is an extra term in the torque formula (Eq. 7); the total torque is zero, there is no hidden angular momentum, and nothing rotates. In the Ampère model there is no third term in the torque formula (Eq. 5)[16]; the torque is *not* zero, and drives the increasing hidden angular momentum—but still nothing rotates.[17] It helps to separate the angular momentum into two types: "overt" (associated with actual rotation) and "hidden" (so called because it is *not* associated with any overt rotation of the object). Torque is the rate of change of the *total* angular momentum:

$$\mathbf{N} = \frac{d\mathbf{L}_o}{dt} + \frac{d\mathbf{L}_h}{dt}. \quad (18)$$

In both models $d\mathbf{L}_o/dt = \mathbf{0}$. In the Gilbert model $\mathbf{N}$ and $d\mathbf{L}_h/dt$ are also zero; in the Ampère model they are equal but non-zero.

IV. MAGNETIZED MATERIALS

It is of interest to see how this plays out in Mansuripur's formulation of the problem. He treats the dipole as magnetized medium, and calculates the torque directly from the Lorentz force law, without invoking $\mathbf{p} \times \mathbf{E}$ or $\mathbf{m} \times \mathbf{B}$. In the proper frame, he takes

$$\mathbf{M}'(x', y', z', t') = m_0\delta(x')\delta(y')\delta(z' - d)\,\hat{\mathbf{x}}. \quad (19)$$

Now, $\mathbf{M}$ and $\mathbf{P}$ constitute an antisymmetric second-rank tensor:

$$P^{\mu\nu} = \begin{pmatrix} 0 & cP_x & cP_y & cP_z \\ -cP_x & 0 & -M_z & M_y \\ -cP_y & M_z & 0 & -M_x \\ -cP_z & -M_y & M_x & 0, \end{pmatrix} \quad (20)$$

and the transformation rule is[18]

$$P_z = P'_z,\ P_x = \gamma(P'_x + \frac{v}{c^2}M'_y),\ P_y = \gamma(P'_y - \frac{v}{c^2}M'_x)$$
$$M_z = M'_z,\ M_x = \gamma(M'_x - vP'_y),\ M_y = \gamma(M'_y + vP'_x)$$

(for motion in the z direction). In the present case, then, the polarization and magnetization in the "lab" frame are

$$\mathbf{M}(x, y, z, t) = m_0\delta(x)\delta(y)\delta\left(z - vt - (d/\gamma)\right)\hat{\mathbf{x}}, \quad (21)$$
$$\mathbf{P}(x, y, z, t) = \frac{m_0 v}{c^2}\delta(x)\delta(y)\delta\left(z - vt - (d/\gamma)\right)\hat{\mathbf{y}}. \quad (22)$$

According to the Lorentz law, the force density is

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}, \quad (23)$$

where $\rho = -\boldsymbol{\nabla} \cdot \mathbf{P}$ is the bound charge density and $\mathbf{J} = \partial\mathbf{P}/\partial t + \boldsymbol{\nabla} \times \mathbf{M}$ is the sum of the polarization current and the bound current density. Using Eqs. 2, 3, 21, and 22, we obtain

$$\mathbf{f} = -(\boldsymbol{\nabla} \cdot \mathbf{P})\mathbf{E} + (\boldsymbol{\nabla} \times \mathbf{M}) \times \mathbf{B} + \frac{\partial \mathbf{P}}{\partial t} \times \mathbf{B}$$
$$= -\frac{qm_0 v}{4\pi\epsilon_0 c^2}\frac{d}{R^3}\delta(x)\delta'(y)\delta(z - vt - d/\gamma)\,\hat{\mathbf{z}} \quad (24)$$

(where a prime denotes the derivative). The net force on the dipole is

$$\mathbf{F} = \int \mathbf{f}\, dx\, dy\, dz = \frac{qm_0 vd}{4\pi\epsilon_0 c^2}\frac{d}{dy}\left[\frac{1}{(y^2 + d^2)^{3/2}}\right]\Bigg|_{y=0}\hat{\mathbf{z}} = \mathbf{0}. \quad (25)$$

Meanwhile, the torque density is

$$\mathbf{n} = \mathbf{r} \times \mathbf{f} = -\frac{qm_0 vd}{4\pi\epsilon_0 c^2}\frac{y}{R^2}\delta(x)\delta'(y)\delta(z - vt - d/\gamma)\,\hat{\mathbf{x}}, \quad (26)$$

so the net torque on the dipole is

$$\mathbf{N} = \int \mathbf{n}\, dx\, dy\, dz$$
$$= -\frac{qm_0 vd}{4\pi\epsilon_0 c^2}\left\{-\frac{d}{dy}\left[\frac{y}{(y^2 + d^2)^{3/2}}\right]\right\}\Bigg|_{y=0}\hat{\mathbf{x}}$$
$$= \frac{qm_0 v}{4\pi\epsilon_0 c^2 d^2}\,\hat{\mathbf{x}}, \quad (27)$$

confirming Eq. 5. This is the torque required to account for the increase in hidden angular momentum.

What if we run Mansuripur's calculation for a dipole made out of magnetic monopoles? The bound charge, bound current, and magnetization current are[19]

$$\rho_b^* = -\boldsymbol{\nabla}\cdot\mathbf{M}, \quad \mathbf{J}_b^* = -c^2\boldsymbol{\nabla}\times\mathbf{P}, \quad \mathbf{J}_p^* = \frac{\partial\mathbf{M}}{\partial t}, \quad (28)$$

so the force density on the magnetic dipole (again invoking Eqs. 2, 3, 21, and 22) is[20]

$$\begin{aligned}\mathbf{f} &= \rho^*\mathbf{B} - \frac{1}{c^2}\mathbf{J}^*\times\mathbf{E} \\ &= -(\boldsymbol{\nabla}\cdot\mathbf{M})\mathbf{B} + (\boldsymbol{\nabla}\times\mathbf{P})\times\mathbf{E} - \frac{1}{c^2}\left(\frac{\partial\mathbf{M}}{\partial t}\right)\times\mathbf{E} \\ &= \mathbf{0}. \end{aligned} \quad (29)$$

The total force is again zero, but this time so too is the torque density ($\mathbf{n} = \mathbf{r}\times\mathbf{f}$), and hence the total torque. As before, the torque is zero in the Gilbert model—and there is no hidden angular momentum.

V. THE EINSTEIN–LAUB FORCE LAW

Having concluded that the Lorentz force law is unacceptable, Mansuripur proposes to replace Eq. 24 with an expression based on the Einstein–Laub law:[21]

$$\begin{aligned}\mathbf{f}_{\rm EL} &= (\mathbf{P}\cdot\boldsymbol{\nabla})\mathbf{E} + \frac{\partial\mathbf{P}}{\partial t}\times(\mu_0\mathbf{H}) + (\mathbf{M}\cdot\boldsymbol{\nabla})\mu_0\mathbf{H} \\ &\quad - \frac{1}{c^2}\frac{\partial\mathbf{M}}{\partial t}\times\mathbf{E} \\ &= \frac{m_0 q v\gamma}{4\pi\epsilon_0 c^2}\frac{1}{R^3}\delta(x)\delta(y)[2\delta(z-vt-d/\gamma) \\ &\quad - (z-vt)\delta'(z-vt-d/\gamma)]\,\hat{\mathbf{y}}. \end{aligned} \quad (30)$$

The total force on the dipole still vanishes:

$$\begin{aligned}\mathbf{F}_{\rm EL} &= \frac{m_0 q v\gamma}{4\pi\epsilon_0 c^2}\,\hat{\mathbf{y}}\left\{\frac{2}{d^3} + \frac{1}{\gamma^3}\frac{d}{dz}\left[\frac{1}{(z-vt)^2}\right]\Bigg|_{z-vt=d/\gamma}\right\} \\ &= \mathbf{0}. \end{aligned} \quad (31)$$

The torque density should be $\mathbf{r}\times\mathbf{f}_{\rm EL}$:

$$\begin{aligned}\mathbf{n}_{\rm EL} = -\frac{m_0 q v\gamma}{4\pi\epsilon_0 c^2}\frac{z}{R^3}\delta(x)\delta(y)[2\delta(z-vt-d/\gamma) \\ - (z-vt)\delta'(z-vt-d/\gamma)]\,\hat{\mathbf{x}}, \end{aligned} \quad (32)$$

giving a total torque

$$\begin{aligned}\mathbf{N}_{\rm EL} &= -\frac{m_0 q v\gamma}{4\pi\epsilon_0 c^2}\,\hat{\mathbf{x}}\left\{\frac{2(vt+d/\gamma)}{d^3} + \frac{1}{\gamma^3}\frac{d}{dz}\left[\frac{z}{(z-vt)^2}\right]\right\} \\ &= -\frac{m_0 q v}{4\pi\epsilon_0 c^2 d^2}\,\hat{\mathbf{x}} \end{aligned} \quad (33)$$

(the derivative is again evaluated at $z - vt = d/\gamma$). It's not zero! In fact, it's *minus* the "Lorentz" torque, Eq. 27. But Mansuripur argues that, "To guarantee the conservation of angular momentum, [Eq. 32] must be supplemented . . . "

$$\mathbf{n}'_{\rm EL} = \mathbf{n}_{\rm EL} + (\mathbf{P}\times\mathbf{E}) + (\mathbf{M}\times\mathbf{B}). \quad (34)$$

In our case the extra terms are

$$(\mathbf{P}\times\mathbf{E}) + (\mathbf{M}\times\mathbf{B}) = \frac{m_0 q v}{4\pi\epsilon_0 c^2 d^2}\delta(x)\delta(y)\delta(z-vt-d/\gamma)\,\hat{\mathbf{x}},$$

and their contribution to the total torque is

$$\int[(\mathbf{P}\times\mathbf{E}) + (\mathbf{M}\times\mathbf{B})]\,dx\,dy\,dz = \frac{m_0 q v}{4\pi\epsilon_0 c^2 d^2}\,\hat{\mathbf{x}}, \quad (35)$$

which is just right to cancel Eq. 33, yielding a net torque of zero (which Mansuripur takes to be the correct answer).

What are we to make of this argument? In the first place, the Einstein–Laub force density was derived assuming that the medium is at rest,[21] which in this case it is not. More important, the magnetization terms implicitly assume a Gilbert model for the magnetic dipole:

$$(\mathbf{M}\cdot\boldsymbol{\nabla})\mathbf{B} = -\boldsymbol{\nabla}\times(\mathbf{M}\times\mathbf{B}) + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{M} - (\boldsymbol{\nabla}\cdot\mathbf{M})\mathbf{B}; \quad (36)$$

as long as the magnetization is localized, the first two terms yield vanishing surface integrals,[22] leaving $-(\boldsymbol{\nabla}\cdot\mathbf{M})\mathbf{B} - (1/c^2)[(\partial\mathbf{M}/\partial t)\times\mathbf{E}]$ for the net force density on the object, the same as in the Gilbert model (Eq. 29).[23] There may be some contexts in which the Einstein–Laub force law is valid and useful, but this is not one of them. Mansuripur is quite explicit in writing that the magnetic dipole he has in mind is "a small, charge neutral loop of current," which is to say, an Ampère dipole.

VI. CONCLUSION

The resolution of Mansuripur's "paradox" depends on the model for the magnetic dipole:

- If it is a Gilbert dipole (made from magnetic monopoles), the third term in Namias's formula (Eq. 7) supplies the missing torque. In Mansuripur's formulation (using a polarizable medium), it comes from a correct accounting of the bound charge/current (Eq. 28). The net torque is zero in the lab frame, just as it is in the proper frame.

- If it is an Ampère dipole (an electric current loop), the third term in Namias's equation is absent, and the torque on the dipole is *not* zero. It is, however, just right to account for the increasing hidden angular momentum in the dipole.

In either model the Lorentz force law is entirely consistent with special relativity.

We thank Kirk McDonald, Daniel Vanzella, and Daniel Cross for useful correspondence. VH coauthored this paper in his private capacity; no official support or endorsement by the Centers for Disease Control and Prevention is intended or should be inferred.

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[1] M. Mansuripur, "Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation," Phys. Rev. Lett. **108**, 193901(4) (2012).

[2] A. Cho, "Textbook Electrodynamics May Contradict Relativity," Science **336**, 404 (2012).

[3] K. T. McDonald, "Mansuripur's Paradox," ⟨ www.physics.princeton.edu/∼mcdonald/examples/mansuripur.pdf⟩ (14 pp); D. A. T. Vanzella, "Comment on 'Trouble with the Lorentz law of force,'" e-print arXiv:1205.1502 (2 pp); D. J. Cross, "Resolution of the Mansuripur Paradox," e-print arXiv:1205:5451 (3 pp); P. L. Saldanha, "Comment on 'Trouble with the Lorentz law of force,'" e-print arXiv:1205:6858 (2 pp); D. J. Griffiths and V. Hnizdo, "Comment on 'Trouble with the Lorentz law of force,'" e-print arXiv:1205.4646 (3 pp). Later critiques include K. A. Milton and G. Meille, "Electromagnetic Angular Momentum and Relativity," e-print arXiv:1208:4826 (4 pp); F. De Zela, "Comment on 'Trouble with the Lorentz law of force,'" e-print arXiv:1210.7344 (2 pp); T. M. Boyer, "Examples and comments related to relativity controversies," Am. J. Phys. **80**, 962-971 (2012); A. L. Kholmetskii, O. V. Missevitch, and T. Yarman, "Torque on a Moving Electric/Magnetic Dipole," Prog. Elecromagnetics Research B, **45**, 83-99 (2012). See also M. Mansuripur, "Trouble with the Lorentz Law of Force: Response to Critics," Proc. SPIE, **8455**, 845512 (2012).

[4] The details are worked out in Cross and Vanzella (ref. 3).

[5] D. A. T. Vanzella,"Comment on 'Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation,'" Phys. Rev. Lett. **110**, 089401-1 (2013); and, under the same title, S. M. Barnett, Phys. Rev. Lett. **110**, 089402-1 (2013); P. L. Saldanha, Phys. Rev. Lett. **110**, 089403-1-2 (2013); M. Khorrami, Phys. Rev. Lett. **110**, 089404-1 (2013).

[6] "Paradox Lost," Science **339**, 496 (2013); A. Cho, "Purported Relativity Paradox Resolved," ⟨http://scim.ag/Lorpara⟩ (2013). Mansuripur himself does not accept this verdict, though he does appear to have softened his assertions somewhat: M. Mansuripur, "Mansuripur replies," Phys. Rev. Lett. **110**, 089405-1 (2013).

[7] V. Namias, "Electrodynamics of moving dipoles: The case of the missing torque," Am. J. Phys. **57**, 171-177 (1989); D. Bedford and P. Krumm, "On the origin of magnetic dynamics," Am. J. Phys. **54**, 1036-1039 (1986).

[8] See, for instance, V. Hnizdo, "Magnetic dipole moment of a moving electric dipole," Am. J. Phys. **80**, 645-647 (2012).

[9] See, for example, D. J. Griffiths, *Introduction to Electrodynamics*, 4th ed. (Pearson, Boston, 2013), Eq. 7.69.

[10] We calculate all torques (in the lab frame) with respect to the origin. But because the net force on the dipole is zero in all cases, it does not matter—we could as well use *any* fixed point, including the (instantaneous) position of the dipole.

[11] W. Shockley and R. P. James, "'Try simplest cases' discovery of 'hidden momentum' forces on 'magnetic currents,'" Phys. Rev. Lett. **18**, 876-879 (1967); W. H. Furry, "Examples of momentum distributions in the electromagnetic field and in matter," Am. J. Phys. **37**, 621-636 (1969); L. Vaidman, "Torque and force on a magnetic dipole," Am. J. Phys. **58**, 978-983 (1990); V. Hnizdo, "Conservation of linear and angular momentum and the interaction of a moving charge with a magnetic dipole," Am. J. Phys. **60**, 242-246 (1992); V. Hnizdo, "Hidden mechanical momentum and the field momentum in stationary electromagnetic and gravitational systems," Am. J. Phys. **65**, 515-518 (1997).

[12] This is of course an unrealistic model for an actual current-carrying wire. Vaidman (Ref. 11) explores more plausible models, but the result is unchanged.

[13] If the center of energy of a closed system is at rest, the total momentum of the system must be zero. See, for example, S. Coleman and J. H. Van Vleck, "Origin of 'Hidden Momentum Forces' on Magnets," Phys. Rev. **171**, 1370-1375 (1968); M. G. Calkin, "Linear momentum of the source of a static electromagnetic field," Am. J. Phys. **39**, 513-516 (1971).

[14] Mansuripur variously calls hidden momentum an "absurdity" (M. Mansuripur, "Resolution of the Abraham-Minkowski controversy," Opt. Commun. **283**, 1997-2005 (2010), p. 1999), a "problem" to be "solved" (Ref. 1) and "as applied to magnetic materials . . . an unnecessary burden" (Ref. 6).

[15] D. J. Griffiths, "Dipoles at rest," Am. J. Phys. **60**, 979-987 (1992). The magnetic field of a point dipole is $\mathbf{B} = (\mu_0/4\pi)(1/r^3)\,[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] + \alpha\mu_0\mathbf{m}\delta^3(\mathbf{r})$, where $\alpha = 2/3$ for Ampère dipoles and $-1/3$ for Gilbert dipoles. The delta function term leads, for example, to hyperfine splitting in the ground state of hydrogen, and provides experimental confirmation that the proton is an Ampére dipole.

[16] We do not know a simple way to prove this directly, but we will confirm it implicitly in Section IV.

[17] This is certainly not the first time such issues have arisen. How can there be a torque in the lab frame, when there is none in the proper frame? See J. D. Jackson, "Torque or no torque? Simple charged particle motion observed in different inertial frames," Am. J. Phys. **72**, 1484-1487 (2004). How can there be a torque, with no accompanying rotation? See D. G. Jensen, "The paradox of the L-shaped object," Am. J. Phys. **57**, 553-555 (1989).

[18] See, for instance, Ref. 9, Eq. 12.118.

[19] The minus sign in $\mathbf{J}_b^*$ is due to the switched sign in "Ampère's law" for magnetic monopoles (see Ref. 9, Eq. 7.44). Note that in Eq. 28 (and hence also Eq. 29), $\mathbf{M}$ and $\mathbf{P}$ are the densities of the dipole moments of magnetic monopoles and magnetic-monopole currents, respectively.

[20] There is some dispute as to the correct form of the Lorentz force law for magnetic monopoles in the presence of polarizable and magnetizable materials, but not when (as here) the polarization/magnetization is

itself due to monopoles. See K. T. McDonald, "Poynting's Theorem with Magnetic Monopoles," (11 pp), ⟨http://puhep1.princeton.edu/∼mcdonald/examples/poynting.pdf⟩ (2013).

[21] A. Einstein and J. Laub, "Über die im elektromagnetischen Felde auf ruhende Körper ausgeübten ponderomotorischen Kräfte," Ann. Phys. (Leipzig) **26**, 541–550 (1908); English translation in *The Collected Papers of Albert Einstein*, Vol. 2 (Princeton University Press, Princeton, NJ, 1989). In evaluating the force, Mansuripur uses the field $\mathbf{H}$ due to q (see Eq. 12b in Ref. 1); his $\mu_0\mathbf{H}$ is our $\mathbf{B}$ (Eq. 3).

[22] The ith component of the second term is $(\mathbf{B}\cdot\boldsymbol{\nabla})M_i = \boldsymbol{\nabla}\cdot(M_i\mathbf{B}) - M_i\boldsymbol{\nabla}\cdot\mathbf{B} = \boldsymbol{\nabla}\cdot(M_i\mathbf{B})$.

[23] B. D. H. Tellegen, "Magnetic-dipole models," Am. J. Phys. **30**, 650–652 (1962). Tellegen's force (6), which assumes a Gilbert magnetic dipole, can be obtained by integrating the magnetization-dependent terms in the Einstein–Laub force density in which it is assumed that $\mathbf{M} = \mathbf{m}\,\delta^3(\mathbf{r}-\mathbf{r}_0)$. Another derivation of the force on a Gilbert magnetic dipole is by A. D. Yaghjian, "Electromagnetic forces on point dipoles," IEEE Anten. Prop. Soc. Symp. **4**, 2868–2871 (1999).

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40.03 Exercises Problems and Physical Applications

40-001 Energy Absorption Versus Linear Momentum Absorption

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|2.41 - Q-15||

Can any object absorb ElectroMagnetic energy in the form of light without absorbing any ElectroMagnetic linear momentum ? If yes do some examples; if not explain why.

Can any object absorb ElectroMagnetic linear momentum in the form of light without absorbing any EM energy ? If yes do some examples; if not explain why.

40-002 Distance Dependence

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|2.41.17||

Walk toward an isotropically emitting lamp by $D = 162$ m. You find that the intensity rises by a factor 1.5. What was the initial distance ? Can one determine the power radiated by the lamp? If yes, find it out. If not explain why.

40-003 Solar Light

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|2.41.20||

The intensity of solar light just ouside the Earth atmosphere is about 1.4 kW/m^2. Find the average values of the ElectroMagnetic fields.

40-004 I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ... 4.210

Demonstrate that at the boundary between two media the normal components of the Poynting vector are continuous.

40-005 I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ... 4.224

Gravitational attraction versus sun radiation pressure on interplanetary particles.

40-006 A Charging Capacitor

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|2.41.31||

Consider a cylindrical capacitor with parallel, plane and circular plates, neglecting edge effects, while it is charging via two long rectilinear wires coaxial to the plates.

Show that the Poynting vector is everywhere radial entering the capacitor on the lateral surface of the cylinder.

Show that the rate of increase of the electric energy stored in the capacitor can be calculated using Poynting vector.

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40-007 Feynman Cylinder Paradox

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|17.4||

©|WEB - URL |||

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CHAPTER 41

ElectroMagnetism - Miscellaneous Complements Applications and Phenomenology

41.01 Electric|Magnetic Dipole Moments 1946
41.02 Forces on Dipoles and Polarized/Magnetized Materials 1947
41.03 Laws of ElectroMagnetism in Presence of Matter 1951
41.04 Constitutive Relations of Electric|Magnetic Materials 1960
41.05 Charges and Currents in External Static ElectroMagnetic Fields 1964
41.06 Quasi-Static ElectroMagnetic Fields 1969
41.07 Phenomenology 1970
41.08 Motion of Particles in ElectroMagnetic Fields 1973
41.09 Applications of EM 1974
41.10 Experimental Results 1974
41.11 Examples and Physical Applications 1976
41.12 Exercises Problems and Physical Applications 1976
41.13 Other Exercises and Problems in ElectroMagnetism 1981

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©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|6|*Good text.*|
©|C.Kittel, *Introduction To Solid State Physics*, 1996, J.Wiley & Sons, ...Ed., ...|||
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|1.4||

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41.01 Electric|Magnetic Dipole Moments

This § is referenced at pages:
[1843, 1865, 1885, 1908, 2690]

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|5.6||

The **Electric Dipole** moment is independent from the origin of the Coordinate System for all systems with zero total charge: $\iiint \rho \, \mathrm{d}V = 0$.

The **Magnetic Dipole** moment is independent from the origin of the Coordinate System for all systems with $\iiint \mathbf{j} \, \mathrm{d}V = 0$. A sufficient condition is a steady current distribution ($\mathbf{div}\mathbf{j} = 0$) with compact support.

The Electric|Magnetic field of ideal **Electric Dipoles** | **Magnetic Dipoles** are identical.

The Electric|Magnetic field of real **Electric Dipoles** | **Magnetic Dipoles** are identical at large distances. At short distances the fields are very different. In particular the fields at and near the center are opposite in the two cases. See figure 41.11.

Read § 67.12 for a general discussion of the energetic aspects.

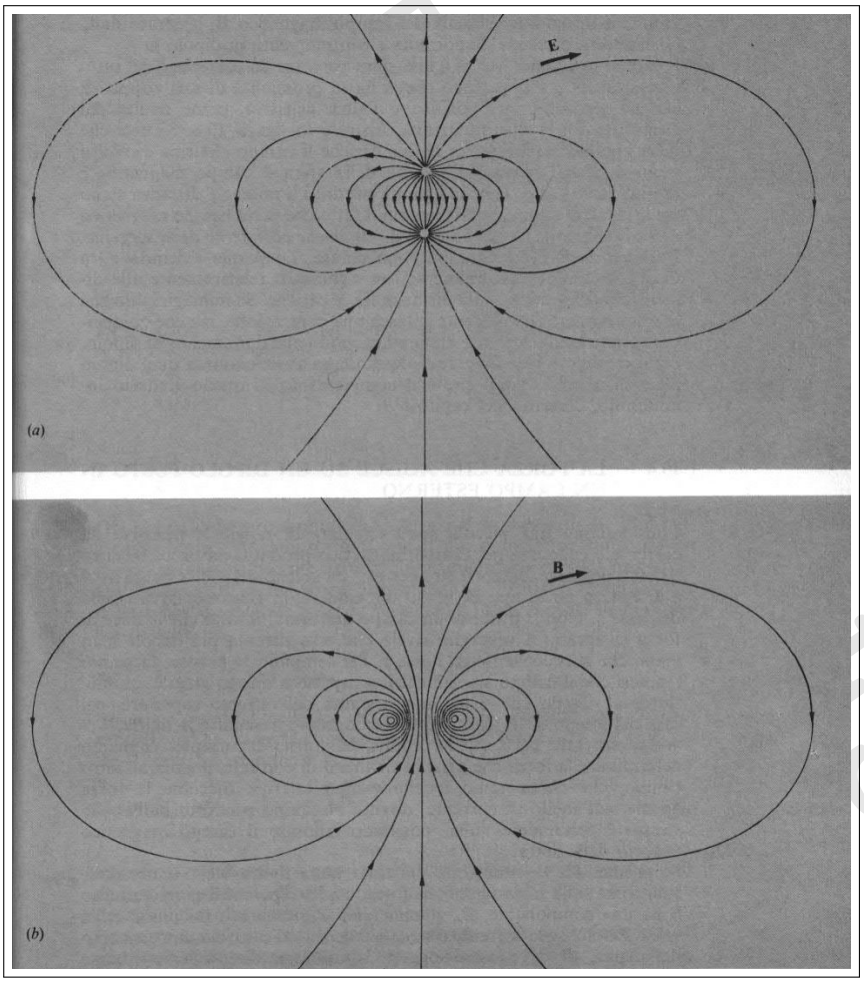

Figure 41.11: Difference of near fields between real Electric|Magnetic Dipoles.

41.02 Forces on Dipoles and Polarized/Magnetized Materials

This § is referenced at pages:
[1856, 1900, 2690]

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|23||

According to the statements in section § 36.05.02 the forces on Dipoles in external (applied) fields, in a static situation and away from the Charge|Current producing the external ElectroMagnetic fields, can be written as the equivalent expressions:

$$\boxed{\mathbf{f}_{\mathrm{E}} = (\mathbf{p}\cdot\mathbf{grad})\mathbf{E} \qquad \mathbf{f}_{\mathrm{E}} = \mathbf{grad}(\mathbf{p}\cdot\mathbf{E})} \quad ,$$

$$\boxed{\mathbf{f}_{\mathrm{M}} = (\mathbf{m}\cdot\mathbf{grad})\mathbf{B} \qquad \mathbf{f}_{\mathrm{M}} = \mathbf{grad}(\mathbf{m}\cdot\mathbf{B})} \quad ,$$

where in the right-most expressions the **Dipole** moments must always be considered fixed with respect to the gradient operator, that is non differentiated.

It must be emphasized that in the above expressions the fields are the external fields, excluding the fields produced by the **Dipoles** themselves.

The equivalence of the two expressions under the given hypotheses is ensured by the fact that the hypotheses imply that both the external Electric|Magnetic fields are irrotational:

$$\frac{\partial E_k}{\partial x_j} = \frac{\partial E_j}{\partial x_k} \qquad \frac{\partial B_k}{\partial x_j} = \frac{\partial B_j}{\partial x_k} \quad ,$$

and by the fact that the **Dipole** moments are considered as fixed constants, that is non differentiated.

Read § 67.12 for a general discussion of the energetic aspects.

41.02.01 Linearly Polarizable|Magnetizable Entities in Static Conditions

This § is referenced at pages:
[1813]

Consider electricly neutral **linearly** Polarizable|Magnetizable entities, with no intrinsic **Dipole** moment and whose induced **Dipole** moment is written in terms of the external (applied) fields, $\mathbf{E}$ and|or $\mathbf{B}$, as:

$$\boxed{\mathbf{p} = (\alpha_{\mathrm{E}}\varepsilon_0)\,\mathbf{E}} \qquad \boxed{\mathbf{m} = (\alpha_{\mathrm{M}}/\mu_0)\,\mathbf{B}} \quad .$$

It must be clear that the external (applied) fields, $\mathbf{E}$ and|or $\mathbf{B}$, are the total fixed fields acting on the entity and produced by external and far-away generators, producing the fixed external fields that is fixed charges and|or fixed currents.

As the generators of the external (applied) fields are outside the region of interest and the situation is a static one the forces can be written as:

$$\mathbf{f}_{\mathrm{E}} = (\mathbf{p}\cdot\mathbf{grad})\mathbf{E} = (\varepsilon_0\alpha_{\mathrm{E}})\,(\mathbf{E}\cdot\mathbf{grad})\mathbf{E} \quad ,$$

$$\mathbf{f}_{\mathrm{M}} = (\mathbf{m}\cdot\mathbf{grad})\mathbf{B} = (\alpha_{\mathrm{M}}/\mu_0)\,(\mathbf{B}\cdot\mathbf{grad})\mathbf{B} \quad ,$$

$$\{\mathbf{f}_{\mathrm{E}}\}_k = \mathbf{p}_r\boldsymbol{\nabla}_r E_k = (\varepsilon_0\alpha_{\mathrm{E}})\,E_r\boldsymbol{\nabla}_r E_k = (\varepsilon_0\alpha_{\mathrm{E}})\,E_r\boldsymbol{\nabla}_k E_r = \frac{\varepsilon_0\alpha_{\mathrm{E}}}{2}\boldsymbol{\nabla}_k\mathbf{E}^2 \Longrightarrow \boxed{\mathbf{f}_{\mathrm{E}} = \frac{\varepsilon_0\alpha_{\mathrm{E}}}{2}\boldsymbol{\nabla}\mathbf{E}^2} \quad ,$$

$$\{\mathbf{f}_{\mathrm{M}}\}_k = \mathbf{m}_r\boldsymbol{\nabla}_r B_k = (\alpha_{\mathrm{M}}/\mu_0)\,B_r\boldsymbol{\nabla}_r B_k = (\alpha_{\mathrm{M}}/\mu_0)\,B_r\boldsymbol{\nabla}_k B_r = \frac{\alpha_{\mathrm{M}}}{2\mu_0}\boldsymbol{\nabla}_k\mathbf{B}^2 \Longrightarrow \boxed{\mathbf{f}_{\mathrm{M}} = \frac{\alpha_{\mathrm{M}}}{2\mu_0}\boldsymbol{\nabla}\mathbf{B}^2} \quad .$$

First note that the gradient of the square of a field is always directed pointing to where the module of the field increases.

The above results imply that:

- Polarizable media are always attracted towards the region where the electric field is most intense, as the **Polarisability** is always positive;
- Magnetizable media are attracted towards the region where the magnetic field is most intense, if their **Magnetisability** is positive (paramagnetic media), while they are repelled away from the region where the magnetic field is most intense, if their **Magnetisability** is negative (diamagnetic media).

The energy energy of a Polarizable|Magnetizable **Dipole** in an external (applied) field is thus:

$$\mathbf{p} \propto \mathbf{E} \implies \Phi_{\mathrm{E}} = -\frac{\mathbf{p}\cdot\mathbf{E}}{2} \qquad \mathbf{m} \propto \mathbf{B} \implies \Phi_{\mathrm{M}} = -\frac{\mathbf{m}\cdot\mathbf{B}}{2} \quad .$$

The factor 2 with respect to the case of a fixed **Dipole** should be noted.

It is also important to stress that:

- in the electric case, this is the total energy, not the potential energy, because, while building the system, some energy is stored as internal energy;
- in the magnetic case, this is not the total energy, because, in order to keep the external magnetic field fixed while building the system, the external generator must do some work which has not been accounted here.

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©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|§ 15.1, 15.2|*Full* ...

The magnetic energy of a **Dipole** $\Phi = -\mathbf{m}\cdot\mathbf{B}$ may not include all the energy of the system (it is a fake kind of energy) but it gives the correct expression for the forces on a steady current distribution!
The total (true) energy of the world is: $+\mathbf{m}\cdot\mathbf{B}$. It is only if we assume that all currents are constant that we can use only a part of the total energy, $-\mathbf{m}\cdot\mathbf{B}$, to find the mechanical forces.

Read § 38.10.08 and section § 39.10.05 for the **Polarisability** |Magnetizability of a spherical piece of material.

41.02.01.01 Energy Balance

This § is referenced at pages:
[2688]

Energy Balance for a Model Polarizable Molecule

Model the Polarized molecule as two opposite-sign charge distributions and the restoring force between the positive and negative charges of the molecule by a linear (elastic spring-like) restoring force, which is actually of ElectroMagnetic origin. In fact, the stable equilibrium condition of the molecule, corresponding to zero Polarization, is distorted by the external electric field, and the restoring force is linear for small displacements.

Define the system as the Polarized molecule with an induced **Electric Dipole** , that is, the two equal and opposite charge distributions have absolute value $|q|$, and centers of charge separated by a distance $\Delta\ell$.

The change of total energy of the system is given by the total work made on the system: the work done by the external forces plus the work done by the internal forces. The spring models the internal restoring force, of ElectroMagnetic origin. The contribution of the spring can be accounted for either as work done by the internal forces, modeled by the elastic spring or as internal (elastic-like) potential energy of the spring, Φ_{k}.

Consider the work, $\mathrm{d}\mathcal{W}_{\mathrm{E}}$, made by an external agent to build the system, by a quasi-static process. The molecule, when non wave-Polarized, has zero internal (elastic-like) potential energy.

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The first principle of thermodynamics gives:

$$\mathrm{d}U_{\mathrm{EM}} = \mathrm{d}\mathcal{W}_{\mathrm{E}} + \mathrm{d}\mathcal{W}_{\mathrm{I}} \implies \mathrm{d}U_{\mathrm{EM}} - \mathrm{d}\mathcal{W}_{\mathrm{I}} = \mathrm{d}U_{\mathrm{EM}} + \mathrm{d}U_{\mathrm{k}} = \mathrm{d}\mathcal{W}_{\mathrm{E}} = \mathbf{f}\cdot\mathrm{d}\mathbf{r} + \boldsymbol{\gamma}\cdot\mathrm{d}\boldsymbol{\theta} \quad ,$$

where $\mathbf{f}$, $\boldsymbol{\gamma}$ and $\mathrm{d}\mathcal{W}_{\mathrm{E}}$ are the force, torque and work by the external agent to keep the system at equilibrium at any time (a quasi-static process), that is to say the opposite of the Force|Torque acted on the system by the external field; $\mathrm{d}U_{\mathrm{EM}}$ is the ElectroStatic interaction energy between the **Dipole** and the external field.

At equilibrium, with the induced **Dipole** aligned with the external field, every charge is in equilibrium under the effect of the electric field (assumed to be the same at the position of the two charges) and the force of the spring. One has:

$$k\Delta\ell = |q|E \quad ,$$
$$\mathbf{p} \parallel \mathbf{E} \quad ,$$
$$\mathcal{W}_{\mathrm{E}} = \frac{1}{2}k\left(\Delta\ell\right)^2 + q\left(\Phi[\mathbf{r}_+] - \Phi[\mathbf{r}_-]\right) = \frac{1}{2}k\left(\Delta\ell\right)^2 - \mathbf{p}E = \frac{1}{2}\frac{\left(k\Delta\ell\right)^2}{k} - \mathbf{p}E = \frac{1}{2}\frac{\left(qE\right)^2}{k} - \mathbf{p}E = \frac{1}{2}\mathbf{p}E - \mathbf{p}E = -\frac{1}{2}\mathbf{p}E \quad ,$$
$$\mathcal{W}_{\mathrm{E}} = -\frac{1}{2}\mathbf{p}E = U_{\mathrm{k}} + U_{\mathrm{EM}} \quad .$$

The pure **ElectroStatic energy is the same for both a rigid and a Polarizable Dipole** : it is, in fact, a purely positional potential energy and it cannot depend on the way the configuration was obtained.

The internal elastic energy, of ElectroMagnetic origin and modeled as spring, is a part of the internal energy.

In the energy balance, the total energy must be considered, as the **Dipole** is unavoidably Polarized when going into the external field. The addition of the spring, modeling the internal energy, provides the overall energy balance.

The above approach is often called the **virtual work** principle. Therefore in the use of the **virtual work** principle the total energy of the system must be used.

The expression above with k at denominator allow to recover the case of the rigid **Dipole** , $k \to \infty$, where the first term (spring potential energy) is negligible.

41.02.02
Forces on a Linearly Polarizable|Magnetizable Entities in Static External (Applied) Fields

This § is referenced at pages:
[1336, 1336, 1710, 1711, 1864, 1964]

Read § 41.05.

Consider a static situation with the generators of the external (applied) fields outside the region of interest.

In a static situation a Polarized/Magnetized medium placed inside an external field, $\mathbf{E}_0$ or $\mathbf{B}_0$, produced by fixed Charge|Currents located far-away from the region where the bodies are placed, is acted on by a total force per unit volume given by:

$$\frac{\mathrm{d}\mathbf{F}_{\mathrm{E}}}{\mathrm{d}V} = (\mathbf{P}\cdot\mathbf{grad})\mathbf{E}_0 \quad ,$$
$$\frac{\mathrm{d}\mathbf{F}_{\mathrm{M}}}{\mathrm{d}V} = (\mathbf{M}\cdot\mathbf{grad})\mathbf{B}_0 \quad .$$

Note that the fields produced by the bodies do not enter the expression for the total force because they give rise to internal forces with zero sum. However, if the bodies are not rigid, the internal forces may contribute to the deformation of the bodies.

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41.02.03 Examples

41.02.03.01 Microscopic Explanation of Attraction of Linear Media Towards Intense Fields

Any piece of LHI Polarizable material is attracted into a free-space capacitor because of the net force on the induced **Electric Dipoles** . In fact the electric field outside the capacitor Polarized the material and every induced microscopic **Electric Dipole** is attracted towards the regions where the field is most intense.

The same explanation obviously apply to attraction of paramagnetic materials towards a magnet.

41.02.03.02 Diamagnetic Levitation

41.02.03.03 ElectroStatics Induction

41.02.03.04 Van Der Waals Forces

©|J.Franklin, *Classical ElectroMagnetism*, 2017, 2ndEd, ...Ed., WEB - URL.|§ 2.3.3||

Read § 38.10.06.

See J.Franklin, *Classical ElectroMagnetism*, 2017, 2ndEd, ...Ed., WEB - URL. for the averaged **Van Der Waals force** .

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41.03 Laws of ElectroMagnetism in Presence of Matter

This § is referenced at pages:
[1846]

41.03.01 Free|Polarization Charges and Free|Polarization|Magnetization Currents

This § is referenced at pages:
[1829, 1832, 1954]

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|pag. 347||
©|F.N.H.Robinson, *Macroscopic ElectroMagnetism*, ..., ..., ...Ed., .|||

Read § 37.03.02.
Read § 41.11.01.
The following properties are assumed.

- The net charge on the body is zero: $0 = \iiint \rho^{\mathrm{F}} \, \mathrm{d}V$.
- $\mathbf{P}$ is zero outside the body.
- The net current leaving the surface of the body is zero: $0 = \oiint \mathbf{j}^{\mathrm{F}} \cdot \mathrm{d}\mathbf{S}$.
- $\mathbf{M}$ is zero outside the body.

The distinction between free and Polarization charges and among free, Polarization and Magnetization currents has some degree of arbitrariness. That is: different choices may be possible, all of them giving rise to the same macroscopic Electric|Magnetic fields. In fact macroscopic Electric|Magnetic fields are observable physical quantities, only depending on the Charge|Currents distributions, not on the way we choose to describe Charge|Current distributions.

Polarization and Magnetization refer to inaccessible charges and|or currents.

Note that after introducing the Polarization charge and Magnetization current, consistency between **Maxwell Equations** and charge conservation **forces to introduce a Polarization current**, as in section § 38.02.04.

In general, one decides which charges are described as Polarization charges and/or Magnetization currents, normally bound, that is inaccessible charges|currents, and all the remaining charges|currents must then be treated as free charges|currents. One has to be careful to count all charges|currents one and only one time. The definition of free and Polarization currents follows consequently.

Consider the following examples.

- Consider for instance a ionic crystal, away from external electric fields.
  It can be described as zero Polarization.
  However, it can be also described in terms of a Polarization vector, that is a density of **Electric Dipole** moment, plus free charges. In fact there can be different groupings of one positive ion with one negative ion to produce an Electric Dipole. Different groupings give rise to different Polarization vectors. All groupings leave some ungrouped ions at the surface, that is a surface distribution of free charges. The volume density is zero for these uniform Polarization. The surface Polarization charge density is calculated from the Polarization and it is at any point equal and opposite to the surface free charge density.
- Consider a solid metallic conductor.
  One possibility is to define all the charges as Polarization charges, with the **free** conduction electrons considered as bound electrons with a vanishingly small bound strength.

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A second possibility is to describe the **free** conduction electrons as free charge. In this case, however, it is appropriate to include in the free charges also the positive charge of the reticular ions necessary to make the overall free charges a neutral one; that is, one positive charge from nuclei for each free electron. In this way also the remaining charges, that is the bound electrons plus the corresponding positive charges, constitute a neutral system, the Electric Dipole moment is independent of the origin. The free current is thus given by the electron drift velocity plus the positive ion charge drift velocity, often equal to zero.

- A similar discussion might apply to an electrolytic solutions, with the positive ion charge drift velocity different from zero.

A similar type of arbitrariness exists for the Magnetization currents, normally bound, that is inaccessible currents, whose charges are not accounted for as free charges. One might, for instance, describe the currents produced by the orbital electrons not only as a Magnetization currents but also as a free currents, if the electrons are considered, and accounted for, as free charges.

41.03.01.01 Model Dependence

In the end it is worth noting that the separation into free/bound Charge|Current ultimately requires a model for the medium, and hence the introduction of $\mathbf{D}$ and $\mathbf{H}$ is model dependent. Accordingly, the validity Maxwell equations written in terms of $\mathbf{D}$ and $\mathbf{H}$ is model dependent, it relies on assumptions about the properties of the medium and requires supplementary information to give it a meaning.

Polarization and Magnetization refer to **unaccessible** charges and|or currents.

41.03.02 Polarization and Magnetization

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|||

It can be shown, as a consistency cross-check, that the identities below are valid. For the sake of simplicity, assume that:

$$\rho_{\mathrm{S}}^{\mathrm{P}} = 0 \qquad \mathbf{j}_{\mathrm{S}}^{\mathrm{M}} = 0 \qquad \text{static conditions} \quad . \tag{41.03.01}$$

With the Polarization defined as it was done, the **Electric Dipole** moment of a volume Polarization distribution is given in terms of the divergence of $\mathbf{P}$. That is to say the **Electric Dipole** moment induced by a volume Polarization charge is, for any volume totally enclosed inside the material medium :

$$\mathbf{p} = \iiint \left(\mathbf{r}\rho^{\mathrm{P}}\right) \, \mathrm{d}V = -\iiint \left(\mathbf{r}\mathbf{div}\mathbf{P}\right) \, \mathrm{d}V \quad .$$

With the Magnetization defined as it was done, the **Magnetic Dipole** moment of a volume Magnetization distribution is given in terms of the rotor of $\mathbf{M}$. That is to say the **Magnetic Dipole** moment induced by a volume Magnetization current is, for any volume totally enclosed inside the material medium ::

$$\mathbf{m} = \frac{1}{2}\iiint \left(\mathbf{r}\wedge\mathbf{j}^{\mathrm{M}}\right) \, \mathrm{d}V = \frac{1}{2}\iiint \left(\mathbf{r}\wedge\mathbf{rot}\mathbf{M}\right) \, \mathrm{d}V \quad .$$

CREATIVE-PRINCIPLES

41.03.03
Conducting Media

It is often the case that the relation between the free current and the electric field can be assumed as a linear one:

$$\boxed{\mathbf{j}^{\mathrm{F}} = \boldsymbol{\sigma}\mathbf{E}} \quad ,$$

where the **conductivity** tensor does not depend on the electric field. This is the expression of Ohm law.

Note that the free current must be used here.

41.03.03.01 Equilibrium Inside a Conductor

Note that in complex situations there might be other forces, than the force caused by the electric field, affecting the mechanics of free charges inside a conductor.

The magnetic field is a second-order relativistic effect. Therefore the magnetic fields associated to the electric fields inside the conductor normally have a negligible effect. However, other, maybe external, magnetic fields my exist of arbitrary strength. Moreover, in non-homogeneous materials effects due to the non-uniformity may arise, for instance diffusive forces. Finally inertial forces may exist, $\mathbf{F}_{\mathrm{I}}$, in conductors in motion.

Therefore in a homogeneous material one can generally write:

$$\boxed{\mathbf{j} = \sigma\left(\mathbf{E} + \mathbf{v}\wedge\mathbf{B} + \mathbf{F}_{\mathrm{I}}\right)} \quad .$$

41.03.04
Maxwell Equations in Presence of Matter

The **Maxwell Equations** in presence of matter, written in differential form and applicable in all the cases when the fields and Charge|Current producing the ElectroMagnetic fields are, from the mathematical point of view, sufficiently well-behaved functions, read:

$$\boxed{\mathbf{div}\mathbf{D} = \rho^{\mathrm{F}}} \quad , \quad , \qquad \rightarrow 1953$$

$$\boxed{\mathbf{div}\mathbf{B} = 0} \quad , \quad , \qquad \rightarrow 1953$$

$$\boxed{\mathbf{rot}\mathbf{E} = -\partial_t\mathbf{B}} \quad , \quad , \qquad \rightarrow 1953$$

$$\boxed{\mathbf{rot}\mathbf{H} = \mathbf{j}^{\mathrm{F}} + \partial_t\mathbf{D}} \quad . \qquad \rightarrow 1953$$

The **Maxwell Equations** must be complemented with the charge conservation law, equation (36.08.09), which is, however, implied by **Maxwell Equations** :

$$\boxed{\mathbf{div}\mathbf{j}^{\mathrm{F}} + \partial_t\rho^{\mathrm{F}} = 0} \quad . \qquad (41.03.02)$$

Moreover, by definition of Polarization Charge|Current and Magnetization currents, one has always the identities:

$$\boxed{\mathbf{div}\mathbf{j}^{\mathrm{P}} + \partial_t\rho^{\mathrm{P}} = 0} \quad ,$$

$$\boxed{\mathbf{div}\mathbf{j}^{\mathrm{M}} = 0} \qquad \text{by definition of } \mathbf{j}^{\mathrm{M}} \text{ as a rotor} \quad .$$

The above relations make clear that all charges must be partitioned into the two sets of free/Polarization charges and can never change it. Moreover, Magnetization currents ar not linked to any Magnetization charge.

Note that when writing Maxwell equations, ((41.03.04), (41.03.04), (41.03.04), (41.03.04)), in terms of $\mathbf{D}|\mathbf{H}$ and free Charge|Current (as in equations (41.03.04), (41.03.04), (41.03.04), (41.03.04)) the charge continuity equation is written in terms for the free Charge|Current and not in terms of the total Charge|Current.

If the media are Linear and Homogeneous, and only if they are, Poisson equation for the **scalar electric potential** and **vector magnetic potentials** are the same as in vacuum with replacement of vacuum **Electric-Permittivity** |**Magnetic-Permeability** with those in the medium: $\varepsilon_0 \rightarrow \varepsilon_{\mathrm{R}}\varepsilon_0$ and $\mu_0 \rightarrow \mu_{\mathrm{R}}\mu_0$. However, in general, in presence of boundary conditions, this does not allow to conclude that the fields in matter are just the re-scaled fields one would find in vacuum. See the neat example § 38.10.10.

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41.03.05
The Auxiliary Electric|Magnetic Fields Vectors

The auxiliary Electric|Magnetic fields quantities D|H are macroscopic fields.

In a more complete treatment, see references, it can be shown that the common definitions for **D**|**H**, equations (38.05.01) and (39.05.01), are just first order approximations, even if the neglected terms are often negligible. It is shown that **D**|**H** also depend on quadrupole and higher order moments, in addition to dipolar moments.

41.03.05.01 Practical Use

The auxiliary vector **D** is of little practical usage because what one normally controls is the macroscopic potential difference, which is linked to the electric field **E**.

The auxiliary vector **H** is more useful because what one normally controls is the macroscopic current, which is linked to the vector field **H**.

41.03.05.02 Case of Magnetic Field: The Two Magnetic Field Vectors

This § is referenced at pages:
[1693]

Several facts support the idea of **B** begin the fundamental magnetic field, justifying naming it **the magnetic field**.

- The fields **E** and **B** appear in the Lorentz force.
- The fields **E** and **B** appear in the two homogeneous Maxwell equations, those defining the structural properties of the ElectroMagnetic fields, with no reference to Charge|Current; as a consequence, **E** and **B** are related to the **ElectroMagnetic potentials** and to the ElectroMagnetic field tensor.
- The appearance of **D** and **H** in Maxwell equations is in the presence of **free** Charge|Current; however, it is known that the separation of Charge|Current into free and bound Charge|Current can be ambiguous, at least in some circumstances; see for instance § 41.03.01. So, for instance, different legitimate choices for what is free/bound may lead to different **D**|**H** fields.

Experimental Results

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|8.2||

Read also § 36.20.82.

Relativistic Covariance

©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|1.1||

The four **Maxwell Equations** can be divided into two sets: the homogeneous equations (involving **E** and **B**) and the non-homogeneous ones (involving **D** and **H** as well as the charge and current densities).

If a **Lorentz Transformation** is carried on, and charge and current densities transform as a four-vector, the two sets of equations keep the same form and do not mix provided the two pairs $\{\mathbf{E}, \mathbf{B}\}$ and $\{\mathbf{D}, \mathbf{H}\}$ form a anti-symmetrical four-tensor of rank two (read § 61).

Since the non-homogeneous set contains Charge|Current, which represent the influence of matter, one is forced to attribute the corresponding pair $\{\mathbf{D}, \mathbf{H}\}$ to the influence of matter.

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41.03.06
In LHI Media - Electric-Permittivity Versus Magnetic-Permeability

The **Electric-Permittivity** | **Magnetic-Permeability** of LHI media show a basic difference.

Electric LHI Materials

It can be shown that the **Electric-Permittivity** must be, for static fields:

$$\boxed{\varepsilon_{\mathrm{R}} \geq 1} \quad .$$

This implies:

$$\boxed{\chi_{\mathrm{E}} \geq 0} \quad .$$

Note that $\mathbf{P}$ and $\mathbf{E}$ are always parallel as $\chi_{\mathrm{E}} \geq 0$.

Note that $\mathbf{D}$ and $\mathbf{E}$ are always parallel as $\varepsilon_{\mathrm{R}} \geq 1$.

Magnetic LHI Materials

It can be shown that the **Magnetic-Permeability** must be, for static fields:

$$\boxed{\mu_{\mathrm{R}} \geq 0} \quad .$$

This implies, with equation (39.06.05):

$$\boxed{\chi_{\mathrm{M}}^{*} \geq -1 \quad \Leftrightarrow \quad \chi_{\mathrm{M}} \leq +1} \quad .$$

Equation (39.06.05) implies that χ_{M} and χ_{M}^{*} always have the same sign:

$$\boxed{\chi_{\mathrm{M}}\chi_{\mathrm{M}}^{*} > 0} \quad .$$

A diamagnetic material has thus:

$$\boxed{-1 \leq \chi_{\mathrm{M}}^{*} \leq 0 \quad \text{and} \quad -\infty \leq \chi_{\mathrm{M}} \leq 0 \quad \mathbf{M} \text{ anti-parallel to both } \mathbf{H} \text{ and } \mathbf{B}, \text{ diamagnetic}} \quad .$$

A paramagnetic material has thus:

$$\boxed{0 \leq \chi_{\mathrm{M}}^{*} \leq +\infty \quad \text{and} \quad 0 \leq \chi_{\mathrm{M}} \leq +1 \quad \mathbf{M} \text{ parallel to both } \mathbf{H} \text{ and } \mathbf{B}, \text{ paramagnetic}} \quad .$$

Note that $\mathbf{H}$ and $\mathbf{B}$ are always parallel as $\mu_{\mathrm{R}} \geq 0$:

$$\boxed{\mathbf{H}\cdot\mathbf{B} \geq 0} \quad .$$

41.03.07
Comparison Between the Equations of ElectroStatics and MagnetoStatics

This § is referenced at pages:
[1734, 1734, 1908, 1908]

41.03.07.01 Rigid Polarization|Magnetization

A genreric material may have a rigid, that is independent of external (applied) fields, Polarization and|or Magnetization (ferro-electric materials and ferro-magnetic materials).

In this case the three electric vectors, ($\mathbf{E}$, $\mathbf{P}$ and $\mathbf{D}$), and|or the three magnetic vectors, ($\mathbf{B}$, $\mathbf{M}$ and $\mathbf{H}$), need not be parallel, and are not parallel in general, like in ferro-electric materials and ferro-magnetic materials.

Let us denote by $\mathbf{F}_{\mathrm{R}} \rightarrow$ rigid field a rigid field, that is independent of external (applied) fields, and by $\mathbf{F}_{\mathrm{I}} \rightarrow$ induced field and induced field.

We can write:

$$\mathbf{P} = \mathbf{P}_{\mathrm{I}} + \mathbf{P}_{\mathrm{R}} \qquad \mathbf{M} = \mathbf{M}_{\mathrm{I}} + \mathbf{M}_{\mathrm{R}} \quad ,$$

$$\mathbf{P}_{\mathrm{I}} = (\varepsilon_0 - 1)\,\varepsilon_0\mathbf{E} \equiv \chi_{\mathrm{E}}\varepsilon_0\mathbf{E} \qquad \mathbf{M}_{\mathrm{I}} = \left(1 - \frac{1}{\mu_{\mathrm{R}}}\right)\frac{\mathbf{B}}{\mu_0} \equiv \chi_{\mathrm{M}}\frac{\mathbf{B}}{\mu_0} \quad ,$$

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_{\mathrm{R}}\varepsilon_0\mathbf{E} + \mathbf{P}_{\mathrm{R}} \qquad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} = \frac{\mathbf{B}}{\mu_{\mathrm{R}}\mu_0} - \mathbf{M}_{\mathrm{R}} \quad .$$

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41.03.07.02 ElectroStatics and MagnetoStatics Without Free Charge|Current

The equations of ElectroStatics and MagnetoStatics in absence of free Charge|Current show a formal correspondence, to within the dimensional constants ε_0 and μ_0. In order to simplify the correspondence getting rid of the irrelevant dimensional constants, one might consider the re-scaled ElectroMagnetic fields:

$$\boxed{\mathbf{E}' \equiv \varepsilon_0 \mathbf{E} \qquad \mathbf{B}' \equiv \mathbf{B}/\mu_0} \quad .$$

The equations of ElectroStatics and MagnetoStatics in absence of free charges and currents read:

$$\boxed{\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \equiv \mathbf{E}' + \mathbf{P}} \implies \varepsilon_0 \mathbf{E} \equiv \mathbf{E}' = \mathbf{D} - \mathbf{P} \quad ,$$

$$\boxed{\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M} \equiv \mathbf{B}' - \mathbf{M}} \implies \mathbf{B}/\mu_0 \equiv \mathbf{B}' = \mathbf{H} + \mathbf{M} \quad ,$$

$$\boxed{\mathbf{rot}\mathbf{E}' = 0} \implies \mathbf{rot}\mathbf{D} = \mathbf{rot}\mathbf{P} \quad , \qquad \rightarrow 1957$$

$$\boxed{\mathbf{rot}\mathbf{H} = 0} \implies \mathbf{rot}\mathbf{B}' = \mathbf{rot}\mathbf{M} \quad , \qquad \rightarrow 1956$$

$$\mathbf{div}\mathbf{D} = 0 \implies \boxed{\mathbf{div}\mathbf{E}' = -\mathbf{div}\mathbf{P}} \quad ,$$

$$\mathbf{div}\mathbf{B}' = 0 \implies \boxed{\mathbf{div}\mathbf{H} = -\mathbf{div}\mathbf{M}} \quad . \qquad \rightarrow 1956\ 1957$$

These equations can be applied at any point except where there are discontinuities in the physical properties of the media and|or discontinuities of the fields.

The formal correspondence is obvious:

$$\boxed{\mathbf{E}' \quad \Leftrightarrow \quad \mathbf{H}} \quad , \qquad \rightarrow 1956$$

$$\boxed{\mathbf{P} \quad \Leftrightarrow \quad \mathbf{M}} \quad ,$$

$$\boxed{\mathbf{D} \quad \Leftrightarrow \quad \mathbf{B}'} \quad .$$

Thanks to the formal parallel between $\mathbf{E}$ and $\mathbf{H}$ the standard definitions of **Electric|Magnetic susceptibility** and **Magnetic-Permeability |Electric-Permittivity** are formally symmetrical even if from the physical point of view the analogy is between the basic fields $\mathbf{E}$ and $\mathbf{B}$.

41.03.07.03 Fictitious Magnetic Charge

The formal parallelism can be pushed forward from equation (41.03.07.02), equation (41.03.07.02) and equations (41.03.07.02). The concept of magnetic charge (source of $\mathbf{H}$) can be formally introduced in parallel with the concept of (total) electric charge (source of $\mathbf{E}$):

$$\boxed{\tilde{\rho} = -\mathbf{div}\mathbf{M}} \quad ,$$

$$\boxed{\tilde{\rho}_S = \mathbf{M}\cdot\mathbf{n}} \quad .$$

This concept is useful, for instance, in order to understand the behavior of field lines in permanent magnets.

One should keep in mind that the magnetic charge is just a fictitious and formal entity. As far as we now today real magnetic charges do not exist.

When the concept of magnetic charge is introduced the concept of de-Magnetizing $\mathbf{H}$ field becomes obvious and parallel to the electric case.

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41.03.07.04 Generators of the Auxiliary Fields in ElectroStatics and MagnetoStatics

Note that equations (41.03.07.02), (41.03.07.02) show clearly that the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$ do not depend only on the free charges and currents but, in general, they also depend on the Polarization and Magnetization. In fact to determine any vector field, in general, the knowledge of both divergence and rotor is required and equations (41.03.07.02), (41.03.07.02) show indeed the dependence of $\mathbf{H}$ on the Magnetization and the dependence of $\mathbf{D}$ on the Polarization.

However, it is possible that some problems with a high degree of symmetry allow their solution by using only one of the two equations: either divergence or rotor equation, in differential form; either flux or circulation equation, in integral form.

41.03.08 ElectroMagnetic Fields Inside a Cavity in a Material Medium

This § is referenced at pages:
[1868, 1868, 1915, 1915]

Note that carving out a cavity in a Polarized/Magnetized medium is equivalent to superimposing an object of the same shape of the cavity but with opposite Polarization|Magnetization.

41.03.08.01 Electric Case

The electric field inside an ideal cavity in a Polarizable medium, $\mathbf{E}_c$, can be expressed in terms of the average macroscopic field in matter, $\mathbf{E}$, as

$$\mathbf{E}_c = \mathbf{E} + \gamma\frac{\mathbf{P}}{\varepsilon_0} \qquad \mathbf{D}_c = \mathbf{D} + \gamma_D\mathbf{P} \quad ,$$

where γ is a factor depending on the shape of the cavity:

long thin cylinder (pipe or needle) parallel to $\mathbf{E}$	$\gamma = 0$	$\gamma_D = -1$ for $\mathbf{D}$
sphere	$\gamma = 1/3$	$\gamma_D = -2/3$ for $\mathbf{D}$
short large cylinder (disk or wafer) perpendicular to $\mathbf{E}$	$\gamma = 1$	$\gamma_D = 0$ for $\mathbf{D}$

.

It can be easily shown that:

$$\gamma_D = \gamma - 1 \quad .$$

41.03.08.02 Magnetic Case

The magnetic field inside an ideal cavity in a Magnetizable medium, $\mathbf{B}_c$, can be expressed in terms of the average macroscopic field in matter, $\mathbf{B}$, as

$$\mathbf{B}_c = \mathbf{B} + \gamma\mu_0\mathbf{M} \qquad \mathbf{H}_c = \mathbf{H} + \gamma_H\mathbf{M} \quad ,$$

where γ is a factor depending on the shape of the cavity:

long thin cylinder (pipe or needle) parallel to $\mathbf{B}$	$\gamma = -1$	$\gamma_H = 0$ for $\mathbf{H}$
sphere	$\gamma = -2/3$	$\gamma_H = 1/3$ for $\mathbf{H}$
short large cylinder (disk or wafer) perpendicular to $\mathbf{B}$	$\gamma = 0$	$\gamma_h = 1$ for $\mathbf{H}$

.

It can be easily shown that:

$$\gamma_D = \gamma + 1 \quad .$$

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41.03.08.03 Analogies and Differences Between the Electric and the Magnetic Case

For a Polarized sphere the electric field generated by the matter inside the sphere is always anti-parallel to the Polarization:

$$\boxed{\mathbf{E}_{\mathrm{I}} = -\frac{\mathbf{P}}{3\varepsilon_0}} \quad .$$

For a Magnetized sphere the magnetic field generated by the matter inside the sphere is always parallel to the Magnetization:

$$\boxed{\mathbf{B}_{\mathrm{I}} = +\frac{2\mu_0\mathbf{M}}{3}} \quad .$$

For a sphere of an LHI material in an external uniform Electric|Magnetic field the field inside the sphere is given by:

$$\boxed{\mathbf{E} = \frac{3}{\varepsilon_{\mathrm{R}}+2}\mathbf{E}_0 \qquad \mathbf{B} = \frac{3\mu_{\mathrm{R}}}{\mu_{\mathrm{R}}+2}\mathbf{B}_0} \quad . \tag{41.03.03}$$

→ 1958

Note the following two limiting cases, from equations (41.03.03).

A perfect conductor can be considered as a electric material with infinite polarisability: equations (41.03.03) implies $\mathbf{E} = 0$, when $\chi_{\mathrm{E}} \to +\infty$ ($\varepsilon_{\mathrm{R}} \to +\infty$). In a perfect conductor free charges move in such a way to cancel the internal electric field.

A perfect diamagnet can be considered as a magnetic material with zero magnetisability: equations (41.03.03) implies $\mathbf{B} = 0$, when $\chi_{\mathrm{M}} \to -\infty$ ($\mu_{\mathrm{R}} \to 0$). In a perfect diamagnet free currents start in such a way to cancel the internal magnetic field.

41.03.09 Properties of LH Media

The following properties only apply to linear and homogeneous media. They do not apply to media with other constitutive relations, such as media with rigid Polarization|Magnetization.

Note that isotropy is not required but we will only consider, for simplicity, the case of an isotropic medium, as the generalization is straightforward.

41.03.09.01 Volume Charge|Current Density in LH Media

In electric LH media one can easly show that:

$$\boxed{\rho^{\mathrm{F}} \propto \rho^{\mathrm{P}} \propto \rho} \quad .$$

In magnetic LH media, in static conditions (this is an essential hypotesis), one can easly show that:

$$\text{in static conditions: } \partial_t = 0 \qquad \Longrightarrow \quad ,$$

$$\boxed{\mathbf{j}^{\mathrm{F}} \propto \mathbf{j}^{\mathrm{M}} \propto \mathbf{j} \qquad \text{in static conditions, such that } \mathbf{j}^{\mathrm{P}} = \partial_t\mathbf{P} = 0 \text{ and } \partial_t\mathbf{D} = 0} \quad .$$

In conducting LH media (homogeneous perfect **perfect Ohmic media**), for harmonic dependence with frequency ω, one can easily show that:

$$\boxed{\mathbf{j}^{\mathrm{P}} \propto \omega\mathbf{E}} \quad .$$

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41.03.10
Non-Isotropic Media

It can be shown that the **Electric|Magnetic susceptibility** tensors are symmetrical. They can be diagonalized but note that a diagonal form of the susceptibility tensors in general does not imply at all that the $\mathbf{E}|\mathbf{B}$ fields are parallel to the $\mathbf{P}|\mathbf{M}$ fields, unless the coefficients are all identical. Actually when starting from non parallel fields a change of bases which just diagonalizes the matrix cannot make the two vector fields parallel.

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41.04 Constitutive Relations of Electric|Magnetic Materials

©|A.Moliton|*Basic Electromagnetism and Materials, § 4 |Many Phenomenological Detalis|*

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41.04.01 Rigidly Polarized/Magnetized Bodies

Rigid constitutive relations: $\mathbf{P}$ and|or $\mathbf{M}$ are fixed, independent of the external (applied) fields (permanent Polarization|Magnetization: ferro-electric and|or ferro-magnetic materials).

41.04.02 Time-Invariant Systems

Even if not often stated, most constitutive relations concern time-invariant systems, read § 19.06. This clearly excludes such phenomena as hysteresis and ageing.

41.04.03 Linearity

Linearity does not require proportionality of the fields at any point is space or time but it is sufficient to have proportionality of the space and time Fourier components, that is to say the result is given by a convolution, for a generic scalar field:

$$W[\mathbf{x},t]=\int\iiint \chi[\mathbf{x}',t']V[\mathbf{x}-\mathbf{x}',t-t']\,\mathrm{d}t'\,\mathrm{d}\mathbf{x}' \quad ,$$

where $\chi[\mathbf{x}',t']$ is non zero only near some region around $\mathbf{x}'=0$ and $t'=0$.

41.04.03.01 Time Dispersion

This § is referenced at pages:
[Never referenced.]

The value of the field W at time t may depend on the values of the field V at all the times before t, $t'<t$. Causality implies:

$$\chi[\mathbf{x}',t'<0]=0 \quad .$$

41.04.03.02 Space Dispersion

This § is referenced at pages:
[Never referenced.]

The value of the field W at point x may depend on the values of the field V at all the points around x, $|\mathbf{x}'-\mathbf{x}|<\Delta$. Spatial dispersion is normally negligible in ElectroMagnetic phenomena, except for special situations.

41.04.04
Hysteresis

There are materials in which the Polarization|Magnetization depends not only on the actual Electric|Magnetic field intensity but on the sequence of preceding states as well: this phenomenon is called hysteresis.

41.04.05
General Constitutive Relations

This § is referenced at pages:
[1216]

41.04.05.01 Macroscopic ElectroMagnetism

In general:

$$\boxed{\mathbf{P}=\mathbf{P}\{\mathbf{E},\mathbf{B}\}} \qquad \boxed{\mathbf{M}=\mathbf{M}\{\mathbf{B},\mathbf{E}\}} \qquad \boxed{\mathbf{j}=\mathbf{j}\{\mathbf{E},\mathbf{B}\} \quad \text{for conducting materials}} \quad , \tag{41.04.01}$$

→ 1961

The rotory brackets mean that the connections are not necessarily simple and may be nonlinear, non local in space or non local in time (for instance they may depend on past history: hysteresis). Read § 21.04.

Therefore, To be generally correct, equations (41.04.01), should be understood as holding for the Fourier transforms in space and time of the field quantities, because the basic linear connections can be non-local.

Read § 21.04.

41.04.05.02 Microscopic ElectroMagnetism

Including **electro-magnetic** and **magneto-electric** effects, equation (38.02.01.01), (39.02.01.01), the relations among local ElectroMagnetic fields and induced dipoles become:

$$\begin{pmatrix}\mathbf{p}/\varepsilon_0\\ \mathbf{m}\mu_0\end{pmatrix}=\begin{pmatrix}\alpha_{\mathtt{ee}} & \alpha_{\mathtt{em}}\\ \alpha_{\mathtt{me}} & \alpha_{\mathtt{mm}}\end{pmatrix}\begin{pmatrix}\mathbf{E}_{\mathtt{LOC}}\\ \mathbf{B}_{\mathtt{LOC}}\end{pmatrix}. \tag{41.04.02}$$

41.04.06
Molecules With a Permanent Electric|Magnetic Dipole Moment - Langevin Equation

This § is referenced at pages:
[1841, 1855, 1883, 1899, 1977, 1977, 1979, 2690]

©|Am.J.Phys, , ..., ..., ...Ed., ..., Vol. 45, No. 10, October 1977 922|Polarization of a polar gas by external and internal ...

The **Clausius-Mossotti equation** (sections § 38.09 and § 39.09) gives the relation between the susceptibility and **Polarisability** |Magnetizability based on the relation between the local field (acting on the single molecule) and the average macroscopic fields. It applies to substances which are isotropic on the macroscopic scale.

The **Langevin equation** gives the susceptibility of substances whose molecules have a permanent **Electric Dipole** | **Magnetic Dipole** moment. If the nearest neighbor interaction energies are small, a material made of molecules with a permanent Electric|Magnetic moments will normally have the **Dipoles** oriented randomly in the absence of an Electric|Magnetic field.

In a very strong field all the **Dipoles** will align with the Electric|Magnetic field, giving a maximum (saturation) Polarization/Magnetization. At smaller field strengths the thermal randomizing will oppose the alignment to some extent. The average Polarization/Magnetization may be found from statistical physics.

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The **Polarisability** |Magnetizability by orientation follows the general definition in equations (38.02.01.01), (39.02.01.01). Clearly, in this case, the induced **Polarisability** |Magnetizability will be parallel to the external local field. In general one should use the local field, not the average macroscopic field, except for very low-density materials. However, for the sake of simplicity, let us consider, for simplicity, the case of very low-density materials: the generalization to dense LI materials is given by **Clausius-Mossotti equations** .

The energy of a single **Dipole** in an external field **E** / **B** is $\varepsilon[\theta,\phi] = -\,\mathbf{p}\cdot\mathbf{E}$ / $\varepsilon[\theta,\phi] = -\,\mathbf{m}\cdot\mathbf{B}$ which ranges from $-\mathbf{p}E$ / $-\mathbf{m}B$ to $+\mathbf{p}E$ / $+\mathbf{m}B$, depending on the orientation of the **Dipole** with respect to the field.

Statistical physics teaches that for any system in equilibrium at absolute temperature T, the probability for having energy $\varepsilon[\theta,\phi]$ is proportional to the Boltzmann factor (Boltzmann statistics):

$$\exp\left[-\varepsilon/\left(k_{\mathrm{B}}T\right)\right] \qquad \text{in terms of the fundamental constant } k_{\mathrm{B}} = 1.38{\cdot}10^{-23}\ \mathrm{J/K} \quad .$$

Consider an isotropic material whose molecules have a permanent **Dipole** moment.

The average energy, for a generic **Dipole** of moment g in an external field **W**, directed along the z axis, is then:

$$\langle\varepsilon\rangle = \frac{\iint \varepsilon \exp\left[-\varepsilon/\left(k_{\mathrm{B}}T\right)\right]\ \mathrm{d}\varepsilon}{\iint \exp\left[-\varepsilon/\left(k_{\mathrm{B}}T\right)\right]\ \mathrm{d}\varepsilon} \qquad \varepsilon = \varepsilon[\theta,\phi] \quad .$$

The average **Dipole** moment along the z axis for a generic **Dipole** of moment g in an external field **W**, directed along the z axis, is then found by integrating on the solid angle:

$$\langle \mathrm{g}_z\rangle = \langle \mathrm{g}\cos\left[\theta\right]\rangle = \frac{\iint \mathrm{g}\cos\left[\theta\right]\exp\left[\mathrm{g}W\cos\left[\theta\right]/\left(k_{\mathrm{B}}T\right)\right]\ \mathrm{d}\phi\ \mathrm{d}(\cos\left[\theta\right])}{\iint \exp\left[\mathrm{g}W\cos\left[\theta\right]/\left(k_{\mathrm{B}}T\right)\right]\ \mathrm{d}\phi\ \mathrm{d}(\cos\left[\theta\right])} \quad .$$

Langevin equation follows:

$$\langle \mathbf{p}_z\rangle = \mathbf{p}\left(\mathrm{cth}\left[\frac{\mathbf{p}E}{k_{\mathrm{B}}T}\right] - \frac{k_{\mathrm{B}}T}{\mathbf{p}E}\right) \qquad \langle \mathbf{m}_z\rangle = \mathbf{m}\left(\mathrm{cth}\left[\frac{\mathbf{m}B}{k_{\mathrm{B}}T}\right] - \frac{k_{\mathrm{B}}T}{\mathbf{m}B}\right) \quad .$$

It is common usage to introduce the dimensionless variable x:

$$\langle \mathrm{g}_z\rangle = \mathrm{g}\left(\mathrm{cth}\left[x\right] - \frac{1}{x}\right) \equiv \mathrm{g}L[x] \qquad \text{where } L \text{ is the so-called } \textbf{Langevin Function} \qquad x = \left\{\frac{\mathbf{p}E}{k_{\mathrm{B}}T}, \frac{\mathbf{m}B}{k_{\mathrm{B}}T}\right\} \quad .$$

At large fields and|or at low temperatures, all the molecules are lined-up and the material has thus a non-linear response. Ordinarily $k_{\mathrm{B}}T$ is much greater than $\mathbf{p}E$ / $\mathbf{m}B$. In this **regime the material has a linear response**, and the first-order development of the **Langevin Function** gives:

$$\langle \mathrm{g}_z\rangle \simeq \frac{\mathrm{g}x}{3} \quad .$$

At low fields (and low-density) the **Polarisability** |Magnetizability follow:

$$\alpha^{(\mathrm{e})} = \frac{\mathbf{p}^2}{3\varepsilon_0 k_{\mathrm{B}}T} \qquad \alpha^{(\mathrm{m})} = \frac{\mu_0\mathbf{m}^2}{3k_{\mathrm{B}}T} \quad .$$

The temperature dependence $1/T$ gives the so-called **Curie law**. Clearly **Polarization|Magnetization by orientation has a strong temperature dependence**, at variance with the Polarization|Magnetization by deformation.

In general the **Polarisability** |Magnetizability is the sum of the one due to deformation (always present) plus the one due to the permanent moment.

At high fields all **Dipoles** are aligned and the Polarization/Magnetization **saturates**.

In dense materials the distinction between the average field and the local field must be accounted for, as for **Clausius-Mossotti equation** .

The total energy, for N **Dipoles** , is given by:

$$\mathcal{U} = N\langle\varepsilon\rangle = -N\langle \mathrm{g}_z\rangle W \implies \begin{cases} \dfrac{\Delta\mathcal{U}}{\Delta V} = -\langle P_z\rangle E \\ \dfrac{\Delta\mathcal{U}}{\Delta V} = -\langle M_z\rangle B \end{cases} \quad , \qquad .$$

→ 2721

MATERIAL IN PROGRESS

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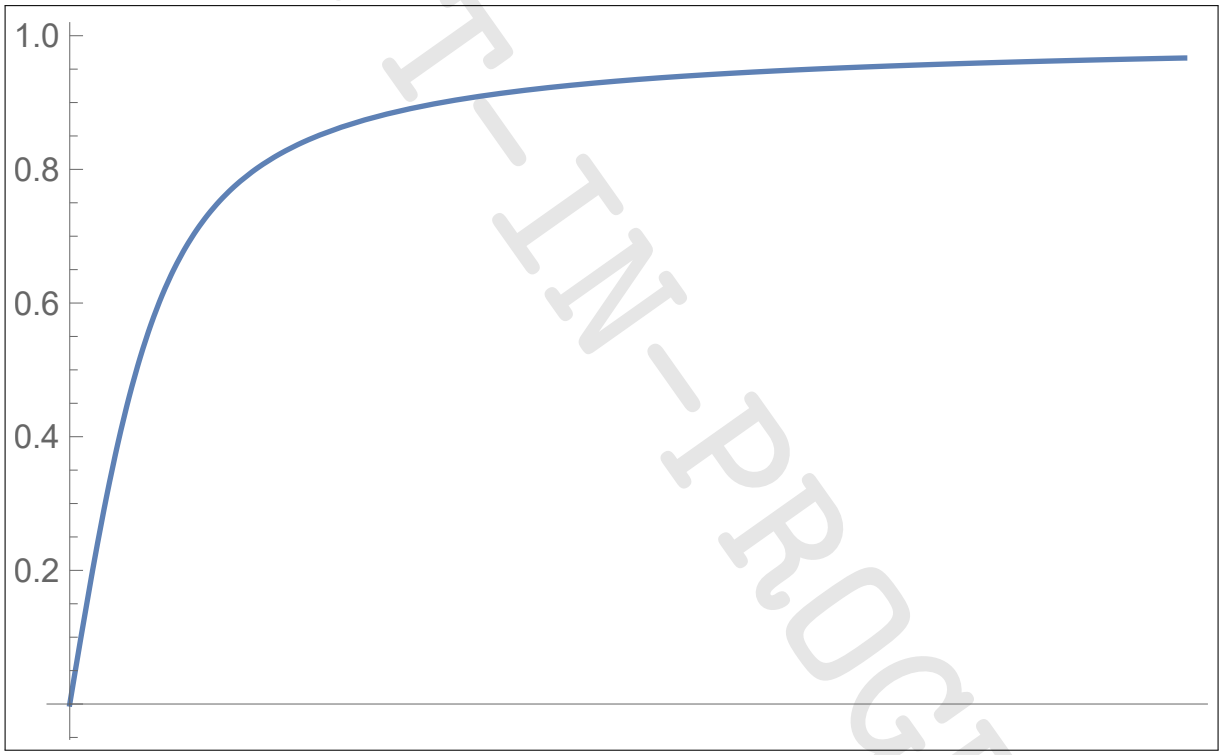

Figure 41.12: Langevin Function

41.05 Charges and Currents in External Static ElectroMagnetic Fields

This § is referenced at pages:
[1856, 1900, 1949]

Read § 41.02.02.

When one considers the effects of external ElectroMagnetic fields on either a charge or a current distribution as a whole one must neglect the effects of the ElectroMagnetic fields produced by the Charge|Current distribution on the same Charge|Current distribution itself. In fact the fields produced by the Charge|Current distribution itself have no effect on the global motion of the system, according to the cardinal equation of mechanics, as the resultant of the internal forces and the resultant of the internal torques are zero. These fields may actually have an effect on the internal dynamics of the Charge|Current distribution, unless the system is a rigid one, but they do not influence the linear momentum nor the angular momentum of the Charge|Current distribution. Their effect, in case of a non-rigid system, may cause deformation of the Charge|Current distribution.

Note also that a static electric field in a region without charges has the same mathematical properties of a static magnetic field in a region without currents. In fact both are fields with zero divergence and zero rotor.

41.05.01 Static Charges in an External ElectroStatic Field

This § is referenced at pages:
[1710]

©|F.Wegner, *Classical Electrodynamics*, 2003, ..., ...Ed., WEB - URL.|§ 4||
©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|§ 3|*Definitive*|

41.05.01.01 Force|Torque on a Electric Dipole in an External ElectroStatic Field

This § is referenced at pages:
[1706, 1707, 1710]

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|1.8||

Consider a localized stationary charge distribution in an external ElectroStatic field. The external ElectroStatic field is supposed to be stationary, that is the generators do what is required to keep it constant, whatever happens.

By direct computation the total force, due to the external electric field $\mathbf{E}[\mathbf{r}]$, acting on a system of two point charges of equal magnitude and opposite sign, separated by $\Delta\mathbf{r}$, and carrying a fixed **Electric Dipole** moment equal to $\mathbf{p}$, is found to be:

$$F_k[\mathbf{r}] = p_s \frac{\partial E_k}{\partial x_s}[\mathbf{r}] \quad \Leftrightarrow \quad \mathbf{F} = (\mathbf{p}\cdot\mathbf{grad})\mathbf{E} \quad . \tag{41.05.01}$$

→ 1706 1965

It is assumed that the two charges are in equilibrium, separated by a fixed distance $\Delta r \equiv |\Delta\mathbf{r}|$, due to the action of some other force, internal to the pair of charges.

Note that the vector expression is not ambiguous if one takes literally the dot of the scalar product, whose meaning is to point out that scalar product is between $\mathbf{p}$ and $\mathbf{grad}(.)$. This is the force exerted by the external electric field $\mathbf{E}$.

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The torque, with respect to an arbitrary pole O, taken as the origin of the Coordinate System, can be evaluated as follows. Let $\mathbf{r}$ the location of the center of the **Dipole** , and let $\mathbf{E}$ the electric field at the center of the **Dipole** , let $\mathbf{r}_+$ the location of the positive charge, let $\mathbf{r}_-$ the location of the negative charge, and let $\mathbf{E}_+$ and $\mathbf{E}_-$ the external electric field at the position of the positive and negative charges respectively. One finds:

$$\begin{aligned}\boldsymbol{\Gamma}_0 &\equiv |q|\,(\,\mathbf{r}_+\wedge\mathbf{E}_+ - \mathbf{r}_-\wedge\mathbf{E}_-) \quad ,\\ &= |q|\,(\,(\mathbf{r}+\Delta\mathbf{r}/2)\wedge\mathbf{E}_+ - (\mathbf{r}-\Delta\mathbf{r}/2)\wedge\mathbf{E}_-) \quad ,\\ &= |q|\,(\,\mathbf{r}\wedge(\mathbf{E}_+ - \mathbf{E}_-) + (\Delta\mathbf{r}/2)\wedge(\mathbf{E}_+ + \mathbf{E}_-)) \quad .\end{aligned}$$

Taking the limit of the ideal (point) **Electric Dipole** one finds:

$$\boxed{\boldsymbol{\Gamma}_0 = |q|\,\big(\mathbf{r}\wedge(\,(\Delta\mathbf{r}\cdot\mathbf{grad})\mathbf{E}) + \Delta\mathbf{r}\wedge\mathbf{E}\big) = \mathbf{r}\wedge(\,(\mathbf{p}\cdot\mathbf{grad})\mathbf{E}) + \mathbf{p}\wedge\mathbf{E} = \mathbf{r}\wedge\mathbf{F} + \mathbf{p}\wedge\mathbf{E}} \quad .$$

For an ElectroStatic external field, as $\mathbf{rot}\mathbf{E} = 0$, the expression of the force (as long as $\mathbf{p}$ is treated as a constant vector) is equivalent to:

$$F_k = p_s\frac{\partial E_s}{\partial x_k} = \frac{\partial}{\partial x_k}(p_s E_s) = \frac{\partial}{\partial x_k}(\,\mathbf{p}\cdot\mathbf{E}) \qquad \Leftrightarrow \qquad \boxed{\mathbf{F} = \mathbf{grad}(\,\mathbf{p}\cdot\mathbf{E})} \quad . \qquad (41.05.02)$$

→ 1965

In fact:

$$0 = \mathbf{p}\wedge\mathbf{rot}\mathbf{E} \qquad \Longrightarrow \qquad p_j\partial_i E_j = p_j\partial_j E_i \quad .$$

Note, however, that equation (41.05.01) applies in all circumstances, including the case of induced **Dipoles** . On the other hand equation (41.05.02) applies to ElectroStatics fields only and, moreover, the expression of the **Electric Dipole** moment must be kept constant when it is taken inside/outside the gradient.

Read § 36-006 for an alternative way of writing the force formula.

In an ElectroStatic field (as long as $\mathbf{p}$ is treated as a constant) the force can be derived from the potential energy:

$$\boxed{\Phi[\mathbf{r}] = -\,\mathbf{p}\cdot\mathbf{E}} \quad . \qquad (41.05.03)$$

→ 1980

Note that the expression (41.05.02) means that force is equal to the gradient of the projection of the magnetic field along the **Dipole** .

Both the Force|Torque can be derived from the potential energy (as long as $\mathbf{p}$ is treated as a constant vector):

$$\boxed{\mathbf{F} = (\mathbf{p}\cdot\mathbf{grad})\mathbf{E} \qquad \boldsymbol{\Gamma}_0 = \mathbf{r}\wedge\mathbf{F} + \mathbf{p}\wedge\mathbf{E}} \quad .$$

Note that in a uniform field there is no net force on the **Electric Dipole** , $\mathbf{F} = 0$, and therefore the moment, $\boldsymbol{\Gamma}_0$, is independent of the pole, O.

41.05.01.02 Force|Torque on a Static Charge Distribution in an External ElectroStatic Field

This § is referenced at pages:
[1707, 1710]

©|F.Wegner, *Classical Electrodynamics*, 2003, ..., ...Ed., WEB - URL.|§ 4.e||
©|J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|||
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|§ 4.2||

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41.05.01.03 Multipole Development of the Energy of a Static Charge Distribution in an External ElectroStatic Field

This § is referenced at pages:
[Never referenced.]

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|4.2, P4.5, P4.6||

In the general case of an arbitrary static charge distribution in an external static electrostatic field, the following approach applies.

Static distribution means that the multipole moments are assumed to be fixed, independent from the external field.

Note that the formula below is the energy in an external fixed field, not the total energy (no factor $1/2$).

Energia della distribuzione di cariche localizzata attorno ad $\mathbf{x}_0$:

$$\Phi = \iiint_V \rho[\mathbf{x}]\Phi[\mathbf{x}]\, \mathrm{d}V_{\mathbf{x}} = q\Phi[\mathbf{x}_0] - \mathbf{p}\cdot\mathbf{E}[\mathbf{x}_0] + \frac{1}{6}Q^{\cdot\cdot}_{jk}\frac{\partial^2\Phi}{\partial x_j \partial x_k}[\mathbf{x}_0] + \ldots = q\Phi[\mathbf{x}_0] - \mathbf{p}\cdot\mathbf{E}[\mathbf{x}_0] - \frac{1}{6}Q^{\cdot\cdot}_{jk}\frac{\partial E_j}{\partial x_k}[\mathbf{x}_0] + \ldots \quad , \tag{41.05.04}$$

dove si è usato il fatto che $\mathbf{div}\mathbf{E} = 0$ per il campo esterno per passare dal tensore di quadrupolo cartesiano a quello a traccia nulla. → 1759

Si ricavano le equazioni per forze e momenti su dipoli e multipoli superiori.

41.05.02 Steady Currents in an External MagnetoStatic Field

©|F.Wegner, *Classical Electrodynamics*, 2003, ..., ...Ed., WEB - URL.|§ 10||

©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|§ 12|*Definitive*|

This § is referenced at pages:
[1711]

41.05.02.01 Force|Torque on a Magnetic Dipole in an External MagnetoStatic Field

This § is referenced at pages:
[1706, 1707, 1711]

©|Am.J.Phys, , ..., ..., ...Ed., ... 67, 45 (1999);|WEB - URL ||

The total force, due to the external magnetic field $\mathbf{B}[\mathbf{r}]$, acting on a plane circuit carrying a steady current of **Magnetic Dipole** moment equal to $\mathbf{m}$ is found to be, by direct computation:

$$F_k[\mathbf{r}] = m_s\frac{\partial B_k}{\partial x_s}[\mathbf{r}] \qquad \Leftrightarrow \qquad \mathbf{F} = (\mathbf{m}\cdot\mathbf{grad})\mathbf{B} \quad .$$

One can limit the derivation to a small plane square circuit and deduce the result for any general shape by decomposing the generic plane circuit into infinitesimal square circuits.

For a MagnetoStatic external field, and away from the currents generating the extrrnal magnetic field, as $\mathbf{rot}\mathbf{B} = 0$, the expression of the force (as long as $\mathbf{m}$ is treated as a constant vector) is equivalent to:

$$F_k = m_s\frac{\partial B_s}{\partial x_k} = \frac{\partial}{\partial x_k}(m_s B_s) \quad .$$

In fact the currents generating the external MagnetoStatic field, most likely, shall have zero current density at the points where the current loop is located.

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Therefore:

$$0 = \mathbf{m}\wedge\mathbf{rot}\mathbf{B} \qquad \Longrightarrow \qquad m_j\partial_i B_j - m_j\partial_j B_i \quad .$$

Read § 36-006 for an alternative way of writing the force formula.

In an MagnetoStatic field (as long as $\mathbf{m}$ is treated as a constant) the force can be derived from the potential energy:

$$\Phi[\mathbf{r}] = -\,\mathbf{m}\cdot\mathbf{B} \quad .$$

Both the Force|Torque can be derived from the potential energy (as long as $\mathbf{m}$ is treated as a constant):

$$\boxed{\mathbf{F} = (\mathbf{m}\cdot\mathbf{grad})\mathbf{B} \qquad \mathbf{\Gamma}_0 = \mathbf{r}\wedge\mathbf{F} + \mathbf{m}\wedge\mathbf{B}} \quad .$$

Note that in a uniform field there is no net force on the **Magnetic Dipole** , $\mathbf{F} = 0$, and therefore the moment, $\mathbf{\Gamma}_0$, is independent of the pole, O.

The torque can be also derived by direct computation.

41.05.02.02 Force|Torque on a Steady Current Distribution in an External MagnetoStatic Field

This § is referenced at pages:
[1707, 1711]

©|F.Wegner, *Classical Electrodynamics*, 2003, ..., ...Ed., WEB - URL.|§ 10.c||
©|J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|13.3.2, 13.5||
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|§ 5.7||

Consider a localized steady current distribution in an external MagnetoStatic field. The external MagnetoStatic field is supposed to be stationary, that is the generators do what is required to keep it constant, whatever happens.

A system of forces acts on the system of currents. The total Force|Torque are given by:

$$\boxed{\mathbf{F} = \iiint \mathbf{j}[\mathbf{x}]\wedge\mathbf{B}[\mathbf{x}]\ \mathrm{d}V_{\mathbf{x}} \qquad \mathbf{\Gamma}_0 = \iiint\ (\mathbf{x}-\mathbf{x}_0)\wedge(\mathbf{j}[\mathbf{x}]\wedge\mathbf{B}[\mathbf{x}])\ \mathrm{d}V_{\mathbf{x}}} \quad .$$

If the magnetic field varies of a little amount inside the volume occupied by the current distribution one can use a series development around a convenient origin, $\mathbf{x}_0$:

$$\mathbf{B}[\mathbf{x}] \simeq \mathbf{B}[\mathbf{x}_0] + (\Delta\mathbf{x}\cdot[\mathbf{grad})\mathbf{B}]_{\mathbf{x}_0} \quad .$$

After some calculations (see [1]), also exploiting the fact that the volume integral of the divergenceless $\mathbf{j}$ is null for steady currents and the fact that the magnetic field is divergence-less, one finds, as a function of the **Magnetic Dipole** moment of the distribution,

$$\boxed{\mathbf{m} = \frac{1}{2}\iiint\ \mathbf{x}\wedge\mathbf{j}[\mathbf{x}]\ \mathrm{d}V_{\mathbf{x}}} \quad , \tag{41.05.05}$$

the expression of the force on the current distribution (as long as $\mathbf{m}$ is treated as a constant):

$$\boxed{\mathbf{F} = \mathbf{grad}(\,\mathbf{m}\cdot\mathbf{B})} \quad . \tag{41.05.06}$$

→ 1706 1968

The previous result is totally general, it also applies to time-dependent external fields.

For a MagnetoStatic external field and away from the currents generating the external magnetic field, as $\mathbf{rot}\mathbf{B} = 0$, the expression of the force (as long as $\mathbf{m}$ is treated as a constant vector) is equivalent to:

$$\boxed{\mathbf{F} = (\mathbf{m}\cdot\mathbf{grad})\mathbf{B}} \tag{41.05.07}$$

→ 1968

In fact the currents generating the external MagnetoStatic field, most likely, shall have zero current density at the points where the current loop is located.

[1] J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...

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Note, however, that equation (41.05.06) applies in all circumstances, including the case of induced **Dipoles** . On the other hand equation (41.05.07) applies to MagnetoStatics fields only and, moreover, the expression of the **Magnetic Dipole** moment must be kept constant when it is taken inside/outside the gradient.

In a similar manner one can deduce the torque on the current distribution (see [1], §5.7):

$$\boxed{\Gamma_{\mathbf{x}_0} = \mathbf{m} \wedge \mathbf{B}[\mathbf{x}_0]} \quad .$$

41.05.02.03 Multipole Development of the Energy of a Steady Current Distribution in an External MagnetoStatic Field

This § is referenced at pages:
[Never referenced.]

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|5.7||

In the general case of an arbitrary steady current distribution in an external static magnetostatic field, the following approach applies.

Static distribution means that the multipole moments are assumed to be fixed, independent from the external field.

Note that the formula below is the energy in an external fixed field, not the total energy (no factor $1/2$).

DRAFT-EN-PROGRESS

[1] J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...

41.06 Quasi-Static ElectroMagnetic Fields

©|J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|||
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||

The concepts used to describe lumped-components circuits, such as resistance, capacitance, inductance and impedance, can be extended to the general case where the dimensions of the circuits are not negligible with respect to the vacuum wavelength, as is the case for high-frequency cases.

They can be defined in a general fashion from the fields concepts. To do that, one starts with the assumption that the Sources|Sinks and fields have a harmonic time dependence, exploiting the linearity of **Maxwell Equations** .

41.06.01
Poynting Theorem for Harmonics Fields

In the case of harmonic fields, assuming that only linear media are present, one finds, with a procedure exactly identical to the one in section § 40.01 an expression for the energy balance written in terms of the complex Poynting vector, $\mathbf{S}_{\text{EM}}$, and the complex harmonic densities of electric, w_e, and magnetic energy, w_m, and starting from the expression of the time-averaged power spent by the ElectroMagnetic field inside the region, W_{EM}:

$$\mathbf{S}_{\text{EM}} \equiv \frac{1}{2}\mathbf{E}\wedge\mathbf{H}^\star \quad ,$$

$$w_e \equiv \frac{1}{2}\mathbf{E}\cdot\mathbf{D}^\star \quad ,$$

$$w_m \equiv \frac{1}{2}\mathbf{B}\cdot\mathbf{H}^\star \quad ,$$

$$W_{EM} = \frac{1}{2}\iiint \mathbf{j}^\star\cdot\mathbf{E}\ \mathrm{d}V_{\mathbf{x}} \quad .$$

The result is:

$$\frac{1}{2}\iiint_V \mathbf{j}^\star\cdot\mathbf{E}\ \mathrm{d}V_{\mathbf{x}} + 2\mathrm{i}\omega\iiint_V (w_e - w_m)\ \mathrm{d}V_{\mathbf{x}} + \oiint_{\partial V} \mathbf{S}_{\text{EM}}\cdot\hat{\mathbf{n}}\ \mathrm{d}\mathbf{S} = 0 \quad . \tag{41.06.01}$$

→ 1969 1970

The real part of equation (41.06.01) gives the time-averaged energy conservation while the imaginary part is connected to the oscillating behavior of the stored energy, the reactive part of the energy.

If the harmonic densities of electric, w_e, and magnetic energy, w_m, have a real volume integral (as it happens with lossless Polarizable material and ideal conductors)

the real part of equation (41.06.01) becomes:

$$\frac{1}{2}\iiint \Re\mathfrak{e}\left[\mathbf{j}^\star\cdot\mathbf{E}\right]\ \mathrm{d}V_{\mathbf{x}} + \oiint \Re\mathfrak{e}\left[\mathbf{S}_{\text{EM}}\cdot\hat{\mathbf{n}}\right]\ \mathrm{d}\mathbf{S} = 0 \quad .$$

This equation means that in steady conditions the work done by the ElectroMagnetic fields inside the volume is equal to the average energy flux entering the surface. If the system is dissipative this is represented by the second term in equation (41.06.01), which has, in this case, a real part also.

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41.06.02
Impedance in Terms of Fields

The complex Poynting theorem, equation (41.06.01), can be used to define the input impedance of a general two-terminal, linear and passive, ElectroMagnetic system.

Suppose that all the system is enclosed inside a volume V, bounded by a surface $S \equiv \partial V$ and that only the two terminal of the **Dipole** are exiting from this volume. Moreover, let S_i be the small portion of the surface S from which the energy is allowed to enter the volume (for instance the section of the connecting cable), the only part of the boundary of V where energy is allowed to be transferred in. Note, however, that energy can exits the surface from the portion $S - S_i$ of the surface S. Let V_i and I_i be the input (harmonic) voltage and current. Apply equation (41.06.01) to all the space external to S one finds:

$$\frac{V_i I_i^\star}{2} = -\oiint_{S_i} \mathbf{S}_{\text{EM}} \cdot \hat{\mathbf{n}} \, \mathrm{d}\mathbf{S} \quad . \tag{41.06.02}$$

→ 1970

One can now use equation (41.06.01) in equation (41.06.02) to find:

$$\frac{V_i I_i^\star}{2} = \frac{1}{2} \iiint_V \mathbf{j}^\star \cdot \mathbf{E} \, \mathrm{d}V_{\mathbf{x}} + 2\mathrm{i}\omega \iiint_V (w_e - w_m) \, \mathrm{d}V_{\mathbf{x}} + \oiint_{S-S_i} \mathbf{S}_{\text{EM}} \cdot \hat{\mathbf{n}} \, \mathrm{d}\mathbf{S} \quad . \tag{41.06.03}$$

→ 1970

If the surface $S - S_i$ is brought to infinity the integral is real.

and it represents the radiated energy. This is normally negligible at low frequencies and moreover no distinction is required any more between the two surfaces S and $S - S_i$. Equation (41.06.03) can be split into its real and imaginary parts, introducing the resistance and reactance and assuming that the energy flux across S is real:

$$R = \frac{1}{I_i^2} \left(\Re\mathfrak{e} \left[\iiint_V \mathbf{j}^\star \cdot \mathbf{E} \, \mathrm{d}V_{\mathbf{x}} \right] + 2 \oiint_{S-S_i} \mathbf{S}_{\text{EM}} \cdot \hat{\mathbf{n}} \, \mathrm{d}\mathbf{S} + 4\omega \, \Im\mathfrak{m} \left[\iiint_V (w_e - w_m) \, \mathrm{d}V_{\mathbf{x}} \right] \right) \quad ,$$

$$X = \frac{1}{I_i^2} \left(4\omega \, \Re\mathfrak{e} \left[\iiint_V (w_e - w_m) \, \mathrm{d}V_{\mathbf{x}} \right] - \Im\mathfrak{m} \left[\iiint_V \mathbf{j}^\star \cdot \mathbf{E} \, \mathrm{d}V_{\mathbf{x}} \right] \right) \quad .$$

The term $\iiint \mathbf{j}^\star \cdot \mathbf{E} \, \mathrm{d}V_{\mathbf{x}}$ represents the radiation resistance, important at high frequencies. At low frequencies and in those cases when the Ohmic losses are the only causes of energy dissipation only the first term in each of the two equations survive.

41.06.03
Lumped Components

41.07
Phenomenology

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||
©|S.Bobbio & E.Gatti, *Elettromagnetismo E Ottica*, 1991, Bollati-Boringhieri, ...Ed., ...|||
©|C.Kittel, *Introduction To Solid State Physics*, 1996, J.Wiley & Sons, ...Ed., ...|||

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41.07.01
Phenomenology of Polarization

Note: linearity requires $E \ll E_{\text{atomic}} \approx 10^{11}$ V/m.

1. Polarization goes via two main phenomena:
   - Deformation.
     Always present. Normally no temperature dependence.
   - Orientation.
     Strong dependence on the temperature. χ_{E} decreases while increasing the temperature.
     Energy scale for thermal agitation effects: $E \approx k_{\text{B}}T \approx 0.025$ eV.
2. Ferro-Electricity. Example: $BaTiO_3$, spontaneous symmetry breaking below the critical temperature. Surface Polarization charges are quickly neutralized. Any electric field produced attracts neutralizing charges. If Polarization is changed the external charges need time for neutralization: an effect is visible. Ferroelectric materials possess domains, called ferroelectric, inside which dipole moments are coupled with each other, thus giving rise to spontaneous Polarization.
3. Piezo-Electricity. A change of the mechanical stress will induce a change of the Polarization and vice-versa. The piezo-electric constant, relating the Polarization vector to the strain tensor, forms a third-rank tensor. Example from: C.Kittel, *Introduction To Solid State Physics*, 1996, J.Wiley & Sons, ...Ed., ...: an elementary cell made as three/four identical **Dipoles** with total **Dipole** moment equal to zero; when stressed the directions deform and do not sum to zero any longer. Most used material: Quartz. Used for piezo-electric transducers (deformation to electric signal). A piezoelectric transducer is a device that uses the piezoelectric effect to convert a deformation to an electric signal. It can be used to measure pressure, strain, force,... If the changes of strain is very fast, such as in a collision, the resulting Polarization might be so high to start sparks.
4. Pyro-Electricity. Changes of temperature imply change of dimensions of the crystalline solid which can change the Polarization of the solid if the change of dimensions shifts the center of charge of the elementary cells. Used for infrared radiation detection (motion sensors).
5. Electro-Striction.
6. **Super-Dielectrics**: artificial materials made of an insulating porous substrate filled with a ionic liquid. It behaves as the model in section § 38-012.

41.07.02
Phenomenology of Magnetization

1. Diamagnetism. Always present. It opposes to the applied external (applied) field. Normally no temperature dependence.
2. Paramagnetism. Strong dependence on the temperature. χ_{M} decreases while increasing the temperature. Energy scale for thermal agitation effects: $E \approx k_{\text{B}}T \approx 0.025$ eV. Curie law:

$$\chi_{\text{M}} = C\frac{\rho}{T - T_{\text{C}}} \quad .$$

   T_{C}, the Curie temperature, is a characteristic of the material. Exception: Paramagnetism from electrons in metal having no temperature dependence.
   - Mu-metal is a nickel-iron alloy (0.75 nickel, 0.15 iron, plus copper and molybdenum) that has very high **Magnetic-Permeability** . The high **Magnetic-Permeability** makes mu-metal effective for screening static or low-frequency magnetic fields, which cannot be attenuated by other methods.
   - Permalloy is the term for a nickel iron magnetic alloy. It refers to an alloy with about 0.20 iron and 0.80 nickel content. Permalloy has a high **Magnetic-Permeability** , low coercivity, near zero magneto-striction, and significant anisotropic magneto-resistance.
3. Ferromagnetic phenomena. Non-linear, history dependence of the field (hysteresis). Strong effect: $\chi^*_{\text{M}} \approx 10^5$. Above the Curie temperature it becomes paramagnetic. The locking process of $BaTiO_3$

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is not possible due to the faint magnetic interaction energy with respect to the electric interaction energy. QM effect.

- Ferromagnetism.
- Ferrimagnetism.
- Antiferromagnetism.

4. MagnetoStriction.

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41.08 Motion of Particles in ElectroMagnetic Fields

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|||

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.29||

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41.09 Applications of EM

41.10 Experimental Results

41.10.01 Inverse Square Law or the Mass of the Photon

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|§ I.2||

41.10.02 Linear Superposition

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|§ I.3||

What evidence do we have to support the idea of linear superposition?
Small non-linear effects are due to quantum phenomena (gamma-gamma scattering).

41.10.03 Maxwell Equations in Macroscopic Media

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|§ I.4||

© - QUOTE

In most materials the electric and higher terms in are completely negligible. Only the Polarization and Magnetization are significant. This does not mean, however, that the constitutive relations are then simple. There is tremendous diversity in the electric and magnetic properties of matter, especially in crystalline solids, with ferroelectric and ferromagnetic materials having nonzero Polarization or Magnetization in the absence of applied fields, as well as more ordinary materials.....

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|§ I.4||

41.10.04 Boundary Conditions at Interfaces Between Different Media

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|§ I.5||

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41.10.05
Some Remarks on Idealizations in ElectroMagnetism

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|§ I.6||

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41.11 Examples and Physical Applications

41.11.01 Arbitrariness of the Distinction Between Free/bound Charges

This § is referenced at pages:
[1951]

Consider a thin and large slab of a ionic crystal, with its parallel faces perpendicular to one of the axes of the crystal (the z axis). Find alternative descriptions of the charge distribution such that:

1. the Polarization vector is parallel to the z axis;
2. the Polarization vector is anti-parallel to the z axis;
3. the Polarization vector is zero.

Show that the total charge densities and macroscopic electric fields are the same in all the cases.

41.11.02 Electric Fields Around Current-Carrying Conductor

41.11.03 DePolarizing Field for an Ellipsoidal Shape

41.12 Exercises Problems and Physical Applications

41-001 Properties of Material Media

Find a few examples of material media which are, from the point of view of their ElectroMagnetic properties:

- homogeneous and isotropic;
- homogeneous and non-isotropic;
- non-homogeneous and isotropic;
- non-homogeneous and non-isotropic,

Explain and motivate your answers.

Try first to find the above examples among materials with a linear response, as for small enough ElectroMagnetic fields the response is always linear.

41-002 Typical Values

Assume the following data: $E = 1.0{\cdot}10^{6}$ V/m; $B = 1.0{\cdot}10^{1}$ T; $\mathbf{p} = 1.0{\cdot}10^{-30}$ C$\cdot$m; $\mathbf{m} = 1\mu_{\mathrm{B}}$; $T = 300$ K; $\eta_{\mathrm{N}} \approx 1.0{\cdot}10^{28}$ molecules/m^3.

One finds:

$$\frac{\mathbf{p}E}{k_{\mathrm{B}}T} = 2{\cdot}10^{-2} \quad ,$$

$$\frac{\mathbf{m}B}{k_{\mathrm{B}}T} = 2{\cdot}10^{-2} \quad ,$$

$$\alpha^{(\mathrm{e})} = 9{\cdot}10^{-30}\ \mathrm{m}^3 \qquad \chi_{\mathrm{E}} \simeq \eta_{\mathrm{N}}\alpha^{(\mathrm{e})} \quad ,$$

$$\alpha^{(\mathrm{m})} = 9{\cdot}10^{-33}\ \mathrm{m}^3 \qquad \chi_{\mathrm{M}} \simeq \eta_{\mathrm{N}}\alpha^{(\mathrm{m})} \quad .$$

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41-003 Determination of Permanent Moments and the Polarisability

Exploiting the fact that the **Polarisability** |Magnetizability is the sum of the one due to deformation (always present) plus the one due to the permanent moment one can measure the value of the **Polarisability** |Magnetizability due to deformation by measuring and plotting the linear dependence on the inverse of the temperature.

Starting from the **Clausius-Mossotti equations** § 38.09 and § 39.09 one should replace the **Polarisability** |Magnetizability as the sum of the one due to deformation and the one due to the permanent **Dipole** moment:

$$\frac{\alpha_g}{3} \longrightarrow \frac{1}{3}\left((\alpha_g)_{\texttt{deformation}} + (\alpha_g)_{\texttt{orientation}}\right) \quad .$$

The orientation term depends on the temperature and therefore one can plot the linear dependence on the inverse of the temperature.

41-004 Order of Magnitude of the Polarisability

1. Estimate the order of magnitude of the **Polarisability** , by charge density deformation, of a molecule. A safe way, if no other method helps, is to deduce it from the measured values of electric susceptibilities. Read § 38.10.03.
2. Estimate the order of magnitude of the **Polarisability** by orientation, read § 41.04.06.
3. Compare the above results.

41-005 Order of Magnitude of the Magnetisability

1. Estimate the order of magnitude of the magnetisability, by current density deformation, of a molecule (diamagnetism). A safe way, if no other method helps, is to deduce it from the measured values of magnetic susceptibilities. See any textbook for semi-classical models of diamagnetism.
2. Estimate the order of magnitude of the magnetisability by orientation, read § 41.04.06.
3. Compare the above results.

41-006 D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...- Problem 5.49

Show that the two forces between two arbitrary current loops are equal and opposite.

41-007 Polarizable Sphere in an External Electric Monopole Field

Show that:

$$|\mathbf{p}| \propto 1/r^2 \qquad \mathbf{p}\cdot\mathbf{E} \propto 1/r^4 \qquad \mathbf{p}\cdot\mathbf{grad}\,\mathbf{E} \propto 1/r^5 \quad .$$

41-008 Polarization Surface Charge Density

Consider a circular cylindrical piece of matter which is Polarized along the axis of the cylinder.

Assume it is made of molecules, with a **Electric Dipole** moment, all perfectly aligned and Polarized along the axis of the cylinder. Assume every molecule carries a Electric Dipole moment as for one molecule of water.

1. Estimate, as a function of the density of the material, the surface charge density at the two bases of the cylinder.
2. If all the molecular **Dipoles** in a cup of water could be made to point down what would be the magnitude of the surface charge density at the upper water surface.

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41-009 Mangetization Surface Current Density

Consider a circular cylindrical piece of matter which is Magnetized along the axis of the cylinder.

Assume it is made of molecules, with a **Magnetic Dipole** moment, all perfectly aligned and Magnetized along the axis of the cylinder. Assume every molecule carries a Magnetic Dipole moment given by one Bohr-magneton.

1. Estimate, as a function of the density of the material, the surface current density at the lateral surface of the cylinder.
2. If all the molecular **Dipoles** in a cup of water could be made to point down what would be the magnitude of the surface current density at the lateral water surface.

41-010 Relaxation Time Inside a Conducting Electric Material

This § is referenced at pages:
[1766]

©|J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|17.1.2||
©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|7.4||

Read also § 36.20.21.

Suppose that at a certain time a free charge density ρ_0 exists inside a small region of a homogeneous and isotropic medium having **conductivity** σ and relative **Electric-Permittivity** ε_{R}. Determine the time development of the free charge density.

WORKED SOLUTION

$$\rho[\mathbf{x},t]=\rho_0[\mathbf{x}]\exp\left[-\frac{\sigma}{\varepsilon_0\varepsilon_{\mathrm{R}}}t\right]\qquad \tau=\frac{\varepsilon_0\varepsilon_{\mathrm{R}}}{\sigma}\quad .$$

41-011 Hall Effect

A steady current I is flowing in a metal strip of rectangular section with width L and thickness D. The metal has n_e free electrons per unit volume. If a magnetic field $\mathbf{B}$ penetrates the metal strip, perpendicular to the width of the strip, determine the expression of the potential difference between the two sides of the strip.

WORKED SOLUTION

$$\Delta\Phi=\frac{IB}{n_e e d}\quad .$$

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41-012 A Rotating Cylinder

A long cylinder made of a conducting material with radius $R = 5\ \mathrm{cm}$ rotates around its axis with an angular velocity $\omega = 100\ \mathrm{turns/s}$ in a stationary and uniform magnetic field $B = 1\ \mathrm{T}$ parallel to the axis. Assume that the fringe effects at the edges of the cylinder are negligible.

1. Compare ω with the cyclotron frequency of the free electrons inside the conductor.
2. Determine, as a function of ω and B, the volume charge density, the surface charge density, the Polarization and the **Electric Dipole** moment per unit length of the cylinder.
3. Determine the volume current density, the surface current density, the Magnetization and the **Magnetic Dipole** moment per unit length of the cylinder.
4. Determine the electric field and the Polarization in all the space.

41-013 Langevin Equation

Refer to § 41.04.06. Show that the **Polarisability** |Magnetizability in any direction perpendicular to the external field is zero.

41-014 A Useful Integral

This § is referenced at pages:
[1866, 1908]

The following integral

$$\mathbf{I}[\mathbf{x}] \equiv \iiint_{\tau} \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\, \mathrm{d}V_{\mathbf{y}} \qquad \text{with volume } \tau \text{ such that } |\mathbf{y}| < R \quad ,$$

appears in some different situations, so that it is useful to calculate it for the simple case of a spherical volume.

Let us calculate the integral in the case of a spherical volume centered at the origin. To do that, note that the integral is just proportional to the electric field produced by a uniform volume charge distribution inside volume τ. If this is a sphere, the result is well-known as it follows from application of Gauss law:

$$\mathbf{I}[\mathbf{x}] \equiv \iiint_{\tau} \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\, \mathrm{d}V_{\mathbf{y}} = \begin{cases} \dfrac{4\pi}{3}\mathbf{x} & \text{for } |\mathbf{x}| \le R \\ \dfrac{4\pi R^3}{3}\dfrac{\mathbf{x}}{x^3} & \text{for } |\mathbf{x}| > R \end{cases} \quad , \qquad \text{with } \tau \textbf{ a sphere of radius } R \textbf{ located at the origin} \quad .$$

WORKED SOLUTION

Let $\mathbf{E}[\mathbf{x}]$ be the electric field produced by a charge, uniformly distributed with volume density ρ, inside a sphere of radius R. We then have:

$$\mathbf{I}[\mathbf{x}] \equiv \iiint_{\tau} \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\, \mathrm{d}V_{\mathbf{y}} = \frac{4\pi\varepsilon_0}{\rho}\mathbf{E}[\mathbf{x}] = \begin{cases} \dfrac{4\pi}{3}\mathbf{x} & \text{for } |\mathbf{x}| \le R \\ \dfrac{4\pi R^3}{3}\dfrac{\mathbf{x}}{x^3} & \text{for } |\mathbf{x}| > R \end{cases} \quad , \quad .$$

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41-015 Torque From the Potential Energy

Show that the torque with respect to the position of the **Dipole** can be derived from the potential energy (41.05.03).

41-016 Magnetic Mirrors

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|5.7||

Consider a charged particle spiraling in a slow-changing magnetic field. That is: the **Larmor Radius** of the particle is much smaller than the distance scale over which the magnetic field changes in a significant way. Under these conditions the time-averaged force on the particle can be evaluated by considering the force over the **Magnetic Dipole** moment generated by the current.

Show qualitatively that, by taking into account the sign of the forces and **Magnetic Dipole** moments, the result is that all particles are pushed away from regions were the flux lines of the magnetic field are denser.

41-017 Polarisability of a Conducting Sphere

Show that the **Polarisability** of a conducting sphere of radius R is $\propto R^3$.

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41.13 Other Exercises and Problems in ElectroMagnetism

41.13.01

Problems From I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...

Some of the following problems require the use of the theory of relativity.

3.108,
3.111,
3.128,
3.129,
3.130,
3.137,
3.153,
3.230,
3.231,
3.239,
3.244,
3.245,
3.250,
3.262,
3.263,
3.290,
3.296,
3.297,
3.322,
3.348,
3.354,
3.355,
3.356,
3.357,
3.374,
3.371,
4.196,
4.204,
4.205,
4.206,
4.210

41.13.02

Problems From JDJ

See J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...:

- P-6.13;
- P-6.14;
- P-6.15.

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CHAPTER 42

Other Topics

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Part E

Basics of Waves

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43 General References 1986
44 General Properties of Waves 1990
45 Mechanical Waves 2025
46 ElectroMagnetic Waves 2061
47 Waves in Bounded Regions 2156
48 Reflection and Transmission | Refraction of Waves 2180
49 Radiation of ElectroMagnetic Waves 2219
50 Interference 2256
51 Diffraction 2291
52 Interference Diffraction Wave-Fields and Coherence 2309
53 Microscopic Classical Model for Interaction of ElectroMagnetic Waves and Matter 2319
54 ElectroMagnetic Waves in Matter 2335
55 Geometrical | Ray Optics 2357
56 Other Topics 2384

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CHAPTER 43

General References

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Pagina Web del Corso

Informazioni sul corso, testi di studio, problemi ed esercizi, fonti di dati, materiale vario di studio e multimediale:
WEB - URL.
Many classical texts are excellent texts, sometimes considered to be a *Bible* in their field, as they often are. However, physics progresses very very fast, so that excellent texts written many years ago remain a milestone, but may become old.

Useful General Mathematical Resources

©||S.Wolfram Mathematics Encyclopedia|*WEB - URL*|

©||J.Wevers Mathematics Formulary|*WEB - URL*|

©||WEB - URL|*Nice summary website.*|

Some Useful General Physics Resources/References

©||D.Halliday & R.Resnick & K.S.Krane, *Physics*, 2001, J.Wiley & Sons, 5thEd., ...|*Excellent, basic, modern; but do not ...*

©||Berkeley Physics Course, in 5 Vol., , ..., McGraw-Hill, ...Ed., ...|*Excellent text, but do not forget it was written many ...*

©||D.V.Sivuchin, , ..., ..., ...Ed., ...|*Excellent text, but do not forget it was written many years ago, and physics progresse- ...*

©||J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|*Excellent, basic, modern.*|

©||R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|*Excellent text, but do not ...*

©||L.D.Landau *et al.*, , ..., ..., ...Ed., ...|*Excellent text, but do not forget it was written many years ago, and physics ...*

©||W.M.Haynes *et al.*, *CRC Handbook of Chemistry and Physics*, 2016, CRC press, 97thEd., WEB - URL.|*Source of Data*|

©||R.Blandford & K.Thorne, *Applications Of Classical Physics*, ..., ..., ...Ed., WEB - URL, WEB - URL.|*Excellent, ...*

Some Useful Physics Specific Resources/References

©||G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|*Excellent text, ...*

©||Barger & Olsson, , ..., ..., ...Ed., ...||

©||M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|*Excellent, advanced text; but do not forget it was ...*

©||H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|*Excellent, advanced text; but do not forget it was ...*

©|M.dePodesta, *Understanding The Properties Of Matter*, 2002, CRC Press, 2ndEd., ...|Now freely available: ...

©||H.Goldstein *et al.*, *Classical Mechanics*, 2014, Pearson Education, 3rdEd., ...|*Excellent, advanced text; but do not ...*

©||D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|*Excellent, basic, modern.*|

©||E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|*Excellent, basic, modern.*|

©||J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|*Excellent, advanced text; but do not forget ...*

©||W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|*Excellent, ...*

©||F.Reif, *Fundamentals Of Statistical And Thermal Physics*, 2009, Waveland Press, ...Ed., ...|*Excellent, basic text; but ...*

©||A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|*Excellent, intermediate, modern.*|

©||M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|*Excellent, basic text; but ...*

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Some Useful Resources/References on Physics and the Real World

©||L.Bloomfield, *How Things Work*, ..., ..., ...Ed., WEB - URL.||

Other references

©|K.McDonald|WEB - URL|*A real treasure of tens of detailed and critical analyses of key physics problems, showing how ...*

The SI of Units of Measure

Read § B.

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References for this section

Some References for this section follow.

- ...

Exercises and problems for this part

PROBLEMS - at the end of G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...

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CHAPTER 44

General Properties of Waves

44.01 Wave Phenomena 1991
44.02 d'Alembert Classical Non-Dispersive Wave Equation 1997
44.03 Understanding Real Wave Phenomena 2007
44.04 Examples and Physical Applications 2018
44.05 Exercises Problems and Physical Applications 2022

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This § is referenced at pages:
[1426, 1426]

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§ outline

- *Different physical systems lead to the same equation: the **D'Alembert non-dispersive wave-equation** .*
- *Examples and generalities on waves.*
- *Formalization: **D'Alembert non-dispersive wave-equation** .*
- *Solutions to **D'Alembert non-dispersive wave-equation** in 1D: modes and traveling waves.*
- *Solutions to **D'Alembert non-dispersive wave-equation** in 3D: modes and traveling waves.*
- *Introduction to other general wave concepts.*

The concept of wave is a difficult one and therefore we spend some time to try explain it.

This section is very general and a bit abstract; it must be studied (again) after carefully studying of § 45, § 46.

Pre-requisites - Review Harmonic oscillations § 25.

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|All arguments ...
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|All arguments about waves.||
©|R.Fitzpatrick|Oscillations and Waves|***WEB - URL*** - *Complete textbook*|
©|A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...||*Complete text with worked ...*
©|R.Knobel|An Introduction to the Mathematical Theory of Waves|***WEB - URL*** - *Advanced*|
©|G.B.Whitham|Linear and Nonlinear Waves|***WEB - URL*** - *Advanced*|
©|SCHOLARPEDIA|Linear and Nonlinear Waves - **WEB - URL** |*Nice overview*|
©|SCHOLARPEDIA|Partial differential equations - **WEB - URL** |*Nice overview*|
©|J.Bell|**WEB - URL** |*Nice Notes onf PDE*|

The concept of wave is fundamental in all physics.

In this section the **general properties of all waves** are discussed, valid for any kind of wave. These properties do not depend on any specific kind of wave, even if different kinds of waves are used as examples. Specific properties, different for different kinds of waves, must be discussed in the specific case. **Most of what is discussed in this section can only be better understood after studying all the following topics in waves as well as specific cases of waves**.

Everywhere, if not specified otherwise, the **wave equation** is implicitly assumed to be the **classical non-dispersive d'Alembert wave partial differential equation** (`[glo:PDE]`PDE),

either homogeneous or non-homogeneous.

Note that an equation describing waves, as it must involve both time and spatial coordinates, is necessarily a PDE, called in short, **the wave equation**; many types of wave equations do exist; read § 44.01.04.

44.01 Wave Phenomena

44.01.01 Physical Systems Leading to d'Alembert Classical Non-Dispersive Wave Equation

Often, **wave phenomena** are described by the ubiquitous **d'Alembert Classical Non-Dispersive wave equation**. For this reason, sometimes, wave phenomena are identified with all and only the phenomena described by **D'Alembert non-dispersive wave-equation** . However, even if d'Alembert wave phenomena is the simplest and a very common case in physics, it is not the full story. Read section § 44.01.04.

The following examples of physical systems lead, at least to a first approximation, to the **D'Alembert non-dispersive wave-equation** .

- From **Maxwell Equations** in vacuum one deduces the ElectroMagnetic wave equation in vacuum (**D'Alembert non-dispersive wave-equation** , read § 46.01).
- From **Maxwell Equations** in a neutral **perfect dielectric medium** one deduces the ElectroMagnetic wave equation in a neutral **perfect dielectric medium** (d'Alembert wave equation, read § 46.02).
- The equations for the **ElectroMagnetic potentials** in the **Lorenz Gauge** (read § 49.01) is **D'Alembert non-dispersive wave-equation** .
- The linear spring-mass chain (read § 25.07.04) gives, in the continuum limit, **D'Alembert non-dispersive wave-equation** .
- The small transverse vibrations of an ideal string (read § 27.07.08) give **D'Alembert non-dispersive wave-equation** , read § 45.03.01.
- The small longitudinal vibrations of any piece of matter (sound) give **D'Alembert non-dispersive wave-equation** .
- Small waves on a liquid surface are described, to a first approximation, by **D'Alembert non-dispersive wave-equation** .

44.01.02 What is a wave?

©|G.B.Whitham|Linear and Nonlinear Waves|*WEB - URL - Advanced*|

©|D.A.Russell|WEB - URL |*Many excellent demos!*|

©|D.A.Russell|WEB - URL |*What is a wave?*|

©|D.A.Russell|WEB - URL |*Wave Motion in Time and Space*|

Waves appear under many aspects in common experience. A few general concepts about waves are summarized in this section. Their understanding will improve when discussing specific examples.

There is no general precise definition of wave in physics, as the broad use of the term wave makes it difficult to produce a precise definition which would not be limiting.

Sort of *definition* of wave can be as follows.

A wave is a signal (or perturbation, or disturbance), possibly (but not necessarily) impulsive or periodical, produced by some well-defined source and traveling with a well-defined kinematics (speed, in particular). The traveling signal, with the above characteristics, is a signal modifying the equilibrium conditions of some physical field, describing some property of some physical system. The signal may distort and|or attenuate/amplify as long as it remains recognizable. Superposition of different signals, not necessarily a linear superposition, can lead to more and more complex behavior, also different from a qualitative point of view. There is no overall movement of matter.

The terms signal, source, traveling, **physical field/system** and the others terms above are used in a sufficiently broad manner to allow a wide range of interpretation of what constitutes a wave.

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This generic definition may seem to be a too vague, but it turns out to be perfectly adequate; moreover, any attempt to be more precise appears to be too restrictive; in fact, different features are important in different types of waves.

The first question to ask when dealing with waves is: **which physical quantity is oscillating**?

44.01.03
Additional Examples of General Wave Phenomena

This § is referenced at pages:
[2007]

- A large crowd of people at a stadium during a soccer, football or baseball game often does **the wave (or the Mexican wave)**: some people jump up and sit down and then some other nearby people see them and they also jump up and sit down, and so on with everybody jumping up and sitting down as soon as they see their neighboring sitting down. We then have a traveling wave around the stadium. The wave is the Perturbation | Disturbance | Signal made by people jumping up and sitting down. The wave travels around the stadium with a definite speed and trajectory. None of the individual people at the stadium is moving while the wave travels: they all remain at their place. Note that it is essential that everybody moves at the right time, by following the kinematics of his/her neighboring spectators. If this is not the case and everybody moves at random we have no wave but just a random movement. Note that nobody is moving, so that the wave may travel at a very large speed with respect to the speed a single person might rush around the stadium. Note also that the speed the wave is moving has nothing to do with the speed each spectator is jumping-up/sitting-down.
- Sound waves in matter: longitudinal oscillations of matter particles with respect to their equilibrium position (music, for instance).
- Elastic waves in solid matter (both longitudinal and transversal): longitudinal and transverse oscillations of matter particles with respect to their equilibrium position (seismic waves, for instance).
- Shock waves in gases: an abrupt change of pressure propagating in the gas.
- EM waves: oscillating ElectroMagnetic fields (light, radio-waves and micro-waves, for instance).
- Surface waves on water: oscillating water particles.
- Gravitational waves: oscillating gravitational field.
- Quantum Physics describe particle mechanics by means of **Schrödinger waves**.
- Some examples from outside physics:
  - Traffic-lights waves: a long straight road having identical traffic-lights at equal distances, with disturbances given by the traffic-lights changing color. If traffic-lights are properly synchronized for a certain speed the wise car-driver can get all greens.
  - Traffic waves [a]: car traffic on a road with disturbances, given by possibly abrupt changes in traffic density, produced by traffic lights or other traffic obstacles. As simple model is described by a first-order PDE.
  - Blood Wave propagation in arteries [b]: pressure waves of blood in arteries.
  - Epizootic waves, that is geographic spread of diseases: a population susceptible to contracting a disease with disturbances given by the density of infectives.
  - Grass waves [c].

[a] WEB - URL
[b] WEB - URL
[c] WEB - URL

PRELIMINARY PROGRESS

44.01.04 Different Wave Equations for Different Physical Systems

This § is referenced at pages:
[1990, 1991]

Waves in physics may either involve oscillations of a material medium (mechanical waves) or not (such as ElectroMagnetic, gravitational and Schrödinger waves).

The term **wave equation**, except when specified otherwise, will mean the **D'Alembert non-dispersive wave-equation** . The one-dimensional and three-dimensional wave equations for a quantity, $\xi[\mathbf{x},t]$, in terms of the propagation speed of waves, v, the so-called **phase-velocity** and for a source function $s[\mathbf{r},t]$, read:

$$\frac{\partial^2\xi}{\partial z^2}-\frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2}=s[z,t] \qquad \nabla^2\xi-\frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2}=s[\mathbf{r},t] \quad . \tag{44.01.01}$$

→ 1995 1997

However, it must be emphasized that different wave phenomena may be described by **other wave equations**, that is the concept of wave is not at all linked to **D'Alembert non-dispersive wave-equation** . Phenomena described by **D'Alembert non-dispersive wave-equation** are very common.

In general, $\xi[\mathbf{x},t]$ is any oscillatory physical quantity associated with the wave.

44.01.04.01 PDE Wave Equations and Conditions for Solutions

Every wave phenomenon is described by its own wave equation, a **partial differential equation** ([glo:PDE]PDE) in space and time coordinates, possibly a non linear one (see section § 44.03). The problem of finding solutions, and understanding their existence and uniqueness, is a very complex mathematical problem, in general. As for ODE one must add conditions, to ensure uniqueness of the solutions, but not too many conditions, to ensure existence of some solutions. Broadly speaking two types of conditions are used:

- **initial conditions**: at a certain time, the values of the unknown function at all points is given, and|or for some of the derivatives;
- **boundary conditions**: for all the times, the values of the unknown function at all points of some surface are given, and|or for some of the derivatives;
- mixed conditions: a mixture of initial and boundary conditions.

44.01.04.02 ElectroMagnetic Waves in perfect perfect Ohmic media

As an example of a wave phenomenon which is not described by **D'Alembert non-dispersive wave-equation** one can mention the case of ElectroMagnetic waves in **perfect Ohmic media** (see section § 46.06). In this case there is one more additional term added to **D'Alembert non-dispersive wave-equation** which introduces energy dissipation in the system:

$$\frac{\partial^2\xi}{\partial z^2}-\frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2}-b^2\partial_t\xi=s[z,t] \qquad \nabla^2\xi-\frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2}-b^2\partial_t\xi=s[\mathbf{r},t] \quad . \tag{44.01.02}$$

The added term has the physical interpretation of being related to some kind of friction.

The name **dissipative d'Alembert wave equation** will be used.

It is, in fact, a special case of the **telegraph wave-equation**.

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44.01.04.03 Schrödinger Wave Equation

In non-relativistic quantum physics the dynamics of a particle of mass m in an external potential $V[\mathbf{r}]$, is given, in terms of the wave-function $\psi[\mathbf{r},t]$, by the Schrödinger wave equation:

$$\mathtt{i}\hbar\partial_t\psi[\mathbf{r},t] = -\frac{\hbar^2}{2m}\nabla^2\psi[\mathbf{r},t] + V[\mathbf{r}]\psi[\mathbf{r},t] \quad . \tag{44.01.03}$$

→ 2013

Schrödinger wave equation is, in general, a non-linear one.

Note also that Schrödinger wave equation is intrinsically a complex equation.

44.01.04.04 Advection Wave Equation

44.01.04.05 Korteweg-deVries Wave Equation

An example of a non-linear wave equation is the Korteweg-deVries **Wave-Equation** . It describes small-amplitude, shallow, confined water waves, called **solitons**. A soliton looks like a wave pulse that retains its shape as it travels.

44.01.04.06 Heat Equation

Read § 68.05.04.

44.01.04.07 Airy Equation

A non-linear PDE with **Dispersion Relation** (read § 44.03.08):

$$\omega/k = -k^2 \quad .$$

44.01.04.08 Vibrating Beam Wave Equation

Read § 45.03.01, § 45.03.02, § 25.07.04, § 25.07.05.

The Vibrating Beam Wave Equation is a fourth-Order PDE.

The major difference between the transverse vibrations of an ideal string (read § 27.07.08) and the transverse vibrations of a thin beam is that the beam offers some resistance to bending, at variance of the ideal string which is assumed to be perfectly flexible.

The wave equation becomes:

$$\boxed{\ddot{\xi}[z,t] \propto -\xi''''[z,t]} \quad .$$

See a vibrating beam at WEB - URL .

See the mode shapes for the first four modes of a vibrating cantilever beam at WEB - URL .

44.01.05 Linear and Non-Linear Wave Equations

Very often, phenomena describe by non-linear **Wave-Equation** can be treated by approximating the wave equation to a linear wave equation, under the appropriate conditions. This is called **linearization**. Typical example: water waves on the surface of sea/lake/pool....

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44.01.06
Traveling Waves and Standing Waves

This § is referenced at pages:
[1995]

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Two broad families of waves, depending on their kinematics, are very important: standing waves and traveling waves. Technically, the difference between traveling and standing waves is in the average, Energy, Linear Momentum, Angular Momentum, transport velocity being either different or equal to zero. Read § 45.08, § 46.05 for the specific case of mechanical and ElectroMagnetic waves.

- **Standing waves (also called modes)**.
  Typically when the region of space is limited and waves are near boundaries and boundaries have a very important effect on the behavior of waves (read § 44.02.03.06). At any point in space all the oscillations have the same time dependence, that is space and time variables are separated:
  $$\boxed{\xi[\mathbf{r},t] = g[\mathbf{r}]f[t]} \quad .$$
  Example, a Sinusoidal|Cosinusoidal standing wave in one dimensional bounded region $0 \leq z \leq L$, with $\xi[z=0,t] = \xi[z=L,t] = 0$:
  $$\xi[z,t] = \xi_0 \sin[\pi z/L]\cos[\omega t] \qquad \text{or} \qquad \xi[z,t] = \xi_0 \sin[2\pi z/L]\cos[\omega t] \quad .$$
  Show that it satisfies the wave equation (44.01.01)
  Example, a attenuating Sinusoidal|Cosinusoidal standing wave in one dimensional bounded region $0 \leq z \leq L$, with $\xi[z=0,t] = \xi[z=L,t] = 0$:
  $$\xi[z,t] = \xi_0 \exp[-t/\tau]\sin[\pi z/L]\cos[\omega t] \qquad \text{or} \qquad \xi[z,t] = \xi_0 \exp[-t/\tau]\sin[2\pi z/L]\cos[\omega t] \quad .$$
  Show that it does not satisfy the wave equation (44.01.01)
  Example: the plucked string of a guitar.
  Standing waves do not propagate at all.
  Standing waves do not transport, on average, Energy, Linear Momentum, Angular Momentum. This means that, while there is a local transport of Energy, Linear Momentum, Angular Momentum, on average, Energy, Linear Momentum, Angular Momentum, move forth and back, so that on average there is no global transport.
- **Traveling waves (also called progressive waves)**.
  Typically when the region of space is large and waves are far away from boundaries (read § 44.02.02 and § 44.02.03).
  In general, the solutions of PDE are functions of space and time variables independently. Traveling waves are so-called whenever they are function of combinations of the variables only:
  $$\boxed{\xi[\mathbf{r},t] = H[k[\mathbf{r}] \pm vt]} \quad .$$
  Example: a Gaussian pulse in one dimension:
  $$\xi[z,t] = \frac{\xi_0}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(z \pm vt)^2}{2\sigma^2}\right] \quad .$$
  Show that it satisfies the wave equation (44.01.01).
  Example: an attenuating Gaussian pulse in one dimension:
  $$\xi[z,t] = \exp[-t/\tau]\,\frac{\xi_0}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(z \pm vt)^2}{2\sigma^2}\right] \quad .$$
  Show that it does not satisfy the wave equation (44.01.01); it satisfies a different **Wave-Equation** .
  Example: waves in open water.
  Traveling waves transport Energy, Linear Momentum, Angular Momentum.

Superposition, not necessarily a linear one, among different waves can lead to more complex behavior,

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also different from a qualitative point of view. For instance, a suitable superposition of different traveling waves may give rise to standing waves (read § 44.01.06), as well as the other way round.

44.01.07
Some Important Types of Simple Waves

Beware that the nomenclature about waves is not universal and often sloppy....

- **Harmonic wave**: any wave whose time dependence is harmonic.
- **Plane Waves**: PW. Waves such that either amplitude and|or phase are uniform on all the planes perpendicular to some fixed direction in space; strictly speaking: harmonic PW with constant phase on all the planes perpendicular to some fixed direction in space; for non-harmonic waves the latter definition does not make any sense, so it is sometimes used to refer to waves such that the amplitude is uniform on all the planes perpendicular to some fixed direction in space; note that MPW do not necessarily have constant amplitude planes corresponding to the constant phase planes.
- **MPW**: harmonic PW with homogeneous amplitude, sometimes, defined just as harmonic PW; the latter have a distinguished importance as in quantum mechanics represent particle with fixed energy and linear momentum.
- **Attenuated|Amplificated MPW**: varying (decreasing|increasing) amplitude; energy is dissipated|absorbed.
- **Non-homogeneous MPW**: constant amplitude planes and constant phase planes are not parallel to each other.
- **Spherical Waves**: SW. Waves such that either amplitude and|or phase are uniform on all the spheres centered at the same point; strictly speaking: harmonic SW with constant phase on all the spheres centered at the same point; for non-harmonic waves the latter definition does not make any sense, so it is is sometimes used to refer to waves such that the amplitude is uniform on all the spheres centered at the same point; note that MSW do not necessarily have constant amplitude spheres corresponding to the constant phase spheres.
- **MSW**: harmonic SW with homogeneous amplitude, sometimes, defined just as harmonic SW; the latter have a distinguished importance as in quantum mechanics represent particle with fixed energy and angular momentum.
- Cylindrical Waves: CW. Waves such that either amplitude and|or phase are uniform at fixed distance from a fixed axis; same distinction as for PW and SW.

Note that all the above names are often used in a sloppy and non universal way.

44.01.08
What Is a Wave?

©|J.A.Scales & R.Snieder|What is a wave?|*NATURE 401, 21 OCTOBER 1999, pag. 739*|

© - QUOTE

Most people assume that a wave, being central to all the phenomena we observe, has a uniform definition. But defining this basic concept isn't so easy.
...
Taking all these examples into account, we stick with our definition of a wave as an organized propagating imbalance; just don't ask us to define 'organized'.

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44.02 d'Alembert Classical Non-Dispersive Wave Equation

©|A.D.Polyanin *et al.*|Handbook of Linear Partial Differential Equations for Engineers and Scientists|*Chapman Hall/CRC ...*

The term **wave equation**, except when specified otherwise, will mean the **d'Alembert Classical Non-Dispersive wave equation**.

The general solution of the wave equation for a **non-dispersive wave** is discussed in this section. A non-dispersive wave is by definition such that the **phase-velocity** , v, is a fixed constant which does not depend, in particular, on the frequency of MW.

Solutions of **D'Alembert non-dispersive wave-equation** that are initially zero outside some limited region propagate out from the region at a fixed speed in all spatial directions, as do physical waves from a localized disturbance: traveling waves. Another class of solutions represent standing waves, or modes, defined inside a limited region.

The equation alone does not specify a solution: a unique solution is usually obtained by setting a problem with further conditions, such as:

1. initial conditions, which prescribe the value and velocity of the wave on a suitable surface;
2. or values on a closed surface (boundary conditions).

The effect of boundaries, giving rise to boundary conditions, reflection and Transmission|Refraction of waves will be introduced later.

The **phase-velocity** must not be confused with the velocity of the movement of the Perturbation|Disturbance|Signal: $\dot{\xi}$.

We consider here the wave equation for a **scalar quantity**, ξ. When the physical field is a vector or tensor field the general discussion in this section is still valid but additional properties may come into play, which are specific of the physical field and cannot be discussed in general.

In all the cases, solutions to the wave equation will be found and studied but in no case the problem of finding all the solutions will be studied, as this is a complex problem left to Mathematics for Physics.

However, it is known that the **Fourier Series | Fourier Integral** (read § 18) makes **Monochromatic Plane Waves** ([glo:MPW]MPW) the basic elements for solving the wave equation. In fact the **Fourier Series | Fourier Integral** allows one to write any and all the solutions as a sum/integral over MPW. It is therefore sensible, thanks to the linearity of d'Alembert wave equation, to consider only MPW solutions, to start with, and study them in detail as the basic building block of any solution to the wave equation.

44.02.01
Linearity and Superposition of Classical d'Alembert Waves

Read also sections (44.02.01), (44.02.07).

The classical homogeneous **D'Alembert non-dispersive wave-equations** (44.01.01) are linear and homogeneous. Therefore any sum of solutions of the equation is still a solution of the equation and the Perturbation|Disturbance|Signal caused by any one solution sum to the Perturbation|Disturbance|Signal produced by all the other Perturbation|Disturbance|Signals in a linear way.

When there is a source the equation becomes a non-homogeneous one (read § 45-004). The classical non-homogeneous **D'Alembert non-dispersive wave-equation** is linear and non-homogeneous. In this case the sum of the Perturbation|Disturbance|Signal caused by two sources is the sum of the two Perturbation|Disturbance|Signals.

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44.02.02
One-Dimensional d'Alembert Non-Dispersive Wave Equation

This § is referenced at pages:
[1995]

The one-dimensional wave equation for the scalar quantity $\xi[z,t]$, whatever the meaning of the physical quantity ξ, is:

$$\boxed{\frac{\partial^2\xi}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2}} \quad , \tag{44.02.01}$$

→ 1997 1998

where v is the propagation speed of the waves, the so-called **phase-velocity** .

Only non-dispersive waves such that the **phase-velocity** , v, is a fixed constant independent of the frequency of MW, will be considered here.

44.02.02.01 General Solution for Non-Dispersive Waves in Unbounded Media

This § is referenced at pages:
[1999]

A traveling one-dimensional solution is a any function with argument $kz-\omega t$.

The general solution of the one-dimensional wave equation in free space without boundaries is easy to write, as long as the **phase-velocity** $v\equiv|\mathbf{v}|>0$ is a fixed number, that is the **phase-velocity** is same for every MW (a so-called non-dispersive wave). In this case the general solution is:

$$\boxed{\xi[z,t]=f_{\oplus}[z-vt]+f_{\ominus}[z+vt]} \quad , \tag{44.02.02}$$

→ 1998 2000 2002

in terms of any two arbitrary functions: $f_{\oplus}$ and $f_{\ominus}$.

It can be easily shown by direct calculation that the above function (44.02.02) satisfy the wave equation for any $f_{\oplus}$ and $f_{\ominus}$.

The solution of equation (44.02.01) can be found by direct construction with the change of variables:

$$w_{\oplus}\equiv z-vt \qquad w_{\ominus}\equiv z+vt \quad ,$$

in terms of which equation (44.02.01) takes the form, in case the **phase-velocity** v is a fixed number:

$$\frac{\partial^2\xi}{\partial w_{\oplus}\partial w_{\ominus}}=0 \quad ,$$

whose solution is equation (44.02.02)

It can be shown that the latter formula gives all the solutions.

The special form of solution (44.02.02) means that, for a non-dispersive wave, any Perturbation|Disturbance|Signal travels keeping the same shape.

Any generic functions $f_{\oplus}$ and $f_{\ominus}$ satisfies the equation. However, any specific physical problem will have **initial/boundary conditions** so that the specific solution will be determined in terms of the functions specifying initial/boundary conditions.

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44.02.02.02 Rigidly Traveling MW Solutions in 1D

This § is referenced at pages:
[1406, 2002]

Read also § 44.02.03.03.

As we have a general solution (read § 44.02.02.01) we can move to studying MW in 1D.

Read § 19.02 for the important and useful relations about the sum of harmonic functions with the same frequency.

Consider non-dispersive waves in unbounded one-dimensional media, that is fixed v, in the whole one-dimensional space.

Guided by the example of the oscillations of of the surface of a water pool Let us try to find a solution such that a space-harmonic profile rigidly moves at constant speed, v. A function of type:

$$\xi[z,t]? =?C\cos\left[\pm z - vt\right] \quad ,$$

cannot be used as the argument of the cosinus must be dimensionless. Therefore we need to introduce a constant, k, with dimensions of inverse length, and obtain (after adding also a generic **phase constant** , ϕ):

$$\boxed{\xi[z,t] = C\cos\left[k\left(\pm z - vt\right) - \phi\right]} \qquad \boxed{v \equiv \omega[k]/k \qquad \textbf{phase-velocity}} \quad , \tag{44.02.03}$$

with the defining relation of the **phase-velocity** in terms of time frequency and wave-number. As in the previous case the quantity k is called the **wave-number** as it is linked to the space behavior of the solution. The constants C and ϕ must be determined from the initial/boundary conditions.

Replacing the above tentative solution in the above equation one finds the necessary conditions for the above function to be a solution, which is the **Dispersion Relation** (read § 44.03.08):

$$\boxed{\omega[k] = kv \qquad \textbf{Dispersion Relation}} \quad .$$

In this case, for waves described by **D'Alembert non-dispersive wave-equation** with fixed v, the **Dispersion Relation** (read § 44.03.08) is somewhat of a trivial dispersion relation, in the sense that there is no dispersion at all: these waves are non-dispersive waves, that is, waves whose **phase-velocity** is independent of ω and k, read § 44.03.08. One should not be confused by the fact that, in this case, the dispersion relation is identical to the definition of the **phase-velocity** : this is because we are dealing with **D'Alembert non-dispersive wave-equation** with fixed v. The two concepts are totally distinct concepts.

As the **Fourier Series** | **Fourier Integral** allows us to use Sinusoid|Cosinusoid to describe any function it is worthwhile to study better the behavior of Sinusoid|Cosinusoid harmonic waves.

In order to use harmonic waves it is necessary to use the dimensionless phase (44.02.10). As a particular case **D'Alembert non-dispersive wave-equation** always admits harmonic solutions:

$$\boxed{\xi[z,t] = \xi_0\cos\left[\pm kz - \omega t - \phi\right] \qquad \text{with } k>0 \text{ and } \omega>0} \quad , \tag{44.02.04}$$

→ 1999 2000

where $\omega > 0$ is the **angular frequency** of the **time oscillation** at a fixed space point, linked to the time period of the oscillation, $k > 0$ is the **wave number**, linked to the **wavelength** λ of the space oscillation at a fixed time and corresponding to the space traveled in a period by any part of the wave at fixed phase.

The corresponding **phase-velocity** , that is the speed of travel of the planes of constant phase (as it is demonstrated in section § 44.02.02.03) is:

$$\boxed{v \equiv \omega[k]/k} \quad . \tag{44.02.05}$$

→ 2000

The Sinusoid|Cosinusoid wave (44.02.04) represents a Sinusoid|Cosinusoid vibration in time, at any fixed location z, and a Sinusoid|Cosinusoid vibration in space, at any fixed time, t.

The following general relations apply:

$$\boxed{\omega = \frac{2\pi}{T} \equiv 2\pi\nu} \quad , \quad ,$$

$$\boxed{k = \frac{2\pi}{\lambda} \equiv 2\pi\sigma} \quad , \quad ,$$

$$\boxed{\lambda = vT} \quad .$$

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The latter relation shows the link among the spatial period, the time period and the **phase-velocity** : it is in fact a direct consequence of the **phase-velocity** equation (44.02.05).

MW are characterized by a **double periodicity**: in time and in space. The **Dispersion Relation** (44.03.04) (read § 44.03.08) is the link among the two periodicities. It depends on the particular physical phenomena under study, that is on the wave equation. The relation (44.03.04) provides, by means of the solution to the wave equation, the relation between ω and k for a given MW (the **Dispersion Relation** , read § 44.03.08). The first task of the solution of any problem with waves is to find the **Dispersion Relation** (read § 44.03.08) between the frequency, ω, and the wave-number, k.

The existence of the **Dispersion Relation** (read § 44.03.08) comes from the fact that the one-dimensional **Wave-Equation** has only one parameter, the velocity v. When one introduces the solution in the form of equation (44.02.04) the two parameters, k and ω, turn out to be linked by the wave equation. The link is actually the relation between the space and time periodicity of the wave.

44.02.02.03 Phase-Velocity

This § is referenced at pages:
[1999]

The special form of equation (44.02.02) means that the Perturbation | Disturbance | Signal $\xi[x,t]$ takes the same value at all times-points where ($v > 0$):

$$\boxed{z - vt = \text{constant} \qquad \text{for the part } f_{\oplus}[z - vt] \qquad \text{and} \qquad z + vt = \text{constant} \qquad \text{for the part } f_{\ominus}[z + vt]} \quad .$$

In other words the two perturbations/disturbances, $f_{\oplus}$ and $f_{\ominus}$ **travel rigidly** with velocity, the **phase-velocity** , $+v$, for the part $f_{\oplus}[z - vt]$, and $-v$, for the part $f_{\ominus}[z + vt]$. In fact:

$$z \pm vt = \text{constant} \implies z_2 \pm vt_2 = z_1 \pm vt_1 \implies \Delta z = \mp v \Delta t \implies \frac{\Delta z}{\Delta t} = \mp v \quad .$$

Clearly the total Perturbation | Disturbance | Signal is not moving rigidly unless either $\xi = f_{\oplus}$ or $\xi = f_{\ominus}$.

Note that it is essential that the **phase-velocity** is a fixed constant in order to have a rigid propagation of the wave-form. The **phase-velocity** could reasonably be called the **argument velocity**, because v is the speed with which a point with constant argument, $z \pm vt$, of the function f moves. However, not all systems have the property that the **phase-velocity** $\pm\omega/k$ is constant, that is, independent of ω and k: waves described by d'Alembert Classical Non-Dispersive equation with fixed v have this property, that is they are non-dispersive waves.

The case of an exactly non-dispersive waves is important, in practice, because it includes many important phenomena, such as ElectroMagnetic waves propagating in vacuum or in a **perfect dielectric medium**.

44.02.02.04 Time-Harmonic Standing waves - Normal Modes

This § is referenced at pages:
[2160, 2161]

Guided by the example of the oscillations of the plucked string of a guitar, Let us try to find a solution such that all the parts, which are labeled by the coordinate z, oscillate with the same time dependence, a harmonic one (this kind of solutions is a standing wave called normal mode):

$$\xi[z,t] \qquad ? = ? \qquad A[z]\cos[\omega t - \phi] \quad .$$

Note: there is no propagation of the wave at all in this case.

In other words the space and time dependence are factorized (see section § 44.02.03.06) and **all oscillators oscillate with the same time dependence**, a time-harmonic one in this case.

Replacing the above tentative solution into the wave equation one finds the necessary conditions for the above function to be a solution:

$$\frac{d^2 A[z]}{\mathrm{d}z^2} = -\frac{\omega^2}{v^2} A[z] \equiv -k^2 A[z] \quad , \tag{44.02.06}$$

→ 2019

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with the necessary condition given by the **Dispersion Relation** (read § 44.03.08):

$$\boxed{\omega = \omega[k] \equiv kv} \quad .$$

In this case, for waves described by **D'Alembert non-dispersive wave-equation** with fixed v, the **Dispersion Relation** (read § 44.03.08) is somewhat of a trivial dispersion relation, in the sense that there is no dispersion at all: these waves are non-dispersive waves, that is, waves whose **phase-velocity** is independent of ω and k, read § 44.03.08.

The quantity k is called the **wave-number** as it is linked to the space behavior of the solution. In fact the general solution for $A[z]$ is given by:

$$A[z] = b_c \cos[kz] + b_s \sin[kz] \quad .$$

In summary:

$$\xi[z,t] = (b_c \cos[kz] + b_s \sin[kz]) \cos[\omega t - \phi] \quad ,$$

$$,$$

$$\omega = \omega[k] \qquad \text{the } \textbf{Dispersion Relation} \quad .$$

The constants b_c, b_s and ϕ must be determined from the initial/boundary conditions.

In this case, for waves described by **D'Alembert non-dispersive wave-equation** with fixed v, the **Dispersion Relation** (read § 44.03.08) is somewhat of a trivial dispersion relation, in the sense that there is no dispersion at all: these waves are non-dispersive waves, that is, waves whose **phase-velocity** is independent of ω and k, read § 44.03.08.

Consider the plucked string of a guitar, whose ends are fixed, so that $\xi[z=0,t]=0$ and $\xi[z=L,t]=0$:

$$\xi[z=0,t]=0 \implies b_c = 0 \quad ,$$

$$\xi[z=L,t]=0 \implies \sin[kL]=0 \quad ,$$

$$\implies k = q\frac{\pi}{L} \qquad q \in \mathbb{Z} \qquad k \text{ is quantized} \implies \omega = qv\frac{\pi}{L} \qquad \omega \text{ is quantized} \quad ,$$

$$b_s \qquad \text{is arbitrary} \quad .$$

44.02.02.05 Modes - Standing Waves With Arbitrary Time Dependence

The phenomena in one dimension described so far concerns waves propagating in an infinite medium. If the system is bounded (closed) in such a way that the energy must remain inside a finite volume some other different phenomena appear.

Standing waves in one dimension are separable solutions of the wave equations, that is solutions such that the space and time parts are factorized:

$$\boxed{\xi[z,t] = f[z]g[t]} \quad .$$

In modes of oscillations **all the oscillators oscillate with the same time dependence but different amplitudes**.

For a separable solution one finds that:

$$\frac{1}{f[z]}\frac{d^2 f}{dz^2} = \frac{1}{v^2 g[t]}\frac{d^2 g}{dt^2} = -k^2 \quad .$$

In fact as the two members depend on a priori different variables they must be constant and the constant is negative because of the oscillatory character of the phenomena.

It follows:

$$\frac{d^2 f}{dz^2} + k^2 f[z] = 0 \qquad \frac{d^2 g}{dt^2} + \omega^2 g[t] = 0 \quad ,$$

so that the solution for the standing wave is:

$$\xi[z,t] = A \cos[kz + \phi_z] \cos[\omega t + \phi_t] \quad .$$

Boundary conditions must be satisfied, imposed by the specific problem.

Examples of standing waves in one dimension include the following.

- Standing waves on a string, like on a guitar.
- In quantum physics wave-functions are solutions to Schrödinger equation.

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44.02.02.06 Oscillations of a Closed System: Standing Waves in Bounded 1D Regions

Read § 47.

Example: the vibrating string with fixed ends.

44.02.03 Three-Dimensional d'Alembert Non-Dispersive Wave Equation

This § is referenced at pages:
[1995]

The three-dimensional wave equation for the scalar quantity $\xi[x,t]$, whatever the meaning of the physical quantity ξ, is:

$$\boxed{\nabla^2\xi = \frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2}} \quad , \tag{44.02.07}$$

→ 1997 2003 2005

where v is the propagation speed of the waves, the so-called **phase-velocity** .

Here, only non-dispersive waves such that the **phase-velocity** , v, is a fixed constant independent of the frequency of monochromatic wave, will be considered here.

44.02.03.01 General Solution for Non-Dispersive Waves in Unbounded Media

In 3D, a simple general solution as in 1D, equation (44.02.02), cannot be written.

44.02.03.02 Rigidly Traveling PW Solutions in 3D

This § is referenced at pages:
[Never referenced.]

Read § 44.02.02.02.

Consider non-dispersive waves in unbounded media, that is fixed v, in the whole three-dimensional space.

Let us consider general PW solutions, not necessarily monochromatic: this is the simplest generalization of the traveling wave in higher dimensions.

Strictly speaking, MPW do not exist in nature as they are made of a Sinusoidal|Cosinusoidal time-space behavior of infinite extent both in space and in time. However, they are often a very useful approximation.

Similarly, strictly speaking, PW do not exist in nature as well, as they are made of fronts of infinite extent. However, they are often a very useful approximation.

In the three-dimensional case any expression of the type

$$\boxed{\xi[\mathbf{x},t] = f[\,\mathbf{n}\cdot\mathbf{x} - vt] \qquad \text{a scalar wave}} \quad , \tag{44.02.08}$$

→ 2002 2004

where f is any arbitrary function, $\widehat{\mathbf{n}}$ is any arbitrary fixed unit vector and is a PW.

In fact:

$$f \equiv f[s] \implies \mathbf{grad}\xi = \widehat{\mathbf{n}}\frac{\mathrm{d}f}{\mathrm{d}s} \qquad \nabla^2\xi \equiv \mathbf{divgrad}\xi = (\,\mathbf{n}\cdot\mathbf{n})\,\frac{d^2f}{\mathrm{d}s^2} = \frac{d^2f}{\mathrm{d}s^2} \quad .$$

Therefore for any function f and any $\widehat{\mathbf{n}}$ the expression (44.02.08) is a solution, as long as v is a fixed constant.

While for the one-dimensional equation two arbitrary functions give all the (PW) solutions, for the three-dimensional equation one can have an infinite number of solutions by choosing any arbitrary function and any arbitrary unit vector, $\widehat{\mathbf{n}}$.

It is more convenient to write the **phase** as a dimensionless quantity, as it was done for the other cases discussed before, also adding a generic **phase constant** , ϕ:

$$\boxed{\phi \qquad \text{a fixed value}} \quad , \tag{44.02.09}$$

$$\boxed{\Phi\,[\mathbf{x},t] \equiv \mathbf{k}\cdot\mathbf{x} - \omega t - \phi \qquad \text{a function of } \mathbf{x} \text{ and } t} \quad . \tag{44.02.10}$$

→ 1999

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As the d'Alembert equation has only one parameter, the velocity v, the two quantities $\mathbf{k}$ and ω must be related in order to represent a solution to the wave equation. By direct substitution in equation (44.02.07) one finds the basic relation between the two quantities k and ω, the **Dispersion Relation** (read § 44.03.08):

$$v^2\,(\mathbf{k}\cdot\mathbf{k}) = v^2k^2 = \omega^2 \implies \boxed{v \equiv \frac{\omega}{k}} \quad .$$

In this case, for waves described by **D'Alembert non-dispersive wave-equation** with fixed v, the **Dispersion Relation** (read § 44.03.08) is somewhat of a trivial **Dispersion Relation** , in the sense that there is no dispersion at all: these waves are non-dispersive waves, that is, waves whose **phase-velocity** is independent of ω and k, read § 44.03.08.

The dimensionless phase of the wave, is the function of position and time

$$\Phi[\mathbf{x},t] \equiv \mathbf{k}\cdot\mathbf{x} - \omega t - \phi = k\,(\mathbf{n}\cdot\mathbf{x} - vt) - \phi \qquad \widehat{\mathbf{n}} \equiv \mathbf{k}/|\mathbf{k}| \quad ,$$

giving, at any time, the locus of the points in space where the value of the Perturbation|Disturbance|Signal has the same fixed value. At any time the locus of constant phase is a plane perpendicular to the unit vector $\widehat{\mathbf{n}}$: the unit vector $\widehat{\mathbf{n}}$ defines the direction in space.

Note: ϕ is the dimensionless **phase constant** , a fixed number; $\Phi[\mathbf{x},t]$ is the dimensionless **phase of the wave**, (phase, in short) a function of space and time coordinates. Obviously one can add any integer multiple of 2π to the **phase constant** without changing the value of the function. Normally one uses a value in the range $0 \le \phi \le 2\pi$.

The **phase-velocity** , that is the velocity of movement of constant-phase planes, is still v.

WORKED SOLUTION

The **phase-velocity** can be calculated as follows.
Consider one specific constant-phase wave-front, defined by a certain values of the phase, c. The phase of the wave will keep the same value, c, after a certain time interval Δt, at all the points of a plane which can be calculated as follows:

$$c = \Phi[\mathbf{x},t] = \mathbf{k}\cdot\mathbf{x} - \omega t - \phi = \mathbf{k}\cdot\mathbf{x} + \mathbf{k}\cdot\Delta\mathbf{x} - \omega t - v\Delta t - \phi = c \quad ,$$
$$\implies \mathbf{k}\cdot\Delta\mathbf{x} - \omega\Delta t = |\mathbf{k}|\,\widehat{\mathbf{n}}\cdot\Delta\mathbf{x} - \omega\Delta t = 0 \quad ,$$
$$\implies v = \omega/|\mathbf{k}| \quad .$$

As the quantity $\widehat{\mathbf{n}}\cdot\Delta\mathbf{x}$ represent the distance between the two planes with the same phase the above relation shows that v is the **phase-velocity** .
In the special case of $\Delta t = T$ one has $\widehat{\mathbf{n}}\cdot\Delta\mathbf{x} = \lambda$.
An alternative derivation follows.
Consider any two points, $\mathbf{x}_1$ and $\mathbf{x}_2$, on two planes with the same phase at different times t_1 and t_2. That is the second plane is the first one moved between t_1 and t_2. One has:

$$\widehat{\mathbf{n}}\cdot(\mathbf{x}_2 - \mathbf{x}_1) - v\,(t_2 - t_1) = 0 \quad ,$$

where $\widehat{\mathbf{n}}\cdot(\mathbf{x}_2 - \mathbf{x}_1)$ is the distance between the two planes and the above equation describes the kinematics of the planes at constant phase. If the above relation is satisfied then ξ has the same values at the two points $\{\mathbf{x}_1, t_1\}$ and $\{\mathbf{x}_2, t_2\}$.
Q.E.D.

Note that at large distance from the source, whatever the source, all waves can be approximated as PW.

44.02.03.03 Rigidly Traveling MPW Solutions in 3D

This § is referenced at pages:
[1999, 2063]

Let us now consider MPW solutions.

Consider non-dispersive waves in unbounded media, that is fixed v, in the whole three-dimensional space.

It is any PW as in equation (44.02.08) with $f[w] = \sin[w - \phi_s]$ or $f[w] = \cos[w - \phi_s]$. It is harmonic in both space and time.

44.02.03.04 Isotropic Cylindrical Waves Solutions for Non-Dispersive Waves in Unbounded Media

This § is referenced at pages:
[2009]

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Consider non-dispersive waves in unbounded media, that is fixed v, in the whole three-dimensional space.

Consider a isotropic two-dimensional traveling wave propagation (water waves on a water surface, for instance): the amplitude decreases as energy is spread along the circumference.

Consider either cylindrical waves in a three-dimensional problem or circular waves in a two-dimensional problem. Assume waves isotropic in the plane perpendicular to the axis of the cylinder.

A isotropic cylindrical wave profile, moving rigidly in the radial direction with constant amplitude, is described by:

$$\xi[r,t] = f[r \pm vt] \quad ,$$

where the $\pm$ sign describes ingoing/outgoing waves. Following the one-dimensional case we can try this kind of solution.

However, if energy is proportional to the square of the amplitude then energy conservation makes the latter relation inconsistent and requires

$$\boxed{\xi[r,t] \approx \frac{\xi_0}{\sqrt{r}}} \quad .$$

The Laplacian in cylindrical coordinates for systems with cylindrical symmetry reads

$$\nabla^2\xi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \xi}{\partial r}\right) = \frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r}\frac{\partial \xi}{\partial r} \quad .$$

The exact solutions in this case are given in terms of the so-called Bessel functions [1], which typically arise in problems with cylindrical symmetry. It can be shown that, for $kr \gg 1$, they can be approximated as

$$\boxed{\xi[r,t] \sim \frac{\xi_0}{\sqrt{r}}\cos[kr \pm \omega t - \phi]} \quad .$$

Note that, unlike the plane and spherical waves, **cylindrical waves cannot assume an arbitrary functional form**.

[1] WEB - URL

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44.02.03.05 Isotropic Spherical Waves Solutions for Non-Dispersive Waves in Unbounded Media

This § is referenced at pages:
[2009]

©|E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|§ 2||

Consider non-dispersive waves in unbounded media, that is fixed v, in the whole three-dimensional space.

Consider a isotropic three-dimensional traveling wave propagation (acoustic waves in three dimensions, for instance): the amplitude decreases as energy is spread along the sphere.

Consider spherical waves in a three-dimensional problem. Assume isotropic waves. A isotropic spherical wave profile, moving rigidly in the radial direction with constant amplitude, is described by:

$$\xi[r,t] = f[r \pm vt] \quad ,$$

where the $\pm$ sign describes ingoing/outgoing waves. Following the one-dimensional case we can try this kind of solution.

If energy is proportional to the square of the amplitude of the wave energy conservation makes the latter relation inconsistent and implies

$$\boxed{\xi[r,t] \approx \frac{\xi_0}{r}} \quad .$$

The latter result is made more precise by solving the three-dimensional wave equation (44.02.07) in spherical coordinates for a isotropic scalar function: $\xi[r,t]$.

The Laplacian in spherical coordinates for systems with spherical symmetry reads

$$\nabla^2 \xi[r,t] \longrightarrow \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\xi[r,t]) = \frac{\partial^2 \xi}{\partial r^2} + \frac{2}{r}\frac{\partial \xi}{\partial r} \quad .$$

The general solution follows from the general solution of the wave equation, in terms of two arbitrary functions:

$$\boxed{\xi[z,t] = \frac{f_{\oplus}[r-vt] + f_{\ominus}[r+vt]}{r}} \quad .$$

44.02.03.06 Oscillations of a Closed System: Standing Waves in Bounded 3D Regions

This § is referenced at pages:
[1995, 2000, 2157]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ... 2.3|||

©|J.P.Pérez *et al.*, *Optique*, ..., DUNOD, ...Ed., WEB - URL.|19.6||

©|A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...|§ 6||

Read § 47.

Examples: transmission lines, ElectroMagnetic cavities, water waves on a swimming pool.

Consider non-dispersive waves in bounded media, that is fixed v, in a bounded (limited) three-dimensional region, such that it is contained inside some sphere.

The phenomena in three dimensions described so far concerns waves propagating in an infinite medium. If the system is bounded (closed) in such a way that the energy must remain inside a finite volume some other different phenomena appear.

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Standing waves in three dimensions are separable solutions of the wave equations, that is solutions such that the space and time parts are separated:

$$\boxed{\xi[\mathbf{r},t] = f[\mathbf{r}]g[t]} \quad .$$

In modes of oscillations **all the oscillators oscillate with the same time dependence but different amplitudes**.

Boundary conditions must be satisfied, imposed by the specific problem.

Examples of standing waves in three dimensions include the following.

- Standing waves on a drum.
- Wave-guides and resonant cavities.
- In quantum physics wave-functions are solutions to Schrödinger equation.

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44.03 Understanding Real Wave Phenomena

This § is referenced at pages:
[1993]

In this section some general properties and phenomena related to waves are just mentioned and introduced, while their study, for the different types of waves, is postponed to the next sections.

Wave phenomena arise everywhere in the world, but the phenomena are often much more complex than the simple **D'Alembert non-dispersive wave-equation** can describe. The main complications include non-linearity, which is absent for linear d'Alembert waves, and dispersion.

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44.03.01 Oscillators at Different Places With Defined Relative Phase

The concept of wave can be better understood starting from **a set of localized oscillating systems** as follows.

An oscillating system is something physically localized in space and characterized by some physical property oscillating in time (a particular and important case is that of harmonic oscillations). Physical waves can be considered to be built from many (possibly infinite) different oscillators, distributed at in some space region, with some oscillating physical property.

Neighboring oscillators oscillate with a well-defined relative relationship among each-other, described by the wave equation.

In fact any wave consists of **oscillations both in space and in time**.

The oscillating physical property might be periodic both in time and in space.

In any case the wave equation implies a well-defined relation between the time-oscillation of oscillators located at neighboring points in space and the variation in space of the oscillating physical property.

The *Mexican wave* (read § 44.01.03) is clarifying: each supporter must stand-up and sit-down at the right moment, depending on what his neighboring supporters are doing.

44.03.02 Scalar Vector and Tensor Waves

The case when the physical quantity propagated by the wave is a scalar quantity will be mainly considered in the exemplifications.

44.03.02.01 MP Scalar Vector and Tensor Waves

Consider a generic wave-field, W, $\mathbf{W}$, $\mathbb{W}$:

$$W[\mathbf{r},t] = W_0 \cos\left[\mathbf{k}\cdot\mathbf{r} - \omega t - \delta\right] \quad ,$$
$$\mathbf{W}[\mathbf{r},t] = \mathbf{W}_0 \cos\left[\mathbf{k}\cdot\mathbf{r} - \omega t - \delta\right] \implies W_i[\mathbf{r},t] = (W_i)_0 \cos\left[\mathbf{k}\cdot\mathbf{r} - \omega t - \delta\right] \quad ,$$
$$\mathbb{W}[\mathbf{r},t] = \mathbb{W}_0 \cos\left[\mathbf{k}\cdot\mathbf{r} - \omega t - \delta\right] \implies W_{ij}[\mathbf{r},t] = (W_{ij})_0 \cos\left[\mathbf{k}\cdot\mathbf{r} - \omega t - \delta\right] \quad .$$

Whenever vector/tensor waves are considered the vector/tensor nature of the physical quantity usually imposes additional relations/constraints. Examples:

- scalar waves: longitudinal elastic waves (acoustic waves);
- vector waves: ElectroMagnetic (transverse) waves, transverse vibrations on a string;
- tensor waves: gravitational waves.

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44.03.03
Velocity of the Oscillating Field

Let the oscillating physical quantity, that is the propagating disturbance, be $\xi[\mathbf{r},t]$: the quantity $\dot{\xi}[\mathbf{r},t]$ represents a velocity. If the quantity $\boldsymbol{\xi}[\mathbf{r},t]$ represents a displacement, this velocity is the velocity associated to the displacement (as for mechanical waves); in other cases the velocity may have a different meaning, for instance in EM waves where it is the time-derivative of the electric or magnetic field.

The $\dot{\xi}[\mathbf{r},t]$ velocity must not be confused with any propagation velocity of the associated wave, such as phase-velocity or group-velocity .

44.03.04
Longitudinal and Transverse Vector Waves

When the physical field is a **vector field** (which includes the two important cases of mechanical waves and ElectroMagnetic waves) one can introduce the distinction between **longitudinal waves** and **transverse waves**. In the simplest case of PW this is defined with respect to the direction of propagation of the wave, $\mathbf{k}$, that is for any given wave-vector $\mathbf{k}$, as follows.

- If the field, $\boldsymbol{\xi}$, of the PW is parallel to $\mathbf{k}$ we say **the wave is longitudinal** (such as: acoustic waves).
- If the field, $\boldsymbol{\xi}$, of the PW is perpendicular to $\mathbf{k}$ we say **the wave is transverse** (such as: transverse waves on a string, plane ElectroMagnetic waves, torsion waves on a bar).
  Transverse waves require the additional concept of **wave-Polarization** to be described, because, for a transverse vector field, all directions in the plane perpendicular to $\mathbf{k}$ are allowed, in principle, so that the concept of **wave-Polarization** is required to fully describe the field (that is, one has to describe the two degrees of freedom in the plane perpendicular to $\mathbf{k}$). Read § 46.03 for the full description of the concept of **wave-Polarization** introduced for the case of ElectroMagnetic waves; the same applies to transverse waves on a string.

There are cases of waves which have both a longitudinal and a transverse component, such as: water waves and elastic waves in solid media. For elastic waves in solid media: In the most general case:

- longitudinal waves are defined as waves such that: $\mathbf{rot}\boldsymbol{\xi} = 0$;
- transverse waves are defined as waves such that: $\mathbf{div}\boldsymbol{\xi} = 0$.

44.03.05
General Time-Harmonic Waves

©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|§ 1.3||

Harmonic waves are waves such that at any fixed point in space the time dependence of the wave disturbance is harmonic

$$\xi[\mathbf{r},t] = G[\mathbf{r}]\exp\left[\mathtt{i}\left(F[\mathbf{r}]-\omega t\right)\right] \equiv G[\mathbf{r}]\exp\left[\mathtt{i}\left(\Phi[\mathbf{r},t]\right)\right] \equiv G[\mathbf{r}]\exp\left[\mathtt{i}\left(F[\mathbf{r}]\right)\right]\exp\left[-\mathtt{i}\omega t\right] \quad , \qquad (44.03.01)$$

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$$\frac{\partial\Phi[\mathbf{r},t]}{\partial t} \equiv -\omega \qquad \text{if and only if time-harmonic wave} \quad , \qquad (44.03.02)$$

$$\mathbf{grad}\Phi[\mathbf{r},t] \equiv \mathbf{k}[\mathbf{r},t] \qquad \text{if and only if time-harmonic wave, local MPW} \quad . \qquad (44.03.03)$$

MPW is a special case of harmonic waves, such that also the space dependence is sinusoidal.

Read § 19.02 for the important and useful relations about the sum of harmonic functions with the same frequency.

Also Read § 44.04.03.

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44.03.05.01 Constant-Phase Surfaces for Harmonic Waves

Consider harmonic waves [1].

A **constant-phase wave-front** ([glo:CPWF]CPWF) is defined as the locus of points in space such that the phase in time of the different oscillators in space is the same. Different CPWF are defined by all possible different values of the phase, between zero and 2π. Harmonic Waves propagates such that any CPWF, moves in space with a well-defined kinematics (speed). The velocity of CPWF is the so-called **phase-velocity** .

Note that the **phase-velocity** does not, in general, represent the velocity information, signals, energy or other physical quantities are moving. The problem of the velocity of transport of information, signals, energy or other physical quantities is a very complex one and the concept of **phase-velocity** is not sufficient to describe all the complexity.

Note that a CPWF is the locus of points with equal phase of the time oscillation, not equal amplitude. In fact, for waves propagating in more than one dimension the amplitude decreases even when there is no absorption.

44.03.05.02 Constant-Amplitude Surfaces for Harmonic Waves

Consider harmonic waves [2].

The surfaces $G[\mathbf{r}] = \text{constant}$ are called **constant-amplitude wave-front** ([glo:CAWF]CAWF).

In general, CPWF and CAWF do not coincide.Such a wave is said to be a non-homogeneous one.

44.03.06
Energy Linear Momentum and Angular Momentum

Waves may carry energy, linear momentum and angular momentum. As a well-known example it is common experience that water waves on the ocean carry energy as they are capable both to set into oscillation a ship (that is they impart kinetic energy to the ship) and to raise the quota of the ship (that is they impart gravitational potential energy to the ship). Water waves can also set in motion surfers, imparting them a significant speed provided the surfer can catch the right kinematics. Energy must be put into the system to cause the initial disturbance and it is this energy that is then transmitted by the wave.

44.03.07
Shape and Energy

Energy is clearly related the amplitude of the oscillations.

In case of two-dimensional and three-dimensional traveling waves it is common experience that the amplitude of the wave decreases with the distance from the source and this is due to the energy spreading over larger and larger regions of space. In fact, in two or three dimensions, while the wave spreads out, its amplitude will decrease in order to conserve the energy.

In presence of absorption, even in one dimension, the wave will decrease its amplitude as it moves. However, waves with absorption are not described by **D'Alembert non-dispersive wave-equation** .

Water waves and the plucked string of a guitar, as a common example, show clearly that often (but not always) waves, at least to a first approximation, are perturbations/disturbances either moving or stationary without changing their shape or keeping the same shape but with a progressively decreasing amplitude or even changing (often slowly) their shape.

A rather reasonable assumption is that a wave carries an energy per unit area per unit time proportional to the square of the amplitude of the wave. In fact we know that kinetic energy is proportional to the square of the speed and elastic potential energy is proportional to the square of the deformation; moreover, the energy of ElectroMagnetic fields depends on the square of the ElectroMagnetic field intensity. Read § 44.02.03.04, § 44.02.03.05.

[1] In the literature there is a variety of nomenclature. Here, we will use wave-front and front as generic nouns, while reserving CPWF and CAWF for a more precise/technical usage.

[2] In the literature there is a variety of nomenclature. Here, we will use wave-front and front as generic nouns, while reserving CPWF and CAWF for a more precise/technical usage.

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44.03.08
Dispersion Relation

This § is referenced at pages:
[1425, 1994, 1999, 1999, 1999, 1999, 1999, 1999, 2000, 2000, 2001, 2001, 2001, 2001, 2002, 2003, 2003, 2164, 2164]

Read § 18.04.02.

The following basic fact must be stressed, as it might be confusing for the special case of the **D'Alembert non-dispersive wave-equation** with fixed v: there is a neat distinction between the **dispersion relation** and the **phase-velocity relation**.

The **Dispersion Relation is always**, for any kind of MPW, whatever its wave equation, the relation between frequency and wave-vector:

$$\boxed{\omega = \omega[\mathbf{k}]} \quad , \tag{44.03.04}$$

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which is obtained replacing any MPW into the wave equation and imposing the MPW to be a solution.

An equation is **dispersive** if $\omega[k]$ is real and its second derivative is different from zero: $\omega''[k] \neq 0$. If $\omega[k]$ is complex, the equation is **diffusive**.

Note that, for a isotropic medium:

$$\boxed{\omega = \omega[|\mathbf{k}|] \qquad \text{isotropic medium}} \quad . \tag{44.03.05}$$

In general, a complex exponential solution must be used to replace in the wave equation and find the **Dispersion Relation** :

$$\boxed{\overline{\xi}[\mathbf{r},t] = \overline{A}\exp\left[\mathtt{i}\left(\overline{\mathbf{k}}\cdot\mathbf{r} - \omega t\right)\right] \implies \omega = \omega[\overline{\mathbf{k}}]} \quad ,$$

where the wave-vector and amplitude are complex (read § 44.04.01 and references therein). A complex wave-vector allows to describe harmonic solutions which are either attenuating or amplificating waves.

The **phase-velocity relation is always**, in 1D, for any kind of MPW, whatever its wave equation:

$$\boxed{(v_p)_z \equiv \frac{\omega[k_z]}{k_z} \qquad \text{one dimension only, } z \text{ axis}} \quad ,$$

$$\boxed{v_p \equiv \frac{\omega[k]}{|\mathbf{k}|} \qquad \text{general, velocity of the constant-phase planes}} \quad . \tag{44.03.06}$$

It is therefore derived from the **Dispersion Relation** . Note that **phase-velocity** is defined and kinematically meaningful for MPW only: velocity of the constant-phase planes, in the direction perpendicular to the planes. For standing waves this kinematic interpretation breaks-down.

The **phase-velocity** is not a vector, even though it has a magnitude and a direction. In fact, it does not transform like a vector under rotations.

In the special case of the **D'Alembert non-dispersive wave-equation** with fixed v the definition of the **phase-velocity** and the **Dispersion Relation** coincide and this may lead to some confusion.

If the wave is dispersive then MPW with different frequencies will travel at different speeds so that a generic waveform might not keep the same shape as it moves.

The **group-velocity** of a wave is defined by:

$$\boxed{(v_g)_z \equiv \frac{\mathrm{d}\omega[k_z]}{\mathrm{d}k_z} \qquad \text{one dimension only, } z \text{ axis}} \quad ,$$

$$\boxed{\mathbf{v}_g \equiv \mathfrak{Re}\left[\mathbf{grad}\omega[\mathbf{k}]\right] \qquad \text{general}} \quad . \tag{44.03.07}$$

and it represents, to a first approximation, the speed of propagation of the energy of a wave, for moderately dispersive media and packets localized enough: the group-velocity represents the change of phase-velocity for the relevant frequencies. Read § 44.03.11 and § 18.04.02.

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44.03.09
Dispersion and Dispersion Relation

©|A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...||*A nice introduction.*|

©|King|§ 8|*A nice introduction.*|

©|L.D.Landau *et al.*, *Electrodynamics of continuous media*, ..., ..., ...Ed., ...|§ 103||

©|D.Morin|WEB - URL OR WEB - URL |*Very nice and clear introduction to dispersion.*|

©|G.B.Whitham|Linear and Nonlinear Waves|*WEB - URL - Advanced*|

The speed of ElectroMagnetic waves in a vacuum does not depend on the frequency of the MPW: ElectroMagnetic waves in vacuum are exactly non-dispersive.

Also ElectroMagnetic waves in a **perfect dielectric medium** are exactly non-dispersive.

In matter, often to a good approximation, almost-MPW ElectroMagnetic waves suffer low dispersion and low attenuation.

However, in general, ElectroMagnetic waves in matter suffer dispersion and attenuation.

To a very good approximation the velocity of sound waves in air is independent of the frequency of the MPW. If that would not be the case, people at different distances from the center of an auditorium would hear a different musical experience.

However, the idealization that MPW of any frequency will travel at the same speed will fail, to some extent, for most real physical systems.

Dispersion is the dependence of the **phase-velocity** on the frequency of the MPW. Dispersion has a profound effect on the behavior of a wave. For instance, dispersion of light causes such effects as the rainbow.

In order to get the **Dispersion Relation** for any linear homogeneous system of partial differential equations: replace a Sinusoid|Cosinusoid into the equation, use the boundary conditions and ask for the relation to be identically satisfied. This requirement implies a relation on some coefficients: this is the **Dispersion Relation** .

Non trivial examples we will study are the following: EM waves in **perfect Ohmic media**; EM waves in plasma; EM waves in a wave-guide.

44.03.09.01 Dispersion Relation for ElectroMagnetic Waves in Vacuum

These are exactly described by the classical **D'Alembert non-dispersive wave-equation** .

They show, exactly, a simple **Dispersion Relation** , the frequency is strictly proportional to the wave-number and the proportionality constant in the speed of light in vacuum:

$$\boxed{\omega = \omega[k] = kc} \quad .$$

44.03.09.02 Dispersion Relation for ElectroMagnetic Waves in Matter

These are not described, in general, by the classical **D'Alembert non-dispersive wave-equation** .

In general, ElectroMagnetic waves in matter are dispersive. In fact, matter is made by charges in motion, which can have a resonant response at different frequencies; therefore it is to be expected that waves of different frequencies travel with different kinematics. Read § 53.

In **perfect Ohmic media**, the ElectroMagnetic wave equation reads (read § 46.06):

$$\boxed{\boldsymbol{\nabla}^2\mathbf{E} - \sigma\mu_{\mathrm{R}}\mu_0\partial_t\mathbf{E} - \varepsilon_{\mathrm{R}}\varepsilon_0\mu_{\mathrm{R}}\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2} = 0} \quad ,$$

$$\boxed{\boldsymbol{\nabla}^2\mathbf{B} - \sigma\mu_{\mathrm{R}}\mu_0\partial_t\mathbf{B} - \varepsilon_{\mathrm{R}}\varepsilon_0\mu_{\mathrm{R}}\mu_0\frac{\partial^2\mathbf{B}}{\partial t^2} = 0} \quad .$$

The resulting **Dispersion Relation** is equation (54.02.04).

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44.03.09.03 Dispersion Relation for ElectroMagnetic Waves in a Plasma

These are not described by the classical **D'Alembert non-dispersive wave-equation** .

The **Dispersion Relation** of ElectroMagnetic waves propagating through an unMagnetized plasma is:

$$\omega = \pm\sqrt{k^2c^2 + \omega_p^2} \quad ,$$

in term of the so-called plasma frequency, ω_p, which depends on the properties of the plasma.

44.03.09.04 Dispersion Relation for ElectroMagnetic Waves in Wave-Guides

Read § 47.04.01, § 47.04.02.

44.03.09.05 Dispersion Relation for Surface Water Waves

This § is referenced at pages:
[2055]

These are not described by the classical **D'Alembert non-dispersive wave-equation** .

For pure gravity waves, that is waves characterized by a negligible effect of the surface tension, the **Dispersion Relation** , in the linear regime, is given by:

$$\omega^2 = gk\tanh[kh] \implies \begin{cases} kh \gg 1 \implies \omega = \sqrt{gk} & \text{deep-water waves, dispersive} \\ kh \ll 1 \implies \omega = k\sqrt{gh} & \text{shallow-water waves, non dispersive} \end{cases} ,$$

as a function of the sea depth, h.

In general, the **Dispersion Relation** , in the linear regime, is given by:

$$\omega^2 = \left(gk + k^3\frac{\gamma}{\rho}\right)\tanh[kh] \quad ,$$

as a function of the sea depth, h, and surface tension γ.

44.03.09.06 Dispersion Relation for Linear Mass-Spring Chain

Consider the linear mass-spring chain of § 25.07.04 and its **Dispersion Relation** (equations (25.07.01), (25.07.02), (25.07.03)). In that case the wavelength, $\lambda_p = 2\pi/k_p$, associated to the mode p, but the corresponding frequency, ω_p, is not simply proportional to p. In fact:

$$\omega_p \propto 2\omega_0 \sin\left[\frac{ak_p}{2}\right] \qquad k_p \propto p/a \qquad p = 1, \dots N \quad ,$$

$$v_p \propto \frac{2\omega_0}{k_p}\sin\left[\frac{ak_p}{2}\right] \quad ,$$

$$p \ll N \implies ak_p \ll 1 \implies v_p \simeq a\omega_0 \quad .$$

The above relations show that in general the **phase-velocity** is different for different modes, that is frequency. Only for low-frequency modes the phase velocity are about the same. It is physically clear that, for a number N of points in the chain, it is not possible to have Sinusoidal|Cosinusoidal waves with $p > N$. This phenomenon is known as cut-off frequency: the maximum possible frequency is $2\omega_0$.

All the previous discussion shows that one has to be careful when modeling a real system made of a large number of entities as a continuous system.

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44.03.09.07 Dispersion Relation for a Real Vibrating String

For a non-ideal string, where stiffness is taken into account, the **Dispersion Relation** is written as:

$$\boxed{\omega^2 = k^2 \left(T/\rho_l + ak^2\right)} \quad ,$$

where a is a constant that depends on the string, T is the tension of the string and ρ_l the linear mass density.

44.03.09.08 Dispersion Relation for Free Schrödinger Equation

The **Dispersion Relation** for a free non-relativistic particle, $V = 0$, is:

$$\boxed{\omega = \frac{\hbar k^2}{2m}} \quad ,$$

see equation (44.01.03).

44.03.09.09 Dispersion Relation for Free Klein-Gordon Equation

The Klein-Gordon equation, which arises in relativistic quantum physics, gives the **Dispersion Relation** for a free particle:

$$\boxed{\omega = \pm\sqrt{\alpha^2 k^2 + \beta^2}} \quad ,$$

with constants α and β.

44.03.09.10 Dispersion and High-Frequency Cut-Off

©|A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...|§ 7||

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A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...
Dispersion is linked to the effect known as cut-off. It is the inability of a dispersive system to transmit waves above or below a certain critical (cut-off) frequency. The effect high-frequency cut-off is clear in the analysis of the normal modes of the spring-mass chain, read § 25.07.04 and in the case of ElectroMagnetic wave-guides, read § 46.09.
No frequency higher that the maximum frequency can exist. However, nothing prevents us from applying a forcing at an higher frequency. What does happen in this case?
It can be shown that for $\omega > \omega_0$ the wave is damped and for $\omega \gg \omega_0$ it is strongly damped so that the structure behaves like a rigid structure.

44.03.10 Attenuation/Dissipation

Attenuation/dissipation refers to the wave loosing energy while it propagates. The wave might possibly increase its energy if something is pumping energy into the wave, such as in a LASER.

Suppose ω is real but the **Dispersion Relation** leads to a complex **k**: the wave is attenuated or, possibly, enhanced depending on the sign of the imaginary part of the complex wave-vector.

If **k** is purely imaginary there is no oscillation at all but the wave is exponentially damped (or, possibly, enhanced).

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44.03.11
Modulation and Beats

This § is referenced at pages:
[1163, 1164, 1165, 2010, 2015, 2023, 2149, 2164, 2261]

Consider the superposition between two one-dimensional MPW with different frequencies and same amplitudes and phase constants traveling in the same direction, in case the **Dispersion Relation** $\omega = \omega[k]$ is just a generic function, depending on the physical problem:

$$\boxed{\omega = \omega[k]} \quad ,$$

$$\xi[z,t] = A\cos[k_1 z - \omega_1 t] + A\cos[k_2 z - \omega_2 t] = \quad ,$$

$$2A\cos\left[\frac{k_1 - k_2}{2}z - \frac{\omega_1 - \omega_2}{2}t\right]\cos\left[\frac{k_1 + k_2}{2}z - \frac{\omega_1 + \omega_2}{2}t\right] \equiv 2A\cos[k_{\text{MOD}}z - \omega_{\text{MOD}}t]\cos\left[\overline{k}z - \overline{\omega}t\right] \quad ,$$

$$\boxed{\xi[z,t] = A_{\text{MOD}}[z,t]\cos\left[\overline{k}z - \overline{\omega}t\right]} \quad ,$$

$$\boxed{k_{\text{MOD}} \equiv \frac{k_1 - k_2}{2} \qquad \omega_{\text{MOD}} \equiv \frac{\omega_1 - \omega_2}{2}} \quad .$$

Note that the assumptions of equal amplitudes and zero **phase constants** are not necessary at all, but only make the mathematics simpler.

If $\omega_{\text{MOD}} \ll \overline{\omega}$, that is $\omega_1 \simeq \omega_2$**, the combination is a fast Sinusoid|Cosinusoid, at the high frequency** $\langle\omega\rangle$**, whose amplitude is modulated by a slowly varying Sinusoidal|Cosinusoidal amplitude** $A_{\text{MOD}}[z,t]$, at a low-frequency ω_{MOD}. The so-called **carrier** frequency, $\overline{\omega} \equiv \langle\omega\rangle$, has about the same frequency as the two components, $\omega_1 \simeq \overline{\omega} \simeq \omega_2$ and a Sinusoidal|Cosinusoidal amplitude slowly varying.

By superposition of many frequencies one can get different wave-forms for the amplitude, thus generating complex signals via the amplitude of a basic frequency, the carrier. This is the basis of **amplitude modulation techniques**.

The amplitude moves at a velocity, the so-called **group-velocity** :

$$\boxed{\frac{\omega_{\text{MOD}}}{k_{\text{MOD}}} = \frac{\omega_2 - \omega_1}{k_2 - k_1} \simeq \frac{\mathrm{d}\omega}{\mathrm{d}k} \equiv v_g} \quad . \tag{44.03.08}$$

→ 1163 1166 2149

If the two disturbances are very close in frequency the combined disturbance exhibits the so-called called beats: the combined vibration is basically a disturbance with a frequency equal to the average of the two frequencies and an amplitude which varies sinusoidally with time with one wavelength including many wavelengths of the fast oscillation.

Note that depending on the **group-velocity** being smaller or larger than the **phase-velocity** the oscillations inside the envelope may go either backward or forward with respect to the envelope. The waveform travels as a rigid profile if and only if the **group-velocity** and **phase-velocity** are identical.

While mathematically the result is always valid the physical interpretation in terms of beats is only meaningful whenever the condition

$$\boxed{\omega_{\text{MOD}} \ll \overline{\omega} \qquad \text{beats}} \quad , \tag{44.03.09}$$

is satisfied.

The intensity is:

$$\boxed{I \propto (A_{\text{MOD}})^2[z,t]\cos^2\left[\overline{k}z - \overline{\omega}t\right] \qquad \text{beats}} \quad .$$

The envelope travels with a frequency $\Delta\omega = |\omega_1 - \omega_2|$, as the square doubles the frequency. This gives rise to beats: at a fixed point in space the intensity increases and decreases with a frequency determined by the difference of the original frequencies.

PREPUBLICATION

44.03.12
Real waves - Wave Packets

Read § 18.04.03.

44.03.13
Wave Impedance

This § is referenced at pages:
[2041, 2041, 2187, 2187]

Typically two properties characterize wave propagation (at least for classical d'Alembert waves).

- EM waves in LHI media:

$$\varepsilon_{\mathrm{R}} \oplus \mu_{\mathrm{R}} \quad \Leftrightarrow \quad Z \oplus v = c/n \quad .$$

  For ElectroMagnetic waves the interpretation is: large/small **wave impedance** means that the electric field **E** is large/small with respect to the field **H**.
- The impedance of mechanical waves is, in all generality, defined as the ratio between the force and the velocity of the oscillating part: it has therefore dimensions of $\mathrm{N/(m/s)} = \mathrm{kg/s}$:
  - Mechanical longitudinal waves:

$$Y \oplus \rho \quad \Leftrightarrow \quad Z = S\sqrt{\rho Y} \oplus v = \sqrt{\frac{Y}{\rho}} \quad .$$

  - Mechanical transverse waves on a string:

$$T \oplus \lambda \quad \Leftrightarrow \quad Z = \sqrt{\lambda T} \oplus v = \sqrt{\frac{T}{\lambda}} \quad .$$

  For mechanical waves, according to definition (45.08.05), a medium with large/small Z is a hard/soft medium as far as the propagation of waves is concerned, because for a given force the resulting velocity is small/large.

In both cases the wave equation gives a relation between the pair of quantities entering the definition of the **wave impedance** and the two quantities enter the expression of the power/intensity carried by the wave. This, in the end, explains the motivation for introducing the concept and its usefulness.

44.03.14
MPW Versus Real Waves

This § is referenced at pages:
[2143]

MPW are the basic building blocks to build any kind of waves, that is any solution to the wave equation. In fact, linearity implies that any superposition of solutions MPW is still a solution. However, real waves are more complicated than MPW.

- The CPWF of real waves are not planes, in general. However, at a distance R very far enough from the source of dimension D, $R \gg D$, any CPWF can be approximated by a spherical wave. Furthermore, if one studies only a small part of the CPWF of size L, with $L \ll R$, it can be approximated by a PW. Therefore, when one studies waves far away from the source, either PW or spherical waves can be assumed.
  Note that a plane/spherical wave is not necessarily monochromatic but the concept of CPWF is defined for any MW.
- Transverse waves are often the superposition of waves with different states of **wave-Polarization** .

Read § 46.15.01.

- A strictly MPW does not exist, as it is infinite both in space and in time. A better model is a finite sinusoid, a so-called wave-train lasting for times $\Delta\tau$. Read § 18.
- A non-MPW can be made by superposition of different frequencies. Superposition of two MPW at two different frequencies may give rise to: non-rigid movement of the waveform, whenever the **phase-velocity** of the two MPW are different; amplitude modulation; the concept of **group-velocity** . Read § 44.03.11.
- In general any waveform can be expressed, via the **Fourier Series** | **Fourier Integral** (see section § 18) as a Series|Integral of MPW with complex coefficients of the different MPW.

44.03.15
Doppler Effect

©|H.J. Pain, The physics of vibrations and waves, John Wiley and Sons, 2005.|§ 15||

The **Doppler effect** is a general wave phenomenon showing up when the source of the waves and the observer/receiver have a relative motion.

The **Doppler effect** consists in the fact that the frequency of the wave perceived by the observer/receiver is, in general, different from the frequency of the wave emitted by the source.

The effect exists for all types of waves but it may differ, from the quantitative point of view, for different types of waves. In particular there is a difference between the case of mechanical waves and the case of ElectroMagnetic waves.

The **Doppler effect** is a very important physical phenomenon, with many applications and uses.

The **Doppler effect** for mechanical waves is presented in section § 45.10.

The **Doppler effect** for ElectroMagnetic waves is presented in section § 46.12.

44.03.16
Interference and Diffraction

Interference and diffraction are closely-related characteristic phenomena of waves.

Interference is discussed in section § 50. Basic evidence: Young experiment and similar.

Diffraction is discussed in section § 51. Basic evidence: limited resolution of optical instruments.

Interference and diffraction have been fundamental concepts in the historical evolution of the wave-particle duality saga.

44.03.17
Classification of PDE Solutions

©|A.D.Polyanin *et al.*|Handbook of non Linear Partial Differential Equations|*Chapman Hall/CRC Press, 2012, 2nd Ed*|

Assume z and t variables (similar discussion for more than one space variable). Some specific types of solutions:

Traveling Wave	$\xi[z,t]=F[w] \qquad w=\alpha z+\beta t \qquad \alpha\beta\neq 0$
Additive separable solution	$\xi[z,t]=\phi[z]+\psi[t]$
Multiplicative separable solution	$\xi[z,t]=\phi[z]\psi[t]$
Steady solution (ψ_T periodic)	$\xi[z,t]=\phi[z]\psi_T[t]$
Generalized traveling wave (1)	$\xi[z,t]=F[w] \qquad w=t\psi[z]+\phi[z]$
Generalized traveling wave (2)	$\xi[z,t]=F[w] \qquad w=z\psi[t]+\phi[t]$
Generalized separable solution	$\xi[z,t]=\phi_1[z]\psi_1[t]+\phi_2[z]\psi_2[t]+\dots$
Functional separable solution	$\xi[z,t]=F[w] \qquad w=\phi_1[z]\psi_1[t]+\phi_2[z]\psi_2[t]+\dots$

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44.03.18
Data Tables

©||WEB - URL |*Useful and extensive data tables*|

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44.04 Examples and Physical Applications

44.04.01 MPW Solutions With the Complex Formalism

This § is referenced at pages:
[1171, 2010]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|Appendix||

Read also sections § 19.03, § 19.04, § 25.06.

MPW are the simplest general wave solutions to **D'Alembert non-dispersive wave-equation** .

It is convenient to introduce the **complex notation** as a convenient tool to simplify the calculations. In fact, once it is understood that we deal with linear systems where **Sinusoids|Cosinusoids** have a distinguished role, it is worth noting that a Sinusoid|Cosinusoid at a given frequency is characterized by two parameters: its amplitude and its phase. It is possible to pack the two parameters together into a single entity by using a complex number.

It has to be clear that the complex number has no physical significance at all, but it is only a useful way to treat two different real quantities as a single complex entity. Other advantages will become clear at the end of this section.

Both the two harmonic functions

$$b_c \cos[\mathbf{k}\cdot\mathbf{x} - \omega t] \qquad b_s \sin[\mathbf{k}\cdot\mathbf{x} - \omega t] \quad ,$$

as well as

$$A\cos[\mathbf{k}\cdot\mathbf{x} - \omega t - \phi] \qquad A\sin[\mathbf{k}\cdot\mathbf{x} - \omega t - \phi] \quad ,$$

where b_c, b_s, A and ϕ are all constants, are solutions of the **D'Alembert non-dispersive wave-equation** , provided the **Dispersion Relation** is satisfied.

Thanks to the linearity also the linear combination with complex coefficients

$$A\exp[\mathtt{i}(\mathbf{k}\cdot\mathbf{x} - \omega t - \phi)] \equiv A\cos[\mathbf{k}\cdot\mathbf{x} - \omega t - \phi] + \mathtt{i}A\sin[\mathbf{k}\cdot\mathbf{x} - \omega t - \phi] \equiv \overline{A}\exp[\mathtt{i}\,(\mathbf{k}\cdot\mathbf{x} - \omega t)] \quad ,$$

is a solution, where the definition

$$\overline{A} \equiv A\exp[-\mathtt{i}\phi]$$

has been introduced.

One can now appreciate two more advantages: the first one is that both time and space derivation became just a multiplication; secondly the time and space parts are factorised.

It is important to stress that one can use the complex formalism without problems as long as only operations which do not mix the real and imaginary parts of complex numbers are involved, such as addiction, subtraction, derivation and integration. In this case the real art of the complex result is the results for the physical fields. When one carries on operations mixing the real and imaginary parts of complex numbers, such as multiplication and division of complex numbers, this is not true any more and the complex formalism cannot be used without taking countermeasures.

LATEX-IN-PROGRESS

44.04.02
Velocity for a General Time-Harmonic Wave

Find the **phase-velocity** for the wave in equation (44.03.01).
Read § 44.04.03.

44.04.03
Velocity for a General Wave

This § is referenced at pages:
[2008, 2019]

©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|§ 1.3||

Consider a generic wave defined by the implicit wave equation:

$$f[\mathbf{r},t]=c \qquad \text{where } c \text{ is a constant} \quad .$$

The time-dependence is not harmonic, in general.

1. Show that the velocity of the constant amplitude surfaces (CAWF) can be expressed as:

$$\boxed{v=-\frac{\partial_t f}{|\mathbf{grad}_{\mathbf{r}} f|}} \quad .$$

2. What happens when applying the formula to a steady solution such as (44.02.06)?

44.04.04
Superposition of many MPW in one dimension

This § is referenced at pages:
[Never referenced.]

Use a suitable software program to calculate the superpistion of MPW in one dimension with equal amplitued and zero phase constant to build a wave-packet.
For instance, superpose $n=30$ waves with (assume arbitrary but consistent SI units):

$$\begin{aligned}
\nu &= 0.550, 0.553, 0.557, 0.560, 0.563, 0.567, 0.570, 0.573, 0.577, 0.580, 0.583, 0.587, 0.590, 0.593, 0.597,\\
&= 0.600, 0.603, 0.607, 0.610, 0.613, 0.617, 0.620, 0.623, 0.627, 0.630, 0.633, 0.637, 0.640, 0.643, 0.647;\\
\omega &= 3.456, 3.477, 3.498, 3.519, 3.540, 3.560, 3.581, 3.602, 3.623, 3.644, 3.665, 3.686, 3.707, 3.728, 3.749,\\
&= 3.770, 3.791, 3.812, 3.833, 3.854, 3.875, 3.896, 3.917, 3.937, 3.958, 3.979, 4.000, 4.021, 4.042, 4.063;\\
v_p &= 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000,\\
&= 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000, 3.000;\\
k &= 1.152, 1.159, 1.166, 1.173, 1.180, 1.187, 1.194, 1.201, 1.208, 1.215, 1.222, 1.229, 1.236, 1.243, 1.250,\\
&= 1.257, 1.264, 1.271, 1.278, 1.285, 1.292, 1.299, 1.306, 1.312, 1.319, 1.326, 1.333, 1.340, 1.347, 1.354. \qquad (44.04.01)
\end{aligned}$$

DRAFT-WORK-IN-PROGRESS

44.04.05
Relation Between Energy Flux and Energy Density

This § is referenced at pages:
[2040, 2080]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|2.2||

It can be shown that, in general, the time-averaged energy volume density at any point in space, $\langle u_{\text{EM}}[\mathbf{r},t]\rangle$, is linked to the time-averaged energy flux at that point (intensity), $I \equiv \langle S[\mathbf{r},t]\rangle$, by the relation:

$$\boxed{\langle \mathbf{S}[\mathbf{r},t]\rangle \equiv \mathbf{v}_E \langle u[\mathbf{r},t]\rangle} \quad , \tag{44.04.02}$$

→ 2020 2150

in terms of the **speed of energy transport**, $\mathbf{v}_E$.

In fact, if energy travels at a speed $\mathbf{v}_E \equiv \hat{\mathbf{n}} v_E$, the time-averaged energy crossing any small plane surface defined by the vector $\mathbb{d}\mathbf{S}$ is:

$$\frac{\mathbb{d}E}{\mathbb{d}t} = \langle u[\mathbf{r},t]\rangle \frac{(\hat{\mathbf{n}}\cdot \mathbb{d}\mathbf{S})\,(v_E\,\mathbb{d}t)}{\mathbb{d}t} \quad .$$

Therefore equation (44.04.02) can be actually taken as the definition of speed of energy transport.

It is important to stress that the speed energy travels is not, in general, the **phase-velocity** and not, in general, the **group-velocity** .

44.04.06
Standing Waves as Superposition of Traveling Waves

Consider the general solution to the one-dimensional wave equation for a on an elastic string with fixed extremes, in the case of small oscillations. Consider two harmonic solutions, with velocity $\pm v$, at fixed frequency and same amplitude:

$$\xi[z,t] = A\cos\left[(kz+\omega t+\phi_{\oplus})\right] + A\cos\left[(kz-\omega t+\phi_{\ominus})\right] = 2A\cos\left[kz+\frac{\phi_{\oplus}+\phi_{\ominus}}{2}\right]\cos\left[\omega t+\frac{\phi_{\oplus}-\phi_{\ominus}}{2}\right] \quad .$$

The solution is a standing wave solution, because there is nothing traveling but only standing solutions.

Standing waves are appropriate in those cases where the wave is limited to a closed volume, that is energy cannot escape.

44.04.07
Monochromatic Quasi-PW Solutions to the d'Alembert Wave Equation

©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|||

©|J.P.Pérez *et al.*, *Optique*, ..., DUNOD, ...Ed., WEB - URL.|||

Write the solution to the **D'Alembert non-dispersive wave-equation** as follows:

$$\Phi[\mathbf{r},t] = \phi[\mathbf{r}]\exp -\mathtt{i}\omega t \implies \nabla^2\phi + \frac{\omega^2}{v^2}\phi = 0 \qquad \text{Helmholtz equation} \qquad \phi[\mathbf{r}] \equiv A[\mathbf{r}]\exp \mathtt{i}\psi[\mathbf{r}] \quad .$$

It is time periodic, but non periodic in space.

Often the amplitude, $A[\mathbf{r}]$ changes less rapidly than the phase, $\psi[\mathbf{r}]$, so that it is sensible to neglect variations of the amplitude with respect to variations of the phase. The resulting **phase-velocity** and wave-vector are:

$$\boxed{v = \frac{\omega}{|\mathbf{grad}\psi|} \qquad \mathbf{k} \equiv \mathbf{grad}\psi} \quad .$$

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44.04.08
PW Approximation of Spherical Waves

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|9.30||

Show that a spherical wave of wavelength λ can be approximated by a plane wave, over a circular fenditure of radius R transverse with respect to the radial direction and placed at distance L from the source, as long as $R^2 \ll \lambda L$, that is roughly the area of the fenditure is small with respect to λL.

44.04.09
Examples of Interference/Diffraction

Read § 50 and § 51.

- Interference causes the superposition of two (or more) waves with the same amplitude at some point to give a resulting wave amplitude ranging from a minimum of zero (totally destructive interference) to a maximum of twice (or the number of sources) the amplitude (maximally constructive interference).
- Interference is particularly interesting for light. In fact if light were similar to the flux of a beam of **classical particles** it would not be possible to add two beams and produce zero effect (that is to cancel the effects). This is what happens in the milestone Young two-slits experiment.
- The beam of a laser maintains, to a first approximation, the same diameter but it actually spreads a little bit. This is not due to any imperfection but to inherent diffraction.
- Any optical instrument has a limiting performance due to diffraction, which might prevent two point sources close enough to be seen as distinct sources by optical instruments.
- Sounds can be heard after obstacles thanks to interference/diffraction.
- Interference/Diffraction just differ for the type of sources:
  - a finite or infinite countable set of sources of finite intensity is called interference (sum over the sources is involved);
  - a continuum infinite set of sources of infinitesimal intensity is called diffraction (integration over the sources is involved).

DRAFT, IN PROGRESS

44.05 Exercises Problems and Physical Applications

44-001 ✓ **I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...**

From: 4.001 - To: 4.015 .

44-002 ✓ **A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...**

Exercises and problems in § 7.

44-003 ✓ **Addition of Sinusoidal|Cosinusoidal Functions**

Show that the equations (19.02.01) make sense that is the two ratios defining $\cos[\phi]$ and $\sin[\phi]$

$$\cos[\phi] = \frac{A_1\cos[\phi_1] + A_2\cos[\phi_2]}{A} \qquad \sin[\phi] = \frac{A_1\sin[\phi_1] + A_2\sin[\phi_2]}{A} \quad ,$$

are always smaller than one.

44-004 ✓ **Synchronization of Traffic Lights**

A straight road has identical traffic lights at equal distance of $D = 300$ m.

If the traffic lights are synchronized at $v = 37$ km/h what is the frequency of **oscillation** of the traffic lights?

What is the **phase-velocity** and wavelength of the **traffic light wave**?

44-005 ✓ **Different expressions for One Wave**

©|A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...|§ 7||

Show that the following expressions all describe the same traveling wave:

$$\xi[z,t] = A\sin[2\pi(z - vt)/\lambda] \quad ,$$
$$\xi[z,t] = A\sin[2\pi(kz - vt))] \quad ,$$
$$\xi[z,t] = A\sin[2\pi((x/\lambda) - (t/T))] \quad ,$$
$$\xi[z,t] = -A\sin[\omega(t - x/v)] \quad ,$$
$$\xi[z,t] = A\,\Im\mathfrak{m}[\exp[2\pi\mathtt{i}(kz - \nu t)]] \quad .$$

44-006 ✓ **Superposition of Two Harmonic Oscillations**

Consider two harmonic oscillations at different frequencies, ω_1 and ω_2, for instance the oscillations in time of two waves at a fixed point is space.

Under what conditions the resulting oscillation is periodic and what is the period?

DRAFT-IN-PROGRESS

44-007 ✓ Modulation With Different Amplitude Waves

Carry on the exercise of § 44.03.11 without assuming that the two amplitudes are equal.

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CHAPTER 45

Mechanical Waves

45.01 Some Mechanical Wave Phenomena 2026
45.02 Longitudinal Waves in an LHI Medium (Acoustic Waves) 2027
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About Mechanical Waves in regions of space far from boundaries.

This § is referenced at pages:
[1990, 2207]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|All arguments ...
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|All arguments about waves.||
©|R.Fitzpatrick|Oscillations and Waves|***WEB - URL*** - *Complete textbook*|
©|A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...|Complete text with worked ...
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Pre-requisites - Review Harmonic oscillations, **§** 25.

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45.01 Some Mechanical Wave Phenomena

This § is referenced at pages:
[2034]

General definition of mechanical wave: a wave propagating as a Perturbation|Disturbance|Signal|Deformation of a material medium.

In any mechanical wave, there are real pieces of matter, particles in the end, oscillating. There is no overall motion of matter, that is the Center-Of-Mass position of any piece of the oscillating matter keeps on average the same position.

In most cases, mechanical waves have a more or less pronounced **energy dissipation** inside the medium.

Only **only non-dispersive and non-dissipative mechanical waves** are considered in this section.

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45.02 Longitudinal Waves in an LHI Medium (Acoustic Waves)

This § is referenced at pages:
[1494]

45.02.01 Elastic Force in a Solid

Consider a cylindrical String|Bar with non uniform deformations.

Imagine to cut the cylindrical String|Bar by an imaginary plane perpendicular to its length. By definition, the stress exerted by one part of the String|Bar on the other nearby part is the tension per unit area, aka the normal stress. The tension can also be defined as the two equal and opposite forces needed to keep the two sides of the cut cylindrical String|Bar in place after cutting.

All the above discussion neglects the sign of the tension, which need careful evaluation as follows.

Consider a small piece of the cylindrical bar, so short that one can assume that its longitudinal deformation is approximately uniform. The rest length, Δz, will be left tend to zero at the end of the calculation. Its generic elongation is: $\Delta\xi[z, z+\Delta z] \equiv \xi[z+\Delta z] - \xi[z]$. The force along z it exerts at its two extremes, which is equal and opposite to the force exerted on it by the nearby parts of the bar, thanks to the **Action-Reaction Principle** , is:

$$F_z = \pm YS\frac{\Delta\xi[z, z+\Delta z]}{\Delta z} = \pm YS\frac{\xi[z+\Delta z] - \xi[z]}{\Delta z} \to \pm YS\frac{\partial\xi}{\partial z}[z, t] \quad ,$$

where the sign is determined by which of the two ends one considers and depending on whether one considers the force on one of the bases of the bar or the force one of the two bases of the bar exerts on the nearby parts.

When the String|Bar is longitudinally deformed, the force acting on the piece of the bar after z due to the part of the bar before z, is:

$$F_\ominus = -YS\frac{\partial\xi}{\partial z}[z, t] \quad \text{force on the piece after } z \text{ due to the part before } z \quad , \qquad (45.02.01)$$

→ 2028

the force acting on the piece of the bar before z due to the part of the bar after z, is:

$$F_\oplus = +YS\frac{\partial\xi}{\partial z}[z+\Delta z, t] \quad \text{force on the piece before } z \text{ due to the part after } z \quad . \qquad (45.02.02)$$

→ 2028

The sign of the two contributions is readily evaluated when considering the piece of elongated/contracted.

45.02.02 Acoustic Waves in a Solid Bar

This § is referenced at pages:
[1661, 1661, 2030]

Read § 32.08.02 and § 45.12.03.

The derivation in this section is based on the Young module: it applies to solids.

Consider a cylindrical solid bar, of rest length L and cross sectional area S, made of a LHI solid. Consider the longitudinal deformations of the bar. Let the z axis be on the axis of the bar. The bar is supposed to undergo longitudinal deformations in a non necessarily uniform way. Assume that the change in the transverse dimensions is negligible, that is the Poisson module is zero.

Denote by $\xi[z, t]$ the displacement from the reference position of the part of the bar which is located, when at rest, at z.

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Note that:

- z is just a label to identify the different elementary constituents of the bar;
- $\xi[z,t]$ is the dynamical variable: an equation for it must be found and solved.

Consider a piece of the bar located, at rest, between z and $z+\Delta z \geq z$ and apply the equations of mechanics to this system. When the bar is longitudinally deformed the force acting on the piece of the bar, due to the part of the bar before z, is:

$$F_{\ominus} = -YS\frac{\partial\xi}{\partial z}[z,t] \qquad \text{force on the piece of the bar at } z \text{ from the part before } z \quad ,$$

while the force acting on the piece of the bar, due to the part of the bar after $z+\Delta z$, is:

$$F_{\oplus} = +YS\frac{\partial\xi}{\partial z}[z+\Delta z,t] \qquad \text{force on the piece of the bar at } z+\Delta z \text{ from the part after } z+\Delta z \quad .$$

as from equation (45.02.02) and (45.02.01).

Note that the two forces will not have the same module unless the deformation is uniform, which would implies:

$$\frac{\partial\xi}{\partial z}[z,t] = \text{constant} \qquad \Longrightarrow \qquad \xi[z,t] = \alpha+\beta z \qquad \text{with } \alpha \text{ and } \beta \text{ constants} \quad .$$

The first **Cardinal Equation** applied to the piece of the bar reads:

$$(f_{\text{V}}[z,t]S\Delta z) + YS\left(\frac{\partial\xi}{\partial z}[z+\Delta z,t] - \frac{\partial\xi}{\partial z}[z,t]\right) = \rho_0 S\Delta z\frac{\partial^2\xi_{\text{CM}}}{\partial t^2}[z_{\text{CM}},t] \quad ,$$

in terms of the rest density of the bar, ρ_0, the location of the Center-Of-Mass of the system when at rest, z_{CM}, with $z \leq z_{\text{CM}} \leq z+\Delta z$ and the displacement of the Center-Of-Mass, $\xi_{\text{CM}}[z_{\text{CM}}]$.

Note that in general:

$$\xi_{\text{CM}}[z_{\text{CM}}] \neq \xi[z_{\text{CM}}] \quad ,$$

because if the Center-Of-Mass corresponds to same piece of matter when at rest, in general the Center-Of-Mass of the deformed piece of bar will correspond to some other piece of matter so that in general the displacement of a piece of matter, $\xi[z]$, is different from the displacement of the Center-Of-Mass, $\xi_{\text{CM}}[z_{\text{CM}}]$. Remember in fact that the Center-Of-Mass is a gometrical point, not a mass-point.

Taking the limit for $\Delta z \longrightarrow 0$ of the first **Cardinal Equation** above one finds:

$$z \leq z_{\text{CM}} \leq z+\Delta z \quad ,$$
$$z+\xi[z] \leq z_{\text{CM}}+\xi_{\text{CM}}[z_{\text{CM}}] \leq z+\Delta z+\xi[z+\Delta z] \quad ,$$
$$\text{for } \Delta z \longrightarrow 0 \implies \quad ,$$
$$z \leq z_{\text{CM}} \leq z \quad ,$$
$$z+\xi[z] \leq z+\xi_{\text{CM}}[z_{\text{CM}}] \leq z+\xi[z] \implies \xi[z] \leq \xi_{\text{CM}}[z_{\text{CM}}] \leq \xi[z] \quad .$$

Therefore the equation becomes:

$$\rho_0\frac{\partial^2\xi}{\partial t^2}[z,t] - Y\frac{\partial^2\xi}{\partial z^2}[z,t] = f_{\text{V}}[z,t] \quad . \qquad (45.02.03) \quad \rightarrow 1503$$

This is the non-homogeneous one-dimensional **D'Alembert non-dispersive wave-equation** with propagation speed:

$$c_{\text{L}} = \sqrt{\frac{Y}{\rho_0}} \quad . \qquad (45.02.04) \quad \rightarrow 1426$$

Note that this kind of derivation is based on the assumption of validity of Hooke law, that is a linear constitutive equation between stress and strain. This is why it gives a linear wave equation. A derivation without this assumption is given in section § 45.02.03.

The expression for c_{L} shows that the stiffer the solid the highest the speed of sound.

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45.02.03
General Acoustic Waves in Fluids/Solids

This § is referenced at pages:
[1657, 1657, 2028]

Read § 32.08.02 and § 45.12.03.

The derivation in this section is based on the normal stress: it applies to both solids and fluids. The general non-linear equation will be derived, with the linear limit at the end.

Consider a long cylindrical tube, in the z direction, filled with a uniform and homogeneous fluid or, alternatively, a long cylindrical solid bar. Let the cross-sectional dimension of the tube be much shorter than its length and let its cross-sectional area be A. Assume that any motion is only along z.

Consider a macroscopically small, but finite, part of the mass delimited by two planes perpendicular to the z axis, of rest length Δz_0, and follow its motion under the action of the pressure (that is the normal stress) exerted by the neighboring masses. The volume is macroscopically small, so that its properties are assumed not to change inside it. This allows to write for the density:

$$\rho_0[z] \equiv \frac{\Delta m}{\Delta z_0} \quad ,$$

which might be in principle non uniform.

The mass one follows is always made by the same particles (this is the so-called Lagrangian approach). In the Lagrangian approach $\mathbf{r}$ is just a label for an infinitesimal mass of the medium and therefore independent of time. It is not the dymamical variable.

Label each infinitesimally small part of fluid by its location in the reference configuration of the fluid, $\mathbf{r}$. Let $\xi_z[z] \equiv \xi[z]$ be the displacement along z from the reference position of the particles of fluid lying at z in the reference configuration. Let $\rho[z]$ and $p[z]$ be, respectively, the linear mass density and pressure, in the disturbed state, of the small region of fluid lying at z in the reference configuration. They are the functions describing the properties of the part of mass which stay at z in the undisturbed state.

We follow the particles, and so the functions do, in a sense.

The value of any quantity in any configuration can be calculated rigorously via the Lagrange theorem of analysis. Infinitesimals of higher order than the lowest order will vanish when letting Δz_0 go to zero.

The following kinematics equations apply, where the time dependence is normally omitted from the formulas. The limit of small displacements or, better, small strains, to be precise,

$$\left|\frac{\partial \xi}{\partial z}\right| \ll 1 \quad ,$$

is considered and the limit for Δz_0 go to zero is taken in the end.

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One has:

$$\Delta z = \Delta z_0 + \xi[z+\Delta z_0] - \xi[z] = \Delta z_0 \left(1 + \frac{\partial \xi}{\partial z}[\overline{z}]\right) \qquad z \leq \overline{z} \leq z + \Delta z_0 \qquad \lim_{\Delta z_0 \longrightarrow 0} \overline{z} = z \quad ,$$

$$\Delta z_0 \longrightarrow \Delta z \equiv \Delta z_0 + \Delta \xi = \Delta z_0 \left(1 + \frac{\partial \xi}{\partial z}[\overline{z}]\right) = \Delta z_0 \left(1 + \frac{\partial \xi}{\partial z}[z]\right) + \mathcal{O}\left(\Delta z_0\right)^2 \quad ,$$

$$\rho_0[z] \equiv \lim_{\Delta z_0 \longrightarrow 0} \frac{\Delta m}{\Delta z_0} \longrightarrow \rho[z] \equiv \lim_{\Delta z_0 \longrightarrow 0} \frac{\Delta m}{\Delta z} \quad ,$$

$$\rho[z] = \lim_{\Delta z_0 \longrightarrow 0} \frac{\rho_0[z]}{\left(1 + \frac{\partial \xi}{\partial z}[z]\right) + \mathcal{O}\left(\Delta z_0\right)^1} = \frac{\rho_0[z]}{1 + \frac{\partial \xi}{\partial z}[\overline{z}]} = \frac{\rho_0[z]}{1 + \frac{\partial \xi}{\partial z}[z]} \qquad \text{for } \Delta z_0 \longrightarrow 0 \text{ an exact equation} \quad , \quad \rightarrow 2030\ 2031$$

$$\rho[z] \simeq \rho_0[z] \left(1 - \frac{\partial \xi}{\partial z}[z]\right) \qquad \text{approximate relation for } \left|\frac{\partial \xi}{\partial z}\right| \ll 1 \text{ and } \Delta z_0 \longrightarrow 0 \quad ,$$

$$\frac{\partial \rho}{\partial z} \simeq -\rho_0 \frac{\partial^2 \xi}{\partial z^2} \qquad \text{in the common case that } \rho_0[z] \equiv \rho_0 \text{ is uniform} \quad .$$

As the medium is homogeneous the Center-Of-Mass, at reference, will be at the middle of the cylinder. It is assumed that Δz_0 is so small that it remains homogeneous at any time, so that the Center-Of-Mass is still in the middle of the piece at any time. The limit for Δz_0 go to zero is taken in the end, in any case. The first **Cardinal Equation** applied to the small piece of mass, as a function of the pressure, gives:

$$z_{\mathtt{CM}} \equiv z + \frac{\Delta z_0}{2} + \xi[z + \frac{\Delta z_0}{2}] \qquad \text{the small piece of mass is so small that it is uniform at any time} \quad ,$$

$$A\left(p[z] - p[z+\Delta z_0]\right) = \rho_0[z] \Delta z_0 A \frac{\partial^2 z_{\mathtt{CM}}}{\partial t^2} \quad , \quad \rightarrow 1657$$

$$z_{\mathtt{CM}} = z + \frac{\Delta z_0}{2} + \xi[z] + \frac{\Delta z_0}{2} \frac{\partial \xi}{\partial z}[\overline{z}] \qquad z \leq \overline{z} \leq z + \Delta z_0 \qquad \lim_{\Delta z_0 \longrightarrow 0} \overline{z} = z \quad ,$$

$$\frac{\partial^2 z_{\mathtt{CM}}}{\partial t^2}[z] = \frac{\partial^2 \xi}{\partial t^2}[z] + \frac{\Delta z_0}{2} \frac{\partial}{\partial z} \left(\frac{\partial^2 \xi}{\partial t^2}\right)[\overline{z}] \quad ,$$

$$-\frac{\partial p}{\partial z}[z] \Delta z_0 = \rho_0[z] \Delta z_0 \frac{\partial^2 \xi}{\partial t^2}[z] + \mathcal{O}\left(\Delta z_0\right)^2 \quad ,$$

$$-\frac{\partial p}{\partial z}[z] = \rho_0[z] \frac{\partial^2 \xi}{\partial t^2}[z] \qquad \text{for } \Delta z_0 \longrightarrow 0 \text{ an exact equation} \quad . \quad \rightarrow 2030\ 2031$$

We have now collected two basic exact equations (that is no approximation in the limit for $\Delta z_0 \longrightarrow 0$), the equation for the density (45.02.03) and the first cardinal equation (45.02.03):

$$-\frac{\partial p}{\partial z}[z,t] = -\frac{\partial p}{\partial \rho}[\rho[z,t]] \frac{\partial \rho}{\partial z}[z,t] = \rho_0 \frac{\partial p}{\partial \rho}[\rho[z,t]] \frac{\partial^2 \xi}{\partial z^2}[z,t] = \rho_0 \frac{\partial^2 \xi}{\partial t^2}[z,t] \quad .$$

We know have three unknown functions: $\xi[z,t]$, $\rho[z,t]$ and $p[z,t]$. We only have two equations: equation (45.02.03) and equation (45.02.03).

The third equation must be in fact the constitutive equation (or equation of state) of the medium, as the relation between $\rho[z,t]$ and $p[z,t]$ is actually the constitutive equation (equation of state). The constitutive equation allows one to calculate the derivative of pressure with respect to density.

In the problem discussed in section § 45.02.02 the third equation, the constitutive equation, is given by the assumption that the solid is elastic, so that normal stress and strain are proportional.

Finally, within the small strain approximation, the derivative of pressure with respect to density is calculated at the reference configuration, thus becoming a constant and the classical wave equation is found:

$$\left.\frac{\partial p}{\partial \rho}\right|_0 \frac{\partial^2 \xi}{\partial z^2}[z,t] = \frac{\partial^2 \xi}{\partial t^2}[z,t] \qquad \text{for } \left|\frac{\partial \xi}{\partial z}\right| \ll 1 \quad . \qquad (45.02.05) \quad \rightarrow 2058$$

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It must be stressed again that, in this Lagrangian approach, the meaning of $p[z,t]$, $\rho[z,t]$ as well as other functions is: the value at time t of the quantity at those infinitesimal piece of matter whose reference position is at z.

It must be stressed again that equations (45.02.03), (45.02.03) are mathematically exact, as it was shown in the derivation, in the limit for $\Delta z_0 \longrightarrow 0$. Derivations of this kind are often carried on without all those precise mathematical passages, by employing the first-order Taylor series development without further details, as it is implicit that the limit process implied will make the approximate relations exact.

Note also that:

$$\frac{\Delta\rho}{\rho} \simeq -\frac{\Delta V}{V} \longrightarrow c_{\text{L}}^2 \simeq \left.\frac{\Delta p}{\Delta\rho}\right|_0 = \frac{B}{\rho_0} \quad ,$$

so that, after adding the external volume forces:

$$\boxed{\frac{\partial^2\xi}{\partial t^2}[z,t] - \left.\frac{B}{\rho}\right|_0 \frac{\partial^2\xi}{\partial z^2}[z,t] = f_{\text{V}} \qquad \text{for} \left|\frac{\partial\xi}{\partial z}\right| \ll 1} \quad . \tag{45.02.06}$$

Read § 45-003, § 45-003, § 45-004.

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45.03 Transverse Waves in an LHI Medium

Transverse waves cannot exist in fluids as the shear modulus is zero.

45.03.01 Transverse Elastic Waves on a Stretched Ideal String

This § is referenced at pages:
[1991, 1994]

Consider a stretched ideal string, with linear mass density at rest λ, whose tension, T, is kept constant by two suitable forces at the extremes. Let the z axis coincide with the string when at rest.

Consider the transverse (but plane) oscillations of the string.

Assume:

- Small displacements, in the transverse dimensions only (no longitudinal displacement).
- Small angles with respect to the z axis. Note that this is not related to the condition of small displacements, but it is an independent assumption. One might in fact have small displacements with large derivatives of the displacement function.
- Constant tension. Actually one may consider the tension changing according to Hooke law. For small displacements and small angles the change of length is of second-order so that to first-order the tension does not change.

Note: the string can only be stretched, not compressed. This is basically different from the case of the elastic solid, and orthogonal to the case of a fluid.

Consider a small piece of the string when deformed in a plane (the zy plane). Let transverse displacement, along y, with respect to the reference position of the piece located at z when the string is at rest be $\xi[z,t]$. In the present problem this is actually a displacement along the y direction.

The first **Cardinal Equation** applied to a small piece of string whose length at rest is Δz, when projected on the z and y directions, read:

$$F_z = T\cos[\alpha+\Delta\alpha] - T\cos[\alpha] = 0 \quad ,$$

$$F_y = T\sin[\alpha+\Delta\alpha] - T\sin[\alpha] = \lambda\Delta z\frac{\partial^2\xi}{\partial t^2}[\overline{z},t] \quad .$$

Under the given assumptions one can approximate the angles to first-order to obtain:

$$F_z = T - T = 0 \qquad \text{as } \cos[\alpha]\simeq 1 \text{ and } \cos[\alpha+\Delta\alpha]\simeq 1 \text{ to first-order} \quad , \quad ,$$

$$F_y = T\Delta\alpha = \lambda\Delta z\frac{\partial^2\xi}{\partial t^2}[\overline{z},t] \qquad \text{as } \sin[\alpha]\simeq\alpha \text{ and } \sin[\alpha+\Delta\alpha]\simeq\alpha+\Delta\alpha \text{ to first-order} \quad .$$

When one takes the limit for $\Delta z \longrightarrow 0$ one obtains (time dependence of all quantities is implicit in all the equations):

$$\alpha[z,z+\Delta z] \simeq \frac{\xi[z+\Delta z]-\xi[z]}{\Delta z} \longrightarrow \frac{\partial\xi}{\partial z}[z] \quad ,$$

$$\Delta\alpha[z,z+\Delta z] \equiv \alpha[z+\Delta z]-\alpha[z] \quad ,$$

$$\frac{\Delta\alpha[z,z+\Delta z]}{\Delta z} = \frac{\alpha[z+\Delta z]-\alpha[z]}{\Delta z} \longrightarrow \frac{\partial\alpha[z]}{\partial\Delta z} = \frac{\partial^2\xi}{\partial z^2}[z] \quad .$$

Therefore:

$$\boxed{\frac{\partial^2\xi}{\partial z^2} = \frac{\lambda}{T}\frac{\partial^2\xi}{\partial t^2}} \quad .$$

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This is the one-dimensional wave equation with propagation speed:

$$\boxed{c_{\mathrm{T}} = \sqrt{\frac{T}{\lambda}}} \quad . \tag{45.03.01}$$

→ 1426

45.03.02
Transverse Elastic Waves on a Elastic Bar

This § is referenced at pages:
[1994]

The **phase-velocity** of transverse elastic waves on a elastic bar is:

$$\boxed{c_{\mathrm{T}} = \sqrt{\frac{G}{\rho}}} \quad .$$

45.03.03
Torsional Vibrations of a Cylindrical Round Homogeneous and Isotropic Elastic Bar

Read § 32.08.07.

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45.04 General Waves in Unbounded Homogeneous Isotropic and Elastic Media

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Read § 32.06 for a more complete discussion.

45.04.01 General Longitudinal Waves in Unbounded Homogeneous Isotropic and Elastic Media

This § is referenced at pages:
[1500]

This type of wave is also often called a dilatation wave, a irrotational wave, a pressure wave, or a P-wave (P for: primary or pressure). The displacement is always parallel to the wave-vector:

$$\boldsymbol{\xi}[\mathbf{r},t] = \boldsymbol{\xi}_0 \exp\left[\mathtt{i}\,(\,\mathbf{k}\cdot\mathbf{r} - \omega t - \phi)\right] \qquad \mathbf{k}\wedge\boldsymbol{\xi}_0 = 0 \quad .$$

The derivations in section § 45.01 were based on the fact that the system was not undergoing any lateral shrinkage/expansion, that is the Poisson module was assumed to be zero. In the more general case it can be shown that the velocity of longitudinal waves is:

$$c_{\mathrm{L}}^2 = \frac{(1-\Sigma)}{(1+\Sigma)\,(1-2\Sigma)}\frac{Y}{\rho_0} \quad , \tag{45.04.01}$$

→ 2034

$$c_{\mathrm{L}}^2 = \frac{3B+4G}{3\rho_0} \quad . \tag{45.04.02}$$

→ 2034 2058

Note that, according to the relations in section § 27.02.04, the two expressions (45.04.01), (45.04.02) are totally equivalent.

The two different ways of writing the same result are more or less suitable depending on the required application. Equation (45.04.01) gives the result for the case when there is no lateral shrinkage/expansion using $\Sigma = 0$. Refer to § 27.07.07 for an example with $\Sigma \neq 0$ but longitudinal only displacements, thanks to the presence of forces perpendicular to the wave-vector which forbid displacements perpendicular to the wave-vector.

Equation (45.04.02) is useful to consider the case of a fluid, which has $G = 0$.

Read also § 32.

45.04.02 General Transverse Waves in Unbounded Homogeneous Isotropic and Elastic Media

This § is referenced at pages:
[Never referenced.]

This type of wave is also often called a rotational wave, a shear wave, or an S-wave (S for: secondary or shear). The displacement is always perpendicular to the wave-vector:

$$\boldsymbol{\xi}[\mathbf{r},t] = \boldsymbol{\xi}_0 \exp\left[\mathtt{i}\,(\,\mathbf{k}\cdot\mathbf{r} - \omega t - \phi)\right] \qquad \mathbf{k}\cdot\boldsymbol{\xi}_0 = 0 \quad .$$

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The displacement can have any direction in a plane normal to the wave-vector, so that they two independent states of **wave-Polarization** are present.

$$\boxed{c_{\mathrm{T}}^2 = \frac{Y}{\rho_0}\frac{1}{2\,(1+\Sigma)}} \quad , \tag{45.04.03}$$

→ 2035

$$\boxed{c_{\mathrm{T}}^2 = \frac{G}{\rho_0}} \quad . \tag{45.04.04}$$

→ 2035 2058

Note that, according to the relations in section § 27.02.04, the two expressions (45.04.03), (45.04.04) are totally equivalent.

The two different ways of writing the same result are more or less suitable depending on the required application.

Equation (45.04.04) is useful to consider the case of a fluid, which has $G = 0$.

Read also § 32.

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45.05 Waves at the Surface of Water

Surface waves, that is 2D waves.

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45.06 Mechanical Waves in Bounded Regions of Space

This § is referenced at pages:
[2160, 2161]

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Standing mechanical waves are used to describe the generic mechanical waves of systems with a finite extent. The following cases are particularly significant in practice.

- A vibrating string with fixed ends, like in musical instruments (piano, guitar).
- A vibrating membrane with fixed boundary, like in musical instruments (drum).

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45.07 Sources for Mechanical Waves

This § is referenced at pages:
[Never referenced.]

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As mechanical waves are oscillations of parts of matter, sources of mechanical waves are in principle any thing which can set matter in motion.

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45.08 Energy Linear Momentum and Angular Momentum of Mechanical Waves

This § is referenced at pages:
[1995]

It is a common experience that **mechanical waves carry energy even if they transport no matter**. For instance a ship is set into motion by water waves, that is it can gain both kinetic and gravitational potential energy. Acoustic waves carry mechanical energy from the source to the ear.

45.08.01 Longitudinal Mechanical Waves in LHI Solids

©|W.C.Elmore & M.A.Heald, , ..., ..., ...Ed., ...|general discussion, § su mezzi elastici||

45.08.01.01 Energy of Longitudinal Mechanical Waves in LHI Solids

This § is referenced at pages:
[2080]

The case of purely longitudinal mechanical waves along the z axis (that is one-dimensional waves) will be considered in this section, for a material with $\Sigma = 0$.

Consider a cylindrical bar with cross-sectional area S, made of an LHI material, with volume mass density ρ and Young modulus Y. Let the deformation along the z direction be $\xi[z,t]$.

The kinetic energy per unit volume/length is:

$$\frac{\mathrm{d}K}{\mathrm{d}V} = \frac{\rho}{2}\left(\dot{\xi}\right)^2 \implies \frac{\mathrm{d}K}{\mathrm{d}z} = \frac{\rho S}{2}(\partial_t \xi)^2 \quad .$$

The general calculation of elastic potential energy in a LHI solid is rather complex and therefore here we limit to a very simplified, far from general, derivation. Consider a short piece of the system such that the deformation can be considered as uniform, that is the normal stress is uniform. Let the rest length of the piece be Δz. The piece of system can be assimilated to a spring and the potential energy written down by analogy:

$$k = \frac{YS}{\Delta z} \qquad \Delta\mathcal{U} = \frac{k}{2}\Delta\xi^2 \implies \frac{\Delta\mathcal{U}}{\Delta V} = \left(\frac{Y}{2\Delta z}\right)\frac{\Delta\xi^2}{\Delta z} \quad ,$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}V} = \frac{Y}{2}\left(\frac{\partial\xi}{\partial z}\right)^2 \implies \frac{\mathrm{d}\Phi}{\mathrm{d}z} = \frac{YS}{2}\left(\frac{\partial\xi}{\partial z}\right)^2 \quad .$$

The elastic potential energy per unit volume|length is thus:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}V} = \frac{Y}{2}\left(\frac{\partial\xi}{\partial z}\right)^2 \implies \frac{\mathrm{d}\Phi}{\mathrm{d}z} = \frac{YS}{2}\left(\frac{\partial\xi}{\partial z}\right)^2 \quad .$$

Note that energies are not expressed in terms of second derivatives but in terms of first derivatives.

DRAFT in Progress

45.08.01.02 Energy of Longitudinal Monochromatic Mechanical Waves in LHI Solids

Consider a MPW with wave-vector module $k > 0$, angular frequency ω and amplitude ξ_0. Let the **phase-velocity** be $v_z \gtrless 0$ and its module be $v \equiv |v_z|$. We have:

$$\xi[z,t] = \xi_0 \cos[\pm kz - \omega t] \qquad \dot{\xi}[z,t] = \omega\xi_0 \sin[\pm kz - \omega t] \qquad \frac{\partial\xi}{\partial z} = \mp k\xi_0 \sin[\pm kz - \omega t] \quad , \quad \rightarrow 2041$$

$$\omega = kv \qquad v^2 = \frac{Y}{\rho} \quad .$$

The kinetic energy per unit volume and its time-average are:

$$\frac{\mathrm{d}K}{\mathrm{d}V} = \frac{1}{2}\rho\xi_0^2\omega^2 \sin^2[\pm kz - \omega t] \implies \langle\frac{\mathrm{d}K}{\mathrm{d}V}\rangle = \frac{1}{4}\rho\xi_0^2\omega^2 \quad . \qquad (45.08.01) \quad \rightarrow 2042$$

The elastic potential energy per unit volume and its time-average are:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}V} = \frac{1}{2}Y\xi_0^2k^2 \sin^2[\pm kz - \omega t] \implies \langle\frac{\mathrm{d}\Phi}{\mathrm{d}V}\rangle = \frac{1}{4}Y\xi_0^2k^2 \quad . \qquad (45.08.02) \quad \rightarrow 2042$$

The mechanical energy per unit volume and its time-average are:

$$\frac{\mathrm{d}E}{\mathrm{d}V} = \frac{\xi_0^2}{2}\left(\rho\omega^2 + Yk^2\right)\sin^2[\pm kz - \omega t] = \frac{\rho\xi_0^2}{2}\left(\omega^2 + v^2k^2\right)\sin^2[\pm kz - \omega t] \quad , \quad \rightarrow 2042$$

$$\langle\frac{\mathrm{d}E}{\mathrm{d}V}\rangle = \frac{1}{4}\xi_0^2\left(\rho\omega^2 + Yk^2\right) = \frac{1}{2}\rho\xi_0^2\omega^2 = \frac{1}{2}Y\xi_0^2k^2 \quad .$$

The power transmitted along the system from a smaller z to a larger z (from left to right, in short) across any surface perpendicular to the z direction is calculated from the expression of the force from left to right of any section of the system,

$$F_\ominus = -YS\frac{\partial\xi}{\partial z} \quad , \qquad (45.08.03) \quad \rightarrow 2041$$

and therefore one finds:

$$\Pi_\ominus \equiv \partial_t\xi F_\ominus = -YS\frac{\partial\xi}{\partial z}\partial_t\xi = \pm YS\xi_0^2k\omega \sin^2[\pm kz - \omega t] \quad .$$

Note that the latter is the power produced by the force $F_\ominus$, that is the force, from left to right, of any section of the system on the piece at its right. **A positive power then means that positive work, and therefore energy, is transmitted from left to right and vice-versa**. The power is positive for waves traveling in the positive direction (that is power runs from left to right) and it is negative for waves traveling in the negative direction (that is power runs from right to left).

The time-averaged power transmitted along the system from left to right across any surface perpendicular to the system is:

$$\langle\Pi_\ominus\rangle = \pm\frac{1}{2}YS\xi_0^2k\omega = \pm\frac{1}{2}YS\xi_0^2\frac{\omega^2}{v} = \pm\frac{1}{2}\rho S\xi_0^2v\omega^2 = \frac{1}{2}\rho S\xi_0^2\omega^2v_z \quad ,$$

in terms of v_z, the signed component of the **phase-velocity** along z. The intensity is thus:

$$I_\ominus \equiv \frac{\langle\Pi_\ominus\rangle}{S} = \pm\frac{1}{2}\rho\xi_0^2\omega^2v \quad . \qquad (45.08.04) \quad \rightarrow 2042$$

The relation between intensity and energy density is thus:

$$I_\ominus = \frac{\langle\Pi_\ominus\rangle}{S} = \frac{v_z}{S}\langle\frac{\mathrm{d}E}{\mathrm{d}z}\rangle = v_z\langle\frac{\mathrm{d}E}{\mathrm{d}V}\rangle \quad ,$$

which shows that, for mechanical MPW in an elastic non-dispersive medium, the intensity is proportional to the volume energy density via the **phase-velocity** v_z (read § 44.04.05 for the general discussion). Read § 46.05.01.01 for the similar relation for ElectroMagnetic waves.

Note that in any one-dimensional problem, where transverse dimensions are neglected, the power is a quantity as useful as the more commonly used intensity.

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45.08.01.03 Linear Momentum of Longitudinal Mechanical Waves in LHI Solids

©|K.Y.Bliokh & F.Nori|Spin and orbital angular momenta of acoustic beams|*DOI: WEB - URL - and references.*|

Linear momentum, along z, of one piece of length Δz:

$$\boxed{\Delta P_z = \rho S \Delta z \partial_t \xi} \quad .$$

The first **Cardinal Equation** of mechanics allows to recover the wave equation:

$$F_{\ominus}[z,t] + F_{\oplus}[z+\Delta z,t] = \partial_t \Delta P_z = \rho S \Delta z \frac{\partial^2 \xi}{\partial t^2} \implies \quad \text{in the limit } \Delta z \to 0 \quad \implies Y \frac{\partial^2 \xi}{\partial z^2} = \rho \frac{\partial^2 \xi}{\partial t^2} \quad .$$

45.08.01.04 Angular Momentum of Longitudinal Mechanical Waves in LHI Solids

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45.08.01.05 Wave Impedance of Longitudinal Mechanical Waves in LHI Solids

This § is referenced at pages:
[2187]

Read also § 44.03.13 for other comments on the **wave impedance** .

As the expression of the power involves the force and the speed it is useful to introduce the wave-impedance of longitudinal mechanical waves, considering all quantities along z:

$$\boxed{\xi \equiv \xi_z[z,t] \implies |\partial_t \xi| Z \equiv |F_z|} \quad . \qquad (45.08.05)$$

→ 2015 2041

It turns out to depend on the properties of the medium only, by using equations (45.08.01.02), (45.08.03):

$$\boxed{Z = \frac{YS}{v} = \rho S v = S\sqrt{Y\rho} \qquad [\mathrm{Z}] = MLT^{-1}} \quad .$$

Note: sometimes in this problem the **wave impedance** is defined as the quantity Z/S above. This happens when in the definition in equation (45.08.05) one uses the stress instead of the force.

It turns out:

$$\boxed{\langle \Pi_{\ominus} \rangle = \pm \frac{1}{2} Z \xi_0^2 \omega^2} \quad .$$

As the mechanical power is force times velocity, the impedance allows one to express the intensity as a function of force/velocity multiplied/divided by the impedance, read § 48.03.01 and § 44.03.13.

DRAFT WORK IN PROGRESS

45.08.02
Transverse Mechanical Waves

Transverse mechanical waves need, for a complete definition, the specification of their **wave-Polarization** state, as for ElectroMagnetic waves § 46.03.

45.08.02.01 Energy and Intensity of Transverse Mechanical Waves

This § is referenced at pages:
[2055]

The same expressions derived for longitudinal mechanical waves, with the suitable reinterpretation of symbols and expressed only in terms of density, frequency wave-number and **phase-velocity** , apply, as in equation (45.08.01), equation (45.08.02), equation (45.08.01.02), and equation (45.08.04), provided the **wave-Polarization** state is taken into account.

45.08.02.02 Linear Momentum of Transverse Mechanical Waves

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45.08.02.03 Angular Momentum of Transverse Mechanical Waves

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45.08.03
General Mechanical Waves

©|L.D.Landau *et al.*, , ..., ..., ...Ed., ...|Theory of Elasticity||

©|R.Blandford & K.Thorne, *Applications Of Classical Physics*, ..., ..., ...Ed., WEB - URL, WEB - URL.|2017|*part 4 and ...*

General mechanical waves have both longitudinal and transverse displacements. There study is considerably more complex than pure longitudinal|transverse waves.

A thermo-mechanical treatment is normally necessary,

45.08.03.01 Energy and Intensity of General Mechanical Waves

45.08.03.02 Linear momentum of General Mechanical Waves

45.08.03.03 Angular Momentum of General Mechanical Waves

45.09
Elements of Acoustics

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|2.2||

PAPER - IN - PROGRESS

45.09.01
Intensity of Acoustic Waves: The Decibel

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|2.2||

Definition of decibel, in terms of a reference intensity, I_0,:

$$\mathcal{D} \equiv 10 \log_{10} \left[\frac{I}{I_0} \right] \qquad \text{with } I_0 = 10^{-12}\ \mathrm{W/m^2} \quad .$$

The value for I_0 is chosen as the typical value at the limit of the ear sensitivity for a signal at 1000 Hz.

Note that the decibel is not an absolute measurement but a relative measurement.

45.09.02
Hearing Range

©||The Human Ear|*WEB - URL* |

©||Sensitivity of Human Ear|*WEB - URL* |

The typical hearing range of humans spans from 20 Hz to 20000 Hz, which corresponds to sound waves in air at $T = 20\ ^\circ\mathrm{C}$ with wavelengths of $17\ \mathrm{m} \div 0.017\ \mathrm{m}$.

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45.10 Doppler Effect for Mechanical Waves

This § is referenced at pages:
[2016, 2136]

45.10.01 Source at Rest With Respect to the Medium and Observer in Motion

Read § 45.12.01 for the conveyor belt analogy, very helpful to understand **Doppler effect** in one dimension and related relations.

In this case the **Doppler effect** is just a matter of relative velocity, because the relative velocity between the observer and any CPWF is just $\mathbf{v}_0 - \mathbf{c}$.

45.10.01.01 Slow Motion of the Observer

Consider a PW with wave-vector $\mathbf{k} \equiv k\hat{\mathbf{n}}$ and **phase-velocity** $\mathbf{c}$. When the source, emitting at frequency ν, is at rest with respect to the material medium and the observer is in motion with velocity $\mathbf{v}_0$ with respect to the material medium, the observer measures a frequency ν':

$$\nu' = \nu\left(1 - \frac{\hat{\mathbf{n}}\cdot\mathbf{v}_0}{c}\right) = \nu\left(1 - \frac{v_0\cos[\theta_0]}{c}\right) \qquad \text{for } v_0|\cos[\theta_0]| \leq c \quad . \tag{45.10.01}$$

→ 2044

This relation is valid in the case $v_0|\cos[\theta_0]| < c$ and, in particular, in the limit $\mathbf{v}_0 \longrightarrow \mathbf{0}$.

Consider in fact the special case of collinear and equi-verse $\mathbf{c}$ and $\mathbf{v}_0$ and any specific phase of the wave (its maximum amplitude, for instance). Start from the instant of time when the observer is coincident with the maximum amplitude of the wave. The next maximum amplitude of the wave will reach the observer after a time

$$T' = \frac{\lambda}{(c - v_0)} = \frac{Tc}{(c - v_0)} = \frac{Tc}{|c - v_0|} \quad ,$$

as given by the relative velocity between the wave and the observer, giving the new frequency of equation (45.10.01).

45.10.01.02 Fast Motion of the Observer

In the case $v_0|\cos[\theta_0]| > c$ the observer moves faster than the wave and the two above relations become:

$$\nu' = \nu\left(\frac{\hat{\mathbf{n}}\cdot\mathbf{v}_0}{c} - 1\right) = \nu\left(\frac{v_0\cos[\theta_0]}{c} - 1\right) \qquad \text{for } v_0|\cos[\theta_0]| > c \quad . \tag{45.10.02}$$

and

$$T' = \frac{\lambda}{(v_0 - c)} = \frac{Tc}{(v_0 - c)} = \frac{Tc}{|c - v_0|} \quad , \tag{45.10.03}$$

In this case the sequence of wave-crests seen by the observer is inverted with respect to the sequence of a fixed observer.

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45.10.01.03 Summary

$$\boxed{\nu' = \nu\left|1 - \frac{\widehat{\mathbf{n}}\cdot\mathbf{v}_0}{c}\right| = \nu\left|1 - \frac{v_0\cos[\theta_0]}{c}\right|} \quad . \tag{45.10.04}$$

→ 2047

$$\boxed{T' = \frac{\lambda}{(v_0 - c)} = \frac{Tc}{(v_0 - c)} = \frac{Tc}{|v_0 - c|}} \quad ,$$

45.10.02
Source in Motion With Respect to the Medium and Observer at Rest

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Read § 45.12.01 for the conveyor belt analogy, very helpful to understand **Doppler effect** in one dimension and related relations.

Consider a PW with wave-vector $\mathbf{k} \equiv k\widehat{\mathbf{n}}$ and **phase-velocity** $\mathbf{c}$. When the source, emitting at frequency ν, is in motion with respect to the material medium with velocity $\mathbf{v}_S$ and the observer is at rest with respect to the material medium, the observer measures a frequency ν':

$$\boxed{\nu' = \nu\frac{1}{(1 - \widehat{\mathbf{n}}\cdot\mathbf{v}_S/c)} = \nu\frac{1}{(1 - v_S\cos[\theta_S]/c)} \quad \text{for } v_S|\cos[\theta_S]| \leq c} \quad . \tag{45.10.05}$$

→ 2045 2047

Consider in fact the special case of collinear and equi-verse $\mathbf{c}$ and $\mathbf{v}_S$ and any specific phase of the wave (the maximum amplitude, for instance).

A first derivation

Start from the instant of time when the source is emitting the maximum amplitude of the wave. After one period the maximum amplitude of the wave will have traveled by cT, where we suppose the observer is located, while the source will have traveled by $v_S T$. The new maximum amplitude of the wave will reach the observer after a time

$$\boxed{T' = T\frac{c - v_S}{c}} \quad ,$$

giving the new frequency of equation (45.10.05).

A second derivation

When the source is moving the wavelength seen at the observer, λ', is changed with respect to the wavelength of the wave when the source is standing, λ. The relations among the two wavelengths are:

$$\boxed{\lambda \equiv cT = \lambda' + v_S T} \quad .$$

45.10.03
Source Faster Than Wave

This § is referenced at pages:
[2045]

If the source moves with a velocity which is larger than the **phase-velocity** of the wave inside the medium peculiar phenomena happen: a shock-wave is generated, read § 45.10.03.

The CPWF have an envelope which is a conical surface. It is the locus of the equal-phase points for the waves emitted by the source during its motion and it constitutes itself a kind of shock-wave. This conical front is the shock-wave. The source lies at the vertex of the conical front.

DRAFT - IN PROGRESS

The ratio between the speed of the source and the speed of the wave defines the Mach ratio (or Mach number):

$$\mathcal{M} \equiv \frac{v_{\mathrm{S}}}{c} \quad .$$

From geometric considerations one can easily show that the front of the shock-wave moves at an angle with the velocity of the source, θ, given by:

$$\cos\left[\theta\right] = \frac{c}{v_{\mathrm{S}}} = \frac{1}{\mathcal{M}} \quad .$$

In the case of mechanical waves, that is waves propagating as Perturbation|Disturbance|Signal of a material medium, the movement itself of some object is the source of the Perturbation|Disturbance|Signal and it is not necessary that a real wave-emitting source exists. This is the case, for instance, of the sonic shock-wave produced by a plane moving at a speed larger than the speed of sound in air.

DRAFT-IN-PROGRESS

45.11 General Case

Read § 45.12.01 for the conveyor belt analogy, very helpful to understand **Doppler effect** in one dimension and related relations.

The **Doppler effect** for mechanical waves is non-symmetrical under the interchange of the velocity of the observer and source. This is due to the fact that the medium sets a kind of privileged Reference Frame. The speed of the wave, as it is seen by the observer, is not the same in the two cases.

In the general case of both source and observer moving at velocity, respectively, $\mathbf{v}_S$ and $\mathbf{v}_0$, with respect to the medium the expression is:

$$\nu' = \nu \frac{(c - \hat{\mathbf{n}}\cdot\mathbf{v}_0)}{(c - \hat{\mathbf{n}}\cdot\mathbf{v}_S)} \quad . \tag{45.11.01}$$

→ 2047

For low velocities the results only depends on the relative velocity of source and observer, as it becomes:

$$\nu' \simeq \nu + \hat{\mathbf{n}} \left(\frac{\mathbf{v}_S - \mathbf{v}_0}{c} \right) \quad . \tag{45.11.02}$$

It is implicit in equation (45.11.01) that the **phase-velocity** of the wave is larger in module of both the module of the source velocity and the module of the observer velocity.

Note that in the case $\hat{\mathbf{n}}\cdot\mathbf{v}_0 = \hat{\mathbf{n}}\cdot\mathbf{v}_S$ there in no change in frequency, showing that only the relative motion is relevant. However, the two different cases in equations (45.10.04), (45.10.05) are not symmetrical due to the presence of the propagation medium. Note, however, that in the case either the source velocity or the observer velocity are very small with respect to the **phase-velocity** the results from the two equations (45.10.04), (45.10.05) are not distinguishable for the same relative velocity.

Table 45.11 summarizes the characteristics of mechanical **Doppler effect** . All velocities are with respect to the medium. In the first case the observer sees a different **phase-velocity** of the wave because

$\nu \neq \nu'$	as seen by the observer		
Source at rest, observer moving with $\mathbf{v}_0$:	$\lambda' = \lambda$	$\mathbf{c}' = \mathbf{c} - \mathbf{v}_0 \neq \mathbf{c}$	$\lvert\mathbf{v}_0\rvert \leq c$
Observer at rest, source moving with $\mathbf{v}_S$:	$\lambda' = T\lvert\mathbf{c} - \mathbf{v}_S\rvert \neq \lambda$	$\mathbf{c}' = \mathbf{c}$	$\lvert\mathbf{v}_S\rvert \leq c$

Table 45.11: **Doppler effect** for mechanical waves; the two cases.

it is moving with respect to the medium which sets the **phase-velocity** . In the second case the observer sees the same **phase-velocity** of the wave because it is not moving with respect to the medium which sets the **phase-velocity** .

In the first case the observer sees the same wavelength of the wave because that is set by the stationary source and **phase-velocity** . The wavelength is measured on water and it does not change. In the second case the observer sees a different wavelength of the wave because that is set by the moving source and **phase-velocity** . The wavelength is measured on water but it does change because the sources emits the different phases of the waves at different points.

Note that **in all cases the physical quantity relevant to the observer is the frequency of the wave it receives because this will set the frequency of the oscillations at the observer**, both in case of a stationary observer or in in the case of observer in motion.

Compare with the **Doppler effect** for ElectroMagnetic waves § 46.12.

DRAFT IN PROGRESS

45.12 Examples and Physical Applications

45.12.01 Conveyor Belt Example

This § is referenced at pages:
[2044, 2045, 2047]

The conveyor belt gives an excellent analogy to **Doppler effect** . The speed of the conveyor belt corresponds to the phase speed of the wave in the medium.

A men putting stones, at regular time intervals, on the belt corresponds to the source.

A men taking out stones corresponds to the observer.

The frequencies corresponds to the rate stones are put on the belt (by the source) and to the rate stones are taken out by the observer.

45.12.02 SONAR

45.12.03 Speed of Sound in a Ideal Gas

This § is referenced at pages:
[2027, 2029]

Calculate the speed of sound in a ideal gas with adiabatic index equal to γ.

Assume both cases of adiabatic and isothermal transformation, and discuss which of them is best suited to the problem.

WORKED SOLUTION

One should assume an adiabatic transformation because, normally, the compression/rarefaction of the gas are so fast and the thermal **conductivity** of the gas so low that there is negligible heat transfer between neighboring parts of the gas.

For adiabatic transformations one finds:

$$p = \rho \frac{RT}{M} \qquad \text{equation of state of a ideal gas} \quad ,$$

$$p = k\rho^{\gamma} \qquad \text{adiabatic transformation, } k \text{ is a constant} \quad ,$$

$$\frac{\partial p}{\partial \rho} = k\gamma\rho^{\gamma-1} = p\rho^{-\gamma}\gamma\rho^{\gamma-1} = \gamma\frac{p}{\rho} = \gamma\frac{RT}{M} \quad ,$$

$$c^2 = p'[\rho_0] = \frac{\partial p}{\partial \rho} = \gamma\frac{p}{\rho} = \gamma\frac{RT}{M} \quad .$$

For isothermal trnasformations one would find:

$$c^2 = p'[\rho_0] = \frac{\partial p}{\partial \rho} = \frac{p}{\rho} = \frac{RT}{M} \qquad \text{as it formally corresponds to } \gamma = 1 \quad .$$

45.12.04
Echography

Medical ultra-sonography.

45.12.05
Doppler Ultra-Sonography

45.12.06
Dynamics of an LHI String|Bar Hanging in the Gravity Field With a Suspended Mass

This § is referenced at pages:
[1502, 1502]

©|W.C.Elmore & M.A.Heald, , ..., ..., ...Ed., ...|4.8||

Read section § 27.07.10 for the statics.

Consider an elastic string, made of an homogeneous and isotropic material, hanging in the gravity field from one extreme. The rest length is L, its sound speed is c and its total mass m. A small mass M is appended to the free extreme.

Determine the configuration of equilibrium assuming small strain and no lateral change of dimensions.

Show that the fundamental angular frequency of the small oscillations, ω, can be approximated, for $m \ll M$, as

$$\frac{\omega^2}{\omega_0^2} = \frac{3M}{(m+3M)} = \frac{1}{(1+\alpha/3)} \quad , \tag{45.12.01}$$

→ 2049 2052

where ω_0 is the fundamental angular frequency of the small oscillations for negligible mass of the spring.

Compare the approximate formula (45.12.01) with the exact result for a few values of the ratio m/M.

WORKED SOLUTION

Introduce a vertical z axis, oriented downwards, with origin at the point where the string is hanging. The equation of motion is equation (45.13.03):

$$\frac{\partial^2 \xi}{\partial t^2}[z,t] - c^2 \frac{\partial^2 \xi}{\partial z^2}[z,t] = g \quad . \tag{45.12.02}$$

let us first determine the equilibrium condition. The boundary conditions at equilibrium are:

$$\xi[z=0] = 0 \qquad \text{fixed extreme} \tag{45.12.03}$$

$$\frac{\partial \xi}{\partial z}[z=L] = \frac{Mg}{YA} \qquad \text{positive because it is elongated, it is required to sustain the mass} \quad .$$

The static solution is found from the static equation

$$-c^2 \frac{\partial^2 \xi}{\partial z^2}[z,t] = g \quad ,$$

→ 2050

in terms of two integration constants, β_0 and β_1, as:

$$\xi[z] = -\frac{g}{2c^2} z^2 + \beta_1 z + \beta_0 \quad . \tag{45.12.04}$$

And the boundary conditions give the static solution:

$$\overline{\xi}[z] = -\frac{g}{2c^2} z^2 + \left(\frac{gL}{c^2} + \frac{Mg}{YA} \right) z = \frac{g}{Y} \left(-\frac{\rho_0}{2} z^2 + \left(L\rho_0 + \frac{M}{A} \right) z \right) \quad .$$

The stress exerted by a part on the part at larger z is:

$$\frac{F}{A} = S_{zz} = -YA \frac{\partial \xi}{\partial z} \quad .$$

The stress exerted by a part on the part at smaller z is:

$$\frac{F}{A} = S_{zz} = +YA\frac{\partial \xi}{\partial z} \quad .$$

We can check the stress at the mass and at the hanging point.

$$\frac{F}{A}[z=0] = S_{zz}[z=0] = -YA\frac{\partial \xi}{\partial z}[z=0] = -\left(M+m\right)g \quad ,$$

$$\frac{F}{A}[z=L] = S_{zz}[z=L] = -YA\frac{\partial \xi}{\partial z}[z=L] = -Mg \quad .$$

Both stresses are negative as the chain is hanging the mass and itself.
The general solution can be found by finding the general solution of the homogeneous equation and adding a particular solution of the complete equation, for instance the static solution determined above.
The generic boundary conditions are:

$$\xi[z=0,t] = 0 \tag{45.12.05}$$

$$M\frac{\partial^2 \xi}{\partial t^2}[z=L,t] = -YA\frac{\partial \xi}{\partial z}[z=L,t] + Mg \qquad \text{the equation of motion for the mass } M \quad . \tag{45.12.06}$$

Consider now the general solution as:

$$\xi[z,t] = \widetilde{\xi}[z,t] + \overline{\xi}[z,t] \quad ,$$

where $\widetilde{\xi}[z,t]$ is the general solution of the homogeneous equation. As $\overline{\xi}[z,t]$ is a particular solution of the full equation $\widetilde{\xi}[z,t]$ satisfies:

$$\frac{\partial^2 \widetilde{\xi}}{\partial t^2}[z,t] - c^2\frac{\partial^2 \widetilde{\xi}}{\partial z^2}[z,t] = 0 \quad . \tag{45.12.07}$$

→ 2050

The frequency spectrum of the homogeneous equation and of the non-homogeneous equations are the same.
We can then solve the homogeneous equation.
The **Dispersion Relation** :

$$\omega = kc \quad ,$$

is derived from the homogeneous equation for $\widetilde{\xi}$, equation (45.12.07).
The boundary conditions for $\widetilde{\xi}$ become:

$$\widetilde{\xi}[z=0] = \xi[z=0] - \overline{\xi}[z=0] = 0 - 0 = 0 \quad ,$$

and

$$M\frac{\partial^2 \xi}{\partial t^2}[z=L,t] = -YA\frac{\partial \xi}{\partial z}[z=L,t] + Mg \quad ,$$

$$M\frac{\partial^2 \overline{\xi}}{\partial t^2}[z=L,t] + M\frac{\partial^2 \widetilde{\xi}}{\partial t^2}[z=L,t] = -YA\frac{\partial \overline{\xi}}{\partial z}[z=L,t] - YA\frac{\partial \widetilde{\xi}}{\partial z}[z=L,t] + Mg \quad ,$$

$$\frac{\partial^2 \overline{\xi}}{\partial t^2}[z,t] = 0 \qquad \overline{\xi} \text{ is a static solution} \quad ,$$

$$-YA\frac{\partial \overline{\xi}}{\partial z}[z=L,t] + Mg = 0 \qquad \text{from the boundary condition} \quad ,$$

$$M\frac{\partial^2 \widetilde{\xi}}{\partial t^2}[z=L,t] = -YA\frac{\partial \widetilde{\xi}}{\partial z}[z=L,t] \quad ,$$

$$\widetilde{\xi}_{\mathtt{M}} \equiv \widetilde{\xi}[z=L] \quad ,$$

having used equation (45.12.06). Consider a solution for $\widetilde{\xi}[z,t]$ of type:

$$\widetilde{\xi}[z,t] = B\sin\left[kz\right]\cos\left[\omega t\right] \quad ,$$

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with B arbitrary constant. This already satisfies the boundary condition:

$$\widetilde{\xi}[z=0] = \xi[z=0] - \overline{\xi}[z=0] = 0 - 0 = 0 \quad .$$

The second boundary condition must take into account the motion of the mass, described by the displacement $\widetilde{\xi}_{\mathtt{M}}$:

$$\widetilde{\xi}_{\mathtt{M}} = \widetilde{\xi}[z=L] = B\sin[kL]\cos[\omega t] \quad ,$$

$$F[z=L] = -YA\frac{\partial\widetilde{\xi}}{\partial z}[z=L] = M\frac{\partial^2\xi_{\mathtt{M}}}{\partial t^2} \quad ,$$

$$-YABk\cos[kL]\cos[\omega t] = -M\omega^2 B\sin[kL]\cos[\omega t] \quad ,$$

$$-YAk\cos[kL] = -M\omega^2\sin[kL] \quad ,$$

$$\omega^2 = k^2c^2 = k^2\frac{Y}{\rho_0} \quad ,$$

$$\tan[kL] = (kL)\frac{YA}{M\omega^2 L} = (kL)\frac{YA\rho_0}{Mk^2YL} = (kL)\frac{YA\rho_0}{Mk^2YL} \quad ,$$

$$\Longrightarrow (kL)\tan[kL] = \frac{m}{M} \quad .$$

Therefore the allowable wave numbers are given by:

$$\boxed{(kL)\tan[kL] = \frac{m}{M}} \quad .$$

Note that, due to the form of the equation, the different modes will not be multiple of each other.
The equation always has one and only one solution in the interval

$$0 \le kL\pi/2 \qquad \text{first mode} \quad ,$$

that is the smallest solution describing the first oscillation mode.
Let $\theta \equiv kL$ and $\alpha \equiv m/M$.
The transcendental equation cannot be solved exactly in closed form. The equation has infinite solutions, corresponding to the infinite normal modes of the oscillations of the chain.
Consider the limiting case $M \ll m$. In this case the solutions can be approximated by:

$$M \ll m \implies \theta \equiv kL \simeq \frac{\pi}{2} + q\pi \qquad \text{with } q \text{ any integer positive number} \quad .$$

This is coherent with the fact that in this case the effect of the mass is negligible, and we have the oscillations of a chain free at its unconstrained extreme. Note that the first mode corresponds to

$$kL = \frac{\pi}{2} \Longrightarrow \lambda = 4L \quad ,$$

as it is expected for a chain with one fixed extreme and the second one free.
Consider the limiting case $m \ll M$. In this case the fundamental mode can be determined by noting that θ must be small and therefore the tangent can be developed in Taylor series around $\theta = 0$:

$$m \ll M \implies \theta\tan[\theta] \simeq \theta\left(\theta + \frac{\theta^3}{3} + \mathcal{O}(\theta)^5\right) = \alpha \ll 1 \quad ,$$

$$\theta^4 + 3\theta^2 - 3\alpha = 0 \implies \theta^2 = \frac{-3 \pm \sqrt{9+12\alpha}}{2} \longrightarrow \theta^2 = \frac{-3+\sqrt{9+12\alpha}}{2} \qquad \text{positive solution acceptable} \quad .$$

As $\alpha \ll 1$ we can develop the solution around $\alpha = 0$:

$$\theta^2 = \frac{-3+\sqrt{9+12\alpha}}{2} = \alpha - \frac{\alpha^2}{3} + \frac{2\alpha^3}{9} + \mathcal{O}(\alpha)^4 \quad .$$

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By taking the lowest order term we find:

$$\theta^2 = \alpha \implies k^2 L^2 = \frac{m}{M} \quad ,$$

$$\implies \omega^2 = k^2 c^2 = \frac{\alpha}{L^2}\frac{Y}{\rho_0} = \frac{m}{M}\frac{Y}{L^2}\frac{AL}{m} = \frac{YA}{ML} = \frac{k_{\text{E}}}{M} \quad ,$$

$$k_{\text{E}} \equiv \frac{YA}{L} \qquad \text{the equivalent elastic constant of the chain} \quad .$$

Note that this level of approximation amount to take: $\tan[\theta] = \theta$.
By taking the second to lowest order term we find:

$$\theta^2 = \alpha - \frac{\alpha^2}{3} \implies k^2 L^2 = \alpha - \frac{\alpha^2}{3} \quad ,$$

$$\implies \omega^2 = k^2 c^2 = \frac{1}{L^2}\left(\alpha - \frac{\alpha^2}{3}\right)\frac{Y}{\rho_0} = \frac{1}{L^2}\alpha\frac{Y}{\rho_0}\left(1 - \frac{\alpha}{3}\right) = \quad ,$$

$$\frac{k_{\text{E}}}{M}\left(1 - \frac{\alpha}{3}\right) = \frac{k_{\text{E}}}{M}\left(1 - \frac{m}{3M}\right) \simeq \frac{k_{\text{E}}}{M}\frac{1}{\left(1 + \dfrac{m}{3M}\right)} = \frac{k_{\text{E}}}{M + m/3} \quad ,$$

$$k_{\text{E}} \equiv \frac{YA}{L} \qquad \text{the equivalent elastic constant of the chain} \quad .$$

Therefore the first fundamental frequency is changed, with respect to the case of zero hanging mass, by the factor:

$$\frac{\omega^2}{\omega_0^2} = \frac{1}{\left(1 + \dfrac{m}{3M}\right)} = \frac{1}{\left(1 + \dfrac{\alpha}{3}\right)} \quad .$$

In general if one measures the oscillation period one will always find the period of the lowest harmonics.
The comparison of the approximate formula (45.12.01) with the exact result for a few values of the ratio $\alpha \equiv m/M$ follows. We compare the approximate result for θ, $\widetilde{\theta}$, with the exact result.
For $\alpha = 1.0$ one finds $\widetilde{\theta} = 0.82$ and $\theta = 0.86$.
For $\alpha = 0.5$ one finds $\widetilde{\theta} = 0.645$ and $\theta = 0.653$.
For $\alpha = 0.1$ one finds $\widetilde{\theta} = 0.3109$ and $\theta = 0.3110$.
For $\alpha = 0.05$ one finds $\widetilde{\theta} = 0.22174$ and $\theta = 0.22176$.

45.12.07
Seismic Waves

Seismic waves allows a tomography of the Earth.
See figures 45.11, 45.12, 45.13.

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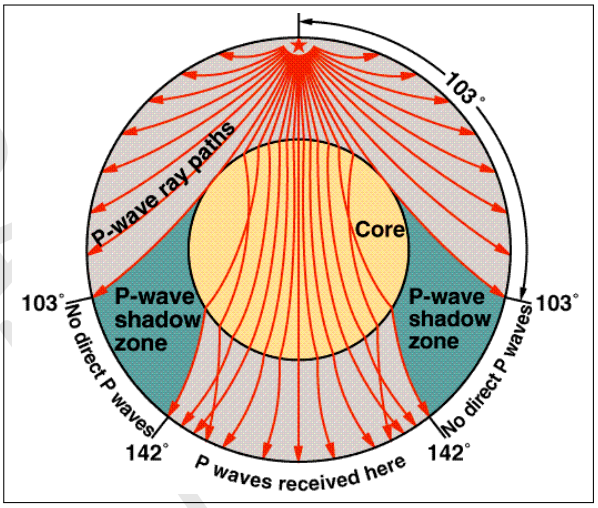

Figure 45.11: Earth interior seismic waves propagation p-shadow

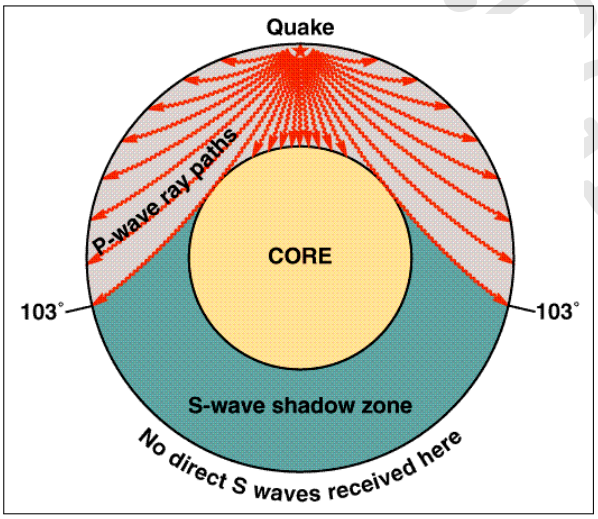

Figure 45.12: Earth interior seismic waves propagation s-shadow

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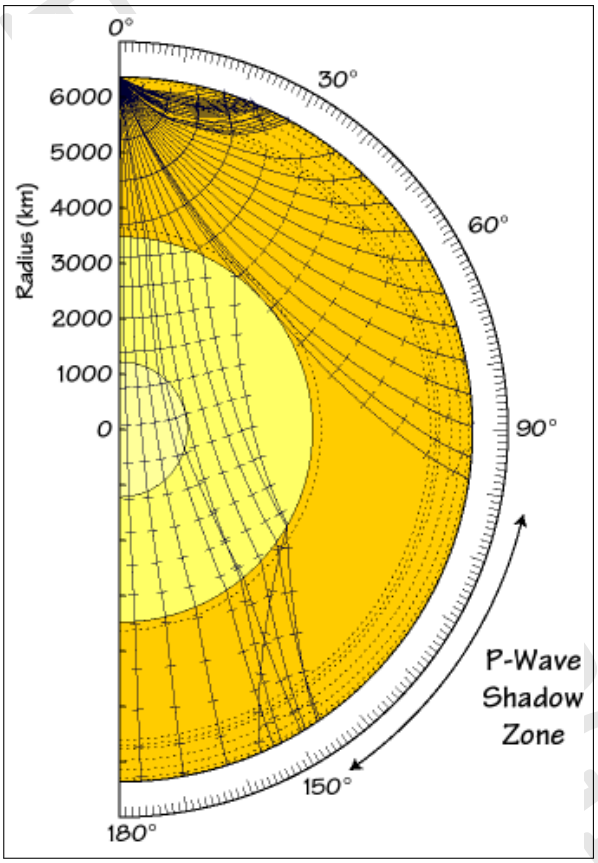

Figure 45.13: Earth interior seismic waves propagation

45.12.08
Tsunamis

Tsunamis cannot, strictly speaking, be described by the classical **D'Alembert non-dispersive wave-equation** , but a more complex nonlinear **Wave-Equation** is required. A very simplified and very approximate qualitative description can be given as if a one-dimensional classical d'Alembert wave were involved.

Tsunamis are born from small amplitude and long wavelength waves in deep-water, generated by Earthquakes. The region where tsunamis are generated is usually large (hundreds of km) with respect to the average sea depth (about four km) and so is the wavelength.

A rapid transient results in gravitational waves with $\lambda \approx 100$ km (long waves).

Water waves are in general strongly dispersive, read § 44.03.09.05.

Tsunamis result in shallow-water waves non dispersive gravity waves. Tsunamis periods results range between $10^2 \div 10^4$ s. Tsunamis amplitudes rarely exceed tens of cm in open seas so they are not noticeable.

The analogy with the longitudinal mechanical waves on a bar must consider a bar whose physical properties are changing along the direction of propagation: in case of water waves the change of depth is a change of the physical properties, changing the **phase-velocity** , and the **wave impedance** .

When these waves come close to the shore, where the depth of the sea is h, the **phase-velocity** is approximated by $v \sim \sqrt{gh}$. The frequency must stay constant at any distance along the propagation direction: see the discussion on kinematic relations leading to Snell laws, § 48). Assume an idealized steady situation: the intensity/power must stay constant at any distance along the propagation direction.

As the **phase-velocity** tends to zero, while getting closer and closer to the shore one can deduce from the equations in section § 45.08.02.01 that the wave amplitude increases. Moreover, as $\lambda = v/\nu$, the wavelength tends to zero.

Example

Near the shore: $h = 0.5$ m gives $v = 2.2$ m/s.

In the Pacific Ocean, where a typical water depth is about $h = 4km$, a tsunami travels at about $v = 200$ m/s (long wavelength with respect to sea-depth).

45.12.09
Blood Speed Analysis (Eco-Doppler) and Auto-Velox

This § is referenced at pages:
[2136]

©|Lim Yung-Kuo, *Problems And Solutions On Mechanics*, 1994, World Scientific, ...Ed., ...|1251||

Consider an emitter-receiver of sound waves who emits a signal and detects the echo of the signal it emitted. The frequency of the acoustic signal received by a stationary device as a function of the emitted frequency ν_0, in terms of the velocity of the object with respect to the air, V, and the speed of sound, c, is given by (one-dimensional problem)

$$\nu = \nu_0 \frac{c - V}{c + V} \quad .$$

In fact the object receives and re-emits, as a moving source, a frequency $\nu = \nu_0 (1 - V/c)$.

Typically a piezo-electric emitter/receiver is used.

Read also § 46.12.

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45.12.10
Phase Between Pressure and Particle Velocity for PW

©|D.A.Russell|WEB - URL ||

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45.13 Exercises Problems and Physical Applications

45-001 ✓ I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...

4.0150, 4.0155, 4.0157, 4.0159, 4.0160, 4.0162, 4.0163, 4.0164, 4.0165, 4.0168, 4.0183, 4.0185, 4.0188.

45-002 ✓ Velocity of Acoustic Waves

This § is referenced at pages:
[Never referenced.]

Show that the velocity of longitudinal mechanical waves can be written as:

$$c_{\text{L}}^2 = \left.\frac{\partial p}{\partial \rho}\right|_0 = \frac{B}{\rho_0} \quad . \tag{45.13.01}$$

→ 1426

WORKED SOLUTION

$$0 = \text{d}m = \rho\,\text{d}V + V\,\text{d}\rho \quad ,$$
$$\frac{1}{B} = -\frac{1}{V}\frac{\partial V}{\partial p} = \frac{1}{\rho}\frac{\partial \rho}{\partial p} \quad .$$

45-003 ✓ Velocity of Acoustic Waves in Elastic Solids

This § is referenced at pages:
[2031, 2031]

Show that for an elastic solid medium the velocity of longitudinal waves propagating along the z axis, in absence of any displacement perpendicular to the z axis, can be written as:

$$c_{\text{L}}^2 = \left.\frac{\partial p}{\partial \rho}\right|_0 = \frac{B}{\rho_0} \longrightarrow \frac{Y}{\rho_0} \quad . \tag{45.13.02}$$

→ 1426

Note that the derivation in section § 27.07.06 cannot be applied without changes, as in the present case it is assumed that the block deforms only in one direction. The fact that **it can deform in one direction only implies that the relative change of volume is the same as the relative change of length, so that** $Y = B$.

WORKED SOLUTION

$$\rho \equiv \frac{\text{d}m}{\text{d}V} \Longrightarrow \frac{\partial \rho}{\partial V} = -\frac{\rho}{V} \quad ,$$
$$-\Delta p \equiv S_{zz} \equiv Y\frac{\Delta \xi_z}{\Delta z} \qquad \text{by the convention on the signs of } p \text{ and } S_{zz} \quad ,$$
$$\frac{\partial p}{\partial \rho} = \frac{\partial p}{\partial V}\frac{\partial V}{\partial \rho} = -\frac{\partial p}{\partial V}\frac{V}{\rho} \equiv \frac{B}{\rho} \quad ,$$
$$\frac{\partial p}{\partial \rho} = -\frac{\partial p}{\partial V}\frac{V}{\rho} \sim -\frac{\Delta p}{\rho}\frac{z_0}{\Delta z} = \frac{Y}{\rho_0} \quad .$$

Partial derivatives were used because one knows that the $\rho = \rho[V, \ldots]$ function depends on other thermodynamical variables also.

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Beware that the pressure in Young law is actually a differential pressure, with respect to the equilibrium configuration.

45-004 ✓ D'Alembert Equation in Presence of Volume Forces

This § is referenced at pages:
[1997, 2031]

Equation (45.02.05) refers to the case when there are no volume forces acting on the medium. Generalize the derivation to the case when a given volume force is acting on the medium.

WORKED SOLUTION

The eqaution is written in terms of the force per unit volume acting on the bulk of the material, f_V:

$$\frac{\partial^2 \xi}{\partial t^2}[z,t] - c^2 \frac{\partial^2 \xi}{\partial x^2}[z,t] = \frac{f_V}{\rho_0} \quad . \tag{45.13.03}$$

→ 2049

45-005 Phase-Velocity of Longitudinal and Transverse Waves in LHI Solids

Show that $c_L \geq c_T$.

WORKED SOLUTION

It is obvious from equations (45.04.02), (45.04.04).

45-006 ✓ Sound and Wind

©|A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...|§ 8.11||

Sound waves travel horizontally from a source to a receiver. Assume that the source has a speed u, that the receiver has a speed v and that a wind of speed w is blowing from the source toward the receiver; all velocities are along the same direction. Assume that the source emits sound of frequency ν_0, and the speed of sound in still air is V.

Calculate the frequency recorded by the receiver.

Note that if the velocities of source and receiver are equal, the existence of the wind makes no difference to the observed frequency of the received signal.

45-007 ✓ One Sound Wave

©|A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...|§ 7||

1. What is the equation for a longitudinal wave traveling in the negative z direction with amplitude $A = 0.003$ m, frequency $\nu = 5$ Hz, and phase speed $v = 3000$ m/s ?
2. Which kind of wave it is? In which material is it traveling?
3. Find the maximum speed of any particle.
4. How far apart are two points whose displacements are 30 ° apart in phase?
5. At a given point, what is the phase difference between two displacements occurring at times separated by $\Delta t = 0.01$ s ?

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45-008 ✓ One water Wave

©|A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...|§ 7||

Two points on the surface of a lake are observed as a traveling water wave passes them. The two points are at $x_1 = 0$ and $x_2 = L$. The transverse motions of the two points are found to be: $y_1 = 0.2 \sin[3\pi t]$ and $y_2 = 0.2 \sin[3\pi t + \pi/8]$.

What is the frequency and wavelength?

What is the phase speed?

Which way is the wave traveling?

45-009 ✓ Acoustic Waves From an Airplane

The noise produced by a plane at a distance $D = 100$ m is 120 db.

What is the mean quadratic value of the pressure variation?

What is the mean quadratic value of the displacement at $\nu = 200$ Hz ?

45-010 ✓ Amplitide for Painfully Loud Sound

From data on the sensitivity of human ears find the oscillation amplitude corresponding to the maximum sensitivity of ears (painfully loud sound).

45-011 ✓ Amplitude for Barely Audible Sound

From data on the sensitivity of human ears find the oscillation amplitude corresponding to the minimum sensitivity of ears.

45-012 ✓ A Siren

An emergency vehicle, whose siren is emitting at a frequency ν, is passing at constant speed, v, nearby you, at a minimum distance H.

Determine the frequency you hear as a function of time.

45-013 ✓ Wave Reflection From a Moving Wall

This § is referenced at pages:
[2184]

45-014 ✓ Standing Waves on an Elastic String With Fixed Extremes

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CHAPTER 46

ElectroMagnetic Waves

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About ElectroMagnetic Waves in regions of space far from boundaries.

This § is referenced at pages:
[1176, 1730, 1814, 1927, 1990, 2148]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|All arguments ...
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|All arguments about waves.||
©|R.Fitzpatrick|Oscillations and Waves|***WEB - URL*** - *Complete textbook*|
©|A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...|Complete text with worked ...

Pre-requisites - Review Harmonic oscillations **§** 25 and complex algebra **§** 19.03, **§** 19.04.

WORK-IN-PROGRESS

46.01 EM Waves in Vacuum Without Boundaries

This § is referenced at pages:
[1991, 2068, 2071]

©|G.S.Smith, *An Introduction To Classical Electromagnetic Radiation*, 1997, CUP, ...Ed., ... 3.1|excellent and clear||

Vacuum means: no sources.

We start by studying **EM waves in free vacuum space**, that is, without matter and without boundaries. The effect of boundary conditions will be considered later (section § 48) when introducing Transmission|Refraction and reflection of waves, and in section § 47. The effect of matter will be studied in detail in section § 54.

For ElectroMagnetic waves in vacuum there is zero charge density and zero current density. Then, **Maxwell Equations** give the **classical d'Alembert wave equation**:

$$\boxed{\boldsymbol{\nabla}^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}} \quad , \tag{46.01.01}$$

→ 2062 2063 2065 2151

$$\boxed{\boldsymbol{\nabla}^2\mathbf{B} = \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}} \quad , \tag{46.01.02}$$

→ 2062 2063 2065

$$\boxed{c^2 \equiv (\varepsilon_0\mu_0)^{-1}} \quad .$$

At this stage, c^2 is just a shortcut for $(\varepsilon_0\mu_0)^{-1}$. Note that, ε_0, μ_0 and c are exact constants, nowadays (read § B).

As often, one starts by studying MPW solutions, as a basis for description of any wave. In fact, the description in terms of MPW waves has its importance in that MPW allow to express any function via the **Fourier Series** | **Fourier Integral** .

Read § 46.15 for an introduction to a realistic description of real EM waves, one which is not limited to MPW.

46.01.01 Maxwell Equations Versus d'Alembert Wave Equation

This § is referenced at pages:
[2068, 2086]

Equations (46.01.01), (46.01.02) are deduced from **Maxwell Equations** by performing some derivatives and therefore they might admit more solutions than the original **Maxwell Equations** .

ElectroMagnetic fields must satisfy the wave equation, which follows from Maxwell Equations , but they must also satisfy Maxwell Equations themselves. Therefore one must investigate the constraints on the solution of the wave equation imposed by **Maxwell Equations** . The generic solution of the wave equation must then be replaced into **Maxwell Equations** to find the additional constraints, if any.

Note that it is rather obvious that there will be additional constraints from **Maxwell Equations** , because we know from Maxwell equations that the Electric|Magnetic fields are linked, while the wave equations (46.01.01), (46.01.02) decouple the two fields providing independent solutions but in non-static situations the Electric|Magnetic field cannot be separated.

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46.01.02
EM MPW in Vacuum

This § is referenced at pages:
[2065]

Read § 44.02.03.03.

The ElectroMagnetic wave equations (46.01.01), (46.01.02) admit MPW solutions, separately and independently for $\mathbf{E}$ and $\mathbf{B}$:

$$\boxed{\mathbf{E}[\mathbf{x},t]=\mathbf{E}_0\cos\left[\mathbf{k}_E\cdot\mathbf{x}-\omega_E t-\phi_E\right]} \quad , \tag{46.01.03}$$

→ 2063 2064

$$\boxed{\mathbf{B}[\mathbf{x},t]=\mathbf{B}_0\cos\left[\mathbf{k}_B\cdot\mathbf{x}-\omega_B t-\phi_B\right]} \quad , \tag{46.01.04}$$

→ 2063 2064

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are constant vectors, carrying the information on both the module, direction and orientation of the ElectroMagnetic fields.

For ElectroMagnetic waves in vacuum the (real) angular frequency ω and the real wave-number, $\mathbf{k}$, are connected, as a consequence of the wave equation, by the **Dispersion Relation** :

$$\boxed{\mathbf{k}\cdot\mathbf{k}=\frac{\omega^2}{c^2}\Longrightarrow\omega=|\mathbf{k}|c} \quad .$$

The ElectroMagnetic fields (46.01.03), (46.01.04) are solutions to the wave equation if and only if:

$$\omega_E=|\mathbf{k}_E|c \qquad \omega_B=|\mathbf{k}_B|c \quad .$$

However, a priori, the wave-vector, $\mathbf{k}$, frequency, ω, and phase, ϕ, might be different for the $\mathbf{E}$ and $\mathbf{B}$ fields. Nevertheless the ElectroMagnetic fields must also satisfy **Maxwell Equations** , from which the wave equation was derived, which might impose additional constraints. First of all note the following identities, which can be used to deduce the required results.

$$\mathbf{V}\text{ independent of }\mathbf{x}\Longrightarrow\mathbf{rot}(f\mathbf{V})=(\mathbf{grad}\,f)\wedge\mathbf{V} \quad ,$$
$$\mathbf{V}\text{ independent of }\mathbf{x}\Longrightarrow\mathbf{div}(f\mathbf{V})=(\mathbf{grad}\,f)\cdot\mathbf{V} \quad .$$

By replacing the ElectroMagnetic fields (46.01.03), (46.01.04) into **Maxwell Equations** one finds the conditions to be satisfied by the fields in order to satisfy not only the wave equation but also **Maxwell Equations** . The two rotor **Maxwell Equations** , those which mix Electric|Magnetic fields, imply, first of all:

$$\boxed{\omega\equiv\omega_E=\omega_B \qquad \mathbf{k}\equiv\mathbf{k}_E=\mathbf{k}_B \qquad \phi\equiv\phi_E=\phi_B} \quad . \tag{46.01.05}$$

→ 2063

In fact the identity between the two members of the **Maxwell Equations** with rotor, at any point and any time, can only be satisfied if frequency, wave-vector and phase are identical for the two fields, because Sinusoid|Cosinusoid functions with different coefficients are linearly independent.

Equation (46.01.05) implies, among other things, that **the Electric|Magnetic fields are in phase, have the same angular frequency and wave-vector**.

The two divergence equations do not mix the Electric|Magnetic fields and they provide the **transversality conditions** of the fields with respect to the wave-vector:

$$\boxed{\mathbf{k}\cdot\mathbf{E}_0=0} \quad , \tag{46.01.06}$$

→ 2064

$$\boxed{\mathbf{k}\cdot\mathbf{B}_0=0} \quad .$$

→ 2064

That is, the Electric|Magnetic field in a MPW are transverse, that is orthogonal to the wave-vector.

On the contrary, the two rotor equations do mix the Electric|Magnetic fields, and give in terms of

$$\boxed{\mathbf{k}\equiv|\mathbf{k}|\widehat{\boldsymbol{\kappa}}\equiv k\widehat{\boldsymbol{\kappa}} \qquad k=\omega/c \qquad \text{where }\mathbf{k}\text{ is a real vector here}} \quad ,$$

the following set of equivalent equations:

$$\boxed{\mathbf{B}_0=+\frac{1}{\omega}(\mathbf{k}\wedge\mathbf{E}_0)=+\frac{1}{c}(\widehat{\boldsymbol{\kappa}}\wedge\mathbf{E}_0)\Longrightarrow E_0=cB_0} \quad , \tag{46.01.07}$$

→ 2064

$$\boxed{\mathbf{E}_0=-\frac{c^2}{\omega}(\mathbf{k}\wedge\mathbf{B}_0)=-c\,(\widehat{\boldsymbol{\kappa}}\wedge\mathbf{B}_0)\Longrightarrow E_0=cB_0} \quad . \tag{46.01.08}$$

→ 2064 2240

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Note that the two equations (46.01.07), (46.01.08) are actually equivalent, as it can be shown by vector multiplying any one of them by $\mathbf{k}$ and using the transversality conditions. On the other hand the two equations (46.01.07), (46.01.08) imply the transversality conditions (46.01.06), (46.01.02). That is, for MPW the two divergence equations do not give any additional information with respect to the two rotor equations.

The two equations (46.01.07), (46.01.08) show that the three vectors $\mathbf{E}_0$, $\mathbf{B}_0$ and $\mathbf{k}$ form a **orthogonal set of real vectors**. In fact, combining the equations, one can obtain:

$$\mathbf{E}_0 \wedge \mathbf{B}_0 = \frac{\widehat{\boldsymbol{\kappa}}}{c}|\mathbf{E}_0|^2 = c\widehat{\boldsymbol{\kappa}}|\mathbf{B}_0|^2 \quad . \tag{46.01.09}$$

→ 2080

Equations (46.01.17), (46.01.16) show that any of the two ElectroMagnetic fields is known when the other is given.

It is customary to treat EM MPW by describing the electric field only, while the magnetic field can be calculated from (46.01.16) once the electric field is known.

46.01.03
Longitudinal Components of the ElectroMagnetic Fields for a MPW in Vacuum

©|J.E.Vitela|Am.J.Phys, , ..., ..., ...Ed., ... 72, 393 (2004) - WEB - URL|*Excellent and complete.*|

While for MPW only transverse ElectroMagnetic fields are allowed, thanks to the divergence equations, **Maxwell Equations** do not forbid a constant and uniform component of the ElectroMagnetic fields along the wave-vector, in general. However, this uniform and constant field is normally not interesting when dealing with ElectroMagnetic waves. It is therefore normally ignored.

Consider any plane ElectroMagnetic wave, not necessarily a monochromatic one, propagating in vacuum along the $\widehat{\mathbf{e}}_3$ direction. The constant phase planes are normal to the z axis so that the fields only depend on z and time:

$$\mathbf{E}[\mathbf{r},t] = E_x[z,t]\widehat{\mathbf{e}}_1 + E_y[z,t]\widehat{\mathbf{e}}_2 + E_z[z,t]\widehat{\mathbf{e}}_3 \quad ,$$
$$\mathbf{B}[\mathbf{r},t] = B_x[z,t]\widehat{\mathbf{e}}_1 + B_y[z,t]\widehat{\mathbf{e}}_2 + B_z[z,t]\widehat{\mathbf{e}}_3 \quad .$$

Actually, one knows that the dependence is not on z and t independently, but on the single combination $z \pm vt$

$$[z,t] \implies [z \pm vt] \quad ,$$

but this is not relevant here, as we deal with **Maxwell Equations** only, not the wave equation.

Moreover, we also know (read § 46.02.03) that:

$$E_x[z \pm vt] \propto B_y[z \pm vt] \qquad E_y[z \pm vt] \propto B_x[z \pm vt] \quad ,$$

but this is not relevant here as well.

Then one has, for every component of the fields:

$$\frac{\partial(\cdots)}{\partial x} = \frac{\partial(\cdots)}{\partial y} = 0 \quad .$$

Therefore:

$$\mathbf{div}\,\mathbf{E} = 0 \qquad \implies \qquad \frac{\partial E_z}{\partial z} = 0 \quad ,$$
$$\mathbf{div}\,\mathbf{B} = 0 \qquad \implies \qquad \frac{\partial B_z}{\partial z} = 0 \quad .$$

Moreover:

$$\frac{1}{c^2}\partial_t \mathbf{E} = \mathbf{rot}\,\mathbf{B} \qquad \implies \qquad \partial_t E_z = 0 \quad ,$$
$$-\partial_t \mathbf{B} = \mathbf{rot}\,\mathbf{E} \qquad \implies \qquad \partial_t B_z = 0 \quad .$$

Therefore if any component of the Electric|Magnetic field exist along the propagation direction it is both time and position independent.

In other words one can add to the MPW solutions (equation (46.01.03) or (46.01.04)) any constant and uniform term along the z axis, thus obtaining a valid solution to the **Maxwell Equations** .

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46.01.04
Longitudinal Components for General ElectroMagnetic Waves in Vacuum

Note, however, that if one does not consider plane ElectroMagnetic waves, but different waves, a longitudinal component of the ElectroMagnetic fields may be present. This is the case of so-called non TEM waves, that is non transverse-ElectroMagnetic waves.

For instance, in wave-guides, where ElectroMagnetic waves propagate in a bounded region, the longitudinal component of the MPW can be different from zero. Read § 47.04.01.

46.01.05
Complex Electric|Magnetic Field Vector

This § is referenced at pages:
[Never referenced.]

In order to describe a generic state of **wave-Polarization** it is possible to use a complex field vector as well, which is capable to describe any **wave-Polarization** state, as described in section § 46.03.03.

46.01.06
EM MPW in Vacuum via the Complex Formalism

This § is referenced at pages:
[2070]

In this section the same calculations done in section § 46.01.02 are done via the complex formalism.

The wave equation admits MPW solutions, which can be written as:

$$\boxed{\mathbf{U}_0 \cos\left[\mathbf{k}\cdot\mathbf{x} - \omega t - \phi\right]} \quad , \tag{46.01.10}$$

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which can be written, in complex form, as:

$$\boxed{\mathbf{U}_0 \cos\left[\mathbf{k}\cdot\mathbf{x} - \omega t - \phi\right] \Longrightarrow \overline{\mathbf{U}}[\mathbf{x},t] = \mathbf{U}_0 \exp\left[\mathtt{i}\left(\overline{\mathbf{k}}\cdot\mathbf{x} - \omega t - \phi\right)\right] \equiv \overline{\mathbf{U}}_0 \exp\left[\mathtt{i}\left(\overline{\mathbf{k}}\cdot\mathbf{x} - \omega t\right)\right]} \quad . \tag{46.01.11}$$

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As long as one considers ElectroMagnetic waves in vacuum the wave-vector $\mathbf{k}$ is a real vector. However, the wave-vector $\mathbf{k}$ can be generalized to a complex wave-vector $\overline{\mathbf{k}}$ (read § 46.06.02).

In this section we still assume $\mathbf{k}$ to be a real vector, as it is appropriate for vacuum and **perfect dielectric media**, read § 46.02.04. A truly real complex vector will be introduced in section § 46.06.

The constant $\overline{\mathbf{U}}_0$ is a complex constant vector carrying the information on both the amplitude and phase of the wave.

The effect of space and time derivation on a complex MPW is the following:

$$\begin{aligned}
\partial_t \left(\overline{\mathbf{U}}_0 \exp\left[\mathtt{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]\right) &= -\mathtt{i}\omega \overline{\mathbf{U}}_0 \exp\left[\mathtt{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right] \quad , \\
\frac{\partial}{\partial x^i}\left(\overline{\mathbf{U}}_0 \exp\left[\mathtt{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]\right) &= +\mathtt{i}k_i \overline{\mathbf{U}}_0 \exp\left[\mathtt{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right] \quad , \\
\Longrightarrow \mathbf{div}\left(\overline{\mathbf{U}}_0 \exp\left[\mathtt{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]\right) &= +\mathtt{i}\left(\mathbf{k}\cdot\overline{\mathbf{U}}_0\right) \exp\left[\mathtt{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right] \quad , \\
\Longrightarrow \mathbf{rot}\left(\overline{\mathbf{U}}_0 \exp\left[\mathtt{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]\right) &= +\mathtt{i}\left(\mathbf{k}\wedge\overline{\mathbf{U}}_0\right) \exp\left[\mathtt{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right] \quad .
\end{aligned}$$

Shortly:

$$\begin{aligned}
\mathbf{grad} \exp\left[\mathtt{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t - \phi\right)\right] &\longrightarrow \mathtt{i}\mathbf{k} \quad , \\
\partial_t \exp\left[\mathtt{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t - \phi\right)\right] &\longrightarrow -\mathtt{i}\omega \quad .
\end{aligned}$$

Introduce a complex MPW solution in equations (46.01.01), (46.01.02), with the usual convention that the physical quantity is given by the real part of the complex number:

$$\boxed{\mathbf{U}[\mathbf{x},t] \equiv \Re\mathfrak{e}\left[\overline{\mathbf{U}}_0 \exp\left[\mathtt{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]\right]} \quad .$$

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A priori the wave-vector, $\mathbf{k}$, frequency, ω, and phase, ϕ, might be different for the $\mathbf{E}$ and $\mathbf{B}$ fields, except that for both fields the **Dispersion Relation** must be obeyed:

$$\omega_E = |\mathbf{k}_E|c \qquad \omega_B = |\mathbf{k}_B|c \quad .$$

The expressions for the fields read:

$$\overline{\mathbf{E}}[\mathbf{x},t] = \overline{\mathbf{E}}_0 \exp\left[\mathtt{i}\left(\mathbf{k}_{\mathtt{E}}{\cdot}\mathbf{x} - \omega_E t\right)\right] = \mathbf{E}_0 \exp\left[\mathtt{i}\left(\mathbf{k}_{\mathtt{E}}{\cdot}\mathbf{x} - \omega_E t - \phi_E\right)\right] \quad , \tag{46.01.12}$$

$$\overline{\mathbf{B}}[\mathbf{x},t] = \overline{\mathbf{B}}_0 \exp\left[\mathtt{i}\left(\mathbf{k}_{\mathtt{B}}{\cdot}\mathbf{x} - \omega_B t\right)\right] = \mathbf{B}_0 \exp\left[\mathtt{i}\left(\mathbf{k}_{\mathtt{B}}{\cdot}\mathbf{x} - \omega_B t - \phi_B\right)\right] \quad , \tag{46.01.13}$$

where $\overline{\mathbf{E}}_0$ and $\overline{\mathbf{B}}_0$ are complex constant vectors carrying the information on the amplitude, direction and phase of the fields.

The four **Maxwell Equations** in vacuum, equations (36.08.06), (36.08.07), (36.08.08), (36.08.02), as described in section § 36.08.02, give two transversality conditions, from the divergence equations, and the relations between the fields and the wave-vector.

The two divergence equations do not mix the Electric|Magnetic fields and they provide the transversality conditions of the fields with respect to the wave-vector:

$$\mathbf{k}{\cdot}\overline{\mathbf{E}}_0 = 0 \tag{46.01.14}$$

$$\mathbf{k}{\cdot}\overline{\mathbf{B}}_0 = 0 \quad . \tag{46.01.15}$$

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The Electric|Magnetic field in a MPW are transversal that is orthogonal to the wave-vector. Note that, as long as the wave-vector $\mathbf{k}$ is real, the two complex equations (46.01.14), (46.01.15) are equivalent to the two real equations:

$$\mathbf{k}{\cdot}\mathbf{E}_0 = 0 \quad ,$$

$$\mathbf{k}{\cdot}\mathbf{B}_0 = 0 \quad .$$

On the contrary, the two rotor equations do mix the Electric|Magnetic fields, and give the following set of equivalent equations:

$$\mathbf{rot}\overline{\mathbf{E}} = +\mathtt{i}\left(\mathbf{k}_{\mathtt{E}}{\wedge}\overline{\mathbf{E}}_0\right)\exp\left[\mathtt{i}\left(\mathbf{k}_{\mathtt{E}}{\cdot}\mathbf{x} - \omega_E t\right)\right] = -\partial_t\overline{\mathbf{B}} = \mathtt{i}\omega_B\overline{\mathbf{B}}_0\exp\left[\mathtt{i}\left(\mathbf{k}_{\mathtt{B}}{\cdot}\mathbf{x} - \omega_B t\right)\right] \quad ,$$

$$\mathbf{rot}\overline{\mathbf{B}} = +\mathtt{i}\left(\mathbf{k}_{\mathtt{B}}{\wedge}\overline{\mathbf{B}}_0\right)\exp\left[\mathtt{i}\left(\mathbf{k}_{\mathtt{B}}{\cdot}\mathbf{x} - \omega_B t\right)\right] = +\varepsilon_0\mu_0\partial_t\overline{\mathbf{E}} = -\mathtt{i}\varepsilon_0\mu_0\omega_E\overline{\mathbf{E}}_0\exp\left[\mathtt{i}\left(\mathbf{k}_{\mathtt{E}}{\cdot}\mathbf{x} - \omega_E t\right)\right] \quad .$$

It is obvious from the two previous equations that a necessary condition to satisfy Maxwell equations is that the wave-vectors and frequencies of the Electric|Magnetic fields are equal, because the equations must be satisfied at any point in space at any time, and this cannot happen unless the two exponential expressions are identical.

Therefore **Maxwell Equations** imply, first of all:

$$\boxed{\omega \equiv \omega_E = \omega_B \qquad \mathbf{k} \equiv \mathbf{k}_E = \mathbf{k}_B} \quad .$$

Secondly, as long as $\mathbf{k}$ is a real number, it follows that the two phases must be equal as well:

$$\boxed{\phi \equiv \phi_E = \phi_B} \quad .$$

In fact the identity between the two members of the **Maxwell Equations** with rotor can only be satisfied if frequency, wave-vector and phase are identical for the two fields. The latter equations shows that the Electric|Magnetic field must have the same frequency and wave-vector and, as long as $\mathbf{k} \equiv k\widehat{\boldsymbol{\kappa}}$ is a real number, they must be in phase. Therefore the **Dispersion Relation** is:

$$\boxed{\omega = c|\mathbf{k}|} \quad .$$

Finally **Maxwell Equations** imply:

$$\overline{\mathbf{B}}_0 = +\frac{1}{\omega}\left(\mathbf{k}{\wedge}\overline{\mathbf{E}}_0\right) = +\frac{1}{c}\left(\boldsymbol{\kappa}{\wedge}\overline{\mathbf{E}}_0\right) \tag{46.01.16}$$

$$\overline{\mathbf{E}}_0 = -\frac{c^2}{\omega}\left(\mathbf{k}{\wedge}\overline{\mathbf{B}}_0\right) = -c\left(\boldsymbol{\kappa}{\wedge}\overline{\mathbf{B}}_0\right) \quad . \tag{46.01.17}$$

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Note that the two equations (46.01.16), (46.01.17) are actually equivalent, as it can be shown by vector multiplying any one of them by $\mathbf{k}$ and using the transversality conditions. Note that, as long as the wave-

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vector $\mathbf{k}$ is real, the two complex equations (46.01.16), (46.01.17) are equivalent to the two real equations:

$$\mathbf{B}_0 = +\frac{1}{\omega}(\mathbf{k}\wedge\mathbf{E}_0) = +\frac{1}{c}(\boldsymbol{\kappa}\wedge\mathbf{E}_0) \quad ,$$
$$\mathbf{E}_0 = -\frac{c^2}{\omega}(\mathbf{k}\wedge\mathbf{B}_0) = -c\,(\boldsymbol{\kappa}\wedge\mathbf{B}_0) \quad .$$

The two equations (46.01.16), (46.01.17) show that the three vectors $\overline{\mathbf{E}}_0$, $\overline{\mathbf{E}}_0$ and $\mathbf{k}$ form a orthogonal set of complex vectors.

Equations (46.01.16), (46.01.17), imply, for real values of $\mathbf{k}$:

$$\mathbf{B}_0 = +\frac{1}{\omega}(\mathbf{k}\wedge\mathbf{E}_0) \qquad \Longrightarrow \qquad B_0 = \frac{kE_0}{\omega} = \frac{E_0}{c} \tag{46.01.18}$$
$$\mathbf{E}_0 = -\frac{c^2}{\omega}(\mathbf{k}\wedge\mathbf{B}_0) \qquad \Longrightarrow \qquad E_0 = \frac{c^2 kB_0}{\omega} = cB_0 \quad . \tag{46.01.19}$$

Equations (46.01.16), (46.01.17), together with the transversality conditions (46.01.14), (46.01.15), imply, for real values of $\mathbf{k}$:

$$\mathbf{k}\frac{cE_0B_0}{\omega} = \widehat{\boldsymbol{\kappa}}E_0B_0 = \mathbf{E}_0\wedge\mathbf{B}_0 \quad ,$$

as one can show by vector multiplication of either equation (46.01.16) for $\overline{\mathbf{E}}_0$ or equation (46.01.17) for $\overline{\mathbf{B}}_0$.

Note that with complex functions products/divisions are not allowed trivially as they mix real ad imaginary parts However, relations such as $\mathbf{k}\wedge\mathbf{E}$ and $\mathbf{k}\wedge\mathbf{B}$ from Maxwell equations are a formal, verbatim, consequence of solutions with complex exponential: so they are correct as they are, that is the scalar product is not the hermitian scalar product of complex vector fields. Read also sections § 19.03, § 19.04, § 25.06.

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46.02 EM Waves in perfect dielectric media Without Boundaries

This § is referenced at pages:
[1991]

The description of ElectroMagnetic fields inside matter in terms of **E** and **B** plus particles is always good under any circumstance: it is the **microscopic description**. However, sometimes it is not necessary to go into the level of detail of the microscopic description. Then one may introduce the **macroscopic description** in terms of **P** and **M** which is always good, even in the realm of modern physics. In this case, however, it is sometimes necessary to make a model of matter.

The detailed treatment of the behavior of ElectroMagnetic waves in matter will be discussed in section § 54.

For the present purposes it is enough to introduce the concept of **perfect dielectric medium** , that is an LHI medium such that the **Electric-Permittivity** | **Magnetic-Permeability** are just fixed (real) numbers independent of anything else than the nature of the material medium with $\varepsilon_{\text{R}} \geq 1$ and $\mu_{\text{R}} > 0$, the **conductivity** is zero, $\sigma = 0$, so the free current density is zero and the free charge density is assumed to be zero, that is electrically neutral at any point:

$$\begin{aligned} &\text{LHI medium} \quad , \\ \varepsilon_{\text{R}} \geq 1 \qquad &\text{fixed, real} \quad , \\ \mu_{\text{R}} > 0 \qquad &\text{fixed, real} \quad , \\ \sigma = 0 \implies \mathbf{j}^{\text{F}} = 0 \quad &, \\ \rho^{\text{F}} = 0 \qquad &\text{electrically neutral at any point} \quad . \end{aligned}$$

Note that if any one of the latter hypotheses is not valid some of the basic relations deduced in this section may be violated.

A **perfect dielectric medium does not absorb ElectroMagnetic energy** from the ElectroMagnetic waves. The meaning of real (as opposed to complex) **Electric-Permittivity** | **Magnetic-Permeability** means that, for harmonic fields, there is no phase-shift between **E**/**P** and **B**/**M**, as discussed in section § 54.

In fact it will become clear that there is **little difference between the behavior of ElectroMagnetic waves in vacuum with respect to a perfect dielectric medium** .

A **perfect dielectric medium** is often **a good approximation for describing ElectroMagnetic waves in matter, as long as the ElectroMagnetic wave is almost monochromatic and absorption is negligible**.

46.02.01 EM MPW in perfect dielectric media

The reasoning in section § 46.01 can be carried on in a similar way for a perfect LHI medium to find (Read also section § 54) the following relations:

$$\boxed{\boldsymbol{\nabla}^2\mathbf{E} = \frac{1}{v^2}\frac{\partial^2\mathbf{E}}{\partial t^2}} \quad ,$$

$$\boxed{\boldsymbol{\nabla}^2\mathbf{B} = \frac{1}{v^2}\frac{\partial^2\mathbf{B}}{\partial t^2}} \quad ,$$

$$\boxed{\omega \equiv kv} \qquad \boxed{c^2k^2 = \varepsilon_{\text{R}}\mu_{\text{R}}\omega^2} \qquad \boxed{k = \frac{\omega}{c}\sqrt{\varepsilon_{\text{R}}\mu_{\text{R}}}} \quad ,$$

$$\boxed{v \equiv \frac{1}{\sqrt{\mu_{\text{R}}\varepsilon_{\text{R}}}}\frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{c}{\sqrt{\mu_{\text{R}}\varepsilon_{\text{R}}}}} \qquad \text{with} \qquad c = 2.998{\cdot}10^8 \text{ m/s} \quad .$$

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Actually, the two equations are decoupled, but the Electric|Magnetic fields are only apparently decoupled. In fact, the equations have been obtained by derivation from Maxwell equations, which still rule, coupling the Electric|Magnetic fields. Therefore, additional constraints must be included on the solutions of d'Alembert equations. Read § 46.01.01. The **index of refraction** of the material is defined as:

$$n \equiv \frac{c}{v} = \frac{ck}{\omega} = \sqrt{\varepsilon_R \mu_R} \quad .$$

The same definition of the **index of refraction** is extended to the more general case when the wave-vector is a complex vector:

$$\overline{n} \equiv \frac{c}{\omega}\overline{\mathbf{k}} \quad . \tag{46.02.01}$$

Maxwell Equations for MPW lead to the following relations:

$$\mathbf{k}\cdot\mathbf{D}_0 = 0 \implies \mathbf{k}\cdot\mathbf{E}_0 = 0 \quad ,$$

$$\mathbf{k}\cdot\mathbf{B}_0 = 0 \implies \mathbf{k}\cdot\mathbf{H}_0 = 0 \quad ,$$

$$\mathbf{B}_0 = +\frac{1}{\omega}(\mathbf{k}\wedge\mathbf{E}_0) \implies \mathbf{H}_0 = +\frac{v^2}{\omega}(\mathbf{k}\wedge\mathbf{D}_0) \implies B_0 = \frac{kE_0}{\omega} = \frac{E_0}{v} \quad ,$$

$$\mathbf{D}_0 = -\frac{1}{\omega}(\mathbf{k}\wedge\mathbf{H}_0) \implies \mathbf{E}_0 = -\frac{v^2}{\omega}(\mathbf{k}\wedge\mathbf{B}_0) \implies D_0 = \frac{kH_0}{\omega} = \frac{H_0}{v} \quad .$$

Note, in the above relations for the amplitudes, the inverted role of the Electric|Magnetic fields $\mathbf{D}$ and $\mathbf{H}$ with respect to $\mathbf{E}$ and $\mathbf{B}$.

$$B_0 = \frac{E_0}{v} \qquad D_0 = \frac{H_0}{v} \quad .$$

Note that, on dimensional grounds, the role of electric and magnetic fields in the above equations are interchanged.

46.02.02
The index of refraction

This § is referenced at pages:
[Never referenced.]

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©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|§ 31||

We have said before that light goes slower in water than in air, and slower, slightly, in air than in vacuum. This effect is described by the **index of refraction** . Now we would like to understand how such a slower velocity could come about. In particular, we should try to see what the relation is to some physical assumptions, or statements, we made earlier, which were the following:

- That the total electric field in any physical circumstance can always be represented by the sum of the fields from all the charges in the universe.
- That the field from a single charge is given by its acceleration evaluated with a retardation at the speed c, always (for the radiation field).

But, for a piece of glass, you might think: 'Oh, no, you should modify all this. You should say it is retarded at the speed c/n.' That, however, is not right, and we have to understand why it is not. It is approximately true that light or any electric wave does appear to travel at the speed c/n through a material whose **index of refraction** is n, but the fields are still produced by the motions

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of all the charges'including the charges moving in the material - and with these basic contributions of the field travelling at the ultimate velocity c. Our problem is to understand how the apparently slower velocity comes about.

46.02.03
General ElectroMagnetic Non-Monochromatic PW in perfect dielectric media

This § is referenced at pages:
[2064]

©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|1.4||

EM PW (non-monochromatic, in general) are described, in terms of the phase function, $\Phi[\mathbf{r},t]$, by the fields:

$$\boxed{\Phi[\mathbf{r},t] \equiv \mathbf{k}\cdot\mathbf{r} - \omega t - \phi} \quad ,$$

$$\boxed{\mathbf{E}[\mathbf{r},t] = \mathbf{E}[\,\mathbf{k}\cdot\mathbf{r} - \omega t - \phi] \equiv \mathbf{E}[\Phi]} \quad ,$$

$$\boxed{\mathbf{B}[\mathbf{r},t] = \mathbf{B}[\,\mathbf{k}\cdot\mathbf{r} - \omega t - \phi] \equiv \mathbf{B}[\Phi]} \quad .$$

Let us introduce some short-hand notations:

$$\mathbf{V}[\mathbf{r},t] \equiv \mathbf{V}\,[\Phi] \equiv \mathbf{V}\,[\,\mathbf{k}\cdot\mathbf{r} - \omega t - \phi] \quad .$$

It can be easily demonstrated that:

$$\partial_t \mathbf{V} = -\omega \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}\Phi} \qquad \frac{\partial \mathbf{V}}{\partial x^s} = k_s \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}\Phi} = -\frac{k_s}{\omega}\partial_t \mathbf{V} \quad .$$

One then finds from **Maxwell Equations** :

$$\mathbf{k}\wedge\frac{\mathrm{d}\mathbf{E}}{\mathrm{d}\Phi} = +\omega\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\Phi} \qquad \mathbf{k}\wedge\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}\Phi} = -\frac{\omega}{c^2}\frac{\mathrm{d}\mathbf{D}}{\mathrm{d}\Phi} \quad .$$

By integrating the previous relations and neglecting possible integration constants, that is constant fields, one then finds:

$$\boxed{\mathbf{B} = +\frac{\mathbf{k}\wedge\mathbf{E}}{\omega} \qquad \mathbf{D} = -c^2\frac{\mathbf{k}\wedge\mathbf{H}}{\omega}} \quad .$$

46.02.04
EM MPW in perfect dielectric media via the Complex Formalism

This § is referenced at pages:
[2065]

It is almost the same as in section § 46.01.06, with the minor changes imposed by the presence of the fixed $\varepsilon_{\mathrm{R}} \geq 1$ and $\mu_{\mathrm{R}} > 0$.

PRELIMINARY PROOFS

46.03 wave-Polarization of ElectroMagnetic Waves

This § is referenced at pages:
[2008, 2041, 2265]

To complete the properties of ElectroMagnetic Waves it is necessary to describe the configuration of ElectroMagnetic fields in the plane perpendicular to the direction of the wave-vector.

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.33||

EM waves are **transverse** waves. In the case of transverse waves different possible configurations are possible for the oscillating physical quantity in the plane perpendicular to the direction of propagation.

In this section the **wave-Polarization of ElectroMagnetic waves** will be described, but it should be kept in mind that a parallel discussion might be carried on for any other transverse wave (for instance transverse elastic waves on a spring or in a solid medium).

It is interesting to note that **the wave-Polarization of ElectroMagnetic waves and its description is perhaps the simplest example of the description of states of a physical system in quantum physics**.

46.03.01
Description of the wave-Polarization of ElectroMagnetic Waves

Maxwell Equations give a well-defined relation between the electric and magnetic field of an Electro-Magnetic MPW. However, they impose no relation at all on the components of the Electric|Magnetic fields in the plane perpendicular to the wave-vector, **k**, that is on two components of the same field. The concept of **wave-Polarization** deals with the different configurations of the Electric|Magnetic fields in the plane perpendicular to the wave-vector, **k**, for a MPW.

When dealing with **wave-Polarization** of ElectroMagnetic waves one normally refers, just by convention, to the electric field vector, because it is the electric field vector which gives, normally, the most significant effects on matter. One knows that the general relations presented in section § 46.01 allow to calculate the magnetic field of a MPW when the electric field is known. Therefore the ElectroMagnetic fields of ElectroMagnetic waves can be fully described from the knowledge of the electric field only.

Let the z axis be the propagation axis. **Maxwell Equations** do not imply any relation between the transverse components of the same field. We then choose two ElectroMagnetic MPW, one with electric fields along x and the other along y, having the same frequency and wave-vector but arbitrary amplitudes and **phase constants** . The **phase-velocity** is assumed here to be fixed.

As **Maxwell Equations** are linear the resultant field is an admissible ElectroMagnetic field.

Let us consider the electric field at some fixed point, $z = \overline{z}$, as a function of time:

$$E_x[t] = E_{x0}\cos[k\overline{z} - \omega t - \phi_x] \qquad \text{with } E_{x0} \geq 0 \text{ and } 0 \leq \phi_x < 2\pi \quad ,$$

$$E_y[t] = E_{y0}\cos[k\overline{z} - \omega t - \phi_y] \qquad \text{with } E_{y0} \geq 0 \text{ and } 0 \leq \phi_y < 2\pi \quad .$$

Note that, provided one chooses the correct **phase constant** $0 \leq \phi < 2\pi$, the amplitudes E_{x0} and E_{y0} can be taken as positive. This convention will be used here.

Different cases may arise, as discussed below, depending of the value of the difference of the phases of the two components. By convention it is defined as the **phase constant** of the y component minus the **phase constant** of the x component:

$$\Delta\phi \equiv (\phi_y - \phi_x) \quad .$$

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We want to describe the behavior of the electric field vector in the xy plane as a function of time. From the point of view of geometry this means to pass from the parametric representation of $E_x[t]$ and $E_y[t]$ as a function of time (a parametric curve in the xy plane) to the Cartesian representation $\mathcal{F}[E_x[t], E_y[t]] = 0$ where time has been eliminated (a Cartesian curve in the xy plane).

46.03.01.01 Linear wave-Polarization

If $\Delta\phi \equiv (\phi_y - \phi_x) = q\pi$, with q any integer number, the ElectroMagnetic wave is said to be **linearly wave-Polarized** and the two components of the field are proportional and either in phase or out-of-phase by π:

$$|E_x|E_{y0} = |E_y|E_{x0} \quad .$$

At any fixed z position the electric field vector runs on a straight line segment, whose orientation depends on the values of E_{y0} and E_{x0}.

A relation carrying more information, namely the relative sign between the two components, is easily deduced:

$$\Delta\phi \equiv (\phi_y - \phi_x) = q\pi \implies E_x E_{y0} = (-1)^q E_y E_{x0} \quad .$$

The electric field of a linearly wave-Polarized ElectroMagnetic wave has a fixed direction but its amplitude and orientation change with time.

46.03.01.02 Components in Quadrature and Circular wave-Polarization

If $\Delta\phi \equiv (\phi_y - \phi_x) = \pi\,(q + 1/2)$, with q any integer number, the ElectroMagnetic wave is said to be **elliptically wave-Polarized with axes parallel to** x **and** y and the two components of the field are out-of-phase by plus/minus one-quarter cycle with the electric field going across an ellipse whose axes are the x and y axes:

$$\left(\frac{E_x}{E_{x0}}\right)^2 + \left(\frac{E_y}{E_{y0}}\right)^2 = 1 \quad .$$

In the special and most important case $E_{x0} = E_{y0}$ the **wave-Polarization is circular** and the electric field goes across a circumference.

The electric field of a circularly wave-Polarized ElectroMagnetic wave has a fixed amplitude but its oriented direction changes with time.

The handedness of rotation can be calculated from the sign of the projection on the z axis of the vector $\mathbf{E}\wedge\dot{\mathbf{E}}$, in a similar way to angular momentum, see section § 46-016.

46.03.01.03 General Elliptic wave-Polarization

This § is referenced at pages:
[1071, 1239]

In the general case one obtains, with a generic $\Delta\phi \equiv (\phi_y - \phi_x)$, a generic ellipse:

$$\left(\frac{E_x}{E_{x0}}\right)^2 + \left(\frac{E_y}{E_{y0}}\right)^2 - 2\left(\frac{E_x}{E_{x0}}\right)\left(\frac{E_y}{E_{y0}}\right)\cos[\Delta\phi] = \sin^2\Delta\phi \quad , \tag{46.03.01}$$

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giving the so-called Lissajous figures for **general elliptical wave-Polarization** .

Equation (46.03.01) actually includes all the possible cases of **wave-Polarization** .

In fact, from the geometrical properties of the equation (46.03.01), one has that the ellipse has its axes parallel to the x and y axes if and only if $\cos[\Delta\phi] = 0$ and it degenerates into a straight line segment if and only if $\sin[\Delta\phi] = 0$.

Also in this general case, the handedness of rotation can be calculated from the sign of the projection on the z axis of the vector

$$\mathbf{E}\wedge\dot{\mathbf{E}} \implies \hat{\mathbf{e}}_3\cdot\left(\mathbf{E}\wedge\dot{\mathbf{E}}\right) \quad ,$$

in a similar way to angular momentum, see section § 46-016.

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46.03.01.04 Handedness and Helicity

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In general one might be interested to know the sense of rotation of the electric field. This is either described in terms of **handedness** or **helicity**. By convention one describes the sense of rotation of the electric field as it is seen by an observer looking at the incoming MPW, that is the MPW is going towards the observer.

The conventional nomenclature is the following one:

as seen by the observer	traditional (optics) name **handedness**	particle physics name **helicity**	$\Delta\phi$
clockwise	right-handed (R)	negative helicity	$\sin[\Delta\phi] > 0$
counter-clockwise	left-handed (L)	positive helicity	$\sin[\Delta\phi] < 0$

.

Note that the traditional (optics) name is just opposite to what one would use by applying the right-hand/left-hand rule along the direction of propagation, as if the observer would look at the back of the MPW.

46.03.02 Combinations of Different wave-Polarization States

Consider ElectroMagnetic MPW traveling along the positive z direction. **wave-Polarization** is described in the xy plane.

It is particularly simple to describe **wave-Polarization** states via the complex formalism.

Note: be careful to get correct the amplitude of the resulting ElectroMagnetic wave as a function of the amplitudes of the constituent waves.

Read § 46.15.01 before for the operative definition of **wave-Polarization** states. In fact, the Quantum nature of the **wave-Polarization** of ElectroMagnetic waves requires careful operative definitions, to match with formalism.

46.03.02.01 Circular wave-Polarization From Linearly wave-Polarized Waves

Circular **wave-Polarization** can be obtained by superposition of two waves linearly wave-Polarized along two perpendicular axes and out-of-phase by $\pm\pi/2$ (plus/minus multiples of π), with totally arbitrary phases ϕ_{R} and ϕ_{L}, as follows:

$$\mathbf{E}_{\mathrm{R}}[z,t] = E_0\left(\cos\left[kz-\omega t-\phi_{\mathrm{R}}\right]\widehat{\mathbf{e}}_1 + \cos\left[kz-\omega t-(\phi_{\mathrm{R}}+\pi/2)\right]\widehat{\mathbf{e}}_2\right) \qquad \Delta\phi = +\pi/2 \quad ,$$
$$\mathbf{E}_{\mathrm{L}}[z,t] = E_0\left(\cos\left[kz-\omega t-\phi_{\mathrm{L}}\right]\widehat{\mathbf{e}}_1 + \cos\left[kz-\omega t-(\phi_{\mathrm{L}}-\pi/2)\right]\widehat{\mathbf{e}}_2\right) \qquad \Delta\phi = -\pi/2 \quad .$$

The two above circularly wave-Polarized ElectroMagnetic MPW traveling along the z direction can also described by the complex relations:

$$\mathbf{E}_{\mathrm{R}} = \mathfrak{Re}\left[\sqrt{2}E_{00}\exp\left[\mathtt{i}\left(kz-\omega t\right)\right]\left(\frac{\widehat{\mathbf{e}}_1-\mathtt{i}\widehat{\mathbf{e}}_2}{\sqrt{2}}\right)\right] = \mathfrak{Re}\left[\sqrt{2}E_{00}\exp\left[\mathtt{i}\left(kz-\omega t\right)\right]\overline{\widehat{\mathbf{e}}}_{\mathrm{R}}\right] \quad ,$$
$$E_0 \equiv E_{00}\exp\left[-\mathtt{i}\phi_R\right] \quad ,$$
$$\mathbf{E}_{\mathrm{L}} = \mathfrak{Re}\left[\sqrt{2}E_{00}\exp\left[\mathtt{i}\left(kz-\omega t\right)\right]\left(\frac{\widehat{\mathbf{e}}_1+\mathtt{i}\widehat{\mathbf{e}}_2}{\sqrt{2}}\right)\right] = \mathfrak{Re}\left[\sqrt{2}E_{00}\exp\left[\mathtt{i}\left(kz-\omega t\right)\right]\overline{\widehat{\mathbf{e}}}_{\mathrm{L}}\right] \quad ,$$
$$E_0 \equiv E_{00}\exp\left[-\mathtt{i}\phi_L\right] \quad .$$

Superposition of two ElectroMagnetic MPW traveling along the z direction with orthogonal **wave-Polarization** and $\pm\pi/2$ phase difference leads to a circularly wave-Polarized wave with the same amplitude as the amplitude of the two original ElectroMagnetic waves.

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46.03.02.02 Linear wave-Polarization From Circularly wave-Polarized Waves

Linear **wave-Polarization** can be obtained by superposition of two waves with opposite circular **wave-Polarization** :

$$\mathbf{E}[z,t]=\mathbf{E}_{\mathrm{R}}[z,t]+\mathbf{E}_{\mathrm{L}}[z,t]=2E_0\cos\left[kz-\omega t-\frac{\phi_{\mathrm{R}}+\phi_{\mathrm{L}}}{2}\right]\left(\cos\left[\frac{\phi_{\mathrm{L}}-\phi_{\mathrm{R}}}{2}\right]\widehat{\mathbf{e}}_1+\sin\left[\frac{\phi_{\mathrm{L}}-\phi_{\mathrm{R}}}{2}\right]\widehat{\mathbf{e}}_2\right)\quad . \tag{46.03.02}$$

→ 2141

It is clear from the above relation that the resulting **wave-Polarization** ia a linear one, because both components have the same phase factor, with a common **phase constant** : $\frac{\phi_{\mathrm{R}}+\phi_{\mathrm{L}}}{2}$.

The characteristics of the resulting linearly wave-Polarized wave is fully determined by the two arbitrary phases ϕ_{R} and ϕ_{L}. Note that the factor:

$$\left(\cos\left[\frac{\phi_{\mathrm{L}}-\phi_{\mathrm{R}}}{2}\right]\widehat{\mathbf{e}}_1+\sin\left[\frac{\phi_{\mathrm{L}}-\phi_{\mathrm{R}}}{2}\right]\widehat{\mathbf{e}}_2\right)\quad ,$$

is in fact a unit vector, whose direction gives the direction of the resulting field.

It then follows that the resulting ElectroMagnetic wave is always linearly wave-Polarized in a direction which depends on the difference between the two phases ϕ_{R} and ϕ_{L}.

Note that the resulting amplitude is twice the amplitude of the two component EM waves.

The x and y components of the resulting field are:

$$E_x=2E_0\cos\left[\frac{\phi_{\mathrm{L}}-\phi_{\mathrm{R}}}{2}\right]\cos\left[kz-\omega t-\frac{\phi_{\mathrm{R}}+\phi_{\mathrm{L}}}{2}\right]\quad ,$$

$$E_y=2E_0\sin\left[\frac{\phi_{\mathrm{L}}-\phi_{\mathrm{R}}}{2}\right]\cos\left[kz-\omega t-\frac{\phi_{\mathrm{R}}+\phi_{\mathrm{L}}}{2}\right]\quad .$$

46.03.03 Generic wave-Polarization States

This § is referenced at pages:
[2065, 2311]

The **wave-Polarization** state of ElectroMagnetic MPW, as described by the complex amplitude in expressions like equation (46.01.11), form **a complex vector space with Hermitian scalar product**.

In fact, the addition of two MPW along the same direction provides a MPW along the same direction with the same frequency which is described by the algebraic sum of the two **wave-Polarization** vectors, as from § **??**.

Moreover, multiplication of MPW by a complex scalar provides a MPW along the same direction with the same frequency, and different amplitude and phase, which is described by the algebraic algebraic product of the complex scalar with the **wave-Polarization** vector, as from § **??**.

In fact, all the requirements for a complex vector space are satisfied.

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46.03.03.01 Bases and Orthogonal wave-Polarization States

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Read § 46.15.01.

- Standard Base for linear **wave-Polarization** :

$$\{\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2\} \quad .$$

- Other Bases for linear **wave-Polarization** : any orthonormal base, $\{\widehat{\mathbf{e}}'_1, \widehat{\mathbf{e}}'_2\}$, coming from any rotation in the xy plane starting from $\{\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2\}$:

$$\left\{\widehat{\mathbf{e}}'_1, \widehat{\mathbf{e}}'_2\right\} = \mathbb{R}\left[\{\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2\}\right] \quad .$$

- Standard Base for circular **wave-Polarization** :

$$\left\{\overline{\widehat{\mathbf{e}}}_{\mathrm{R}}, \overline{\widehat{\mathbf{e}}}_{\mathrm{L}}\right\} \equiv \left\{\frac{\widehat{\mathbf{e}}_1 - \mathrm{i}\widehat{\mathbf{e}}_2}{\sqrt{2}}, \frac{\widehat{\mathbf{e}}_2 + \mathrm{i}\widehat{\mathbf{e}}_2}{\sqrt{2}}\right\} \qquad \overline{\widehat{\mathbf{e}}}_{\mathrm{R}} \cdot \overline{\widehat{\mathbf{e}}}^{\star}_{\mathrm{R}} = 1 \qquad \overline{\widehat{\mathbf{e}}}_{\mathrm{L}} \cdot \overline{\widehat{\mathbf{e}}}^{\star}_{\mathrm{L}} = 1 \qquad \overline{\widehat{\mathbf{e}}}_{\mathrm{R}} \cdot \overline{\widehat{\mathbf{e}}}^{\star}_{\mathrm{L}} = 0 \qquad \overline{\widehat{\mathbf{e}}}_{\mathrm{L}} \cdot \overline{\widehat{\mathbf{e}}}^{\star}_{\mathrm{R}} = 0 \quad .$$

- Other Bases for circular **wave-Polarization** : any base obtained from $\left\{\overline{\widehat{\mathbf{e}}}_{\mathrm{R}}, \overline{\widehat{\mathbf{e}}}_{\mathrm{L}}\right\}$ as the standard bases for linear **wave-Polarization** are obtained from $\{\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2\}$:

$$\left\{\mathbf{e}'_{\mathrm{R}}, \mathbf{e}'_{\mathrm{L}}\right\} = \mathbb{R}\left[\left\{\overline{\widehat{\mathbf{e}}}_{\mathrm{R}}, \overline{\widehat{\mathbf{e}}}_{\mathrm{L}}\right\}\right] \quad .$$

In general:

$$\widehat{\mathbf{e}}_{\mathrm{k}} \cdot \widehat{\mathbf{e}}^{\star}_{\mathrm{j}} = \delta^{\cdot\cdot}_{kj} \quad .$$

It can be shown that in general two **wave-Polarization** states are orthogonal if and only if their complex vectors are orthogonal, as for base states above:

$$\mathbf{E}_1 \cdot \mathbf{E}^{\star}_2 = 0 \qquad \text{if and only if } \mathbf{E}_1 \text{ and } \mathbf{E}_2 \text{ are orthogonal} \quad .$$

Any **wave-Polarization** state can be described by a linear combination of the base vectors of any orthonormal base.

46.03.03.02 Pure States

The generic complex field of a MPW propagating along the z axis can be written as:

$$\mathbf{E} = \mathbf{E}_{\mathrm{R}} + \mathrm{i}\mathbf{E}_{\mathrm{I}} \quad ,$$

with the usual convention that the physical field is the real part of the complex field.

It can be also written, terms of the real positive amplitudes, E_{0x} and E_{0y}, and the two real phases ϕ_x and ϕ_y, as:

$$\begin{aligned}\mathbf{E}[\mathbf{r}, t] &= \widehat{\mathbf{e}}_1 E_{0x} \exp\left[-\mathrm{i}\phi_x\right] \exp\left[\mathrm{i}\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right] + \widehat{\mathbf{e}}_2 E_{0y} \exp\left[-\mathrm{i}\phi_y\right] \exp\left[\mathrm{i}\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right] \quad , \\ &= \left(\widehat{\mathbf{e}}_1 E_{0x} \exp\left[-\mathrm{i}\phi_x\right] + \widehat{\mathbf{e}}_2 E_{0y} \exp\left[-\mathrm{i}\phi_y\right]\right) \exp\left[\mathrm{i}\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right] \quad .\end{aligned}$$

The latter relation shows that a generic complex vector, $\mathbf{E}_0$, multiplying the MPW factor can describe a generic MPW with any given state of **wave-Polarization** as:

$$\mathbf{E}[\mathbf{r}, t] = \mathbf{E}_0 \exp\left[\mathrm{i}\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right] \quad .$$

Two MPW propagating along the same direction and characterized by the vectors $\mathbf{E}_{0|1}$ and $\mathbf{E}_{0|2}$ are orthogonal if and only if:

$$\left\langle \mathbf{E}_{0|1} \cdot \mathbf{E}_{0|2} \right\rangle = \mathbf{E}^{\star}_{0|1} \cdot \mathbf{E}_{0|2} = 0 = \mathbf{E}_{0|1} \cdot \mathbf{E}^{\star}_{0|2} = \left\langle \mathbf{E}_{0|2} \cdot \mathbf{E}_{0|1} \right\rangle \quad .$$

This can be checked by direct calculation, motivating the use of complex base vectors in a complex vector space with **hermitian scalar product**.

In the complex formalism any **wave-Polarization** state can be described as a superposition of a pair of **complex base vectors in a complex vector space with hermitian scalar product** satisfying the complex orthonormality condition:

$$\mathbf{e}_{\mathrm{i}} \cdot \mathbf{e}^{\star}_{\mathrm{j}} = \delta^{\cdot\cdot}_{ij} \qquad i, j = 1, 2 \quad .$$

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Any **wave-Polarization** state can be written in terms of the base vectors as:

$$\boxed{\mathbf{E} = E_1\mathbf{e}_1 + E_2\mathbf{e}_2} \quad .$$

This is the basis of the so-called **Jones-calculus** capable to deal in a compact way with pure **wave-Polarization** states and instruments changing the **wave-Polarization** states.

Note that the description of **wave-Polarization** states is in terms of a complex vector space, as it is appropriate for a two-states quantum system. This shall not be confused with the **algebra of complex vectors** of § 19.04.

46.03.03.03 Stokes Parameters

Stokes parameters are used to describe the **wave-Polarization** state of a MPW.

Three independent parameters are required to characterize the **wave-Polarization** ellipse such as the two amplitudes and the phase ϕ. It is often convenient to use a different set of parameters, having the same physical dimensions.

The Stokes parameters of a MPW are defined as:

$$\begin{aligned} s_0 &= E_{0x}^2 + E_{0y}^2 \quad , \\ s_1 &= E_{0x}^2 - E_{0y}^2 \quad , \\ s_2 &= 2E_{0x}E_{0y}\cos\left[\Delta\phi\right] \quad , \\ s_3 &= 2E_{0x}E_{0y}\sin\left[\Delta\phi\right] \quad . \end{aligned}$$

Only three of the Stokes parameters are independent as they are related by:

$$s_0^2 = s_1^2 + s_2^2 + s_3^2 \quad .$$

The parameter s_0 is clearly proportional to the intensity of the wave. This allows to express the three parameters s_1, s_2 and s_3 as:

$$\begin{aligned} s_1 &= s_0\cos\left[2\chi\right]\cos\left[2\psi\right] \quad , \\ s_2 &= s_0\cos\left[2\chi\right]\sin\left[2\psi\right] \quad , \\ s_3 &= s_0\sin\left[2\chi\right] \quad . \end{aligned}$$

These relations point to a simple geometrical interpretation: the three parameters s_1, s_2 and s_3 can be interpreted as the three orthonormal Cartesian Coordinates Coordinate System of a point on the surface of a sphere of radius s_0 such that the angle 2χ may be regarded as the polar angle while 2ψ may be regarded as the azimuth angle.

Thus to any given **wave-Polarization** state of a MPW of intensity s_0 can be associated a point at the surface of the sphere, called the Poincaré sphere.

The following properties can be demonstrated.

- Define the auxiliary quantity:

$$\boxed{\tan\left[\alpha\right] \equiv \frac{E_{0y}}{E_{0x}} \qquad 0 \leq \alpha \leq \pi/2} \quad .$$

- The half azimuth angle ψ ($0 \leq \psi \leq \pi$) characterizes the orientation of the ellipse:

$$\boxed{\tan\left[2\psi\right] = \frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}\cos\left[\Delta\phi\right] = \tan\left[2\alpha\right]\cos\left[\Delta\phi\right]} \quad .$$

- Let a and b be the major and minor axes of the ellipse. Define the auxiliary quantity:

$$\boxed{\tan\left[\chi\right] \equiv \pm\frac{b}{a}} \quad ,$$

  where the sign in the definition may be used to distinguish the two senses in which the ellipse may be described.
- The half polar angle χ ($-\pi/4 \leq \chi \leq +\pi/4$) characterizes the eccentricity of the ellipse and the sense in which the ellipse is being described.

$$\left|\tan\left[\chi\right]\right| = \frac{b}{a} \leq 1 \quad .$$

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46.04 Absorption Scattering Extinction of ElectroMagnetic Waves

This § is referenced at pages:
[2319]

EM waves are absorbed and scattered; phenomenological introduction.

46.04.01 Phenomenological Introduction to the Beer-Lambert Law

This § is referenced at pages:
[2328]

In this section a **phenomenological introduction** to absorption scattering and extinction of ElectroMagnetic waves (such as light) is presented.

A phenomenological approach is such that phenomena are just described without any attempt to explain, interpret nor model the phenomena themselves.

The term **extinction** is used to indicate the complex of **absorption** and **scattering**, that is light removal from the beam. Absorption implies transfer of energy into matter. Scattering implies removal of energy from the beam and re-direction of ElectroMagnetic waves and energy into a direction different from the original one.

Both **scattering and absorption remove ElectroMagnetic waves and energy from a beam** traversing a medium, so that the beam is attenuated and **this attenuation is called extinction**. Extinction is seen when looking directly at the light source. The Sun, for instance, is fainter and redder at sunset than at noon due to extinction along the air path. Extinction is strong in all colors, but it is stronger in blue light than in red light. Whether scattering or absorption is mainly responsible for this extinction cannot be judged from this observation alone. However, when looking sideways at the air through which the Sun shines one can see that actually blue light is scattered more strongly. Measurements show that all light taken away from the original beam reappears sideways as scattered light. Therefore in this case it is scattering and not absorption causing the extinction.

Consider a cylindrical light beam across matter. The amount of light undergoing absorption/scattering is normally proportional to the actual beam intensity.

This follows easily from the corpuscular interpretation of ElectroMagnetic radiation in terms of photons and from the fact that the number of photons absorbed/scattered is proportional to the number of photons present in the beam, for truly independent absorption/scattering events and in absence of any coherence/correlation effect among different scattering centers.

Moreover, the number of photons absorbed/scattered is proportional to the thickness of the matter traversed, at least for infinitesimally thin layers of matter. The fact that the number of photons absorbed/scattered is proportional to the thickness of the infinitesimal layer of matter but this is not true for a finite piece of matter can be understood as follows. If a certain intensity enters a disk of certain thickness the extinction will be proportional to the original intensity. If another identical disk of matter follows the extinction will be again proportional to the entering intensity, which is the extincted intensity from the first disk, that is smaller than the original intensity. Therefore the extinction caused by the second disk will be smaller than the one caused by the first disk and the total extinction caused by the two disks less that twice the extinction caused by each single disk on the original intensity.

The proportionality to the intensity and infinitesimal disk thickness gives the Beer-Lambert law in optics, for the intensity of a beam as a function of the position z inside an homogeneous and isotropic

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absorbing medium, in terms of the quantity λ, the **attenuation length** (or **extinction length** or **mean free path**), Read also section § 30.06.01.

The Beer-Lambert law is easily derived by assuming that the intensity change is proportional to the intensity itself as well as to the (infinitesimal) length of the path traversed as follows:

$$\boxed{\frac{\mathrm{d}I[z]}{\mathrm{d}z} = -\frac{I[z]}{\lambda} \qquad \Longrightarrow \qquad I[z] = I_0 \exp\left[-\frac{z}{\lambda}\right]} \quad . \tag{46.04.01}$$

→ 2138 2726

The mean free path provides the length over which the beam is attenuated. It can be expressed in terms of the **cross-section** for the interaction (either absorption or scattering) and the number density of targets, $\eta_{\mathbb{N}}$, as (compare with equation (30.06.04)):

$$\boxed{\frac{1}{\lambda} \equiv \eta_{\mathbb{N}}\sigma} \quad ,$$

in case of fixed targets.

Read § 30.06 for further details and developments.

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46.05 Energy Linear Momentum Angular Momentum of ElectroMagnetic Waves

This § is referenced at pages:
[1995, 2079, 2140]

*After discussing **wave-Polarization** , the description of ElectroMagnetic MPW is complete and the dynamic quantities associated to ElectroMagnetic MPW can be introduced.*

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©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|||

When an ElectroMagnetic wave is impinging onto a piece of matter it can give energy to the matter and it can also give linear momentum along its propagation direction and angular momentum around its propagation direction.

46.05.01 Intensity of ElectroMagnetic MPW

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 2, § 27.2||

©|K.T.McDonald|WEB - URL |*Extensive Recent Review - for instance see about Feynman, Stratton, Bossavit.*|

The time-averaged value of the Poynting vector across any surface is usually called the intensity of the radiation across the surface. It is a measure of the energy flux across any surface, that is energy per unit time per unit area:

$$I \equiv |\langle \mathbf{S}_{\text{EM}} \rangle| \quad .$$

Note that, even if the Poynting vector can be safely used only when integrated on a closed surface to describe the total power, in the present case the following arguments can be given in favor of interpreting it as the local energy flux, point by point.

- Experimentally the local energy flux via ElectroMagnetic waves described by the Poynting vector is consistent with measurements.
- In general many alternatives have been proposed to the Poynting vector, normally more complex than the Poynting vector itself, but none of them has been shown to be a better alternative to it, nor it has been able to replace the Poynting vector.
- A microscopic model of the energy given to a piece of matter, even if with the possible limits of any model, confirms, within the model, the use of the Poynting vector (read § 46.05).

In fact the energy flux involves **some subtleties**: see the references.

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46.05.01.01 Intensity of ElectroMagnetic MPW in Vacuum

This § is referenced at pages:
[2040]

Note that all the expressions in this section are only valid for **E** and **B** in-phase; all quantities are real quantities.

Consider a ElectroMagnetic MPW in vacuum:

$$\omega = |\mathbf{k}|c \quad ,$$
$$\mathbf{k} \equiv k\widehat{\boldsymbol{\kappa}} \quad ,$$
$$\mathbf{E}[\mathbf{x},t] = \mathbf{E}_0 \cos[\mathbf{k}\cdot\mathbf{x} - \omega t - \phi] \quad ,$$
$$\mathbf{B}[\mathbf{x},t] = \mathbf{B}_0 \cos[\mathbf{k}\cdot\mathbf{x} - \omega t - \phi] \quad ,$$
$$\mathbf{k}\cdot\mathbf{E}_0 = 0 \quad ,$$
$$\mathbf{k}\cdot\mathbf{B}_0 = 0 \quad ,$$
$$\boxed{\mathbf{B}'_0 \equiv c\mathbf{B}_0} \quad ,$$
$$\mathbf{B}_0 = +\frac{1}{\omega}(\mathbf{k}\wedge\mathbf{E}_0) \Longrightarrow \boxed{\mathbf{B}'_0 = +(\widehat{\boldsymbol{\kappa}}\wedge\mathbf{E}_0)} \quad ,$$
$$\mathbf{E}_0 = -\frac{c^2}{\omega}(\mathbf{k}\wedge\mathbf{B}_0) \Longrightarrow \boxed{\mathbf{E}_0 = -(\widehat{\boldsymbol{\kappa}}\wedge\mathbf{B}'_0)} \quad .$$

The time-averaged energy density is:

$$\boxed{\langle\frac{\mathrm{d}U_{\mathrm{EM}}}{\mathrm{d}V}\rangle \equiv \langle u_{\mathrm{EM}}\rangle = \frac{\varepsilon_0}{2}\langle|\mathbf{E}|^2\rangle + \frac{1}{2\mu_0}\langle|\mathbf{B}|^2\rangle = 2\langle u_{\mathrm{EM}E}\rangle = 2\langle u_{\mathrm{EM}B}\rangle = \frac{\varepsilon_0}{2}|\mathbf{E}_0|^2 = \frac{1}{2\mu_0}|\mathbf{B}_0|^2} \quad . \qquad (46.05.01)$$

→ 2080

The time-averaged energy flux is (using equation (46.01.09)):

$$\boxed{\langle\mathbf{S}_{\mathrm{EM}}\rangle \equiv \frac{1}{\mu_0}\langle\mathbf{E}\wedge\mathbf{B}\rangle = \frac{1}{\mu_0}\widehat{\boldsymbol{\kappa}}\langle|\mathbf{E}||\mathbf{B}|\rangle = \frac{1}{\mu_0}\frac{\widehat{\boldsymbol{\kappa}}}{c}\langle|\mathbf{E}|^2\rangle = \frac{1}{\mu_0}c\widehat{\boldsymbol{\kappa}}\langle|\mathbf{B}|^2\rangle = \frac{\varepsilon_0 c}{2}\widehat{\boldsymbol{\kappa}}|\mathbf{E}_0|^2 = \frac{c}{2\mu_0}\widehat{\boldsymbol{\kappa}}|\mathbf{B}_0|^2} \quad . \qquad (46.05.02)$$

→ 2080

Finally:

$$\boxed{|\langle\mathbf{S}_{\mathrm{EM}}\rangle| \equiv \frac{1}{\mu_0}|\langle\mathbf{E}\wedge\mathbf{B}\rangle| = \frac{\varepsilon_0 c}{2}|\mathbf{E}_0|^2 = \frac{c}{2\mu_0}|\mathbf{B}_0|^2 = c\langle u_{\mathrm{EM}}\rangle} \quad , \qquad (46.05.03)$$

→ 2080

which shows that, for ElectroMagnetic MPW in vacuum, the intensity is proportional to the volume energy density via the **phase-velocity** c (read § 44.04.05 for the general discussion). Read § 45.08.01.01 for the similar relation for mechanical waves.

Note the suggestive relation (46.05.03) lining the energy flux with the energy density via the **phase-velocity** is valid in this special case with no dispersion, that is with fixed **phase-velocity** . In presence of dispersion the velocity might be not the **phase-velocity** , see section § 47.04.01.

46.05.01.02 Intensity of ElectroMagnetic MPW in perfect dielectric media

Note that all the expressions in this section are only valid for **E** and **B** in-phase; all quantities are real quantities.

The above relations (46.05.01), (46.05.02), (46.05.03) can be translated for a **perfect dielectric medium** by replacing:

$$\boxed{\varepsilon_0 \longrightarrow \varepsilon_0\varepsilon_{\mathrm{R}}} \quad ,$$
$$\boxed{\mu_0 \longrightarrow \mu_0\mu_{\mathrm{R}}} \quad ,$$
$$\boxed{c \longrightarrow v} \quad .$$

Read § 46.06.02.08 for the more general case.

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46.05.02
Linear Momentum and Relation to the Energy Flux

This § is referenced at pages:
[2319, 2321, 2508]

This is one of the rare cases when the effects of magnetic fields of ElectroMagnetic waves are noticeable. It is in fact the magnetic part of the Lorentz force which provokes the pressure. In fact for an ElectroMagnetic wave the ratio of the magnetic to electric force is:

$$\frac{f_{\text{mag}}}{f_{\text{ele}}} \approx \frac{vB}{E} = \frac{v}{c} \quad ,$$

So that the effects of magnetic fields are often negligible, for non relativistic speeds of particles. However, under some circumstances, they are important, as in the case of **radiation pressure**.

46.05.02.01 An Approach Based on the Particle Nature at Normal Incidence

For every amount of energy ϵ in a traveling ElectroMagnetic MPW there is an amount of linear momentum $\mathbf{p}$ such that (read § 37.02):

$$\epsilon = c|\mathbf{p}| \equiv cp \quad . \tag{46.05.04}$$

→ 2081

Let p_0 be the module of the momentum of the incoming bit of wave. Let P be the module of the total final momentum. Assume the wave is fully absorbed: if the amount of energy given is ϵ, the linear momentum given along the propagation direction is ϵ/c (from linear momentum conservation and $\epsilon = pc$):

$$p_0 + 0 = P \implies P = \epsilon/c \quad .$$

Assume the wave is fully reflected, elastically, and let p' be the module of the momentum of the bit of reflected wave: if the amount of energy given is ϵ, the linear momentum given along the propagation direction is $2\epsilon/c$ (from linear momentum conservation and $\epsilon = pc$):

$$p_0 + 0 = -p' + P \qquad p' = p_0 \implies P = 2\epsilon/c \quad .$$

Nevertheless the result can be demonstrated via classical arguments only, at least in a simplified configuration, with the advantage that the demonstration makes clear the physical processes leading to radiation pressure. Actually the fact that ElectroMagnetic waves are quantised has nothing to do with equation (46.05.04).

46.05.02.02 A Simplified Microscopic Argument for Total Absorbtion at Normal Incidence

Interaction between light and matter is a complex phenomenon, where the incident and re-radiated radiation enter. The interference between incoming an re-radiated field to give rise to the **index of refraction** , see [1].

We want to compare the average force along the z direction, that is the linear momentum transferred by the wave to the matter per unit time, with the energy transferred by the wave to the matter per unit time. Total absorption of the ElectroMagnetic wave in matter is assumed. Consider the following simplified argument.

Consider an ElectroMagnetic wave whose propagation direction is along the z axis and the **wave-Polarization** is along the x axis:

$$E_x = E_{x0} \cos\left[k\overline{z} - \omega t\right] \quad ,$$
$$B_y = B_{y0} \cos\left[k\overline{z} - \omega t\right] \quad .$$

Note that **wave-Polarization** state was chosen arbitrarily as it has no effect.

The two fields are in-phase, as it must be, in vacuum.

The ElectroMagnetic MPW is impinging perpendicularly on a flat piece of material, with no free charges and no free currents. Assume that a very thin and rigid sheet of matter is present at a fixed $z = \overline{z}$, kept there by some external force. **Strictly speaking the above are the ElectroMagnetic fields outside matter:**

[1] R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...

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they are the same inside matter because of the conservation of tangential components for no free currents (also there are no free charges but this does not matter here). Actually we use here the real Electric|Magnetic fields plus charges/currents. The charges inside the sheet of matter cannot move in the z direction ($v_z = 0$), because the thin sheet is constrained to stay there by the external forces.

The average force on any charge is:

$$\langle \mathbf{F} \rangle = q\widehat{\mathbf{e}}_1 \left(\langle E_x \rangle - \langle v_z B_y \rangle \right) + q \langle v_x B_y \rangle \widehat{\mathbf{e}}_3 \quad .$$

In view of the fact that, by hypothesis, the particle cannot move in the z direction the term $v_z B_y$ vanishes. Moreover, $\langle E_x \rangle$ vanishes as well. The ElectroMagnetic fields are the ElectroMagnetic fields of the incoming wave, in vacuum. Actually if the particles in matter re-radiate the total ElectroMagnetic fields will be the sum of the incoming plus the radiated field.

It follows that:

$$\langle F_z \rangle \equiv \langle F_z \rangle_{\texttt{EM} \longrightarrow \texttt{MATTER}} \simeq q \langle v_x B_y \rangle \quad .$$

It is assumed that the ElectroMagnetic waves is totally absorbed by the sheet of matter. Therefore the rate at which work is done on the charge by the traveling wave is exactly:

$$\langle \Pi \rangle \equiv \langle \Pi \rangle_{\texttt{EM} \longrightarrow \texttt{MATTER}} = \langle \, \mathbf{v} \cdot \mathbf{F} \, \rangle = q \langle \, \mathbf{v} \cdot \mathbf{E} \rangle = q \langle v_x E_x \rangle \quad .$$

Therefore:

$$\langle F_z \rangle = q \langle v_x B_y \rangle = q \frac{\langle v_x E_x \rangle}{c} = \frac{\langle \Pi \rangle}{c} \quad .$$

Finally:

$$\boxed{\langle F_z \rangle = \frac{\langle \Pi \rangle}{c} \qquad \Leftrightarrow \qquad p_F \equiv \frac{\langle F_z \rangle}{A} = \frac{\langle \Pi \rangle}{Ac} \equiv \frac{I}{c}} \quad .$$

46.05.02.03 Radiation Force-Pressure at Normal Incidence

Let us keep in mind that the layer of matter has no recoil.

The general expression for the radiation force-pressure (that is force per unit area) at normal incidence is easily found, via the particle-like interpretation, in terms of the absorbed plus reflected intensity (as it is obvious that the transmitted energy gives no contribution):

$$\boxed{p_F \equiv \frac{d^2 F_z}{\mathrm{d}A_z \, \mathrm{d}t} = \frac{I_{\texttt{A}}}{c} + 2\frac{I_{\texttt{R}}}{c}} \quad . \tag{46.05.05}$$

→ 2082

While the expression (46.05.05) can be easily deduced via the corpuscular interpretation of ElectroMagnetic radiation (linear momentum conservation), it is more difficult to justify the factor 2 via the simplified in the classical model presented above, because in case of reflection one should account for the energy re-radiated back as ElectroMagnetic radiation by the charges in the sheet of matter.

In terms of the corpuscular interpretation of ElectroMagnetic waves one would write:

$$\epsilon = |\mathbf{p}| c \quad ,$$

and therefore:

$$\boxed{\Delta P = \frac{E_{\texttt{A}}}{c} + 2\frac{E_{\texttt{R}}}{c}} \quad . \tag{46.05.06}$$

Read also § 40.01.03.

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46.05.02.04 Force-Pressure From Linear Momentum Transport

Use equation (40.01.08).

46.05.03 Angular Momentum and Relation to the Energy Flux

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©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|3.7||

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|P-7.19, P-7.20, P-7.21||

Circularly wave-Polarized ElectroMagnetic waves carry angular momentum along the propagation direction.

Right-handed circularly wave-Polarized ElectroMagnetic waves have negative helicity and therefore carry angular momentum anti-parallel to the propagation direction.

Left-handed circularly wave-Polarized ElectroMagnetic waves have positive helicity and therefore carry angular momentum parallel to the propagation direction.

In fact the angular momentum carried by MPW involves **many subtleties**: see the references.

A similar reasoning to the one carried on for the radiation force-pressure can be carried on for the angular momentum transfer.

46.05.03.01 An Approach Based on the Particle Nature of ElectroMagnetic Waves

Assume that a ElectroMagnetic MPW is impinging at normal incidence on the surface of separation between two media. Assume that the wave is fully absorbed: if the amount of energy given is E, the angular momentum given along the propagation direction by a circularly wave-Polarized ElectroMagnetic wave is E/ω.

This is self-evident from the concept of photons, read § 37.02. In fact every amount of energy ϵ in a traveling monochromatic ElectroMagnetic PW has a (quantised) amount of angular momentum around the propagation direction $|\mathbf{J}| = \hbar$ such that:

$$\boxed{\epsilon = |\mathbf{j}|\omega = \hbar\omega} \quad . \tag{46.05.07}$$

46.05.03.02 A Simplified Microscopic Argument for Total Absorbtion at Normal Incidence

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...||*Clear ...*

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...||*Clear derivation*|

46.05.03.03 Radiation Torque-Pressure at Normal Incidence

The boundary condition, as discussed in section § 48.03, implies what follows.

Assume the wave is fully reflected: it can be easily shown that the circular **wave-Polarization** state is inverted by the reflection and therefore the helicity changes its sign so that the reflected wave carries the same angular momentum along z as the incoming wave. In fact the boundary conditions at the interface impose conditions on the tangential components of the ElectroMagnetic fields, that is the transverse ElectroMagnetic fields of the EM wave. The ElectroMagnetic fields of circularly wave-Polarized waves at the interface describe a circumference, for both the incoming and reflected wave. Any continuity condition will require, to be satisfied, that the ElectroMagnetic fields of the incoming and reflected waves run along their circumference in the same way. This immediately implies that the incident and reflected waves have opposite handedness (and helicity) as they travel in opposite directions. Note that this is true also in the case when the reflected field is opposed to the incoming field.

In general: any continuity relation among incident, reflected and transmitted waves is such that all three waves rotate the same way, that is with the same angular momentum component along the direction of

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motions. This is to be contrasted to the case of the linear momentum component, which has the same sign for the transmitted wave but opposite sign for the reflected wave, both with respect to the incoming wave.

The general expression for the radiation torque-pressure (that is torque per unit area) at normal incidence from an elliptically (circularly) wave-Polarized ElectroMagnetic wave is given in terms of the absorbed intensity.

$$\boxed{p_\Gamma \equiv \frac{d^2 J}{\mathrm{d}A\ \mathrm{d}t} = \frac{I_{\mathrm{A}}}{\omega}} \quad . \tag{46.05.08}$$

The reflected intensity does not contribute as the reflected wave will carry the same angular momentum as the incoming wave.

In terms of the corpuscular interpretation of ElectroMagnetic waves one would write:

$$\epsilon = |\mathbf{j}|\omega = \hbar\omega \quad .$$

and therefore:

$$\boxed{\Delta J = \frac{E_{\mathrm{A}}}{\omega}} \quad . \tag{46.05.09}$$

Read also § 40.01.04.

46.05.03.04 Torque-Pressure From Angular Momentum Transport

Use equation (40.01.15).

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46.06 EM Waves in perfect Ohmic media Without Boundaries

This § is referenced at pages:
[1993, 2011, 2065, 2323, 2327]

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*After vacuum and **perfect dielectric media**, **perfect Ohmic media**. Absorption appears.*

After describing ElectroMagnetic waves in vacuum and **perfect dielectric media**, which feature no energy absorption, and having introduced absorption phenomenologically, it is possible to discuss absorptive media. Only dissipative media are discussed here, while also media amplificating energy, in principle, do exist (active/gain media).

Moreover, after having discussed **wave-Polarization** of ElectroMagnetic waves it is possible to discuss the full complex representation of MPW.

perfect Ohmic media include plasmas, liquid metals as well as solid metals in most cases. In fact solid metals may be microscopically non isotropic but they are normally built of different small crystals with random orientation. As long as these crystals are small with respect to the wavelength of the ElectroMagnetic waves, the metal is macroscopically isotropic.

46.06.01
Equations for ElectroMagnetic MPW in perfect Ohmic media

A **perfect Ohmic medium** is different from a **perfect dielectric medium** only in the fact that its **conductivity** is fixed (real) number independent of anything else than the nature of the material medium with $\sigma > 0$, instead of being zero:

$$\begin{aligned} &\text{LHI medium} \quad , \\ \varepsilon_{\mathrm{R}} \geq 1 \qquad &\text{fixed, real} \quad , \\ \mu_{\mathrm{R}} > 0 \qquad &\text{fixed, real} \quad , \\ \sigma > 0 \implies \mathbf{j}^{\mathrm{F}} \neq 0 \quad &, \\ \rho^{\mathrm{F}} = 0 \qquad &\text{electrically neutral at any } \textbf{internal point} \text{ after relaxation time} \quad . \end{aligned}$$

It should be kept in mind that linearity is normally only justified for small enough fields and independence of frequecy only in a limited range of frequencies.

If the **conductivity** is different from zero **any free charge will decay to zero at any point inside the medium in a finite time** and will accumulate either at the boundary of the medium or at infinity. The time-scale of the free charge decay depends on the **conductivity** . A large number of cases is covered by assuming $\rho^{\mathrm{F}} = 0$, that is **conductivity** large enough that free charges decay fast enough with respect to the rate of change of the ElectroMagnetic fields. **The hypothesis of good enough conductor will be assumed**, unless specified otherwise.

The free current density, $\mathbf{j}^{\mathrm{F}}$, is linked to the electric field (by definition for any **perfect Ohmic medium**):

$$\boxed{\mathbf{j}^{\mathrm{F}} = \sigma \mathbf{E}} \quad .$$

Moreover, (by definition for any **perfect Ohmic medium**) one has:

$$\boxed{\mathbf{D} = \varepsilon_{\mathrm{R}} \varepsilon_0 \mathbf{E}} \qquad \boxed{\mathbf{B} = \mu_{\mathrm{R}} \mu_0 \mathbf{H}} \quad .$$

The ElectroMagnetic fields in matter will then move the charges and the motion of charges will transfer energy to matter. Therefore, in general, energy will be transferred by ElectroMagnetic fields to matter, that is ElectroMagnetic energy is dissipated into internal energy of matter and the ElectroMagnetic fields attenuate.

Maxwell Equations in a in **perfect Ohmic medium**, in the absence of free charges, read:

$$\mathbf{div}\mathbf{E} = 0 \quad ,$$
$$\mathbf{div}\mathbf{B} = 0 \quad ,$$
$$\mathbf{rot}\mathbf{E} = -\partial_t \mathbf{B} \quad ,$$
$$\mathbf{rot}\mathbf{B} = +\sigma\mu_{\mathrm{R}}\mu_0\mathbf{E} + \varepsilon_{\mathrm{R}}\mu_{\mathrm{R}}\varepsilon_0\mu_0\partial_t\mathbf{E} \quad ,$$

where **the free charge density has been set to zero as it is appropriate for any good enough conductor**.

One derives, from **Maxwell Equations** , the following two decoupled **modified d'Alembert Wave equations**:

$$\boxed{\boldsymbol{\nabla}^2\mathbf{E} - \sigma\mu_{\mathrm{R}}\mu_0\partial_t\mathbf{E} - \varepsilon_{\mathrm{R}}\varepsilon_0\mu_{\mathrm{R}}\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2} = 0} \quad , \qquad (46.06.01)$$

→ 2090

$$\boxed{\boldsymbol{\nabla}^2\mathbf{B} - \sigma\mu_{\mathrm{R}}\mu_0\partial_t\mathbf{B} - \varepsilon_{\mathrm{R}}\varepsilon_0\mu_{\mathrm{R}}\mu_0\frac{\partial^2\mathbf{B}}{\partial t^2} = 0} \quad . \qquad (46.06.02)$$

→ 2090

Exactly the same equations are found for $\mathbf{D}$ and $\mathbf{H}$.

Actually, the two equations are decoupled, but the Electric|Magnetic fields are only apparently decoupled. In fact, the equations have been obtained by derivation from Maxwell equations, which still rule, coupling the Electric|Magnetic fields. Therefore, additional constraints must be included on the solutions of the wave equations. Read § 46.01.01.

Therefore all ElectroMagnetic fields satisfy a modified **D'Alembert non-dispersive wave-equation** (the **dissipative d'Alembert Classical Non-Dispersive Wave equation**) **with the addition of a term linear in the first time-derivative** which has deep consequences on the behavior, as discussed in section § 54.

It should be noted that the present case of **perfect Ohmic media** includes, as special cases, the cases of vacuum and **perfect dielectric media**.

46.06.02
EM MPW in perfect Ohmic media With the Complex Formalism

This § is referenced at pages:
[2065]

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Another step forward for studying ElectroMagnetic waves is done by considering the possibility of a complex solution to **Maxwell Equations** , with a complex wave-vector.

In general the **Wave-Equation** might admit MPW solutions with complex wave-vector:

$$\boxed{\overline{\mathbf{U}}[\mathbf{r},t] = \overline{\mathbf{U}}_0 \exp\left[\mathtt{i}\left(\overline{\mathbf{k}}\cdot\mathbf{r} - \omega t\right)\right]} \quad , \qquad (46.06.03)$$

→ 2087

where $\overline{\mathbf{U}}_0$ is a complex constant vector carrying the information on both the amplitude and phase of the wave and the complex wave-vector, is written in terms of the two real vectors $\mathbf{k}_{\mathrm{R}}$ and $\mathbf{k}_{\mathrm{I}}$, as:

$$\boxed{\overline{\mathbf{k}} = \mathbf{k}_{\mathrm{R}} + \mathtt{i}\mathbf{k}_{\mathrm{I}}} \quad .$$

The use of a complex wave-vector is motivated by the following physical considerations. We know that absorption of ElectroMagnetic waves in matter exists and therefore ElectroMagnetic waves may be attenuated under certain circumstances, amplified under other circumstances. Therefore a simple sinusoidal/cosinusoidal time dependence is not general enough to describe cases when absorption/amplification exists. Therefore, one possible generalization is to think of an exponentially damped harmonic field with a complex wave-vector and the formalism of complex fields.

Read § 19.03, § 19.03 for the algebra of complex scalars and vectors.

Use the convention that real fields are described via the real part of a complex field:

$$\boxed{\mathbf{U}[\mathbf{x},t] \equiv \mathfrak{Re}\left[\overline{\mathbf{U}}[\mathbf{x},t]\right]} \quad ,$$

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also exploiting Euler formula.

As long as we perform linear operations, which keep the real and imaginary parts of complex numbers separated, we can use the formalism of complex numbers without troubles, remembering that the true physical quantity is the real part of the complex number.

The complex number in itself has no physical meaning; it is only a way to pack two real numbers, into one single complex number.

It is clear that the complex exponential dependence for the fields leads to a particularly simple form of the **Maxwell Equations** .

46.06.02.01 Absorption and dispersion

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It is clear that the real part of the wave-vector, $\mathbf{k}_{\mathrm{R}}$, is linked to the **phase-velocity** and the imaginary part of the wave-vector, $\mathbf{k}_{\mathrm{I}}$, is linked to the extinction.

Both absorption and dispersion are phenomena described by the complex wave-vector and they are therefore linked. The link is given by the so-called **Kramers-Krönig** relations (actually the Hilbert transform). In fact the knowledge of one of the two quantities allows to recover the other one, thanks to the analiticity of the functions involved and the causality requirement. So, it turns out the real and imaginary parts of both $\varepsilon_{\mathrm{R}}[\omega]$ and|or $\mu_{\mathrm{R}}[\omega]$, analytic response functions for a causal system, are connected and each part can be derived from the other.

Causality means that Polarization|Magnetization of matter cannot precede the electric and magnetic fields, respectively. This is described by the Kramers-Krönig relations.

46.06.02.02 Operations with the complex formalism

It is useful to determine the effects of common vector operations on the expression (46.06.03), **in Cartesian Coordinates**:

$$\mathbf{grad}\exp\left[\mathrm{i}\left(\overline{\mathbf{k}}\cdot\mathbf{r}-\omega t\right)\right]=\left(\mathrm{i}\overline{\mathbf{k}}\right)\exp\left[\mathrm{i}\left(\overline{\mathbf{k}}\cdot\mathbf{r}-\omega t\right)\right] \quad , \tag{46.06.04}$$

$$\nabla^2\exp\left[\mathrm{i}\left(\overline{\mathbf{k}}\cdot\mathbf{r}-\omega t\right)\right]=\mathbf{divgrad}\exp\left[\mathrm{i}\left(\overline{\mathbf{k}}\cdot\mathbf{r}-\omega t\right)\right]=-\overline{\mathbf{k}}\cdot\overline{\mathbf{k}}\exp\left[\mathrm{i}\left(\overline{\mathbf{k}}\cdot\mathbf{r}-\omega t\right)\right] \quad , \tag{46.06.05}$$

$$\boldsymbol{\nabla}^2\overline{\mathbf{U}}_0\exp\left[\mathrm{i}\left(\overline{\mathbf{k}}\cdot\mathbf{r}-\omega t\right)\right]=-\overline{\mathbf{U}}_0\left(\overline{\mathbf{k}}\cdot\overline{\mathbf{k}}\right)\exp\left[\mathrm{i}\left(\overline{\mathbf{k}}\cdot\mathbf{r}-\omega t\right)\right] \quad \text{Cartesian Coordinates only} \quad , \tag{46.06.06}$$

where the latter is obtained by observing that in Cartesian Coordinates the vector Laplacian is just the scalar Laplacian applied to the three components, as from equation (16.08.09)

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46.06.02.03 Complex Direction Angles

It is often useful to introduce **direction angles** to define the direction of the wave-vector $\mathbf{k}$. When the wave-vector is complex, direction angles are complex as well.

Read § 19.04.01.

46.06.02.04 Homogeneous MPW

One first case is the case when the wave-vector is complex and the real and imaginary parts of the complex wave-vector are two real vectors parallel to each other. In this case one can write:

$$\boxed{\overline{\mathbf{k}} = \mathbf{k}_{\mathtt{R}} + \mathtt{i}\mathbf{k}_{\mathtt{I}} = \overline{k}\widehat{\boldsymbol{\kappa}} \qquad \text{with } \widehat{\boldsymbol{\kappa}} \text{ a real unit vector} \qquad \widehat{\boldsymbol{\kappa}}\cdot\widehat{\boldsymbol{\kappa}} = 1 \qquad \text{as} \qquad \mathbf{k}_{\mathtt{R}} \parallel \mathbf{k}_{\mathtt{I}}} \quad .$$

The MPW is called homogeneous because the constant amplitude planes are parallel to the constant phase planes.

For homogeneous MPW the direction angles are real. This is in fact a point in favor of defining them in terms of $\overline{k} \equiv \sqrt{\overline{\mathbf{k}}\cdot\overline{\mathbf{k}}}$ and not $|\overline{\mathbf{k}}|$. In fact,

$$\boxed{\overline{\mathbf{k}} \equiv \overline{k}\widehat{\boldsymbol{\kappa}} \qquad \widehat{\boldsymbol{\kappa}}\cdot\widehat{\boldsymbol{\kappa}} = 1 \qquad \overline{k} \in \mathbb{C} \qquad \widehat{\boldsymbol{\kappa}} \in \mathbb{R}^3} \quad ,$$

implies that the direction angles are the direction cosines of the real unit vector $\widehat{\boldsymbol{\kappa}}$:

$$\overline{\mathbf{k}} \equiv \overline{k}\widehat{\boldsymbol{\kappa}} = \rho_k e^{\mathtt{i}\theta_k}\widehat{\boldsymbol{\kappa}} \tag{46.06.07}$$

$$\overline{\mathbf{k}}\cdot\overline{\mathbf{k}} = \rho_k^2 e^{2\mathtt{i}\theta_k} \tag{46.06.08}$$

$$\overline{k} = \sqrt{\overline{\mathbf{k}}\cdot\overline{\mathbf{k}}} = \rho_k e^{\mathtt{i}\theta_k} \tag{46.06.09}$$

$$\overline{\mathbf{k}} \equiv \overline{k}\widehat{\boldsymbol{\kappa}} \quad . \tag{46.06.10}$$

Therefore, the direction angles are the real direction cosines of the real unit vector $\widehat{\boldsymbol{\kappa}}$.

The unit vector $\widehat{\boldsymbol{\kappa}}$ gives the direction of the time-averaged energy flow. Moreover, it points in the direction of the phase and attenuation vectors: all three vectors point in the same direction for a homogeneous monochromatic PW. This direction is, unambiguously, the direction of propagation of the MPW. The planes of constant phase are also the planes of constant amplitude, and all of them are perpendicular to $\widehat{\boldsymbol{\kappa}}$. That is, the MPW has a uniform amplitude and phase across the plane perpendicular to the direction of propagation.

46.06.02.05 Non-Homogeneous MPW

One second case general case is the case when the wave-vector is complex and the real and imaginary parts of the complex wave-vector are two real vectors not parallel to each other. In this case one can write:

$$\boxed{\overline{\mathbf{k}} = \mathbf{k}_{\mathtt{R}} + \mathtt{i}\mathbf{k}_{\mathtt{I}} \qquad \text{with} \qquad \mathbf{k}_{\mathtt{R}} \wedge \mathbf{k}_{\mathtt{I}} \neq 0} \quad .$$

The MPW is called non-homogeneous because the constant amplitude planes and the constant phase planes are not parallel to each other.

Constant-phase planes are orthogonal to the real part of the wave-vector: $\mathbf{k}_{\mathtt{R}}$. Constant-amplitude planes are orthogonal to the imaginary part of the wave-vector: $\mathbf{k}_{\mathtt{I}}$. Constant-phase planes and constant-amplitude planes are not, in general, orthogonal between each other. MPW whose planes of constant amplitude are not coincident with the planes of constant phase are called non-homogeneous or non-uniform MPW.

If the MPW is not homogeneous, corresponding to complex direction angles, then the physical interpretation of the direction angles is not so clear.

U.A.E. Electromagnetics

46.06.02.06 ElectroMagnetic Waves in Vacuum

For ElectroMagnetic waves in vacuum the (real) angular frequency ω and the complex wave-vector, $\overline{\mathbf{k}}$, are connected, as a consequence of the wave equation, by the relation:

$$k^2 \equiv \overline{\mathbf{k}}\cdot\overline{\mathbf{k}} = \frac{\omega^2}{c^2} \Longrightarrow \omega = kc \qquad k \equiv \sqrt{\overline{\mathbf{k}}\cdot\overline{\mathbf{k}}} \quad .$$

This implies $\overline{\mathbf{k}}_{\mathrm{R}}\cdot\overline{\mathbf{k}}_{\mathrm{I}} = 0$, that is the two real vectors, $\mathbf{k}_{\mathrm{R}}$ and $\mathbf{k}_{\mathrm{I}}$, are orthogonal.

46.06.02.07 ElectroMagnetic Waves in perfect dielectric media

For ElectroMagnetic waves in prefect media the (real) angular frequency ω and the complex wave-vector, $\overline{\mathbf{k}}$, are connected, as a consequence of the wave equation, by the relation:

$$\overline{\mathbf{k}}\cdot\overline{\mathbf{k}} = \frac{\omega^2}{c^2}\varepsilon_{\mathrm{R}}\mu_{\mathrm{R}} \qquad k \equiv \sqrt{\overline{\mathbf{k}}\cdot\overline{\mathbf{k}}} \quad .$$

This implies $\overline{\mathbf{k}}_{\mathrm{R}}\cdot\overline{\mathbf{k}}_{\mathrm{I}} = 0$, that is the two real vectors, $\mathbf{k}_{\mathrm{R}}$ and $\mathbf{k}_{\mathrm{I}}$, are orthogonal. An example is given by evanescent waves (read § 48.03.06).

46.06.02.08 Dispersion Relation for perfect Ohmic media

This § is referenced at pages:
[2080]

From this section on, complex quantities will not explicitly indicated any more in systematic way, in order to simplify the notations, unless it is necessary to avoid misunderstandings. It is implicit that wave-vectors and vector fields are in general complex quantities.

One of the main interests is in determining the energy flow which is given by the Poynting vector expressed in terms of $\mathbf{H}$. Therefore in this section $\mathbf{H}$ will be mostly used instead of $\mathbf{B}$.

Consider a ElectroMagnetic MPW with complex wave-vector: $\overline{\mathbf{k}} \equiv \mathbf{k}_{\mathrm{R}} + \mathrm{i}\mathbf{k}_{\mathrm{I}}$.

The most general case of arbitrary $\mathbf{k}_{\mathrm{R}}$ and $\mathbf{k}_{\mathrm{I}}$ (non-homogeneous MPW) gives a rather cumbersome treatment and it is more general than what is needed in most cases. Therefore we will limit to the particular case, sufficient in most practical cases, of parallel $\mathbf{k}_{\mathrm{R}}$ and $\mathbf{k}_{\mathrm{I}}$:

$$\overline{\mathbf{k}} \equiv \mathbf{k}_{\mathrm{R}} + \mathrm{i}\mathbf{k}_{\mathrm{I}} \qquad \mathbf{k}_{\mathrm{R}} \parallel \mathbf{k}_{\mathrm{I}} \quad .$$

In this case one can write:

$$\overline{\mathbf{k}} \equiv \mathbf{k}_{\mathrm{R}} + \mathrm{i}\mathbf{k}_{\mathrm{I}} = \overline{k}\widehat{\boldsymbol{\kappa}} \qquad \text{with } \widehat{\boldsymbol{\kappa}} \text{ a real unit vector} \quad .$$

The real and imaginary parts of the wave-vector, $\mathbf{k}$, define the phase vector, $\mathbf{k}_{\mathrm{R}}$, and an attenuation vector $\mathbf{k}_{\mathrm{I}}$.

The phase vector has units of radians/meter and gives the direction of most rapid phase change.

The attenuation vector has units of nepers/meter and gives the direction of most rapid attenuation of amplitude.

The magnitude of the phase vector gives the phase change per unit length along the direction of the phase vector, while the magnitude of the attenuation vector determines the rate of attenuation along the direction of the attenuation vector.

Consider ElectroMagnetic MPW, implicitly suppressing the common time phase factor $\exp[-\mathrm{i}\omega t]$, expressed in the form:

$$\mathbf{E}[\mathbf{r},t] = \mathbf{E}_0 \exp[\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}] \equiv \mathbf{E}_0\Phi[\mathbf{r}] \quad ,$$

$$\mathbf{D}[\mathbf{r},t] = \mathbf{D}_0 \exp[\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}] \equiv \mathbf{D}_0\Phi[\mathbf{r}] \quad ,$$

$$\mathbf{B}[\mathbf{r},t] = \mathbf{B}_0 \exp[\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}] \equiv \mathbf{B}_0\Phi[\mathbf{r}] \quad ,$$

$$\mathbf{H}[\mathbf{r},t] = \mathbf{H}_0 \exp[\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}] \equiv \mathbf{H}_0\Phi[\mathbf{r}] \quad .$$

The complex vectors $\mathbf{E}_0$, $\mathbf{B}_0$, $\mathbf{D}_0$, $\mathbf{H}_0$ define the **wave-Polarization** of the wave.

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As we are dealing with a **perfect Ohmic medium**, we have the constitutive equations:

$$\boxed{\mathbf{j}^{\mathrm{F}}[\omega]=\sigma[\omega]\mathbf{E}[\omega]} \qquad \boxed{\mathbf{D}[\omega]=\varepsilon_0\varepsilon_{\mathrm{R}}[\omega]\mathbf{E}[\omega]} \qquad \boxed{\mathbf{B}[\omega]=\mu_0\mu_{\mathrm{R}}[\omega]\mathbf{H}[\omega]} \quad , \tag{46.06.11}$$

→ 2090

in Fourier space, including, in the more general case, also **time dispersion**.

Actually, if also **space dispersion** in accounted for:

$$\boxed{\mathbf{j}^{\mathrm{F}}[\mathbf{k},\omega]=\sigma[\mathbf{k},\omega]\mathbf{E}[\mathbf{k},\omega]} \qquad \boxed{\mathbf{D}[\mathbf{k},\omega]=\varepsilon_0\varepsilon_{\mathrm{R}}[\mathbf{k},\omega]\mathbf{E}[\mathbf{k},\omega]} \qquad \boxed{\mathbf{B}[\mathbf{k},\omega]=\mu_0\mu_{\mathrm{R}}[\mathbf{k},\omega]\mathbf{H}[\mathbf{k},\omega]} \quad . \tag{46.06.12}$$

Read § 21.04.

Replacing the ElectroMagnetic MPW for the **E/B/D/H** fields into the dissipative **D'Alembert non-dispersive wave-equation** , equations (46.06.01) (46.06.02) and the similar for the auxiliary fields, one finds that is satisfied if and only if the complex wave-vector satisfies:

$$\boxed{|\mathbf{k}|^2 \neq \mathbf{k}\cdot\mathbf{k} \equiv k^2 = \varepsilon_{\mathrm{R}}\mu_{\mathrm{R}}\varepsilon_0\mu_0\omega^2 + \mathrm{i}\omega\mu_{\mathrm{R}}\mu_0\sigma \qquad k \equiv \mathfrak{Re}\,[k] + \mathrm{i}\,\mathfrak{Im}\,[k] \qquad \mathfrak{Im}\,[k] > 0} \quad , \tag{46.06.13}$$

that is the **Dispersion Relation** for these waves.

Ohm law, $\mathbf{j}^{\mathrm{F}} = \sigma\mathbf{E}$, can be also accounted for together with the **Electric-Permittivity** , ε_{R}, which is often is a way to simplify the expressions, as shown in equation (46.06.02.08). In fact, it is clear that in static conditions there is a qualitative difference between the dynamics of free and bound charges but, when the fields oscillate harmonically, the difference becomes a quantitative one only.

The **conductivity** can be included in the **Electric-Permittivity** , as described in detail in section § 54.02, by replacing the **Electric-Permittivity** , ε_{R}, which does not include the contribution of free charges, with the **Electric-Permittivity** , $\varepsilon'_{\mathrm{R}}$, which also includes the effect of free charges, according to equation (53.01.10), so that:

$$\boxed{\overline{\varepsilon}_{\mathrm{R}} = \varepsilon_{\mathrm{R}} + \mathrm{i}\frac{\sigma}{\omega\varepsilon_0}} \quad , \tag{46.06.14}$$

→ 2090

$$\boxed{|\mathbf{k}|^2 \neq \mathbf{k}\cdot\mathbf{k} = k^2 = \varepsilon_{\mathrm{R}}\mu_{\mathrm{R}}\varepsilon_0\mu_0\omega^2 + \mathrm{i}\omega\mu_{\mathrm{R}}\mu_0\sigma \equiv \overline{\varepsilon}_{\mathrm{R}}\mu_{\mathrm{R}}\varepsilon_0\mu_0\omega^2} \quad , \tag{46.06.14}$$

→ 2090

$$\boxed{k = \frac{\omega}{c}\sqrt{\overline{\varepsilon}_{\mathrm{R}}\mu_{\mathrm{R}}} \equiv \mathfrak{Re}\,[k] + \mathrm{i}\,\mathfrak{Im}\,[k] \qquad \mathfrak{Im}\,[k] > 0} \quad . \tag{46.06.14}$$

It is important to be careful with the signs of $\mathfrak{Re}\,[k]$ and $\mathfrak{Im}\,[k]$, due to different conventions in the literature. Some authors, in fact, use a convention such that:

$$\cancel{k \equiv \mathfrak{Re}\,[k] - \mathrm{i}\,\mathfrak{Im}\,[k]} \qquad \boxed{k \equiv \mathfrak{Re}\,[k] + \mathrm{i}\,\mathfrak{Im}\,[k]} \quad .$$

Starting from the convention for the complex notation, equations (46.01.10), an attenuating wave requires a positive $\mathfrak{Im}\,[k]$.

A further extension is implicit from the constitutive relations (46.06.11), where all parameters

$$\boxed{\sigma[\omega] \longrightarrow \overline{\sigma}[\omega] \qquad \varepsilon_{\mathrm{R}}[\omega] \longrightarrow \overline{\varepsilon_{\mathrm{R}}}[\omega] \qquad \mu_{\mathrm{R}}[\omega] \longrightarrow \overline{\mu_{\mathrm{R}}}[\omega]} \quad , \tag{46.06.15}$$

are, a priori, complex functions of the frequency, as motivated by equation (46.06.11), by moving **from the time-domain to the frequency domain** with the **Fourier transform**.

At the end, it is worth recalling the logical steps followed:

1. One starts with LHI media with parameters, ε_{R}, μ_{R}, σ, all fixed real numbers, bounded from below.
2. The **Dispersion Relation** , equation (46.06.02.08), implies that also with the above conditions, the wave-vector can become complex and frequency-dependent; imaginary just means that there is a time phase-shift between the time-harmonic field vectors linked by ε_{R}, μ_{R}, σ.
3. Equation (46.06.02.08) actually makes $\varepsilon'_{\mathrm{R}}$ complex and frequency-dependent.
4. Given all that we are suggested that all three parameters, ε_{R}, μ_{R}, σ, can be generalized to complex and frequency-dependent parameters, because Maxwell equations themselves are producivn complex and frequency-dependent parameters startgin from fixed real ones.

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46.06.02.09 Consequences From Maxwell Equations

This § is referenced at pages:
[Never referenced.]

$$\boxed{\mathbf{E}[\mathbf{x},t]=\mathbf{E}_0\cos\left[\mathbf{k}\cdot\mathbf{x}-\omega t-\phi\right]} \quad ,$$
$$\boxed{\mathbf{H}[\mathbf{x},t]=\mathbf{H}_0\cos\left[\mathbf{k}\cdot\mathbf{x}-\omega t-\phi\right]} \quad .$$

By replacing the field expressions in **Maxwell Equations** one finds that are still valid, also for complex $\mathbf{k}$, the following equations, even if there are products: in fact, they **come from Maxwell Equations by direct algebraic calculation**:

$$\boxed{\mathbf{k}\cdot\mathbf{E}_0=0} \quad ,$$
$$\boxed{\mathbf{k}\cdot\mathbf{H}_0=0} \quad ,$$
$$\boxed{\mathbf{k}\wedge\mathbf{H}_0=-\left(\mathrm{i}\sigma[\omega]+\omega\varepsilon_0\varepsilon_{\mathrm{R}}[\omega]\right)\mathbf{E}_0} \quad ,$$
$$\boxed{\mathbf{k}\wedge\mathbf{E}_0=+\mu_0\mu_{\mathrm{R}}\omega\mathbf{H}_0} \quad .$$

The above equations can then be cast in the more suitable and symmetrical form:

$$\boxed{\mathbf{k}\cdot\mathbf{E}_0=0} \quad , \qquad (46.06.16) \quad \rightarrow 2091\ 2092\ 2093$$
$$\boxed{\mathbf{k}\cdot\mathbf{H}_0=0} \quad , \qquad (46.06.17) \quad \rightarrow 2091\ 2092\ 2093$$
$$\boxed{\mathbf{k}\wedge\mathbf{E}_0=+\omega\mu_0\mu_{\mathrm{R}}\mathbf{H}_0} \quad , \qquad (46.06.18) \quad \rightarrow 2091\ 2092\ 2093$$
$$\boxed{\mathbf{k}\wedge\mathbf{H}_0=-\omega\varepsilon_0\overline{\varepsilon}_{\mathrm{R}}\mathbf{E}_0} \quad . \qquad (46.06.19) \quad \rightarrow 2091\ 2092\ 2093$$

The real and imaginary parts of the wave-vector are real vectors, not necessarily orthogonal, at variance from vacuum and **perfect dielectric media**.

46.06.02.10 Complex Wave Impedance and Wave Admittance

This § is referenced at pages:
[Never referenced.]

The above equations (46.06.16), (46.06.17), (46.06.18), (46.06.19), imply that $\mathbf{E}_0$, $\mathbf{H}_0$ **and** $\mathbf{k}$ **form an orthogonal set of complex vectors**.

By scalar multiplying (46.06.18) by $\mathbf{H}_0$, scalar multiplying (46.06.19) by $\mathbf{E}_0$ and comparing the results, the definitions of the **complex wave impedance** $\overline{Z}$ and **complex wave admittance** $\overline{Y}$ is possible:

$$\boxed{\mathbf{E}_0\cdot\mathbf{E}_0\equiv\overline{Z}^2\,\mathbf{H}_0\cdot\mathbf{H}_0 \qquad \overline{Z}\equiv\sqrt{\frac{\mu_0\mu_{\mathrm{R}}}{\varepsilon_0\overline{\varepsilon}_{\mathrm{R}}}} \qquad \text{all complex, } \overline{Z} \text{ chosen so that } \Re\mathfrak{e}\left[\overline{Z}\right]\geq 0} \quad , \qquad (46.06.20) \quad \rightarrow 2091\ 2092\ 2342\ 2343$$

$$\boxed{\mathbf{H}_0\cdot\mathbf{H}_0\equiv\overline{Y}^2\,\mathbf{E}_0\cdot\mathbf{E}_0 \qquad \overline{Y}\equiv\sqrt{\frac{\varepsilon_0\overline{\varepsilon}_{\mathrm{R}}}{\mu_0\mu_{\mathrm{R}}}} \qquad \text{all complex, } \overline{Y} \text{ chosen so that } \Re\mathfrak{e}\left[\overline{Y}\right]\geq 0} \quad . \qquad (46.06.21) \quad \rightarrow 2091\ 2092$$

Equation (46.06.20) implies:

$$\boxed{E_0^2\equiv\mathbf{E}_0\cdot\mathbf{E}_0=\overline{Z}^2\,\mathbf{H}_0\cdot\mathbf{H}_0\equiv\overline{Z}^2H_0^2\implies E_0=\overline{Z}H_0 \qquad \text{with } E_0,\ H_0 \text{ and } \overline{Z} \text{ complex numbers}} \quad .$$

The **wave impedance** is thus defined as:

$$\boxed{\overline{Z}\equiv\frac{E_0}{H_0}} \quad . \qquad (46.06.22) \quad \rightarrow 2187$$

Equation (46.06.21) implies:

$$\boxed{H_0^2\equiv\mathbf{H}_0\cdot\mathbf{H}_0=\overline{Y}^2\,\mathbf{E}_0\cdot\mathbf{E}_0\equiv\overline{Y}^2E_0^2\implies H_0=\overline{Y}E_0 \qquad \text{with } H_0,\ E_0 \text{ and } \overline{Y} \text{ complex numbers}} \quad .$$

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The **wave admittance** is thus defined as:

$$\boxed{\overline{Y} \equiv \frac{H_0}{E_0}} \quad . \tag{46.06.23}$$

For homogeneous MPW only, equations (46.06.16), (46.06.17), (46.06.18), (46.06.19), imply that:

$$\boxed{|\mathbf{E}_0| = |\overline{Z}||\mathbf{H}_0| \qquad |\mathbf{H}_0| = |\overline{Y}||\mathbf{E}_0|} \quad . \tag{46.06.24}$$

→ 2092 2154

Read § 46-029. All MPW obey equations (46.06.20), (46.06.21), which is not in general the same as equation (46.06.24) because for a complex vector, in general,

$$|\mathbf{V}| \quad \text{is a real number} \quad ,$$
$$\sqrt{\mathbf{V}\cdot\mathbf{V}} \quad \text{is a complex number} \quad ,$$
$$\boxed{\sqrt{\mathbf{V}\cdot\mathbf{V}^\star} \equiv |\mathbf{V}| \neq \sqrt{\mathbf{V}\cdot\mathbf{V}} = \sqrt{\mathbf{V}^\star\cdot\mathbf{V}^\star}} \quad . \tag{46.06.25}$$

Any homogeneous MPW can be decomposed into a sum of two PW, with electric field vectors wave-Polarized perpendicular to each other, and with electric field vectors that are real, to within a multiplicative complex constant. This follows from a simple rotation of coordinates: calling x and y the direction of the two electric field vectors one has, for a wave propagating in the positive z direction:

$$\boxed{\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \overline{Z} \qquad \frac{H_y}{E_x} = -\frac{H_x}{E_y} = \overline{Y}} \quad .$$

EM homogeneous MPW have real direction angles and obey

$$\boxed{|\mathbf{E}| = |\overline{Z}||\mathbf{H}|} \quad , \tag{46.06.26}$$

which is more than equation (46.06.20), (46.06.21) obeyed by any ElectroMagnetic MPW.

46.06.03
Energy Flow

The average energy flux for a MPW is given by any of the two equivalent equations:

$$\boxed{\langle \mathbf{S}_{\text{EM}} \rangle = \langle \Re\mathfrak{e}\,[\,\mathbf{E}\wedge\mathbf{H}^\star]\rangle = \frac{1}{2}\Re\mathfrak{e}\,[\,\mathbf{E}_0\wedge\mathbf{H}_0^\star]} \quad , \tag{46.06.27}$$

→ 2092

$$\boxed{\langle \mathbf{S}_{\text{EM}} \rangle = \langle \Re\mathfrak{e}\,[\,\mathbf{E}^\star\wedge\mathbf{H}]\rangle = \frac{1}{2}\Re\mathfrak{e}\,[\,\mathbf{E}_0^\star\wedge\mathbf{H}_0]} \quad . \tag{46.06.28}$$

→ 2093

46.06.03.01 Energy Flow in Terms of the Electric Field

In the most common case when either $\mathbf{k}$ or $\mathbf{E}_0$ are proportional to a real vector (that is all the components of the complex vector have the same phase), one has:

$$\mathbf{k}^\star\cdot\mathbf{E}_0 = 0 \qquad \Leftrightarrow \qquad \mathbf{k}\cdot\mathbf{E}_0^\star = 0 \quad ,$$

so that equations (46.06.16), (46.06.17), (46.06.18), (46.06.19), by developing the triple vector product, allows to write (46.06.27) as:

$$\boxed{\langle \mathbf{S}_{\text{EM}} \rangle = \frac{1}{2}\Re\mathfrak{e}\,[\,\mathbf{E}_0\wedge\mathbf{H}_0^\star] = \frac{1}{2\omega}\Re\mathfrak{e}\left[\frac{1}{\mu_0\mu_{\text{R}}{}^\star}|\mathbf{E}_0|^2\mathbf{k}^\star\right] = \frac{1}{2\omega}\Re\mathfrak{e}\left[\frac{1}{\mu_0\mu_{\text{R}}}|\mathbf{E}_0|^2\mathbf{k}\right]} \quad ,$$

where the last passage follows from the fact the real part of a complex number and the real part of its complex conjugate are identical.

It can be also rewritten as:

$$\boxed{\langle \mathbf{S}_{\text{EM}} \rangle = \frac{1}{2\omega}\Re\mathfrak{e}\left[\frac{1}{\mu_0\mu_{\text{R}}}|\mathbf{E}_0|^2\mathbf{k}\right] = \frac{1}{2\omega}|\mathbf{E}_0|^2\,\Re\mathfrak{e}\left[\frac{k}{\mu_0\mu_{\text{R}}}\right]\widehat{\boldsymbol{\kappa}} = \frac{1}{2\omega}|\mathbf{E}_0|^2\,\Re\mathfrak{e}\left[\frac{\omega\sqrt{\varepsilon_0\mu_0\overline{\varepsilon}_{\text{R}}\mu_{\text{R}}}}{\mu_0\mu_{\text{R}}}\right]\widehat{\boldsymbol{\kappa}} = \frac{1}{2}|\mathbf{E}_0|^2\,\Re\mathfrak{e}\,[Y]\,\widehat{\boldsymbol{\kappa}}} \quad .$$

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46.06.03.02 Energy Flow in Terms of the Magnetic Field

In the most common case when either $\mathbf{k}$ or $\mathbf{H}_0$ are proportional to a real vector (that is all the components of the complex vector have the same phase), one has:

$$\mathbf{k}^\star \cdot \mathbf{H}_0 = 0 \qquad \Leftrightarrow \qquad \mathbf{k} \cdot \mathbf{H}_0^\star = 0 \quad ,$$

so that equations (46.06.16), (46.06.17), (46.06.18), (46.06.19), by developing the triple vector product, allows to write (46.06.28) as:

$$\boxed{\langle \mathbf{S}_{\text{EM}} \rangle = \frac{1}{2} \mathfrak{Re} \left[\mathbf{H}_0 \wedge \mathbf{E}_0^\star \right] = \frac{1}{2\omega} \mathfrak{Re} \left[\frac{1}{\varepsilon_0 \bar{\varepsilon}_{\text{R}}{}^\star} |\mathbf{H}_0|^2 \mathbf{k}^\star \right] = \frac{1}{2\omega} \mathfrak{Re} \left[\frac{1}{\varepsilon_0 \bar{\varepsilon}_{\text{R}}} |\mathbf{H}_0|^2 \mathbf{k} \right]} \quad ,$$

where the last passage follows from the fact the real part of a complex number and the real part of its complex conjugate are identical.

It can be also rewritten as:

$$\boxed{\langle \mathbf{S}_{\text{EM}} \rangle = \frac{1}{2\omega} \mathfrak{Re} \left[\frac{1}{\varepsilon_0 \bar{\varepsilon}_{\text{R}}} |\mathbf{H}_0|^2 \mathbf{k} \right] = \frac{1}{2\omega} |\mathbf{H}_0|^2 \mathfrak{Re} \left[\frac{k}{\varepsilon_0 \bar{\varepsilon}_{\text{R}}} \right] \widehat{\boldsymbol{\kappa}} = \frac{1}{2\omega} |\mathbf{H}_0|^2 \mathfrak{Re} \left[\frac{\omega \sqrt{\varepsilon_0 \mu_0 \bar{\varepsilon}_{\text{R}} \mu_{\text{R}}}}{\varepsilon_0 \bar{\varepsilon}_{\text{R}}} \right] \widehat{\boldsymbol{\kappa}} = \frac{1}{2} |\mathbf{H}_0|^2 \mathfrak{Re} \left[Z \right] \widehat{\boldsymbol{\kappa}}} \quad .$$

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PLANE WAVE PROPAGATION AND REFLECTION

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Abstract

The basic properties of plane waves propagating in a homogeneous isotropic region are reviewed, for both lossy and lossless media. These basic properties include the nature of the electric and magnetic fields, the properties of the wavenumber vector, and the power flow of the plane wave. Special attention is given to the specific case of a homogenous (uniform) plane wave, i.e., one having real direction angles, since this case is most often met in practice (for example, in the far-field of an antenna). Properties such as wavelength, phase and group velocity, penetration depth, and polarization are discussed for these plane waves.

The theory of reflection of plane waves from layered media is then discussed in a unified manner, using a transmission line model called the transverse equivalent network. This method decomposes the incident plane wave into TM and TE parts, and models each separately using a transmission line equivalent circuit. Basic reflection phenomena such as the law of reflection, Snell's law and the Brewster angle effect come directly from this model. More involved reflection problems are also easily treated using this approach. As an example, the reflected and transmitted fields due to a circular-polarized wave incident on the surface of the ocean are calculated.

1. INTRODUCTION

Planes waves are the simplest solution of Maxwell's equations in a homogeneous region of space, such as free-space (vacuum). In spite of their simplicity, plane waves have played an important role throughout the development of electromagnetics, starting from the time of the earliest radio transmissions through the development of modern communications systems. Plane waves are important for several reasons. First, the far-field radiation from any transmitting antenna has the characteristics of a plane wave sufficiently far from the antenna. The incoming wavefield impinging on a receiving antenna can therefore usually be approximated as a plane wave. Second, the exact field radiated by any source in a region of space can be constructed in terms of a continuous spectrum of plane waves via the Fourier transform. Understanding the nature of plane waves is thus important for understanding both the far-field and the exact radiation from sources.

The theory of plane wave reflection from layered media is also a well-developed area, and relatively simple expressions suffice for understanding reflection and transmission effects when layers are present. Problems involving reflections from the earth or sea, for example, are easily treated using plane-wave theory. Even when the incident wavefront is actually spherical is shape, as from a transmitting antenna, plane-wave theory may often be approximately used with accurate results.

Throughout this article, it will be assumed that the regions of interest are homogeneous (the material properties are constant) and isotropic, which covers most cases of practical interest.

2. BASIC PROPERTIES OF A PLANE WAVE

Definition of a plane wave

The most general definition of a plane wave is an electromagnetic field having the form

$$\boldsymbol{E} = \boldsymbol{E_0}\,\psi(x,y,z) \tag{1}$$

$$\boldsymbol{H} = \boldsymbol{H_0}\,\psi(x,y,z) \tag{2}$$

where $\boldsymbol{E_0}$ and $\boldsymbol{H_0}$ are constant vectors, and the wavefunction ψ is defined as

$$\begin{aligned}\psi(x,y,z) &= e^{-j\left(k_x x + k_y y + k_z z\right)} \\ &= e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}\end{aligned} \tag{3}$$

where

$$\boldsymbol{k} = \hat{\boldsymbol{x}}k_x + \hat{\boldsymbol{y}}k_y + \hat{\boldsymbol{z}}k_z = \boldsymbol{\beta} - j\boldsymbol{\alpha} \tag{4}$$

$$\boldsymbol{r} = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z\ , \tag{5}$$

with k_x, k_y, and k_z being complex constants that define a wavenumber vector $\boldsymbol{k}$. (A time-harmonic dependence of $e^{j\omega t}$ is assumed and suppressed.) The vector $\boldsymbol{E_0}$ defines the polarization of the plane wave. The real and imaginary parts of the wavenumber vector $\boldsymbol{k}$ define the phase vector $\boldsymbol{\beta}$ and an attenuation vector $\boldsymbol{\alpha}$. The phase vector has units of radians/meter and gives the direction of most rapid phase change, while the attenuation vector has units of nepers/meter and gives the direction of most rapid attenuation. The magnitude of the phase vector gives the phase change per unit length along the direction of the phase vector, while the magnitude of the attenuation vector determines the rate of attenuation along the direction of the attenuation vector.

Basic properties

In a homogeneous lossless space a plane wave must satisfy Maxwell's equations, which in the time-harmonic form are [1]

$$\nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} \tag{6}$$

$$\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \tag{7}$$

$$\nabla \cdot \boldsymbol{E} = \boldsymbol{0} \tag{8}$$

$$\nabla \cdot \boldsymbol{H} = \boldsymbol{0} \tag{9}$$

where ε and μ are the permittivity and permeability of the space. For free space, $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$, where μ_0 is defined to be $4\pi \times 10^{-7}$ Henry/meter and ε_0 is determined from the *defined* velocity of light (or any plane wave) in vacuum [2], $c = 2.99792458 \times 10^8$ meters/second, since $c = 1/\sqrt{\varepsilon_0 \mu_0}$. This gives the approximate value $\varepsilon_0 = 8.85418781762039 \times 10^{-12}$ Farads/meter. A lossy medium can be modeled using a complex effective permittivity, accounting for conduction loss and/or polarization loss [1]. The complex effective permittivity is expressed as

$$\varepsilon = \hat{\varepsilon} - j\left(\frac{\sigma}{\omega}\right). \tag{10}$$

In this equation $\hat{\varepsilon}$ is the complex permittivity of the material, accounting for polarization loss (if any), and σ is the conductivity of the medium.

Taking the curl of Eq. (7) and then substituting in Eq. (6) gives the vector wave equation

$$\nabla \times (\nabla \times \boldsymbol{E}) - k^2 \boldsymbol{E} = \boldsymbol{0}, \tag{11}$$

where k is the (possibly complex) wavenumber defined by

$$k = k' - jk'' = \omega\sqrt{\mu\varepsilon}, \tag{12}$$

with the square root chosen so that k lies in the fourth quadrant on the complex plane. Using the definition of the vector Laplacian,

$$\nabla^2 \boldsymbol{E} \equiv \nabla(\nabla \cdot \boldsymbol{E}) - \nabla \times (\nabla \times \boldsymbol{E}), \tag{13}$$

and the fact that the divergence of the electric field is zero for a time-harmonic field in a homogeneous region [1], results in the vector Helmholtz equation

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = \boldsymbol{0}. \tag{14}$$

In rectangular coordinates, the vector Laplacian is expressed as

$$\nabla^2 \boldsymbol{E} = \hat{\boldsymbol{x}} \nabla^2 E_x + \hat{\boldsymbol{y}} \nabla^2 E_y + \hat{\boldsymbol{z}} \nabla^2 E_z. \tag{15}$$

Hence, all three rectangular components of the electric field satisfy the scalar Helmholtz equation

$$\nabla^2 \Psi + k^2 \Psi = 0 \tag{16}$$

in a homogeneous region.

Substituting Eq. (3) into Eq. (16) gives the result

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

or

$$\boldsymbol{k} \cdot \boldsymbol{k} = k^2. \tag{17}$$

This is the "separation equation" that relates the components of the wavenumber vector $\boldsymbol{k}$ defined in Eq. (4). Note that the term on the left side of Eq. (17) is not in general equal to $|\boldsymbol{k}|^2$, since $\boldsymbol{k}$ may be complex.

Other fundamental relations for a plane wave may be found by substituting Eq. (1) and (2) into Maxwell's equations (6)-(9). Noting that $\nabla \rightarrow -j\boldsymbol{k}$ for a plane wave, Maxwell's equations reduce to

$$\boldsymbol{k} \times \boldsymbol{H} = -\omega\varepsilon \boldsymbol{E} \tag{18}$$

$$\boldsymbol{k} \times \boldsymbol{E} = \omega\mu \boldsymbol{H} \tag{19}$$

$$\boldsymbol{k} \cdot \boldsymbol{E} = 0 \tag{20}$$

$$\boldsymbol{k} \cdot \boldsymbol{H} = 0 . \tag{21}$$

Equations (18) and (19) each imply that $\boldsymbol{E} \cdot \boldsymbol{H} = 0$, and together they also imply that $\boldsymbol{k} \cdot \boldsymbol{E} = 0$ and $\boldsymbol{k} \cdot \boldsymbol{H} = 0$. That is, all three vectors $\boldsymbol{k}$, $\boldsymbol{E}$, and $\boldsymbol{H}$ are mutually orthogonal. Another interesting property that is true for any plane wave, which may be derived directly from Eq. (18) or (19), is that

$$\boldsymbol{E} \cdot \boldsymbol{E} = \eta^2 \, \boldsymbol{H} \cdot \boldsymbol{H} , \tag{22}$$

where η is the intrinsic impedance of the space (possibly complex), defined from

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{23}$$

(the principle branch of the square root is chosen so that the real part of η is nonnegative). For vacuum, the intrinsic impedance is often denoted as η_0, and has a value of approximately 376.7303 Ω.

Power flow

The complex Poynting vector for a plane wave, giving the complex power flow, is (assuming peak notation for phasors)

$$\boldsymbol{S} = \frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^* . \tag{24}$$

Using Eq. (19), the Poynting vector for a plane wave can be written as

$$\boldsymbol{S} = \frac{1}{2\omega\mu^*}|\psi|^2 \boldsymbol{E_0} \times \left(\boldsymbol{k}^* \times \boldsymbol{E_0^*}\right). \tag{25}$$

Using the triple product rule $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$, this can be rewritten as

$$\boldsymbol{S} = \frac{1}{2\omega\mu^*}|\psi|^2 |\boldsymbol{E_0}|^2 \boldsymbol{k}^* - \frac{1}{2\omega\mu^*}|\psi|^2 \left(\boldsymbol{E_0} \cdot \boldsymbol{k}^*\right)\boldsymbol{E_0^*}. \tag{26}$$

The second term in the above equation is not always zero for an arbitrary plane wave, even though $\boldsymbol{E_0} \cdot \boldsymbol{k} = 0$, since $\boldsymbol{k}$ may be complex in the most general case. If either $\boldsymbol{k}$ or $\boldsymbol{E_0}$ is proportional to a real vector (i.e., all of the components of the vector have the same phase angle), then it is easily demonstrated that the second term vanishes. In this case the Poynting vector becomes

$$\boldsymbol{S} = \frac{1}{2\omega\mu^*}|\psi|^2 |\boldsymbol{E_0}|^2 \boldsymbol{k}^* . \tag{27}$$

If the medium is also lossless (μ is real), the time-average power flow (coming from the real part of the complex Poynting vector) is in the direction of the phase vector $\boldsymbol{\beta}$. A similar derivation, casting the Poynting vector in terms of the $\boldsymbol{H_0}$ vector, yields

$$\boldsymbol{S} = \frac{1}{2\omega\varepsilon}|\psi|^2 |\boldsymbol{H_0}|^2 \boldsymbol{k}, \tag{28}$$

provided either $\boldsymbol{k}$ or $\boldsymbol{H_0}$ is proportional to a real vector. If the region is lossless (ε is real), the time-average power flow is then in the direction of the phase vector. Hence, for a lossless region, the time-average power flow is in the direction of the phase vector $\boldsymbol{\beta}$ if any of the three vectors

$\boldsymbol{k}$, $\boldsymbol{E_0}$, or $\boldsymbol{H_0}$ is proportional to a real vector. In many practical cases of interest, one of the three vectors will be proportional to a real vector, and hence the conclusion will be valid. However, it is always possible to find exceptions, even for free space. One such example is the plane wave defined by the vectors $\boldsymbol{k} = (1, j, 1)$, $\boldsymbol{E_0} = (2, 1, -2-j)$, and $\boldsymbol{H_0} = (1/(\omega\mu))(-2j, 4+j, 1-2j)$, at a frequency $\omega = c$ ($k_0 = 1$). This plane wave satisfies Maxwell's equations (18)-(21). However for this plane wave the vectors $\boldsymbol{\beta}$, $\boldsymbol{\alpha}$, and $\boldsymbol{p} = \mathrm{Re}(\boldsymbol{S})$ are all in different directions, since the power flow is in the direction of the vector $(5, 0, 4)$ One must then be careful to define what is meant by the "direction of propagation" for such a plane wave.

Direction angles

One way to characterize a plane wave is through direction angles (θ, ϕ) in spherical coordinates. The direction angles (which are in general complex) are defined from the relations

$$k_x = k \sin\theta \cos\phi \tag{29}$$

$$k_y = k \sin\theta \sin\phi \tag{30}$$

$$k_z = k \cos\theta . \tag{31}$$

Homogeneous (uniform) plane wave

One important class of plane waves is the class of *homogeneous* or *uniform* plane waves. A homogeneous plane wave is one for which the direction angles are, by definition, real. A

homogeneous plane wave enjoys certain special properties that are not true in general for all plane waves. For such a plane wave the wavenumber vector can be written as

$$\boldsymbol{k} = k\,\hat{\boldsymbol{R}} = \hat{\boldsymbol{R}}\left(k' - jk''\right), \tag{32}$$

where $\hat{\boldsymbol{R}}$ is a real unit vector defined from

$$\hat{\boldsymbol{R}} = \hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta\,. \tag{33}$$

In this case the $\boldsymbol{k}$ vector is proportional to the real vector $\hat{\boldsymbol{R}}$, so the result of Eq. (27) applies. The unit vector $\hat{\boldsymbol{R}}$ then gives the direction of time-average power flow, and also points in the direction of the phase and attenuation vectors. That is, all three vectors point in the same direction for a homogeneous plane wave. This direction is, unambiguously, the direction of propagation of the plane wave. The planes of constant phase are also then the planes of constant amplitude, being the planes perpendicular to the $\hat{\boldsymbol{R}}$ vector. That is, the plane wave has a uniform amplitude across the plane perpendicular to the direction of propagation. If the plane wave is not homogeneous, corresponding to complex direction angles, then the physical interpretation of the direction angles is not clear.

From Eq. (18) or (19) it may be easily proven that the fields of a homogeneous plane wave obey the relation

$$|\boldsymbol{E}| = |\eta||\boldsymbol{H}|\,. \tag{34}$$

(Recall that all plane waves obey Eq. (22), which is not in general the same as the above equation.)

Any homogeneous plane wave can be decomposed into a sum of two plane waves, with electric field vectors polarized perpendicular to each other, and with electric field vectors that are real, to within a multiplicative complex constant. This follows from a simple rotation of coordinates (the direction of propagation is then z', with electric field vectors in the x' and y' directions. the two plane wave then have the fields $\left(E_x, H_y\right)$ and $\left(E_y, H_x\right)$, with $E_x / H_y = -E_y / H_x = \eta$. This decomposition is used later in the discussion of polarization.

Lossless media

Another important special case is when the medium is lossless, so that $k'' = 0$. If Eq. (4) is substituted into the separation equation (17), the imaginary part of this equation immediately yields the relation

$$\boldsymbol{\beta} \cdot \boldsymbol{\alpha} = 0 . \tag{35}$$

Hence, for a lossless region, the phase and attenuation vectors are always perpendicular. In some applications (e.g., a Fourier transform solution of radiation from an aperture in a ground plane at $z = 0$, or from a planar current source at $z = 0$ [3]), a plane wave propagating in a lossless region has the characteristic that two of the wavenumbers, e.g., k_x and k_y, are real (corresponding to the transform variables). In this case the third wavenumber k_z will be real if $k_x^2 + k_y^2 < k^2$ and will be imaginary if $k_x^2 + k_y^2 > k^2$. That is, all transverse wavenumbers $\left(k_x, k_y\right)$ that lie within a circle of radius k in the wavenumber plane will be propagating, while all wavenumber outside the circle will be evanescent. In the first case the power flow is in the direction of the (real) $\boldsymbol{k}$ vector $\left(k_x, k_y, k_z\right)$, so that power leaves the aperture from this plane wave. In the second case

the power flow is in the direction of the transverse wavenumber vector $\left(k_x, k_y, 0\right)$, so that no power leaves the aperture for this plane wave component. If the medium is complex, there is no sharp distinction between propagating and evanescent plane waves. In this case all plane waves carry power, in the direction of the vector $\left(k_x, k_y, \operatorname{Im} k_z\right)$.

Finally, it can be noted that if a homogeneous plane wave is propagating in a lossless region, then the attenuation vector $\boldsymbol{\alpha}$ must be zero, and all wavenumber components k_x, k_y, and k_z are real.

Summary of Basic Properties

A summary of the basic properties of a plane wave, discussed in the previous sections, is given in Table 1.

Table 1. Summary of the basic properties of a plane wave.

	INHOMOGENEOUS		HOMOGENEOUS	
PROPERTY	**Lossy**	**Lossless**	**Lossy**	**Lossless**
Direction angles	complex	complex	real	real
Relation between phase and attenuation vectors	$\boldsymbol{\beta}\cdot\boldsymbol{\alpha} = k'k''$	$\boldsymbol{\beta}\cdot\boldsymbol{\alpha} = 0$	$\boldsymbol{\beta} \parallel \boldsymbol{\alpha}$	$\boldsymbol{\alpha} = \boldsymbol{0}$
Impedance relations	$\boldsymbol{E}\cdot\boldsymbol{E} = \eta^2 \boldsymbol{H}\cdot\boldsymbol{H}$	$\boldsymbol{E}\cdot\boldsymbol{E} = \eta^2 \boldsymbol{H}\cdot\boldsymbol{H}$	$\boldsymbol{E}\cdot\boldsymbol{E} = \eta^2 \boldsymbol{H}\cdot\boldsymbol{H}$ $\lvert\boldsymbol{E}\rvert = \lvert\eta\rvert\lvert\boldsymbol{H}\rvert$	$\boldsymbol{E}\cdot\boldsymbol{E} = \eta^2 \boldsymbol{H}\cdot\boldsymbol{H}$ $\lvert\boldsymbol{E}\rvert = \eta\lvert\boldsymbol{H}\rvert$
Direction of power flow vector	not necessarily in $\boldsymbol{\beta}$ direction	$\boldsymbol{\beta}$ if $\boldsymbol{k}$, $\boldsymbol{E_0}$ or $\boldsymbol{H_0}$ is proportional to a real vector	$\boldsymbol{\beta}$	$\boldsymbol{\beta}$
Wavenumber vector	$\boldsymbol{k}\cdot\boldsymbol{k} = k^2$	$\boldsymbol{k}\cdot\boldsymbol{k} = k^2$	$\boldsymbol{k}\cdot\boldsymbol{k} = k^2$ $\boldsymbol{k}\cdot\boldsymbol{k}^* = \lvert k^2\rvert$	$\boldsymbol{k}\cdot\boldsymbol{k} = k^2$ $\boldsymbol{k}\cdot\boldsymbol{k}^* = k^2$
Orthogonality of vectors	$\boldsymbol{E}\cdot\boldsymbol{H} = 0$ $\boldsymbol{E}\cdot\boldsymbol{k} = 0$ $\boldsymbol{H}\cdot\boldsymbol{k} = 0$	$\boldsymbol{E}\cdot\boldsymbol{H} = 0$ $\boldsymbol{E}\cdot\boldsymbol{k} = 0$ $\boldsymbol{H}\cdot\boldsymbol{k} = 0$	$\boldsymbol{E}\cdot\boldsymbol{H} = 0$ $\boldsymbol{E}\cdot\boldsymbol{k} = 0$ $\boldsymbol{H}\cdot\boldsymbol{k} = 0$	$\boldsymbol{E}\cdot\boldsymbol{H} = 0$ $\boldsymbol{E}\cdot\boldsymbol{k} = 0$ $\boldsymbol{H}\cdot\boldsymbol{k} = 0$

3. PROPAGATION OF A HOMOGENEOUS PLANE WAVE

The case of a homogeneous plane wave is important enough to warrant further attention. Far away from a transmitting antenna, the radiation field always has behaves as a homogeneous plane wave (if the spherical wavefront is approximated as planar). As mentioned, a homogeneous plane wave is one propagating with real direction angles in space, with a direction of propagation described by a real unit vector. By a suitable rotation of coordinate axis, the direction of propagation may be assumed to be along the *z* axis. Furthermore, the plane way can always be represented as a sum of two separate plane waves, one with the electric field polarized

in the *x* direction, and the other polarized in the *y* direction. Considering the wave polarized in the *x* direction, the fields are

$$E_x = E_0 e^{-jkz} \tag{36}$$

$$H_y = \frac{E_0}{\eta} e^{-jkz} . \tag{37}$$

The wavenumber *k* and the intrinsic impedance η of the space are given by Eqs. (12) and (23), respectively.

Wavelength, phase velocity, and group velocity

The wavelength λ is defined at the distance required for the plane wave to change phase by 2π radians, and is given by

$$\lambda = \frac{2\pi}{k'} . \tag{38}$$

The phase velocity of the plane wave is defined as the velocity at which a point of constant phase travels (such as the crest of the wave, where the electric field is the maximum). The phase velocity is given by

$$v_p = \frac{\omega}{k'} = \frac{\omega}{\mathrm{Re}\left(\omega\sqrt{\mu\varepsilon}\right)} . \tag{39}$$

If the permittivity ε is not a function of frequency (true for vacuum, and often approximately true for nonconducting media, over a certain frequency range), then the phase velocity is constant with frequency (the plane wave propagates without dispersion). For vacuum the phase velocity is

c = 2.99792458 × 10^8 meters/second. The group velocity (velocity of energy flow, and often a good approximation to the velocity of signal propagation [4], is given by

$$v_g = \frac{d\omega}{dk'} = \frac{1}{\frac{d}{d\omega}\mathrm{Re}\left(\omega\sqrt{\mu\varepsilon}\right)} . \tag{40}$$

For a nondispersive lossless media (constant real-valued permittivity) the group velocity is equal to the phase velocity, and both are equal to

$$v_p = v_g = \frac{1}{\sqrt{\mu\varepsilon}} . \tag{41}$$

Depth of penetration (skin depth)

For a lossy medium, the plane wave decays as it propagates, since $k'' > 0$. The depth of penetration d_p is defined as the distance required to reduce the field level by a factor of $e \approx 2.718\,28$, so that the field is 36.788 % of the starting value (the power is reduced by a factor of e^2, or is at a level of 13.534 % of the starting value). The depth of penetration is given as

$$d_p = \frac{1}{k''} = \frac{-1}{\mathrm{Im}\left(\omega\sqrt{\mu\varepsilon}\right)} . \tag{42}$$

The permittivity is complex for a lossy media, and is represented by Eq. (10). Special cases are of particular interest. For a low loss medium, $\varepsilon'' << \varepsilon'$, a simple binomial approximation of the square root in Eq. (42) gives

$$d_p = \frac{1}{k''} \approx \frac{1}{0.5\,k' \tan\delta} \tag{43}$$

where

$$k' \approx \omega\sqrt{\mu\varepsilon'} \tag{44}$$

and the loss tangent is defined as

$$\tan\delta = \frac{\varepsilon''}{\varepsilon'} . \tag{45}$$

If the loss is due purely to medium conductivity (no polarization loss, so that $\hat{\varepsilon} = \varepsilon'$ is a real number), the penetration depth formula (43) for the low-loss case becomes

$$d_p \approx \frac{2}{\sigma}\sqrt{\frac{\varepsilon'}{\mu}} . \tag{46}$$

For a highly lossy medium, $\varepsilon' << \varepsilon''$, in which case we have

$$k' \approx k'' \approx \omega\sqrt{\frac{\mu\varepsilon''}{2}} . \tag{47}$$

For a good conductor the conductivity is very large, so that $\varepsilon'' \approx \sigma/\omega$ from Eq. (10). In this case the penetration depth is more commonly referred to as the "skin depth" δ, since the value may be very small. (There is no relation to the δ symbol appearing in the loss tangent symbol.) For a good conductor,

$$\delta = d_p \approx \sqrt{\frac{2}{\omega\mu\sigma}} . \tag{48}$$

For pure copper ($\sigma = 5.8\times10^7$ S/m , $\mu \approx \mu_0$) at 2.45 GHz, $\delta = 1.335\,\mu$m .

Whether a particular material falls into the "low loss" or "highly lossy" categories may depend on frequency. For example, consider typical seawater, which has roughly $\hat{\varepsilon}_r = 78$

(ignoring polarization losses) and $\sigma = 4.0 \text{ S/m}$. Table 2 shows the loss tangent versus frequency. It is seen that the loss tangent is on the order of unity at microwave frequencies. For much lower frequencies the seawater is "highly lossy", and for much higher frequencies it is a "low loss" media. Note that the depth of penetration continues to decrease as the frequency is raised, even though the medium becomes more of a "low loss" material at higher frequencies, according to the definition used (a low loss tangent). At low frequencies the penetration depth is increasing inversely as the square root of frequency, according to Eq. (48). At high frequencies the penetration depth approaches a constant, according to Eq. (46).

Table 2. Loss tangent and penetration depth for typical seawater, versus frequency.

frequency (GHz)	**loss tangent** ($\varepsilon'' / \varepsilon'$)	**penetration depth** d_p **(meters)**
0.000001	921800	7.958
0.00001	92180	2.5165
0.0001	9218	0.7958
0.001	921.8	0.2518
0.01	92.18	0.0800
0.1	9.218	0.0266
1.0	0.9218	0.0127
10.0	0.09218	0.01173
100.0	0.009218	0.01172

Summary of Homogeneous Plane Wave Properties

A summary of the properties relating to wavelength, velocity, and depth of penetration for a homogeneous plane wave is given in Table 3.

Table 3. Propagation Properties of a homogeneous plane wave.

	General	Highly Conducting	Low Loss	Lossless
wavelength	$\frac{2\pi}{k'}$	$2\pi\sqrt{\frac{2}{\omega\mu\sigma}}$	$\frac{2\pi}{\omega\sqrt{\mu\varepsilon'}}$	$\frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$
phase velocity	$\frac{\omega}{k'}$	$\sqrt{\frac{2\omega}{\mu\sigma}}$	$\frac{1}{\sqrt{\mu\varepsilon'}}$	$\frac{1}{\sqrt{\mu\varepsilon}}$
group velocity	$\frac{d\omega}{dk'}$	$2\sqrt{\frac{2\omega}{\mu\sigma}}$	$\frac{1}{\sqrt{\mu\varepsilon'}}$	$\frac{1}{\sqrt{\mu\varepsilon}}$
depth of penetration	$1/k''$	$\sqrt{\frac{2}{\omega\mu\sigma}}$	$\frac{2}{\sigma}\sqrt{\frac{\varepsilon'}{\mu}}$	∞

Polarization

The above discussion has assumed a homogeneous plane wave propagating in the *z* direction, polarized with the electric field in the *x* direction. The most general polarization of a wave propagating in the *z* direction is one having both *x* and *y* components of the field,

$$\boldsymbol{E} = \left(\hat{\boldsymbol{x}}\, E_{x0} + \hat{\boldsymbol{y}}\, E_{y0}\right) e^{-j(kz)} . \tag{49}$$

The field components are represented in polar form as

$$E_{x0} = \left|E_{x0}\right| e^{j\phi_x} , \tag{50}$$

$$E_{y0} = \left|E_{y0}\right| e^{j\phi_y} . \tag{51}$$

The phase difference between the two components is defined as

$$\phi = \phi_y - \phi_x \,. \tag{52}$$

Without any real loss of generality, ϕ_x may be chosen as zero, so that $\phi = \phi_y$. In the time domain, the field components are then

$$\mathcal{E}_x = \left|E_{x0}\right| \cos(\omega t) \tag{53}$$

$$\mathcal{E}_y = \left|E_{y0}\right| \cos(\omega t + \phi) \,. \tag{54}$$

Using trigonometric identities, Eq. (54) may be expanded into a sum of $\sin(\omega t)$ and $\cos(\omega t)$ functions, and then Eq. (53) may be used to put both $\cos(\omega t)$ and $\sin(\omega t)$ in terms of $\mathcal{E}_x$ (using $\sin^2 = 1 - \cos^2$). After simplification, the result is

$$A\mathcal{E}_x^2 + B\mathcal{E}_x\mathcal{E}_y + C\mathcal{E}_y^2 = D \tag{55}$$

where

$$A = \left|\frac{E_{y0}}{E_{x0}}\right|^2 \tag{56}$$

$$B = -2\left|\frac{E_{y0}}{E_{x0}}\right| \cos\phi \tag{57}$$

$$C = 1 \tag{58}$$

$$D = \left|E_{y0}\right|^2 \sin^2\phi \,. \tag{59}$$

After simplification, the discriminant of this quadratic curve is

$$\Delta = B^2 - 4AC = -4\left|\frac{E_{y0}}{E_{x0}}\right|^2 \sin^2\phi \,, \tag{60}$$

which is always negative. This curve thus always represents an ellipse. The general form of the ellipse is shown in Fig. 1. The tilt angle of the ellipse is τ, and the axial ratio AR is defined as the ratio of the major axis of the ellipse to the minor axis ($AR \geq 1$). In the time domain the electric field vector rotates with the tip of the vector lying on the ellipse. Right-handed elliptical polarization, or RHEP, corresponds to counterclockwise rotation (the thumb of the right hand aligns with the direction of propagation, the fingers of the right hand align with the direction of rotation in time). (This is the IEEE definition, opposite to the usual optics convention.) LHEP corresponds to rotation in the opposite direction.

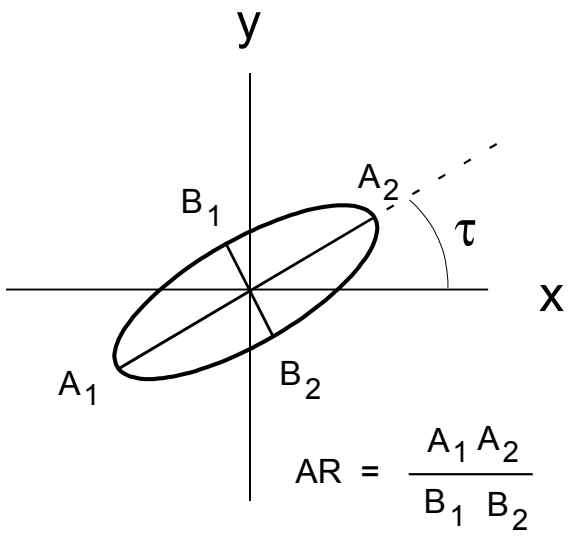

Figure 1. Geometry of the polarization ellipse.

A convenient way to represent the polarization state is with the Poincaré sphere [5]. Using spherical trigonometric relations, the following results may be derived for the tilt angle of the ellipse and the axial ratio.

$$\tan 2\tau = \tan 2\gamma \cos\phi \tag{61}$$

$$\sin 2\xi = \sin 2\gamma \sin\phi\,, \tag{62}$$

where the parameter ξ is related to the axial ratio as

$$\begin{aligned} &\xi = \cot^{-1}\left(\mp AR\right), \qquad -45^0 \le \xi \le +45^0 \\ &\quad + \text{ for LHEP} \\ &\quad - \text{ for RHEP}. \end{aligned} \tag{63}$$

The phase angle ϕ is defined in Eq. (52). The parameter γ characterizes the ratio of the fields along the *x* and *y* axes, and is defined from

$$\gamma = \tan^{-1}\left|\frac{E_{y0}}{E_{x0}}\right|, \qquad 0 \le \gamma \le 90^{\circ}\,. \tag{64}$$

There is no ambiguity in Eq. (62) for ξ, since $-90^0 \le 2\xi \le +90^0$. However, Eq. (61) gives an ambiguity for τ, since adding multiples of 180° does not change the tangent. To resolve this ambiguity, Table 4 may be used to determine the appropriate quadrant (1, 2, 3 or 4) that the angle 2τ is in, based on the Poincaré sphere [5].

Table 4. Quadrant that the angle 2τ is in.

	$\cos\phi > 0$	$\cos\phi < 0$
$\cos 2\gamma > 0$	1	4
$\cos 2\gamma < 0$	2	3

The following special cases are important.

1) Linear polarization: $\phi = 0$ or 180^0, or either E_{x0} or E_{y0} is zero.

In this case AR $= \infty$ (the ellipse degenerates to a straight line).

2) Circular polarization: $E_{y0} = E_{x0}$ and $\phi = \pm 90^0$ (to within any multiple of 180°).

In this case the ellipse becomes a circle (either RHCP or LHCP), and AR = 1.

It can also be noted that a wave of arbitrary polarization can be represented as a sum of RHCP and a LHCP waves, by noting that the arbitrarily polarized wave in Eq. (49) can be written as

$$\boldsymbol{E} = \hat{\boldsymbol{r}}\left[\frac{1}{\sqrt{2}}\left(E_{x0} + jE_{y0}\right)\right] + \hat{\boldsymbol{l}}\left[\frac{1}{\sqrt{2}}\left(E_{x0} - jE_{y0}\right)\right] \tag{65}$$

where the unit-amplitude RHCP and LHCP waves are

$$\hat{\boldsymbol{r}} = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{x}} - j\hat{\boldsymbol{y}}\right)e^{-jkz} \tag{66}$$

$$\hat{\boldsymbol{l}} = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{x}} + j\hat{\boldsymbol{y}}\right)e^{-jkz}. \tag{67}$$

5. PLANE WAVE REFLECTION AND TRANSMISSION

A general plane wave reflection/transmission problem consists of an incident plane wave impinging on a multilayer structure, as shown in Fig. 2. An important special case is the two-region problem shown in Fig. 3. The incident plane wave may be either homogeneous or inhomogeneous, and any of the regions may have an arbitrary amount of loss. The vector

representing the direction of propagation for the incident wave in Figs. 2 and 3 is a real vector (as shown) if the incident wave is homogeneous, but the analysis is valid for the general case.

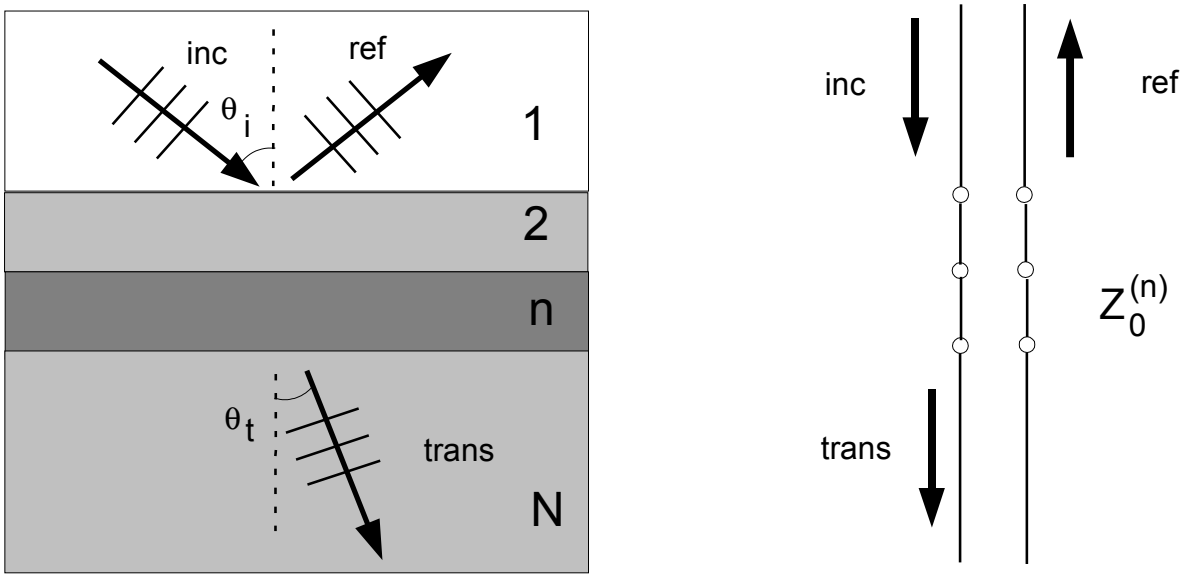

Figure 2. A plane wave reflecting from a multilayer stack of different materials. Each layer has a uniform (constant) set of material parameters. On the right side the equivalent transmission-line model (transverse equivalent network) is shown.

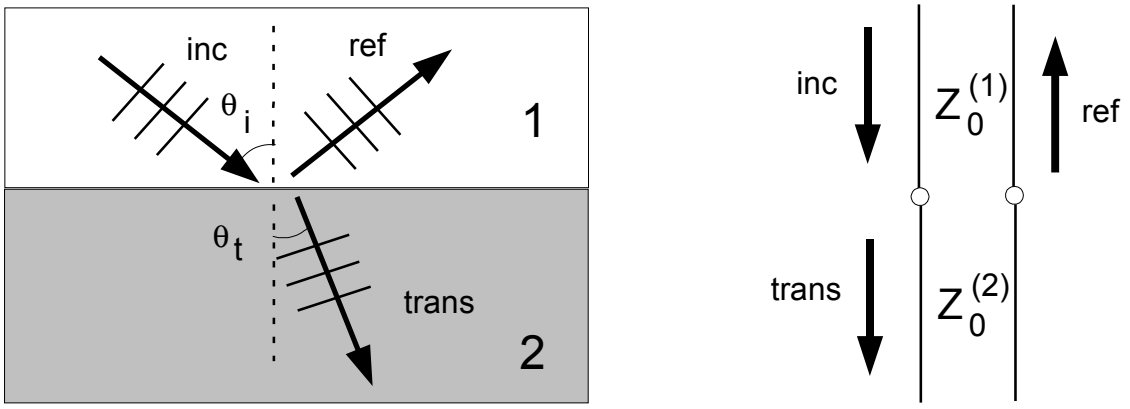

Figure 3. Reflection and transmission from a single interface between two different media. The transmission-line model (transverse equivalent network) for the two-region reflection problem is shown on the right.

The key to obtaining a simple solution to such reflection and transmission problems is the use of a transmission line model of the layered structured, which is based on TE-TM decomposition of the plane waves, discussed next.

TE-TM Decomposition

According to a basic electromagnetic theorem [1], the fields in a source-free homogeneous region can be represented as the sum of two types of fields, a field that is transverse magnetic to *z* (TM_z) and a field that is transverse electric to *z* (TE_z). The TM_z field is defined as one that has $H_z = 0$, while the TE_z field by definition has $E_z = 0$. A general plane wave field may therefore be written as the sum of a TM_z plane wave and a TE_z plane wave. In free space, the direction *z* is rather arbitrary. When a layered media is present, the preferred direction *z* is perpendicular to the layers, because this allows for the TM_z and TE_z plane waves in each region to be modeled as waves on a transmission line. Standard transmission-line theory may then be conveniently used to solve plane-wave reflection and transmission problems in a relatively simple manner, without having to solve the electromagnetic boundary-value problem of matching fields at the interfaces. The TM_z plane wave is commonly referred to as one that is polarized with the electric field "in the place of incidence", while the TE_z place wave is polarized with the electric field "perpendicular to the plane of incidence". (The plane of incidence is the *y*-*z* plane.) For a homogeneous plane wave with an incident wave vector in the *y*-*z* plane, the polarizations are shown in Fig. 4.

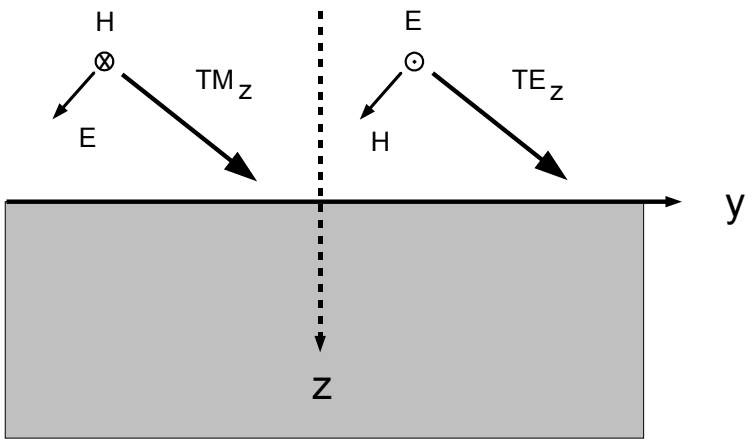

Figure 4. Polarizations of incident TM_z and TE_z plane waves.

The field components may be found by using Maxwell's equations to write the transverse (*x* and *y*) field components in terms of the longitudinal components E_z and H_z. The nonzero longitudinal component is assumed to be proportional to the wavefunction

$$\psi = \exp\left(-j(k_x x + k_y y \pm k_z z)\right). \tag{68}$$

The plus sign in this equation is chosen for plane waves propagating or decaying in the positive *z* direction, while the negative sign is for propagation or decay in the minus *z* direction. This representation is convenient for plane-wave reflection problems, since the characteristic impedance of a transmission line that models a particular region will then have a positive characteristic impedance for both upward and downward propagating plane waves, in agreement with the usual transmission line convention.

For the TM_z plane wave, the normalized field components may then be written as

$$
\begin{aligned}
E_x &= \mp k_x k_z \Psi(x,y,z) & H_x &= \omega\varepsilon\, k_y \Psi(x,y,z) \\
E_y &= \mp k_y k_z \Psi(x,y,z) & H_y &= -\omega\varepsilon\, k_x \Psi(x,y,z) \\
E_z &= \left(k^2 - k_z^2\right)\Psi(x,y,z) & H_z &= 0.
\end{aligned}
\tag{69}
$$

For a TE_z plane wave the corresponding results are

$$
\begin{aligned}
E_x &= -\omega\mu\, k_y \Psi(x,y,z) & H_x &= \mp k_x k_z \Psi(x,y,z) \\
E_y &= \omega\mu\, k_x \Psi(x,y,z) & H_y &= \mp k_y k_z \Psi(x,y,z) \\
E_z &= 0 & H_z &= \left(k^2 - k_z^2\right)\Psi(x,y,z).
\end{aligned}
\tag{70}
$$

Note that a plane wave propagating in the z direction ($k_z = k$) has both longitudinal field components that are zero. Such a plane wave is TEM (transverse electric and magnetic) to the z direction)

It may be seen from the above equations that the transverse fields obey the relations

$$
\boldsymbol{E}_t^{TM} = \mp Z_0^{TM}\left(\hat{\boldsymbol{z}} \times \boldsymbol{H}_t^{TM}\right) \tag{71}
$$

$$
\boldsymbol{E}_t^{TE} = \mp Z_0^{TE}\left(\hat{\boldsymbol{z}} \times \boldsymbol{H}_t^{TE}\right) \tag{72}
$$

where

$$
Z_0^{TM} = \frac{k_z}{\omega\varepsilon} \tag{73}
$$

$$
Z_0^{TE} = \frac{\omega\mu}{k_z}. \tag{74}
$$

Transverse Equivalent Network

From the above relations (71) and (72), the transverse (perpendicular to *z*) field components behave as voltages and currents on a transmission line model – called the transverse equivalent network. In particular, the correspondence is given through the relations (*i* = TM or TE)

$$\boldsymbol{E}_t^i = \hat{\boldsymbol{e}}\,\psi_t(x,y)V^i(z) \tag{75}$$

$$\boldsymbol{H}_t^i = \hat{\boldsymbol{h}}\,\psi_t(x,y)I^i(z) \tag{76}$$

where the unit vectors are chosen so that

$$\hat{\boldsymbol{e}}\times\hat{\boldsymbol{h}} = \pm\hat{\boldsymbol{z}} \tag{77}$$

for a plane wave propagating in the $\pm\hat{z}$ direction. The transverse wavefunction

$$\psi_t(x,y) = e^{-j\left(k_x x + k_y y\right)} \tag{78}$$

is a common transverse phase term that must be the same for all regions (if the transverse wavenumbers k_x or k_y were different between two regions, a matching of transverse fields at the boundary would not be possible). The fact that the transverse wavenumbers are the same in all regions leads to the "law of reflection", which states that the direction angle θ for a reflected plane wave must equal that for the incident plane wave. It also leads to Snell's law, which states that the direction angles θ inside each of the regions are related to each other, through

$$n_i \sin\theta_i = n_1 \sin\theta_1, \qquad i = 1,2,\ldots N, \tag{79}$$

where n_i is the index of refraction (possibly complex) of region *i*, defined as $n_i = k_i / k_0 = \sqrt{\varepsilon_{ri}\mu_{ri}}$. It is often convenient to express the characteristic impedance for region *i* from Eqs. (73) and (74) in terms of the medium intrinsic impedance η_i as

$$Z_0^{TM} = \eta_i \cos\theta_i \tag{80}$$

$$Z_0^{TE} = \eta_i \sec\theta_i \ . \tag{81}$$

Using Snell's law, the impedances then become

$$Z_0^{TM} = \eta_i \sqrt{1-\left(\frac{n_1}{n_i}\right)^2 \sin^2\theta_1} \tag{82}$$

$$Z_0^{TE} = \frac{\eta_i}{\sqrt{1-\left(\frac{n_1}{n_i}\right)^2 \sin^2\theta_1}} . \tag{83}$$

The above expressions remain valid for lossy media. The square roots are chosen so that the real part of the characteristic impedances are positive.

The functions *V(z)* and *I(z)* behave as voltage and current on a transmission line, with characteristic impedance Z_0^{TM} or Z_0^{TE}, depending on the case. Hence any plane-wave reflection and transmission problem reduces to a transmission line problem, giving the exact solution that satisfies all boundary conditions. One consequence of this is that TM_z and TE_z plane waves do not couple at a boundary. If the incident plane wave is TM_z, for example, the waves in all regions will remain TM_z plane waves. Hence, the motivation for the TM_z-TE_z decomposition. The transverse equivalent network for the multilayer and two-region problems are shown on the right sides of Figs. 2 and 3, respectively. The network model is the same for either TM_z or TE_z polarization, except that the characteristic impedances are different. If an incident plane wave is a combination of both TM_z and TE_z waves, the two parts are solved separately and then summed to get the total reflected or transmitted field.

Special Case: Two-Region Problem

For the simple two-region problem of Fig. 3, the reflected and transmitted voltages (modeling the transverse electric fields) are represented as

$$V^{\text{ref}}(z) = \Gamma V^{\text{inc}}(0) e^{+jk_z^{(1)}z} \tag{84}$$

$$V^{\text{trans}}(z) = TV^{\text{inc}}(0) e^{-jk_z^{(2)}z} \tag{85}$$

where the reflection and transmission coefficients are given by the standard transmission line equations

$$\Gamma = \frac{Z_0^{(2)} - Z_0^{(1)}}{Z_0^{(2)} + Z_0^{(1)}} \tag{86}$$

$$T = 1 + \Gamma \,. \tag{87}$$

Note that a plus sign is used in the exponent of Eq. (84) to account for upward propagation of the reflected wave.

A. Critical angle

If regions 1 and 2 are lossless, and region 1 is more dense than region 2 ($n_1 > n_2$), an incident angle $\theta_1 = \theta_c$ will exist for which $k_z^{(2)} = 0$. From Snell's law, the angle is

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right). \tag{88}$$

When $\theta_1 > \theta_c$, the wavenumber $k_z^{(2)}$ will be purely imaginary, of the form $k_z^{(2)} = -j\alpha_z^{(2)}$. In this case there is no power flow into the second region, since the phase vector in region 2 has no z component. In this case 100% of the incident power is reflected back from the interface. There

are, however, still fields present in the second region, decaying exponentially with distance *z*. In the second region the power flow is in the horizontal direction only.

B. Brewster angle

For lossless layers, it is possible to have 100% of the incident power transmitted into the second region, with no reflection. This corresponds to a matched transmission line circuit, with

$$Z_0^{(1)} = Z_0^{(2)} . \tag{89}$$

For nonmagnetic layers, it may be easily shown that this matching equation can only be satisfied in the TM_z case (there is always a nonzero reflection coefficient in the TE_z case, unless the trivial case of identical medium is considered). The angle θ_b at which no reflection occurs is called the Brewster angle. For the TM_z case, a simple algebraic manipulation of Eq. (89) yields the result

$$\tan\theta_b = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} . \tag{90}$$

Orthogonality

When treating reflection problems, there is more than one plane wave in at least one of the regions, a wave traveling in the positive *z* direction (focusing on the *z* variation) and a wave traveling in the negative *z* direction. Often, it is desired to calculate power flow in such a region. For a single plane wave, the complex power density in the *z* direction (the *z* component of the complex Poynting vector) is equal to the complex power flowing on the corresponding transmission line of the transverse equivalent network. When both an incident and a reflected wave are present, or both TM_z and TE_z waves are present the following orthogonality results are

useful. These theorems related to power flow in the z direction may be proven by direct calculation using the field components in rectangular coordinates.

1. An orthogonality exists between a homogeneous TM_z plane wave and a homogeneous TE_z plane wave propagating in the same direction, in the sense that the complex power density in the z direction, S_z, is the sum of the two complex power densities S_{1z} and S_{2z}. This is true for a lossy or lossless media.

2. An orthogonality exists between an incident wave and a reflected wave in a lossless media (both are either TE_z or TM_z), provided the wavenumber component k_z of the two waves is real. The two waves are orthogonal in the sense that the time-average power density in the z direction, $\mathrm{Re}\, S_z$, is the sum of the two time-average power densities $\mathrm{Re}\, S_{1z}$ and $\mathrm{Re}\, S_{2z}$.

To illustrate property 2, consider an incident plane wave traveling in a glass region, impinging on an air gap that separates the glass region from another identical glass region, as shown in Fig. 5.

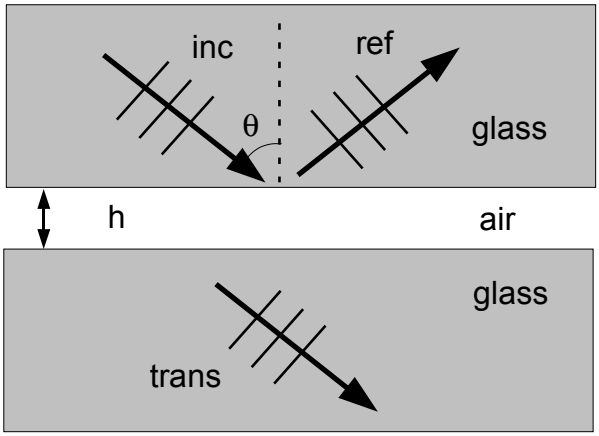

Figure 5. An incident wave traveling in a semi-infinite region of lossless glass impinges on an air gap separating the glass region from an identical region below. Incident, reflected and transmitted plane waves are shown.

If the incident plane wave is beyond the critical angle, the plane waves in the air region will be evanescent, with an imaginary vertical wavenumber k_z. Each of the two plane waves in the air region (upward and downward) that constitute a standing-wave field do not, individually, have a time-average power flow in the z direction, since k_z is imaginary. However, there is an overall power flow in the z direction inside the air region, since there is a transmitted field in the lower region. The total power flow in the z direction is thus not the sum of the two individual power flows. The two waves in the air region are not orthogonal, and property 2 does not apply since the wavenumber k_z is not real.

VI. EXAMPLE

A RHCP plane wave at a frequency of 1.0 GHz is incident on the surface of the ocean at an angle of $\theta = 30^{\circ}$. Determine the percentage of power that is reflected from the ocean, and characterize the polarization of the reflected wave (axial ratio, tilt angle, and handedness). Also, determine the field of the inhomogeneous transmitted plane wave. The parameters of the ocean water are assumed to be $\hat{\varepsilon} = 78$ and $\sigma = 4$ S/m.

Solution

The geometry of the incident and reflected waves is shown in Fig. 6. The incident plane wave is represented as

$$\boldsymbol{E}^{\text{inc}} = E_0\left[\hat{\boldsymbol{x}} + \hat{\boldsymbol{u}}(-j)\right]e^{-j\left(k_y y + k_{z1} z\right)} \tag{91}$$

where

$$k_y = k_0 \sin\theta = \frac{1}{2} k_0 \tag{92}$$

$$k_{z1} = k_0 \cos\theta = \frac{\sqrt{3}}{2} k_0 \tag{93}$$

$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{y}} \cos\theta - \hat{\boldsymbol{z}} \sin\theta . \tag{94}$$

The reflected wave is represented as

$$\boldsymbol{E}^{\text{ref}} = E_0 \left[A\, \hat{\boldsymbol{x}} + B\, \hat{\boldsymbol{v}} \right] e^{-j\left(k_y y - k_{z0} z\right)} \tag{95}$$

where

$$\hat{\boldsymbol{v}} = -\hat{\boldsymbol{y}} \cos\theta - \hat{\boldsymbol{z}} \sin\theta . \tag{96}$$

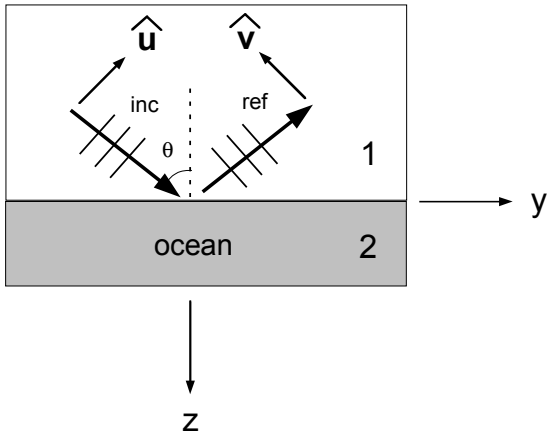

Figure 6. Geometry that defines the coordinate system for the example of a plane wave reflecting from the surface of the ocean.

The *x* component of the incident and reflected waves corresponds to TE_z waves, while the *u* and *v* components correspond to TM_z waves. (The *u* and *v* directions substitute for the *y* direction in the previous discussion on polarization.) The transverse equivalent network is shown in Fig. 3. From Eqs. (82) and (83), the impedances are $Z_1^{TM} = 326.258\ \Omega$, $Z_1^{TE} = 435.011\ \Omega$,

$Z_2^{TM} = 34.051 + j(13.269)\ \Omega$, $Z_2^{TE} = 34.089 + j(13.346)\ \Omega$. The reflection coefficients are then $\Gamma^{TM} = -0.808535 + j(0.066600)$ and $\Gamma^{TE} = -0.853161 + j(0.052724)$.

The coefficient A in Eq. (94) is determined directly from

$$A = \Gamma^{TE} = -0.853161 + j(0.052724). \tag{97}$$

To determine the coefficient B, recall that the voltages in the transverse equivalent network model only the transverse (horizontal) component of the electric field (the y component in the TM case). Hence,

$$-BE_0 \cos\theta^{\text{ref}} = \Gamma^{TM} E_0(-j)\cos\theta^{\text{inc}}. \tag{98}$$

Since $\theta^{\text{inc}} = \theta^{\text{ref}} = \theta$, B is determined directly as

$$B = \Gamma^{TM}(j) = \Gamma^{TM} = -0.066600 - j(0.808535). \tag{99}$$

The percent power reflected is calculated from

$$P_{\%}^{\text{ref}} = 100\left(\frac{\frac{1}{2}\left[\frac{|A|^2}{Z_1^{TE}} + \frac{|B\cos\theta|^2}{Z_1^{TM}}\right]}{\frac{1}{2}\left[\frac{1}{Z_1^{TE}} + \frac{|-j\cos\theta|^2}{Z_1^{TM}}\right]}\right) \tag{100}$$

which yields

$$P_{\%}^{\text{ref}} = 69.441. \tag{101}$$

Hence 30.559 % of the incident power is transmitted into the ocean.

The phase angle between the v and x components of the reflected field is

$$\phi = \angle B - \angle A = 88.827^{\circ} \tag{102}$$

Hence, the reflected wave is a left-handed elliptically polarized wave.

The parameter γ is determined from the ratio of the magnitudes of the *v* and *x* components of the reflected field as

$$\gamma = \tan^{-1}\left(\frac{|B|}{|A|}\right) = 43.504^{\circ}\,. \tag{103}$$

From the formulas in the Polarization section, the tilt angle τ of the polarization ellipse from the *x* axis (towards the positive *v* axis) is then

$$\tau = 10.690^{\circ}\,, \tag{104}$$

and the polarization parameter ξ is

$$\xi = 43.393^{\circ}\,. \tag{105}$$

The axial ratio of the reflected wave is then

$$AR = +\cot\xi = 1.058\,. \tag{106}$$

The reflected wave is nearly CP, since the ocean is highly reflecting (if the ocean were replaced by a perfect conductor, the reflected wave would be LHCP).

The transmitted field in the ocean is represented as

$$\boldsymbol{E}^{\text{trans}} = E_0\left[\hat{\boldsymbol{x}}C + \hat{\boldsymbol{y}}\,D + \hat{\boldsymbol{z}}\,E\right]e^{-j\left(k_y y - k_{z2} z\right)}\,. \tag{107}$$

where k_{z2} is the vertical wavenumber in the ocean, found from Eq. (31) as

$$k_{z2} = k_2\cos\theta_t = k_0 n_2\sqrt{1-\left(\frac{1}{n_2}\right)^2\sin^2\theta} = k_0\left(9.582527 - j3.751642\right). \tag{108}$$

From the transverse equivalent network,

$$C = T^{TE} = 1 + \Gamma^{TE} = 0.146839 + j(0.052724) \tag{109}$$

and

$$D = T^{TM}\left(-j\cos\theta\right) = \left(1+\Gamma^{TM}\right)\left(-j\cos\theta\right) = 0.057678 - j(0.165813) \tag{110}$$

The z component of the transmitted field can be determined from the electric Gauss law (20), which gives

$$E = -\frac{1}{k_{z2}}\left(k_y D\right) \ . \tag{111}$$

Hence,

$$E = -0.0055466 + j(0.00648030) \ . \tag{112}$$

From the wavenumbers k_y and k_{z2} the direction angles of the transmitted wave may be determined from Eqs. (29) – (31). The results are

$$\phi = 90^{\circ} \tag{113}$$

$$\theta_t = 0.045227 + j(0.017679) \ \text{radians} \ . \tag{114}$$

The transmitted plane wave is inhomogeneous, since the angle θ_t is complex.

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GO TO BEFORE included File

46.06.04
Generic Solution for ElectroMagnetic MPW in perfect Ohmic media

Here a different way to solve for ElectroMagnetic MPW is presented.

Look for MPW solutions, that is solutions that only depends on time and on the coordinate measured along some fixed arbitrary direction, ξ, along the unit vector $\widehat{\mathbf{n}}$. This means that at any time the vectors $\mathbf{E}$ and $\mathbf{B}$ are constant over planes normal to $\widehat{\mathbf{n}}$. These planes are defined by the relation:

$$\mathbf{x}\cdot\mathbf{n} = \text{constant} \quad .$$

The fields assume therefore the following form:

$$\mathbf{E} = \mathbf{E}[\xi, t] \qquad \mathbf{B} = \mathbf{B}[\xi, t] \quad .$$

One also has the following relations, for any vector field, $\mathbf{V}$:

$$\boldsymbol{\nabla}\mathbf{V} = \widehat{\mathbf{n}}\frac{\partial \mathbf{V}}{\partial \xi} \quad ,$$

$$\mathbf{rot}\,\mathbf{V} = \frac{\partial}{\partial \xi}(\mathbf{n}\wedge\mathbf{V}) \quad ,$$

$$\mathbf{div}\,\mathbf{V} = \frac{\partial}{\partial \xi}(\mathbf{n}\cdot\mathbf{V}) \quad .$$

One then finds:

$$\frac{\partial^2 \mathbf{E}}{\partial \xi^2} - \varepsilon_{\mathrm{R}}\mu_{\mathrm{R}}\varepsilon_0\mu_0\frac{\partial^2 \mathbf{E}}{\partial t^2} - \sigma\mu_{\mathrm{R}}\mu_0\partial_t\mathbf{E} = \mathbf{0} \quad , \qquad (46.06.29)$$

$$\frac{\partial^2 \mathbf{B}}{\partial \xi^2} - \varepsilon_{\mathrm{R}}\mu_{\mathrm{R}}\varepsilon_0\mu_0\frac{\partial^2 \mathbf{B}}{\partial t^2} - \sigma\mu_{\mathrm{R}}\mu_0\partial_t\mathbf{B} = \mathbf{0} \quad .$$

→ 2339

From the 3rd **Maxwell Equation** and 4th **Maxwell Equation** it follows that:

$$\widehat{\mathbf{n}}\cdot(\varepsilon_{\mathrm{R}}\varepsilon_0\partial_t\mathbf{E} + \sigma\mathbf{E}) = 0 \qquad \widehat{\mathbf{n}}\cdot\partial_t\mathbf{B} = 0 \quad ,$$

→ 2339

while the first and second **Maxwell Equation** imply:

$$\widehat{\mathbf{n}}\cdot\frac{\partial \mathbf{E}}{\partial \xi} = 0 \qquad \widehat{\mathbf{n}}\cdot\frac{\partial \mathbf{B}}{\partial \xi} = 0 \quad .$$

Therefore

$$\boxed{\widehat{\mathbf{n}}\cdot\left(\mathrm{d}\mathbf{E} + \frac{\sigma}{\varepsilon_{\mathrm{R}}\varepsilon_0}\mathbf{E}\,\mathrm{d}t\right) = 0 \qquad \widehat{\mathbf{n}}\cdot\mathrm{d}\mathbf{B} = 0} \quad . \qquad (46.06.30)$$

→ 2130

Equations (46.06.30) prove that:

- transversality of the ElectroMagnetic waves;
- the ξ -component of the magnetic field cannot vary in time nor along $\widehat{\mathbf{n}}$, as a consequence of the fact that the fields are assumed to be constant over planes perpendicular to the $\widehat{\mathbf{n}}$ axis at any time;
- the ξ-component of the electric field satisfies the equation

$$\frac{\mathrm{d}E_\xi}{\mathrm{d}t} + \frac{\sigma}{\varepsilon_{\mathrm{R}}\varepsilon_0}E_\xi = 0 \quad .$$

- if the **conductivity** is non zero, the longitudinal component of the electric field decays exponentially with a decay time given by the relaxation constant;
- a static electric field cannot be maintained inside a conductor;
- inside a perfect Polarizable material medium (that is perfectly insulator) a longitudinal component may be present, but it does not depend on time.

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46.07 Summary for ElectroMagnetic MPW Without Boundaries

46.08 EM Waves in Generic Passive Media Without Boundaries

This § is referenced at pages:
[2350]

©|J.E.Vitela|Am.J.Phys, , ..., ..., ...Ed., ... 72, 393 (2004) - WEB - URL|*Excellent and complete.*|

©|P.Halevi|Plane ElectroMagnetic waves in material media: Are they transverse waves?|*Am.J.Phys, , ..., ..., ...Ed., ... 48, ...*

©|L.D.Landau *et al.*, *Electrodynamics of continuous media*, ..., ..., ...Ed., ...|§ 80||

©|J.Schwinger|§ 7, 51|*extensive and detailed*|

©|JChenEtAl|ExpressionsOfStoredAndDissipatedElectroMagneticEnergyDensities - WEB - URL |*Definitive!*|

In generic material media the simple picture of mutually perpendicular electric and magnetic field, perpendicular to the direction of propagation, must be abandoned, in general.

In general the transversality property of ElectroMagnetic waves no longer holds when they propagate in dissipative media, including LHI dissipative media. In this respect dissipative media can be either conducting media and|or dielectric (that is insulator) media with an imaginary part of the **Electric-Permittivity** . Read § 54.

Both the **Electric-Permittivity** and the **Magnetic-Permeability** depend in general on the frequency, in addition to be complex:

$$\varepsilon_{\mathrm{R}}[\omega] \equiv \mathfrak{Re}\left[\varepsilon_{\mathrm{R}}[\omega]\right] + \mathrm{i}\,\mathfrak{Im}\left[\varepsilon_{\mathrm{R}}[\omega]\right] \qquad \mu_{\mathrm{R}}[\omega] \equiv \mathfrak{Re}\left[\mu_{\mathrm{R}}[\omega]\right] + \mathrm{i}\,\mathfrak{Im}\left[\mu_{\mathrm{R}}[\omega]\right] \quad . \tag{46.08.01}$$

Therefore, also the **index of refraction** and **wave impedance** are in general complex.

The real and imaginary parts of the **Electric-Permittivity** and the **Magnetic-Permeability** are not independent of each other. They are related by the Kramer-Krönig relations. Moreover, the imaginary part of the **Electric-Permittivity** is closely related to the current inside the material via the Ohm relation $\mathbf{j}^{\mathrm{F}} = \sigma\mathbf{E}$.

For an ElectroMagnetic MPW the **power loss per unit volume** is:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}V} = \frac{\omega}{2}\left(\mathfrak{Im}\left[\varepsilon_{\mathrm{R}}[\omega]\right]|\mathbf{E}|^2 + \mathfrak{Im}\left[\mu_{\mathrm{R}}[\omega]\right]|\mathbf{H}|^2\right) = \omega\left(\mathfrak{Im}\left[\varepsilon_{\mathrm{R}}[\omega]\right]\langle|\mathbf{E}|^2\rangle + \mathfrak{Im}\left[\mu_{\mathrm{R}}[\omega]\right]\langle|\mathbf{H}|^2\rangle\right) \quad , \tag{46.08.02}$$

→ 2131

where $\mathbf{E}$ and $\mathbf{H}$ are the real fields, and the bar denotes an average with respect to time.

In general, the absorption (dissipation) of energy is determined by the imaginary parts of $\varepsilon_{\mathrm{R}}[\omega]$ and $\mu_{\mathrm{R}}[\omega]$. The two terms in (46.08.02) are called the electric and magnetic losses.

In passive media, the sign of these losses is determinate: the energy is dissipated, therefore:

$$\mathfrak{Im}\left[\varepsilon_{\mathrm{R}}[\omega]\right] \geq 0 \qquad \mathfrak{Im}\left[\mu_{\mathrm{R}}[\omega]\right] \geq 0 \quad , \tag{46.08.03}$$

for all substances and at all (positive) frequencies.

The signs of the real parts of $\varepsilon_{\mathrm{R}}[\omega]$ and $\mu_{\mathrm{R}}[\omega]$ are subject to no physical restriction. The so-called left-handed materials, have both the real part of the **Electric-Permittivity** and magnetic negative § 54.03.

For passive materials the imaginary part of the **index of refraction** has to be positive

$$\mathfrak{Im}\left[n\right] \geq 0 \quad , \tag{46.08.04}$$

which guarantees that the amplitude of the wave decreases in an absorbing material.

Any non-steady process in an actual body is to some extent thermodynamically irreversible. The electric and magnetic losses in a variable ElectroMagnetic field therefore always occur to some extent, However, slight. That is, the functions $\mathfrak{Im}\left[\varepsilon_{\mathrm{R}}[\omega]\right]$ and $\mathfrak{Im}\left[\mu_{\mathrm{R}}[\omega]\right]$ are not exactly zero for any frequency other than

zero, although it does not exclude the possibility of only very small losses in certain frequency ranges. Such ranges, in which the imaginary parts of the **Electric-Permittivity** and **Magnetic-Permeability** are very small in comparison with the real parts, are called transparency ranges. It is possible to neglect the absorption in these ranges and to introduce the concept of the internal energy of the body in the ElectroMagnetic field, in the same sense as in a static field. To determine this quantity, it is not sufficient to consider a purely monochromatic field, since the strict periodicity results in no steady accumulation of ElectroMagnetic energy. Let us therefore consider a field whose components have frequencies in a narrow range about some mean value ω_0.

The steady rate of change of the energy in unit volume neglecting the imaginary parts of the **Electric-Permittivity** and **Magnetic-Permeability** is given by:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}V} = \frac{1}{2}\left(\frac{\mathrm{d}(\omega\,\Re\mathfrak{e}\,[\varepsilon_{\mathrm{R}}[\omega]])}{\mathrm{d}\omega}\langle|\mathbf{E}|^2\rangle + \frac{\mathrm{d}(\omega\,\Re\mathfrak{e}\,[\mu_{\mathrm{R}}[\omega]])}{\mathrm{d}\omega}\langle|\mathbf{H}|^2\rangle\right) \quad , \tag{46.08.05}$$

→ 2132

where $\mathbf{E}$ and $\mathbf{H}$ are the real fields. This is the required result: the mean value of the ElectroMagnetic part of the internal energy per unit volume of a transparent medium. If there is no dispersion, **Electric-Permittivity** and **Magnetic-Permeability** are constants, and equation (46.08.05) becomes the mean value of the rate of change of ElectroMagnetic energy:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}V} = \frac{1}{2}\left(\varepsilon_{\mathrm{R}}[\omega]\varepsilon_0\langle|\mathbf{E}|^2\rangle + \mu_{\mathrm{R}}[\omega]\mu_0\langle|\mathbf{H}|^2\rangle\right) \quad , \tag{46.08.06}$$

as it should.

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46.09 EM Waves in Bounded Regions of Space

This § is referenced at pages:
[2013]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|||

Read § 47.

Beware that **EM Waves in Bounded Regions of Space, in one, two, three dimensions, may have properties very different from ElectroMagnetic waves in unbounded/infinite regions**.

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46.10 Generation of ElectroMagnetic Waves

This § is referenced at pages:
[Never referenced.]

ElectroMagnetic Waves, in vacuum, are generated by, and only by, accelerated charges. However, in presence of matter, other effects can give rise to EM waves generation, read § 46.13.

Read § 49.

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46.11 Maxwell Equations With a Generic Harmonic Time Dependence

Consider the ElectroMagnetic fields with harmonic time-dependence:

$$\overline{\mathbf{E}}[\mathbf{x},t] \equiv \overline{\mathbf{E}}[\mathbf{x}]\exp[-\mathtt{i}\omega t] \qquad \overline{\mathbf{B}}[\mathbf{x},t] \equiv \overline{\mathbf{B}}[\mathbf{x}]\exp[-\mathtt{i}\omega t] \quad .$$

Consider also the sources with harmonic time dependence:

$$\overline{\rho}[\mathbf{x},t] \equiv \overline{\rho}[\mathbf{x}]\exp[-\mathtt{i}\omega t] \qquad \overline{\mathbf{j}}[\mathbf{x},t] \equiv \overline{\mathbf{j}}[\mathbf{x}]\exp[-\mathtt{i}\omega t] \quad .$$

Note that, even if the same symbols are used to denote the general (time-harmonic) field/source and the space-dependent part of the harmonic field/source itself, this should cause no confusion because the first one depends on $[\mathbf{x},t]$, while the second one only depends on time.

Maxwell Equations with harmonic time-dependence read:

$$\begin{aligned}
\mathbf{div}\overline{\mathbf{E}}[\mathbf{x}] &= \overline{\rho}[\mathbf{x}]/\varepsilon_0 \quad ,\\
\mathbf{div}\overline{\mathbf{B}}[\mathbf{x}] &= 0 \quad ,\\
\mathbf{rot}\overline{\mathbf{E}}[\mathbf{x}] &= +\mathtt{i}\omega\overline{\mathbf{B}}[\mathbf{x}] \quad ,\\
\mathbf{rot}\overline{\mathbf{B}}[\mathbf{x}] &= -\mathtt{i}\varepsilon_0\mu_0\omega\overline{\mathbf{E}}[\mathbf{x}] + \mu_0\overline{\mathbf{j}}[\mathbf{x}] \quad .
\end{aligned}$$

These are sometimes called the time-independent **Maxwell Equations** , implicitly referring to a harmonic time dependence. The equations can be combined to give the vector Helmholtz equation for the Electric|Magnetic fields:

$$\begin{aligned}
\boldsymbol{\nabla}^2\overline{\mathbf{E}}[\mathbf{x}] + \frac{\omega^2}{c^2}\overline{\mathbf{E}}[\mathbf{x}] = 0 \quad ,\\
\boldsymbol{\nabla}^2\overline{\mathbf{B}}[\mathbf{x}] + \frac{\omega^2}{c^2}\overline{\mathbf{B}}[\mathbf{x}] = 0 \quad .
\end{aligned}$$

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46.12 Doppler Effect for ElectroMagnetic Waves

This § is referenced at pages:
[2016, 2047]

Let us only consider ElectroMagnetic waves in vacuum.

Only the relative motion between the generator and the observer is relevant for ElectroMagnetic waves. EM waves in vacuum have the same speed for any observer. This is one of bases of the special theory of relativity.

Read § 59.02.

Compare with the **Doppler effect** for mechanical waves § 45.10.

Radar

The **Doppler effect** can be used by a radar to measure the velocity of detected moving objects.

Auto-Velox

This § is referenced at pages:
[2055]

©|Lim Yung-Kuo, *Problems And Solutions On Mechanics*, 1994, World Scientific, ...Ed., ...|1249||

Real AutoVelox system are based on ElectroMagnetic waves. The principle is the same as Eco-**Doppler** while the quantitative relation is determined by ElectroMagnetic **Doppler effect** .

Read also § 45.12.09.

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46.13 Cherenkov and Transition Radiation for ElectroMagnetic Waves

This § is referenced at pages:
[2134]

46.13.01 Cherenkov Radiation

This § is referenced at pages:
[2222]

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It appears when a charged particle moves in a medium with a speed, $\mathbf{V}$, larger than the phase velocity of ElectroMagnetic waves in the medium (**index of refraction** of the medium: n). It is in fact a kind of shock-wave producing a kind of shock-wave with axial symmetry around the velocity of the particle at an angle from the velocity given by:

$$\cos\left[\theta_c\right] = \frac{c}{nV} \quad .$$

It is produced by the time-changing polarization of matter induced by the moving charge.

46.13.02 Transition Radiation

This § is referenced at pages:
[2222]

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It appears when a charged particle crosses the boundary between two media with different characteristics.

It is produced by the time-changing polarization of matter induced by the moving charge.

DRAFT IN PROGRESS

46.14 Phenomenological Aspects of wave-Polarization of ElectroMagnetic Waves

This § is referenced at pages:
[2144, 2144]

In this section the word **light** will be often use as a synonym of **electromagnetic waves**.

Many important effects depend on the **wave-Polarization** status of ElectroMagnetic radiation.

ElectroMagnetic MPW will be typically considered.

It should be noted that the effects studied in this section require a material medium which fails to fulfill some of the assumptions commonly used so far: homogeneity, isotropy and linearity, zero **conductivity** , the constraint that the **Electric-Permittivity** | **Magnetic-Permeability** are fixed (real) numbers only dependent of the material.

The media considered here cannot be modeled as a perfect dielectric media . Therefore the phenomena discussed in this section, strictly speaking, fall outside the domain of validity of the theory developed so far. However, thanks, basically, to the assumed linearity they may be understood, in the simple form presented here, by using in a appropriate way the theory developed so far.

It is clear that removing some of the properties of a **perfect dielectric medium** may lead to new effects as **wave-Polarization** of light induces a preferential direction and|or a preferential handedness. In fact since linearly wave-Polarized light itself is endowed with a preferred direction, it may interact in a different way with systems which are not isotropic in the plane perpendicular to the wave-vector. If that is the case light with different linear **wave-Polarization** may have different absorption and|or phase speed, for instance. Moreover, since circularly wave-Polarized light itself is endowed with a chiral character, it may interact in a different way with chiral molecules. If that is the case light with different circular **wave-Polarization** may have different absorption and|or phase speed, for instance.

46.14.01 Non-perfect dielectric media: Dichroism (Selective Absorption)

The absorption coefficient (see equation (46.04.01)) depends on the **wave-Polarization** state of light, that is on the direction of the ElectroMagnetic fields (for linear **wave-Polarization**) or on the handedness of the ElectroMagnetic fields (for circular **wave-Polarization**).

46.14.01.01 Linear Dichroism

Typically there exists a preferred direction such that light impinging on the material at right angles with respect to the preferred direction suffers little (minimum) attenuation, when it is linearly wave-Polarized along the preferred direction, while suffers a large (maximum) attenuation while it is wave-Polarized along an orthogonal direction to the preferred direction. It is actually a linear polariser. The transmission axis is defined as the direction of the electric field when light suffers the minimum absorption.

The existence of dichroism experimentally demonstrates that ElectroMagnetic waves are (at least partially) transverse waves, because for longitudinal waves one would not expect a dependence of the transmission on the angle of rotation of the transmission axis in a plane perpendicular to the propagation axis of the wave.

Work in Progress

Examples

Polaroid: preferred direction created by the manufacturing process. It is made of highly-dichroic crystals inside a plastic material.

Tourmaline: little efficiency.

Malus law

The transmitted intensity when light is linearly wave-Polarized at an angle θ with the transmission axis is given by:

$$\boxed{I[\theta] = I_0 \cos^2[\theta]} \quad ,$$

in terms of the incoming light intensity, I_0.

In case of non wave-Polarized light the transmitted intensity is half of the incoming intensity, as it is clear by averaging the $\cos^2[\theta]$.

Examples and Applications

- Two linear polarisers at $\pi/2$ each other: no ElectroMagnetic radiation is transmitted, whatever the **wave-Polarization** of the incident beam.
- non wave-Polarized light going across two linear polarisers at $\pi/2$ each other; add another one in between, not parallel to any of the other two: some ElectroMagnetic radiation is now transmitted. The intensity of the transmitted radiation is: $I_0 \cos^2[\alpha] \sin^2[\alpha]$ as a function of the angle α between the direction of the intermediate polariser and the direction of one of the others two.
  This is a **quantum-like** phenomenon.

46.14.01.02 Circular Dichroism

Circular dichroism, is the differential absorption of left-handed and right-handed circularly wave-Polarized light.

Circular dichroism makes a linearly wave-Polarized ElectroMagnetic wave to become elliptically wave-Polarized.

See section § 46-019.
See § **??**.

46.14.02 Non-perfect dielectric media: Birefringence

46.14.02.01 Linear Birefringence

The phase-velocity of the wave depends on the linear wave-Polarization state of the wave, that is on the direction of the ElectroMagnetic field.

The different phase-velocity can introduce a difference of the phases of the x and y axes components, so that, while passing the medium, the wave-Polarization state changes.

The simplest type of linear birefringence is that of materials with so-called uni-axial anisotropy: the structure of the material is such that it has an axis of symmetry with all perpendicular directions optically equivalent. This axis is known as the optical axis of the material. The two different indices of refraction, along and perpendicular to the optical axis, are known as extraordinary and ordinary.

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Quarter-Wave and Half-Wave Lamina

With a suitable preparation (that is a suitable cutting) of the birefringent material one can build a slab such that two orthogonal axes there exist in the plane of the slab, (the fast and slow axis), such that ElectroMagnetic radiation linearly wave-Polarized along one of the two axes impinging on the slab will be still linearly wave-Polarized along the same direction at the exit of the slab, but the phase velocities are different along the two axes.

Assume a MPW, linearly wave-Polarized, propagating along the z direction. At $z = 0$ it meets a planar lamina perpendicular to the z axis and ending at $z = H$. Let the x axis be the fast axis and the y axis be the slow axis ($n_y > n_x$ and $v_y < v_x$) and let $\Delta n \equiv (n_y - n_x) \geq 0$. The phase of each wave is:

$$\Phi = kz - \omega t - \phi = \frac{\omega}{v} z - \omega t - \phi = \frac{\omega}{c} n z - \omega t - \phi \quad .$$

Assuming that the x and y components are in phase at $z = 0$, the difference in the phase of the two x and y components at the exit of the lamina, $z = H$, at any time, is the time-independent quantity:

$$\Delta\Phi \equiv \Phi_y - \Phi_x = \frac{\omega}{c} n_y H - \frac{\omega}{c} n_x H = \frac{\omega}{c} H (n_y - n_x) \equiv \frac{\omega}{c} H \Delta n = k_0 H \Delta n = \frac{2\pi}{\lambda_0} H \Delta n \quad . \qquad (46.14.01)$$

In the general case of **phase constants** ϕ_y and ϕ_x at $z = 0$ the above relation simply generalizes to:

$$\Delta\Phi \equiv \Phi_y - \Phi_x = k_0 H \Delta n + \Delta\phi = \frac{2\pi}{\lambda_0} H \Delta n + \Delta\phi \quad . \qquad (46.14.02)$$

→ 2140

In the most general case a lamina of birefringent material thus setup will change an elliptic **wave-Polarization** into another elliptic **wave-Polarization** with a different $\Delta\phi$. Exchanging linear and circular **wave-Polarization** are special cases, all cases being described by relation (46.14.02).

Note that the change of phase depends on the wavelength.

Quarter-Wave Lamina

Consider a linearly wave-Polarized MPW with **wave-Polarization** at $\theta = +\pi/4$ or $\theta = -3\pi/4$ ($\theta = -\pi/4$ or $\theta = +3\pi/4$) on the xy plane: the two components E_x and E_y have the same amplitude and same phase (opposite phase) at the entrance. A quarter-wave lamina is such that:

$$|\Delta\Phi| = \pi/2 + q\pi \qquad q \text{ any natural number (non-negative integer number)} \quad .$$

In order to have the solution with minimum thickness one has to choose $q = 0$ and then:

$$\frac{\omega}{c} H \Delta n = \pi/2 \implies H = \frac{\lambda_0}{4\Delta n} \quad , \qquad (46.14.03)$$

→ 2140

where λ_0 is the vacuum wavelength. Under the previous conditions the wave is circularly wave-Polarized at the exit of the lamina. Different choices of the natural number q will give a lamina thicker than the minimum possible thickness given by equation (46.14.03).

In the general case when θ is arbitrary the light exits elliptically wave-Polarized, except for $\theta = \pm\pi/4$ and $\theta = \pm 3\pi/4$ (circular **wave-Polarization**) and $\theta = q\pi/2$ with q any integer (linear **wave-Polarization**).

If a quarter-wave lamina is reached by a circularly wave-Polarized MPW a linear **wave-Polarization** is found at the exit.

Stacking two quarter-wave laminae one after the other one finds at the exit a linear **wave-Polarization** perpendicular to the **wave-Polarization** at the entrance. The device is known as a half-wave lamina.

A quarter-wave lamina which converts linear to circular **wave-Polarization** (or vice-versa) receives an angular momentum from the ElectroMagnetic wave as described in section § 46.05.

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Half-Wave Lamina

A half-wave lamina is such that:

$$|\Delta\Phi| = (q+1)\,\pi \qquad q \text{ any natural number (non-negative integer number)} \quad .$$

In order to have the solution with minimum thickness one has to choose $q = 0$ and then:

$$\frac{\omega}{c} H \Delta n = \pi \implies H = \frac{\lambda_0}{2\Delta n} \quad , \tag{46.14.04}$$

A half-wave lamina will change a circularly wave-Polarized MPW into a circularly wave-Polarized MPW of opposite handedness.

46.14.02.02 Circular Birefringence

This § is referenced at pages:
[1783]

The phase-velocity of the wave depends on the circular wave-Polarization state of the wave, that is on the chirality of the ElectroMagnetic fields and molecules. See § ??.

The different phase-velocity can introduce a difference of the phases of the left/right components, so that, while passing the medium, the wave-Polarization state changes.

Circular birefringence is often known as **optical activity**. It may exist in isotropic media as well.

Circular birefringence depends on the existence of a **Quadrupole** moment of the elementary microscopic charge distribution: therefore **it cannot be described simply in terms of the Polarization|Magnetization of the matter**, which only describe **Dipole** moments of the elementary microscopic charge distribution. It appears in materials such that the **index of refraction** for right-handed circularly wave-Polarized waves is different from the **index of refraction** for left-handed circularly wave-Polarized waves:

$$\boxed{n_R \neq n_L} \quad .$$

It is due to the handedness of elementary constituents of matter (for instance sugar molecules).

The effect is to **rotate the direction of wave-Polarization of linearly wave-Polarized ElectroMagnetic waves**. The rotation does not depend on the initial orientation of the linear **wave-Polarization** . The effect is obvious from equation (46.03.02). It must not be confused with circular **wave-Polarization** : in circular **wave-Polarization** the electric field of the ElectroMagnetic wave at the same place rotates in the plane perpendicular to the direction of propagation while in circular birefringence the ElectroMagnetic wave is linearly wave-Polarized at any point but the direction of **wave-Polarization** changes at different points.

circular birefringence is often masked by linear birefringence.

46.14.03 Electro-Optics and Magneto-Optics Effects

46.14.03.01 Linear Birefringence Induced by Electric Fields

The Pockels and Kerr effects are the linear birefringence induced by an external electric field.

Certain materials exhibit a linear (Pockels) electro-optic effect, linear in the applied electric field. Many isotropic transparent materials and all crystals exhibit a quadratic (Kerr) electro-optic effect, quadratic in the applied field:

$$\boxed{\Delta n = K_E \lambda E^2} \quad .$$

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46.14.03.02 Linear Birefringence Induced by Magnetic Fields

The Voigt and Cotton-Mouton effects are the birefringence induced by an external magnetic field.

Some materials exhibit a linear (Voigt) magneto-optic effect, linear in the applied magnetic field. Some materials exhibit a quadratic (Cotton-Mouton) magneto-optic effect, quadratic in the applied field:

$$\boxed{\Delta n = K_B \lambda B^2} \quad .$$

46.14.03.03 Linear Birefringence Induced by Mechanical Stresses

Birefringence is also induced by mechanical stresses.

46.14.03.04 Induced Circular Birefringence

The Faraday effect is circular birefringence induced by an external magnetic field.

The Verdet law apply:

$$\boxed{\Delta \alpha = V B \Delta z \cos\left[\theta_B\right]} \quad .$$

46.14.04 wave-Polarization of ElectroMagnetic Waves in Nature and Technology

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46.15 Natural Light

This § is referenced at pages:
[2062]

©|E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|||

As often, **light is used as a synonym of ElectroMagnetic radiation**.

An ordinary and natural light source typically consists of a huge number of atomic/molecular light emitters whose positions and orientations are random in space. Excited Atoms|Molecules typically radiate wave-Polarized wave-trains for times of the order of $\Delta\tau \approx 10^{-8}$ s, the so-called coherence time of the source (read § 52.02).

Different Atoms|Molecules and|or the same atom/molecule can emit at different frequencies. All the emitted ElectroMagnetic waves with the same frequency combine to form a single resultant wave-Polarized ElectroMagnetic wave, which persists for times as long as $\approx \Delta\tau$.

Wave-trains are continuously emitted so that the overall **wave-Polarization** changes in a completely unpredictable way.

If these changes take place so quickly that any single resultant **wave-Polarization** state cannot be identified, the ElectroMagnetic wave is referred to as natural light or non wave-Polarized light. However, the name non wave-Polarized light is misleading, since the light is composed of a rapidly varying succession of the different **wave-Polarization** states. Therefore the name randomly wave-Polarized is another way to name it.

As a results natural light is made of finite-length wave-trains with different frequencies, random relative **wave-Polarization** and phases.

All in all, naturally produce ElectroMagnetic waves have the following characteristics different from the simple wave-Polarized MPW discussed so far and requiring a more complex analysis (Read also § 44.03.14):

- real ElectroMagnetic waves may be approximated by plane ElectroMagnetic waves only at large distances from the sources.
- ElectroMagnetic waves at different frequencies may be present and the principle of superposition apply, read § 18;
- at any given frequency, instead of monochromatic (that is infinite length perfect sinusoids) wave-trains there exist finite length wave-trains, that is sinusoids, lasting for times $\Delta\tau$, read § 18;
- each wave-train has a random **wave-Polarization** (see section § 46.15.01);

46.15.01 Polarizers States and Degree of wave-Polarization

This § is referenced at pages:
[2015, 2073, 2075, 2143]

Solutions to Maxwell Equations are always wave-Polarized ElectroMagnetic waves. Any superposition of solutions with arbitrary **wave-Polarization** is a solution thanks to the linearity. A superposition of wave-Polarized solutions with arbitrary **wave-Polarization** gives **partially wave-Polarized** (or totally non wave-Polarized) light.

The formalism describing wave-Polarization states is the same formalism describing states in Quantum Physics.

The wave-Polarization of ElectroMagnetic waves is perhaps the simplest example of quantum two-states system. Its full understanding provides a strong bridge to Quantum Physics.

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46.15.01.01 Polarizers

The degree of **wave-Polarization** of non wave-Polarized radiation can be measured by means of polarisers, instruments capable to select a specific well-defined **wave-Polarization** state from a beam of arbitrary **wave-Polarization** . **A Polarizer only transmits one specific wave-Polarization state while absorbs all the rest**.

A Polarizer is an instrument such that there exist at least one beam which is fully transmitted and at least one beam which is fully absorbed; moreover, if the same beam is passed in sequence through two identical Polarizers the same intensity is found after that first Polarizer and after the sequence of two Polarizers:

$$\boxed{\text{there exist at least one beam such that} \qquad \mathcal{P}I_0 = I = I_0} \quad ,$$

$$\boxed{\text{there exist at least one beam such that} \qquad \mathcal{P}I_0 = 0} \quad ,$$

$$\boxed{\text{for any beam} \qquad \mathcal{P}\mathcal{P}I_0 = \mathcal{P}I_0 \implies \mathcal{P}^2 = \mathcal{P}} \quad .$$

46.15.01.02 States of wave-Polarization

Two Polarizers are said to select two orthogonal **wave-Polarization** states if there exists at least one beam such that the first Polarizer leaves the beam un-attenuated while the second one absorbs it completely and there exists at least another beam such that the second Polarizer leaves the beam un-attenuated while the first one absorbs it completely. Then the two beams are said to carry orthogonal **wave-Polarization** states. Therefore, if $\mathcal{P}_A$ and $\mathcal{P}_B$ are two orthogonal Polarizers

$$\boxed{\mathcal{P}_A^2 = \mathcal{P}_A \qquad \mathcal{P}_B^2 = \mathcal{P}_B \qquad \mathcal{P}_A\mathcal{P}_B = \mathcal{P}_B\mathcal{P}_A = 0} \quad .$$

One example of orthogonal **wave-Polarization** states are two radiation beams linearly wave-Polarized along any two orthogonal directions.

A second example of orthogonal **wave-Polarization** states are two radiation beams one carrying left circular **wave-Polarization** and the other one carrying right circular **wave-Polarization** .

46.15.01.03 Degree of wave-Polarization

Assuming to have Polarizers (read § 46.14) the degree of **wave-Polarization** can be measured, at lest in principle, by considering the minimum and maximum intensities transmitted by all possible Polarizers and defining:

$$\boxed{\mathcal{P} \equiv \frac{I_{\text{MAX}} - I_{\text{MIN}}}{I_{\text{MAX}} + I_{\text{MIN}}}} \quad .$$

Actually: $I_{\text{MIN}} = 0 \implies \mathcal{P} = 1$ and for $I_{\text{MAX}} = I_{\text{MIN}} \implies \mathcal{P} = 0$.

46.15.01.04 Real Polarizers

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How to construct a Polarizer is suggested by the discussion of the phenomenology of the **wave-Polarization** of light in section § 46.14. It is worth remembering the result of problem § 46-020.

Linear Polarizers: a sheet of linearly dichroic material (Polaroid, for instance).

Circular homogeneous Polarizer: a sheet of linearly dichroic material (Polaroid, for instance) sandwiched between two quarter-wave laminas with axes rotated by $\pm\pi/2$ with respect to one another with the transmission axis of the linearly dichroic material at $\pm\pi/4$ with respect to the fast/slow axes of the quarter-wave lamina.

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wave-Polarization of real ElectroMagnetic waves

The degree of **wave-Polarization** of a general ElectroMagnetic wave is defined as the fraction of intensity of wave-Polarized light with respect to the total light intensity:

$$\mathcal{P} \equiv \frac{I_{\mathrm{P}}}{I_{\mathrm{P}} + I_{\mathrm{NP}}} \quad .$$

wave-Polarization of non wave-Polarized light (in general: change of the degree of **wave-Polarization**) can happen (among others means) by:

- reflection;
- scattering;
- polarisers.

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46.16 ElectroMagnetic Spectrum and the Speed of Light

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©|Mazzoldi & Nigro & Voci, , 1998, EdiSES, 2ndEd., ...|13.12|*Extensive discussion on the ElectroMagnetic ...*

The observation of stellar phenomena allows to quantify to a high degree the fact that vacuum is non dispersive at optical wavelengths.

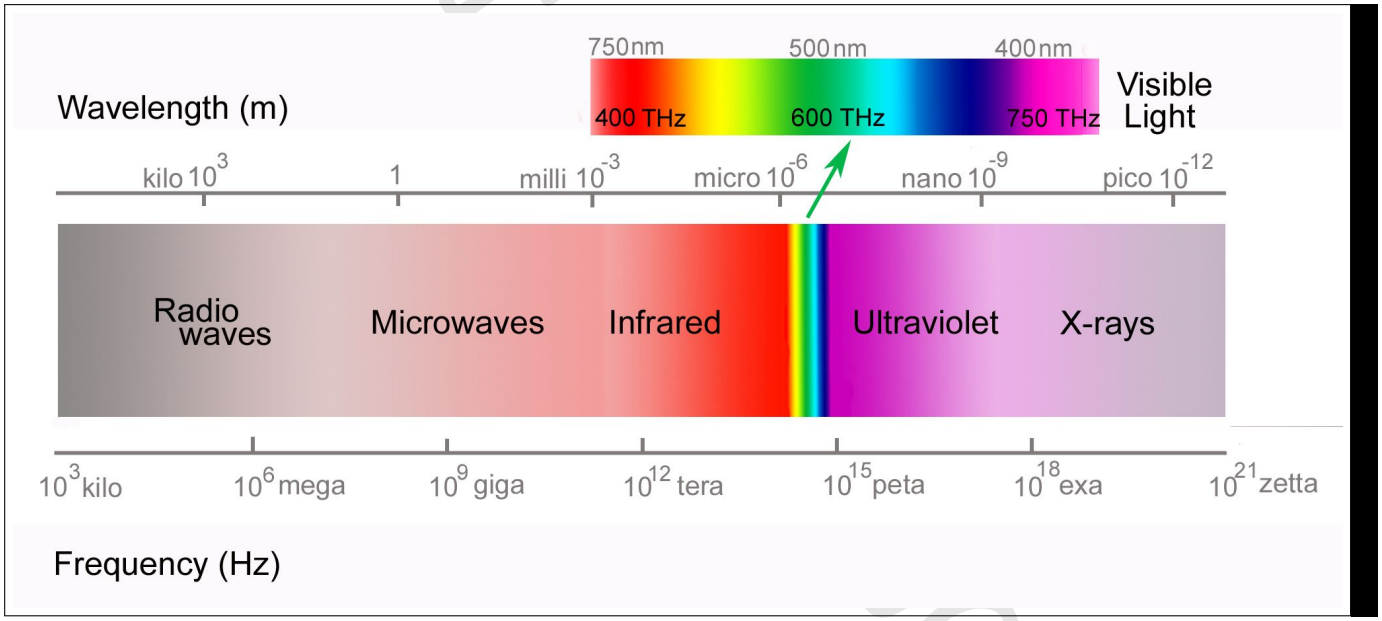

Figure 46.11: EM spectrum - Source: NIST.

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46.17 Examples and Physical Applications

46.17.01
Capacitor with varying currents

©|WEB - URL |ElectroMagnetic Field and Energy - H.A.Haus & J.R.Melcher|*Example 11.2.1*|

46.17.02
Solenoid with varying currents

©|WEB - URL |ElectroMagnetic Field and Energy - H.A.Haus & J.R.Melcher|*Example 11.2.2*|

46.17.03
Energy dissipation due to polarization currents

©|WEB - URL |ElectroMagnetic Field and Energy - H.A.Haus & J.R.Melcher|*§ 11.5 and Example 11.5.2*|

46.17.04
Energy dissipation due to magnetization currents

©|WEB - URL |ElectroMagnetic Field and Energy - H.A.Haus & J.R.Melcher|*§ 11.5 and Example 9.5.2 and Demonstrati- ...*

46.17.05
Complex Formalism for wave-Polarization

Show that a ElectroMagnetic MPW is linearly wave-Polarized if and only if:

$$\mathbf{E}\wedge\mathbf{E}^{\star}=0 \quad .$$

Any real vector gives a linearly wave-Polarized wave.

Show that a MPW is circularly wave-Polarized if and only if:

$$\mathbf{E}\cdot\mathbf{E}=0 \quad .$$

Note the for complex vectors this does not imply at all that the vector is zero but only that the real and imaginary parts are orthogonal each other.

46.17.06
Cellular Phones

A cell phone handset transmits $P=1$ W of power. What is the average Electric|Magnetic field at a distance of $d=0.1$ m from the handset? Assume the ElectroMagnetic wave propagates in air, whose ElectroMagnetic properties are quite similar to vacuum.

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46.17.07
Average Value of the Poynting Vector and ElectroMagnetic Fields Near a Bulb-Lamp

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|||

Consider 200 W lamp isotropically emitting a fraction $\eta = 0.5$ of power, $\Pi = 100$ W, in the visible. The light from the lamp is carried by ElectroMagnetic radiation (see section § 46) and the energy flux (energy per unit time per unit area) is thus given by the Poynting vector.

Calculate the average value of the Poynting vector at a distance $d = 5$ m from the lamp.

EM waves in vacuum have ElectroMagnetic fields linked by $E = cB$ and there is little difference in air: determine the time average of the square of the Electric|Magnetic fields.

WORKED SOLUTION

$$I = \langle S \rangle = \frac{\Pi}{4\pi d^2} = 0.31 \text{ W/m}^2 \quad ,$$

$$I = \langle S \rangle = \frac{\langle E^2 \rangle}{\mu_0 c} = \frac{c\langle B^2 \rangle}{\mu_0} \Longrightarrow \langle E^2 \rangle = \mu_0 c I \quad ,$$

$$\langle B^2 \rangle = \mu_0 I / c \quad ,$$

$$\langle E \rangle \equiv \sqrt{\langle E^2 \rangle} = 1.1 \cdot 10^1 \text{ V/m} \quad ,$$

$$\langle B \rangle \equiv \sqrt{\langle B^2 \rangle} = 3.6 \cdot 10^{-8} \text{ T} \quad .$$

Some comments are required.
$\langle E \rangle$ is an appreciable macroscopic field, easly detectable in a Laboratory.
$\langle B \rangle$ is very small, from the point of view of a Laboratory, because it has to be compared with the Earth magnetic field of the order of 0.5 gauss $= 0.5 \cdot 10^{-4}$ T.
In practice instruments usually measure the electric field rather than the magnetic field.
However, the average energy stored by Electric|Magnetic fields in ElectroMagnetic waves are the same and $\langle E \rangle = c\langle B \rangle$. One cannot say the electric field in ElectroMagnetic waves is stronger than the magnetic field. One can only say that, when compared to **typical** fields on the Earth electric fields of ElectroMagnetic waves are relatively bigger than magnetic fields of EM waves are.

46.17.08
Sun-Bath

The intensity of the solar radiation on top of the atmosphere is about $I = 1.2$ kW/m^2. Neglecting the atmospheric absorption the radiation pressure is thus: $p_F \approx 3 \cdot 10^{-6}$ Pa.

46.17.09
Tails of Comets

Small dust speckles are subject to both the gravitational attraction of the comet nucleus and the Sun radiation pressure: they are thrown away if they are too small as the gravitational force goes like the cube of the size of the particle, while the radiation pressure force goes like the square of of the size of the particle.

Both the radiation pressure force and the gravitational force goes like the inverse square distance from the Sun: depending on the size of the dust speckle one wins over the other.

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46.17.10
Superposition of Two Monochromatic ElectroMagnetic MPW Along the Same Direction

Consider the superposition of two monochromatic ElectroMagnetic MPW along the same direction with the same frequency in vacuum.

The resulting ElectroMagnetic wave is the linear superposition of the ElectroMagnetic fields.

Note, however, that **the energy density and the Poynting vector do not sum, in general**:

$$u \equiv \frac{\varepsilon_0}{2}(\mathbf{E}_1+\mathbf{E}_2)^2+\frac{1}{2\mu_0}(\mathbf{B}_1+\mathbf{B}_2)^2 \neq u_1+u_2 \quad ,$$

$$\mathbf{S}_{\text{EM}} \equiv \frac{1}{\mu_0}(\mathbf{E}_1+\mathbf{E}_2)\wedge(\mathbf{B}_1+\mathbf{B}_2) \neq \mathbf{S}_{\text{EM}1}+\mathbf{S}_{\text{EM}2} \quad .$$

The equality may hold in specific cases.

46.17.11
One Wave-Guide

Visible light: $\lambda \in [300\ \text{nm}; 700\ \text{nm}]$ with $\nu = c/\lambda \approx 10^{15}$ Hz.

Propagation in a wave-guide: $\omega = 500$ MHz and $\omega \geq \omega_c = pc\pi/\ell$ giving $\ell \geq 0.3$ m.

46.17.12
Calculation of group-velocity from the index of refraction

This § is referenced at pages:
[1163, 1165]

Read § 18.04.02.

Read § 18.05.05.

Read § 44.03.11 and equation (44.03.08).

The index of refraction for a gas can be often expressed in terms of the Sellmeier formula:

$$n[\lambda] = 1+\frac{a}{\frac{1}{b^2}-\frac{1}{\lambda^2}} \quad , \tag{46.17.01}$$

with a and b suitable constants depending on the gas.

1. Show that the **phase-velocity** is:

$$\frac{\lambda^2-b^2}{b^2\left(a\lambda^2-1\right)+\lambda^2} \quad . \tag{46.17.02}$$

2. Show that the **group-velocity** is:

$$\frac{b^4\left(1-3a\lambda^2\right)+b^2\lambda^2\left(a\lambda^2-2\right)+\lambda^4}{\left(b^2\left(a\lambda^2-1\right)+\lambda^2\right)^2} \quad . \tag{46.17.03}$$

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46.17.13
Kerr cell

46.17.14
Measurement of cosmic magnetic fields

46.17.15
wave-Polarization in 3D cinema

46.17.16
Upper limit on the velocity of ElectroMagnetic energy flow

Show that, from the definition (44.04.02), the speed of energy transport for ElectroMagnetic fields is always less than the speed of light in vacuum.

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46.18 Exercises Problems and Physical Applications

46-008 ✓ **I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...**

Problems on ElectroMagnetic energy:
4.203, 4.204, 4.205, 4.206, 4.207, 4.208, 4.209, 4.210.

46-009 ✓ **I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...**

Problems on ElectroMagnetic waves:
4.189, 4.190, 4.192, 4.193, 4.194, 4.195, 4.196, 4.197, 4.198, 4.199, 4.200, 5.159, 5.161, 5.162, 5.164, 5.178, 5.199.

46-010 ✓ **I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...**

Problems on ElectroMagnetic attenuation:
5.213, 5.214, 5.215, 5.216, 5.217, 5.218, 5.219, 5.220, 5.221, 5.222, 5.223.

46-011 ✓ **A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...**

8.08, 8.09, 8.11.

46-012 ✓ **Example of a EM Solution to d'Alembert Equation**

Consider a divergence-less electric field which satisfies equation (46.01.01) and add an electric field with non-zero divergence and zero rotor. The resulting field, clearly, does not satisfy **Maxwell Equations** .
Show that the resulting field it nevertheless satisfies equation (46.01.01).

46-013 ✓ **Most General One-Dimensional ElectroMagnetic Wave**

Consider a plane ElectroMagnetic wave propagating, in vacuum, in the positive z direction, with $\mathbf{E}$ along the x axis (that is ElectroMagnetic wave wave-Polarized along x). Show that the most general solution to this problem is:

$$\mathbf{E}[\mathbf{x},t] = \widehat{\mathbf{e}}_1 f[kz-\omega t] \quad ,$$
$$c\mathbf{B}[\mathbf{x},t] = \widehat{\mathbf{e}}_2 f[kz-\omega t] \quad ,$$

with f an arbitrary function.

46-014 ✓ **General Elliptic wave-Polarization**

Demonstrate equation (46.03.01).

WORKED SOLUTION

The electric field at some fixed point at $z=\overline{z}$, as a function of time, is:

$$E_x = E_{x0}\cos[k\overline{z}-\omega t-\phi_x] \equiv E_{x0}\cos[\mathcal{Z}] \quad ,$$
$$E_y = E_{y0}\cos[k\overline{z}-\omega t-\phi_y] \equiv E_{y0}\cos[\mathcal{Z}+\Delta\phi] = E_{y0}\left(\cos[\mathcal{Z}]\cos[\Delta\phi]-\sin[\mathcal{Z}]\sin[\Delta\phi]\right) \quad ,$$

with $\mathcal{Z}\equiv(k\overline{z}-\omega t-\phi_x)$ and $\Delta\phi\equiv\phi_y-\phi_x$. Then

$$\frac{E_x}{E_{x0}} = \cos[\mathcal{Z}] \quad ,$$

$$\frac{E_y}{E_{y0}} - \cos[\mathcal{Z}]\cos[\Delta\phi] = -\sin[\mathcal{Z}]\sin[\Delta\phi] \quad ,$$

and summing after squaring both equations one finds equation (46.03.01).

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46-015 ✓ General Elliptic wave-Polarization : Special Cases

Start from the general equation for elliptic **wave-Polarization** , equation (46.03.01), and demonstrate under which conditions the **wave-Polarization** is linear and circular.

46-016 ✓ Left and Right Elliptic wave-Polarization

This § is referenced at pages:
[2072, 2072]

Consider a MPW propagating along the positive direction of the z axis. The electric field at some fixed point at $z = \overline{z}$, as a function of time, is:

$$E_x = E_{x0} \cos\left[k\overline{z} - \omega t - \phi_x\right] \quad ,$$
$$E_y = E_{y0} \cos\left[k\overline{z} - \omega t - \phi_y\right] \quad .$$

Show that an observer looking the incoming wave, that is looking towards the negative z axis, sees the electric field (and therefore also the magnetic field) vector rotating clockwise (negative helicity, right-handed **wave-Polarization**) if $\sin\left[\Delta\phi\right] \equiv \sin\left[\phi_y - \phi_x\right] > 0$ and counter-clockwise (positive helicity, left-handed **wave-Polarization**) if $\sin\left[\Delta\phi\right] \equiv \sin\left[\phi_y - \phi_x\right] < 0$.

WORKED SOLUTION

First Solution

Consider the vector $\mathbf{E}$ in the xy plane. In analogy to the position vector, whose sense of rotation is given by the component along z of the angular momentum, one can calculate the projection on the z axis of the vector $\mathbf{E} \wedge \dot{\mathbf{E}}$. The result is:

$$\widehat{\mathbf{e}}_3 \cdot \left(\mathbf{E} \wedge \dot{\mathbf{E}} \right) = E_{x0} E_{y0} \omega \sin\left[-\Delta\phi\right] \quad .$$

Second Solution

Consider the time-derivative of the electric field at a time when $k\overline{z} - \omega t - \phi_x = 0$. Its x component, at the chosen time, is zero, while its y component is:

$$\dot{E}_y = +\omega E_{y0} \sin\left[k\overline{z} - \omega t - \phi_x - \Delta\phi\right] = +\omega E_{y0} \sin\left[-\Delta\phi\right] \quad .$$

At the chosen time the x component of the electric field is positive and it takes its maximum value and its time derivative is zero.
At the chosen time the y component of the electric field has a time derivative which is positive if $\Delta\phi < 0$ (the sense of rotation is therefore counter-clockwise), while it is negative if $\Delta\phi > 0$ (the sense of rotation is therefore clockwise).

46-019 Circular dichroism on a linearly wave-Polarized ElectroMagnetic wave

This § is referenced at pages:
[2139]

Show that circular dichroism makes a linearly wave-Polarized ElectroMagnetic wave to become elliptically wave-Polarized.

46-020 ✓ Intensity of an ElectroMagnetic Wave in Vacuum With Arbitrary wave-Polarization

This § is referenced at pages:
[2144]

Consider a ElectroMagnetic MPW moving in vacuum along the positive z direction, with electric field having the x and y components E_x and E_y and generic **wave-Polarization** . Show that the intensity of the ElectroMagnetic wave can be written as:

$$I = \frac{c\varepsilon_0}{2}\left(E_{x0}^2 + E_{y0}^2\right) \equiv I_x + I_y \quad ,$$

whatever the phase relation between the x and y components of the fields.

WORKED SOLUTION

Use the real representation of the fields.

$$I = \langle|\mathbf{S}_{\text{EM}}|\rangle = c\varepsilon_0\langle\,\mathbf{E}\cdot\mathbf{E}\rangle = c\varepsilon_0\left(E_{x0}^2\cos^2[\ldots] + E_{y0}^2\cos^2[\ldots]\right) = \frac{c\varepsilon_0}{2}\left(E_{x0}^2 + E_{y0}^2\right) = I_x + I_y \quad .$$

Note that any pair of orthogonal **wave-Polarization** states will give the same decomposition so that:

$$\mathbf{E}_{01}\cdot\mathbf{E}_{02} = 0 \implies I = I_1 + I_2 \quad .$$

46-021 ✓ Radiation Pressure From the Sun at the Earth During a Sun-Bath

Calculate the average pressure on a person taking a Sun-bath if the Sunlight intensity is of the order of $I \approx 10^1 \ \text{kW/m}^2$.

46-022 ✓ Pressure From a Laser Pointer

Estimate the radiation pressure from a typical Laser pointer.

46-023 ✓ A Laser Pointer and Sun Radiation

Assume that the same intensity of sunlight at the top of atmosphere, $1.4 \ \text{kW/m}^2$, reaches the Earth surface.

If you could safely look directly the sun at noon, estimate the power entering your eye. Assume the eye pupil can become as small as one mm diameter.

Compare the power with a typical safe laser pointer of a few mW.

46-024 ✓ Radiation Pressure From non wave-Polarized EM radiation at Oblique Incidence

Consider a beam of non wave-Polarized MPW (this is an approximate description) at incidence θ absorbed by a plane surface. Show that the normal radiation pressure (force per unit area perpendicular to the surface of separation) is given, in terms of the pressure the wave would give at normal incidence, as:

$$p[\theta] = p_0\cos^2[\theta] \quad . \tag{46.18.01}$$

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46-025 ✓ Radiation Pressure From non wave-Polarized Isotropic EM Radiation

Consider non wave-Polarized isotropic ElectroMagnetic radiation, with intensity I, either absorbed or reflected by a plane surface. Show that the normal radiation pressure (force per unit area perpendicular to the surface of separation) is given, in terms of the pressure the wave wopuld give at normal incidence, as:

$$p = \frac{I}{3c} \quad \text{total absorption} \qquad p = \frac{2I}{3c} \quad \text{total reflection} \quad .$$

46-026 Polar Navigation Exploiting the wave-Polarization of the Sky

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46-027 Finding the Solar Constant

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46-028 Lambertian Diffuser

This § is referenced at pages:
[2589]

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46-029 Impedence for Homogeneous EM MPW

This § is referenced at pages:
[2092]

Demonstrate equation (46.06.24).

46-030 ✓ A Simple Model for Faraday Effect

Consider an elementary classical model for the microscopic interpretation of the Faraday effect. A constant and uniform magnetic field, $\mathbf{B}$, is present along the z direction. A plane circularly wave-Polarized ElectroMagnetic wave propagates along the z axis:

$$\mathbf{E}[\mathbf{r}] = E_0 \left(\sin\left[kz - \omega t\right] \widehat{\mathbf{e}}_1 + \sin\left[kz - \omega t + \epsilon^{\pi}/2\right] \widehat{\mathbf{e}}_2\right) \quad ,$$

where $\epsilon = -1$ corresponds to positive helicity and $\epsilon = +1$ corresponds to negative helicity.

Assume one single resonance frequency at ω_0. Show that the elementary classical theory of dispersion leads to the formula:

$$n_+^2 - n_-^2 = \frac{e^2}{\varepsilon_0 m_e} \eta_{\mathbb{N}} \left(\frac{1}{\omega_0^2 - \omega^2 + eB\omega/m} - \frac{1}{\omega_0^2 - \omega^2 - eB\omega/m} \right) \quad .$$

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CHAPTER 47

Waves in Bounded Regions

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This § is referenced at pages:
[2002, 2005, 2062, 2133]

Resonant Cavities Transmission Lines and wave-guides

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47.01 Standing Waves

Standing waves, as opposed to traveling waves, are **waves existing in at least partially bounded regions of space with all the points in space oscillating with the same time dependence**. **The wave is standing wave along the directions such that the time-averaged energy flux is zero**.

Read § 47.04.01 for a wave-guide with standing waves in the directions perpendicular to the direction of propagation along the guide, along which direction the wave is a traveling wave: a mixed standing-traveling wave.

Waves such that the time and space dependence are factorized, as described in section § 44.02.03.06, are standing waves:

$$\boxed{\xi[\mathbf{r},t] = f[\mathbf{r}]g[t]} \quad ,$$

as at every point in space the time dependence is identical.

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47.02 Standing Mechanical Waves

47.02.01 The Plucked|Struck|Bowed String

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- The Plucked string: guitar.
- The Struck string: piano.
- The bowed string: violin.

47.02.02 The Oscillating Membrane

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47.02.03 Mechanical Oscillations of 3D Objects

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47.03 Standing ElectroMagnetic Waves in one dimension

Examples of ElectroMagnetic waves traveling in vacuum with boundaries made of **perfect conductors** are studied.

Note that if the conductor is not perfect the ElectroMagnetic wave will drop to zero in the skin-depth length as discussed in section § 54.02.

Note that the behavior of ElectroMagnetic waves in vacuum is quite different from the behavior in a wave-guide or cavity. In fact, in bounded regions, the governing equation is not only the **D'Alembert non-dispersive wave-equation** , Maxwell equation have to be satisfied as well, one has a system of PDE different from the one in vacuum. Therefore, the **Dispersion Relation** is different.

47.03.01 Creation by Reflection from a perfect conductor

This § is referenced at pages:
[2160, 2207, 2208, 2253, 2253]

In this section an example of one-dimensional **standing ElectroMagnetic wave** is built.

Consider a ElectroMagnetic MPW traveling along the positive z direction, in the $z<0$ half-space, with **wave-Polarization** along the x axis, impinging on a perfect conductor filling the half-space $z\geq 0$. The ElectroMagnetic fields, neglecting a non-essential phase-constant, are:

$$E_x = +E_{x0}\cos\left[+kz-\omega t\right] \quad ,$$
$$B_y = +B_{y0}\cos\left[+kz-\omega t\right] \quad .$$

As the conductor is a perfect conductor the electric field inside must be zero.

Moreover, the tangential component of the electric field must be conserved at the boundary and the electric field of the ElectroMagnetic MPW is tangential to the boundary.

As the above incoming electric field is not necessarily zero at $z=0$ another electric field must be generated such that the sum of the two electric fields is zero at $z<0$ near $z=0$ in order to satisfy the continuity of the tangential component of the electric field.

As a result one finds that in order to satisfy the continuity of the tangential component of the electric field a reflected wave must be generated, which is assumed to be a ElectroMagnetic MPW traveling along the negative z direction, in the $z<0$ half-space. This is a second ElectroMagnetic wave traveling in the negative direction of the z axis in the $z<0$ half-space which sums to the incoming ElectroMagnetic wave. In order to satisfy the boundary condition for $\mathbf{E}$ the reflected wave must have an electric field equal and opposite to the incoming wave at $z=0$ and therefore its **wave-Polarization** must be along the x axis. As a consequence of the requirements on the electric field the magnetic field will be identical to the incoming magnetic field, at $z=0$, because $\mathbf{k}$ will be reversed too. Therefore:

$$E_x^R = -E_{x0}\cos\left[-kz-\omega t\right] \quad ,$$
$$B_y^R = +B_{y0}\cos\left[-kz-\omega t\right] \quad .$$

The reflection from the perfect conductor is exactly what happens when light is reflected at normal incidence from a mirror.

In the $z<0$ half-space the electric field will be the sum of the incoming plus reflected wave:

$$z\leq 0 \implies E_x = E_{x0}\cos\left[+kz-\omega t\right] - E_{x0}\cos\left[-kz-\omega t\right] = 2E_{x0}\sin\left[\omega t\right]\sin\left[kz\right] \quad ,$$
$$z\leq 0 \implies B_y = B_{y0}\cos\left[+kz-\omega t\right] + B_{y0}\cos\left[-kz-\omega t\right] = 2B_{y0}\cos\left[\omega t\right]\cos\left[kz\right] \quad .$$

Note that the magnetic field inside the perfect conductor must be constant, that is zero in absence of other static magnetic fields.

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At $z = 0$ the electric field is zero but the magnetic field, in general, it is not, except at times such that $\cos[\omega t] = 0$. As we are supposing that at $z > 0$ there is a perfect conductor **a surface current will arise** at $z = 0$ to fulfill the boundary condition for the tangential components of the magnetic field.

This **plane infinite surface current sheet is the real generator of the reflected and part of the transmitted ElectroMagnetic wave**. In fact, it generates a plane ElectroMagnetic wave in both directions, going towards positive z ($\mathbf{E}^{g+}/\mathbf{B}^{g+}$) and negative z ($\mathbf{E}^{g-} \equiv \mathbf{E}^{R}/\mathbf{B}^{g-} \equiv \mathbf{B}^{R}$), as shown in section § 49-004. Therefore, the reflected EM wave sums to the incident ElectroMagnetic wave; the ElectroMagnetic wave going in the direction of positive z also sums and perfectly cancels the incident wave in the $z > 0$ semi-space so that there is no transmitted ElectroMagnetic wave.

The electric field is always zero at points where $\sin[kz] = 0$ (for $z < 0$), while the magnetic field is always zero at points where $\cos[kz] = 0$ (for $z < 0$). The Electric|Magnetic fields, therefore, are not in phase in space: they are out of phase by $\lambda/4$. Moreover, also the time dependence is out of phase by $T/4$.

47.03.02
One dimensional EM cavity

This § is referenced at pages:
[2161, 2181, 2207, 2208]

Consider section § 47.03.01. If one imagines to put another perfectly conducting wall at any $\overline{z} < 0$ such that $\sin[k\overline{z}] = 0$ the ElectroMagnetic waves are trapped between $-L \equiv \overline{z} \leq z \leq 0$. The system is now a very simple example of what is known as a one-dimensional ElectroMagnetic cavity. If the frequency of the wave is fixed and the length of the one-dimensional cavity is L, steady solutions can exist if and only if:

$$k = q\frac{\pi}{L} \qquad q \text{ any integer, } k \text{ is quantized; in this problem } q < 0 \quad . \tag{47.03.01}$$

→ 2160

Note the analogy with the problem of the vibrating string (read § 45.06, § 44.02.02.04): in fact the equation is the same. So the current problem is totally analogous to the transverse vibrations of a string with the two ends fixed, where the electric field is the analogous of the transverse displacement of the string.

As the real generator of the reflected ElectroMagnetic wave is the infinite plane surface current sheet at $z = 0$, excited by the incoming ElectroMagnetic wave, there will be also a ElectroMagnetic wave, at $z > 0$ going towards positive z ($\mathbf{E}^{g+}/\mathbf{B}^{g+}$), which exactly cancel the incoming ElectroMagnetic wave at $z > 0$, so that the transmitted wave is zero:

$$E_x^{g+} = -E_{x0}\cos[+kz - \omega t] \Longrightarrow E_x^T = 0 \quad ,$$
$$B_y^{g+} = -B_{y0}\cos[+kz - \omega t] \Longrightarrow B_y^T = 0 \quad .$$

Al the combinations of sinus and cosinus are allowed for both Electric|Magnetic fields:

$$E_x \sim \sin[\omega t]\sin[kz] \qquad E_x \sim \sin[\omega t]\cos[kz] \qquad E_x \sim \cos[\omega t]\sin[kz] \qquad E_x \sim \cos[\omega t]\cos[kz] \quad ,$$

with different values of k, quantised similarly to equation (47.03.01).

It can be shown by direct calculation that the above expressions solve Maxwell equations, as it is necessary to be qualified as ElectroMagnetic waves.

These kind of waves are known as standing waves, as opposed to traveling waves. **The time-average of the energy flux at any point is zero, as it can be shown from the ElectroMagnetic fields expressions**. That is, the energy flux is not zero but its time average is zero at any point.

Even if these standing waves have been built by superposing traveling MPW, they show a type of solutions to Maxwell equations which are very different from MPW.

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47.04 Standing ElectroMagnetic Waves in two dimensions

47.04.01 Mixed Standing-Traveling Waves in Rectangular Wave-Guides

This § is referenced at pages:
[2012, 2065, 2080, 2157]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|5.3||

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|9.5.2|*general solution*|

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|24.2||

In this section an example of ElectroMagnetic waves in a wave-guide is presented. Ordinary electronic circuits fails at high frequencies because the effects of **distributed capacitance and inductance** become severe. In this case, wave-guides can be used.

While transmission line is a pair of electric conductors carrying electric signals from one place to another, a wave guide is a device that guides waves, with minimal loss of energy while restricting the transmission of energy into one direction. It may refer to any kind of wave.

This kind of ElectroMagnetic waves are **standing waves** in two dimensions, x and y, and **traveling waves** in the third dimension. In other word this kind of wave is a **mixed standing-traveling wave**.

Wave-guides propagate ElectroMagnetic waves whose characteristics are very different and more complex from those of MPW in a **perfect dielectric medium** or vacuum. These characteristics are: **dispersion**, **absorption**, **frequency cut-off**, **non pure transversality** and **complex wave-vector**.

47.04.01.01 Modes TE(px,0) as Special Solutions

Consider a rectangular wave-guide along z with its sides made of a perfect conductor and its section a rectangle with sides ℓ_x and ℓ_y and vacuum inside. Let us study ElectroMagnetic wave propagation along the z axis.

Boundary conditions must be satisfied at the perfect conductor, such that the tangential component of the electric field at the boundary is zero (as it is zero inside the conductor) and the normal component of the magnetic field at the boundary is zero (as it is zero inside the conductor).

EM Fields

Let us try a simple solution with time-harmonic behavior: electric field along y and independent of y, which is transverse to the direction of propagation as it happens for ElectroMagnetic MPW, for instance:

$$\mathbf{E}[\mathbf{r},t] = E_y \widehat{\mathbf{e}}_2 \quad ,$$

$$E_y[x,z,t] = E_{0y} \sin[k_x x] \cos[k_z z - \omega t] \implies E_y[x,z,t] = \mathfrak{Re}\left[E_{0y} \sin[k_x x] \exp[\mathtt{i}\,(k_z z - \omega t)]\right] \quad ,$$

whose divergence is zero everywhere, as demanded by **Maxwell Equations** . The replacement of the above tentative solution for the electric field into **D'Alembert non-dispersive wave-equation** gives the following condition:

$$E_y \left(-k_x^2 - k_z^2 + \frac{\omega^2}{c^2}\right) = 0 \implies \boxed{k_z = \pm\sqrt{\frac{\omega^2}{c^2} - k_x^2}} \quad .$$

However, also all the boundary conditions must be satisfied. The Sinusoidal|Cosinusoidal term allows to satisfy at any point the boundary condition for the tangential component of the electric field provided:

$$\boxed{k_x = \frac{p_x \pi}{\ell_x} \qquad p_x \in \mathbb{Z} \qquad k_x \text{ is quantized; in this problem } p_x > 0} \quad ,$$

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that is a **quantization condition on** k_x, similar to those for the vibrating string with fixed ends (read § 45.06, § 44.02.02.04) or the one-dimensional ElectroMagnetic cavity described in section § 47.03.02.

The quantization condition on k_x translates into a **quantization condition for** k_z:

$$-k_x^2 - k_z^2 + \frac{\omega^2}{c^2} = 0 \implies \boxed{k_z = \pm\sqrt{\frac{\omega^2}{c^2} - k_x^2} = \pm\frac{1}{c}\sqrt{\omega^2 - \omega_c^2}} \quad ,$$

where the **critical|cut-off frequency**, ω_c, has been introduced:

$$\boxed{\omega_c = p_x \frac{\pi c}{\ell_x}} \quad ,$$

whose meaning will become clear shortly.

The magnetic field can be calculated from the electric field and from the third Maxwell equation:

$$\partial_t \mathbf{B} = -\mathbf{rot}\mathbf{E} = +\widehat{\mathbf{e}}_1 \mathtt{i} E_y k_z - \widehat{\mathbf{e}}_3 E_{0y} k_x \cos[k_x x] \exp[\mathtt{i}(k_z z - \omega t)] \quad ,$$

$$\text{with the complex notation: } \partial_t \{\ldots\} \quad \Leftrightarrow \quad -\mathtt{i}\omega\{\ldots\} \quad ,$$

$$B_x = -\frac{E_y k_z}{\omega} \implies \boxed{B_x[x,y,z,t] = -\frac{E_{0y} k_z}{\omega}\sin[k_x x]\cos[k_z z - \omega t]} \quad ,$$

$$B_z[x,y,z,t] = -\mathtt{i}\frac{E_{0y} k_x}{\omega}\cos[k_x x]\exp[\mathtt{i}(k_z z - \omega t)] \implies \boxed{B_z[x,y,z,t] = +\frac{E_{0y} k_x}{\omega}\cos[k_x x]\sin[k_z z - \omega t]} \quad ,$$

which is divergenceless, by direct calculation. **The magnetic field has two components**, even if the electric field has only one. **A longitudinal component of the magnetic field exists**, B_z, which is not constant and not uniform. Therefore, even if the starting point was a purely transverse electric field, it turned out that the magnetic field has also a longitudinal component.

The components E_y and B_x are in phase between each other but they are both out-of-phase by $\pi/2$ both in time and space with respect to B_z.

Some Properties of the Solutions

The wavelength corresponding to vacuum is:

$$\boxed{\lambda_0 = \frac{2\pi c}{\omega}} \quad .$$

The **phase-velocity** depends on the frequency of the wave so that there is **dispersion**:

$$\boxed{v = \frac{\omega}{k_z} = \pm\frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - k_x^2}} = \pm\frac{c}{\sqrt{1 - \frac{\omega_c^2}{\omega^2}}} \qquad \text{with } |v| \geq c} \quad .$$

The **phase-velocity is larger than the speed of light in vacuum**, but this is no problem as **phase-velocity** is not the speed of signal, information or energy transport. Therefore, in a wave-guide, monochromatic traveling waves with **phase-velocity** larger than the speed of light exists.

The wavelength is always larger than it would be in vacuum:

$$\boxed{\lambda = vT = \frac{\omega}{k_z}\frac{2\pi}{\omega} = \frac{2\pi}{k_z} = \frac{\lambda_0}{\sqrt{1 - \frac{\omega_c^2}{\omega^2}}} \geq \lambda_0} \quad .$$

Let us discuss some limiting cases. Consider a wave-guide very large with respect to the wavelength:

- the critical frequency goes to zero:
$$\ell_x \to \infty \implies k_z = \pm\omega/c \qquad \lambda = \lambda_0 \qquad v = c \qquad \omega_c = 0 \quad ,$$
as it should;
- the quantization of k_x disappears, as the minimum absolute value of k_x is zero and the spacing

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between different allowed values of k_x, tends to zero:

$$\Delta k_x = \frac{\pi}{\ell_x}\Delta p_x \quad ,$$

as it should;

- the longitudinal component of the magnetic field goes to zero.

The MPW is thus recovered.

Consider high frequencies. If ω increases then:

- k_z also increases and tends to $k_z = \pm\omega/c$;
- λ_0 and λ both decrease and tend to the same value λ_0;
- the ratio between the amplitudes of the Electric|Magnetic fields tends to c as for plane ElectroMagnetic waves.

In fact for high-frequency waves with small wavelength the boundary have no effect. This is why light passes by long holes and pipes.

Consider high frequencies. If ω decreases then:

- k_z also decreases, until it reaches zero, at the critical frequency;
- while λ_0 λ both increase and λ tends to infinity at the critical frequency.

In fact, for low-frequency waves k_z becomes a pure imaginary number. The meaning of an imaginary wave-number can be understood as follows:

$$k_z = \pm \mathtt{i} k'_z \qquad \text{with } k'_z \equiv \pm\frac{1}{c}\sqrt{\omega_c^2 - \omega^2} \in \mathbb{R} \quad ,$$

$$E_y[x,z,t] = E_{0y}\sin[k_x x]\exp\left[\mp k'_z z\right]\exp[-\mathtt{i}\omega t] \quad .$$

Therefore, **below the critical frequency** the wave equation is satisfied by a wave **exponentially decreasing** along z and with a time harmonic behavior and the **phase-velocity** is a pure imaginary number like k_z.

It is worth remembering that, by definition, the frequency is always a real number, so that consistently with the relation $\omega = k_z v$, the **phase-velocity** is a real number if and only if k_z is a real number.

The sign in front of k'_z must be such that the wave amplitude decreases while going far away from the source, in any passive medium which cannot provide energy to the ElectroMagnetic wave [1]. In these conditions propagation is impossible and the ElectroMagnetic wave is just attenuated, and its intensity is decreased. This happens at and below the critical frequency. The ElectroMagnetic wave penetrates by a length $L \approx 1/|k'_z|$. There is no energy propagation along z. The energy must be necessarily reflected back.

Energy

The energy flux above the critical frequency can be calculated as follows, from the direct general definitions for ElectroMagnetic fields. The Poynting vector has both the x and z components. However, when time-averaged, only the z component is non-zero with the present ElectroMagnetic fields. As the time-averaged z component of the Poynting vector depends on x and y it makes sense to average it on the cross-section of the wave-guide, using the following results:

$$\ell \equiv \frac{q\pi}{k} \qquad \int_{w=0}^{q\pi/k} \sin^2[kw]\ \mathrm{d}w = \int_{w=0}^{q\pi/k} \cos^2[kw]\ \mathrm{d}w = \frac{\pi q}{2k} = \frac{\ell}{2} \quad .$$

The Poynting vector is:

$$\mathbf{S}_{\mathtt{EM}} = \frac{1}{\mu_0}(\widehat{\mathbf{e}}_1 E_y B_z - \widehat{\mathbf{e}}_3 E_y B_x) \quad ,$$

$$\langle S_x\rangle = 0 \quad ,$$

$$\langle S_z\rangle = -\frac{1}{\mu_0}\langle E_y B_x\rangle = \frac{1}{2}\varepsilon_0 c E_{0y}^2 \frac{ck_z}{\omega}\sin^2[k_x x] \qquad \left[\mathtt{W}/\mathtt{m}^2\right] \quad ,$$

$$\langle \Pi_z\rangle = \frac{1}{4}\varepsilon_0 c \ell_x \ell_y E_{0y}^2 \frac{ck_z}{\omega} \qquad [\mathtt{W}] \quad .$$

[1] A notable example of active medium is a Laser, which can amplify the ElectroMagnetic wave by taking energy from the medium.

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The energy density is made of the three terms corresponding to the three non zero components of the ElectroMagnetic fields: E_y, B_x and B_z. We are interested in the time-averaged energy density. Moreover, we can integrate over x and y at fixed z to get the average energy density stored at a certain z.

$$u = \frac{\varepsilon_0}{2}E_y^2 + \frac{1}{2\mu_0}B_x^2 + \frac{1}{2\mu_0}B_z^2 \quad ,$$

$$\langle u_{\mathtt{EM}}\rangle = \frac{\varepsilon_0}{4}E_{0y}^2\sin^2[k_x x] + \frac{1}{4\mu_0}\frac{E_{0y}^2 k_z^2}{\omega^2}\sin^2[k_x x] + \frac{1}{4\mu_0}\frac{E_{0y}^2 k_x^2}{\omega^2}\cos^2[k_x x] \quad ,$$

$$\langle\frac{\Delta\mathcal{U}}{\Delta z}\rangle \equiv \int_{x=0}^{\ell_x}\int_{y=0}^{\ell_x}\langle u_{\mathtt{EM}}\rangle\,\mathtt{d}x\,\mathtt{d}y \quad ,$$

$$\langle\frac{\Delta\mathcal{U}}{\Delta z}\rangle = \frac{\varepsilon_0}{4}\ell_x\ell_y E_{0y}^2 \qquad [\mathtt{J/m}] \quad .$$

The relation between power and energy density follows:

$$\boxed{\langle\Pi_z\rangle = \frac{c^2 k_z}{\omega}\langle\frac{\Delta\mathcal{U}}{\Delta z}\rangle \equiv v_g\langle\frac{\Delta\mathcal{U}}{\Delta z}\rangle} \quad ,$$

$$\boxed{v_g = \frac{c^2 k_z}{\omega} \le c} \quad ,$$

where the **group-velocity** , v_g, has been introduced, showing that **the energy transport has the speed given by the group-velocity , in this problem**. Read § 44.03.08, § 18.04.02.

Phase-Velocity and group-velocity

Read § 44.03.11, § 18.04.02, § 44.03.08.

The **phase-velocity** can be larger than c:

$$\boxed{v \equiv \frac{\omega}{k}} \quad .$$

It has been shown that the energy propagates along the guide with the **group-velocity** which is smaller than the speed of light in vacuum:

$$\boxed{v_g \equiv \frac{\mathtt{d}\omega}{\mathtt{d}k} = \frac{c^2 k_z}{\omega} \le c} \quad .$$

Note, however, that in general it is possible for the **group-velocity** to exceed the speed of light in vacuum. As for the **phase-velocity** , in this case, this is not related to any sort of information transport and does not violate relativity.

The special solution found is a so-called TE, solution, that is transverse electric field, because the electric field is purely transverse and $B_z \neq 0$.

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47.04.01.02 TE Modes

A solution such that the electric field is purely transverse is known as a TE solution, with $B_z \neq 0$.

47.04.01.03 TM Modes

A solution such that the magnetic field is purely transverse is known as a TM solution, with $E_z \neq 0$.

47.04.01.04 Other Modes

Many other solutions with different combinations of Electric|Magnetic fields exist.

The general solution with transverse electric field is known as the TE(p_x, p_y) mode, where the first index is conventionally associated with the larger dimension of the rectangular wave-guide, so we assume $\ell_x > \ell_y$ and at least one of the indices must be non-zero.

Similarly, the general solution with transverse magnetic field is known as the TM(p_x, p_y) mode, where the first index is conventionally associated with the larger dimension of the rectangular wave-guide, so we assume $\ell_x > \ell_y$ and at least one of the indices must be non-zero.

The mode with the lowest cutoff frequency is termed the dominant mode of the wave-guide. Often, one tries to choose the dimensions of the guide such that only this one mode can exist in the operating frequency band. No wave with frequency below it cutoff frequency can propagate (that is transport energy) in the wave-guide and the cutoff frequencies of all other modes are higher than the one for the dominant mode.

Note that in general the critical frequency depend on the mode, that is on the p_x and p_y integer values characterizing the solution.

47.04.02 Mixed Standing-Traveling Waves in Circular Wave-Guides

This § is referenced at pages:
[2012]

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47.05 Standing ElectroMagnetic Waves in Three Dimensions

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|9.5.2|*general solution*|

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|24.2||

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EM cavities.

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47.06 Transmission Lines

This § is referenced at pages:
[Never referenced.]

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A transmission line is a pair of electric conductors carrying electric signals from one place to another.

47.06.01 A Coaxial-Cable Transmission Line

This § is referenced at pages:
[Never referenced.]

Consider a lossless homogeneous coaxial-cable transmission line, that is with zero resistance.

The voltage and current at a certain position z along the coaxial-cable satisfy the **D'Alembert non-dispersive wave-equation** , in case of zero resistance. Let, for a coaxial cable, $L|C$ be the Inductance|Capacitance per unit length.

The characteristic wave impedance is:

$$Z = \sqrt{\frac{L}{C}} \quad . \tag{47.06.01}$$

The **phase-velocity** is:

$$v_p = \sqrt{\frac{1}{LC}} \quad , \tag{47.06.02}$$

and is fixed in a lossless homogeneous coaxial-cable transmission line, that is therefore non-dispersive.

For lossy homogeneous coaxial-cable transmission lines, the resulting equation is the dissipative **D'Alembert non-dispersive wave-equation** .

47.06.02 Detailed calculations

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6.013 - Electromagnetics and Applications Fall 2005

Lecture 10 - Transmission Lines

Prof. Markus Zahn October 13, 2005

I. Transmission Line Equations

A. Parallel Plate Transmission Line

$\bar{E}$ must be perpendicular to the electrodes and $\bar{H}$ must be tangential, so

$$\bar{E} = E_x(z,t)\bar{i}_x$$
$$\bar{H} = H_y(z,t)\bar{i}_y$$

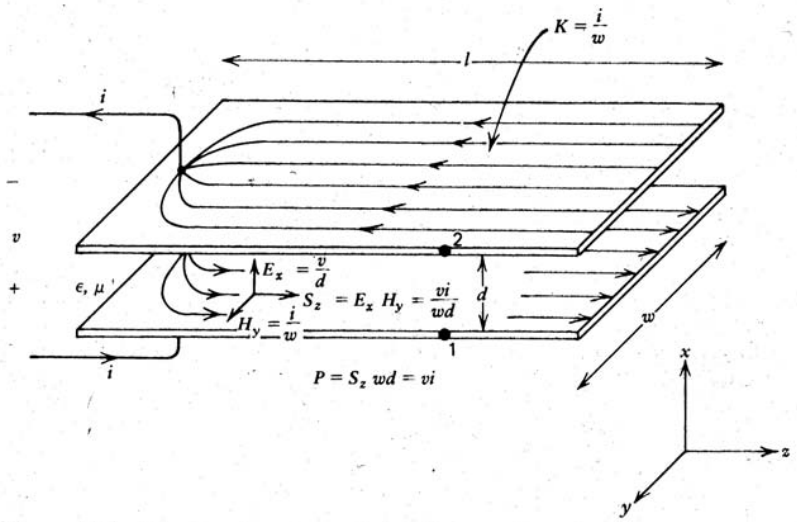

Figure 8-1 The simplest transmission line consists of two parallel perfectly conducting plates a small distance d apart.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \Rightarrow \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$
$$\nabla \times \bar{H} = \epsilon \frac{\partial \bar{E}}{\partial t} \Rightarrow \frac{\partial H_y}{\partial z} = -\epsilon \frac{\partial E_x}{\partial t}$$

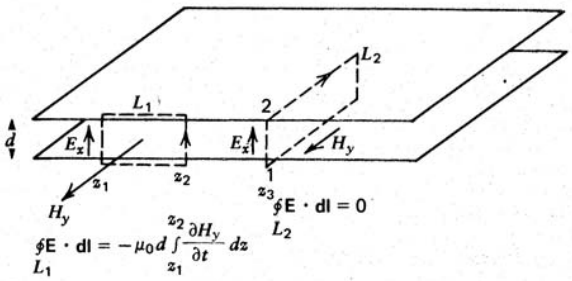

Figure 8-2 The potential difference measured between any two arbitrary points at different positions z_1 and z_2 on the transmission line is not unique—the line integral L_1 of the electric field is nonzero since the contour has magnetic flux passing through it. If the contour L_2 lies within a plane of constant z such as at z_3, no magnetic flux passes through it so that the voltage difference between the two electrodes at the same value of z is unique.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$v(z,t) = \int_{\substack{1 \\ z = \text{constant}}}^{2} \bar{E} \cdot d\bar{l} = E_x(z,t)d$$

$$i(z,t) = K_z(z,t)w = H_y(z,t)w$$

$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} \quad L = \frac{\mu d}{w} \text{ henries / meter} \quad \text{Inductance per unit length}$$

$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} \quad C = \frac{\epsilon w}{d} \text{ farads / meter} \quad \text{Capacitance per unit length}$$

$$LC = \frac{\mu \cancel{d}}{\cancel{w}} \frac{\epsilon \cancel{w}}{\cancel{d}} = \mu\epsilon = \frac{1}{c^2}$$

B. Transmission Line Structures

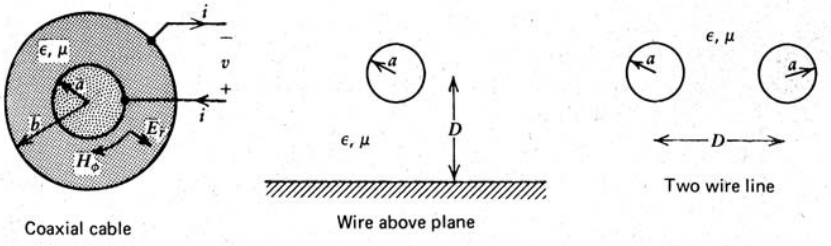

Figure 8-3 Various types of simple transmission lines.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

EXAMPLE 8-1 THE COAXIAL TRANSMISSION LINE

Consider the coaxial transmission line shown in Figure 8-3 composed of two perfectly conducting concentric cylinders of radii a and b enclosing a linear medium with permittivity ϵ and permeability μ. We solve for the transverse dependence of the fields as if the problem were static, independent of time. If the voltage difference between cylinders is v with the inner cylinder carrying a total current i the static fields are

$$E_r = \frac{v}{r \ln (b/a)}, \qquad H_\phi = \frac{i}{2\pi r}$$

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The surface charge per unit length q and magnetic flux per unit length λ are

$$q = \varepsilon E_r(r=a)2\pi a = \frac{2\pi\varepsilon v}{\ln(b/a)}$$

$$\lambda = \int_a^b \mu H_\phi \, dr = \frac{\mu i}{2\pi} \ln\frac{b}{a}$$

so that the capacitance and inductance per unit length of this structure are

$$C = \frac{q}{v} = \frac{2\pi\varepsilon}{\ln(b/a)}, \qquad L = \frac{\lambda}{i} = \frac{\mu}{2\pi} \ln\frac{b}{a}$$

where we note that as required

$$LC = \varepsilon\mu$$

Substituting E_r and H_ϕ into (12) yields the following transmission line equations:

$$\frac{\partial E_r}{\partial z} = -\mu\frac{\partial H_\phi}{\partial t} \Rightarrow \frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t}$$

$$\frac{\partial H_\phi}{\partial z} = -\varepsilon\frac{\partial E_r}{\partial t} \Rightarrow \frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t}$$

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C. Distributed Circuit Representation with Losses

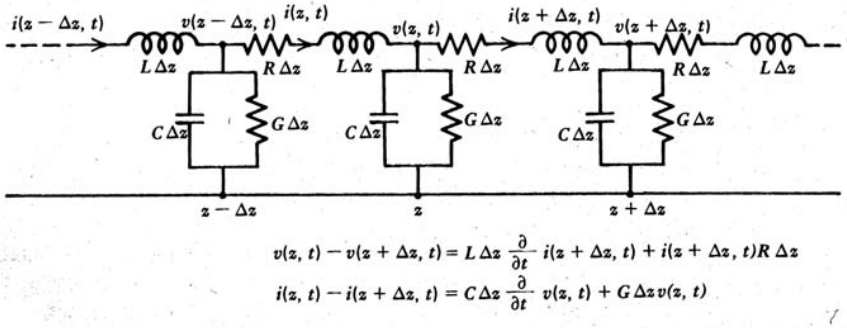

Figure 8-5 Distributed circuit model of a transmission line including small series and shunt resistive losses.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$i(z,t) - i(z+\Delta z,t) = C\Delta z\frac{\partial v(z,t)}{\partial t} + G\Delta z\, v(z,t)$$

$$v(z,t) - v(z+\Delta z,t) = L\Delta z\frac{\partial i(z+\Delta z,t)}{\partial t} + i(z+\Delta z,t)R\Delta z$$

$$\lim_{\Delta z\to 0}\frac{i(z+\Delta z,t)-i(z,t)}{\Delta z} = \frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} - Gv$$

$$\lim_{\Delta z\to 0}\frac{v(z+\Delta z,t)-v(z,t)}{\Delta z} = \frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} - iR$$

R is the series resistance per unit length, measured in ohms/meter, and G is the shunt conductance per unit length, measured in siemens/meter.

If the line is lossless ($R = G = 0$), we have the Telegrapher's equations:

$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t}$$
$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t}$$

Including loss, Poynting's theorem for the circuit equivalent form is:

$$v \cdot \left| \frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} - Gv \right.$$
$$i \cdot \left| \frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} - iR \right.$$
$$\text{Add: } v\frac{\partial i}{\partial z} + i\frac{\partial v}{\partial z} = \frac{\partial (vi)}{\partial z} = -\frac{\partial}{\partial t}\left[\frac{1}{2}Cv^2 + \frac{1}{2}Li^2\right] - Gv^2 - i^2R$$

D. Wave Equation (Lossless, $R = 0, G = 0$)

$$\frac{\partial}{\partial t}\left| \frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} \right. \quad \Rightarrow \quad \frac{\partial^2 i}{\partial z \partial t} = -C\frac{\partial^2 v}{\partial t^2}$$
$$\frac{\partial}{\partial z}\left| \frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} \right. \quad \frac{\partial^2 i}{\partial z \partial t} = -\frac{1}{L}\frac{\partial^2 v}{\partial z^2}$$
$$-\frac{1}{L}\frac{\partial^2 v}{\partial z^2} = -C\frac{\partial^2 v}{\partial t^2} \quad \Rightarrow \quad \underbrace{\frac{\partial^2 v}{\partial z^2} = LC\frac{\partial^2 v}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}}_{\text{Wave equation}}$$

II. Sinusoidal Steady State

A. Complex Amplitude Notation

$$v(z,t) = \text{Re}\left[\hat{v}(z)e^{j\omega t}\right]$$
$$i(z,t) = \text{Re}\left[\hat{i}(z)e^{j\omega t}\right]$$

Substitute into the wave equation:

$$\frac{\partial^2 v}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} \Rightarrow \frac{d^2\hat{v}}{dz^2} = -\frac{\omega^2}{c^2}\hat{v}(z), \text{ let } k = \frac{\omega}{c}$$
$$\frac{d^2\hat{v}}{dz^2} + k^2\hat{v} = 0 \Rightarrow \hat{v}(z) = \hat{V}_+ e^{-jkz} + \hat{V}_- e^{+jkz}$$
$$\frac{d\hat{v}}{dz} = -Lj\omega\hat{i} \Rightarrow \hat{i}(z) = -\frac{1}{Lj\omega}\left(-jk\hat{V}_+ e^{-jkz} + jk\hat{V}_- e^{+jkz}\right)$$
$$\frac{k}{\omega} = \frac{\omega}{c\omega} = \sqrt{LC} \Rightarrow \frac{k}{\omega L} = \frac{\sqrt{LC}}{L} = \sqrt{\frac{C}{L}} = Y_0 \text{ is the Line Admittance}$$
$$Z_0 = \frac{1}{Y_0} = \sqrt{\frac{L}{C}} \text{ is the Line Impedance}$$
$$\hat{i}(z) = Y_0\left(\hat{V}_+ e^{-jkz} - \hat{V}_- e^{+jkz}\right)$$

$$\hat{v}(z) = \hat{V}_+ e^{-jkz} + \hat{V}_- e^{+jkz}$$
$$v(z,t) = \mathrm{Re}\left[\hat{V}_+ e^{j(\omega t - kz)} + \hat{V}_- e^{j(\omega t + kz)}\right]$$
$$i(z,t) = \mathrm{Re}\, Y_0 \left[\hat{V}_+ e^{j(\omega t - kz)} - \hat{V}_- e^{j(\omega t + kz)}\right]$$
$$k = \frac{\omega}{c} = \omega\sqrt{LC} = \omega\sqrt{\epsilon\mu}$$

B. Short Circuited Line ($v(z=0,t) = 0, v(z=-l,t) = V_0\cos(\omega t)$)

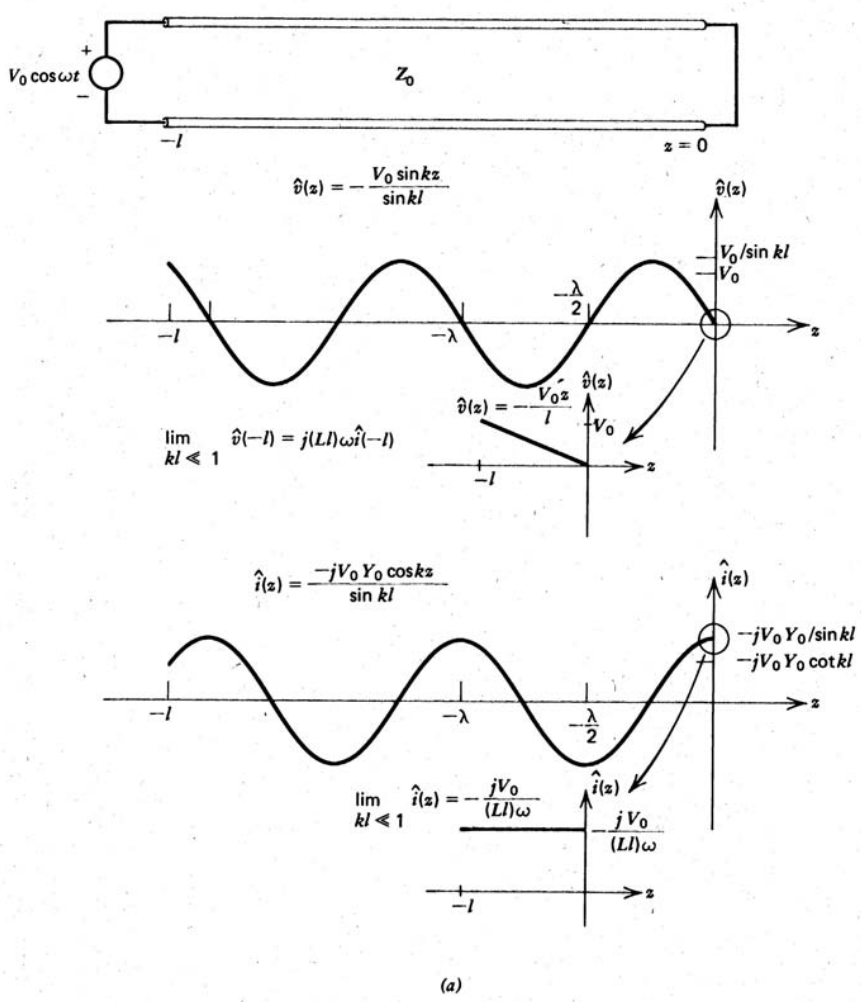

Figure 8-15 The voltage and current distributions on a (*a*) short circuited and (*b*) open circuited transmission line excited by sinusoidal voltage sources at $z = -l$. If the lines are much shorter than a wavelength, they act like reactive circuit elements. (*c*) As the frequency is raised, the impedance reflected back as a function of z can look capacitive or inductive making the transition through open or short circuits.

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$$\hat{v}(z) = \hat{V}_+ e^{-jkz} + \hat{V}_- e^{+jkz} \Rightarrow \hat{v}(z=0) = 0 = \hat{V}_+ + \hat{V}_- \Rightarrow \hat{V}_+ = -\hat{V}_-$$
$$\hat{v}(z=-l) = V_0 = \hat{V}_+ e^{+jkl} + \hat{V}_- e^{-jkl} = \hat{V}_+ \left(e^{jkl} - e^{-jkl}\right)$$

$$= 2j\sin(kl)\hat{V}_+$$
$$\hat{V}_+ = -\hat{V}_- = \frac{V_0}{2j\sin(kl)}$$

$$\hat{v}(z) = \frac{V_0}{2j\sin(kl)}\left(e^{-jkz} - e^{+jkz}\right) = \frac{V_0(-2j)\sin(kz)}{2j\sin(kl)}$$
$$= -\frac{V_0\sin(kz)}{\sin(kl)}$$
$$\hat{i}(z) = Y_0\left(\hat{V}_+e^{-jkz} - \hat{V}_-e^{jkz}\right) = \frac{Y_0V_0}{2j\sin(kl)}\left(e^{-jkz} + e^{+jkz}\right)$$
$$= \frac{2Y_0V_0\cos(kz)}{2j\sin(kl)}$$
$$= -\frac{jY_0V_0\cos(kz)}{\sin(kl)}$$
$$v(z,t) = \mathrm{Re}\left[\hat{v}(z)e^{j\omega t}\right] = \mathrm{Re}\left[-\frac{V_0\sin(kz)}{\sin(kl)}e^{j\omega t}\right] = -\frac{V_0\sin(kz)\cos(\omega t)}{\sin(kl)}$$
$$i(z,t) = \mathrm{Re}\left[\hat{i}(z)e^{j\omega t}\right] = \mathrm{Re}\left[-\frac{jY_0V_0\cos(kz)}{\sin(kl)}e^{j\omega t}\right] = \frac{Y_0V_0\cos(kz)\sin(\omega t)}{\sin(kl)}$$

We have resonance when $\sin(kl) = 0 \Rightarrow kl = n\pi = \frac{\omega l}{c} \Rightarrow \omega = \omega_n \equiv \frac{n\pi c}{l}, n = 1, 2, 3, \ldots$
Complex impedance: $Z(z) = \frac{\hat{v}(z)}{\hat{i}(z)} = -jZ_0\tan(kz)$

$$Z(z = -l) = +jZ_0\tan(kl)$$

In the following, take $n = 1, 2, 3, \ldots$:

$kl = n\pi$	$Z(z=-l) = 0$	short circuit
$kl = (2n-1)\frac{\pi}{2}$	$Z(z=-l) = \infty$	open circuit
$(n-1)\pi < kl < (2n-1)\frac{\pi}{2}$	$Z(z=-l) = +jX, X > 0$	(positive reactance, inductive)
$(n-\frac{1}{2})\pi < kl < n\pi$	$Z(z=-l) = -jX, X > 0$	(negative reactance, capacitive)
$kl \ll 1 \Rightarrow$	$Z(z) = -jZ_0k$	
	$= -j\sqrt{\frac{L}{\mathscr{C}}}\omega\sqrt{L\mathscr{C}}$	
	$= -jLZ$	
	$Z(z=-l) = j(Ll)$ inductive	
$\lvert kz\rvert \ll 1$	$v(z,t) = -\frac{V_0z}{l}\cos(\omega t) \Rightarrow$	$v(z=-l,t) = V_0\cos(\omega t)$
		$= (Ll)\frac{di}{dt}(z=-l,t)$
	$i(z,t) = \frac{V_0Y_0}{kl}\sin(\omega t)$	$i(z=-l,t) = \frac{V_0\sin(\omega t)}{(Ll)\omega}$

C. Open Circuited Line ($i(z=0,t)=0$)

$$
\begin{aligned}
v(z=-l,t) &= V_0\sin(\omega t)\\
\hat{i}(z) = Y_0\left[\hat{V}_+e^{-jkz} - \hat{V}_-e^{+jkz}\right] \Rightarrow \hat{i}(z=0) &= 0 = Y_0\left[\hat{V}_+ - \hat{V}_-\right] \Rightarrow \hat{V}_+ = \hat{V}_-\\
\hat{v}(z=-l) = -jV_0 &= \hat{V}_+e^{+jkl} + \hat{V}_-e^{-jkl} = \hat{V}_+\left(e^{jkl} + e^{-jkl}\right) = 2\hat{V}_+\cos(kl)\\
\hat{V}_+ = \hat{V}_- &= -\frac{jV_0}{2\cos(kl)}\\
\hat{v}(z) &= -\frac{jV_0}{2\cos(kl)}\left(e^{-jkz} + e^{+jkz}\right)\\
&= -\frac{jV_0\cdot \cancel{2}\cos(kz)}{\cancel{2}\cos(kl)}\\
&= -\frac{jV_0\cos(kz)}{\cos(kl)}\\
\hat{i}(z) &= -\frac{jY_0V_0}{2\cos(kl)}\left(e^{-jkz} - e^{+jkz}\right)\\
&= \frac{(-jY_0V_0)(-2j)\sin(kz)}{2\cos(kl)}\\
&= -\frac{Y_0V_0\sin(kz)}{\cos(kl)}
\end{aligned}
$$

$$
\begin{aligned}
v(z,t) &= \mathrm{Re}\left[\hat{v}(z)e^{j\omega t}\right] = \frac{V_0\cos(kz)}{\cos(kl)}\sin(\omega t)\\
i(z,t) &= \mathrm{Re}\left[\hat{i}(z)e^{j\omega t}\right] = \frac{-V_0Y_0}{\cos(kl)}\sin(kz)\cos(\omega t)\\
\text{Resonance: } \cos(kl) &= 0 \Rightarrow (kl) = (2n-1)\frac{\pi}{2}, n = 1,2,3,\ldots\\
\omega_n &= \frac{(2n-1)\frac{\pi}{2}}{2l}
\end{aligned}
$$

Complex Impedance

$$
\begin{aligned}
Z(z) &= \frac{\hat{v}(z)}{\hat{i}(z)} = Z_0 j\cot(kz)\\
Z(z=-l) &= -jZ_0\cot(kl)\\
kl \ll 1 \Rightarrow v(z,t) &= V_0\sin(\omega t)\\
i(z,t) &= -V_0Y_0kz\cos(\omega t)\\
i(z=-l,t) &= (Cl)\omega V_0\cos(\omega t) = (Cl)\frac{dv}{dt}(z=-l,t)
\end{aligned}
$$

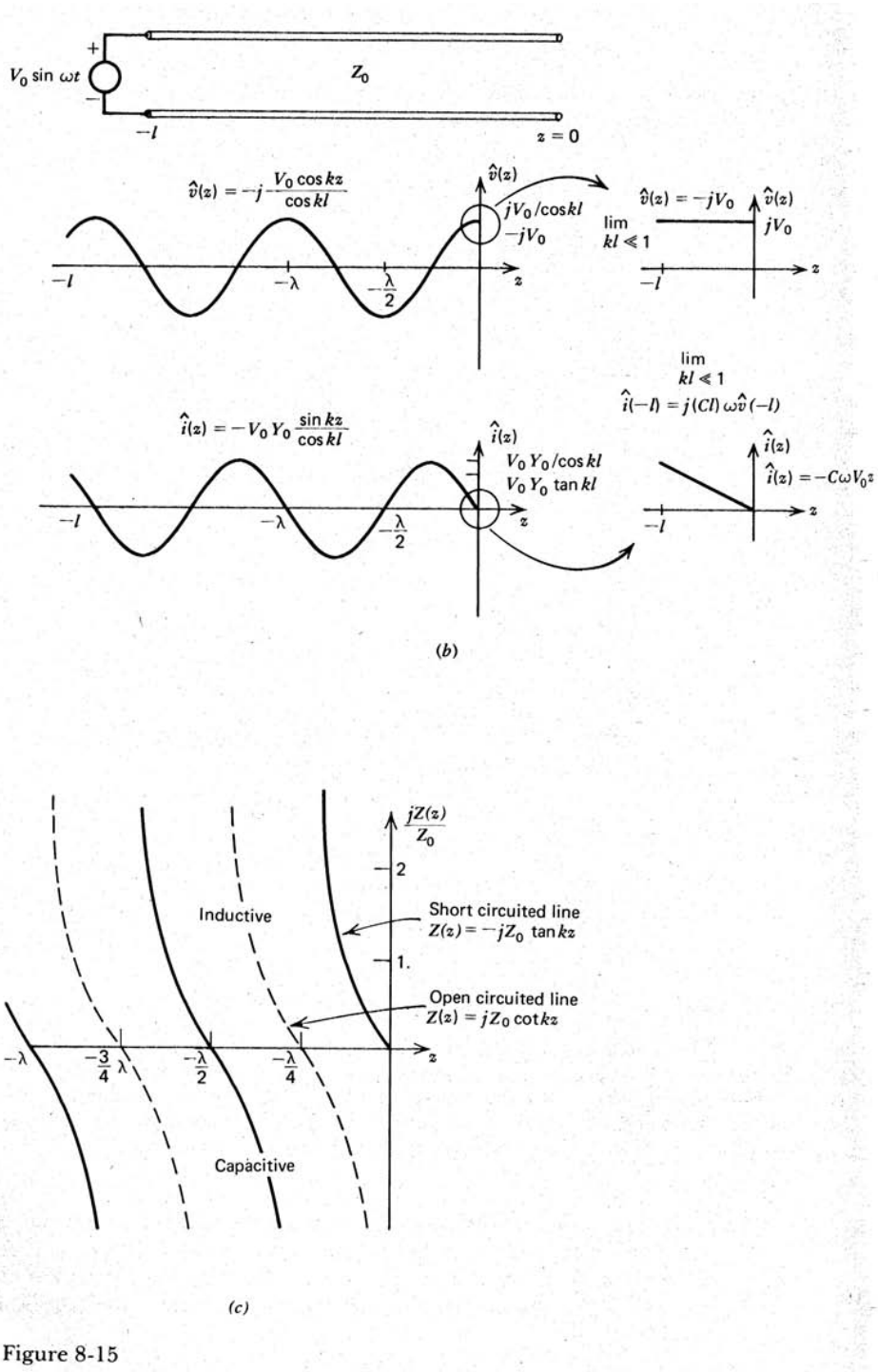

Figure 8-15

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Impedance for short and open circuited wires

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47.07 Examples and Physical Applications

47.08 Exercises Problems and Physical Applications

47-005 ✓ Standing ElectroMagnetic waves

Consider the standing wave:

$$E_x = E_{x0} \sin[\omega t] \sin[kz] \quad ,$$
$$B_y = B_{y0} \cos[\omega t] \cos[kz] \quad .$$

- Show that satisfy **Maxwell Equation** in vacuum and determine the **Dispersion Relation** .
- Determine the energy flux as a function of position and time as well as the time-averaged energy flux.

Start from either the electric or the magnetic field expression above and derived the other field by direct integration of **Maxwell Equations** .

47-006 ✓ BB

Problems on EM waves in bounded regions.
5.1, 5.3, 5.4, 5.5, 5.6, 5.7.

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CHAPTER 48

Reflection and Transmission|Refraction of Waves

48.01 Introduction 2181
48.02 Snell Laws for Reflection and Transmission|Refraction of Waves 2182
48.03 Reflection and Transmission|Refraction of ElectroMagnetic Waves 2187
48.04 Reflection and Transmission|Refraction of Longitudinal Mechanical Waves 2207
48.05 Reflection and Transmission|Refraction of Transverse Mechanical Waves 2209
48.06 Examples and Physical Applications 2210
48.07 Exercises Problems and Physical Applications 2214

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Waves in regions of space near boundaries and the effect of boundaries.

This § is referenced at pages:
[2055, 2062]

Note: **Refraction == Transmission**.
Note: **reflectivity == reflectance** and **transmissivity == transmittance**.

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48.01 Introduction

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After having studied the properties of waves in spatially unlimited media one must study what happens when **physical boundaries are present**.

Let us always consider the problem **locally, on a small part of the interface surface** such that it can be considered a plane surface.

The propagation **speed of waves depend on the physical properties of the medium**. Therefore one expects some change to happen at the interfaces between media with different physical properties, as a consequence of the change of the propagation speed. Note that in the case of ElectroMagnetic waves also vacuum must be considered, in this respect, a **medium**, as the properties of ElectroMagnetic waves change passing from vacuum to another medium.

When waves reach a surface which separates two media with different physical properties one has to take into account the change in the physical properties of the medium. Some kind of **discontinuity** in the wave property (such as in some derivative) may arise at these boundaries, as a consequence of the discontinuities of the physical properties of the media, even in the case when the waves are not material (that is mechanical) waves.

In general when waves reach a surface which separates two media with different physical properties there is both a **reflected** wave and a **transmitted** (also equivalently known as **refracted**) wave.

Reflection and Transmission|Refraction are phenomena characteristic of all the types of waves. **The directional properties of reflected and refracted waves do not depend on the specific kind of wave**. The amplitude relations do depend on the specific kind of wave.

We must therefore distinguish between:

- **kinematic properties**: relation between the directions of the incoming, reflected and refracted wave-vectors, which are the same for every wave;
- **dynamical properties**: intensity of the incoming, reflected and refracted waves and changes in the states of **wave-Polarization** , which may be different for different types of waves.

It is very important to stress that, at each interface, **one should always consider also the change of phase of the transmitted|refracted waves with respect to the incident wave**. It is obvious that the amplitude transmission coefficient must be always positive: if it is not consider a smoothly varying velocity and model it as infinitesimal steps: the transmitted wave should change its phase at each infinitesimal step. On the other hand, on reflection, a change of phase may happen.

48.01.01 Some Examples

- When ElectroMagnetic waves from vacuum reach a piece of matter the charges are set into motion by the ElectroMagnetic fields of the wave and they re-radiate ElectroMagnetic waves, giving rise to reflected and transmitted ElectroMagnetic waves.
- See the example in section § 47.03.02 for one example with the precise mechanism for reflection.
- .....

DRAFT, In Progress

48.02 Snell Laws for Reflection and Transmission|Refraction of Waves

This § is referenced at pages:
[1067, 1067, 1079, 2187, 2207]

The laws of reflection and Transmission|Refraction for waves, which will be derived in this section, are **kinematic laws which do not depend on the specific type of wave**.

Consider a MW moving in medium (1) where the **phase-velocity** is v_1. The wave meets a surface separating the first medium from a second medium (2) where the **phase-velocity** is v_2 and let's consider MW only.

Consider a small part of the surface such that it can be considered a plane surface. Consider a small part of the CPWF (which is well-defined as the wave is monochromatic) such that it can be considered locally plane.

Note that, in general, the two media might be dispersive media so that the **phase-velocity** may depend on the frequency of the wave, but this is not a problem as long as one considers MW.

Note that, at the boundary, the frequency of the waves cannot change because it is linked to the frequency of the source incoming wave which excites a force re-emission at the same frequency. This is clearly obvious, in steady conditions, for mechanical waves, as motion is produced by motion of pieces of matters, and also for ElectroMagnetic waves, as the incoming ElectroMagnetic fields excite motion of charges at the same frequency. That is, as long as in steady conditions the reflected and transmitted waves are monochromatic waves, their frequency must be the same as that of the incoming wave. Therefore the wavelength and **phase-velocity** has both to change. However, this argument comes more from dynamical properties. A cleaner argument, only based on kinematic arguments and therefore a more general one, will bring to the same conclusion in equation (48.02.02).

The expression of the incident wave (I), reflected wave (R) and transmitted wave (T) are:

$$\psi_{\mathrm{I}}[\mathbf{r},t]=\psi_{\mathrm{I0}}\cos\left[\mathbf{k}_{\mathrm{I}}\cdot r-\omega_{\mathrm{I}}t-\phi_{\mathrm{I}}\right]\qquad\text{with}\qquad\nu_{\mathrm{I}}\lambda_{\mathrm{I}}=v_{\mathrm{I}}\quad,$$

$$\psi_{\mathrm{R}}[\mathbf{r},t]=\psi_{\mathrm{R0}}\cos\left[\mathbf{k}_{\mathrm{R}}\cdot r-\omega_{\mathrm{R}}t-\phi_{\mathrm{R}}\right]\qquad\text{with}\qquad\nu_{\mathrm{R}}\lambda_{\mathrm{R}}=v_{\mathrm{R}}\quad,$$

$$\psi_{\mathrm{T}}[\mathbf{r},t]=\psi_{\mathrm{T0}}\cos\left[\mathbf{k}_{\mathrm{T}}\cdot r-\omega_{\mathrm{T}}t-\phi_{\mathrm{T}}\right]\qquad\text{with}\qquad\nu_{\mathrm{T}}\lambda_{\mathrm{T}}=v_{\mathrm{T}}\quad.$$

The complex representation for the waves might be used, but it is not necessary here.

Note that the field $\psi[\mathbf{r},t]$ might be a scalar field or any component of a vector or tensor field. Moreover, its interpretation depends on the specific physical problem.

Let the interface between the two media be the plane $z=0$ with medium 1 at $z>0$ and medium 2 at $z<0$. Therefore the incoming wave is traveling with $k_z<0$.

Assume that it is legal to admit that the wave and/or its derivatives and/or some linear combination of them satisfy some condition of continuity across the boundary. Moreover, assumed that the superposition principle applies.

- This is certainly the case, usually, for mechanical waves.
- It is also the case for ElectroMagnetic waves when there is no free Charge|Current at the boundary, and for the always valid continuity conditions for the tangential component of the electric field and the normal component of the magnetic field.

The continuity condition at $z=0$ reads something like:

$$\psi_{\mathrm{I}}[x,y,0,t]+\psi_{\mathrm{R}}[x,y,0,t]=\psi_{\mathrm{T}}[x,y,0,t]\quad. \tag{48.02.01}$$

→ 2182 2183

For different linear combinations also with derivatives, the rest of the demonstration is equally valid. Note that the arbitrary constant term in the phases allow to account for any possible choice of the signs in equation (48.02.01).

In order to satisfy the continuity condition (48.02.01) at any point at $z=0$ and at any time it is necessary, even if not sufficient in general, that the $\mathbf{k}\cdot\mathbf{r}-\omega t$ part of the phases of the three components at any point at $z=0$ are identically equal. In fact, only if the space-dependent and time-dependent parts of

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the three phases are identical at any point at $z = 0$ at any time, there is some hope to satisfy the full continuity condition (48.02.01) at any point at $z = 0$ at any time; it is necessary, but not sufficient. Basically this necessity comes from the fact that Sinusoidal|Cosinusoidal functions with different wave-number, and/or angular frequency, are linearly independent.

Note, on the contrary, that the constant term in the phase, as it is a constant term not dependent on space nor on time, is not constrained by this kind of argument on the phase, but maybe constrained by others arguments depending on the specific type of waves.

First of all the continuity condition (48.02.01) constrains the angular frequencies to be identical, in order to guarantee that, at any arbitrary point $\mathbf{r} = \{x, y, 0\}$, the condition is satisfied at any time:

$$\boxed{\omega_{\mathrm{I}} = \omega_{\mathrm{R}} = \omega_{\mathrm{T}} \equiv \omega} \quad . \tag{48.02.02}$$

→ 2182 2183 2184

Secondly the continuity condition (48.02.01) at $z = 0$ sets a constraint on the projection of the wave-vector $\mathbf{k}$ on the $z = 0$ plane:

$$\boxed{\left[\mathbf{k}_{\mathrm{I}}\cdot\mathbf{r}\right]_{z=0} = \left[\mathbf{k}_{\mathrm{R}}\cdot\mathbf{r}\right]_{z=0} = \left[\mathbf{k}_{\mathrm{T}}\cdot\mathbf{r}\right]_{z=0}} \quad . \tag{48.02.03}$$

→ 2183

The two equations (48.02.02), (48.02.03) summarize the purely kinematic conditions at the interfaces for waves. Equation (48.02.03) has a number of consequences.

Due to the arbitrariness of the position vector $\mathbf{r}$ in the $z = 0$ plane, the latter equations imply that the projection of the three wave-vectors on the $z = 0$ plane must be identical. Equation (48.02.03) can be written in explicit form in the xy plane for any of the wave-vectors (with $\mathbf{r} = \{x, y, z\}$):

$$x k_{\mathrm{I}x} + y k_{\mathrm{I}y} = x k_{\mathrm{R}x} + y k_{\mathrm{R}y} = x k_{\mathrm{T}x} + y k_{\mathrm{T}y} \quad .$$

In fact equation (48.02.03) is just the equality of the three scalar products on the xy plane, that is the equality of the three scalar products is valid after projecting all the vectors on the xy plane, and not valid for the original vectors with three components. The equality of the projection on the $z = 0$ plane of the scalar products implies that the three wave-vectors all lye in a plane:

$$\mathbf{k}_{\mathrm{I}}, \mathbf{k}_{\mathrm{R}}, \mathbf{k}_{\mathrm{T}} \qquad \text{are all co-planar} \quad .$$

Define the **incidence plane** as the plane containing the incident wave-vector and the normal to the surface at the point of incidence, $\widehat{\mathbf{N}}$, oriented along the positive direction of the z axis. The plane of incidence is thus the plane normal to the unit vector:

$$\widehat{\mathbf{p}} \equiv \frac{\widehat{\mathbf{N}}\wedge\mathbf{k}_{\mathrm{I}}}{\left|\widehat{\mathbf{N}}\wedge\mathbf{k}_{\mathrm{I}}\right|} \quad . \tag{48.02.04}$$

Therefore:

$$\boxed{\widehat{\mathbf{N}}, \mathbf{k}_{\mathrm{I}}, \mathbf{k}_{\mathrm{R}}, \mathbf{k}_{\mathrm{T}} \qquad \text{are all co-planar}} \quad . \tag{48.02.05}$$

The co-planarity condition can be expressed as:

$$\boxed{\widehat{\mathbf{p}} \equiv \frac{\widehat{\mathbf{N}}\wedge\mathbf{k}_{\mathrm{I}}}{\left|\widehat{\mathbf{N}}\wedge\mathbf{k}_{\mathrm{I}}\right|} \qquad \widehat{\mathbf{p}}\cdot\mathbf{k}_{\mathrm{I}} = \widehat{\mathbf{p}}\cdot\mathbf{k}_{\mathrm{R}} = \widehat{\mathbf{p}}\cdot\mathbf{k}_{\mathrm{T}} = 0} \quad , \tag{48.02.06}$$

→ 2184

from (48.02.03). Note that, for ElectroMagnetic waves, the Snell conditions are just the conservation of the component of the momentum of the photons tangential to the interface plane, because $\mathbf{k} \propto \mathbf{p}$.

It follows:

$$\left(k_{\mathrm{I}} \sin\left[\theta_{\mathrm{I}}\right]\right)\left[|\mathbf{r}|\right]_{z=0} = \left(k_{\mathrm{R}} \sin\left[\theta_{\mathrm{R}}\right]\right)\left[|\mathbf{r}|\right]_{z=0} = \left(k_{\mathrm{T}} \sin\left[\theta_{\mathrm{T}}\right]\right)\left[|\mathbf{r}|\right]_{z=0} \implies k_{\mathrm{I}} \sin\left[\theta_{\mathrm{I}}\right] = k_{\mathrm{R}} \sin\left[\theta_{\mathrm{R}}\right] = k_{\mathrm{T}} \sin\left[\theta_{\mathrm{T}}\right] \quad ,$$

$$\boxed{k_{\mathrm{I}} = k_{\mathrm{R}} \qquad \text{as the media are the same}} \quad , \tag{48.02.07}$$

→ 2184

$$\boxed{\omega = v_{\mathrm{I}} k_{\mathrm{I}} = v_{\mathrm{R}} k_{\mathrm{R}} = v_{\mathrm{T}} k_{\mathrm{T}} \qquad \text{the three products are all equal to the angular frequency}} \quad . \tag{48.02.08}$$

→ 2184

Alternatively one can reach the same conclusion introducing the projected vectors on the xy plane, $\widetilde{\mathbf{k}}_{\mathrm{I}}$, $\widetilde{\mathbf{k}}_{\mathrm{R}}$, $\widetilde{\mathbf{k}}_{\mathrm{T}}$ and $\widetilde{\mathbf{r}}$ (all bi-dimensional vectors). The arbitrariness of $\widetilde{\mathbf{r}}$ and the fact that equation (48.02.03) implies that the scalar product of $\widetilde{\mathbf{r}}$ with all the three differences of any pair of the three projected wave-vectors is zero implies that all the three projected wave-vectors are identical.

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Finally, one has the full set of Snell laws, from equations (48.02.02), (48.02.06), (48.02.07), (48.02.08):

$$\boxed{\omega_{\mathrm{I}} = \omega_{\mathrm{R}} = \omega_{\mathrm{T}} \equiv \omega} \quad , \tag{48.02.09}$$

$$\boxed{\widehat{\mathbf{p}} \equiv \frac{\widehat{\mathbf{N}} \wedge \mathbf{k}_{\mathrm{I}}}{\left| \widehat{\mathbf{N}} \wedge \mathbf{k}_{\mathrm{I}} \right|} \qquad \widehat{\mathbf{p}} \cdot \mathbf{k}_{\mathrm{I}} = \widehat{\mathbf{p}} \cdot \mathbf{k}_{\mathrm{R}} = \widehat{\mathbf{p}} \cdot \mathbf{k}_{\mathrm{T}} = 0} \quad , \tag{48.02.10}$$

$$\boxed{k_{\mathrm{I}} = k_{\mathrm{R}}} \quad , \tag{48.02.11}$$

$$\boxed{\omega = v_{\mathrm{I}} k_{\mathrm{I}} = v_{\mathrm{R}} k_{\mathrm{R}} = v_{\mathrm{T}} k_{\mathrm{T}}} \quad , \tag{48.02.12}$$

$$\boxed{\sin\left[\theta_{\mathrm{I}}\right] = \sin\left[\theta_{\mathrm{R}}\right]} \quad \text{Snell law of reflection (first Snell law)} \quad , \tag{48.02.13}$$

→ 2184

$$\boxed{\frac{\sin\left[\theta_{\mathrm{I}}\right]}{v_{\mathrm{I}}} = \frac{\sin\left[\theta_{\mathrm{T}}\right]}{v_{\mathrm{T}}}} \quad \text{Snell law of Transmission|Refraction (second Snell law)} \quad . \tag{48.02.14}$$

→ 2184

The laws (48.02.13), (48.02.14), (48.02.06) have been deduced by just using arguments of continuity of the phase of the wave. Therefore they are valid for **any wave**, as long as some continuity condition at the interface can be assumed, as it is usually the case.

Note that while the law of Transmission|Refraction (48.02.14) involves the properties of both media, the law of reflection (48.02.13) does not involve any properties of the two media as it concerns properties of the waves in the same medium.

In case of a MPW incident on a plane, the reflected and refracted waves are still MPW, because the reflected and refracted rays are parallel. In case of incidence on a non-plane surface, the shape of the CPWF, in general, is not conserved.

If the continuity conditions apply on the derivatives, one still finds Sinusoid|Cosinusoids and therefore Snell laws, as necessary conditions, apply as well.

Snell laws, in general are not sufficient to fully determine the properties of the reflected and refracted wave, as dynamical conditions apply as well.

Note that **a possible absorption inside the two media will not affect the Snell laws** because the boundary conditions are to be applied at the boundary, while absorption will affect the waves in the bulk of the medium.

Note that the Snell laws apply in the Rest Frame of the reflecting/refracting surface: in other Reference Frame the expressions change. Read § 24-001, and § 45-013.

48.02.01
Mechanical Waves

The case of purely longitudinal waves will be only considered, that is a purely uni-dimensional waves.

The general case of mechanical waves is complex and upon Reflection and/or Transmission|Refraction transverse and longitudinal waves may interchange.

Read § 48.04.

48.02.02
EM Waves

In the case of ElectroMagnetic waves the second Snell law is usually re-written in terms of the **index of refraction** es of the two media:

$$\boxed{n_{\mathrm{I}} \sin\left[\theta_{\mathrm{I}}\right] = n_{\mathrm{T}} \sin\left[\theta_{\mathrm{T}}\right]} \quad .$$

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48.02.03
Total Internal Reflection

Snell law for Transmission|Refraction can be written as:

$$\sin[\theta_T] = \frac{v_T}{v_I}\sin[\theta_I] \quad .$$

In the case $v_I > v_T$ the wave-vector of the transmitted wave gets closer to the normal.

In the case $v_I < v_T$ the wave-vector of the transmitted wave gets farther away from the normal. In this case it is possible that Snell law for Transmission|Refraction gives $\sin[\theta_T] > 1$. If this occurs there is no transmission but only reflection, the so-called total internal reflection. The condition for total internal reflection is:

$$\frac{v_I}{v_T} \leq \sin[\theta_I] \leq 1 \qquad \text{that is } v_I < v_T \text{ is required} \quad . \tag{48.02.15}$$

Conversely, if $v_I > v_T$, waves can enter the second medium at a maximum angle given by:

$$\sin[\theta_T] = \frac{v_T}{v_I} < 1 \qquad \text{for } \theta_I = \pi/2 \quad .$$

48.02.04
Reflection and Transmission|Refraction Laws From Fermat Principle

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The reflection and Transmission|Refraction Snell laws can be easily derived from kinematic considerations from Fermat principle, read § 55.02.

48.02.05
Reflection and Transmission|Refraction Laws From Huygens-Fresnel Principle

The reflection and Transmission|Refraction Snell laws can be easily derived from kinematic considerations from Huygens-Fresnel principle, read § 51.03.

48.02.06
Reflection of Waves From Rough Surfaces and Rayleigh Criterion

In the general case when the wave is not a MPW and the interface is not a plane one it is possible to subdivide the CPWF and the interface surface in small parts such that both can be considered as plane. This works as long as the dimensions of the plane piece of CPWF and|or surface is large with respect to the wavelength. If this condition is not satisfied diffraction phenomena (section § 51) might become relevant.

The condition to be satisfied in order to treat the piece of CPWF and|or surface as plane is that the radius of curvature of the CPWF/surface is very much larger than the dimensions of the piece of CPWF/surface considered

The reflection from rough surfaces is known as non-specular reflection. The reflection from any real surface is, in general, a combination of **specular** and **diffuse** (non-specular) reflections.

The relevant **smoothness of surface** depends on the wavelength and angle of incidence of the PW. The **Rayleigh criterion** is usually adopted as a rough guideline. The Rayleigh criterion is based on the fact that different parallel rays will have different paths when reflected from a non planar surface, even if they are reflected into parallel directions.

A simplified argument to estimate to what extent reflection will be specular/diffuse is the following one. Assume a step-like surface, with steps of random length and equal height d. Two parallel rays from MPW, with angle of incidence θ, reflected one by the upper layer and the other one by the lower layer, will gain a path/phase difference (see the calculations as in section § 50.10.05 and equation (50-009) with $n = 1$):

$$\Delta s = 2d\cos[\theta] \implies \Delta\delta = \frac{4\pi d\cos[\theta]}{\lambda} \quad .$$

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The geometrical derivation is the same as in section § 50.10.05 with the simplifying assumption, here with respect to section § 50.10.05, that the **index of refraction** es are the same, so that the difference in the optical paths is only a geometrical difference. Also note that the relation $\Delta s = 2d\cos[\theta]$ is trivially deduced when one considers two parallel rays one reflected by the upper layer and the second one reflected by the lower layer with the two reflection points lying on the same perpendicular line to the two layers.

A surface is defined as planar if and only if:

$$\Delta\delta \ll 2\pi \implies 2d\cos[\theta] \ll \lambda \quad .$$

A rough surface is defined as a surface such that:

$$\Delta\delta > \frac{\pi}{2} \implies d > \frac{\lambda}{8\cos[\theta]} \quad .$$

The Rayleigh criterion can only be a generic guide, as in general the angle of incidence will vary because surfaces are not, in general, parallel.

The result implies that the effect of roughness is less important for high-incidence angles than it is for low incidence angles, as it is turns out from the fact that at grazing incidence the difference of paths tends to zero (the two paths tends to become parallel paths) while at normal incidence it tends to $\Delta s = 2d$.

At grazing incidence surface tend to behave as specular surfaces. This result is made stringer by the fact that at grazing incidence the amplitude reflection coefficients tend to one.

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48.03 Reflection and Transmission|Refraction of ElectroMagnetic Waves

This § is referenced at pages:
[2083, 2197]

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In this section the intensity of reflected and refracted waves will be studied for the case of Electro-Magnetic waves.

EM MPW will be assumed except when specified otherwise.

Both media are assumed to be LHI media; they might be dispersive media as one will consider MPW.

In terms of the **index of refraction** Snell law of Transmission|Refraction can be written as:

$$n_{\mathrm{I}} \sin[\theta_{\mathrm{I}}] = n_{\mathrm{T}} \sin[\theta_{\mathrm{T}}] \quad .$$

Thanks to the **superposition principle** , for linear systems, one can split the problem by examining the two cases of **wave-Polarization in the incidence plane** and **wave-Polarization normal to the incidence plane**.

In addition to the conventions used in section § 48.02 it is convenient to set the incidence plane as the $y = 0$ plane. The geometry can thus be reduced to the xz plane.

Let the complex amplitude of the Electric|Magnetic field of the MPW be $\overline{\mathbf{V}}_{\mathrm{I}}$ (incident wave), $\overline{\mathbf{V}}_{\mathrm{R}}$ (reflected wave) and $\overline{\mathbf{V}}_{\mathrm{T}}$ (transmitted/refracted wave).

It will be assumed, as it is appropriate for most cases, that there are no free charges nor free currents at the interface.

The results will be presented in terms of complex quantities in order to be readily applicable to the case of MPW with complex wave-vector.

48.03.01 Impedance|Admittance of ElectroMagnetic MPW

This § is referenced at pages:
[2041]

Read also § 44.03.13 for other comments on the **wave impedance** .

As it is useful to deal with the intensity of reflected and refracted waves and the intensity is written in terms of the Poynting vector it is useful to introduce the concept of EM-**wave impedance** as in equation (46.06.22). The impedance of the ElectroMagnetic MPW is defined as in equation (46.06.22). Note that the **wave impedance** is a property of the medium only.

Note that the definitions involve the modules, as Electric|Magnetic fields have different directions in an ElectroMagnetic wave.

For LHI media it becomes:

$$Z \equiv \frac{E}{H} = \mu_0 \mu_{\mathrm{R}} \frac{E}{B} = \mu_0 \mu_{\mathrm{R}} v = \sqrt{\frac{\mu_0 \mu_{\mathrm{R}}}{\varepsilon_0 \varepsilon_{\mathrm{R}}}} \quad . \tag{48.03.01}$$

The reason why the vector field **H** appears instead of the magnetic field is that **H appears in the expression of the Poynting vector**, not **B**.

Therefore the impedance allows one to express the intensity as a function of one of the ElectroMagnetic fields multiplied/divided by the impedance, see section § 45.08.01.05 and § 44.03.13.

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48.03.01.01 Case of Vacuum

In vacuum:

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \mu_0 c = \frac{1}{\varepsilon_0 c} = 377\ \Omega \quad .$$

48.03.01.02 Case of a perfect dielectric medium

The following relations are useful to interchange the pair of parameters ε_R and μ_R with the pair of parameters n and Z:

$$\varepsilon_R = \frac{n}{Z}\sqrt{\frac{\mu_0}{\varepsilon_0}} \quad , \qquad \rightarrow 2189\ 2191$$

$$\mu_R = nZ\sqrt{\frac{\varepsilon_0}{\mu_0}} \quad . \qquad \rightarrow 2189\ 2195$$

These relations come from the fact that both the **index of refraction** and the wave impedance are expressed in terms of the **Electric-Permittivity** | **Magnetic-Permeability** . Therefore it is possible to invert the equations and express the **Electric-Permittivity** | **Magnetic-Permeability** in terms of the **index of refraction** and the **wave impedance** .

It is also often useful to remind the relation between the modules of the various fields for MPW, useful when dealing with boundary conditions:

$$B = E\frac{n}{c} \quad , \qquad \rightarrow 2192$$

$$D = \frac{n}{cZ}E \quad ,$$

$$H = \frac{1}{Z}E \quad ,$$

$$B = \frac{nZ}{c}H \quad ,$$

$$D = \frac{n}{c}H \quad .$$

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case of a non-magnetic perfect dielectric medium

Note that for most materials $\mu_R \simeq 1$. In this case the **wave impedance** and **index of refraction** are not independent quantities any longer and the following relation apply:

$$\boxed{nZ = \sqrt{\frac{\mu_0}{\varepsilon_0}} \qquad Z = \frac{1}{n}\sqrt{\frac{\mu_0}{\varepsilon_0}} \equiv \frac{Z_0}{n}} \quad .$$

48.03.01.03 General Case of a Complex Quantities

The same relations, equation (48.03.01.02), (48.03.01.02), can be extended to the general case to swap the pair $\{\overline{\varepsilon_R}, \overline{\mu_R}\}$ (more useful in ElectroMagnetism) with the pair $\{\overline{n}, \overline{Z}\}$, (more useful when dealing with waves):

$$\boxed{\{\overline{\varepsilon_R}, \overline{\mu_R}\} \leftrightarrow \{\overline{n}, \overline{Z}\}} \quad , \tag{48.03.02}$$

$$\boxed{\overline{\varepsilon_R} = \frac{\overline{n}}{\overline{Z}}\sqrt{\frac{\mu_0}{\varepsilon_0}}} \quad ,$$

$$\boxed{\overline{\mu_R} = \overline{n}\overline{Z}\sqrt{\frac{\varepsilon_0}{\mu_0}}} \quad ,$$

$$\boxed{\overline{n}\overline{Z} = \sqrt{\frac{\mu_0}{\varepsilon_0}}} \quad , \boxed{\overline{Z} = \frac{1}{\overline{n}}\sqrt{\frac{\mu_0}{\varepsilon_0}} \equiv \frac{\overline{Z}_0}{\overline{n}}} \quad .$$

→ 2207

See equation (48.04.02) for the similar two pairs of parameters for mechanical waves.

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48.03.02
wave-Polarization Parallel to the Incidence Plane

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Aka p-wave-Polarization .

It is assumed that the density of free charges and currents is negligible at the surface.

In this section we explicitly denote physical quantities which might be complex as complex numbers, while normally this is not done.

Consider an ElectroMagnetic MPW wave-Polarized parallel to the incidence plane, that is with the three (incident, reflected and transmitted) electric field vectors in the incidence plane; the three electric field vectors are co-planar but non collinear. Assume, by convention, that the components of the three electric field vectors along the normal to the interface surface (z axis, oriented from medium 2 towards medium 1) are all positive. That is, imagine to follow the incident wave-vector with its orthogonal triplet of vectors $\{\mathbf{E},\mathbf{B},\mathbf{k}\}$ and after Reflection|Transmission: both orthogonal terns of Reflected|Transmitted vectors $\{\mathbf{E},\mathbf{B},\mathbf{k}\}$ have the same orientation as the incident one, whose component on the z axis is positive. Equivalently any two vector fields in the incidence plane are in-phase if their z -components are parallel while they are out-of-phase if their z -components are anti-parallel. Obviously in case the Reflected|Transmitted components turn out to be negative than the real Reflected|Transmitted electric field vectors would be inverted.

Continuity of the tangential component of $\mathbf{E}$ and the tangential component of $\mathbf{H}$ give:

$$\overline{E}_{\mathrm{I}}\cos\left[\theta_{\mathrm{I}}\right]-\overline{E}_{\mathrm{R}}\cos\left[\theta_{\mathrm{R}}\right]=\overline{E}_{\mathrm{T}}\cos\left[\theta_{\mathrm{T}}\right]\quad,$$
$$\overline{H}_{\mathrm{I}}+\overline{H}_{\mathrm{R}}=\overline{H}_{\mathrm{T}}\quad.$$

→ 2190 2192

Note that equation (48.03.02) can be obtained from equation (48.03.03) by exchanging $\mathbf{H}$ with $\mathbf{E}$.

Introducing the **wave impedance** $\overline{Z}$ and the notations, limited to the wave impedance and angles, $\mathrm{I}\longrightarrow 1$, $\mathrm{R}\longrightarrow 1$ and $\mathrm{T}\longrightarrow 2$, and using explicitly first Snell law, one finds:

$$\overline{E}_{\mathrm{I}}\cos\left[\theta_{1}\right]-\overline{E}_{\mathrm{R}}\cos\left[\theta_{1}\right]=\overline{E}_{\mathrm{T}}\cos\left[\theta_{2}\right]\quad,$$
$$\frac{\overline{E}_{\mathrm{I}}}{\overline{Z}_1}+\frac{\overline{E}_{\mathrm{R}}}{\overline{Z}_1}=\frac{\overline{E}_{\mathrm{T}}}{\overline{Z}_2}\quad.$$

The solution for the **amplitude reflection coefficient** and **amplitude transmission coefficient** is given by Fresnel equations:

$$\boxed{\overline{r}_{\parallel}\equiv\frac{\overline{E}_{\mathrm{R}}}{\overline{E}_{\mathrm{I}}}=\left(\frac{\overline{Z}_1\cos\left[\theta_1\right]-\overline{Z}_2\cos\left[\theta_2\right]}{\overline{Z}_1\cos\left[\theta_1\right]+\overline{Z}_2\cos\left[\theta_2\right]}\right)}\qquad\boxed{\overline{t}_{\parallel}\equiv\frac{\overline{E}_{\mathrm{T}}}{\overline{E}_{\mathrm{I}}}=\left(\frac{2\overline{Z}_2\cos\left[\theta_1\right]}{\overline{Z}_1\cos\left[\theta_1\right]+\overline{Z}_2\cos\left[\theta_2\right]}\right)}\quad.\qquad(48.03.03)$$

→ 2190 2192 2211

It must be kept in mind that in equation (48.03.03) **the angles θ_1 and θ_2 are not independent** but they are linked by Snell law.

Note that **the amplitude transmission coefficient can be larger than one**.

The **reflectivity** (also known as **reflectance**) and **transmissivity** (also known as **transmittance**) follow (Read also § 48.03.05):

$$\boxed{\mathcal{R}_{\parallel}\equiv\left|\overline{r}_{\parallel}\right|^2\equiv\overline{r}_{\parallel}\overline{r}_{\parallel}^{\star}=\left|\frac{\overline{E}_{\mathrm{R}}}{\overline{E}_{\mathrm{I}}}\right|^2=\left|\frac{\overline{Z}_1\cos\left[\theta_1\right]-\overline{Z}_2\cos\left[\theta_2\right]}{\overline{Z}_1\cos\left[\theta_1\right]+\overline{Z}_2\cos\left[\theta_2\right]}\right|^2}\qquad\boxed{\mathcal{T}_{\parallel}=1-\mathcal{R}_{\parallel}\neq\left|\overline{t}_{\parallel}\right|^2}\quad.\qquad(48.03.04)$$

Note that continuity of the normal component of $\mathbf{B}$ is meaningless in this case because it is identically zero.

The continuity of the normal component of $\mathbf{D}$ gives:

$$\overline{D}_{\mathrm{I}}\sin\left[\theta_{\mathrm{I}}\right]+\overline{D}_{\mathrm{R}}\sin\left[\theta_{\mathrm{R}}\right]=\overline{D}_{\mathrm{T}}\sin\left[\theta_{\mathrm{T}}\right]\quad.$$

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This is equivalent to the equation for the tangential component of **H** by using the Snell law and the expression of ε_R in terms of n and Z (equation (48.03.01.02)).

The **reflectivity** and **transmissivity** at normal incidence are (Read also § 48.03.05):

$$\mathcal{R}_{\parallel} = \left|\frac{\overline{E}_R}{\overline{E}_I}\right|^2 = \left|\frac{\overline{Z}_1 - \overline{Z}_2}{\overline{Z}_1 + \overline{Z}_2}\right|^2 \qquad \mathcal{T}_{\parallel} \equiv 1 - \mathcal{R}_{\parallel} \quad . \tag{48.03.05}$$

→ 2194

The **transmissivity** at normal incidence can be re-written as follows

$$\mathcal{T}_{\parallel} = 1 - \mathcal{R}_{\parallel} = \frac{4\left|\overline{Z}_1\overline{Z}_2\right|}{\left|\overline{Z}_1 + \overline{Z}_2\right|^2} \qquad \mathcal{T}_{\parallel} = \frac{\overline{Z}_1}{\overline{Z}_2}\left|\frac{\overline{E}_T}{\overline{E}_I}\right|^2 = \frac{\overline{Z}_1}{\overline{Z}_2}\left|\frac{2\overline{Z}_2}{\overline{Z}_1 + \overline{Z}_2}\right|^2 \quad ,$$

where the first equation above is from direct calculation from $\mathcal{R}$ while the second one is the application of (48.03.19).

The transmitted part of the electric field never inverts its sign.

Assume: $Z_1 > Z_2$. With this conventions, at normal incidence and for real impedance, the reflected electric field does not change its sign for $Z_1 > Z_2$, which means that the electric field is inverted. Moreover, with this conventions, at grazing incidence ($\theta = \pi/2$) and for real impedance, the reflected electric field changes its sign for $Z_1 > Z_2$, which means that the electric field is inverted as well.

For a non-magnetic medium when the incident medium has a lower **index of refraction** than the transmitting medium, the component of the electric field parallel to the incidence will be in-phase from normal incidence until the Brewster angle and out-of-phase by π radians thereafter. However, the transition is not actually discontinuous, since the electric field goes to zero at the Brewster angle. In case the incident medium has a larger **index of refraction** than the transmitting medium, the analysis is more complex and can be found in the references.

With this convention at normal incidence the convention for the components parallel and perpendicular to the incidence plane are inverted.

Never confuse the conventions about the sign of the component of the vector field with the real orientation of the vector field.

48.03.02.01 Definition of Components for P-wave-Polarization

The conventions used can be summarized as:

$$\overline{\mathbf{H}}_{I|R|T} \propto \overline{\mathbf{k}}_{I|R|T} \wedge \widehat{\mathbf{N}} \qquad \overline{\mathbf{E}}_{I|R|T} \propto \overline{\mathbf{H}}_{I|R|T} \wedge \overline{\mathbf{k}}_{I|R|T} \quad . \tag{48.03.06}$$

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48.03.03
wave-Polarization Perpendicular to the Incidence Plane

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|7|*Good text.*|
©|E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|4.6|*Excellent*|
©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|||

Aka s-wave-Polarization .

It is assumed that the density of free charges and currents is negligible at the surface.

In this section we explicitly denote physical quantities which might be complex as complex numbers, while normally this is not done.

Consider an ElectroMagnetic MPW wave-Polarized perpendicular to the incidence plane, that is with the three (incident, reflected and transmitted) electric field vectors perpendicular to the incidence plane; the three electric field vectors are collinear. Assume, by convention, that the components of the three electric field vectors perpendicular to the incidence plane, are positive when they have the same orientation with respect to the incidence plane. Obviously in case the Reflected|Transmitted components turn out to be negative than the real Reflected|Transmitted electric field vectors would be inverted.

Continuity of the tangential component of **E** and the tangential component of **H** give:

$$\overline{E}_{\mathrm{I}} + \overline{E}_{\mathrm{R}} = \overline{E}_{\mathrm{T}} \quad ,$$
$$\overline{H}_{\mathrm{I}} \cos[\theta_{\mathrm{I}}] - \overline{H}_{\mathrm{R}} \cos[\theta_{\mathrm{R}}] = \overline{H}_{\mathrm{T}} \cos[\theta_{\mathrm{T}}] \quad .$$

→ 2190 2192

Note that equation (48.03.03) can be obtained from equation (48.03.02) by exchanging **E** with **H**.

Introducing the **wave impedance** $\overline{Z}$ and the notations, limited to the wave impedance and angles, I $\longrightarrow$ 1, R $\longrightarrow$ 1 and T $\longrightarrow$ 2, and using explicitly first Snell law, one finds:

$$\overline{E}_{\mathrm{I}} + \overline{E}_{\mathrm{R}} = \overline{E}_{\mathrm{T}} \quad ,$$
$$\frac{\overline{E}_{\mathrm{I}} \cos[\theta_1]}{\overline{Z}_1} + \frac{\overline{E}_{\mathrm{R}} \cos[\theta_1]}{\overline{Z}_1} = \frac{\overline{E}_{\mathrm{T}} \cos[\theta_2]}{\overline{Z}_2} \quad .$$

The solution for the **amplitude reflection coefficient** and **amplitude transmission coefficient** is given by Fresnel equations:

$$\overline{r}_{\perp} \equiv \frac{\overline{E}_{\mathrm{R}}}{\overline{E}_{\mathrm{I}}} = \left(\frac{\overline{Z}_2 \cos[\theta_1] - \overline{Z}_1 \cos[\theta_2]}{\overline{Z}_2 \cos[\theta_1] + \overline{Z}_1 \cos[\theta_2]} \right) \qquad \overline{t}_{\perp} \equiv \frac{\overline{E}_{\mathrm{T}}}{\overline{E}_{\mathrm{I}}} = \left(\frac{2\overline{Z}_2 \cos[\theta_1]}{\overline{Z}_2 \cos[\theta_1] + \overline{Z}_1 \cos[\theta_2]} \right) \quad . \tag{48.03.07}$$

→ 2211

It must be kept in mind that in equation (48.03.03) **the angles θ_1 and θ_2 are not independent** but they are linked by Snell law.

Note that **the amplitude transmission coefficient can be larger than one**.

The **reflectivity** (also known as **reflectance**) and **transmissivity** (also known as **transmittance**) follow (Read also § 48.03.05):

$$\mathcal{R}_{\perp} \equiv |\overline{r}_{\perp}|^2 \equiv \overline{r}_{\perp} \overline{r}^{\star}_{\perp} = \left| \frac{\overline{E}_{\mathrm{R}}}{\overline{E}_{\mathrm{I}}} \right|^2 = \left| \frac{\overline{Z}_2 \cos[\theta_1] - \overline{Z}_1 \cos[\theta_2]}{\overline{Z}_2 \cos[\theta_1] + \overline{Z}_1 \cos[\theta_2]} \right|^2 \qquad \mathcal{T}_{\perp} = 1 - \mathcal{R}_{\perp} \neq \left| \overline{t}_{\perp} \right|^2 \quad . \tag{48.03.08}$$

Note that continuity of the normal component of **D** is meaningless in this case because it is identically zero.

The continuity of the normal component of **B** gives:

$$\overline{B}_{\mathrm{I}} \sin[\theta_{\mathrm{I}}] + \overline{B}_{\mathrm{R}} \sin[\theta_{\mathrm{R}}] = \overline{B}_{\mathrm{T}} \sin[\theta_{\mathrm{T}}] \quad .$$

This is equivalent to the equation for the tangential component of **E** by using the Snell law and the expression of B in terms of E, n and c (equation (48.03.01.02)).

The **reflectivity** and **transmissivity** at normal incidence are (Read also § 48.03.05):

$$\mathcal{R}_{\perp} = \left| \frac{\overline{E}_{\mathrm{R}}}{\overline{E}_{\mathrm{I}}} \right|^2 = \left| \frac{\overline{Z}_2 - \overline{Z}_1}{\overline{Z}_1 + \overline{Z}_2} \right|^2 \qquad \mathcal{T}_{\perp} \equiv 1 - \mathcal{R}_{\perp} \quad . \tag{48.03.09}$$

→ 2194

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The **transmissivity** at normal incidence can be re-written as follows

$$\mathcal{T}_\perp = 1 - \mathcal{R}_\perp = \frac{2\left(\overline{Z}_1\overline{Z}_2^\star + \overline{Z}_1^\star\overline{Z}_2\right)}{\left|\overline{Z}_1 + \overline{Z}_2\right|^2} \qquad \mathcal{T}_\perp = \frac{\overline{Z}_1}{\overline{Z}_2}\left|\frac{\overline{E}_\mathrm{T}}{\overline{E}_\mathrm{I}}\right|^2 = \frac{\overline{Z}_1}{\overline{Z}_2}\left|\frac{2\overline{Z}_2}{\overline{Z}_1 + \overline{Z}_2}\right|^2 \quad ,$$

where the first equation above is from direct calculation from $\mathcal{R}$ while the second one is the application of (48.03.19).

The transmitted part of the electric field never inverts its sign.

Assume: $Z_1 > Z_2$. With this conventions, at normal incidence and for real impedance, the reflected electric field components changes its sign for $Z_1 > Z_2$, which means that the electric field is inverted. Moreover, with this conventions, at grazing incidence ($\theta = \pi/2$) and for real impedance, the reflected electric field changes its sign for $Z_1 > Z_2$, which means that the electric field is inverted as well.

For a non-magnetic medium when the incident medium has a lower **index of refraction** than the transmitting medium, the component of the electric field normal to the incidence undergoes a phase shift of π radians upon reflection.

In case the incident medium has a larger **index of refraction** than the transmitting medium, the analysis is more complex and can be found in the references.

On the other hand **the transmitted part of the electric field never inverts its sign**.

With this convention at normal incidence the convention for the components parallel and perpendicular to the incidence plane are inverted.

Never confuse the conventions about the sign of the component of the vector field with the real orientation of the vector field.

48.03.03.01 Definition of Components for S-wave-Polarization

The conventions used can be summarized as:

$$\overline{\mathbf{E}}_{\mathrm{I|R|T}} \propto \overline{\mathbf{k}}_{\mathrm{I|R|T}} \wedge \widehat{\mathbf{N}} \qquad \overline{\mathbf{H}}_{\mathrm{I|R|T}} \propto \overline{\mathbf{k}}_{\mathrm{I|R|T}} \wedge \overline{\mathbf{E}}_{\mathrm{I|R|T}} \quad . \tag{48.03.10}$$

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48.03.04
Properties of Reflected|Transmitted Waves

48.03.04.01 Some Properties Derived From Fresnel Laws

The amplitude reflection coefficient $r_\perp$ never changes its sign.

The absolute values of $r_\parallel$ and $r_\perp$ are symmetrical in θ_1 and θ_2.

Reflection can produce totally wave-Polarized light but Transmission|Refraction can only produce partially wave-Polarized light.

It is always:

$$\boxed{|r_\parallel| \leq |r_\perp| \qquad \text{the equality applies if and only if } \theta = 0 \text{ and } \theta = \pi/2} \quad ,$$

showing that the partial **wave-Polarization** perpendicular to the incidence plane always increases. In particular, for incident non wave-Polarized light, the reflected light is always partially wave-Polarized perpendicular to the incidence plane. This effect is exploited in Polaroid glasses to reduced the glaring effect of specular reflection.

On the other hand the refracted light will experience an increase of **wave-Polarization** parallel to the incidence plane.

The coefficient $r_\perp$ is such that $|r_\perp|$ increases monotonically with increasing θ_1, starting from the value $\dfrac{n_1 - n_2}{n_1 + n_2}$. On the other hand the coefficient $r_\parallel$ is such that $|r_\parallel|$ starts from the value $\dfrac{n_1 - n_2}{n_1 + n_2}$, as $|r_\perp|$, and decreases until the Brewster angle, then it rises monotonically.

For the behavior at $\theta = \pi/2$ (specular reflection) two cases must be distinguished: in the case the second medium is optically denser ($n_2 > n_1$) both $|r_\parallel|$ and $|r_\perp|$ rises up to one at $\theta = \pi/2$; in the case the first medium is optically denser ($n_1 > n_2$) both $|r_\parallel|$ and $|r_\perp|$ become one at the angle of total internal reflection and θ can never reach $\theta = \pi/2$.

Consider the component of the field in the incidence plane: in the case $n_2 > n_1$ the direction of the electric field is inverted both at $\theta_\text{I} = 0$ and $\theta_\text{I} = \pi/2$; in the case $n_1 > n_2$ the direction of the electric field is conserved both at $\theta_\text{I} = 0$ and at $\theta_\text{I} = \pi/2$.

Consider the component of the field perpendicular to the incidence plane: it always changes sign for $n_2 > n_1$; it always keeps the same sign for $n_2 < n_1$.

The transmitted electric field, on the other hand, is always in phase with the incident electric field.

At grazing incidence, $\theta_\text{I} = \pi/2$, the reflectance goes to unity in any case. This is well known from common experience where one observes strong reflection at $\theta_\text{I} = \pi/2$.

48.03.04.02 Reflected|Transmitted Intensities at Normal Incidence

This § is referenced at pages:
[2208]

At normal incidence one finds, from both equations (48.03.05), (48.03.09):

$$\boxed{|r| = \left|\frac{\overline{Z}_1 - \overline{Z}_2}{\overline{Z}_1 + \overline{Z}_2}\right| \qquad |t| = \left|\frac{2\overline{Z}_2}{\overline{Z}_1 + \overline{Z}_2}\right|} \quad . \tag{48.03.11}$$

The different signs of $r_\parallel$ and $r_\perp$ at normal incidence is just an artifact of the conventions chosen.

DRAFT - In Progress

48.03.04.03 Media With Real Impedance

Assume that all the **wave impedances** are real: $\overline{Z} \to Z$, for the rest of this section.

The intensity for MPW is:

$$I_0 = \frac{1}{2}\frac{|\mathbf{E}_0|^2}{Z} = \frac{1}{2}Z|\mathbf{H}_0|^2 \qquad \text{for real } \overline{Z} \quad .$$

Energy conservation reads:

$$I_{\Re} + I_{\Im} = \frac{1}{2Z_1}|\mathbf{E}_{\mathrm{R}}|^2 + \frac{1}{2Z_2}|\mathbf{E}_{\mathrm{T}}|^2 = \frac{|\mathbf{E}_{\mathrm{I}}|^2}{2}\left(\frac{1}{Z_1}\left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 + \frac{1}{Z_2}\left(\frac{2Z_2}{Z_1 + Z_2}\right)^2\right) = \frac{1}{2Z_1}|\mathbf{E}_{\mathrm{I}}|^2 = I_0 \quad .$$

At normal incidence one can notice that on entering an optically denser medium ($Z_1 > Z_2$) the electric field inverts its orientation while the magnetic field keeps the same orientation. The reverse happens on exiting an optically denser medium ($Z_1 > Z_2$).

Note that the amplitude transmission coefficient is different from the one for mechanical waves (read § 48.04.01) ■ However, the reflected and transmitted intensities are, exactly as for mechanical waves:

$$\frac{I_{\mathrm{R}}}{I_{\mathrm{I}}} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 \quad ,$$

$$\frac{I_{\mathrm{T}}}{I_{\mathrm{I}}} = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2} \quad .$$

48.03.04.04 Brewster Angle

In the case of ElectroMagnetic wave wave-Polarized in the plane of incidence there is one angle, the so-called Brewster angle, $\theta_1 \equiv \theta_B$, such that reflection is zero:

$$\overline{Z}_1 \cos[\theta_1] = \overline{Z}_2 \cos[\theta_2] \implies \tan[\theta_B] = \sqrt{\frac{1 - \frac{Z_1^2}{Z_2^2}}{\frac{n_1^2}{n_2^2} - 1}} \quad . \tag{48.03.12}$$

In fact:

$$n_1^2 \sin^2[\theta_1] = n_2^2 \sin^2[\theta_2] \quad ,$$

$$Z_1^2 \cos^2[\theta_1] = Z_2^2 \cos^2[\theta_2] \quad ,$$

$$1 = \cos^2[\theta_2] + \sin^2[\theta_2] = \frac{Z_1^2}{Z_2^2}\cos^2[\theta_1] + \frac{n_1^2}{n_2^2}\sin^2[\theta_1] = \cos^2[\theta_1] + \sin^2[\theta_1] \quad ,$$

$$\left(\frac{Z_1^2}{Z_2^2} - 1\right)\cos^2[\theta_1] + \left(\frac{n_1^2}{n_2^2} - 1\right)\sin^2[\theta_1] = 0 \tag{48.03.13}$$

If, as it is usual, the two media have negligible magnetic properties equation (48.03.01.02) implies $Z \propto n^{-1}$ and the above formula reduces to the usual expression:

$$\tan[\theta_B] = \frac{n_2}{n_1} \qquad \text{if and only if } \mu_{\mathrm{R}1} = \mu_{\mathrm{R}2} \quad . \tag{48.03.14}$$

Note that Brewster angle exists in both cases of $n_1 > n_2$ and $n_1 < n_2$.

It can be shown that Brewster angle never exists for the component of the electric field perpendicular to the incidence plane as the corresponding equation has no solution.

Brewster angle is such that:

$$\theta_{\mathrm{R}} + \theta_{\mathrm{T}} = \pi/2 \quad .$$

Light reflected at the Brewster angle is totally wave-Polarized in the plane perpendicular to the incidence angle. This gives the chance to build a linear Polarizer: Brewster angle Polarizer.

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48.03.04.05 Impedance Matching|Mismatch

If the impedances are very different there is large reflection (impedance mismatch).

If the impedances are very similar there is large transmission (impedance matching). If $Z_1 = Z_2$ the reflected wave is zero and all the energy passes into the second medium. In this case the impedances of the two media are said to be matched. For a medium which is both Polarizable and Magnetizable, the matching condition is:

$$\frac{\varepsilon_{\mathrm{R}1}}{\mu_{\mathrm{R}1}} = \frac{\varepsilon_{\mathrm{R}2}}{\mu_{\mathrm{R}2}} \quad . \tag{48.03.15}$$

Note that in this case, the condition is obtained without the need to insert a anti-reflection coating made of a third material, because both media are both Polarizable and Magnetizable. Read § 48.06.10.

Note that a perfect conductor has zero impedance, as it can be considered a medium with infinite **Electric-Permittivity** . Therefore ElectroMagnetic waves are reflected from a perfect conductor as there is a huge impedance mismatch.

48.03.04.06 Reflection Matrix

The change of the **wave-Polarization** state on reflection is described by the Reflection Matrix.

$$\begin{pmatrix} \overline{\mathbf{E}}_p\big|_{\mathrm{R}} \\ \overline{\mathbf{E}}_s\big|_{\mathrm{R}} \end{pmatrix} = \begin{pmatrix} \overline{r}_{pp} & \overline{r}_{ps} \\ \overline{r}_{sp} & \overline{r}_{ss} \end{pmatrix} \begin{pmatrix} \overline{\mathbf{E}}_p\big|_{\mathrm{I}} \\ \overline{\mathbf{E}}_s\big|_{\mathrm{I}} \end{pmatrix}$$

Experimental technique: ellipsometry.

48.03.04.07 Fresnel Equations for Non-Magnetic Media

This is by far the most common case for most materials. In this case:

$$Z \propto \frac{1}{n} \quad .$$

The amplitude reflection coefficient and the amplitude transmission coefficient in the case of **wave-Polarization** parallel to the incidence plane become:

$$r_{\parallel} = +\frac{\tan[\theta_1 - \theta_2]}{\tan[\theta_1 + \theta_2]} \qquad t_{\parallel} = \frac{2\sin[\theta_2]\cos[\theta_1]}{\sin[\theta_1 + \theta_2]\cos[\theta_1 - \theta_2]} \quad . \tag{48.03.16}$$

The amplitude reflection coefficient $r_{\parallel}$ changes its sign at the Brewster angle, when $\theta_1 + \theta_2 = \pi/2$, where $r_{\parallel} = 0$, because the denominator changes its sign.

The amplitude reflection coefficient and the amplitude transmission coefficient in the case of **wave-Polarization** perpendicular to the incidence plane become:

$$r_{\perp} = -\frac{\sin[\theta_1 - \theta_2]}{\sin[\theta_1 + \theta_2]} \qquad t_{\perp} = \frac{2\sin[\theta_2]\cos[\theta_1]}{\sin[\theta_1 + \theta_2]} \quad . \tag{48.03.17}$$

When we have Fresnel equations for non magnetic media, the **Magnetic-Permeability** can be put back into any of the equations by making the replacement:

$$n \to n/\mu_{\mathrm{R}} \quad .$$

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48.03.04.08 Non perfect dielectric media and Complex Reflection|Transmission Coefficients

Fresnel formulas derived in section § 48.03 are still valid for non **perfect dielectric media**, whose **wave impedance** and **index of refraction** may be complex numbers. In this case the **amplitude reflection coefficient** and **amplitude transmission coefficients** have a modulus given by the modulus of the coefficient and a change of phase given by the phase of the complex number.

48.03.04.09 Plots of Coefficients in Various Cases

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Plots of the Ratio of the Amplitudes: Amplitude Reflection Coefficient

Non-magnetic media: $\mu_{R1} = \mu_{R2} = 1$ and $Z \propto 1/n$.

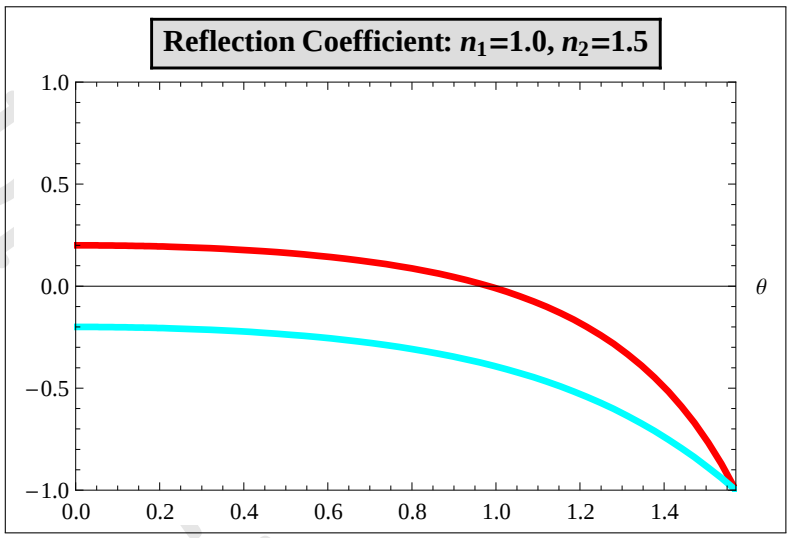

Figure 48.11: amplitude reflection coefficients: parallel to the incidence plane (red) and perpendicular to the incidence plane (blue); $n_1 = 1.0$ and $n_2 = 1.5$.

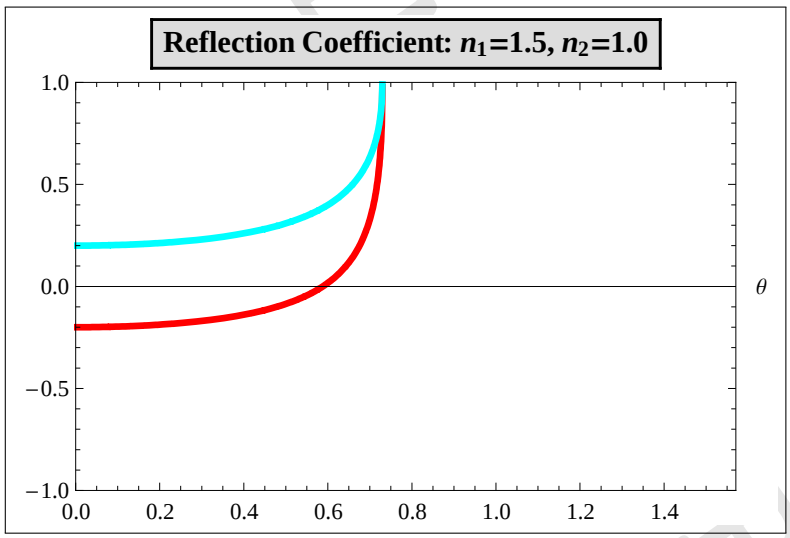

Figure 48.12: amplitude reflection coefficients: parallel to the incidence plane (red) and perpendicular to the incidence plane (blue); $n_1 = 1.5$ and $n_2 = 1.0$.

Plots of the Ratio of the Amplitudes: Amplitude Transmission Coefficient

Non-magnetic media: $\mu_{R1} = \mu_{R2} = 1$ and $Z \propto 1/n$.

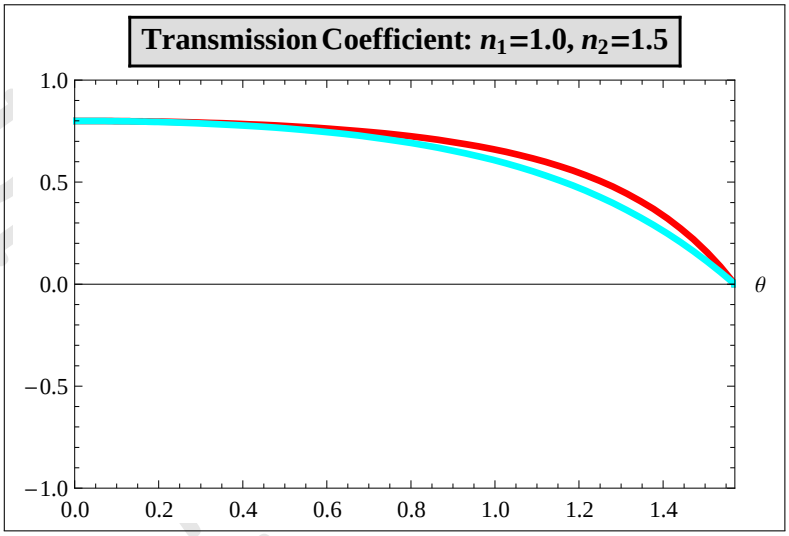

Figure 48.13: Amplitude transmission coefficients: parallel to the incidence plane (red) and perpendicular to the incidence plane (blue); $n_1 = 1.0$ and $n_2 = 1.5$.

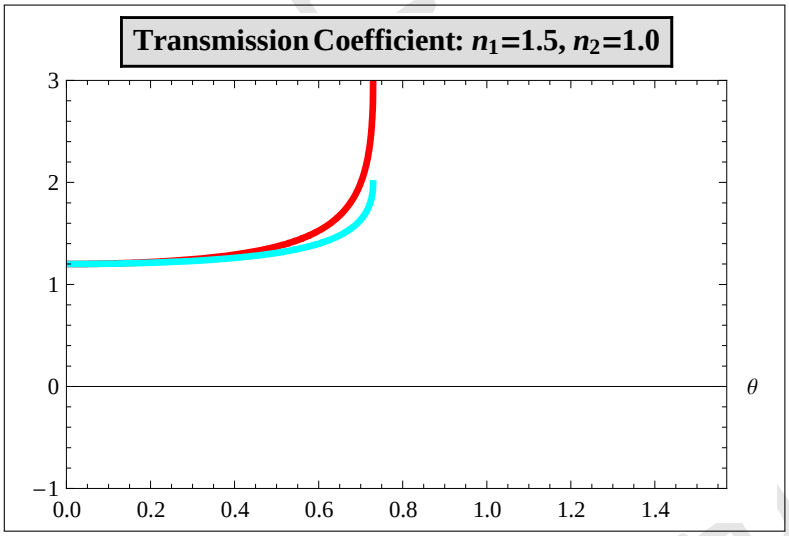

Figure 48.14: Amplitude transmission coefficients: parallel to the incidence plane (red) and perpendicular to the incidence plane (blue); $n_1 = 1.5$ and $n_2 = 1.0$.

Plots for Incidence Onto a Medium With Larger index of refraction (ratio close to one)

Non-magnetic media: $\mu_{R1} = \mu_{R2} = 1$ and $Z \propto 1/n$.

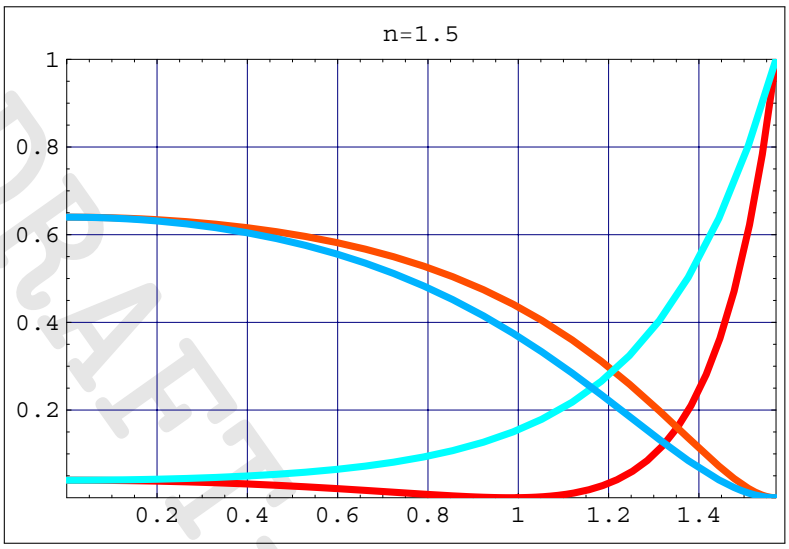

Figure 48.15: Amplitude reflection coefficient squared: parallel to the incidence plane (dark red) and perpendicular to the incidence plane (light blue); amplitude transmission coefficient squared: parallel to the incidence plane (light red) and perpendicular to the incidence plane (dark blue); **index of refraction** es: $n_2/n_1 = 1.5$.

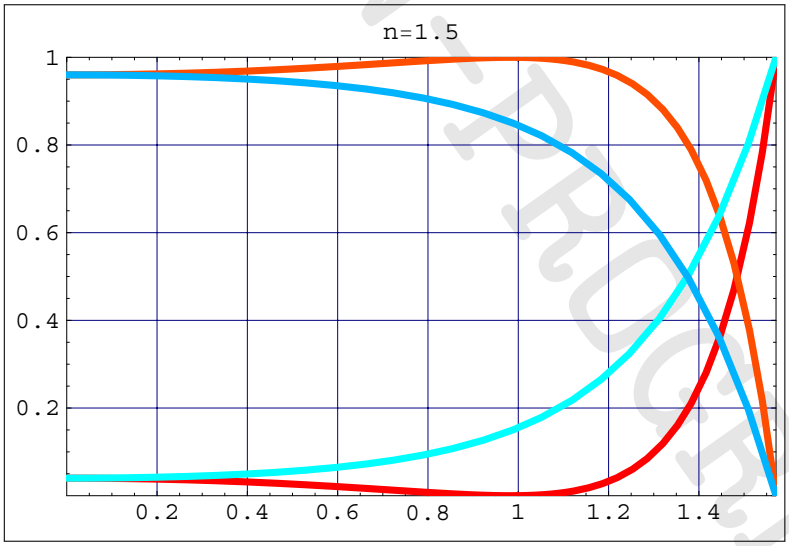

Figure 48.16: Reflectivity: parallel to the incidence plane (dark red) and perpendicular to the incidence plane (light blue). Transmissivity coefficient parallel to the incidence plane (light red) and perpendicular to the incidence plane (dark blue); **index of refraction** es: $n_2/n_1 = 1.5$.

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Plots for Incidence Onto a Medium With Larger index of refraction (ratio much larger than one)

Non-magnetic media: $\mu_{R1} = \mu_{R2} = 1$ and $Z \propto 1/n$.

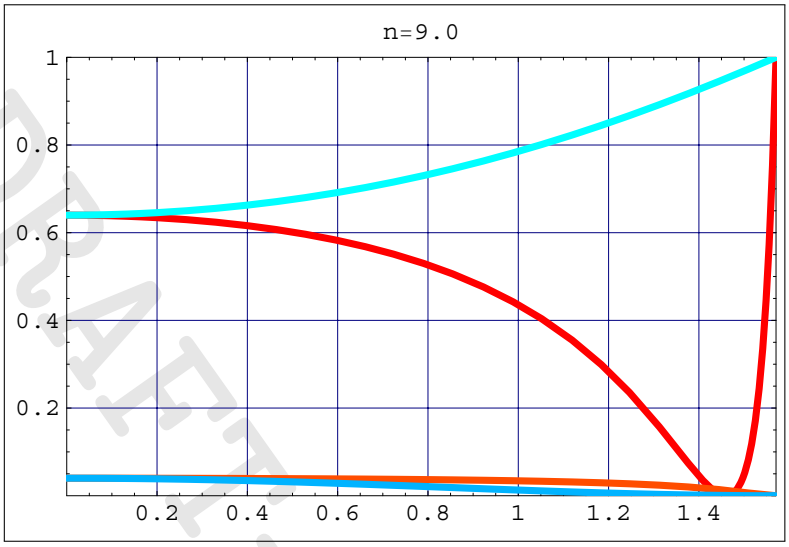

Figure 48.17: Amplitude reflection coefficient squared: parallel to the incidence plane (dark red) and perpendicular to the incidence plane (light blue); amplitude transmission coefficient squared: parallel to the incidence plane (light red) and perpendicular to the incidence plane (dark blue); **index of refraction** es: $n_2/n_1 = 10.0$.

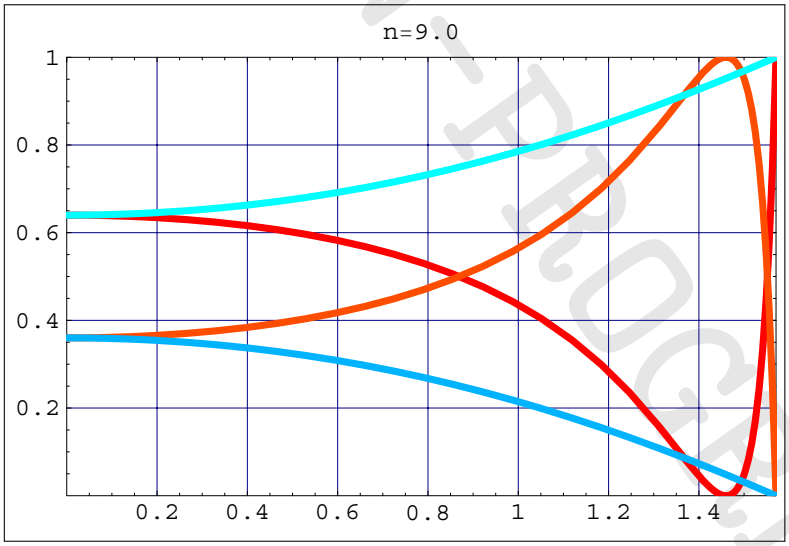

Figure 48.18: Reflectivity coefficient: parallel to the incidence plane (dark red) and perpendicular to the incidence plane (light blue). Transmissivity coefficient parallel to the incidence plane (light red) and perpendicular to the incidence plane (dark blue); **index of refraction** es: $n_2/n_1 = 10.0$.

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Plots for Incidence Onto a Medium With Smaller index of refraction

Non-magnetic media: $\mu_{R1} = \mu_{R2} = 1$ and $Z \propto 1/n$.

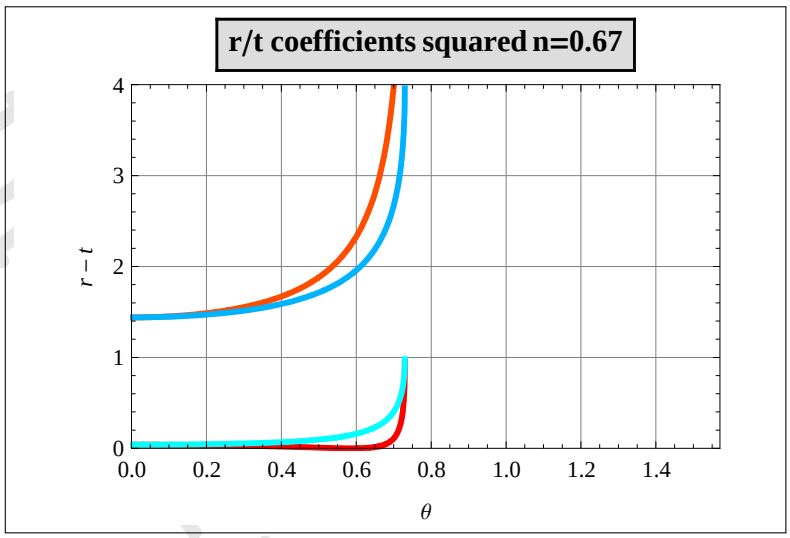

Figure 48.19: Amplitudereflection coefficient squared: parallel to the incidence plane (dark red) and perpendicular to the incidence plane (light blue); amplitude transmission coefficient squared: parallel to the incidence plane (light red) and perpendicular to the incidence plane (dark blue); **index of refraction** es: $n_2/n_1 = 0.67$.

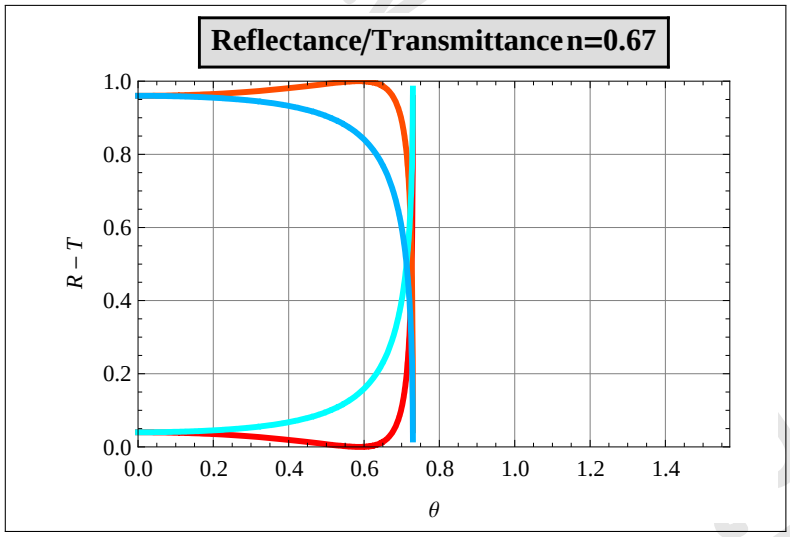

Figure 48.20: Reflectivity:parallel to the incidence plane (dark red) and perpendicular to the incidence plane (light blue). Transmissivity coefficient parallel to the incidence plane (light red) and perpendicular to the incidence plane (dark blue); **index of refraction** es: $n_2/n_1 = 0.67$.

48.03.05
Energy Balance

This § is referenced at pages:
[2190, 2190, 2192, 2192, 2216]

The reflected energy flux can be determined by just comparing the square of the ratio of the reflected and incident amplitudes.

On the other hand the transmitted energy flux cannot be determined by simply comparing the square of the ratio of the transmitted and incident amplitudes. In fact in this case the direction of the incident and transmitted waves are different and therefore a geometrical factor must be introduced to account for the fact that energy is exchanged at a fixed area at the interface but the incident and transmitted beams have different cross-sectional areas because the waves have different directions. Moreover, the intensity depends on the **wave impedance** giving factors which do not cancel in the ratio of the transmitted to incident intensities as they do in the ratio of the reflected to incident intensities. This is also evident from the fact that:

$$\left|r_{\parallel}\right|^2+\left|t_{\parallel}\right|^2\neq 1 \qquad |r_{\perp}|^2+|t_{\perp}|^2\neq 1 \quad .$$

The correct way to proceed is to write the balance of the power at a piece of surface of area A of the interface. **Let J be the energy per unit time per unit surface area of the interface** (note: per unit surface area of the interface!): **not the intensity**. For any piece of surface of the interface, A_0, and any beam with Incidence/Reflection|Transmission angle θ, one can write the expression of the energy incident, reflected or transmitted as:

$$S\equiv\langle|\mathbf{S}_{\text{EM}}|\rangle \qquad \frac{\Delta E}{\Delta t}=SA_{\perp}=SA_0\cos\left[\theta\right]\equiv A_0J\implies J\equiv S\cos\left[\theta\right] \quad .$$

The equality in terms of the energy per unit time per unit surface area of the interface, J, for any given piece of the surface, can be translated into an expression involving the Poynting vector and the relevant angles:

$$J_{\text{I}}=J_{\text{R}}+J_{\text{T}}\implies S_{\text{I}}\cos\left[\theta_{\text{I}}\right]=S_{\text{R}}\cos\left[\theta_{\text{R}}\right]+S_{\text{T}}\cos\left[\theta_{\text{T}}\right] \quad .$$

The following results for the reflectivity and the transmissivity can be found, for any of the two states of **wave-Polarization** parallel and perpendicular to the incidence plane:

$$\mathcal{R}\equiv\frac{J_{\text{R}}}{J_{\text{I}}}=|r|^2 \quad , \tag{48.03.18}$$

$$\mathcal{T}\equiv\frac{J_{\text{T}}}{J_{\text{I}}}=|t|^2\frac{\mathfrak{Re}\left[Z_{\text{I}}\right]}{\mathfrak{Re}\left[Z_{\text{T}}\right]}\frac{\cos\left[\theta_{\text{T}}\right]}{\cos\left[\theta_{\text{I}}\right]} \quad . \tag{48.03.19}$$

→ 2191 2193

Note that **the absorption of energy inside the two media does not enter the previous energy balance**, because Fresnel relations strictly refer to just the boundary between the two media. Therefore the absorbed energy inside the second medium will be taken from the energy transmitted at the interface, determined by Fresnel relations. Obviously if the first medium is absorbing one should use for the incoming energy the energy reaching the surface: the energy reaching the interface has been already subtracted of the energy absorbed in the first medium.

PRELIMINARY PROGRESS

48.03.06
Total Internal Reflection

This § is referenced at pages:
[2089, 2215]

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©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||
©|J.E.Vitela|Am.J.Phys, , ..., ..., ...Ed., ... 72, 393 (2004) - WEB - URL|*Excellent and complete.*|

The two **media are two perfect dielectric media** , that is the **index of refractions** are real. In fact total internal reflection is a kinematical property, so it must be possible to describe it for ElectroMagnetic waves also in the simple case of **perfect dielectric media**.

The interface plane is the plane xy.

The incidence plane is the plane $y = 0$, that is the xz plane. All the problem is in this plane.

The projected vectors on the xy plane, $\widetilde{\mathbf{k}}_{\mathrm{I}}$, $\widetilde{\mathbf{k}}_{\mathrm{R}}$, $\widetilde{\mathbf{k}}_{\mathrm{T}}$ (all bi-dimensional vectors) are all parallel. Let $\widetilde{\mathbf{r}}$ be the position vector projected on the xy plane:

$$\overline{k}_{\mathrm{Iy}} = \overline{k}_{\mathrm{Ry}} = \overline{k}_{\mathrm{Ty}} = 0 \tag{48.03.20}$$

The incident wave-vector $\mathbf{k}_{\mathrm{I}}$, and therefore $\widetilde{\mathbf{k}}_{\mathrm{I}}$, are real vectors, by hypothesis.

Snell law implies:

$$\widetilde{\mathbf{k}_{\mathrm{I}}}\cdot\widetilde{\mathbf{r}} = \widetilde{\mathbf{k}_{\mathrm{R}}}\cdot\widetilde{\mathbf{r}} = \widetilde{\mathbf{k}_{\mathrm{T}}}\cdot\widetilde{\mathbf{r}} \quad .$$

Snell law, in particular, does not forbid an imaginary part of the z component of $\widetilde{\mathbf{k}}_{\mathrm{T}}$, as it disappears when projecting onto the incidence plane to get of $\widetilde{\mathbf{k}}_{\mathrm{T}}$ and it does not change $\widetilde{\mathbf{k}}_{\mathrm{T}}\cdot\widetilde{\mathbf{r}}$.

The condition for total internal reflection is:

$$\frac{n_{\mathrm{T}}}{n_{\mathrm{I}}} \leq \sin[\theta_{\mathrm{I}}] \leq 1 \qquad \frac{n_{\mathrm{T}}}{n_{\mathrm{I}}} \leq \sin[\theta_{\mathrm{T}}] \leq 1 \quad . \tag{48.03.21}$$

Actually the ElectroMagnetic fields in the second medium may be not exactly zero: the wave-vector in the second medium is complex with its imaginary part perpendicular to the surface.

Let the boundary between the two media be the plane xy with medium 1 at $z > 0$ and medium 2 at $z < 0$. Let the incidence plane be the plane xz. An ElectroMagnetic MPW is traveling with $\mathbf{k}_{\mathrm{I}} = \{k_{\mathrm{Ix}}, 0, k_{\mathrm{Iz}}\}$, $\mathfrak{Re}\left[\overline{k}_{\mathrm{Iz}}\right] < 0$, with the wave-vector $\mathbf{k}_{\mathrm{I}}$ real. Assume that total internal reflection apply:

$$\sin[\theta_{\mathrm{I}}] > \frac{n_2}{n_1} \equiv n \qquad n_1 > n_2 \text{ and } n < 1 \qquad \mathfrak{Re}\left[\overline{k}_{\mathrm{Iz}}\right] < 0 \quad .$$

The transmitted wave can be written in terms of the, a priori complex, transmitted wave-vector, $\overline{\mathbf{k}_{\mathrm{T}}}$, with $\overline{k}_{\mathrm{Tx}}$ and $\overline{k}_{\mathrm{Tz}}$, a priori, complex numbers, with $\mathfrak{Re}\left[\overline{k}_{\mathrm{Tx}}\right] > 0$:

$$\overline{\mathbf{k}_{\mathrm{T}}} = \left\{\overline{k}_{\mathrm{Tx}}, 0, \overline{k}_{\mathrm{Tz}}\right\} \qquad \mathfrak{Re}\left[\overline{k}_{\mathrm{Tz}}\right] = 0 \quad ,$$

$$\overline{\psi}_{\mathrm{T}}[\mathbf{r}, t] = \overline{\psi}_{0\mathrm{T}} \exp\left[\mathtt{i}\left(\overline{\mathbf{k}}_{\mathrm{T}}\cdot\mathbf{r} - \omega t\right)\right] = \qquad \text{with } \mathfrak{Re}\left[\overline{k}_{\mathrm{Tx}}\right] > 0 \quad ,$$

$$\overline{\psi}_{0\mathrm{T}} \exp[-\mathtt{i}\omega t] \exp\left[\mathtt{i}\left(x\overline{k}_{\mathrm{Tx}} + z\overline{k}_{\mathrm{Tz}}\right)\right] = \quad ,$$

$$\overline{\psi}_{0\mathrm{T}} \exp[-\mathtt{i}\omega t] \exp\left[\mathtt{i}\overline{k}_{\mathrm{T}}\left(x\sin[\theta_{\mathrm{T}}] \pm z\cos[\theta_{\mathrm{T}}]\right)\right] \quad ,$$

$$\text{with} \qquad \overline{k}_{\mathrm{T}} \equiv \sqrt{\overline{\mathbf{k}_{\mathrm{T}}}\cdot\overline{\mathbf{k}_{\mathrm{T}}}} \quad ,$$

where the sign along the z axis still need to be determined. In fact, the sign of $\sin[\theta_{\mathrm{T}}] > 0$ is determined by Snell laws. But, a priori, there exist two opposite values for $\cos[\theta_{\mathrm{T}}]$, so that further investigation is needed. Moreover:

$$\overline{k}_{\mathrm{Tx}} = \overline{k}_{\mathrm{T}}\sin[\theta_{\mathrm{T}}] \qquad \overline{k}_{\mathrm{Tx}} > 0 \qquad \sin[\theta_{\mathrm{T}}] > 0 \quad , \tag{48.03.22}$$

so that we expect $\overline{k}_{\mathrm{T}} > 0$.

UNDER PEER REVIEW IN PROGRESS

Note that, for any complex vector $\overline{\mathbf{k}}$:

$$\overline{k} \equiv \sqrt{\overline{\mathbf{k}}\cdot\overline{\mathbf{k}}} \neq \sqrt{\overline{\mathbf{k}}\cdot\overline{\mathbf{k}}^{\star}} = \sqrt{\overline{\mathbf{k}}^{\star}\cdot\overline{\mathbf{k}}} \equiv \left|\overline{\mathbf{k}}\right| \quad .$$

Up to now it is standard: the fact that $\overline{\mathbf{k}_\mathrm{T}}$ is complex may enter from now on.

However, in the present problem, $\overline{\mathbf{k}_\mathrm{T}}\cdot\overline{\mathbf{k}_\mathrm{T}}$ is real and positive. In fact, the value of the square of $\overline{\mathbf{k}_\mathrm{T}}$ is given by the relation:

$$\overline{\mathbf{k}_\mathrm{T}}\cdot\overline{\mathbf{k}_\mathrm{T}} = \frac{\omega^2}{v_2^2} \geq 0 \implies \overline{k}_\mathrm{T} > 0 \quad .$$

Therefore:

$$\sin\left[\theta_\mathrm{T}\right] = \frac{n_1}{n_2}\sin\left[\theta_\mathrm{I}\right] \equiv \frac{\sin\left[\theta_\mathrm{I}\right]}{n} > \sin\left[\theta_\mathrm{I}\right] \qquad \sin\left[\theta_\mathrm{T}\right] \text{ is real but larger than one} \quad ,$$

$$\cos\left[\theta_\mathrm{T}\right] = \pm\mathtt{i}\sqrt{\frac{\sin^2\left[\theta_\mathrm{I}\right]}{n^2} - 1} \qquad k_{\mathrm{T}z} \text{ is pure imaginary} \quad ,$$

$$\overline{\psi}_\mathrm{T}[\mathbf{r},t] = \overline{\psi}_{0\mathrm{T}}\exp\left[-\mathtt{i}\omega t\right]\exp\left[\mathtt{i}\overline{k}_\mathrm{T}\left(x\sin\left[\theta_\mathrm{T}\right] \pm z\mathtt{i}\sqrt{\frac{\sin^2\left[\theta_\mathrm{I}\right]}{n^2} - 1}\right)\right] = \quad ,$$

$$\overline{\psi}_{0\mathrm{T}}\exp\left[-\mathtt{i}\omega t\right]\exp\left[\mathtt{i}\overline{k}_\mathrm{T}x\sin\left[\theta_\mathrm{T}\right]\right]\exp\left[\mp\overline{k}_\mathrm{T}z\sqrt{\frac{\sin^2\left[\theta_\mathrm{I}\right]}{n^2} - 1}\right] \quad .$$

Actually, **Snell laws also applies to complex wave-vectors**, as no reality hypothesis was made in the derivation.

In this problem, as the incoming $\mathbf{k}$ is real, it implies that the imaginary part of that the transmitted wave-vector must be perpendicular to the surface, as second Snell law deals with the projection of the wave-vectors on the plane, which must be real in this problem, as the incoming $\mathbf{k}$ is real.

48.03.06.01 Evanescent Wave

The resulting wave is exponentially damped along the negative z direction, when the plus sign is taken in the last expression, as it is appropriate for physical reasons. Therefore constant amplitude planes are perpendicular to the z axis, while constant phase planes are perpendicular to the x axis. The amplitude of the wave decreases along the negative z direction while the wave propagates along the x direction. The evanescent wave, therefore does not penetrate into the second medium, in fact, due the decreasing amplitude: it is a surface wave.

The attenuation length in the negative z direction, Λ, is of the order of λ_2, the wavelength of the harmonic oscillation in the second medium along x, because:

$$\frac{2\pi}{\lambda_2} = \overline{k}_\mathrm{T}\sin\left[\theta_\mathrm{T}\right] \qquad \Lambda^{-1} \equiv \overline{k}_\mathrm{T}\sqrt{\frac{\sin^2\left[\theta_\mathrm{I}\right]}{n^2} - 1} \quad .$$

The application of Fresnel relations shows that $\left|r_{\parallel}\right| = 1$ and $\left|r_{\perp}\right| = 1$.

However, $\left|t_{\parallel}\right| \neq 0$ and $\left|t_{\perp}\right| \neq 0$. The energy flux in the second medium must be calculated from scratch from the Poynting vector, as there is no MPW.

In fact, it can be shown that while the ElectroMagnetic fields in the second medium is not zero, **there is no average energy flow in the second medium** as the time-average of the z component of the Poynting vector is zero.

In short the appearance of an imaginary component perpendicular to the interface can be understood as follows. When passing into the second medium $\overline{k}\cdot\overline{k}$ must decrease while its component parallel to the interface must be conserved. Therefore, at a certain point while increasing the incidence angle, the z component must decrease to zero; after that an imaginary component along z must appear, to allow the $\overline{k}\cdot\overline{k}$ to become as short as required.

DRAFT - IN PROGRESS

48.03.06.02 Some More Technical Details

The value of the squares of $\mathbf{k}_{\mathtt{I}}$ and $\mathbf{k}_{\mathtt{T}}$ are given by the relations:

$$\mathbf{k}_{\mathtt{I}}\cdot\mathbf{k}_{\mathtt{I}} = \frac{\omega^2}{v_1^2} \quad , \qquad \mathbf{k}_{\mathtt{I}} \quad \text{is real, by hypothesis} \quad ,$$

$$\overline{\mathbf{k}_{\mathtt{T}}}\cdot\overline{\mathbf{k}_{\mathtt{T}}} = \frac{\omega^2}{v_2^2} \quad , \qquad \overline{\mathbf{k}_{\mathtt{T}}} \quad \text{may be complex, a priori} \quad .$$

By hypothesis as the velocities are real and $\mathbf{k}_{\mathtt{I}}\cdot\mathbf{k}_{\mathtt{I}}$ is real as well, it follows that $\overline{\mathbf{k}_{\mathtt{T}}}\cdot\overline{\mathbf{k}_{\mathtt{T}}}$ must be real. However $\overline{\mathbf{k}_{\mathtt{T}}}\cdot\overline{\mathbf{k}_{\mathtt{T}}}$ can be real also, for instance, in case $\overline{k}_{\mathtt{Tz}}$ is pure imaginary. This allows the quantity $\overline{\mathbf{k}_{\mathtt{T}}}\cdot\overline{\mathbf{k}_{\mathtt{T}}}$ to decrease with respect to $\left(\overline{k}_{\mathtt{Tx}}\right)^2$ itself in order to satisfy:

$$\frac{\left(\overline{k}_{\mathtt{Tx}}\right)^2 + \left(\overline{k}_{\mathtt{Tz}}\right)^2}{\left(k_{\mathtt{Ix}}\right)^2 + \left(k_{\mathtt{Iz}}\right)^2} = \frac{v_1^2}{v_2^2} \quad .$$

Snell law implies, among other things:

$$\overline{k}_{\mathtt{Tx}} = k_{\mathtt{Ix}} \quad .$$

It follows:

$$\left(\overline{k}_{\mathtt{Tz}}\right)^2 = \left(\overline{k}_{\mathtt{Ix}}\right)^2 \left(\frac{v_1^2}{v_2^2} - 1\right) + \left(\overline{k}_{\mathtt{Iz}}\right)^2 \left(\frac{v_1^2}{v_2^2}\right) = \quad ,$$

$$\left(\overline{k}_{\mathtt{Tz}}\right)^2 = \left(n^2 - 1\right)\left(\overline{k}_{\mathtt{Ix}}\right)^2 + n^2\left(\overline{k}_{\mathtt{Iz}}\right)^2 \quad .$$

48.03.06.03 Frustrated Total Internal Reflection

This § is referenced at pages:
[2211]

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Adding a third layer of material parallel, in and past medium 2, close to the first boundary, the boundary conditions dramatically change, and the ElectroMagnetic wave is revived in the third medium. In this case, some energy is transmitted.

As for the example § 48.06.07, this example very nicely shows the effects on ElectroMagnetic waves reflection|transmission when changing boundary conditions: it is, in fact, the standard total internal reflection stuff modified by addition of another parallel boundary which causes a qualitative change of behavior.

48.03.07
Non Monochromatic Radiation Spectral Analysis and Dispersion of Light

Fresnel formulae were deduced for monochromatic radiation. However, there is no change in the frequency at the interface and, moreover, the frequency does not appear in the Fresnel formulae. Therefore they can be applied to non-monochromatic radiation without any change by considering the different frequencies.

Note that Snell Transmission|Refraction law actually depends on the frequency, which gives rise to the phenomenon of **light dispersion** .

DRAFT PREPRINT

48.04 Reflection and Transmission|Refraction of Longitudinal Mechanical Waves

This § is referenced at pages:
[2184, 2209]

The case of purely longitudinal mechanical waves along the z axis (that is one-dimensional waves) will be considered in this section. Read also § 45.

Let the volume mass density be ρ, the Young modulus be Y and the cross-sectional area of the system be S. Let the displacement along the z direction be $\xi[z,t]$.

Consider a MPW with wave-vector module $k>0$, angular frequency ω and amplitude ξ_0:

$$\xi_{\pm}[z,t]=\xi_0\cos\left[\pm kz-\omega t\right] \qquad \frac{\partial\xi_{\pm}}{\partial z}=\mp k\xi_0\sin\left[\pm kz-\omega t\right] \qquad \omega=kv \qquad v^2=\frac{Y}{\rho} \quad .$$

The intensity of the reflected and refracted (transmitted) waves at $z=0$ is calculated from the boundary conditions. The relevant boundary conditions for longitudinal mechanical waves are the continuity of the displacement and of the partial derivative with respect to z multiplied by the Young modulus, as the latter is the force per unit area.

$$\xi_{\mathrm{I}}[0,t]+\xi_{\mathrm{R}}[0,t]=\xi_{\mathrm{T}}[0,t]\implies \xi_{\mathrm{I}0}+\xi_{\mathrm{R}0}=\xi_{\mathrm{T}0} \quad ,$$

$$-Y_1\frac{\partial\xi_{\mathrm{I}}}{\partial z}[0,t]-Y_1\frac{\partial\xi_{\mathrm{R}}}{\partial z}[0,t]=-Y_2\frac{\partial\xi_{\mathrm{T}}}{\partial z}[0,t]\implies k_1Y_1\xi_{\mathrm{I}0}-k_1Y_1\xi_{\mathrm{R}0}=k_2Y_2\xi_{\mathrm{T}0} \quad .$$

The second boundary condition is the condition that the force per unit area must be a continuous function across the boundary. It can be derived by considering a small cylinder around the interface between the two media, writing the first cardinal equation for the enclosed piece of matter and letting the cylinder become infinitesimally small:

$$-Y_1\frac{\partial\xi_1}{\partial z}[z=-\epsilon/2,t]+Y_2\frac{\partial\xi_2}{\partial z}[z=+\epsilon/2,t]=S\epsilon\rho\frac{\partial^2\xi[\widetilde{z},t]}{\partial t^2}\longrightarrow -Y_1\frac{\partial\xi_1}{\partial z}[0,t]=-Y_2\frac{\partial\xi_2}{\partial z}[0,t] \quad .$$

Read § 48.06.12.

It follows:

$$r\equiv\frac{\xi_{\mathrm{R}0}}{\xi_{\mathrm{I}0}}=\frac{k_1Y_1-k_2Y_2}{k_1Y_1+k_2Y_2} \quad ,$$

$$t\equiv\frac{\xi_{\mathrm{T}0}}{\xi_{\mathrm{I}0}}=\frac{2k_1Y_1}{k_1Y_1+k_2Y_2} \quad .$$

Moreover, the following relations apply:

$$v^2=\frac{Y}{\rho} \qquad k=\frac{\omega}{v}=\omega\sqrt{\frac{\rho}{Y}} \qquad kY=\omega\sqrt{\rho Y} \qquad Z\equiv S\sqrt{\rho Y} \quad . \tag{48.04.01}$$

Here two pairs of parameters can be used to describe the wave:

$$\{\rho,Y\}\leftrightarrow\{v,Z\} \quad , \tag{48.04.02}$$

→ 2189

as for ElectroMagnetic waves, see equation (48.03.02).

Note that the frequencies are not changed, as this is a kinematic boundary condition (read § 48.02).

Therefore:

$$r\equiv\frac{\xi_{\mathrm{R}0}}{\xi_{\mathrm{I}0}}=\frac{\sqrt{\rho_1Y_1}-\sqrt{\rho_2Y_2}}{\sqrt{\rho_1Y_1}+\sqrt{\rho_2Y_2}}=\frac{Z_1-Z_2}{Z_1+Z_2} \qquad t\equiv\frac{\xi_{\mathrm{T}0}}{\xi_{\mathrm{I}0}}=\frac{2\sqrt{\rho_1Y_1}}{\sqrt{\rho_1Y_1}+\sqrt{\rho_2Y_2}}=\frac{2Z_1}{Z_1+Z_2}\geq 0 \quad . \tag{48.04.03}$$

→ 2208

The points at the interface, made oscillating by the incident waves, generate a wave going in both directions: the reflected and the transmitted waves. The resulting wave in the first medium is the superposition of the incident and the reflected waves, as in sections § 47.03.02 and § 47.03.01.

WORK IN PROGRESS

The transmitted wave is always in-phase with the incident wave.

The reflected wave is either in-phase or out-of-phase by π according to the relative magnitude of Z_1 and Z_2.

48.04.01
Reflected|Transmitted Intensities

This § is referenced at pages:
[2195]

Note that the amplitude transmission coefficient is different from the one for ElectroMagnetic waves at normal incidence, (read § 48.03.04.02).

The reflected and transmitted intensities are, exactly as for the ElectroMagnetic waves at normal incidence:

$$\frac{I_{\mathrm{R}}}{I_{\mathrm{I}}} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 \quad ,$$

$$\frac{I_{\mathrm{T}}}{I_{\mathrm{I}}} = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2} \quad .$$

48.04.02
Impedance Matching/Mismatch

As a general result, from formulas (48.04.03):

- if the impedances are very different there is large reflection (impedance mismatch);
- if the impedances are very similar there is large transmission (impedance matching).

Consider now some limiting cases.

If $(Z_2/Z_1) \to \infty$ (infinitely hard second medium) the transmitted wave is zero while the reflected wave has the same amplitude of the incident wave but opposite sign (it is out-of-phase by π). In fact, the incident and reflected wave must cancel each other, at the interface, to get zero transmitted wave, by continuity. Moreover, the wave generated in the second medium must cancel the incident wave.

If $(Z_1/Z_2) \to \infty$ (infinitely soft second medium) the transmitted wave is twice the incident wave while the reflected wave has the same amplitude of the incident wave and the same sign (it is in-phase).

If $Z_1 = Z_2$ the reflected wave is zero and all the energy passes into the second medium. In this case the impedances of the two media are said to be matched.

If $Z_2 > Z_1$ the reflected wave is out-of-phase by π with respect to the incident wave to partly cancel, in the boundary condition, the incident wave as the transmitted wave has a smaller amplitude than the incident wave in this case.

If $Z_1 > Z_2$ the reflected wave is in-phase with respect to the incident wave to add to the incident wave, in the boundary condition, as the transmitted wave has a larger amplitude than the incident wave in this case.

Note that in this problem it is not necessary to add a phase-shift between the incident wave and the Reflected|Transmitted waves because the parameters, in particular Z, are real numbers. Therefore it turned out that we are able to solve the problem without an additional phase-shift between the incident wave and the Reflected|Transmitted wave and the only possible phase-shifts are either $\Delta\phi = 0$ or $\Delta\phi = \pm\pi$ which can be simply described as a change of sign.

Compare the similar discussion in sections § 47.03.02 and § 47.03.01.

DRAFT-IN-PROGRESS

48.05 Reflection and Transmission|Refraction of Transverse Mechanical Waves

The case of purely transverse mechanical waves along the z axis (that is two-dimensional waves) will be considered in this section.

Let the linear mass density be λ, the string tension be T and the cross-section of the system having area cross-sectional area S. Let the displacement along the $\{x, y\}$ direction be $\xi[x, y, t]$.

The wave characteristics are the following ones:

$$v^2 = \frac{T}{\lambda} \qquad k = \frac{\omega}{v} = \omega\sqrt{\frac{\lambda}{T}} \qquad kT = \omega\sqrt{\lambda T} \qquad Z \equiv \sqrt{\lambda T} \quad . \tag{48.05.01}$$

The properties of reflection and Transmission are given by the same results as for longitudinal waves, section § 48.04.

DRAFT-IN-PROGRESS

48.06 Examples and Physical Applications

48.06.01
Light Guides

Built from, for instance, flexible glass fibres: optical fibres. Light trapped inside the light guide can run a long way with, ideally, no losses.

Applications: endoscopy, data-transmission.

48.06.02
Air-Glass Interface

Interface air-glass. **index of refraction** of glass $n \simeq 1.5$. **index of refraction** of air $n \simeq 1.0$.

The amplitude reflection coefficient at normal incidence is about 0.04. The tiny reflection from glass is the responsible of the reflected image which can be seen under low illumination when looking across a window.

The total internal reflection angle from glass to air is $\simeq 42\,^{\circ}$.

The Brewster angle is $\simeq 57\,^{\circ}$.

48.06.03
Dispersion of Light

Many different phenomena, such as rainbows, show that Transmission|Refraction of light depends on the light color, that is wavelength. Therefore one is compelled to conclude that the **index of refraction** depends on the wavelength. Read § 54.

48.06.04
Light propagation in atmosphere

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Atmospheric Transmission|Refraction is responsible for the bending of light-rays in the atmosphere. It is a result of the continuous decrease in the **index of refraction** of the air as a function of altitude. A well-known consequence of this phenomenon is the apparently elliptic shape of the setting or rising Sun (or full-Moon).

DRAFT - IN-PROGRESS

48.06.05
Sun at Sunset

Demonstrate, using Snell laws, that the image of Sun at sunset is shorter, in the vertical direction, than in the horizontal direction so that the image is not circular.

Use Transmission|Refraction into a medium with larger **index of refraction** to show:

$$\Delta\theta_2 \simeq \frac{n_1}{n_2}\frac{\cos\theta_1}{\cos\theta_2}\Delta\theta_1 \quad ,$$

$$n_1 < n_2 \implies \theta_1 > \theta_2 \implies \cos\theta_1 < \cos\theta_2 \quad ,$$

$$\Delta\theta_2 < \Delta\theta_1 \quad .$$

48.06.06
Mirage

A mirage is a naturally occurring optical phenomenon in which light rays are bent to produce a displaced image of distant objects or the sky.

Inferior mirage: hot air near the Earth has a lower **index of refraction** than cooler air above. The image of an object appears to come from below the Earth thanks to the curvature of the rays (a convex function).

Superior mirage: the other way round, caused by temperature inversion (including the so-called Fata Morgana).

Mirages also happen for sound waves.

48.06.07
Multiple Reflections Between Layers of Dielectrics (Anti-Reflection Coatings)

This § is referenced at pages:
[2206, 2216, 2216]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|||

Consider three LHI materials separated by three plane surfaces at $z = 0$ and $z = L$. A MPW, linearly wave-Polarized, is traveling in the positive direction of the z axis inside medium 1, and reaches medium 2 at $z = 0$. It is partially reflected and the transmitted part travels, inside medium 2, until it reaches the interface with medium 3 at $z = L$, where it is partially reflected and transmitted.

This example very nicely shows the effects on ElectroMagnetic waves reflection|transmission when changing boundary conditions: it is, in fact, the standard reflection|transmission stuff giving Fresnel equations (48.03.03), (48.03.07), modified by addition of another parallel boundary. Under this respect, it is similar to frustrated internal reflection in (48.03.06.03).

Let us use the convention that all the electric fields are conventionally oriented as the direction of the incoming electric field. Then the conventional orientation for the magnetic field of waves traveling in the positive direction is opposite to that of the magnetic field of waves traveling in the negative direction.

The Electric|Magnetic fields of the MPW are:

$$\overline{\mathbf{E}} = \overline{\mathbf{E}}_0 \exp\left[\mathtt{i}\left(\pm kz - \omega t\right)\right] \qquad \overline{\mathbf{H}} = \overline{\mathbf{H}}_0 \exp\left[\mathtt{i}\left(\pm kz - \omega t\right)\right] \qquad \text{with } \overline{\mathbf{E}} = \overline{Z}\overline{\mathbf{H}} \quad ,$$

where any phase is included in the complex constant vector.

The waves are defined from $z = -\infty$ to $z = +\infty$, but they are made of different pieces to be joined at the two interfaces.

The common $\exp\left[-\mathtt{i}\omega t\right]$ factor can be removed.

The explicit use of $\overline{\cdots}$ to denote complex quantities will be avoided here, because all quantities are considered a priori as complex quantities.

DRAFT LINE PROGRESS

The continuity equations at the interfaces for the electric field and for **H**, are:

$$E_{\mathrm{I}} + E_{\mathrm{R}} = {E_2}^{+} + {E_2}^{-} \quad ,$$
$$\frac{E_{\mathrm{I}}}{Z_1} - \frac{E_{\mathrm{R}}}{Z_1} = \frac{{E_2}^{+}}{Z_2} - \frac{{E_2}^{-}}{Z_2} \quad ,$$
$${E_2}^{+} \exp\left[+\mathtt{i} k_2 L\right] + {E_2}^{-} \exp\left[-\mathtt{i} k_2 L\right] = E_{\mathrm{T}} \exp\left[+\mathtt{i} k_3 L\right] \quad ,$$
$$\frac{{E_2}^{+}}{Z_2} \exp\left[+\mathtt{i} k_2 L\right] - \frac{{E_2}^{-}}{Z_2} \exp\left[-\mathtt{i} k_2 L\right] = \frac{E_{\mathrm{T}}}{Z_3} \exp\left[+\mathtt{i} k_3 L\right] \quad .$$

Note that the phase of $E_{\mathrm{T}} \exp\left[+\mathtt{i} k_3 L\right]$ might be just ignored by re-absorbing it into the definition of E_{T}.

The solution of the system of four equations in four unknown variables, E_{R}, E_2^+, E_2^- and E_{T}, gives the **complex amplitude reflection coefficient**:

$$r = \frac{E_{\mathrm{R}}}{E_{\mathrm{I}}} = \left(\frac{Z_2 \left(Z_3 - Z_1\right) \cos\left[k_2 L\right] - \mathtt{i} \left(Z_2^2 - Z_1 Z_3\right) \sin\left[k_2 L\right]}{Z_2 \left(Z_3 + Z_1\right) \cos\left[k_2 L\right] - \mathtt{i} \left(Z_2^2 + Z_1 Z_3\right) \sin\left[k_2 L\right]} \right) \quad .$$

This result shows that **the Reflection|Transmission coefficients can be complex numbers**, even for real wave impedance.

It can be verified that for either $L = 0$ or $Z_2 = Z_3$ or $Z_1 = Z_2$ one obtains the same results as the standard Fresnel formulas for two media.

The reflectivity is given by:

$$\mathcal{R} = \left| \frac{E_{\mathrm{R}}}{E_{\mathrm{I}}} \right|^2 = \left(\frac{Z_2^2 \left(Z_3 - Z_1\right)^2 \cos^2\left[k_2 L\right] + \left(Z_2^2 - Z_1 Z_3\right)^2 \sin^2\left[k_2 L\right]}{Z_2^2 \left(Z_3 + Z_1\right)^2 \cos^2\left[k_2 L\right] + \left(Z_2^2 + Z_1 Z_3\right)^2 \sin^2\left[k_2 L\right]} \right) \quad .$$

It can be shown that the **reflectivity has extrema for**:

$$k_2 L = m\pi/2 \implies \frac{2L}{\lambda_2} = m/2 \implies \lambda_2 = \frac{2L}{m/2} \quad , \tag{48.06.01}$$

but this is obvious, as it corresponds to an half-integer number of wavelengths inside the path of length $2L$, that is corresponds to a travel forward and backward inside medium 2. Under these conditions, a stationary wave is formed in medium 2.

In the case $Z_1 < Z_2 < Z_3$:

- the points at $k_2 L = m\pi$ are maxima;
- the points at $k_2 L = \pi/2 + m\pi$ are minima.

In the case of minima, the reflectivity is identically zero for $Z_2 = \sqrt{Z_1 Z_3}$. The reflectivity is thus zero (as the numerator of r is clearly zero) for:

$$k_2 L = (2m+1)\,\frac{\pi}{2} \quad \text{and} \quad \left\{ Z_2 = \sqrt{Z_1 Z_3} \quad \Leftrightarrow \quad \left\{ n_2 = \sqrt{n_1 n_3} \quad \text{for non-magnetic media } (\mu_{\mathrm{R}} = 1) \right\} \right\} \quad .$$

The condition on the thickness of the layer correspond, for the minimum thickness $m = 0$, to a **thickness of one quarter of wavelength**.

The case when reflection is zero is the case of **optical impedance matching** between media 1 and 3 by using the adapter medium 2. In practical cases it might be impossible to obtain exact matching, that is zero reflection, but only a reduced reflection. Moreover, the condition depends on the frequency.

The reflected wave is canceled if and only if the wave reflected from the first surface combines to exactly cancel the wave first refracted/transmitted by the first surface and then reflected by the second surface and finally refracted/transmitted again by the first surface to go back in the first medium. Therefore the exact optical impedance matching is the case of destructive interference, in fact (read § 50).

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Examples and Applications

- Multi-layers dielectric coatings.
- Multi-layers dielectric mirrors.

48.06.08
Normal and Anomalous Dispersion

48.06.09
Case of Magnetic Media

©|Am.J.Phys, , ..., ..., ...Ed., ... 63, 471 (1995)|||

©|Am.J.Phys, , ..., ..., ...Ed., ... 56, 555 (1988)|||

©|Am.J.Phys, , ..., ..., ...Ed., ... 55, 277 (1987)|||

©|Am.J.Phys, , ..., ..., ...Ed., ... 54, 87 (1986)|||

©|Am.J.Phys, , ..., ..., ...Ed., ... 53, 916 (1985)|||

©|Am.J.Phys, , ..., ..., ...Ed., ... 52, 660 (1984)|||

48.06.10
The Anti-Echo Condition - Stealth objects

This § is referenced at pages:
[2196]

©|A.Moliton|*Basic Electromagnetism and Materials, § 11.5 |Reflection from a Magnetic Layer; a Study of an Antiradar ...*

48.06.11
Snell laws for a Moving interface

Study Snell laws in case the interface is moving at constant velocity.

48.06.12
Continuity for mechanical waves

This § is referenced at pages:
[2207]

Demonstrate that the force per unit area must be a continuous function. Consider, in fact, a small cylinder across the interface: it is subject to the two forces by the neighboring pieces on the two sides, giving the first **Cardinal Equation** and its limit for $\Delta z \implies 0$.

Note that $F_{\oplus}[z+\Delta z] = -F_{\ominus}[z+\Delta z]$ is always valid as it is the **Action-Reaction Principle** .

DRAFT - IN PROGRESS

48.07 Exercises Problems and Physical Applications

48-003 ✓ **I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...**

5.166, 5.167, 5.168, 5.169, 5.170, 5.171, 5.172, 5.173, 5.174, 5.175.

48-004 ✓ **A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...**

8.08, 8.09, 8.31.

48-005 ✓ **index of refraction and density**

Explain, qualitatively, why the index of refraction increases with density of the material.

48-006 Radio-Telescopes and Optical Telescopes

The surfaces of a radio-telescope, operating in radio-waves with wavelengths of the order of meters, do not need the geometrical precision of optical telescopes, operating with wavelengths of the order of hundreds of nanometers.

48-007 Typical Glasses

Take a typical glass with $n = 1.5$. At normal incidence:

$$\mathcal{R}_{\parallel} = \mathcal{R}_{\perp} \simeq 0.04 \quad .$$

Note: typical glasses show dispersion, that is dependence of the phase velocity (and therefore **index of refraction**) on the frequency of the monochromatic PW.

Double image from the train window: double reflection, the second one with reduced intensity; better seen when outside it is dark.

48-008 ✓ **Maximum Exit Angle from a cylindrical fiber**

Consider a cylindrical optical fiber with circular cross-section and index of refraction n.
Determine the maximum exit angle from one base.

48-009 ✓ **Maximum Entrance Angle into a cylindrical fiber**

Consider a cylindrical optical fiber with circular cross-section and index of refraction n.
Determine the maximum entrance angle into one base.

48-010 ✓ **Reflectivity and transmissivity in TIR for ElectroMagnetic Waves**

Verify that in presence of total internal reflection the reflectivity is one and and the transmissivity is zero using Fresnel equations for ElectroMagnetic waves.

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48-011 ✓ Energy Flux in TIR for ElectroMagnetic Waves

Verify that in presence of total internal reflection the average value of the Poynting vector in the second medium is zero using the results in section § 48.03.06.

48-012 ✓ Reflection of ElectroMagnetic Waves and ElectroMagnetic Fields in the First Medium

Study the total ElectroMagnetic field in the first medium, by superposing the incident and reflected ElectroMagnetic waves. In particular show that a standing wave is present in addition to a traveling wave and that the total field corresponds to a non homogeneous plane ElectroMagnetic wave.

48-013 Reflected and Refracted Flux

Consider ElectroMagnetic waves at normal incidence at the plane interface between two LHI media. Determine if a condition exists on the **index of refraction** es of the two materials such that the reflected energy is larger that the transmitted energy.

48-014 ✓ Reflection of Circularly wave-Polarized Waves

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|8.27||

Determine how the handedness of a circularly wave-Polarized waves changes at normal reflection with two non-magnetic media with $\chi_{\mathrm{M}}^{*} = 1$ in the two cases: $n_2 > n_1$ and $n_1 > n_2$.

48-015 ✓ Curvature and trajectory of optical rays in Atmosphere

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Study the problem of Transmission|Refraction of light rays in the atmosphere.

Assume first that the Earth is flat and let z be the height above the Earth. Consider a light ray going up the Earth atmosphere, starting at an height z_0 where the light ray has a incidence angle, that is the angle with respect to the normal to the Earth, θ_0. Let the atmosphere **index of refraction** be varying as a function of z, $n[z]$, such that: $n[z = z_0] = n_0$ and $n[z = z_1] = n_1$. Consider a large but finite number of layers, M, of thickness $\Delta z \equiv (z_1 - z_0)/M$ and calculate the light reflected and transmitted at every layer. Finally, let the number of layers go to infinity and find the continuous limit.

Use now the results derived above to describe in a qualitative way for a spherical Earth the fact that astronomical bodies can be seen even when they are slightly below the horizon and, in general, the fact that the real position of astronomical bodies is slightly different from their apparent position.

PREPRINT-IN-PROGRESS

48-016 ✓ Shift of Light Rays on Passing a Plane Transparent Lamina

Light rays with incidence angle θ_0 on a transparent lamina of thickness D and **index of refraction** n pass across the lamina. Calculate the shift of the rays exiting the lamina with respect to the incident rays and its dependence on the thickness of the lamina and on the refractive index.

48-017 Snell Laws From Huygens-Fresnel Principle

Derive Snell laws from Huygens-Fresnel principle.

48-018 ✓ Snell Laws From Fermat Principle

Derive Snell laws from Fermat principle.

48-019 ✓ Anti-Reflection Coatings

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|P-5.21||

Read also § 48.06.07.

48-020 Magnesium-Fluoride Anti-Reflection Coating

Magnesium-Difluoride (MgF_2) has $n_2 = 1.38$. Determine the amplitude reflection coefficient for a quarter-wave coating on glass ($n_3 = 1.5$).

Read § 48.06.07.

48-021 Maximum Deviation Prism

A prism is made of a material of **index of refraction** n and has the shape of a cylinder with a isosceles triangle as a basis. The apex angle, that is the angle opposite to the base of the isosceles triangle, is α.

Consider a light ray parallel the bases of the prism with incidence angle θ_0 on one of the two lateral faces corresponding to the two equal sides of the isosceles triangle.

Determine the angular deviation of the ray exiting from the opposite lateral face, that is the angle between the exiting and the entering ray.

Determine the incidence angle giving the minimum deviation angle.

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48-022 ✓ Energy Balance at Reflection|Transmission for ElectroMagnetic Waves

©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|||

Consider EM waves and the plane interface separating two LHI media. Show explicitly that the energy balance presented in section § 48.03.05 is consistent, at arbitrary incidence angles, with Fresnel relations.

48-023 ✓ Brewster Angle

Consider a flat uniform lamina of thickness d and **index of refraction** n_2 inside a medium of **index of refraction** n_1. Assume that light is directed towards the lamina at the Brewster angle relative to the passage from medium 1 to medium 2. Show that the light transmitted inside the lamina and reaching the second face of the lamina will be at the Brewster angle relative to the passage from medium 2 to medium 1.

48-024 ✓ Cherenkov Light and TIR

A plane transparent lamina with parallel faces is made of a material with **index of refraction** n. Cherenkov light is emitted inside the lamina by a particle crossing the lamina perpendicularly at a speed close to the speed of light in vacuum.

Determine the minimum value of n such that all Cherenkov light suffers TIR instead of exiting the lamina.

DRAFT - IN-PROGRESS

CHAPTER 49

Radiation of ElectroMagnetic Waves

49.01 ElectroMagnetic Potentials for Time-Dependent Fields 2220
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This § is referenced at pages:
[2134, 2268, 2268, 2501, 2502]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.28; 1.34; 2.20; 2.21||
©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|4|*Good text.*|
©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|||
©|T.Padmanabhan, *Theoretical Astrophysics, Vol. 1,2,3*, 2000/2001/2002, CUP, ...Ed., ...|Vol 1 § 4|*Excellent*|
©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|10||
©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|16||

After discussing ElectroMagnetic waves regardless of their sources, that is solutions to the homogeneous PDE, in this section generators of ElectroMagnetic waves will be introduced, that is, solutions to the non-homogeneous PDE.

In this chapter, the word radiation will typically mean radiation with energy transport to infinity.
Radiation (to infinity) == Energy transported irreversibly out to infinity.
ElectroMagnetic radiation is produced by accelerated charges, with many examples/applications such as:

- the production of x-rays;
- the acceleration of charged particles to relativistic velocities in particle-accelerators;
- the radiation from antennas;
- the reception of ElectroMagnetic waves by antennas;
- heat and light from the Sun;
- light emitted by a bulb-lamp;
- .....

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49.01 ElectroMagnetic Potentials for Time-Dependent Fields

This § is referenced at pages:
[1731, 1735, 1991]

49.01.01 Lorenz Gauge

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©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|2.1.2||
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||

A most useful gauge choice is the **Lorenz Gauge** , defined in equation (36.12.04).
In fact the **Lorenz Gauge** gives rise to two uncoupled non-homogeneous wave equations:

$$\boxed{\nabla^2\Phi - \varepsilon_0\mu_0\frac{\partial^2\Phi}{\partial t^2} = -\rho/\varepsilon_0} \quad , \tag{49.01.01}$$

→ 1734 2220 2222 2224 2254

$$\boxed{\boldsymbol{\nabla}^2\mathbf{A} - \varepsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j}} \quad . \tag{49.01.02}$$

→ 1734 2220 2222 2224 2254

Therefore, one can find solutions for $\Phi/\mathbf{A}$ independently from one another. However, the sources are non independent, as they must be linked by the charge continuity equation (68.05.01), which is in turn is consistent with the Lorenz **gauge fixing condition** (read § 49-014).

Note that, in static conditions, the **Lorenz Gauge** reduces to:

$$\boxed{\mathbf{div}\,\mathbf{A} = 0 \qquad \text{in static conditions}} \quad ,$$

while the wave equations (49.01.01), (49.01.02) reduce to the **Poisson equation** . Read § 49.03 for discussion of the solutions.

49.01.02 Coulomb Gauge

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Another **useful gauge is the Coulomb gauge, or radiation gauge or transverse gauge**. It is defined as:

$$\boxed{\mathbf{div}\,\mathbf{A} = 0} \quad .$$

This is especially useful in regions without charges and currents ($\rho[\mathbf{x},t] = 0$ and $\mathbf{j}[\mathbf{x},t] = 0$), hence the name: radiation gauge. In fact in this case, the two equations (36.12.09), (36.12.09) become:

$$\boxed{\nabla^2\Phi = 0} \quad ,$$

$$\boxed{\boldsymbol{\nabla}^2\mathbf{A} - \varepsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} = 0} \quad .$$

The **Laplace equation** for the **scalar electric potential** has the particular solution, $\Phi[\mathbf{x},t] = 0$, as $\rho[\mathbf{x},t] = 0$. The wave equation for the **vector magnetic potential** is just an homogeneous wave equation. Both Electric|Magnetic fields can be calculated from the **vector magnetic potential** only:

$$\boxed{\mathbf{E} = -\partial_t\mathbf{A}} \quad ,$$

$$\boxed{\mathbf{B} = +\mathbf{rot}\,\mathbf{A}} \quad .$$

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It can be shown that in the transverse gauge there is no instantaneous propagation of signals, even if the equation for the **scalar electric potential** is a static one, that is based on the Laplace, time independent, operator.

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49.02 Heuristic Introduction to ElectroMagnetic Radiation

This § is referenced at pages:
[2224]

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©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|14, 19||

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|11||

In this chapter, except when noted otherwise, the prescribed generators, charge|current densities exist in otherwise empty space, that is, that no material medium is present. Moreover, it will be always assumed that the generators of ElectroMagnetic fields are localized near the origin of some suitable Coordinate System.

Maxwell Equations in vacuum will be always used, to emphasize their fundamental meaning. Matter will be treated as a set of particles with some specific properties (mass, spin, charge, **Electric Dipole** | **Magnetic Dipole** , ...), the way it is. The macroscopic description of ElectroMagnetic fields in matter will never be used in this section.

When the full set of **Maxwell Equation** are solved one finds that, in the non-static case, solutions exist capable to carry energy to infinity, as they are made of ElectroMagnetic fields decreasing, at large distance from the sources, as r^{-1}, as opposed to static fields, decreasing, at large distance from the sources, as r^{-2}. ElectroMagnetic fields decreasing, at large distance from the sources, as r^{-1} are called **radiation fields**. The flux of the Poynting vector at large distances, in presence of radiation fields, does not go necessarily to zero, leaving room to **energy transport to infinity**. In this case, the **radiated power** is the power crossing a gigantic sphere surrounding the sources when the radius of the sphere goes to infinity.

49.02.01 Physical Origin of Radiation of ElectroMagnetic Fields

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The non-homogeneous wave equations for the **ElectroMagnetic potentials** , equations (49.01.01) and (49.01.02), show that the **vector magnetic potential** is a linear function of the current density, **j**, as the wave equation is a linear equation. We also know that the electric field contains a term which is the time derivative of the **vector magnetic potential** . Therefore one expects that the electric field will contain a term depending on the acceleration of the charges. Similarly, the magnetic field will depend on the acceleration, as it can be shown from Jefimenko equations (49.03.05), (49.03.06).

Dimensional considerations imply that a term depending on the acceleration of the charges must go as r^{-1} at large distances. In fact, assuming that **E** depends on q (linearly, as expected from **Maxwell Equations**), on **a** (linearly, as expected from the above considerations), and on ε_0, c, r, it turns out that the dependence on the distance is proportional to r^{-1}, for a term depending on the acceleration. At large enough distances from the generators of the ElectroMagnetic waves can be approximated by a MPW, so that the magnetic field must have the same r^{-1} dependence on the distance.

The study of ElectroMagnetic radiation involves therefore studying the terms of the electric and magnetic fields going as r^{-1} at large distances. The radiation field is therefore capable to carry ElectroMagnetic energy to infinity.

In order to have **emission of ElectroMagnetic radiation in vacuum**, therefore, the source charges must be accelerated. Important apparent exceptions to the requirement of acceleration of the source charges

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are given by Cherenkov radiation and transition radiation, (read § 46.13.01, § 46.13.02), which, however, involve motion of charges inside some material medium. Actually this point shows how misleading the macroscopic description in terms of averaged Polarization|Magnetization and/or auxiliary vectors **D**|**H** can be. In fact, it hides that Cherenkov radiation and transition radiation are due to time-varying Polarization, producing accelerated Polarization charges. Only apparently, it gives radiation production with non accelerated charges.

The fact that a charge moving at constant speed in vacuum does not emit ElectroMagnetic radiation is consistent with relativity, as there is a Reference Frame where the charge is stationary.

It can be shown that, in case the motion of the particle is periodic, the emitted radiation has a discrete energy spectrum with frequencies multiples of the fundamental frequency [1].

49.02.02
Retarded Time

Maxwell Equations in vacuum give a precise value, c, for the speed of light in vacuum.

Therefore ElectroMagnetic fields move in vacuum at speed c and the time-delay between the event producing the ElectroMagnetic radiation at $\mathbf{r}_1$ at t_1 and the observation of the ElectroMagnetic radiation produced by that event at $\mathbf{r}_2$ at t_2 is:

$$\Delta t \equiv t_2 - t_1 = \frac{|\mathbf{r}_2 - \mathbf{r}_1|}{c} \quad . \tag{49.02.01}$$

This first introduces the concept of retarded time : what the observer sees now is what was emitted exactly a time ago equal to the propagation time from the emission point to the observation point.

The notation $\Big[\ldots\Big]_{\mathrm{R}}$ will denote that the expression must be calculated at the **retarded time** , t_{R}.

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[1] J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...

49.03 General Solution for Spatially-Limited Distribution of the Sources

This § is referenced at pages:
[2220]

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©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|16||
©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|2.1.2||

49.03.01 Retarded Potentials Solution

©|D.Griffiths, M.Heald|Time-dependent generalizations of the Biot-Savart and Coulomb laws|*Am.J.Phys*, , ..., ..., ...*Ed.*, ...

In case of localized Charge|Current, equations (49.01.01), (49.01.02) (the non-homogeneous wave equations for the **ElectroMagnetic potentials** in the **Lorenz Gauge**) have the following solutions (the **retarded potentials solution**):

$$\Phi[\mathbf{x},t]=\frac{1}{4\pi\varepsilon_0}\iiint\frac{\rho[\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}V_{\mathbf{y}}\qquad\mathbf{A}[\mathbf{x},t]=\frac{\mu_0}{4\pi}\iiint\frac{\mathbf{j}[\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}V_{\mathbf{y}}\quad,\tag{49.03.01}$$

→ 2224 2227 2229 2235 2239

the integration being carried throughout the whole space.

The retarded potentials reduce appropriately to equations (36.13.02) and (36.13.03) in the static case, for which charge and currents are independent from time.

The solution equation (49.03.01) can be quickly found from the Green function [1] of the 3D wave operator:

$$G[\mathbf{y},t,\mathbf{x},t_{\mathrm{R}}]=\frac{\delta[t_{\mathrm{R}}-t+|\mathbf{x}-\mathbf{y}|/c]}{|\mathbf{x}-\mathbf{y}|}\quad.$$

It can be verified, by a direct and cumbersome calculations, that equations (49.03.01) are the solutions of the inhomogeneous wave equation (49.01.01), (49.01.02), by substituting into equation (49.01.01), (49.01.01). One has to remember that the **retarded time** brings into a dependence on the position via the **retarded time** .

Equations (49.03.01) have a very simple physical interpretation as they just show that **one may regard the potentials as arising from infinitesimal contributions from each infinitesimal volume element of space, each one time-retarded** and contributing the amounts:

$$\frac{\rho[\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}\qquad\frac{\mathbf{j}[\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}\quad.\tag{49.03.02}$$

The **retarded potentials solution** in equation (49.03.01) represent a **particular solution to the non-homogeneous wave equations** (49.01.01), (49.01.02). Any solution of the homogeneous part of equations (49.01.01), (49.01.02), that is the wave equation, can be added to the **retarded potentials solution** , in order to satisfy the appropriate boundary conditions. The general solution to the equation is obtained by adding to the particular solution, given by the **retarded potentials solution** , the general solution of the homogeneous equation, in the **Lorenz Gauge** .

The **retarded potentials solution** in equation (49.03.01) can be justified by the heuristic argument given in section § 49.02, assuming the speed of propagation is c. However, this arguments must be taken with care. In fact if one were to apply the same argument (the **retardation**) to the electric or magnetic field one would find a totally wrong result. In fact the solution for the ElectroMagnetic fields is provided by Jefimenko equations (49.03.05), (49.03.06).

[1] TO DO ADD REFERENCE

DRAFT - In Progress

© - QUOTE

D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...
...
Well, that all sounds reasonable - and surprisingly simple. But are we sure it's right? I didn't actually derive these formulas for Φ and $\mathbf{A}$; all I did was invoke a heuristic argument ("electromagnetic news travels at the speed of light") to make them seem plausible.
...
In case you think I'm being fussy, let me warn you that if you apply the same argument to the fields you'll get entirely the wrong answer.
...

In fact, d'Alembert equations for the potentials are the same d'Alembert equations as for the fields, so the question arise on why one does not get the retarded solution for the ElectroMagnetic fields? The answer is that, for fields, the d'Alembert equations need to be complemented with additional conditions from Maxwell equations and, in particular, one knows that Electric|Magnetic fields are not really independent. On the other hand, the two equations for potentials are independent, apart from Lorenz gauge condition which is, however, satisfied thanks to charge continuity equation. The two d'Alembert equations for the potentials, in the Lorenz gauge, do not need any additional constraint to be satisfied, while the solutions to the two d'Alembert equations for the fields need additionally to satisfy Maxwell equations. In fact, the wrong time-retarded Coulomb and Biot-Savart laws would imply a total decoupling of Electric|Magnetic fields, which is implicit in the d'Alembert equations for Electric|Magnetic fields, but not consistent with Maxwell equations.

49.03.01.01 Retarded Time and Advanced Time

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The name **retarded potentials solution** (retarded with respect to the timing of Charge|Current), comes from the fact that the effect of the Charge|Current configuration on the ElectroMagnetic fields at position-time $\{\mathbf{x}, t\}$, is due to the configuration of the Charge|Current at an anticipated time, t_{R}, (called the retarded-time), that is, it is *retarded* with respect to what happened at the source, by a *retardation* given by the travel-time of the ElectroMagnetic radiation:

$$\boxed{t_{\mathrm{R}} \equiv t - |\mathbf{x}-\mathbf{y}|/c} \quad . \tag{49.03.03}$$

→ 2235

The difference between the time and the **retarded time** is just the time that EM radiation takes to go from $\mathbf{x}$ to $\mathbf{y}$.

It is important to stress that the **retarded time** has nothing to do with changes of Reference Frame. Times and **retarded times** are always measured in the same Laboratory Frame and therefore no relativistic transformation of times is involved. In fact, in the definition of retarded-time, there is no reference to typical relativistic factors, such as β and γ.

Similarly, it can be shown that there exist also the solution with the **retarded time** replaced by the **advanced time** :

$$\boxed{t_{\mathrm{A}} \equiv t + |\mathbf{x}-\mathbf{y}|/c} \quad .$$

However, even if this is a solution from the mathematical point of view, it has to be discarded on the basis of causality, because it represents the case of radiation being received before being emitted.

ARTEN IN PROGRESS

The Andromeda Galaxy

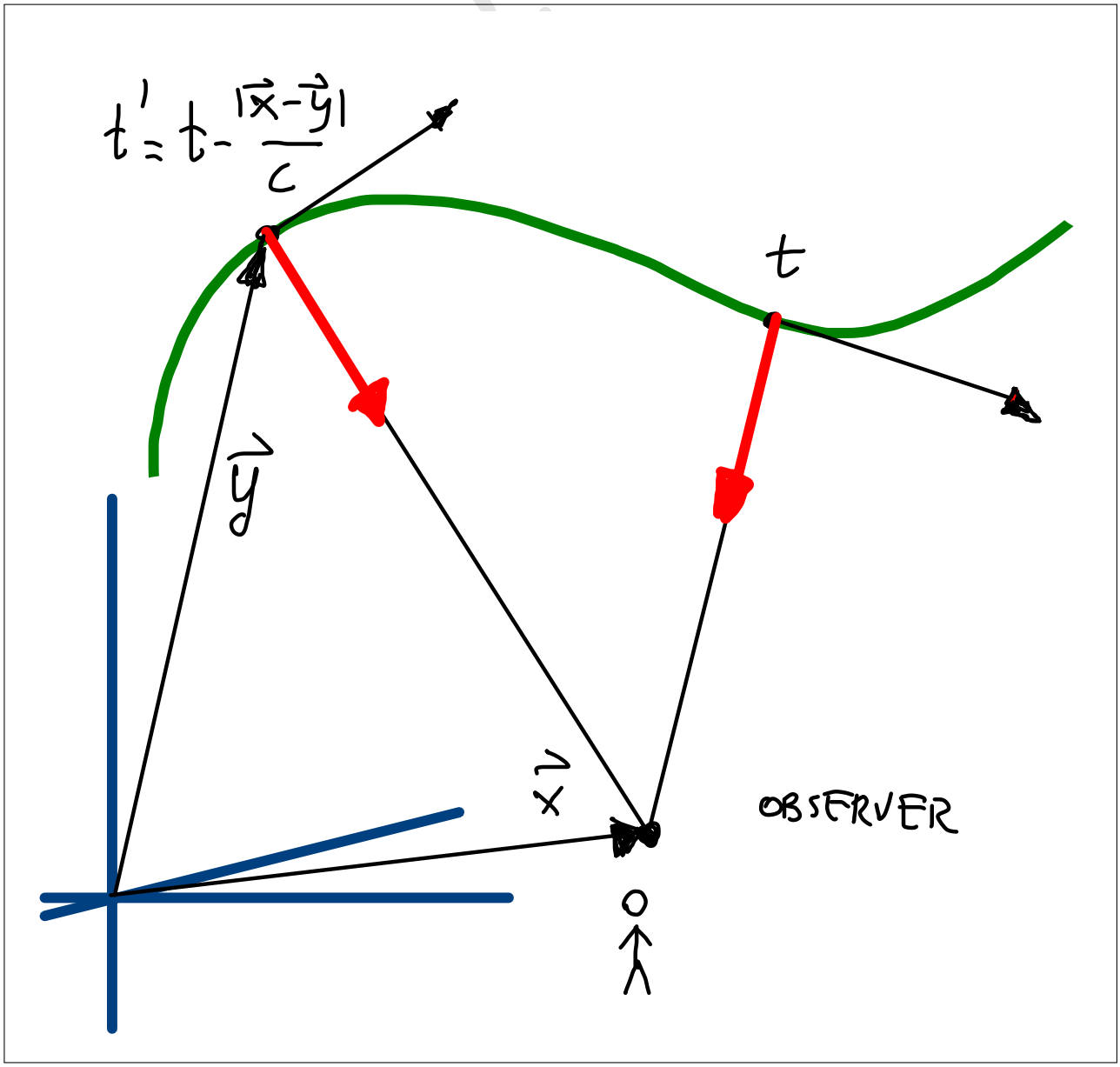
The concept of **retarded time** is well-known in astrophysics; for instance, with respect to The Andromeda Galaxy, which is approximately 2.5 million light-years distant from Earth. We are seeing today how it was 2.5 million years ago, as we receive today the ElectroMagnetic fields generated by charges | currents there.

Similarly, the concept of **advanced time** would be the, clearly non-physical, possibility that we see today the Andromeda Galaxy as it would be in the future, 2.5 million years after now.

DRAFT

49.03.01.02 Information-Collecting Sphere

The **retarded potentials solution** may be visualized as follows. Consider an observer located at $\mathbf{x}$ in space. Let a sphere centered at $\mathbf{x}$ contract with radial speed c such that it has collapsed into $\mathbf{x}$ at the time of observation t. The time t_{R} at which this information-collecting sphere passes by the generators of ElectroMagnetic radiation at $\mathbf{y}$ is then the time at which the source produced the effect which is felt at $\mathbf{x}$ at time t.



49.03.01.03 Quasi-Static Approximation

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The quasi-static approximation involves neglecting the propagation time of the ElectroMagnetic fields in the expressions (49.03.01) of the retarded **ElectroMagnetic potentials** , such that **there is little difference between time and retarded time , when compared to the characteristic time, T, of change of the sources**:

$$|t_{\mathrm{R}} - t| = \frac{|\mathbf{x}-\mathbf{y}|}{c} \ll T \implies \Phi[\mathbf{x},t] \simeq \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho[\mathbf{y},t]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}V_{\mathbf{y}} \qquad \mathbf{A}[\mathbf{x},t] \simeq \frac{\mu_0}{4\pi}\iiint \frac{\mathbf{j}[\mathbf{y},t]}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}V_{\mathbf{y}} \quad . \tag{49.03.04}$$

The condition $|t_{\mathrm{R}} - t| = |\mathbf{x}-\mathbf{y}|/c \ll T$, **in case of harmonic time dependence of the sources, expresses the fact that the distance between the points where the potentials are calculated and the sources is much shorter than the wavelength of the monochromatic components of the ElectroMagnetic fields**:

$$|\mathbf{x}-\mathbf{y}| \ll cT \equiv \lambda \quad .$$

49.03.02 Jefimenko Solution for ElectroMagnetic Fields

This § is referenced at pages:
[Never referenced.]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|10.2.2||

©|D.Griffiths & M.Heald|Time-dependent generalizations of the Biot-Savart and Coulomb laws|*Am.J.Phys, , ..., ..., ...Ed., ...*

©|J.A.Heras||*Am.J.Phys, , ..., ..., ...Ed., ... 62, 525 (1994) and Am.J.Phys, , ..., ..., ...Ed., ... 62, 1109 (1994)*|

©|Tran-Cong Ton|On the time-dependent, generalized Coulomb and Biot-Savart laws|*Am.J.Phys, , ..., ..., ...Ed., ... 59, ...*

Jefimenko equations express the ElectroMagnetic fields in terms of the Charge|Current densities directly. Jefimenko equations are derived by direct calculation from the **retarded potentials solution** :

$$\mathbf{r} \equiv \mathbf{x} - \mathbf{y} \quad ,$$

$$\mathbf{E}[\mathbf{x},t] = \frac{1}{4\pi\varepsilon_0}\iiint \left(\rho[\mathbf{y},t_{\mathrm{R}}]\frac{\mathbf{r}}{r^3} + \frac{\partial\rho}{\partial t}[\mathbf{y},t_{\mathrm{R}}]\frac{\mathbf{r}}{cr^2} - \frac{1}{c^2 r}\frac{\partial\mathbf{j}}{\partial t}[\mathbf{y},t_{\mathrm{R}}]\right)\mathrm{d}V_{\mathbf{y}} \quad , \tag{49.03.05}$$

→ 2222 2224 2227

$$\mathbf{B}[\mathbf{x},t] = \frac{\mu_0}{4\pi}\iiint \left(\left(\frac{1}{r^2}\mathbf{j}[\mathbf{y},t_{\mathrm{R}}] + \frac{1}{cr}\frac{\partial\mathbf{j}}{\partial t}[\mathbf{y},t_{\mathrm{R}}]\right)\wedge\frac{\mathbf{r}}{r}\right)\mathrm{d}V_{\mathbf{y}} \quad . \tag{49.03.06}$$

→ 2222 2224 2227

The equations contain retarded versions of the Coulomb and Biot-Savart laws as their leading terms. **They show explicitly that the Coulomb and Biot-Savart laws fail for time-dependent sources. The equations can be named: generalized Coulomb law, equation** (49.03.05)**, and generalized Biot-Savart law, equation** (49.03.06).

In the quasi-static situation, developing a slowly varying current as

$$\mathbf{j}[t_{\mathrm{R}}] \simeq \mathbf{j}[t] + (t_{\mathrm{R}} - t)\,\partial_t\mathbf{j}[t] \quad ,$$

there is a fortuitous cancellation such that neglecting retardation and dropping the second term in the expression of the magnetic field gives the Biot-Savart law.

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...
Use a Taylor series to express the charge and current-density at the present (un-retarded) time in terms of the source and its derivatives at the retarded time.
...
The integrand of Jefimenko equation for magnetic field neatly embraces the first two terms of this series. That is, the static Biot-Savart formula, evaluated at present time, contains only a second-order error due to retardation when applied to time-dependent currents. The same argument applies to the Jefimenko equation for electric field: the first-order retardation cancels out in the Coulomb portion of the electric field.
On the other hand, when the time variation is rapid, both fields are dominated by their all at terms, which represent accelerated charge and fall off with the inverse first power of distance. These are the terms that represent ElectroMagnetic radiation. These generalized formulas make clear that, fundamentally, it is charges and currents that produce electric and magnetic fields. Because the Maxwell rotor equations relate the rotor of one field to the time-derivative of the other, it is common to describe the phenomenon of Faraday induction, for instance, by saying that a time-varying magnetic field causes a (spatially varying) electric field. But this statement is open to challenge by the argument that in the relativistic formulation the two fields are elements of a single four-dimensional tensor. From this view neither field can cause the other (any more than one component of $\mathbf{E}|\mathbf{B}$ can cause another component). Rather, both fields are caused by charges and currents. Because Maxwell's (differential) equations are local (they apply at a point), their time-derivative induction terms are, in effect, proxies for the charges and currents located elsewhere.
...
The generalized field formulas show an intriguing asymmetry. Using the electric field potential formula, we can trace the $\dot{\mathbf{j}}$ term back to the $\dot{\mathbf{B}}$ induction term in Faraday's law, while the ρ and $\dot{\rho}$ terms are traceable to the Coulomb charge sources of Gauss' law. But the magnetic field potential formula, traces the entire integrand back to the Biot-Savart current sources in the original Ampere's law. There is no explicit manifestation here of Maxwell's $\dot{\mathbf{E}}$ induction term; it is hidden in the retardation.
...

49.03.03
Heaviside-Feynman Solution for a Point Charge

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.28, 2.21||
©|S.Bobbio & E.Gatti, *Elettromagnetismo E Ottica*, 1991, Bollati-Boringhieri, ...Ed., ...|||
©|G.S.Smith, *An Introduction To Classical Electromagnetic Radiation*, 1997, CUP, ...Ed., ...|||

The Heaviside-Feynman formula, is a special case of Jefimenko equations obtained when the source is a single point charge and generalizes Coulomb law for a moving point charge.

The Electric|Magnetic fields produced by a point charge q moving in a totally arbitrary way can be expressed by means of Heaviside-Feynman Formula:

$$\mathbf{E}=\frac{q}{4\pi\varepsilon_0}\left(\left[\frac{\widehat{\mathbf{e}}[\mathbf{r}]}{r^2}\right]_{\mathrm{R}}+\left[\frac{r}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\widehat{\mathbf{e}}[\mathbf{r}]}{r^2}\right)\right]_{\mathrm{R}}+\left[\frac{1}{c^2}\frac{d^2}{\mathrm{d}t^2}\widehat{\mathbf{e}}[\mathbf{r}]\right]_{\mathrm{R}}\right)\qquad\mathbf{B}=\frac{1}{c}\mathbf{E}\wedge\left[\widehat{\mathbf{e}}[\mathbf{r}]\right]_{\mathrm{R}}\quad.\tag{49.03.07}$$

→ 2229

It is often known as Heaviside-Feynman Formula. The expression is written in terms of the **retarded position** of the point charge, with respect to the observer, defined by the unit vector $\widehat{\mathbf{e}}[\mathbf{r}]$. That is, the unit vector $\widehat{\mathbf{e}}[\mathbf{r}]$ points to the observer starting form the retarded (that is apparent) position of the charge: the position the charge had at the time when the radiation now reaching the observer left the charge.

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At variance from equation (49.03.01), here we must use the **retarded position** of the point charge also, because we have just a point radiating particle.

The radiation field, that is the asymptotic field at large distances, is given by just the last term, as the first two terms decrease at infinity more rapidly than the r^{-1} of the last term.

The derivation of equation (49.03.07) is mathematically cumbersome; see the references.

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49.04 ElectroMagnetic Fields of a Point Charge Moving at Constant Velocity

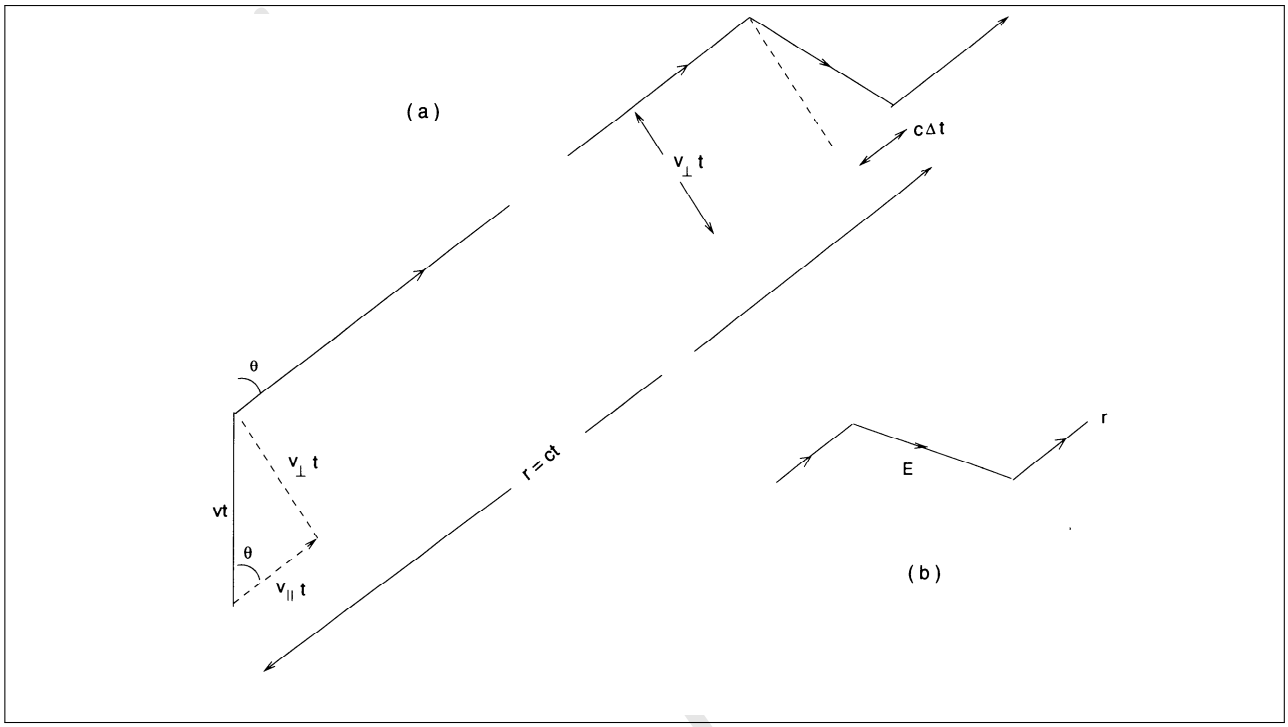

Figure 49.11: FIGURE

Consider a point charge moving at constant velocity, $\mathbf{v}$.

The electric field at a fixed observation point located at $\mathbf{r}$ relative to the instantaneous position of a point charge moving with constant velocity $\mathbf{v}$ is directed along the line **from the instantaneous position of the charge to the observation point**. Its is given by:

$$\mathbf{E}[\mathbf{r}] = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{1-\beta^2}{\left(1-\beta^2\sin^2[\theta]\right)^{3/2}}\frac{\mathbf{r}}{r} \qquad \text{with } \boldsymbol{\beta} \equiv \mathbf{v}/c \quad , \tag{49.04.01}$$

→ 2231 2232

where θ is the angle between the direction of the velocity, $\mathbf{v}$, and the direction from the instantaneous position of the charge to the fixed observation point. The corresponding magnetic field is:

$$\mathbf{B} = \frac{1}{c^2}\mathbf{v}\wedge\mathbf{E} = \frac{1}{c}\boldsymbol{\beta}\wedge\mathbf{E} \quad . \tag{49.04.02}$$

→ 2231

According to special relativity the field is just the Lorentz transformed field of the field of a point charge standing still with respect the inertial Reference Frame moving with the charge.

Note that **the charge is assumed to be in motion at constant velocity for an infinite time**. Therefore the fact that the electric field points to the instantaneous position of the charge does not imply any instantaneous transmission of information. The charge, in fact, has been moving for a long time at constant speed before reaching the current instantaneous position and therefore it has been its position known along its path.

Preliminary Release

49.05 Radiation at large distance from a Non Relativistic Point Charge

This § is referenced at pages:
[2238, 2240]

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|7.5||
©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|||
©|T.Padmanabhan, *Theoretical Astrophysics, Vol. 1,2,3*, 2000/2001/2002, CUP, ...Ed., ...|Vol 1, chapt 4||
©|J.R.Tessman & J.T.Finnell|Electric Field of an Accelerating Charge|*Am.J.Phys, , ..., ..., ...Ed., ..., v. 35, p. 523 (1967)*|

49.05.01 Derivation of ElectroMagnetic Radiation with Thomson Method

©|T.Padmanabhan, *Theoretical Astrophysics, Vol. 1,2,3*, 2000/2001/2002, CUP, ...Ed., ...|Vol 1, § 4||
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.34.1||

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A simplified derivation of the ElectroMagnetic radiation emission will be presented in this section, for a very special case of a charge in accelerated motion for a very short time and moving at constant velocity before and after the short period of acceleration and with all speeds assumed to be non relativistic.

Even if this example is of limited generality, it is very **useful to understand the physical mechanism of ElectroMagnetic radiation generation**.

The formulas (49.04.01), (49.04.02) will be used together with the knowledge that ElectroMagnetic Perturbation|Disturbance|Signals move with the speed of light, c. In fact ElectroMagnetic fields in vacuum satisfy the **D'Alembert non-dispersive wave-equation** with speed c, and we know that, also in dispersive media, no signal can travel faster than c, read § 18.04.04. That is: whatever the shape of the signal, nothing arrives at any point before light in vacuum:

$$\psi[\mathbf{x},t]=0 \qquad \text{for } |\mathbf{x}|>ct\text{; signal generated at } \mathbf{x}=0 \text{ and } t=0 \quad . \tag{49.05.01}$$

Consider a point charge, q, which is standing still at the origin of an Inertial Reference Frame from time $t=-\infty$ until time $t=0$.

At $t=0$ the charge starts to move in the positive direction of the z axis with constant acceleration $a\equiv a_z$, for a short time Δt.

Afterwards it stops accelerating and continues to move at constant velocity $v=a\Delta t$.

The motion of the charge is always in the direction fo the z axis.

The conditions are therefore:

$$|\mathbf{v}|\ll c \qquad \Delta t\ll t \qquad (1/2)\,a\Delta t^2\ll r\equiv ct \qquad (1/2)\,a\Delta t^2\ll vt \quad , \tag{49.05.02}$$

$$\text{constant acceleration: } \{0,\Delta t\} \text{ (duration: } \Delta t\ll t) \quad , \tag{49.05.03}$$

$$\text{constant speed: } \{\Delta t, t\} \text{ (duration: } t-\Delta t\simeq t) \quad . \tag{49.05.04}$$

Note that as a consequence of the above assumptions the charge stays in a limited region of space, that is it is a localized source.

At $t<0$ the electric field is the electric field of a stationary point charge at the origin, as given by Coulomb law, while the magnetic field is zero:

$$\mathbf{E}[\mathbf{r}]=\frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{r}}{r^3} \qquad \mathbf{B}[\mathbf{r}]=0 \quad . \tag{49.05.05}$$

→ 2232

WORK IN PROGRESS

The electric field lines are radially directed from the origin, isotropically distributed at $t < 0$.

Kinks in the electric field lines are created when the charge suddenly starts to accelerate at $t = 0$, and, moreover, the change of the electric field creates a magnetic field. The kinks of the electric field lines and the created magnetic field lines propagate outward at the speed of light: we know from Maxwell equations, in fact, that ElectroMagnetic fields propagate in vacuum at the speed of light.

Note that, due to the geometry, the kinks of the electric fields have a higher field line density, so that one expects that the electric field there is larger than outside.

As we are looking for the fields at large distance, the PW approximation will be adequate, unless we are going to correlate ElectroMagnetic fields at large distances between one another. Using the PW approximation we can simply calculate the magnetic field from the electric field as:

$$\mathbf{B} = \frac{\mathbf{k} \wedge \mathbf{E}}{\omega} \quad .$$

Consider now a time t very much larger than Δt: $t \gg \Delta t$. As the speed of the charge is always non relativistic one can consider that the charge remains confined into a negligibly small region near the origin of the Coordinate System when observing from a very large distance. Therefore one can assume that the information about the charge propagates as spherical waves centered at the origin.

At any time t, at any location at distance $r > ct$ from the origin the news that the charge has started to move (that is the kinks of the electric field lines and the newly created magnetic field) has not arrived yet. Therefore the electric field lines are the same as they were at $t < 0$ while the magnetic field is still zero.

At any time t, at any location at distance $r < c\,(t - \Delta t)$ from the origin the news that the charge has stopped accelerating has already arrived and the kinks of the electric field lines and the newly created magnetic field have already passed away. Therefore the electric field and magnetic fields are the ones due to the charge moving at constant speed v. This is a radial electric field pointing away from the present position of the charge; the position is the present position of the charge, because the motion of the charge is non-relativistic.

According to equation (49.04.01), for non relativistic velocities, the electric field is thus:

$$\mathbf{E}'[\mathbf{r}'] = \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{r}'}{r'^3} \quad . \tag{49.05.06}$$

→ 2232

We are interested in the kinks of the electric field which propagate outward at the speed of light. We can use the first **Maxwell Equation** at a given instant to relate the fields just ahead of the kink (given by equation (49.05.05)) with the fields just behind the kinks (given by equation (49.05.06)) by conserving the flux of the electric field.

Consider a time $t \gg \Delta t$. At this time the kink is very far away from the charge. We can neglect the distance that the charge traveled during the acceleration phase, $a\,(\Delta t)^2/2$, with respect to the much larger distance vt that it has traveled at constant velocity after stopping acceleration. In this way we can consider that the radiation was effectively produced at a point, the origin of the Coordinate System.

Consider an observation point $\mathbf{r}$ whose position vector makes an angle θ with the velocity $\mathbf{v}$. Choose the time t such that the kink is starting to sweep past the observation point at time t: $r = ct$. Consider now $\mathbf{r}'$ at the rear of the kink. Since $v \ll c$ the charge has traveled a distance vt that is much shorter than $r = ct$. Therefore the directions of $\mathbf{r}$ and $\mathbf{r}'$ are essentially parallel to each other. Therefore:

$$\mathbf{r} \parallel \mathbf{r}' \qquad r' = r - vt\cos[\theta] - c\Delta t = ct - vt\cos[\theta] - c\Delta t = r\left(1 - \frac{v}{c}\cos[\theta] - \frac{\Delta t}{t}\right) \approx r \quad .$$

In fact, the wave-front corresponding to the charge startup is described by the sphere $r[t] = ct$, for $t > 0$; the wave-front corresponding to the charge stopping accelerating is described by the sphere: $r[t] = ct - c\Delta t$, for $t > \Delta t$.

The electric field lines come from the line integral of the electric field, by definition, therefore they have to be continuous functions. In fact, as long as the electric field is continuous or has only discontinuity of first kind, the field line is continuous.

Let us consider, for the time being, only the modules of the physical quantities, because all directions and orientations are anyway known.

DRAFT - IN - PROGRESS

Consider, in the region of the kink, the components of the electric field parallel and perpendicular to the radial direction: $E_\perp$ and $E_\parallel$. From geometrical considerations only one can deduce:

$$\frac{E_\perp}{E_\parallel}=\frac{v_\perp t}{c\Delta t}=\frac{(a_\perp\Delta t)\,(r/c)}{c\Delta t}=\frac{a_\perp r}{c^2}\quad . \tag{49.05.07}$$

We need the value of $E_\parallel$. let us apply the first **Maxwell Equation** to a small cylinder box whose axis is parallel to $\mathbf{r}$ while one of the bases is just outside the kink (region not yet reached by the kink, radial field E_r) and the other one is inside the kink (radial component of the field $E_\parallel$). As there is no charge inside we conclude that:

$$E_\parallel=E_r=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\quad .$$

Note that we have applied, in fact, conservation of the normal component of the electric field across the two moving surfaces of discontinuity: $r=ct$ and $r=c\,(t-\Delta t)$. This is possible because conservation of the normal component of the electric field also apply at moving surfaces while the same is not true for the conservation of the tangential component of the electric field (which is not conserved, in fact, in this case), see section § 36.16.02.

If the same argument is applied to the rear of the kink one finds:

$$E_\parallel=E'_r\simeq E_r\qquad \text{because } r\simeq r'\quad .$$

It should be kept in mind that one always has in mind, implicitly, **taking the limit for** $\Delta t\to 0$ which implies an infinitesimally small cylinder such that the contribution of the lateral surface to the flux of the electric field is zero.

The radiation field is thus:

$$\boxed{E_\perp=\frac{a_\perp r}{c^2}E_\parallel=\frac{1}{4\pi\varepsilon_0}\frac{qa_\perp}{rc^2}}\quad .$$

Let us re-introduce vector notation.

In order to calculate the direction of the electric field $\mathbf{E}_\perp$ one should note that $\mathbf{E}_\perp$ at the position $\mathbf{r}$ and time t has the opposite direction of the acceleration. The acceleration relevant is the one at time $t_{\mathtt{R}}=t-(r/c)$, due to retardation. Note that, while in general it is not possible to obtain the solution just by retarding the static solutions for the fields, it this case it was demonstrated the the solution is just the retarded one.

Assuming that, in a limited region of space, the ElectroMagnetic spherical wave can be treated as a plane wave, all the results for PW can be used and summarized in the **radiation formula for a time-infinitesimal acceleration**:

$$\boxed{\mathbf{E}_{\mathtt{R}}[\mathbf{r},t]=-\frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{a}_\perp[t_{\mathtt{R}}]}{rc^2}\qquad \mathbf{B}_{\mathtt{R}}[\mathbf{r},t]=\frac{\widehat{\mathbf{e}}_r\wedge\mathbf{E}_{\mathtt{R}}[\mathbf{r},t]}{c}\qquad \widehat{\mathbf{e}}_r\equiv\mathbf{r}/r\qquad t_{\mathtt{R}}=t-(r/c)=\text{propagation delay}}\quad . \tag{49.05.08}$$

→ 2233 2234 2238 2239 2240

normally used under the condition specified in equation (49.05.10) for non time-infinitesimal motions. Note that, in the above formulas, there is time retardation between both $\mathbf{a}\to\mathbf{E}$ and $\mathbf{a}\to\mathbf{B}$, but no time retardation between $\mathbf{E}$ and $\mathbf{B}$.

Note the useful identity:

$$\mathbf{a}_\perp\equiv\mathbf{n}\wedge(\mathbf{a}\wedge\mathbf{n})\qquad \mathbf{r}\equiv r\widehat{\mathbf{n}}\qquad \widehat{\mathbf{n}}=\frac{\mathbf{k}}{k}\quad ,$$

where the component of the acceleration perpendicular to the line of sight has been introduced:

$$|\mathbf{a}_\perp|=a\sin[\theta]\qquad 0\le\theta\le\pi\qquad \sin[\theta]\ge 0\quad .$$

Thus one can rewrite the above equations as:

$$\boxed{\mathbf{E}_{\mathtt{R}}[\mathbf{r},t]=\frac{1}{4\pi\varepsilon_0}q\left.\frac{\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{a})}{rc^2}\right|_{t_{\mathtt{R}}=t-r/c}\qquad \mathbf{B}_{\mathtt{R}}[\mathbf{r},t]=\frac{\mathbf{k}\wedge\mathbf{E}_{\mathtt{R}}[\mathbf{r},t]}{\omega}\qquad t_{\mathtt{R}}=t-(r/c)<t}\quad . \tag{49.05.09}$$

→ 2252

The results of equation (49.05.08) show three basic features:

DRAFT-IN-PROGRESS

- The Electric|Magnetic fields must be evaluated using the distance and acceleration (actually $\mathbf{a}_\perp$) of the charge at the so-called **retarded position** of the point charge (its apparent position), that is the position the charge had at the time t_R.
- The Electric|Magnetic fields decrease at large distances as r^{-1}, at variance from the static fields.
- The electric field is anti-parallel to the component of the acceleration perpendicular to the line of sight, and the magnetic field is calculated from the electric field according to the PW approximation.

The radial component inside the kink is the same as the radial field ahead and behind the kink: therefore it carries no news, it is not radiation and it is not part of the traveling wave. A detector of ElectroMagnetic fields which could only detect the radial part of the field would not notice anything at the passage of the kink. That is why we will only include the transverse field into the radiation field. This result is to be expected, to some extent, because one knows that for PW the longitudinal component of the fields are constant both in space and in time, for a PW, and, therefore, they are not to be considered part of the time-varying ElectroMagnetic fields. Obviously only in a limited region of space the present ElectroMagnetic wave can be considered as a PW.

The electric field is in the plane determined by the acceleration and the radial position vector such that any charge moving produces **EM radiation instantaneously wave-Polarized in the plane containing the instantaneous acceleration and radial position vectors**. In particular, rectilinear motion produces linearly wave-Polarized radiation.

PRELIMINARY - UNDER REVISION

49.05.02
Spatially Limited Source of Non-Relativistic Point Charges Seen From a Large Distance

We shall assume that, in a limited region of space, at large distance from the origin, all the results for PW can be used.

Consider a charge moving in an arbitrary way at non-relativistic velocity. Assume it stays near the origin of the Coordinate System, such that the position vector from the charge to the observation point can be considered nearly constant. Assume the motion is harmonic in time with angular frequency ω.

Under these conditions there is a one-to-one correspondence between the radiation detected at time t and the transverse acceleration of the radiating charge at the single **retarded time** t_R.

It can be shown in a rigorous way that equation (49.05.08) can be used provided:

$$v \ll c \qquad r \gg \lambda \equiv c2\pi/\omega \gg \{\text{dimension of the source}\} \quad . \tag{49.05.10}$$

→ 2233 2238 2240 2269

49.05.02.01 Power Radiated by a Non-Relativistic Charge: Larmor Radiation Formula

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From equation (49.05.08) the Poynting vector at distance $\mathbf{r}$ and time t turns out to be:

$$|\mathbf{S}_{EM}[\mathbf{r},t]| = \frac{q^2 a^2[t_R]\sin^2[\theta]}{16\pi^2 r^2 \varepsilon_0 c^3} \quad ,$$

which depends on the acceleration at the **retarded time** t_R

The total radiated power crossing a spherical surface of radius r, centered on the charge, at time t can be obtained by integrating over the sphere at radius r (r is arbitrary, fixed and not integrated):

$$P[r,t] = r^2 \iint |\mathbf{S}_{EM}[\mathbf{r},t]| \, d\Omega \quad .$$

The total instantaneous radiated power by a non-relativistic charge (**Larmor Radiation Formula**) is thus:

$$P[r,t] = \frac{q^2 a^2[t_R]}{6\pi\varepsilon_0 c^3} = \frac{q^2 a^2[t-r/c]}{6\pi\varepsilon_0 c^3} \equiv \frac{2e^2 a^2[t_R]}{3c^3} \qquad e^2 \equiv \frac{q^2}{4\pi\varepsilon_0} \quad . \tag{49.05.11}$$

→ 2234 2248

The latter expression is written in terms of the charge, e, in the Gauss system of units.

Note that the power in **Larmor Radiation Formula** , equation (49.05.11), is crossing the surface at distance r from the charge at time t, but it comes from radiation actually emitted by the charge at time $t_R \equiv t - r/c$.

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Note that if one further assumes a Sinusoidal|Cosinusoidal dependence of the acceleration (harmonic motion) one additional factor $1/2$ will appear.

49.05.03
Near-Zone ElectroMagnetic Fields

The exact solution for the ElectroMagnetic fields produced by a moving charge contain, in addition to radiation ElectroMagnetic fields varying as r^{-1}, also ElectroMagnetic fields varying as r^{-2} and r^{-3}. These ElectroMagnetic fields dominate at sufficiently small distances. In fact they are called near-zone ElectroMagnetic fields. They do not contribute any net outgoing energy flux at large distances but they are relevant at small distances.

Their calculation is much more difficult than the calculation of the large-distance ElectroMagnetic fields.

49.05.04
Decoupling of radiation fields from the source

49.06 Radiation from a Point Charge

49.06.01
Radiation at large distance from a Point Charge Moving at Arbitrary Speed

©|J.R.Tessman & J.T.Finnell|Electric Field of an Accelerating Charge|*Am.J.Phys*, , ..., ..., ...Ed., ..., *v. 35, p. 523 (1967)*|

The Thomson method has been generalized to a rigorous solution for charges in arbitrary motion, see the references.

49.06.02
Lienard-Wieckert Potentials for a Point Charge

This § is referenced at pages:
[Never referenced.]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2.21||
©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|||
©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|11||

See the very important comment in section § 36.05.03 about the modeling of matter with point charges, **Dipoles** and higher-order **Multipoles** .

The **ElectroMagnetic potentials** for a point charge moving with arbitrary velocity is derived from the general expressions (49.03.01) once the Charge|Current velocity for a point charge is written in terms of the Dirac delta:

$$\rho[\mathbf{y},t]=q\delta^3[\mathbf{y}-\mathbf{r}_{\mathrm{Q}}[t]] \qquad \mathbf{j}[\mathbf{y},t]=q\mathbf{v}[t]\delta^3[\mathbf{y}-\mathbf{r}_{\mathrm{Q}}[t]] \quad , \qquad \rightarrow 2236$$

$$\Phi[\mathbf{x},t]=\frac{1}{4\pi\varepsilon_0}\Big[\frac{q}{r-\boldsymbol{\beta}\cdot\mathbf{r}}\Big]_{\mathrm{R}} \qquad \mathbf{A}[\mathbf{x},t]=\frac{\mu_0}{4\pi}\Big[\frac{qc\boldsymbol{\beta}}{r-\boldsymbol{\beta}\cdot\mathbf{r}}\Big]_{\mathrm{R}} \qquad \Big[\boldsymbol{\beta}\Big]_{\mathrm{R}}\mathbf{A}[\mathbf{x},t]=c\Phi[\mathbf{x},t] \quad .$$

The fact that the expressions must be calculated at the **retarded time** , t_{R}, comes from the effect of the Dirac delta. In fact, it can be shown that at any time, t, and any observation location, $\mathbf{x}$, there is one and only one solution, for t_{R}, to the equation of **retarded time** , equation (49.03.03), calculated at

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the position of the charge:

$$t_{\mathrm{R}} \equiv t - \frac{|\mathbf{x}-\mathbf{y}[t_{\mathrm{R}}]|}{c} \qquad \Big[\dots\Big]_{\mathrm{R}} := t \to t_{\mathrm{R}} \quad , \tag{49.06.01}$$

→ 2236

where the notation for calculating the expressions at the **retarded time** , t_{R}, instead of the current time, t has been emphasized.

Since no particle can travel at the speed of light nor faster, only one retarded point contributes to the potentials, at any given time.

In equation (49.06.01) the position of the charge, $\mathbf{y}$, must be evaluated at the **retarded time** , t_{R}, because equation (49.06.01) expresses the time when the ElectroMagnetic radiation emitted by the charge at $\mathbf{y}$ was emitted so that it reaches the observation point, $\mathbf{x}$, at time t.

The factor at the denominator can be seen as a purely geometrical factor, due to the particle moving closer/farther from the observer.

49.06.03
Lienard-Wieckert Fields for a Point Charge

This § is referenced at pages:
[2253]

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|14||

The ElectroMagnetic fields derived from the potentials (49.06.02) are made of a part depending on the velocity of the charge (the velocity field) and a part depending on the acceleration of the charge (the acceleration field). Consider a point charge moving with position $\mathbf{r}_{\mathrm{Q}}[t]$, velocity $\mathbf{v} \equiv c\beta$ and acceleration $\mathbf{a}$. The ElectroMagnetic fields are:

$$r \equiv |\mathbf{x}-\mathbf{r}_{\mathrm{Q}}| \quad ,$$

$$\widehat{\mathbf{n}} \equiv \frac{\mathbf{x}-\mathbf{r}_{\mathrm{Q}}}{|\mathbf{x}-\mathbf{r}_{\mathrm{Q}}|} \equiv \frac{\mathbf{x}-\mathbf{r}_{\mathrm{Q}}}{r} \quad ,$$

$$\mathbf{E}[\mathbf{x},t] = \frac{q}{4\pi\varepsilon_0}\Big[\frac{\widehat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma^2 r^2\,(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\Big]_{\mathrm{R}} + \frac{q}{4\pi\varepsilon_0}\Big[\frac{\widehat{\mathbf{n}}\wedge((\widehat{\mathbf{n}}-\boldsymbol{\beta})\wedge\mathbf{a})}{c^2 r\,(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\Big]_{\mathrm{R}} \quad , \tag{49.06.02}$$

→ 2237

$$\mathbf{B}[\mathbf{x},t] = \frac{1}{c}\Big[\mathbf{n}\wedge\mathbf{E}\Big]_{\mathrm{R}} \quad . \tag{49.06.03}$$

Note that the velocity field is apparently different from the field derived for a moving charge in section § 61.09.06. However, the difference is only apparent because in section § 61.09.06 the ElectroMagnetic fields were referred to the current position of the charge while in this section they are refereed to the **retarded position** of the point charge.

DRAFT - IN PROGRESS

49.06.03.01 Velocity Field

It is the part of field depending on the velocity and as the inverse square distance, in equation (49.06.02). Therefore at large enough distance it goes to zero and does not affect radiation fields.

49.06.03.02 Acceleration (Or Radiation) Field

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|20.2||

The acceleration field is also called the **radiation field** as it is the only part of the field surviving at large distances.

The acceleration fields, both $\mathbf{E}$ and $\mathbf{B}$, is an odd function for changing $\mathbf{a} \to -\mathbf{a}$, at fixed β. Therefore the Poynting vector is an even function for changing $\mathbf{a} \to -\mathbf{a}$, at fixed β. In particular, in the instantaneous Rest Frame of the particle the energy flux and linear momentum density are an even function for changing $\mathbf{a} \to -\mathbf{a}$, as $\beta = 0$ and therefore the total linear momentum is zero.

The simple expression of the radiation field in the instantaneous Rest Frame is of special interest, as it often appears:

$$\boxed{r \equiv |\mathbf{x} - \mathbf{r}_Q|} \quad ,$$

$$\boxed{\widehat{\mathbf{n}} \equiv \frac{\mathbf{x} - \mathbf{r}_Q}{|\mathbf{x} - \mathbf{r}_Q|} = \frac{\mathbf{x} - \mathbf{r}_Q}{r}} \quad ,$$

$$\boxed{\mathbf{E}[\mathbf{x}, t] = \frac{q}{4\pi\varepsilon_0}\Big[\frac{\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{a})}{c^2 r}\Big]_R \qquad \text{when } \beta \simeq 0} \quad ,$$

$$\boxed{\mathbf{B}[\mathbf{x}, t] = \frac{q}{4\pi\varepsilon_0}\Big[\frac{\mathbf{a}\wedge\mathbf{n}}{c^3 r}\Big]_R \qquad \text{when } \beta \simeq 0} \quad .$$

49.06.04 Radiation Reaction

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DRAFT WORK IN PROGRESS

49.07 Oscillating Ideal Electric|Magnetic Dipoles - Dipole Radiation

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Oscillating ideal **Electric Dipole** | **Magnetic Dipole** produce ElectroMagnetic radiation (**Electric Dipole** | **Magnetic Dipole** radiation). They represent a particularly significant example because real antennas as well as the emission of ElectroMagnetic radiation by atoms and molecules can often be represented as oscillating ideal **Electric Dipole** | **Magnetic Dipole** .

The ElectroMagnetic **Dipole** radiation produced at large distance from the source is a non isotropic vector (transverse) spherical wave.

See [1] for a different derivation of the dipole radiation, parallel between electric and magnetic cases, with a clean presentation of the different approximations.

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49.07.01 Oscillating Ideal Electric Dipole (Electric Dipole Radiation)

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The **Electric Dipole** radiation comes from a globally neutral system of charges oscillating in such a way to produce an oscillating **Electric Dipole** . A very simple model is that of two equal and opposite charges harmonically oscillating back and forth along the z axis around the origin with a half-cycle de-phasing between each other.

The qualitative behavior of the radiation ElectroMagnetic fields can be easily found from the basic radiation fields relations (49.05.08). In fact we expect zero EM fields on the axis, because the projected acceleration is zero there. On the other hand we expect a maximum electric field, directed along z, on the xy plane, because the projected acceleration is maximum there.

In order to apply the previous results of section § 49.05 the conditions (49.05.10) must be satisfied. In particular the **Electric Dipole** must be short, with respect to the wavelength.

49.07.01.01 Derivation From the Radiation Formula

Consider a point charge undergoing an harmonic motion along the z axis with a component of the acceleration a_z:

$$z[t] = z_0 \cos[\omega t + \phi] \qquad \dot{z}[t] = -z_0 \omega \sin[\omega t + \phi] \qquad \ddot{z}[t] = -z_0 \omega^2 \cos[\omega t + \phi] \equiv a_z \quad ,$$

where a **phase constant** is not relevant. One should note that, as the $\widehat{\mathbf{e}}_\theta$ unit vector is directed as the z axis at $\theta = 0$, we have:

$$E_\theta > 0 \quad \Leftrightarrow \quad a_z > 0 \quad \Leftrightarrow \quad \mathbf{a}_\perp \cdot \widehat{\mathbf{e}}_\theta < 0 \quad ,$$
$$E_\theta < 0 \quad \Leftrightarrow \quad a_z < 0 \quad \Leftrightarrow \quad \mathbf{a}_\perp \cdot \widehat{\mathbf{e}}_\theta > 0 \quad .$$

In short one can simply notice that a positive a_z gives rise to a $\mathbf{a}_\perp$ whose projection along $\widehat{\mathbf{e}}_\theta$ is negative.

Therefore:

$$E_\theta[r,\theta,t] = -\frac{q}{4\pi\varepsilon_0 c^2}\left(\frac{-a_z[t_R]\sin[\theta]}{r}\right) = \frac{q}{4\pi\varepsilon_0 c^2}\left(\frac{a_z[t_R]\sin[\theta]}{r}\right) = \frac{q}{4\pi\varepsilon_0 c^2}\left(\frac{-\omega^2 z_0 \cos[\omega(t - r/c) + \phi]\sin[\theta]}{r}\right) \quad ,$$

[1] ©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 11.1||

where the time retardation has been introduced:

$$\boxed{t_{\mathrm{R}} = t - r/c} \quad .$$

Note that it is exactly the time retardation introduced into the acceleration that gives rise to the progressive outgoing wave factor: $\cos[\omega(t - r/c)]$.

If one also had another charge, with equal magnitude but opposite sign (that is $q \to -q$) oscillating with opposite phase with respect to the first one (that is $z[t] \to -z[t]$ and $a[t] \to -a[t]$) then the radiation Electric|Magnetic fields would be the same. The two charges would make an oscillating **Electric Dipole** .

Therefore, in terms of the **Electric Dipole** moment,

$$\mathbf{p}[t] \equiv \mathbf{p}_0 \cos[\omega t + \phi] \equiv |q|\,(2z_0)\cos[\omega t + \phi] \quad ,$$

the electric and magnetic fields, from equations (49.05.08), can be written as:

$$\boxed{E_\theta[r,\theta,t] = -\frac{\omega^2 \mathbf{p}_0}{4\pi\varepsilon_0 c^2}\frac{\cos[\omega t - kr]}{r}\sin[\theta]} \quad , \qquad (49.07.01) \quad \to 2240\ 2241\ 2269$$

$$\boxed{B_\psi[r,\theta,t] = -\frac{\omega^2 \mathbf{p}_0}{4\pi\varepsilon_0 c^3}\frac{\cos[\omega t - kr]}{r}\sin[\theta] = \frac{E_\theta}{c}} \quad , \qquad (49.07.02) \quad \to 2240\ 2241\ 2269$$

$$\boxed{cB_\psi[r,\theta,t] = E_\theta[r,\theta,t]} \quad .$$

The ElectroMagnetic radiation fields of the oscillating **Electric Dipole** are **non-isotropic spherical waves**.

These ElectroMagnetic fields are in phase, mutually perpendicular, and transverse to the direction of propagation; the ratio of their amplitudes is $E/B = c$, as expected for ElectroMagnetic waves.

Note the geometry:

$$\mathbf{k} \parallel \widehat{\mathbf{e}}_r \qquad \mathbf{E} \parallel \widehat{\mathbf{e}}_\theta \qquad \mathbf{B} \parallel \widehat{\mathbf{e}}_\psi \quad .$$

When a **Electric Dipole** antenna is used as a receiving antenna it detects the electric field of the incoming ElectroMagnetic wave.

49.07.01.02 Direct Derivation From the Potentials

Use the **Lorenz Gauge** . Consider a very small **Electric Dipole** at the origin, oscillating at fixed frequency:

$$\mathbf{p}[t] = \mathbf{p}_0 \exp[-\mathrm{i}\omega t] \qquad \mathbf{p}_0 \parallel \widehat{\mathbf{e}}_3 \quad .$$

Assume the oscillating **Dipole** occupies a small cylindrical region, at the origin of the Coordinate System, of length L and base area S and that is it contained inside a sphere of diameter D.

The current and current density at the origin are given by:

$$\mathbf{j}[t] = I[t]\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}V} = I[t]\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}L\,\mathrm{d}S} \implies \frac{1}{LS}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}[t] = \frac{(-\mathrm{i}\omega)\,\mathbf{p}_0}{LS}\exp[-\mathrm{i}\omega t] \quad ,$$

while the current density anywhere else is zero.

Assume that the ElectroMagnetic wave is observed at a position $\mathbf{r}$ at a distance, $r \equiv |\mathbf{r}|$, very much larger than the wavelength, λ, which is, in turn, very much larger than the dimensions of the source:

$$\boxed{r \gg \lambda \equiv \frac{2\pi c}{\omega} \equiv \frac{2\pi}{k} \gg D/2} \quad . \qquad (49.07.03) \quad \to 2239$$

The latter condition ensures that the retardation is the same for all points of the source.

The **retarded potentials solution** , equation (49.03.01), gives, under the hierarchical approximation (49.07.03):

$$\mathbf{A}[\mathbf{x},t] = \frac{\mu_0}{4\pi}\int \frac{I[t_{\mathrm{R}}]}{r}\,\mathrm{d}\mathbf{L} = \frac{\mu_0}{4\pi}\int \frac{I[t - r/c]}{r}\,\mathrm{d}\mathbf{L} =$$

$$= \frac{\mu_0}{4\pi}L\frac{(-\mathrm{i}\omega)\,\mathbf{p}_0}{L}\exp[-\mathrm{i}\omega(t - r/c)]\frac{1}{r} = \frac{\mu_0}{4\pi}(-\mathrm{i}\omega)\,\mathbf{p}_0\frac{\exp[-\mathrm{i}\omega(t - r/c)]}{r} =$$

$$= \frac{\mu_0}{4\pi}(-\mathrm{i}\omega)\,\mathbf{p}_0\frac{\exp[\mathrm{i}(kr - \omega t)]}{r} \quad .$$

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The magnetic field at large distance is calculated by retaining only the largest terms at large distance, that is the radiation term going as r^{-1}. It is:

$$\mathbf{B}[\mathbf{x},t] = \mathbf{rot}\,\mathbf{A} =$$

$$= \frac{\mu_0}{4\pi}(-\mathtt{i}\omega)\,\mathbf{p}_0\mathbf{rot}\left(\frac{\exp\left[\mathtt{i}\,(kr-\omega t)\right]}{r}\right) = \frac{\mu_0}{4\pi}(\mathtt{i}\omega)\;\mathbf{p}_0\wedge\frac{\exp\left[\mathtt{i}\,(kr-\omega t)\right]}{r} =$$

$$= \frac{\mu_0}{4\pi}\omega^2\frac{\mathbf{r}\wedge\mathbf{p}_0}{cr^2}\exp\left[\mathtt{i}\,(kr-\omega t)\right] \quad .$$

As we are looking for the large distance fields, the ElectroMagnetic waves can be treated as PW. Therefore the electric field can be simply calculated from the well-known relations for MPW (equation (46.01.08)):

$$\mathbf{E} = \frac{c^2}{\omega}(\mathbf{B}\wedge\mathbf{k}) = c\,(\mathbf{B}\wedge n) = \frac{\mu_0}{4\pi}\omega^2\frac{\mathbf{r}\wedge(\mathbf{p}_0\wedge\mathbf{r})}{r^3}\exp\left[\mathtt{i}\,(kr-\omega t)\right] \quad .$$

The non-zero (real) components of the oscillating ideal electric dipole fields in spherical coordinates turn out to be the equations (49.07.01) and (49.07.02).

49.07.02
Oscillating Ideal Magnetic Dipole (Magnetic Dipole Radiation)

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|9.2||

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|||

The **Magnetic Dipole** radiation is very similar to the **Electric Dipole** radiation. It comes from an oscillating ideal **Magnetic Dipole** , that is a globally neutral current loop carrying an oscillating current. A very simple model is that of a plane circular loop with axis along the z direction and centered at the origin carrying an harmonic current.

In order to apply the previous results of section § 49.05 the conditions (49.05.10) must be satisfied. In particular the **Magnetic Dipole** must be short, with respect to the wavelength.

Some qualitative behavior of the radiation ElectroMagnetic fields can be easily found from the basic radiation fields relations (49.05.08). In fact we expect zero ElectroMagnetic fields on the axis, because the projected acceleration of all the charges of the current loop sum to zero there at any time. Note that the accelerations of the charges, as well as the current and velocities of the charges, is changing to produce a changing Magnetic Dipole, and includes both a centripetal and a tangential acceleration. In the $x-y$ plane the contributions do not sum to zero because of the different distances from the observation point of the different current elements. We expect, on the $x-y$ plane the electric field to be in the $x-y$ plane and therefore the magnetic field perpendicular to it.

49.07.02.01 Derivation by Analogy With the Electric Dipole

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 7.64||

It can be derived from the duality transformation, under which Maxwell's equations with magnetic charge are invariant, read the reference.

Consider a small current loop (magnetic **Dipole**), small with respect to the wavelength.

It can be shown that the ElectroMagnetic fields far away from the location of the **Dipole** can be obtained from the **Electric Dipole** fields with the following replacements:

$$\boxed{\mathbf{E}\longrightarrow c\mathbf{B} \qquad \mathbf{B}\longrightarrow -\mathbf{E}/c \qquad \mathbf{p}\longrightarrow \mathbf{m}/c} \quad . \tag{49.07.04}$$

→ 2241 2242

When a **Magnetic Dipole** antenna is used as a receiving antenna it detects the magnetic field of the incoming ElectroMagnetic wave.

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49.07.02.02 Direct Derivation From the Potentials

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Read the reference; there is not so much physics in the calculations.
The non-zero (real) components of the oscillating ideal magnetic dipole fields in spherical coordinates are:

$$E_\psi = +\frac{\mu_0\omega^2\mathbf{m}_0}{4\pi c}\frac{\cos[kr-\omega t]}{r}\sin[\theta] \quad , \tag{49.07.05}$$

$$B_\theta = -\frac{\mu_0\omega^2\mathbf{m}_0}{4\pi c^2}\frac{\cos[kr-\omega t]}{r}\sin[\theta] \quad . \tag{49.07.06}$$

These ElectroMagnetic fields are in phase, mutually perpendicular, and transverse to the direction of propagation; the ratio of their amplitudes is $E/B = c$, as expected for ElectroMagnetic waves.

49.07.03
EM Ideal Dipole Radiation: ElectroMagnetic Spherical Waves

The expression of a generic **monochromatic spherical wave** is:

$$\psi[r,\theta,\psi] = A[\theta,\psi]\frac{\cos[kr-\omega t-\phi]}{r} \quad .$$

The ElectroMagnetic radiation fields produced by both an oscillating **Electric Dipole** and an oscillating **Magnetic Dipole** are non-isotropic spherical waves with no azimuthal dependence, that is:

$$\psi[r,\theta] = A[\theta]\frac{\cos[kr-\omega t-\phi]}{r} \quad .$$

with a dependence from the polar angle, as it is clear from equations (49.07.01), (49.07.02) as well as from the correspondence relation (49.07.04) while there is no dependence on the azimuthal angle.
Note the **strong frequency dependence of the emitted power**.

49.07.04
Oscillating Ideal Dipoles: The Electric Versus the Magnetic Dipole

49.07.05
Radiated Power by Electric Dipole Versus Magnetic Dipole

This § is referenced at pages:
[2319, 2321]

Show that the ratio between radiated power from **Electric Dipoles** | **Magnetic Dipoles**,

$$\frac{P_{\mathrm{E}}}{P_{\mathrm{M}}} = \left(\frac{c\mathbf{p}_0}{\mathbf{m}_0}\right)^2 \quad ,$$

can be written:

1. for a dilute LHI medium with a given Electric|Magnetic susceptibilities is:

$$\frac{P_{\mathrm{E}}}{P_{\mathrm{M}}} \approx \left(\frac{\chi_{\mathrm{E}}}{\chi_{\mathrm{M}}}\right)^2 \gg 1 \quad ,$$

   which is typically much larger than one;
2. for an ideal **Electric Dipole** | **Magnetic Dipole** of size $\approx d$ at frequency ω, as:

$$\frac{P_{\mathrm{E}}}{P_{\mathrm{M}}} \approx \left(\frac{c}{d\omega}\right)^2 \gg 1 \quad ,$$

   which is typically much larger than one.

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Only systems carefully constrained to exclude any **Electric Dipole** radiation will exhibit **Magnetic Dipole** radiation.

WORKED SOLUTION

1. In the first case, use equations (49.07.07), (49.07.08) and:
$$\frac{P_{\mathrm{E}}}{P_{\mathrm{M}}} = \left(\frac{c\mathbf{p}_0}{\mathbf{m}_0}\right)^2 \qquad \mathbf{p}_0 = (\varepsilon_0\chi_{\mathrm{E}})\, E_0\Delta V \qquad \mathbf{m}_0 = (\chi_{\mathrm{M}}/\mu_0)\, B_0\Delta V \quad .$$
Remember that ElectroMagnetic waves are radiated in vacuum, and therefore in this problem all relations are to be computed in vacuum and the electric over magnetic field module ratio is c.
2. In the second case:
$$\mathbf{p}_0 \approx qd \implies I_0 = q\omega \qquad \text{and} \qquad \mathbf{m}_0 \approx \pi d^2 \quad .$$

49.07.06
Radiated Power of Dipole Antennas

The instantaneous and time-averaged radiated energy per unit area and time is:
$$S[r,\theta,t] \propto \frac{\omega^4 \sin^2[\theta]}{r^2}\cos^2[kr-\omega t-\phi] \quad ,$$
$$\boxed{\langle S[r,\theta,t]\rangle \propto \frac{\omega^4 \sin^2[\theta]}{r^2} \equiv S_0\frac{\sin^2[\theta]}{r^2}} \quad .$$

49.07.06.01 Electric Dipole

The total averaged radiated power is found by integrating on the solid angle:
$$\boxed{P_{\mathrm{E}} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}_0^2\omega^4}{3c^3} = \frac{\mathbf{p}_0^2\omega^4}{12\pi\varepsilon_0 c^3}} \quad . \qquad (49.07.07)$$
→ 2242 2328

Note that a factor $1/2$ is introduced with respect to **Larmor Radiation Formula** due the time-averaging.
At fixed $\mathbf{p}_0$ the radiated power depends on the fourth power of the frequency.

49.07.06.02 Magnetic Dipole

The total averaged radiated power is found by replacement from (49.07.04):
$$\boxed{P_{\mathrm{M}} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}_0^2\omega^4}{3c^3} = \frac{\mu_0\mathbf{m}_0^2\omega^4}{12\pi c^3}} \quad . \qquad (49.07.08)$$
→ 2242

Note that a factor $1/2$ is introduced with respect to **Larmor Radiation Formula** due the time-averaging.
At fixed $\mathbf{m}_0$ the radiated power depends on the fourth power of the frequency.

49.07.06.03 Radiation Diagram for Dipole Radiation

See figure 49.12.

WORK-IN-PROGRESS

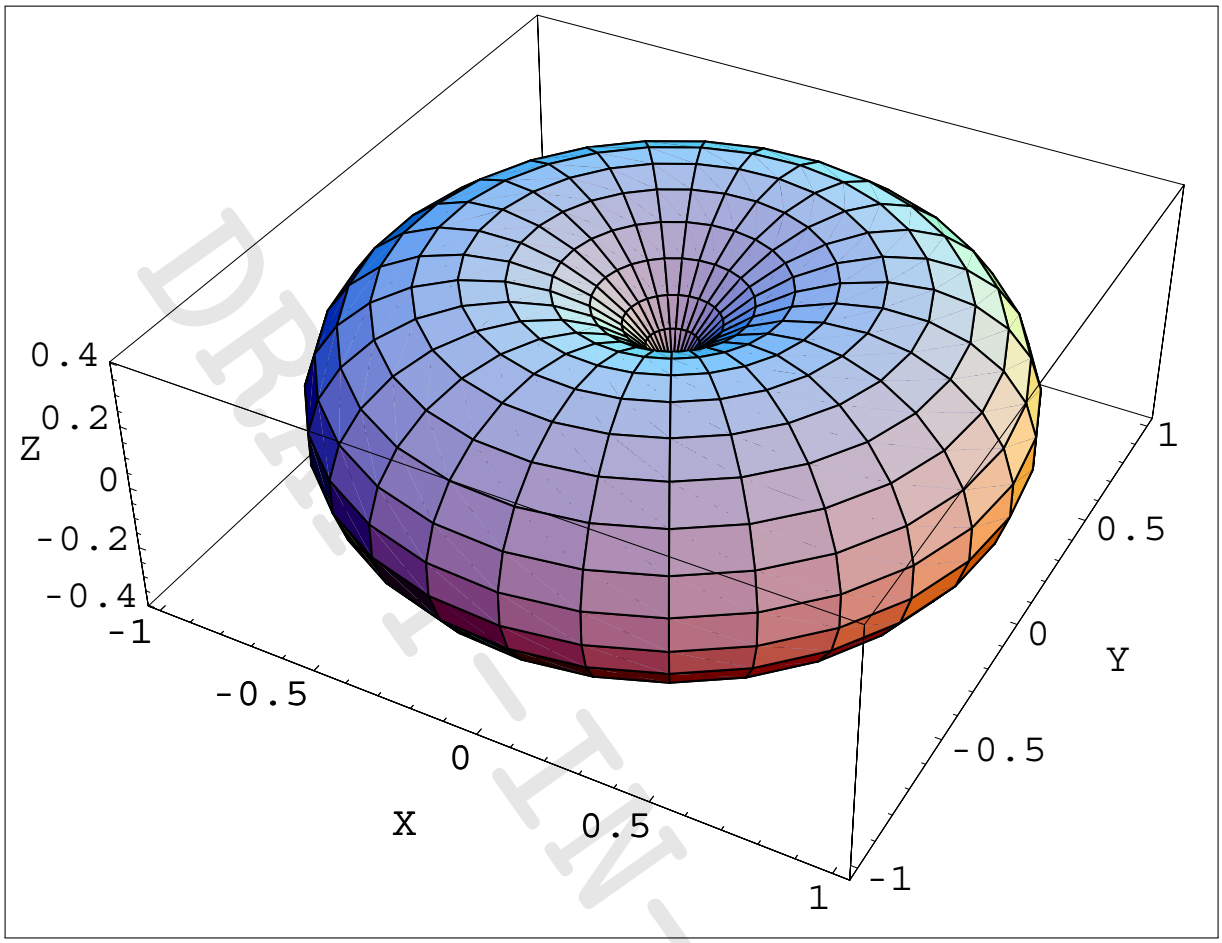

Figure 49.12: FIGURE

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49.07.07
EM Dipole Radiation: Current Elements

Both **Electric Dipole** | **Magnetic Dipole** radiation expressions can be expressed in terms of the currents which power the real antenna.

49.07.07.01 Electric Dipole

Consider the following simple model for an oscillating **Electric Dipole** : two small electrodes connected by a straight conductor with length L along the z axis. Assume that a harmonic current

$$I[t] = I_0 \cos[\omega t + \phi] = \frac{\mathrm{d}q}{\mathrm{d}t} \quad ,$$

crosses any section of the straight conductor. The current is positive when it flows in the positive z direction.

Charge is input/output from the two electrodes: in the circuit language the antenna is a capacitor. The charge on the electrode at positive z is:

$$q[t] = \frac{I_0}{\omega} \sin[\omega t + \phi] \quad .$$

The **Electric Dipole** moment as a function of time is:

$$\mathbf{p}[t] = Lq[t] = L\frac{I_0}{\omega} \sin[\omega t + \phi] \equiv \mathbf{p}_0 \sin[\omega t + \phi] \implies \mathbf{p}_0 = L\frac{I_0}{\omega} \quad .$$

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The total radiated power is thus:

$$\boxed{P_{\mathrm{E}} = \frac{L^2 I_0^2 \omega^2}{12\pi\varepsilon_0 c^3}} \quad . \tag{49.07.09}$$

→ 2244

It shows a dependence on the second power of the frequency (compare with (49.07.10)).

49.07.07.02 Magnetic Dipole

Consider the following simple model for an oscillating **Magnetic Dipole** : a plane circular loop of area A where a current

$$I[t] = I_0 \cos[\omega t + \phi] \quad ,$$

flows.

The **Magnetic Dipole** moment as a function of time is:

$$\boxed{\mathbf{m}[t] = AI[t] = AI_0 \cos[\omega t + \phi] \equiv \mathbf{m}_0 \sin[\omega t + \phi] \implies \mathbf{m}_0 = AI_0} \quad .$$

The total radiated power is thus:

$$\boxed{P_{\mathrm{M}} = \frac{\mu_0 A^2 I_0^2 \omega^4}{12\pi c^3}} \quad . \tag{49.07.10}$$

→ 2244

It shows a dependence on the fourth power of the frequency (compare with (49.07.09)).

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49.08 Multipole Development of the Radiation From a Limited Source

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Electric charge is conserved and therefore the electric monopole does not radiate; in fact, the monopole term is proportional to the total charge, which cannot change. If charge were not conserved, we would get a monopole field proportional to $1/r$.

Note that an electricly charged spherical distribution whose radius oscillates (a monopole) would not radiate, as the field outside is exactly the same at any time. On the other hand, for acoustic waves, monopoles do radiate. In fact, in addition to being both waves, ElectroMagnetic waves and acoustic waves, there are differences due to physical equations describing ElectroMagnetic versus acoustic systems [1].

If the electric dipole moment vanishes then there is no electric dipole radiation. The next term in the development is made of the magnetic dipole term and the electric quadrupole term.

If the magnetic dipole and electric quadrupole contributions vanish, the next term in the development is made of the magnetic quadrupole term and electric octupole term.

[1] It can be shown that a spherically symmetric solution of Maxwell equations, for a system with a spherically symmetric charge and current distributions, is necessarily static in a region with no charges/currents (electromagnetic version of Birkhoff's theorem).

49.09 Antennas

This § is referenced at pages:
[Never referenced.]

A system which converts charges/currents into radiated ElectroMagnetic waves is generically called an **emitting antenna**. Similarly, a system which converts ElectroMagnetic waves into charges/currents is generically called an **receiving antenna**.

49.09.01 Radiation Resistance/Impedance of an Antenna

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ problem 11.3|*Electric dipole antenna*|

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ problem 11.6|*Magnetic dipole antenna*|

The radiation resistance, R, (or radiation impedance, Z) is defined in term of the radiated power and current as:

$$P \equiv R\frac{I_0^2}{2} \quad .$$

The radiation resistance of a small linear **Electric Dipole** antenna of length L is:

$$R_{\text{E}} = 80\pi^2 \left(\frac{L}{\lambda}\right)^2 \simeq 790 \left(\frac{L}{\lambda}\right)^2 \quad \Omega \propto \lambda^{-2} \qquad L \ll \lambda \quad . \tag{49.09.01}$$

The radiation resistance of a small circular **Magnetic Dipole** antenna of radius R is:

$$R_{\text{M}} = \frac{8\pi^5}{3}\mu_0 c \left(\frac{R}{\lambda}\right)^4 \simeq 307432 \left(\frac{R}{\lambda}\right)^4 \quad \Omega \propto \lambda^{-4} \qquad R \ll \lambda \quad . \tag{49.09.02}$$

Note that thanks to the smallness of the antennas the radiation resistance is very small.

Moreover, the radiation resistance of the magnetic dipole antenna is smaller than the electric dipole antenna, for comparable sizes, making it a less efficient antenna.

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49.09.02
Mutual Coupling of Antennas

Antennas located in close proximity one another interact in a complicated manner. This interaction is the so-called mutual coupling and the effect is to change the current on an antenna from the one that the antenna would have if it were isolated in free space. Therefore the current on an antenna not only depends on the voltage source of the antenna itself but also on the current on all nearby antennas.

49.09.03
Dipole Antennas in Practice

They include: **Electric Dipole** antennas and **Magnetic Dipole** antennas.
They are often used in arrays to exploit interference.

49.09.04
Directional Gain

The directional gain of an antenna is defined as:

$$G[\theta,\psi] = \frac{\langle S[r,\theta,\psi]\rangle}{P} 4\pi r^2 \quad .$$

49.09.05
Antennas in Reception

49.10 Power Radiated by ElectroMagnetic Radiation

This § is referenced at pages:
[Never referenced.]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|11||

©|G.S.Smith, *An Introduction To Classical Electromagnetic Radiation*, 1997, CUP, ...Ed., ...|6||

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|20||

©|Am.J.Phys, , ..., ..., ...Ed., ... 47(7), July 1979 602|-||

EM waves in vacuum propagate out to infinity, carrying energy, linear momentum and angular momentum with them. Radiation carries a irreversible flow of energy away from the source. The source is normally assumed to be localized near the origin of a suitable Coordinate System of same Inertial Reference Frame .

If one imagines a large spherical shell of radius R containing all the sources, one can calculate the total power passing out through this surface, $W[R]$, by integrating the Poynting vector. The radiated power is the limit of this quantity as the radius of the sphere goes to infinity:

$$\boxed{W_{\mathtt{RAD}} \equiv \lim_{R\to\infty} W[R]} \quad .$$

This is the energy per unit time that is transported out to infinity, and never comes back.

The power radiated by a charge is actually the amount of energy lost by the charge in a **retarded time** interval $\mathrm{d}t_{\mathtt{R}}$ during the emission of the signal. There is a subtle reason why the rate at which energy per unit area passes through the sphere (at infinity), is not, in general, the same as the rate at which energy per unit area left the particle.

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In fact, let $\frac{\mathrm{d}W}{\mathrm{d}\Omega}$ be the energy per unit time per unit solid angle that passes through a large sphere of radius r. The rate at which energy per unit solid angle left the charge is related to it, in general, by the relation:

$$\frac{\mathrm{d}W}{\mathrm{d}t_{\mathrm{R}}} = \left(1 - \frac{(\mathbf{r}-\mathbf{x})\cdot\mathbf{u}}{|\mathbf{r}-\mathbf{x}|c}\right)\frac{\mathrm{d}W}{\mathrm{d}t} \quad , \tag{49.10.01}$$

→ 2248

with $\mathbf{r}$ the location of the point on the sphere at a large distance r from the source of radiation and $\mathbf{x}$ is the position of the source of radiation moving with velocity $\mathbf{u}$.

The factor relating the two powers per unit solid angle is just a geometrical factor, the same which comes out in **Doppler effect** .

49.10.01
Relativistic Emitted Power

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 11.2||

The generalization of Larmor formula, equation (49.05.11) requires a clear distinction between the rate at which energy passes through the large sphere, the rate at which energy left the particle, according to equation (49.10.01).

49.10.02
Lorentz Transformation of the Power Angular Distribution

©|T.Padmanabhan, *Theoretical Astrophysics, Vol. 1,2,3*, 2000/2001/2002, CUP, ...Ed., ...|1.4.3||

Consider a set of charged particles moving as a whole with velocity $\mathbf{u}$. In the Rest Frame the emitted ElectroMagnetic radiation will have some angular distribution, $\frac{d^2E'}{\mathrm{d}t'\,\mathrm{d}\Omega'}$. One might be interested in the angular distribution in the Laboratory frame, $\frac{d^2E}{\mathrm{d}t\,\mathrm{d}\Omega}$. Consider the special **Lorentz Transformation** between the Laboratory frame, $\mathcal{I}$, and the Rest Frame of the system, $\mathcal{I}'$, moving with velocity $\mathbf{u}$ with respect to the Laboratory frame.

The relativistic aberration law gives:

$$\cos\theta' = \frac{\cos\theta - u}{1 - u\cos\theta} \qquad \psi' = \psi \quad .$$

The transformation laws are:

$$\begin{aligned}
\mathrm{d}\Omega' &= \frac{1}{\gamma^2\left(1-u\cos\theta\right)^2}\,\mathrm{d}\Omega \quad , \\
\mathrm{d}E' &= \gamma\,\mathrm{d}E\left(1-u\cos\theta\right) \quad , \\
\mathrm{d}t_{\mathrm{R}} &= \mathrm{d}t/\gamma \quad .
\end{aligned}$$

It follows:

$$\boxed{\frac{d^2E}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{1}{\gamma^4\left(1-u\cos\theta\right)^3}\frac{d^2E'}{\mathrm{d}t_{\mathrm{R}}\,\mathrm{d}\Omega'}} \quad .$$

DRAFT – IN PROGRESS

49.11 Examples and Physical Applications

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.34|*Examples and applications*|

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|4.3; 4.4|*Examples ...*

49.11.01 ✓ Classical Bohr atom catastrophe

This § is referenced at pages:
[Never referenced.]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 11.14||

©|WEB - URL |||

In Bohr theory of hydrogen, the electron in its ground state was supposed to travel in a circle of radius $a_0 = 0.5 \cdot 10^{-10}$ m, held in orbit by the Coulomb attraction of the proton. According to classical electrodynamics, this electron should radiate, and hence spiral into the nucleus.

1. Show that $v \ll c$ for most of the trip.
2. Calculate the lifespan of Bohr atom using Larmor formula, assuming that each revolution is essentially circular.

49.11.02 ✓ Charged Point Particle in a Uniform and Constant Magnetic Field

The motion of a charged point particle in a uniform and constant magnetic field is a uniform circular motion with axis parallel to the magnetic field. If the particle is losing energy the particle is actually spiraling inward. The motion has a constant velocity (possibly zero) along the direction of the magnetic field but this component of the motion has no acceleration and therefore does not contribute to ElectroMagnetic radiation and can be ignored in the following discussion. The motion in the plane perpendicular to the magnetic field is a uniform circular motion (if the energy is constant), with centripetal acceleration contributing to ElectroMagnetic radiation. Note that the charge in uniform circular motion has a **Magnetic Dipole** moment. However, there is no emission of **Magnetic Dipole** radiation because the magnetic Dipole moment is a constant one.

49.11.02.01 Cyclotron Radiation

Cyclotron radiation is the ElectroMagnetic radiation emitted by a particle moving, at non relativistic velocities, in a uniform circular motion. The name comes from cyclotrons, a kind of particle accelerator, where charged particles move at non relativistic velocities, in a uniform circular motion.

The emitted radiation is at any time a **Electric Dipole** radiation with axis along the radius from the center to the instantaneous position of the particle. Emission is maximum tangentially to the orbit and it is symmetrical with respect to the plane of the orbit.

Due to the features of **Electric Dipole** radiation an observer at large distances (with respect to the orbit radius) will see:

- circularly wave-Polarized radiation with constant intensity, if the observer is along the axis of the orbit (along the magnetic field, when moving from the center of the orbit);
- linearly wave-Polarized radiation with variable intensity, if the observer is in the plane of the orbit (perpendicularly to the magnetic field, when moving from the center of the orbit);
- the observer will see elliptic **wave-Polarization** at any intermediate location.

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The frequency of the emitted radiation, for this periodic motion, is the same cyclotron frequency of the motion of the particle. In fact the generator of ElectroMagnetic radiation is just equivalent to two **Electric Dipoles** located at the center of the orbit and directed at $\pi/2$ rad on to the other, both lying in the plane of the magnetic field and with frequency given by the cyclotron frequency which emit exactly at that frequency.

49.11.02.02 Synchrotron Radiation

Synchrotron radiation is the ElectroMagnetic radiation emitted by a particle moving, at relativistic velocities, in a uniform circular motion. The name comes from the synchrotron, a kind of particle accelerator, where charged particles move at relativistic velocities, in a uniform circular motion.

At relativistic velocities the ElectroMagnetic radiation emission is directed along the velocity of the particle, in the forward direction, inside a cone of half-angle $\Delta\theta \approx 1/\gamma = \sqrt{1-\beta^2}$. That is the emission is concentrated near the velocity $\mathbf{v}$ of the particle, only in the forward direction. An observer in the plane of the orbit will see periodically a bunch of synchrotron radiation, at periodic intervals given by the orbital period.

The concentration of the ElectroMagnetic radiation in the forward direction has significant effects on the energy spectrum. In fact the observed radiation is now an impulsive one. Therefore the radiation spectrum is enlarged to a large range of frequencies and it is not centered any more around the frequency of the circular motion. The spread of frequencies is roughly given by the inverse of the time duration of the pulse and it is roughly given by: $\omega_{\text{M}} = \omega_c \gamma^2$.

49.11.03

✓ Bremsstrahlung

It is the term denoting in general EM radiation emitted by a charged particle under acceleration. The word is a German word for braking radiation. It is in particular used referring to the ElectroMagnetic radiation caused by accelerations of charged particle when passing through the field of another particle, typically an atomic nucleus.

The energy emitted by an accelerated particle is, non relativistically, proportional to m^{-2}, with m the mass of the particle. Bremsstrahlung therefore plays a particularly important role for light particles.

A typical example is X-ray bremsstrahlung from interaction between electrons and nuclei. In this case the classical calculation is easily done by computation of the electron trajectory in the Coulomb field of the nucleus.

Another example is the bremsstrahlung produced in the plasma of the solar corona from free electrons scattering off the free ions.

Other astrophysical examples exist.

49.11.04

Linear/Angular Momentum Radiated by a Non-Relativistic Point Charge

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 11.27||

The energy per unit time radiated by a non-relativistic point charge is given by the Larmor formula.

1. The **linear momentum radiated per unit time** is:

$$\frac{\mathrm{d}\mathbf{p}_{\text{RAD}}}{\mathrm{d}t} = \frac{\mu_0 q^2}{6\pi c^3} a^2 \mathbf{v} \quad . \tag{49.11.01}$$

2. The **angular momentum radiated per unit time** is:

$$\frac{\mathrm{d}\mathbf{j}_{\text{RAD}}}{\mathrm{d}t} = \frac{\mu_0 q^2}{6\pi c} \mathbf{v} \wedge \mathbf{a} \quad . \tag{49.11.02}$$

DRAFT-IN-PROGRESS

49.11.05
Linear/Angular Momentum Radiated by a Moving Point Charge

©|R.Napolitano & S.Ragusa|A simple calculation of the rate of emission of energy and of linear and angular momentum by ...

The energy per unit time radiated by a non-relativistic point charge is given by the Larmor formula.

1. The **linear momentum radiated per unit proper-time** is:

$$\boxed{\frac{\mathrm{d}\mathbf{p}_{\mathrm{RAD}}}{\mathrm{d}\tau}=\frac{\mu_0 q^2}{6\pi c^3}\frac{\mathbf{a}^2-(\mathbf{a}\wedge\boldsymbol{\beta})^2}{(1-\beta^2)^3}\boldsymbol{\beta}} \quad . \tag{49.11.03}$$

2. The **angular momentum radiated per unit proper-time** is:

$$\boxed{\frac{\mathrm{d}\mathbf{j}_{\mathrm{RAD}}}{\mathrm{d}\tau}=\frac{\mu_0 q^2}{6\pi c}\frac{\boldsymbol{\beta}\wedge\mathbf{a}}{1-\beta^2}} \quad . \tag{49.11.04}$$

49.11.06
Radiated fields from a time-varying electric dipole at fixed location

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 10.34||

In the Lorenz gauge, find the potentials and fields of a time-dependent ideal electric dipole at the origin. It is stationary, but its magnitude and/or direction are changing with time. Ignore the contact term.

DRAFT-IN-PROGRESS

49.12 Exercises Problems and Physical Applications

49-002 ✓ **I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...**

4.0210, 4.0212, 4.0213, 4.0217, 4.0218, 4.0221, 4.0222, 4.0223.

49-003 Radiation Fields From A Non Relativistic Distant Charge

Simplify the vector operations of equation (49.05.09) to express in an alternative way the direction of the fields, by using equation (15.06.10).

49-004 Radiating Infinite Neutral Plane Current Sheet (1)

This § is referenced at pages:
[2160]

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|34.5||

Show that an infinite neutral plane current sheet at $z=0$ with surface current $\mathbf{j}_S[x,y,t]=\mathbf{j}_{S0}\cos[\omega t]$, where $\mathbf{j}_{S0}$ is a fixed vector in the xy plane, produces ElectroMagnetic MPW propagating in the $\pm z$ direction.
Show that the **wave-Polarization** of these ElectroMagnetic MPW is along $\mathbf{j}_{S0}$.

WORKED SOLUTION

Choose the direction of $\mathbf{j}_{S0}$ as the x direction.
There can be no dependence on x and y but only on z (measuring the signed distance from the infinite sheet of current), so that partial derivatives with respect to x and y. As a consequence, from the two Maxwell rotor equations at $z\neq 0$, any component of the Electric|Magnetic field along z, if it exists, is uniform and constant.
In fact, using the two Maxwell rotor equations at $z\neq 0$:

$$\frac{\partial()}{\partial x}=\frac{\partial()}{\partial y}=0 \quad ,$$
$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{\partial E_y}{\partial z}=-\partial_t B_x \quad ,$$
$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=+\frac{\partial E_x}{\partial z}=-\partial_t B_y \quad ,$$
$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=0=-\partial_t B_z \quad ,$$
$$\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}=-\frac{\partial B_y}{\partial z}=+\frac{1}{c^2}\partial_t E_x \quad ,$$
$$\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}=+\frac{\partial B_x}{\partial z}=+\frac{1}{c^2}\partial_t E_y \quad ,$$
$$\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=0=-\partial_t E_z \quad .$$

Assume MPW along $\pm z$, as required by symmetry.
At $z=0$ the appropriate boundary conditions (equation (36.09.04)) must be satisfied:

$$\hat{\mathbf{n}}\wedge\Delta\mathbf{B}_{\parallel}=\hat{\mathbf{n}}\wedge\Delta\mathbf{B}=\mu_0\mathbf{j}_S \quad ,$$

with all the other boundary conditions automatically satisfied by the present choice of the Electro-

Magnetic fields.
Therefore, immediately near the current sheet, on the two sides, one has the same electric field vector, at any point and any time, opposite wave-vector and opposite magnetic field vector, at any point and any time.
The conditions on the discontinuity of the magnetic field implies that the magnetic field vector is perpendicular to the current $\mathbf{j}_{s0}$, that is along y. Moreover, its module must be:

$$B_0 = \mu_0 |\mathbf{j}_{s0}|/2 \quad .$$

Therefore one can assume $E_x \neq 0$ and $E_y = 0$:

$$E_x = E_{x0} \cos[kz - \omega t - \phi] \qquad E_y = 0 \quad .$$

The above **Maxwell Equations** imply $B_x =$ constant and

$$-\frac{\partial B_y}{\partial z} = +\frac{1}{c^2}\partial_t E_x \implies B_y = \dots \quad .$$

Read § 36.20.06 and § 47.03.01.

49-005 Radiating Infinite Neutral Plane Current Sheet (2)

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 11.28||

Derive the ElectroMagnetic wave produced by the infinite plane current sheet in case of:
1. a uniform constant surface current, switched on at some time;
2. a uniform surface current, linearly increasing with time, switched on at some time;
3. a time-harmonic surface current, switched on a long time ago.

Read § 36.20.06 and § 47.03.01.

49-006 Velocity Field

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||

©|G.S.Smith, *An Introduction To Classical Electromagnetic Radiation*, 1997, CUP, ...Ed., ...|||

Show that the velocity field derived in section § 61.09.06 are equivalent to the velocity field in section § 49.06.03.

49-007 ✓ Energy Loss of a Charged Particle in a Circular Uniform Motion

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|4.216||

49-008 EM Energy Loss per Turn in a Synchrotron

Consider a synchrotron with orbit radius equal to ρ where independent charged point particles are circulating with an orbital period T. Calculate the energy loss per particle per turn.

WORKED SOLUTION

Use **Larmor Radiation Formula** in equation (??), exploiting the fact that for a circular orbit $\mathbf{v}\cdot\mathbf{a} = 0$. Remind that the acceleration for a uniform circular motion is v^2/ρ in relativistic kinematics as well as in non-relativistic one. Note that the dynamics will not enter, as we are just interested to know the emitted power from a given kinematics, and we don't care about the force which is providing

E.T.E.P. Project

the kinematics. Therefore, omitting the **retarded time** :

$$-W_q[t_R] = \frac{q^2}{6\pi\varepsilon_0 c^3}\Big[\gamma^4 \mathbf{a}^2\Big]_{t_R} \quad ,$$

$$-\Delta E_q^{\text{turn}} = -W_q T = \frac{q^2}{6\pi\varepsilon_0 c^3}\gamma^4 \mathbf{a}^2 T = \frac{q^2}{6\pi\varepsilon_0 c^3}\gamma^4 \frac{v^4}{\rho^2}\frac{2\pi\rho}{v} = \frac{q^2}{3\varepsilon_0 c^3}\frac{v^3\gamma^4}{\rho} \quad .$$

49-009 EM Energy Loss in Circular Versus Linear Accelerators

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|14.2||

Compare form the point of view of energy losses linear versus circular (synchrotron) accelerators.

49-010 Isotropic Emitter

Show that in the case of an isotropic emitter, in the Rest Frame, one can easily calculate the power in the Laboratory frame, $\frac{\mathrm{d}E}{\mathrm{d}t}$, and obtain, as expected from the Lorentz invariance of the emitted power,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E'}{\mathrm{d}t'} \quad .$$

49-011 Electric Quadrupole Radiation

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 11.24||

49-012 Earth as a magnetic dipole radiator

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 11.25||

49-013 Power radiated by a pulsar

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 11.25||

49-014 Charge Continuity And Lorenz Condition For Potentials

This § is referenced at pages:
[2220]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 10.10||

1. Consider equation (49.01.01), (49.01.02) and show that the Charge continuity relations is consistent with the Lorenz **gauge fixing condition** .
2. Confirm that the retarded potentials satisfy the Lorenz **gauge fixing condition** .

DRAFT-LN-PROGRESS

CHAPTER 50

Interference

50.01 Interference (And Diffraction) . . . 2257
50.02 Optical Path Length . . . 2263
50.03 Localization and Visibility of the Interference Fringes . . . 2264
50.04 Interference of ElectroMagnetic MPW . . . 2265
50.05 Interference Among Monochromatic Point Sources of Spherical Waves . . . 2268
50.06 Interference in the Generalized Young Experiment . . . 2279
50.07 Interference From Bi-Dimensional Sources . . . 2280
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50.09 Approximations In Interference and Realistic Interference situations . . . 2282
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DRAFT-IN-PROGRESS

About interference, that is the key properties of wave-fields, for a countable number of sources.

This § is referenced at pages:
[2016, 2021, 2212, 2257, 2292, 2292]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|8|*Good text.*|
©|E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...||*Very detailed*|
©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.29||
©|J.P.Pérez *et al.*, *Optique*, ..., DUNOD, ...Ed., WEB - URL.|||

Pre-requisites - Review properties of sum of sinusoidal functions with the same frequency but arbitrary amplitudes and phases, read **§** 19.02.

SHEET-IN-PROGRESS

50.01 Interference (And Diffraction)

This § is referenced at pages:
[2257, 2292, 2292]

Interference demonstrates the **wave-like character** of a phenomenon.

It is specific of the wave-like character of wave-phenomena that the combined effect of two or more waves (sources) may produce no signal (no effect at all) in a region where one single wave (source) would have produced some signal (effect). Thus the spatial distribution of energy is changed by interference phenomena, which cause a **rearranging/redistribution** in space of the energy density/flux.

Interference is basically an effect derived from the linearity of the equations (**superposition principle**) and from the fact that energy density/flux is typically given by the **square of the sum of the amplitudes**. Thus interference comes from the fact that on squaring a sum of amplitudes cross-products appear.

Interference applies to **any kind of wave phenomenon**.

Quantum physics shows that interference phenomena also apply to particles.

Interference leads to a **redistribution of energy in space**, with respect to the location where energy would be with single sources.

Since now on we will consider cases when either two or more different waves have **different phases at the source** and|or they reach an observation point following, possibly, **different paths**: the problem, most often, is not just a one-dimensional problem. This introduces peculiar effects due to the different paths followed by the different waves from the source to the observation point.

Interference arises when the sources form either a finite or a countable infinite set of finite-intensity (that is non zero and non infinite) sources: it is discussed in section § 50. The mathematical model, based on the linear superposition principle, corresponds to calculate interference effects by means of either finite sums or infinite series. Interference, is a useful preliminary to the study of diffraction, and, in addition is an important topic in itself.

Diffraction arises when the sources form an infinite continuum of infinitesimal-intensity sources: it is discussed in section § 51. The mathematical model, based on the linear superposition principle, corresponds to calculate diffraction effects by means of integrals. In this sense it is only **technically, but not conceptually, different from interference** and the conceptual discussion made in section § 50.01.

In this section, as in the forthcoming § 51 on diffraction, unless specified otherwise, the discussion will be limited to observations at **very large distances from the sources**, in such a way that: in the case of longitudinal waves, the interfering waves are just parallel waves and can be summed as longitudinal components; in the case of transverse waves, the fields lie in the plane perpendicular to the wave-vector and can be summed as components as long as they have the same **wave-Polarization** , or must be summed as 2D vectors in the perpendicular plane, otherwise. Note that **sum the Perturbation|Disturbance|Signals** means to sum the components along any fixed direction.

The very large distances from the sources **is implicit in the so-called Fraunhofer regime for interference/diffraction**.

It is always assumed that different sources do not interact nor affect each other, so that they are **truly independent sources**. This might not be the case in practice but, in this case, no progress is possible without modeling the interactions among sources|fields.

The case of observations near the sources and|or superposition of waves traveling in different directions is considerably more complex from the computational point of view.

Our concerns are only with the classical limit, where **many photons are present simultaneously and their associated ElectroMagnetic fields can be described by Maxwell equations**.

While interference and diffraction are phenomena proper to any kind of waves, the discussion will be limited to ElectroMagnetic waves, due to their huge interest.

WORK IN PROGRESS

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Until coherence is presented, in section § 52, **all sources are assumed to be strictly monochromatic point sources**. Therefore, they appear as generators of spherical (possibly non isotropic) waves. At large enough distance, waves appear as MPW.

50.01.01
Some Interference Phenomena and Applications

Some common phenomena and applications of interference are the following:

- the colored fringes produced by a thin film of oil floating on water and in soap bubbles are due to interference;
- the colored fringes when light is reflected by a CD/DVD;
- anti-reflection coatings on lenses and mirrors exploit interference;
- the classical experiment showing interference is Young experiment;
- radio jamming;
- coherent constructive interference occurs in reflection from a smooth surface when the angle of reflection equals the angle of incidence but it does not occur in reflection when the angle of reflection is different from the angle of incidence when, in fact, coherent destructive superposition is in action.
- ...

50.01.02
Some Toy Examples

This § is referenced at pages:
[2260, 2268, 2268, 2269, 2270]

Interference, is in a nutshell, the fact that the intensity is typically proportional to the time-average of the square of the total field; the total field is the linear superposition of the different fields (for linear systems) and therefore the cross-products show up in the intensity formula.

50.01.02.01 Scalar MPW With the Same Frequencies in 1D

This § is referenced at pages:
[2271, 2271]

Consider two different scalar-fields MPW propagating along the z direction, with the same frequency and **phase-velocity** :

$$\psi_1[z,t] = \psi_{1|0}\cos\left[kz-\omega t-\phi_1\right] \qquad \psi_2[z,t] = \psi_{2|0}\cos\left[kz-\omega t-\phi_2\right] \quad . \tag{50.01.01}$$

→ 2258

For the sake of generality one might assume that the **phase constants** , ϕ_1 and ϕ_2, might be time dependent, but approximately constant on time scales much longer than the oscillation period, $T \equiv \nu^{-1}$:

$$\phi_i = \phi_i[t] \qquad \text{piece-wise constant for time-intervals } \Delta t \gg T \equiv \nu^{-1} \quad . \tag{50.01.02}$$

Let us calculate the intensity, to within a proportionality constant which is irrelevant in the forthcoming derivation. In fact all the results will be expressed in terms of the intensities and any proportionality constant might be included in the definition of the field (50.01.01)

The intensity, after averaging over an integer number of periods, is thus given, to within a proportionality constant, by:

$$I = \langle\psi^2[z,t]\rangle \propto \frac{1}{2}\psi_0^2 \qquad \psi_0 \propto \sqrt{2I} \quad ,$$

that is independent of the position z.

The **superposition principle** implies that, in the case of **two independent sources which do not affect each other**, the intensity is calculated by first summing the fields, squaring afterwards and finally taking the time-average, to obtain:

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$$
\begin{aligned}
I &= \langle(\psi_1[z,t]+\psi_2[z,t])^2\rangle \quad , \\
&= I_1 + I_2 + 2\langle\psi_1[z,t]\psi_2[z,t]\rangle \quad , \\
&= I_1 + I_2 + 4\sqrt{I_1}\sqrt{I_2}\langle(\cos[kz-\omega t-\phi_1]\cos[kz-\omega t-\phi_2])\rangle \quad , \\
&= I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2}\langle(\cos[2kz-2\omega t-(\phi_1+\phi_2)]+\cos[\phi_2-\phi_1])\rangle \quad .
\end{aligned}
$$

When taking into account the hypothesis that the **phase constants** are piece-wise constant for time-intervals $\Delta t \gg T \equiv \nu^{-1}$ one obtains:

$$\langle\cos[2kz-2\omega t-(\phi_1+\phi_2)]\rangle = 0 \quad .$$

Finally it follows:

$$\boxed{I = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2}\langle\cos[\phi_2-\phi_1]\rangle \qquad \text{time-averaged energy flux}} \quad .$$

The term

$$\boxed{2\sqrt{I_1}\sqrt{I_2}\langle\cos[\phi_2-\phi_1]\rangle} \quad ,$$

is called the interference term. Its presence might give rise to either **increased or decreased total intensity**, when compared to the sum of the intensities of the single sources, depending on the quantity

$$\boxed{\langle\cos[\phi_2-\phi_1]\rangle \equiv \langle\cos[\Delta\phi]\rangle \equiv \langle\cos[\phi_1-\phi_2]\rangle} \quad ,$$

which determines the interference characteristics of the phenomenon.

By definition we have interference if and only if:

$$\boxed{I \neq I_1 + I_2 \qquad \Leftrightarrow \qquad \langle\cos[\Delta\phi]\rangle \neq 0} \quad .$$

Interference just moves the distribution is space of the energy.

The limiting cases are the following:

$$\langle\cos[\Delta\phi]\rangle = \begin{cases} +1 \implies I = I_{\text{MAX}} = \left(\sqrt{I_1}+\sqrt{I_2}\right)^2 \quad , & (= 4I_0 \quad \text{for } I_1 = I_2 \equiv I_0\ ) \quad , \\ 0 \implies I = I_{\text{AVE}} = I_1 + I_2 \quad , & (= 2I_0 \quad \text{for } I_1 = I_2 \equiv I_0\ ) \quad , \\ -1 \implies I = I_{\text{MIN}} = \left(\sqrt{I_1}-\sqrt{I_2}\right)^2 \quad , & (= 0 \quad \text{for } I_1 = I_2 \equiv I_0\ ) \quad . \end{cases}$$

→ 2264

Consider, as an example, the simple case $\psi_{1|0} = \psi_{1|0} = \psi_0$; one has:

$$\begin{cases} \Delta\phi = 0 \implies \psi_1[z,t] + \psi_2[z,t] = 2\psi_1 = 2\psi_2 \quad , \\ \Delta\phi = \pm\pi/2 \implies \text{Cosinusoid of amplitude } \sqrt{2}\psi_0 \quad , \\ \Delta\phi = \pm\pi \implies \text{zero} \end{cases} \quad .$$

When the intensity is increased, with respect to the sum of the intensities of the individual sources, one speaks of **constructive interference**. When the intensity is decreased, with respect to the sum of the intensities of the individual sources, one speaks of **destructive interference**.

Two sources having a fixed interference factor, $\langle\cos[\Delta\phi]\rangle$, are said to be **mutually coherent sources** (section § 52.02).

Two sources having an interference factor equal to zero, $\langle\cos[\phi_2-\phi_1]\rangle \equiv \langle\cos[\Delta\phi]\rangle = 0$, are said to be **mutually incoherent sources** (section § 52.02).

50.01.02.02 Scalar MPW With the Same Frequencies in 3D

This § is referenced at pages:
[2267]

In a general 3D configuration, the phase difference at any point in space comes both from the **intrinsic phase difference of the sources** and the possible **difference of phase gained by the two waves when traveling along different paths** from the sources to the observation point. The latter might be both geometrical and|or due to different properties of the crossed media.

Consider two different scalar-fields MPW propagating in 3D, with wave-vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, with the same frequency and **phase-velocity** :

$$\psi_1[\mathbf{r},t] = \psi_{1|0}\cos[\mathbf{k}_1\cdot\mathbf{r}-\omega t-\phi_1] \qquad \psi_2[\mathbf{r},t] = \psi_{2|0}\cos[\mathbf{k}_2\cdot\mathbf{r}-\omega t-\phi_2] \quad . \tag{50.01.03}$$

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One finds:

$$I = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2}\cos\left[(\mathbf{k}_2 - \mathbf{k}_1)\cdot\mathbf{r} - (\phi_2 - \phi_1)\right] \quad .$$

Note that, for vector waves, the computations might be more complex, when accounting for the directions of the vectors, read § 50.04.

The minima and maxima of intensity are located in parallel planes, perpendicular to the difference of the wave-vectors, $\mathbf{k}_\Delta \equiv \mathbf{k}_2 - \mathbf{k}_1$, and separated by the distance $2\pi/|\mathbf{k}_2 - \mathbf{k}_1|$.

In case the minima and maxima of intensity are observed on an arbitrarily oriented observation plane, the fringes can be described in terms of a fringe vector $\mathbf{k}_f$ that is the projection of $\mathbf{k}_\Delta$ into the observation plane whose normal is $\hat{\mathbf{n}}$:

$$\mathbf{k}_f \equiv \hat{\mathbf{n}}\wedge(\mathbf{k}_\Delta\wedge\hat{\mathbf{n}}) = \mathbf{k}_\Delta - (\mathbf{k}_\Delta\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}} \equiv \frac{2\pi}{\Lambda}\widehat{\mathbf{k}_f} \qquad \Lambda \equiv \frac{2\pi}{|\mathbf{k}_f|} \qquad \mathbf{k}_\Delta \equiv \mathbf{k}_2 - \mathbf{k}_1 \quad . \tag{50.01.04}$$

→ 2283

The modulated intensity bands are called interference fringes.

The fringe vector, $\mathbf{k}_f$, describes both the direction of the fringes, and the fringe period Λ

50.01.02.03 Vector MPW With the Same Frequencies in 3D

A less trivial example, exactly calculated at large distances is given by equation (50.05.01) in section § 50.05.

50.01.02.04 Scalar MPW at Different Frequencies

Consider again the same 1D tool example of § 50.01.02 of two different one-dimensional scalar-field MPW, with **phase constants** , ϕ_1 and ϕ_2, piece-wise constant on time scales much longer that the oscillation period $T \equiv \nu^{-1}$, but assume now that the two waves have different frequencies. The wave-vectors will be, a priori, different. Therefore:

$$\psi_1[z,t] = \psi_{1|0}\cos\left[k_1 z - \omega_1 t - \phi_1\right] \qquad \psi_2[z,t] = \psi_{2|0}\cos\left[k_2 z - \omega_2 t - \phi_2\right] \quad .$$

Now the interference term in the intensity s:

$$\propto \langle\cos\left[(k_1 + k_2)\,z - (\omega_1 + \omega_2)\,t - (\phi_1 + \phi_2)\right] + \cos\left[(k_1 - k_2)\,z - (\omega_1 - \omega_2)\,t - (\phi_1 - \phi_2)\right]\rangle = 0 \quad ,$$

averaging to zero, because the time-averaging now is done on two co-sinusoids, while in the previous case the equality of the frequencies left one time-constant term: $\langle\cos\left[\phi_2 - \phi_1\right]\rangle \equiv \langle\cos\left[\Delta\phi\right]\rangle$.

Clearly the same result applies for Scalar MPW at Different Frequencies in 3D.

Therefore, there is **no interference among MW at difference frequencies**: the intensity is just the sum of the two intensities at any location and time.

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50.01.02.05 Scalar Time-Harmonic Waves at different Frequencies in 3D

In the general case, with time-harmonic waves traveling along arbitrary paths in 3D, what affects the existence of interference is the phase difference of the two paths from their respective fixed arbitrary starting points to the point where the superposition of the fields is considered. Therefore, interference is affected by:

- the phases at the starting points;
- the difference of paths in terms of difference of phase change, the optical path, read § 50.02.

50.01.03 Necessary Conditions for Interference to Occur

This § is referenced at pages:
[2293]

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Consider time-harmonic, that is monochromatic waves., MW. In order for interference to occur between two MW the following conditions must apply as **necessary, albeit not sufficient, conditions**.

- The two MW must have **almost the same frequency**. In fact, if two waves are to interfere producing a stable interference pattern, they must have very nearly the same frequency. Otherwise any significant frequency difference would produce a rapidly varying, time-dependent phase difference, which would cause the interference term in the intensity to average to zero during the observation. This is to be compared to the discussion about beats in section § 44.03.11, where it is shown that the closer the two frequencies, the slower are varying the intensity patterns.
- If both sources have a non-monochromatic spectrum the **components at different frequencies will interfere with the waves of the same frequencies** produced by the other source. As a consequence a great many of fairly similar and slightly displaced overlapping monochromatic interference patterns will tend to wash-out the interference pattern. The interference pattern will not be as sharp nor as extensive as an almost monochromatic interference pattern, but some observable interference may be present.
- The two MW must have **the same wave-Polarization** (if transverse waves are involved), or, to be more precise, have non orthogonal **wave-Polarization** , to have some interference.
- The clearest interference pattern (that is the pattern with maximum contrast) exist when the interfering waves have equal or nearly equal amplitudes. In fact, in this case, the central regions of the dark|bright interference fringes correspond to almost complete destructive|constructive interference, respectively, thus yielding maximum contrast of the interference pattern.
- The two sources must be **mutually coherent** (read § 52.02). In fact, for an interference pattern to be observed, the two sources need to have a constant phase-difference, for times long enough with respect to the averaging time. Actually interference patterns shifted in some way will occur whenever there is some phase difference between the sources, as long as it remains constant. Such sources are called **coherent sources**.

The previous conditions are necessary in order to obtain an intensity (irradiance) which is not the sum of the intensities (irradiance) of the two MW.

Interference arises from two different origins:

- a **different phase-constant** of the involved sources;
- a **difference of phases gained between the two sources due to different paths** from the source to the observation point.

Note that, as far as interference is concerned, **what is really important is the time-average of the interference factor**.

In many cases one might actually have interference fringes which changes so quickly, and possibly randomly, such that one observes, in practice, a uniform illumination unless one has a detector capable

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to follow the fast changes.

In general, especially for non scalar fields and observations not far-away from the sources, the addition of the waves might be something more complex than the simple scalar sum envisaged in these tool examples.

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50.02 Optical Path Length

This § is referenced at pages:
[2260, 2359]

Consider any monochromatic ElectroMagnetic wave. The **optical path length** (**OPL**) between two points in a isotropic linear medium is defined as proportional to the travel time:

$$\Lambda \equiv \int_C n[\mathbf{x}]\, \mathbb{d}s = c^2 \int_C \frac{\mathbb{d}t}{v[\mathbf{x}]} \quad . \tag{50.02.01}$$

It is the key quantity, determining the conditions for constructive|destructive interference. In the simplest case, the optical path length is just the change of phase of a MPW

See its use in section § 55.02.

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50.03 Localization and Visibility of the Interference Fringes

We mostly limit the discussion to fringes forming at infinity (Fraunhofer regime) so that some kind of instrument is required to converge the rays into the same point in order for interference to occur.

50.03.00.01 Observation of Haidinger Fringes

In all the cases presented, the superposition of the waves to provide interference takes place, strictly speaking, at infinity because the superposition is assumed to take place at large distance among waves produced by different sources running parallel to each other but, in general, along different straight lines.

An optical instrument capable to let parallel rays converge at a point without adding any relative phase shift (read § 55 for light) is practically required to cause interference. In practical cases, for light, the human eye can do this, allowing to see interference fringes from the interference on the retina of incident parallel rays.

Technical details and realistic configurations, for light, are the subject of **Geometrical Optics | Ray Optics** (read § 55). Here it will be just assumed that such a device exists and is being used. Typically, for light, a suitable lens works well: a lens which makes all parallel rays converge into the same point keeping the same optical path, that is to say without adding any phase-shift.

50.03.00.02 Contrast of the Interference Fringes

It is important to emphasize that, when superposing two different waves, a phase difference at some point might arise from different distances/paths from the sources. Therefore the phase difference, $\Delta\phi$, might be position dependent. This gives rise to the so-called interference fringes: different light intensity at different points in space.

The contrast between the interference fringes is measured by the visibility factor, $\mathcal{V}$, (that is the contrast) which can be expressed via equation (50.01.02.01) as:

$$\text{contrast} \equiv \mathcal{V} \equiv \frac{I_{\texttt{MAX}} - I_{\texttt{MIN}}}{I_{\texttt{MAX}} + I_{\texttt{MIN}}} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} \leq 1 \quad .$$

The maximum visibility factor, $\mathcal{V} = 1$, is obtained when $I_1 = I_2$.

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50.04 Interference of ElectroMagnetic MPW

This § is referenced at pages:
[2260]

Fresnel and Arago demonstrated that **orthogonally wave-Polarized ElectroMagnetic waves never interfere**, confirming the transversal character of ElectroMagnetic waves.

50.04.01 EM MPW Propagating Along the Same Direction

Consider an arbitrary number of different ElectroMagnetic MPW propagating along the z direction, with the same frequency and **phase-velocity** . That is: all the ElectroMagnetic MPW have the same wave-vector and frequency but arbitrary amplitudes and phases.

50.04.01.01 EM MPW in the Same Direction With the Real Formalism

First Analysis

The electric field components along the x and y directions will be the sum of the x and y components of all the ElectroMagnetic waves. These are all Sinusoidal|Cosinusoidal functions with the same frequency (but arbitrary amplitudes and phases) and therefore the sum is still a Sinusoidal|Cosinusoidal function with the same frequency, and a suitable amplitude and phase, according to section § 19.02. The electric field will be therefore the electric field of a general ElectroMagnetic MPW propagating in the z direction (read § 46.03). That is the electric field will have in general both x and y components, with arbitrary amplitudes and phases. The magnetic fields, along the x and y directions, will be given, thanks to **Maxwell Equations** , by:

$$\mathbf{B} = \frac{\mathbf{k} \wedge \mathbf{E}}{\omega} .$$

Therefore the resulting ElectroMagnetic wave will be a generic ElectroMagnetic MPW propagating along the z direction, with generic (in general elliptic) **wave-Polarization** ($\mathbf{k}$ may be complex).

Second Analysis

The resulting Electric|Magnetic field is the sum of all the Electric|Magnetic fields.

The magnetic field of each component ElectroMagnetic MPW is given by

$$\mathbf{B}_j = \frac{\mathbf{k} \wedge \mathbf{E}_j}{\omega} ,$$

and the frequency and wave-vector of every ElectroMagnetic MPW are the same.

Therefore, as also the resulting magnetic field is the sum of all the magnetic fields, the total magnetic field can be written as:

$$\mathbf{B} = \frac{\mathbf{k} \wedge \mathbf{E}}{\omega} .$$

Therefore the resulting ElectroMagnetic wave will be a generic ElectroMagnetic MPW propagating along the z direction, with generic (in general elliptic) **wave-Polarization** ($\mathbf{k}$ may be complex).

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50.04.01.02 EM MPW in the Same Direction With the Complex Formalism

A ElectroMagnetic MPW, with arbitrary **wave-Polarization** , can be written with the complex formalism as:

$$\overline{\mathbf{E}}[z,t] = \overline{\mathbf{E}_0}\exp\left[\mathtt{i}\,(kz-\omega t)\right] \quad . \tag{50.04.01}$$

→ 2266

The complex constant vector $\overline{\mathbf{E}_0}$ contains all the information about amplitudes and phases of the x and y components, that is about the state of **wave-Polarization** of the ElectroMagnetic wave. It fully describes a ElectroMagnetic MPW of a given frequency and wave-vector.

The superposition of many ElectroMagnetic MPW along the z direction can be done easily with the complex formalism and without any problem because it is a linear operation. The superposition of N waves is just obtained by summing the $\overline{\mathbf{E}_0}$ complex vectors of all the waves, because the exponential factors in equation (50.04.01) factorize:

$$\boxed{\overline{\mathbf{E}}[z,t] = \left(\overline{\mathbf{E}}_{0|1} + \overline{\mathbf{E}}_{0|2} + \ldots + \overline{\mathbf{E}}_{0|N}\right)\exp\left[\mathtt{i}\,(kz-\omega t)\right]} \quad .$$

50.04.01.03 No Interference Between ElectroMagnetic Waves With Orthogonal wave-Polarization

This § is referenced at pages:
[Never referenced.]

The relation (50.04.02) shows that if one superimposes two ElectroMagnetic MPW with orthogonal **wave-Polarization** the resulting intensity will be the sum of the two intensities: two ElectroMagnetic MPW with orthogonal **wave-Polarization** do not interfere.

In fact two ElectroMagnetic MPW propagating in the same direction have orthogonal **wave-Polarization** if and only if:

$$\boxed{\left(\overline{\mathbf{E}}_1\right)\cdot\left(\overline{\mathbf{E}}_2\right)^\star = \left(\overline{\mathbf{E}}_1\right)^\star\cdot\left(\overline{\mathbf{E}}_2\right) = 0} \quad .$$

The fact that two EM MPW with orthogonal **wave-Polarization** do not interfere can be also shown directly as follows.

Linear wave-Polarization

Consider two MPW propagating along the positive z direction, $\mathbf{k} = k\widehat{\mathbf{e}}_3$, linearly wave-Polarized, respectively, along the x and y direction. The Electric|Magnetic fields are given by:

$$\mathbf{E}_1 = E_{0|1}\cos\left[\mathbf{k}\cdot\mathbf{r} - \omega t - \phi_1\right]\widehat{\mathbf{e}}_1 \quad ,$$
$$\mathbf{E}_2 = E_{0|2}\cos\left[\mathbf{k}\cdot\mathbf{r} - \omega t - \phi_2\right]\widehat{\mathbf{e}}_2 \quad ,$$
$$\mathbf{H}_1 = \frac{\widehat{\mathbf{e}}_3\wedge\mathbf{E}_1}{Z} \propto \widehat{\mathbf{e}}_2 \quad ,$$
$$\mathbf{H}_2 = \frac{\widehat{\mathbf{e}}_3\wedge\mathbf{E}_2}{Z} \propto \widehat{\mathbf{e}}_1 \quad .$$

Note that the arbitrary relative phases of the x and y components of the electric field imply that the resulting ElectroMagnetic wave has an arbitrary **wave-Polarization** .

The resulting Poynting vector is:

$$\mathbf{S}_{\mathtt{EM}} = (\mathbf{E}_1+\mathbf{E}_2)\wedge(\mathbf{H}_1+\mathbf{H}_2) = \mathbf{E}_1\wedge\mathbf{H}_1 + \mathbf{E}_2\wedge\mathbf{H}_2 \quad ,$$

where the last equality follows from the fact that, thanks to the orthogonality of the **wave-Polarization** : $\mathbf{E}_1 \parallel \mathbf{H}_2$ and $\mathbf{E}_2 \parallel \mathbf{H}_1$. Therefore two waves linearly wave-Polarized in orthogonal directions do not interfere as the intensity (that is the average value of the Poynting vector) sum.

Note that Z might be complex and in this case it would be better to use the complex notation. However, in this problem only the directions of the vectors are relevant to the result.

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Note that the result might have been obtained in a easier way noting that, thanks to the fact the the two waves have identical wave-vectors, one can write:

$$\mathbf{H}_1+\mathbf{H}_2=\frac{\widehat{\mathbf{e}}_3\wedge\mathbf{E}_1}{Z}+\frac{\widehat{\mathbf{e}}_3\wedge\mathbf{E}_2}{Z}=\frac{\widehat{\mathbf{e}}_3\wedge(\mathbf{E}_1+\mathbf{E}_2)}{Z} \quad .$$

This implies:

$$I\propto(\mathbf{E}_1+\mathbf{E}_2)^2 \quad ,$$

because everything goes as in the case of one single MPW with $\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2$.

Circular wave-Polarization

This § is referenced at pages:
[Never referenced.]

In the case the two **wave-Polarization** states are given by left and right circularly wave-Polarized waves one knows that the left and right circularly wave-Polarized states can be expressed as a linear superposition of x and y linearly wave-Polarized states. This fact provides an alternative route to the demonstration that left and right circularly wave-Polarized waves do not interfere, alternative to the direct computation of the Poynting vector.

50.04.01.04 Intensity

Consider a generic set of homogeneous ElectroMagnetic MPW with possibly complex wave-vector of type $\overline{\mathbf{k}}=\mathbf{k}_{\mathrm{R}}+\mathtt{i}\mathbf{k}_{\mathrm{I}}=\overline{k}\widehat{\boldsymbol{\kappa}}$, with $\widehat{\boldsymbol{\kappa}}$ a real unit vector.

The reasoning of the previous sections allows the following conclusions.

The energy flux of ElectroMagnetic MPW will be given, in terms of either the total electric field, $\overline{\mathbf{E}}$, or the total magnetic field, $\overline{\mathbf{H}}$, by the complex Poynting vector:

$$\langle\mathbf{S}_{\mathrm{EM}}\rangle=\frac{\widehat{\boldsymbol{\kappa}}}{2}\mathfrak{Re}\left[\frac{\overline{\mathbf{E}}_0^{\star}\cdot\overline{\mathbf{E}}_0}{\overline{Z}}\right]=\frac{\widehat{\boldsymbol{\kappa}}}{2}\mathfrak{Re}\left[\overline{Z}\,\overline{\mathbf{H}}_0^{\star}\cdot\overline{\mathbf{H}}_0\right] \quad , \tag{50.04.02}$$

→ 2266

in terms of the total Electric|Magnetic fields, as for one single ElectroMagnetic wave.

In vacuum it becomes:

$$I[\mathbf{x}_{\mathrm{P}}]=c\varepsilon_0\langle\overline{\mathbf{E}}[\mathbf{x}_{\mathrm{P}},t]\cdot\overline{\mathbf{E}}^{\star}[\mathbf{x}_{\mathrm{P}},t]\rangle=\frac{c\varepsilon_0}{2}|\overline{\mathbf{E}_0}|^2 \quad .$$

As we have demonstrated that any sum of ElectroMagnetic MPW with the same wave-vector can be reduced to a single ElectroMagnetic MPW with the same wave-vector we get the interference formula for superposition of two waves:

$$I=I_1+I_2+\frac{\widehat{\boldsymbol{\kappa}}}{2}\mathfrak{Re}\left[\frac{\overline{\mathbf{E}}_1^{\star}\cdot\overline{\mathbf{E}}_2}{\overline{Z}}\right] \quad .$$

50.04.02 EM MPW Crossed Beams

Study the interference between two ElectroMagnetic MPW with either electric or magnetic field in the same direction but different wave-vectors. Read § 50.01.02.02.

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50.05 Interference Among Monochromatic Point Sources of Spherical Waves

This § is referenced at pages:
[2260]

The discussion will be based on the ElectroMagnetic waves radiated by an oscillating ideal **Electric Dipole** | **Magnetic Dipole** , considered as a point source (antenna) emitting **Dipole** ElectroMagnetic radiation (read § 49). The advantage of this approach is that **the fields produced by the sources are perfectly known** in the ideal case. In addition to that, real EM antennas are often exploiting interference.

It is assumed that **the two antennas do not affect each other**. For instance, there si a suitable ElectroMagnetic screen between them.

When one needs to be more specific, for the sake of clarity, we will assume the case of the oscillating ideal **Electric Dipole** , but it is obvious that a parallel discussion might be done for a oscillating ideal **Magnetic Dipole** . Moreover, it is obvious that the same reasoning would apply to any interference setup for light, like the one of the Young experiment, as long as one can consider the light sources as point sources. Similarly, the same reasoning also apply to sound sources.

In case the sources cannot be considered point sources one must take into account diffraction phenomena, as discussed in section § 51.

The treatment will be limited to distances from the sources very large with respect to the distances among the sources (the so-called Fraunhofer regime). In this regime the spherical waves can be considered MPW and all wave-vectors can be considered parallel among each other. For longitudinal waves this reduces the problem to exactly the simple tool example discussed in section § 50.01.02. For ElectroMagnetic waves, that are transverse waves, one must take into account also the **wave-Polarization** of the wave. Unless specified otherwise, it will be implicitly assumed that the interfering transverse waves have the same **wave-Polarization** which, again, reduces the problem to exactly the simple tool example discussed in section § 50.01.02.

According to the previous discussion the following hypothesis will be assumed. The point sources are emitters of spherical waves (not necessarily isotropic waves):

$$\psi[\mathbf{r},t]=\frac{A[\theta,\psi]}{r}\cos\left[kr-\omega t-\phi\right]=\mathfrak{Re}\left[\frac{\overline{A}}{r}\exp\left[\mathtt{i}\left(kr-\omega t\right)\right]\right]\qquad\overline{A}=A\exp\left[-\mathtt{i}\phi\right]A>0\qquad A=A[\theta,\psi]\quad,\tag{50.05.01}$$

→ 2268

where in general the amplitude might depend on the angle, $A=A[\theta,\psi]$, and $\psi[\mathbf{r},t]$ must satisfy the appropriate wave equation as well as any other equation for the systems, if any. In the case of the oscillating ideal **Electric Dipole** | **Magnetic Dipole** fields, located at the origin of a spherical Coordinate System and oriented along the $\theta=0$ direction, $\psi[\mathbf{r},t]$ is either the θ or the ψ component of the Electric|Magnetic field. Note the for ideal oscillating (point) dipoles, the field (50.05.01) is exact.

The oscillating ideal **Electric Dipole** | **Magnetic Dipole** fields are taken for a quantitative example, based on known physics, read § 49,

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50.05.01
Two Identical Parallel Oscillating Ideal Electric Dipoles With Arbitrary Phases

This § is referenced at pages:
[2272, 2297]

Different arrangements can be considered, as for instance:

1. two identical (harmonic) oscillating ideal **Electric Dipoles** | **Magnetic Dipoles** , located on the z axis, at $\{0, 0, +d/2\}$ and $\{0, 0, -d/2\}$, and oriented along the positive z axis and with arbitrary phases;
2. two identical (harmonic) oscillating ideal **Electric Dipoles** | **Magnetic Dipoles** , located on the xy plane, at $\{0, +d/2, 0\}$ and $\{0, -d/2, 0\}$, and oriented along the positive z axis with arbitrary phases.

In any case, if the two dipoles are parallel and if $\lambda \gg d$, the field at large distance is just the field of a single **Dipole** , given by the sum of the two identical dipoles, with a fixed phase constant difference depending on the phase constants of the two sources. Therefore, the case interesting for interference is only: $\lambda \lessapprox d$ for otherwise we have interference independent from position.

Note that the small-wavelength condition above is not in conflict with conditions (49.05.10), which are the basis of all our formulas, because the latter refers to the size of the single source. It is the real dipole antenna which has to be smaller than the wavelength.

Let us consider the case two identical ideal (harmonic) oscillating **Electric Dipoles** , located on the xy plane, at $\{0, +d/2, 0\}$ and $\{0, -d/2, 0\}$, and oriented along the positive z axis. Assume they emit ElectroMagnetic waves in vacuum, for simplicity. The results obtained in terms of relative intensities will be valid in any other case, for any LHI medium.

The oscillating ideal **Electric Dipoles** emit non-isotropic spherical waves, as in equation (49.07.01), (49.07.02), which will be described in terms of spherical coordinates $\{r, \theta, \psi\}$:

$$E_\theta[r,\theta,\psi] = -\frac{\mu_0}{4\pi}\frac{\omega^2 \mathbf{p}_0 \sin[\theta]}{r}\cos[kr-\omega t-\phi] \propto \frac{\sin[\theta]}{r}\cos[kr-\omega t-\phi] \quad , \tag{50.05.02}$$

$$E_r[r,\theta,\psi] = E_\psi[r,\theta,\psi] = 0 \quad , \tag{50.05.03}$$ → 2270

$$B_\psi[r,\theta,\psi] = -\frac{\mu_0}{4\pi}\frac{\omega^2 \mathbf{p}_0 \sin[\theta]}{cr}\cos[kr-\omega t-\phi] \propto \frac{\sin[\theta]}{r}\cos[kr-\omega t-\phi] \quad , \tag{50.05.04}$$

$$B_r[r,\theta,\psi] = B_\theta[r,\theta,\psi] = 0 \quad . \tag{50.05.05}$$ → 2270

The emission of the single sources in the xy plane, that is at $\theta = \pi/2$, is totally isotropic, that is independent from ψ. Interference, it will be shown, will make the emission of the pair of sources not isotropic anymore.

Assume that the two sources have a priori different relative phases, ϕ_1 and ϕ_2. In case of real ElectroMagnetic antennas this is easy controlled by the antenna circuitry.

Consider only the fields produced on the xy plane, $\theta = \pi/2$, **at large distance $R \gg d$ from the two Dipoles** . Thanks to the large distance, one can neglect the difference between the distances at the denominator in the expressions of the fields of the two **Dipoles** and one can assume that the ElectroMagnetic fields produced by the two **Dipoles** are parallel to each other, that is they have the same polarization. However, the difference of the two distances cannot be neglected inside the phase of the fields, because its product with k enters as the argument of a cosinus, so that the product can produce large changes of phases. This can be made rigorous by expanding the r^{-1}: the zero order term in r^{-1} is exactly the expression to be used here, as shown in equation (50.05.01).

In this case it is useful to sum the fields, first, and to square afterwards. The sum of the fields is made easy, in this case, by the fact that the amplitude of the two Sinusoidal|Cosinusoidal fields is the same. Compare

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with § 50.01.02, where the fields are first squared, as it is easier in case the amplitudes are different.

$$\boxed{R \equiv \frac{r_1 + r_2}{2}} \quad , \qquad \rightarrow 2269$$

$$\psi[r,\theta,\psi] \equiv \{E_\theta[r,\theta,\psi]; B_\psi[r,\theta,\psi]\} \equiv \frac{A[\theta]}{r}\cos[kr - \omega t - \phi] \qquad \text{independent from } \psi \quad ,$$

$$\psi_1[r_1,\theta] + \psi_2[r_2,\theta] = \frac{A[\theta]}{r_1}\cos[kr_1 - \omega t - \phi] + \frac{A[\theta]}{r_2}\cos[kr_2 - \omega t - \phi] \quad ,$$

$$\psi_1[r_1,\theta] + \psi_2[r_2,\theta] = \frac{A[\theta]}{R+\Delta r_1}\cos[kr_1 - \omega t - \phi] + \frac{A[\theta]}{R+\Delta r_2}\cos[kr_2 - \omega t - \phi] \quad ,$$

$$\psi_1[r_1,\theta] + \psi_2[r_2,\theta] \simeq \frac{A[\theta]}{R}\left(1 - \frac{\Delta r_1}{R}\right)\cos[kr_1 - \omega t - \phi] + \frac{A[\theta]}{R}\left(1 - \frac{\Delta r_2}{R}\right)\cos[kr_2 - \omega t - \phi] \quad ,$$

$$\simeq \frac{A[\theta]}{R}\cos[kr_1 - \omega t - \phi] + \frac{A[\theta]}{R}\cos[kr_2 - \omega t - \phi] - \frac{\Delta r_1}{R}(\ldots) - \frac{\Delta r_2}{R}(\ldots) \quad ,$$

$$\simeq \frac{A[\theta]}{R}(\cos[kr_1 - \omega t - \phi] + \cos[kr_2 - \omega t - \phi]) \quad .$$

At $\theta = \pi/2$, the exact result for large distances is:

$$\psi[r_1,r_2,\theta=\pi/2] \simeq \frac{1}{R}(\cos[kr_1 - \omega t - \phi_1] + \cos[kr_2 - \omega t - \phi_2]) \quad ,$$

$$\simeq \frac{2}{R}\cos\left[kR - \omega t - \frac{(\phi_1+\phi_2)}{2}\right]\cos\left[\frac{k(r_1-r_2) - (\phi_1-\phi_2)}{2}\right] \quad , \qquad \rightarrow 2260\ 2270$$

$$\psi[r_1,r_2,\theta=\pi/2] \boxed{\propto \frac{1}{R}\cos\left[kR - \omega t - \frac{(\phi_1+\phi_2)}{2}\right]\cos\left[\frac{kd\sin[\chi] - (\phi_1-\phi_2)}{2}\right]} \quad .$$

The angle χ has been introduced as the angle between the direction of observation and the median axis between the two sources in the xy plane, as it is customary when studying interference:

$$\boxed{r_1 - r_2 \equiv d\sin[\chi]} \quad . \qquad (50.05.06)$$

In this specific case, the median axis is the x axis. Beware: it has no relation at all with the polar angle in equations (50.05.03) and (50.05.05).

For two identical oscillating dipoles, located on the xy plane and oriented along the z axis, observing on the xy plane at large distance with respect to the distance among the two dipoles and large with respect to the wavelength, equation (50.05.01) give the exact superposition of the two files, showing in a neat way, that **interference depends on both the phase constants of the sources and on the different paths followed from source to observation point**. Note that the condition $\lambda \lessapprox d$ has not been used to derive equation (50.05.01), which implies that, if $\lambda \gg d$, there is no interference. In fact, if $\lambda \gg d$, then $kd \ll 1$ and the interference factor only depends on the difference of the phase constants of the sources, not on the position. We therefore have interference independent from position.

The result is **spherical wave, as a function of R, originating at the origin and modulated in amplitude** by the factor:

$$\boxed{\cos\left[\frac{kd\sin[\chi] - (\phi_1-\phi_2)}{2}\right]} \quad .$$

The emission has become non isotropic in the xy plane (that is at $\theta = \pi/2$). This factor is **position dependent but time-independent**.

The difference of the phases of the two sources is the significant quantity which will be called δ:

$$\boxed{\delta \equiv \frac{k(r_1-r_2) - (\phi_1-\phi_2)}{2} \equiv \frac{kd\sin[\chi] + (\phi_2-\phi_1)}{2} \equiv \frac{kd\sin[\chi] + \Delta\phi}{2}} \quad . \qquad (50.05.07) \qquad \rightarrow 2271$$

Note that in this bi-dimensional problem **the phase difference was born from different path lengths from the source to the observation point in addition to a possible phase difference between the sources**, $\Delta\phi$.

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This is markedly different from the one-dimensional toy-model described in section § 50.01.02, because in that 1D case, the paths were identical (unless considering forth-back changes of direction).

Again, equation (50.05.07) shows that, if $\lambda \gg d$, the interference is only determined by $\Delta\phi$. And, this examples shows a possible origin for the toy example of section § 50.01.02.01 as what an observer far-way on the x axis would see.

Beware that the sign of $(r_1 - r_2) = kd\sin[\chi]$ depends on the sign convention for χ and the definition of source one and two. The present convention is that χ is positive in the counterclockwise direction and source 1 is the one at $\{0, -d/2, 0\}$ and source 2 is the one at $\{0, +d/2, 0\}$.

When time-averaging the electric field the cosinusoidal term will provide the usual $1/2$ factor. The intensity for the two sources, as a function of the intensity of a single source in the same conditions, I_0, is thus, in vacuum:

$$I = c\varepsilon_0\langle E^2\rangle = 4I_0\cos^2[\delta] = I_0\left(\frac{\sin[2\delta]}{\sin[\delta]}\right)^2 \quad . \tag{50.05.08}$$

→ 2271 2287

Note that an essential assumption is that **the two antennas do not interact**, that is do not affect, each other.

Discussion of the Results

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The superposition of the fields produced by two isotropic sources is not isotropic any more. Energy is conserved, but re-distributed in space.

Note that the simple toy example in section § 50.01.02.01 corresponds to the present situation when observing from a large distance on the x axis.

In-phase sources

Consider the case of in-phase sources, $\phi_1 = \phi_2$. The result (50.05.08) shows that the intensity has equal height maxima at

$$\frac{kd\sin[\chi]}{2} = m\pi \qquad \Leftrightarrow \qquad d\sin[\chi] = m\lambda \quad ,$$

that is the difference in the two paths from the source to the receiver, $|r_1 - r_2|$, is equal to a multiple of the wavelength. This is the **constructive interference**.

The result (50.05.08) shows that the intensity has zeros at

$$\frac{kd\sin[\chi]}{2} = (2m+1)\frac{\pi}{2} \qquad \Leftrightarrow \qquad d\sin[\chi] = (2m+1)\frac{\lambda}{2} \quad ,$$

that is the difference in the two paths from the source to the receiver, $|r_1 - r_2|$, is equal to an odd multiple of half-wavelength. This is the **destructive interference**.

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In-quadrature sources

Study the case of in-quadrature sources, $|\phi_2 - \phi_1| = \pi/2$.

Plots of the radiation diagram

The **radiation diagram** for a set of antennas is a graphics, in spherical coordinates, plotting the function $r[\theta \equiv \chi] \propto I[\chi]$, that is to say: in the direction $\theta \equiv \chi$ plot a point at a distance proportional to the intensity emitted by the antennas in that direction.

Some results are plotted in figures 50.11, 50.12, 50.13, 50.14.

50.05.02
Linear Array of Identical Oscillating Electric Dipoles: In-Phase Sources

This § is referenced at pages:
[2275, 2275, 2277, 2278, 2279, 2305]

Consider now the problem similar to the one in section § 50.05.01 but with a linear array of N **Dipoles** , instead of two, in row of length L centered at the origin.

Let d be the distance between neighboring the **Dipoles** , located one the y axis, symmetrically with respect to the origin. The positions are then

$$\boxed{L = (N-1)\,d \qquad y_i = d\,(i-1) - L/2 \qquad x_i = z_i = 0 \qquad i = 1, \dots N} \quad . \tag{50.05.09}$$

→ 2278

Assume that all the **Dipoles** are all in phase and then set $\phi = 0$. It is useful to use in this problem the complex representation. Therefore, putting into a single constant $\overline{A}$ the constant terms but leaving explicitly expressed the distance R, one finds:

$$\begin{aligned}
\overline{\psi}[r_1, r_2, \dots r_N, t] &= \frac{\overline{A}}{R}\left(\exp\left[\mathtt{i}\,(kr_1 - \omega t)\right] + \exp\left[\mathtt{i}\,(kr_2 - \omega t)\right] + \dots + \exp\left[\mathtt{i}\,(kr_N - \omega t)\right]\right) \quad , \\
&= \frac{\overline{A}}{R}\exp\left[\mathtt{i}\,(kr_1 - \omega t)\right]\left(1 + \exp\left[\mathtt{i}\,(kr_2 - kr_1)\right] + \dots + \exp\left[\mathtt{i}\,(kr_N - kr_1)\right]\right) \quad , \\
&= \frac{\overline{A}}{R}\exp\left[\mathtt{i}\,(kr_1 - \omega t)\right]\left(1 + \exp\left[-\mathtt{i}kd\sin[\chi]\right] + \dots + \exp\left[-\mathtt{i}k\,(N-1)\,d\sin[\chi]\right]\right) \quad , \quad . \\
&= \frac{\overline{A}}{R}\exp\left[\mathtt{i}\,(kr_1 - \omega t)\right]\sum_{j=0}^{N-1}\left(\exp\left[-\mathtt{i}kd\sin[\chi]\right]\right)^j \quad , \\
&= \frac{\overline{A}}{R}\exp\left[\mathtt{i}\,(kr_1 - \omega t)\right]\exp\left[-\mathtt{i}\,(N-1)\,\delta\right]\frac{\sin[N\delta]}{\sin[\delta]}
\end{aligned}$$

→ 2277 2278

The expression for the sum of the terms of a geometrical progression (14.04.02) was used.

Finally:

$$\boxed{|\psi| = \frac{|\overline{A}|}{R}\left|\frac{\sin[N\delta]}{\sin[\delta]}\right|} \quad .$$

The result is again a spherical wave whose amplitude is modulated by a position-dependent factor depending again on the quantity:

$$\boxed{\delta \equiv \frac{kd\sin[\chi]}{2}} \quad .$$

Beware that the sign of

$$(r_N - r_1) = -\,(N-1)\,d\sin[\chi] \qquad (r_i - r_1) = -\,(i-1)\,d\sin[\chi] \quad ,$$

$$\boxed{(r_i - r_j) = -\,(i-j)\,d\sin[\chi] \qquad i, j = 1, \dots N} \quad , \tag{50.05.10}$$

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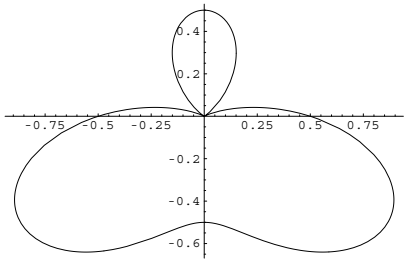
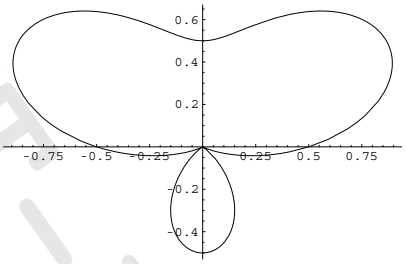


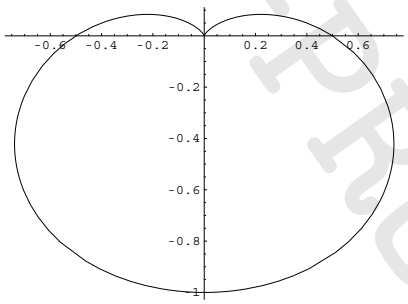
(a) Parameters: $d = \lambda/2$ and $\Delta\phi \equiv \phi_2 - \phi_1 = +\pi/2$ (horizontal x axis, vertical y axis).

Figure 50.11: ...



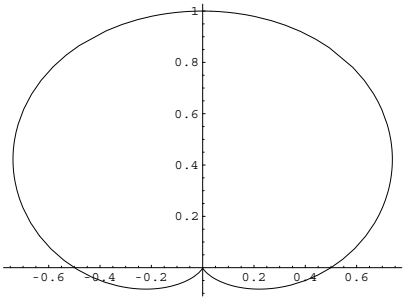
(a) Parameters: $d = \lambda/2$ and $\Delta\phi \equiv \phi_2 - \phi_1 = -\pi/2$ (horizontal x axis, vertical y axis).

Figure 50.12: ...



(a) Parameters: $d = \lambda/4$ and $\Delta\phi \equiv \phi_2 - \phi_1 = +\pi/2$ (horizontal x axis, vertical y axis).

Figure 50.13: ...



(a) Parameters: $d = \lambda/4$ and $\Delta\phi \equiv \phi_2 - \phi_1 = -\pi/2$ (horizontal x axis, vertical y axis).

Figure 50.14: ...

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depends on the sign convention for χ and the definition of source number. The present convention is that χ is positive in the counter-clockwise direction and source 1 is the one at negative y while source N is the one at positive y, that is the numbering of sources increases with y.

The intensity for the N sources, as a function of the intensity of a single source in the same conditions, I_0, is thus:

$$I = I_0 \left(\frac{\sin[N\delta]}{\sin[\delta]} \right)^2 \quad . \tag{50.05.11}$$

→ 2274 2287 2305

In practice the intensity decreases away from the center due to the effect of diffraction from the finite size sources.

Discussion of the Results

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The result (50.05.11) shows that the intensity is an even periodic function in δ with period π, thanks to the $\sin^2[\delta]$ functions. In fact, the periodicity is given by the $\sin^2[\delta]$ at the denominator because the numerator, $\sin^2[N\delta]$, has shorter period π/N.

As the function is an even function and periodic with period π, it is symmetrical about any $\delta = m\pi$.

As a function of χ the intensity function is a even function, periodic with period π, thanks to the $\sin^2[(\ldots)\sin[\delta]]$ functions.

Maxima and Minima of intensity

The intensity has equal height absolute maxima at:

$$\delta = \frac{kd\sin[\chi]}{2} = m\pi \qquad \Leftrightarrow \qquad d\sin[\chi] = m\lambda \quad , \tag{50.05.12}$$

→ 2275

where the denominator vanishes. In this case the difference in the two paths from the source to the receiver of any two adjacent sources is equal to a multiple of the wavelength. This is the constructive interference condition, with all the sources interfering constructively. In fact the maximum value cannot be anywhere else than the direction where all the waves interfere constructively. The value of the intensity at the absolute maxima is: $I = N^2 I_0$.

The result (50.05.11) shows that the intensity has zeros at

$$N\delta = N\frac{kd\sin[\chi]}{2} = m\pi \qquad \Leftrightarrow \qquad d\sin[\chi] = m\frac{\lambda}{N} \quad . \tag{50.05.13}$$

→ 2274 2275

There are $N-1$ zeros between any two absolute maxima. The zeros are missing at the position of the absolute maxima. It is rather clear that the above condition (50.05.13) gives destructive interference: in fact one is summing N sinusoids with a relative phase-shift of: 0, λ/N, $2\lambda/N$, $3\lambda/N$, ..., $(N-1)\lambda/N$. There are $N-2$ relative maxima (of low amplitude) between any two absolute maxima.

It is important to stress that the geometry of the interference figure strongly depends on the wavelength of the waves. In particular the position of the maxima and minima depends on the wavelength

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Angular Spread of the interference figure

Equation (50.05.12) shows that for large λ/d, the two secondary maxima adjacent to the primary maximum, those corresponding to $m = \pm 1$, would have a large value, possibly greater than one. If $\lambda/d > 1$, there is no secondary maxima. However, in practice, in most realistic situations, it is reasonable to doubt that, unless $\lambda/d \ll 1$, some of the approximations fail.

Therefore, it is normally assumed that **the angles with respect to the direction of the primary maximum, the one such that all paths have the same length, are small**, so that $\sin[\chi] \simeq \chi$.

See § 50.09.

Width of the peak

The width of the peak is conventionally measured as the distance between the first two zeros adjacent to the absolute maximum.

Let us first assume that the angles are small, so that $\sin[\chi] \simeq \chi$. A simple derivation is the following. The zeros in the intensity are given by equation (50.05.13):

$$d \sin[\chi] = m \frac{\lambda}{N} \quad .$$

The two zeros adjacent to a maximum are separated by $|\Delta m| = 2$ and therefore equation (50.05.13) implies:

$$\left|\sin[\chi_+] - \sin[\chi_-]\right| \simeq |\chi_+ - \chi_-| \equiv \Delta\chi = \frac{2\lambda}{Nd} \quad . \tag{50.05.14}$$

→ 2301

If one does not assume that the angles are small one obtains:

$$\delta = \frac{kd \sin[\chi_{\text{M}}]}{2} = m\pi \quad ,$$

$$N\delta_- = \frac{Nkd \sin[\chi_+]}{2} = (mN - 1)\,\pi \quad ,$$

$$N\delta_+ = \frac{Nkd \sin[\chi_-]}{2} = (mN + 1)\,\pi \quad ,$$

$$\boxed{\left|\sin[\chi_+] - \sin[\chi_-]\right| \simeq |\cos[\chi_{\text{M}}]||\chi_+ - \chi_-| \equiv |\cos[\chi_{\text{M}}]|\Delta\chi = \frac{2\lambda}{Nd}} \quad ,$$

with the correction factor $|\cos[\chi_{\text{M}}]|$ taking into account the non linear relation between χ and its sinus.

The result shows that **increasing the number of sources the angular width of the peak decreases** (while the value of the maximum intensity increases).

Plots

Plots of the radiation diagram as a function of δ, not χ.

50.05.03
Linear Array of Identical Oscillating Electric Dipoles: Linear Phase Relation

This § is referenced at pages:
[2279]

Consider now the problem similar to the one in section § 50.05.02 but with a linear array of N **Dipoles** with a **phase constant** varying linearly:

$$\phi_n = \phi_0 + n\Delta\phi \qquad \Delta\phi_n \equiv \phi_n - \phi_1 = (n-1)\,\Delta\phi \quad .$$

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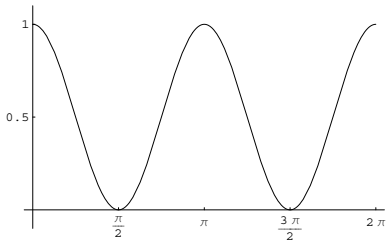
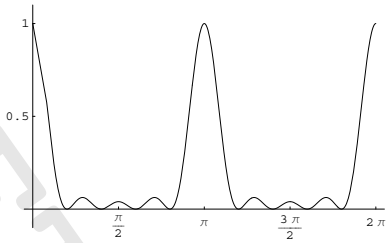
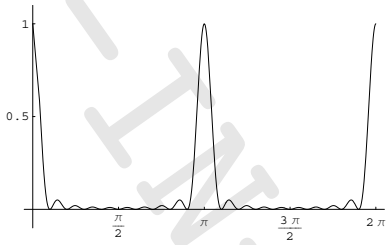


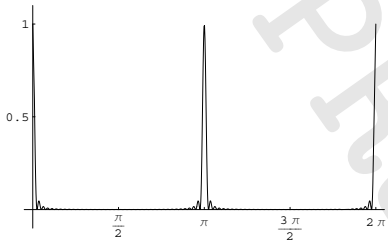
(a) N=2



(b) N=5



(c) N=10



(d) N=40

Figure 50.15: ...

Then the calculation in section § 50.05.02 is modifies as follows:

$$
\begin{aligned}
\overline{\psi}[\mathbf{r},t] &= \frac{\overline{A}}{R}\left(\exp\left[\mathtt{i}\left(kr_1-\omega t-\phi_1\right)\right]+\exp\left[\mathtt{i}\left(kr_2-\omega t-\phi_2\right)\right]+\ldots+\exp\left[\mathtt{i}\left(kr_N-\omega t-\phi_N\right)\right]\right) \quad , \\
&= \frac{\overline{A}}{R}\exp\left[\mathtt{i}\left(kr_1-\omega t-\phi_1\right)\right]\left(1+\exp\left[\mathtt{i}\left(kr_2-kr_1-\phi_2+\phi_1\right)\right]+\ldots+\exp\left[\mathtt{i}\left(kr_N-kr_1-\phi_N+\phi_1\right)\right]\right) \quad , \\
&= \frac{\overline{A}}{R}\exp\left[\mathtt{i}\left(kr_1-\omega t-\phi_1\right)\right]\left(1+\exp\left[-\mathtt{i}\left(kd\sin\left[\chi\right]+\Delta\phi_2\right)\right]+\ldots+\exp\left[-\mathtt{i}\left(k\left(N-1\right)d\sin\left[\chi\right]+\Delta\phi_N\right)\right]\right) \quad , \\
&= \frac{\overline{A}}{R}\exp\left[\mathtt{i}\left(kr_1-\omega t-\phi_1\right)\right]\left(1+\exp\left[-\mathtt{i}\left(kd\sin\left[\chi\right]+\Delta\phi\right)\right]+\ldots+\exp\left[-\mathtt{i}\left(k\left(N-1\right)d\sin\left[\chi\right]+\left(N-1\right)\Delta\phi\right)\right]\right) \quad , \\
&= \frac{\overline{A}}{R}\exp\left[\mathtt{i}\left(kr_1-\omega t-\phi_1\right)\right]\sum_{j=0}^{N-1}\left(\exp\left[-\mathtt{i}\left(kd\sin\left[\chi\right]+\Delta\phi\right)\right]\right)^j \quad , \\
&= \frac{\overline{A}}{R}\exp\left[\mathtt{i}\left(kr_1-\omega t-\phi_1\right)\right]\exp\left[-\mathtt{i}\left(N-1\right)\delta\right]\frac{\sin\left[N\delta\right]}{\sin\left[\delta\right]}
\end{aligned}
$$

. → 2277 2278

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The expression for the sum of the terms of a geometrical progression (14.04.02) was used.

Finally:

$$|\psi| = \frac{|\overline{A}|}{R}\left|\frac{\sin[N\delta]}{\sin[\delta]}\right| \quad .$$

The result is again a spherical wave whose amplitude is modulated by a position-dependent factor depending again on the quantity modified by the phase shift among two neighboring sources:

$$\delta \equiv \frac{kd\sin[\chi] + \Delta\phi}{2} \quad . \tag{50.05.15}$$

The intensity for the N sources, as a function of the intensity of a single source in the same conditions, I_0, is thus:

$$I = I_0\left(\frac{\sin[N\delta]}{\sin[\delta]}\right)^2 \quad . \tag{50.05.16}$$

The phase $\Delta\phi$ can be used to shift the central maximum away from $\chi = 0$: for in-phase sources the maximum is at $\chi = 0$, while for a linear phase-shift $\Delta\phi$ it is at: $\sin[\chi] = -\Delta\phi/(kd)$.

This result may be applied, for instance, to steer ElectroMagnetic waves produced by a set of antennas, not by moving the antennas, but by introducing an appropriate delay between the different transducers in the circuitry. Similarly, if the antenna is used to receive, its sensitivity can be steered introducing an appropriate delay between the different transducers in the circuitry. This can be extended to bi-dimensional arrays and steering, read § 50.07.

Discussion of the Results

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All the discussion done for the case of sources in-phase ($\Delta\phi = 0$) can be repeated here with the result that the plots are shifted by $\Delta\phi$.

50.05.04
Linear Array of Identical Oscillating Electric Dipoles: Random Phases

This § is referenced at pages:
[2278, 2330]

Read also § 53.03.

Consider now the problem similar to the one in section § 50.05.02 but with a linear array of N **Dipoles** , with overall fixed length $(N-1)\,d$, but each source with random phases.

As for the two cases of equations (50.05.02), (50.05.03) one now starts from:

$$\overline{\psi}[\mathbf{r},t] = \frac{\overline{A}}{R}(\exp[\mathtt{i}\,(kr_1 - \omega t - \phi_1)] + \exp[\mathtt{i}\,(kr_2 - \omega t - \phi_2)] + \ldots + \exp[\mathtt{i}\,(kr_N - \omega t - \phi_N)]) \quad ,$$

where the phase factors, ϕ_k, are now random numbers and the observation point is at large distance, $r \gg (N-1)\,d$.

For a very large number of sources, $N \to \infty$ such that $(N-1)\,d$ remains constant, the sum tends to zero. In fact, the sum in parenthesis is a sum of independent random variables with zero average value

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and variance, σ_0, of the order of unity:

$$\Lambda \equiv \sum_{p=0}^{N} \exp\left[+\mathtt{i}\alpha p\right] \exp\left[+\mathtt{i}\phi_p\right] \equiv \sum_{p=0}^{N} \lambda_p[\alpha,\phi_p] \sim 0 + \sigma_0\sqrt{N} \quad ,$$

$$\langle \lambda_p[\alpha,\phi_p] \rangle = 0 \qquad \sigma^2[\lambda_p[\alpha,\phi_p]] \equiv \sigma_0^2 \propto \left|\overline{\psi}[\mathbf{r},t]\right|^2 \quad ,$$

$$\langle \Lambda \rangle = 0 \qquad \sigma^2[\Lambda] = N\sigma_0^2 \qquad \sigma[\Lambda] = \sqrt{N}\sigma_0 \quad .$$

The result is that the resultant electric field scales as $\sqrt{N}$ and the intensity as N, so that interference does not exist: that is the total intensity is the sum of intensities.

An alternative calculations starts from squaring the summed amplitudes. to find a factor N from the square of the single amplitudes plus $N(N-1)/2$ terms with random phase differences, averaging to zero. These terms are not independent, as any phase appears in $N-1$ terms but, as a sum of terms, the mean value is the sum of mean values, no matter independence, so the sum of intensities is found as well.

Note: the average field is zero, while the average squared field is non-zero. But the field has zero average and $\sim \sqrt{N}$ fluctuations.

50.05.05
Linear Array of Identical Oscillating Electric Dipoles: Random Positions

This § is referenced at pages:
[2330]

Read also § 53.03.

Consider now the problem similar to the one in section § 50.05.02 but with a linear array of N **Dipoles** , with overall fixed length $(N-1)\,d$, but each source with random positions.

As for the two cases of equations (50.05.02), (50.05.03) one now starts from:

$$\overline{\psi}[\mathbf{r},t] = \frac{\overline{A}}{R}(\exp\left[\mathtt{i}\left(kr_1 - \omega t - \phi\right)\right] + \exp\left[\mathtt{i}\left(kr_2 - \omega t - \phi\right)\right] + \ldots + \exp\left[\mathtt{i}\left(kr_N - \omega t - \phi\right)\right]) \quad ,$$

where every r_i is now uniformly random in a interval of width d centered at the positions defined by equation (50.05.09). The same result then obtained as in section § 50.05.04 is obtained and interference does not exist: that is the total intensity is the sum of intensities.

Note: the average field is zero, while the average squared field is non-zero. But the field has zero average and $\sim \sqrt{N}$ fluctuations.

DRAFT - IN PROGRESS

50.06 Interference in the Generalized Young Experiment

Consider the interference among long thin linear equi-spaced parallel slits on a plane opaque screen when the opaque screen is illuminated by a MPW at normal|oblique incidence, the **generalized Young experiment**.

This is actually a **bi-dimensional problem** corresponding exactly to § 50.05.02 and § 50.05.03 considered in the plane perpendicular to the dipoles, at $\theta = \pi/2$.

DRAFT-IN-PROGRESS

50.07 Interference From Bi-Dimensional Sources

This § is referenced at pages:
[2277, 2298]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|8.4||

For a bi-dimensional set of identical sources located in the xy plane the expression of the relative intensity is just a generalization of the expression for a linear array. Assume a rectangular array of identical point sources in the xy plane with spacing d_x and d_y and centered at the origin.

This gives, in terms of the angles χ_x and χ_y that the position vector from the origin to the observation point forms with the zy and zx planes from equation (51.04.04):

$$\delta_x \equiv \frac{kd_x \sin[\chi_x]}{2} \qquad \delta_y \equiv \frac{kd_y \sin[\chi_y]}{2} \quad ,$$

the expression:

$$I = I_0 \left(\frac{\sin^2[N_x \delta_x]}{\sin^2[\delta_x]} \right) \left(\frac{\sin^2[N_y \delta_y]}{\sin^2[\delta_y]} \right) \quad . \tag{50.07.01}$$

In practice the intensity decreases away from the center due to the effect of diffraction from the finite size sources (read § 51).

DRAFT - IN - PROGRESS

50.08 Interference From Three-Dimensional Sources

©|E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|§ 10.2||

Typical sources are atoms in a crystal. The treatment is somewhat more complex than the case of one-dimensional and two-dimensional arrays.

DRAFT-IN-PROGRESS

50.09 Approximations In Interference and Realistic Interference situations

This § is referenced at pages:
[2275, 2304]

So far, it has been implicitly assumed that**all sources are mutually coherent, that is, whatever the phase difference between any two sources, it remains constant in time, on average of time periods long with respect to the period of the monochromatic signal**. Read § 51.06.

DRAFT-IN-PROGRESS

50.10 Examples and Physical Applications

50.10.01
Interference Between Two arbitrary MPW

Demonstrate equation (50.01.04).

50.10.02
Young Experience with out-of-phase sources

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|||

In the standard Young experience the two slits are in-phase sources.
Different phase can be obtained as follows.

- Use an incoming plane wave not perpendicular to the screen,
- To give arbitrary phases, just put in front of the slit, before the screen, a piece of glass with suitable thickness and index of refraction.

50.10.03
Lloyd Mirror

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|||

50.10.04
Interference From a Prismatic Glass With Isosceles Trapezium Section

This § is referenced at pages:
[Never referenced.]

50.10.05
Interference From a Thin Lamina

This § is referenced at pages:
[2185, 2185, 2185]

Interference effects are often observed in thin films, such as a thin layer of oil on water or the thin surface of a soap bubble. The colors observed when white light is incident on such films come from the interference of waves reflected from the two surfaces of the film.

Consider a plane transparent lamina made of a non magnetic medium with **index of refraction** n immersed in a non magnetic medium with **index of refraction** n_0. Light is incident on the lamina at a certain angle. This is actually a problem of multiple interference, but it will be treated in a simplified form as a problem of interference between two sources, neglecting the effects of multiple reflections.

Let the layer have **index of refraction** n and thickness H. Let the **index of refraction** of the outside medium be $n_0 = 1$.

The difference in the phases is due to the optical paths inside the layer, for one ray, and the geometrical difference in the paths to the receiver, outside the layer.

DRAFT-IN-PROGRESS

Moreover, a $\pm\pi$ phase change has to be accounted for, possibly, due to the two reflections. This can be easily seen for **wave-Polarization** perpendicular to the incidence plane: there is always change of phase of $\pm\pi$ on reflection at the first surface and there is never change of phase on reflection at the second surface.

The case of **wave-Polarization** parallel to the incidence plane is similar with the additional complication carried by the different behavior of the change of phase according to the value of the angle of incidence with respect to the Brewster angle. The two cases, incidence angle larger/smaller than the Brewster angle, must be treated separately, but they give rise to the same final result, that an overall change of phase of $\pm\pi$ is present, as one can see inspecting the plots 48.11 and 48.12. First note that if light is incident at the Brewster angle on the lamina, the refraction angle exactly corresponds to Brewster angle for the passage from the lamina medium to the external medium. Secondly note that the change of phase by $\pm\pi$ depends on the convention chosen when deriving Fresnel equation, even if the final result does not. Therefore if the incidence angle is smaller than the Brewster angle, there is no phase change at the reflection on the first surface, $r_{\parallel} > 0$, but there is a $\pm\pi$ change of phase on reflection at the second surface, $r_{\parallel} < 0$, such that, if the two optical paths are identical, the two fields recombine out-of-phase of $\pm\pi$. On the other hand, if the incidence angle is larger than the Brewster angle, the change of phase of $\pm\pi$ happens at the first surface.

There is never change of phase in transmission.

We only consider, for simplicity, the case of almost normal incidence.

Let the ray 1 be the reflected only ray and let the ray 2 be the refracted-reflected-refracted ray. The change of phase, $\Delta\Phi$, is given, by the difference of the optical paths, $\Delta\Phi'$, plus the $\pm\pi$ change of phase due to reflections:

$$\Delta\Phi = \Delta\Phi' \pm \pi \quad ,$$

$$k = n\frac{\omega}{c} \quad ,$$

$$\Delta\Phi_1 = 2kH\tan[\theta_T]\sin[\theta_I] = \frac{2\omega H}{c}\tan[\theta_T]\sin[\theta_I] \quad ,$$

$$\Delta\Phi_2 = 2kH\frac{1}{\cos[\theta_T]} = \frac{2\omega H}{c}\frac{n}{\cos[\theta_T]} \quad ,$$

$$\Delta\Phi' \equiv \Delta\Phi_2 - \Delta\Phi_1 = \frac{2\omega H}{c}\left(\frac{n}{\cos[\theta_T]} - \tan[\theta_T]\sin[\theta_I]\right) = \frac{2\omega n H}{c}\cos[\theta_T] \quad ,$$

where the last passage comes from Snell law: $\sin[\theta_I] = n\sin[\theta_T]$. Constructive interference happens for $\Delta\Phi = 2m\pi \quad \Leftrightarrow \quad \Delta\Phi' = (2m+1)\pi$.

Note that when H tends to zero destructive interference can be observed at any angle with $\Delta\Phi' = 0$. This is caused by the $\pm\pi$ change of phase. If the $\pm\pi$ would not be included one would expect constructive interference for a vanishingly thin lamina, which is not observed experimentally.

The interfering rays will not have the same amplitude and therefore the minimum intensity will not be exactly zero. In reality, there are infinite multiple reflection, with decreasing successive amplitudes.

In case the lamina has a large thickness it is difficult to observe the interference fringes because for non perfectly monochromatic light there are different frequencies interfering destructively/constructively in such a way that the overall effect is negligible.

As the constructive interference depends on the wavelength one will observe constructive interference at different angles for different wavelength. This explains the colored fringes which are seen in white light.

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50.10.06
Holography

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|||

50.10.07
On Energy Conservation Mutual Coupling of the Sources and Interference

This § is referenced at pages:
[2305]

©|R.C.Levine|False paradoxes of superposition in electric and acoustic waves|*Am.J.Phys, , ..., ..., ...Ed., ... 48, 28 (1980);* ...

©|N.Gauthier|What happens to energy and momentum when two oppositely-moving wave pulses overlap?|*Am.J.Phys, , ..., ...*

©|D.R.Rowland|Comment on 'What happens to energy and momentum when two oppositely-movingwave pulses overlap?,' ...

©|A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...|§ 7, pp. 228 230||

©|H.J.Pain, *The Physics Of Vibrations And Waves*, 2005, J.Wiley & Sons, 6thEd., ...|||

©|N.Gauthier|Am.J.Phys, , ..., ..., ...Ed., ... 71 (8), August 2003|*DOI: 10.1119/1.1575765*|

©|D.R. Rowland|Am.J.Phys, , ..., ..., ...Ed., ... 72 (11), November 2004|*DOI: 10.1119/1.1783903*|

©|Conservation of energy and wave interference|J. L. Ferguson|*Am.J.Phys, , ..., ..., ...Ed., ... 55, 970 (1987)*|

©|On the ElectroMagnetic wave omnidirectional interference phenomena|Yan-Shek Hoh|*Am.J.Phys, , ..., ..., ...Ed., ... 55,* ...

©|False paradoxes of superposition in electric and acoustic waves|R. C. Levine|*Am.J.Phys, , ..., ..., ...Ed., ... 48, 28 (1980)*|

The argument is often tricky.

When there are two sources, it is possible that the sources interact each-other, changing the emission characteristics and/or the power used by the source generator.

Note that the **superposition principle** at fixed sources, strictly speaking, can only be applied when it can be assumed that none of the sources and none of the waves affects in any way the behavior of any other source. In many cases this might be a good approximation.

One typical example might be interference between different nearby antennas radiating ElectroMagnetic waves: the application of the **superposition principle** implicitly assumes that the ElectroMagnetic waves radiated by one antenna do not affect the radiation properties of the other antenna.

A second typical example is two sound sources, where the sound waves emitted by one source may affect the other emitter.

Additionally whenever one source/wave affects another source, it may happen that the power emitted by each source changes with respect to the power which would be emitted by the source itself alone. As a consequence the energy density/flux carried away by the waves may be different from the sum of the power produced by the sources alone.

© - QUOTE

©|R.C.Levine|False paradoxes of superposition in electric and acoustic waves|*Am.J.Phys, , ..., ..., ...Ed., ... 48, 28* ...

...

When acoustic or ElectroMagnetic waves cancel by destructive interference, the **wave impedance** reflected to the sources of the wave energy changes so that the input power is reduced correspondingly. Because this rather subtle point is not discussed in most instructional treatments of wave physics, misconceptions concerning the apparently "missing" energy are widespread, as reflected in semi-popular science writing.

...

© - QUOTE

©|N.Gauthier|What happens to energy and momentum when two oppositely-moving wave pulses overlap- ...

...
As a result, the net energy density for the superposed solution is always equal to the sum of the energy densities of the right-moving and the left-moving pulses, considered separately. From the perspective of energy conservation, the two pulses are transparent to one another.
...
We again see that the cross terms, which represent the overlapping of the pulses, cancel exactly, just as they did in the total energy density of the superposed pulses. The total momentum density is equal to the algebraic sum of the local momentum densities of each pulse everywhere in space, just as if they didn't overlap at all. In this configuration, the total momentum of the system is always conserved and there is no need to evaluate complicated spatial integrations to prove it.
...

Read § 51.07.02.

50.10.08
Applications - Phased Array

The possibility to have a high intensity in a well defined direction in space (in two dimensions) is used in Radio-astronomy to point arrays of Radio-telescopes in a certain direction with high directionality.

It is also being used in 5G technology for beam steering.

WORK IN PROGRESS

50.11 Exercises Problems and Physical Applications

50-008 I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...

5.064, 5.065, 5.066, 5.068, 5.069, 5.070, 5.071, 5.076, 5.077, 5.079, 5.081, 5.085.

50-009 Geometrical Relations

Show that the following geometrical relations are valid, useful for the precise treatment of the above problem:

$$r_1^2 = R^2 + \frac{d^2}{4} - 2R\frac{d}{2}\cos\left[\frac{\pi}{2} + \chi\right] \quad ,$$
$$r_2^2 = R^2 + \frac{d^2}{4} - 2R\frac{d}{2}\cos\left[\frac{\pi}{2} - \chi\right] \quad ,$$
$$r_1^2 - r_2^2 = 2Rd\sin[\chi] = (r_1 - r_2)(r_1 + r_2)$$

→ 2185

50-010 Geometrical Relations and Fraunhofer Approximation

This § is referenced at pages:
[2307]

©|E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|9.15||

50-011 Emitting Preferentially on One Side

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|8.1||

Show that it is possible to emit most of the energy within and angular region of $\Delta\chi \approx \pi$ rad by using: $d = \lambda/4$, $\Delta\phi = \pm\pi/2$. Draw the radiation plot.

50-012 N=2 Case

Show that the result (50.05.11) reduces to the result (50.05.08) in case of two in-phase sources.

50-013 Other Cases of Interference for N=2

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|8.1||

Study other cases, varying both $\Delta\phi$ and the ratio λ/d.

DRAFT - IN PROGRESS

50-014 A N=4 Case With Half-Wavelength Distance

Draw the radiation diagram for $N = 4$ and $d = \lambda/2$.

50-015 A Radio-Telescope

The radiation diagram of a series of emitting antennas is the same as the sensitivity diagram of the antennas when used as receivers, with the given phase relation.

Consider a radio-telescope with $N = 11$, $d = 1$ km and $\lambda = 6$ cm. Determine the angular resolution.

50-016 Very Large Baseline Interferometry Radio-Telescopes

Angular resolution of a single radio-telescope (separation between two point sources): $\Delta\chi \simeq \dfrac{\lambda}{D}$; D is the radio-telescope diameter.

Angular resolution of a linear array of N radio-telescopes (separation between two point sources): $\Delta\chi \simeq \dfrac{\lambda}{Nd}$; d is the distance between the radio-telescopes: $D \ll d$.

50-017 Linear Array of N Sources With Random Positions

©|J.P.Pérez *et al.*, *Optique*, ..., DUNOD, ...Ed., WEB - URL.|21.4.3||

Study the interference pattern produced in the xy plane by a linear array of N identical sources located along the y axis, centered at the origin with average spacing among the sources d, such that the distance between the two extreme sources is $(N-1)\,d$.

WORKED SOLUTION

No interference at all: sum of the intensity everywhere.

50-018 Linear Array of N Sources With Random Phases

©|J.P.Pérez *et al.*, *Optique*, ..., DUNOD, ...Ed., WEB - URL.|||

Study the interference pattern produced in the xy plane by a linear array of N identical sources located along the y axis, centered at the origin with spacing among the sources d and random phases among each other.

WORKED SOLUTION

No interference at all: sum of the intensity everywhere.

DRAFT - NOT FOR PUBLICATION

50-019 Brewster Angles and a Plane Lamina

Consider a plane transparent lamina made of a non magnetic medium with **index of refraction** n_2 immersed in a non magnetic medium with **index of refraction** n_1.

Show that if the incidence angle on the lamina is the Brewster angle then the refraction angle exactly corresponds to Brewster angle for the passage from the lamina to the external medium.

50-020 Young Experience With Out-of-Phase Slits

A two-slit Young interference experiment with $\lambda = 500\ \mathrm{nm}$ is arranged such that a thin film of transparent material ($n = 1.2$) is placed in front of one of the slits. As consequence the zero order fringe moves to the position previously occupied by the fourth-order bright fringe. Calculate the thickness of the film.

50-021 Other Problems

Read also sections from E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ..., § 9.

50-022 Other Problems

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|§ 9|*A lot of problems.*|

DRAFT-IN-PROGRESS

CHAPTER 51

Diffraction

51.01 Diffraction (And Interference) 2292
51.02 Wave-Optics Versus Geometrical|Ray Optics 2294
51.03 Huygens Principle and the Fresnel-Kirchhoff Formula 2295
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DRAFT - IN-PROGRESS

About diffraction, that is the key properties of wave-fields, for a continuous distribution of infinitesimal sources.

This § is referenced at pages:
[2016, 2021, 2185, 2257, 2257, 2268, 2280, 2292]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|6|*Good text.*|

©|E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...||*Very detailed*|

©|J.P.Pérez *et al.*, *Optique*, ..., DUNOD, ...Ed., WEB - URL.|||

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|1.30||

DRAFT-IN-PROGRESS

51.01 Diffraction (And Interference)

Diffraction demonstrates the **wave-like character** of a phenomenon.

Interference arises when the sources form either a finite or a countable infinite set of finite-intensity (that is non zero and non infinite) sources: it is discussed in section § 50. The mathematical model, based on the linear superposition principle, corresponds to calculate interference effects by means of either finite sums or infinite series. Interference, is a useful preliminary to the study of diffraction, and, in addition is an important topic in itself.

Diffraction arises when the sources form an infinite continuum of infinitesimal-intensity sources: it is discussed in section § 51. The mathematical model, based on the linear superposition principle, corresponds to calculate diffraction effects by means of integrals. In this sense it is only **technically, but not conceptually, different from interference** and the conceptual discussion made in section § 50.01.

In this section, as in the previous § 50 on interference, the discussion will be limited to **observations at very large distances from the sources**, in such a way that, in both the longitudinal and the transverse wave cases, one can often just **sum the components of the Perturbation|Disturbance|Signals**, as explained in section § 50.01. Large distance means: large distance with respect to the dimensions of the sources.

In this section, as in the previous section on interference, we are **more interested in the relative intensity than in the absolute intensity** of the diffraction pattern. Therefore proportionality constants will not be taken seriously into account. However, while in the case of interference it is customary to express the resulting intensity in terms of the intensity of the single source, which is a well-defined concept, in the case of diffraction there is a continuous distribution of infinitesimal sources and therefore there is no such a thing as the intensity of the single source to use as a reference. Therefore the resulting **intensity is usually normalized such that it is one at maximum**.

The analytical treatment of diffraction is a very difficult problem; because of mathematical difficulties, approximate methods must be used in most practical cases.

Our concerns are only with the classical limit, where**many photons are present simultaneously and their associated ElectroMagnetic fields can be described by Maxwell equations**.

The distinguishing feature of diffraction is the **deviation from rectilinear propagation** when the wave is obstructed in some way.

While interference and diffraction are phenomena proper to any kind of waves, the discussion will be limited to ElectroMagnetic waves, due to their huge interest.

Until coherence is presented, in section § 52, **all sources are assumed to be strictly monochromatic point sources**. Therefore, they appear as generators of spherical (possibly non isotropic) waves. At large enough distance, waves appear as MPW.

51.01.01 Some Diffraction Phenomena and Applications

- William Herschel knew as early as 1779 that stars appeared much larger in telescopes than they really were, but he did not know why. When Thomas Young demonstrated interference and the wave nature of light this was unambiguously explained.
- Arago/Poisson spot is the luminous point which appears at the center of the shadow of a circular object illuminated by a point source. It is due to constructive interference caused by the cylindrical symmetry of the problem.
- The common experience that sound can be heard also after obstacles is due to diffraction phenomena which allow long-wavelength waves to travel past obstacles with dimensions smaller that the wavelength. Beware that in the case of sound, in everyday experience, not only diffraction but also reflection effects may be important.
- A common experience with diffraction is provided by water waves passing around obstacles.

DRAFT IN PROGRESS

© - QUOTE

Adapted from E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...
Imagine to have an opaque flat screen, Σ, containing one single small aperture, of any shape, which is illuminated by MPW perpendicular to the screen, coming from a very distant point source. The plane of observation is a screen parallel with Σ.

- When the plane of observation is very close to Σ, an **image of the aperture** is projected onto the screen, which is clearly recognizable despite some slight fringing around its periphery.
- When the **plane of observation is moved away** from Σ, the image of the aperture, though still easily recognizable, becomes increasingly more structured as the fringes become more prominent. This phenomenon is known as **Fresnel or near-field diffraction**.
- If the plane of observation is moved out still farther, a continuous change in the fringes results. At a very great distance from Σ the projected pattern will have spread out considerably, bearing little or no resemblance to the actual aperture. This is **intermediate-field diffraction**.
- Thereafter moving the screen essentially changes only the size of the pattern and not its shape. This is **Fraunhofer or far-field diffraction**.
- **Reduce the wavelength** of the source.
- If at that point we could sufficiently reduce the wavelength of the incoming radiation, the pattern would revert to the **Fresnel case**.
- If the wavelength were decreased even more, so that it approached zero, the fringes would disappear, and the image would take on the limiting shape of the aperture, as predicted by **Geometrical Optics** | **Ray Optics** .
- Returning to the original setup, **if the point source was now moved toward Σ, spherical waves would impinge on the aperture, and a Fresnel pattern would exist, even on a distant plane of observation**.

51.01.02
Some Toy Examples

This § is referenced at pages:
[Never referenced.]

51.01.03
Conditions for Diffraction to Occur

This § is referenced at pages:
[Never referenced.]

Read § 50.01.03

IN PROGRESS

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51.02 Wave-Optics Versus Geometrical|Ray Optics

Geometrical Optics | Ray Optics is the regime when the propagation of waves can be described in terms of light rays, propagating rectilinearly in homogeneous media and satisfying Snell laws at boundaries.

It is typically characterized by:

$$\lambda \ll D \qquad D \text{ dimensions of the obstacles in the path} \quad .$$

Geometrical Optics | Ray Optics is thus the limit of wave-optics when the wavelength is much smaller than the relevant dimensions. This is confirmed by the results of diffraction, showing that diffraction, which implies non-rectilinear light-ray propagation, is only relevant when sizes of obstacles are small with respect to wavelength.

All these qualitative considerations can be made precise.

DRAFT-IN-PROGRESS

51.03 Huygens Principle and the Fresnel-Kirchhoff Formula

This § is referenced at pages:
[2185]

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|9.8||
©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|8.3.2||

The mathematical theory of diffraction is really complex and one should not be carried away from the physical phenomena by the mathematics. Several approximations and simplifications must be accounted for.

Huygens principle, in the improved **Fresnel-Kirchhoff diffraction formula**, provides an **approximate method** to solve wave-propagation problems which works very well in practice.

The Huygens-Fresnel-Kirchhoff principle makes the basic approximation that the amplitude and phase of the waves involved can be described by a scalar, not vector, field, and that any effect arising from the **wave-Polarization** of transverse waves can be neglected. It is in fact called the **scalar-wave approximation**.

Huygens principle states that every point on a wave-front can be considered as a **source of secondary wavelets** that combine so that the envelope of the secondary wavelets constitutes the advancing wave-front. In order to make quantitative statements, the principle must be supplemented by other quantitative conditions, to ensure that the secondary waves are preferentially produced in the direction of wave propagation and that the wave-front is the result of interference among the secondary waves. All this is made quantitative by Fresnel-Kirchhoff diffraction formula.

Huygens principle implemented by the **Fresnel-Kirchhoff formula** can be used for computing the diffraction patterns from an illuminated aperture. It is normally used in two complementary limits, called the Fraunhofer and Fresnel regimes.

It actually applies to any point of the wave field: the screen does not matter, it is just an help to understanding and visualization.

51.03.01 Fresnel-Kirchhoff Scalar Diffraction Formula for MPW Incident Perpendicular to a Flat Opaque Screen

©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|§ 8.3.2||

In the case of a **plane aperture on a flat opaque screen, oriented as $\hat{\mathbf{n}}$, invested by a MPW perpendicularly to the screen**, the **Huygens-Fresnel-Kirchhoff scalar diffraction formula** takes, for a generic **scalar field** or for any **component of the field**, the simplified form:

$$\psi[\mathbf{x},t] \propto k \exp[-\mathrm{i}\omega t] \iint_{\Sigma} \frac{\exp[\mathrm{i}k|\mathbf{x}-\mathbf{w}|]}{|\mathbf{x}-\mathbf{w}|} \frac{1+\cos[\chi]}{2} \mathrm{d}\mathbf{S}[\mathbf{w}] \qquad \cos\chi \equiv \frac{\hat{\mathbf{n}}\cdot(\mathbf{x}-\mathbf{w})}{|\mathbf{x}-\mathbf{w}|} \qquad \mathbf{r} \equiv \mathbf{x}-\mathbf{w} \quad , \tag{51.03.01}$$

where $\mathbf{x}$ is the observation point and $\mathbf{w}$ is the position of the elementary source on the screen aperture and the integral is extended to the aperture. → 2295 2297

Obliquity factor is the name given to the term $(1+\cos[\chi])/2$.

Depending on the scalar/vector nature of the diffracting field,

As the validity of equation (51.03.01) is limited to very large distances, where the addition of different wavelets consists of adding the components, either in the radial direction or in the plane perpendicular

to it, because all components are parallel to each other. In case of transverse waves they are assumed to have the same **wave-Polarization** , or, if not, they have to be added accordingly.

The incoming MPW, perpendicular to the screen, has the same intensity at all the points of the aperture, that is the source surface density is uniform on the aperture.

Every point on the aperture is a secondary emitter of non isotropic spherical waves.

The obliquity factor gives the angular dependence to the spherical wave, that is maximum in the forward direction and zero backwards.

DRAFT-IN-PROGRESS

51.04 Fraunhofer Diffraction

This § is referenced at pages:
[2300]

©|J.P.Pérez *et al.*, *Optique*, ..., DUNOD, ...Ed., WEB - URL.|21||

TRAFFIC PROOFS

51.04.01 The Fraunhofer regime

Under the conditions of equation (51.03.01), one can further simplify the formulas in the so-called **Fraunhofer regime**, the most common situation used in Interference|Diffraction where the detector is set at a very large distance from the screen.

It will be assumed that the following **large-distance/small-obliquity** conditions apply.

- **The obliquity factor can be set constant and equal to one** and taken out of the integral, that is the angular spread of the observed rays, with respect to the normal to the screen, is small: $\cos[\chi] \simeq 1$. That is, everything happens at a small angle in front of the aperture on the diffracting screen.
- **The variation with distance of the outgoing spherical wave comes primarily from the exponential factor and the denominator can be taken outside the integral as a constant**.

The approximation that the variation of the outgoing spherical wave due to the denominator can be neglected while the one coming from the exponential factor cannot be neglected, is the same approximation as in section § 50.05.01. It is due to the fact that, inside the phase the factor, r appears multiplied by the wave-number k and therefore any change of r of the order of the wavelength λ changes the phase by a large amount. On the other hand, as normally $r \gg \lambda$, a change of the order of the wavelength λ gives a very small relative change to the denominator. Consider, in fact, the effect of a change of one wavelength in distance for the exponential and for the factor r at denominator:

$$r \longrightarrow r \pm \lambda \Longrightarrow \begin{cases} \exp[\mathtt{i}kr] \longrightarrow \exp[\pm 2\pi\mathtt{i}]\exp[\mathtt{i}kr] \\ \dfrac{1}{r} \longrightarrow \dfrac{1}{r \pm \lambda} \simeq \dfrac{1}{r}\left(1 \mp \dfrac{\lambda}{r}\right) \end{cases} \quad , \quad \Delta\left(\frac{1}{r}\right) = \left|\frac{1}{r} - \frac{1}{r+\lambda}\right| \sim \frac{\lambda}{r^2} \qquad \frac{\Delta(1/r)}{1/r} = \frac{\lambda}{r} \quad .$$

In the first case the change is big, due to the large relative change in the phase $\exp[\pm 2\pi\mathtt{i}]$ which can bring a change of sign of the term, with big effects, for instance, in the case of oscillating terms. In the second case it is just an effect of the order λ/r.

Consequently the Huygens-Fresnel-Kirchhoff formula for normally incident MPW, in the large-distance/small-obliquity regime, takes the very simple form:

$$\psi[\mathbf{x},t] = C \iint_{\Sigma} \exp[\mathtt{i}kr] \; \text{đ}S \qquad \text{large-distance/small-obliquity regime with perpendicularly incident MPW} \quad .$$

(51.04.01) → 2307

51.04.02
Geometry of Fraunhofer regime

The **Fraunhofer regime** is defined by the additional condition: in the exponential factor only the linear term is relevant for the calculation of kr, equivalent to

$$D^2 \ll R\lambda \quad . \tag{51.04.02}$$

→ 2298

On the other hand, the **Fresnel regime** is defined by the condition:

$$D^2 > R\lambda \quad , \tag{51.04.03}$$

meaning that the quadratic in the exponential is not negligible with respect to the linear term.

In fact, consider a flat screen described by the $\{x, y\}$ orthonormal Cartesian Coordinates Coordinate System; let the z axis be oriented in the direction of propagation of the diffracted light. Look at the diffraction pattern (the intensity) on a parallel screen, described by the $\{X, Y\}$ orthonormal Cartesian Coordinates Coordinate System with its axis, Z, coincident with the z axis. Let the origin of the orthonormal Cartesian Coordinates Coordinate System be on the flat screen inside the aperture (at the geometrical center, or Center-Of-Mass, for instance).

The distance between the generic elementary source (at coordinates $\mathbf{w} \equiv \{x, y, 0\}$) and the generic observation point (at coordinates $\mathbf{x} \equiv \{X, Y, Z\}$ is:

$$\begin{aligned}
R &\equiv \sqrt{X^2 + Y^2 + Z^2} \quad , \\
r &\equiv \left((X-x)^2 + (Y-y)^2 + Z^2 \right)^{+1/2} \quad , \\
&= \left(X^2 + Y^2 + Z^2 + x^2 + y^2 - 2xX - 2yY \right)^{+1/2} \equiv \left(R^2 + x^2 + y^2 - 2xX - 2yY \right)^{+1/2} \quad , \\
&= R \left(1 - 2\frac{xX + yY}{R^2} + \frac{x^2 + y^2}{R^2} \right)^{+1/2} = R \left(1 - \frac{xX + yY}{R^2} + \mathcal{O}\left(\frac{x^2 + y^2}{R^2} \right) \right) \quad , \\
&\simeq R - \frac{xX + yY}{R} = R - (x \sin\chi_x + y \sin\chi_y) \quad ,
\end{aligned}$$

all coherent with equations (51.04.02), (51.04.04) and truncation of the exponential to the linear term, as from equation (51.04.02). In fact, the second-order term is of the order $\sim kRD^2/R^2 \simeq 2\pi D^2/(R\lambda)$. Therefore the condition to neglect it inside the phase of the exponential is $D^2 \ll R\lambda$, that is the condition for Fraunhofer regime (51.04.02). Note that the comparison between the linear and quadratic terms provides the condition $D \ll R$, which has already been assumed.

As in section § 50.07, the variables X and Y are linked to the angles χ_x and χ_y, the position vector of the observation point with the planes zy and zx:

$$\sin\chi_x \equiv \frac{X}{R} \qquad \sin\chi_y \equiv \frac{Y}{R} \qquad\qquad k\alpha_x \equiv k\frac{X}{R} \equiv k\sin\chi_x \equiv u \qquad k\alpha_y \equiv k\frac{Y}{R} \equiv k\sin\chi_x \equiv v \quad . \tag{51.04.04}$$

→ 2280 2298

In the slightly more general case that the incoming wave is a MPW, but not perpendicular to the screen, the formula can be expanded to:

$$\begin{aligned}
\psi[X,Y] \propto \iint_\Sigma \exp\left[-\mathtt{i}k\left(\alpha_x x + \alpha_y y\right)\right] \, \mathtt{d}x \, \mathtt{d}y = \quad , \\
\iint_\Sigma \exp\left[-\mathtt{i}k\left(x\sin\chi_x + y\sin\chi_y\right)\right] \, \mathtt{d}x \, \mathtt{d}y = \quad , \\
\iint_\Sigma \exp\left[-2\pi\mathtt{i}\left(x\sin\chi_x/\lambda + y\sin\chi_y/\lambda\right)\right] \, \mathtt{d}x \, \mathtt{d}y = \quad , \\
\iint \exp\left[-2\pi\mathtt{i}\left(u[X]x + v[Y]y\right)\right] \, \mathtt{d}x \, \mathtt{d}y \quad .
\end{aligned}$$

→ 2298

which is the Fraunhofer regime from a plane aperture for normally incident PW.

It clearly **corresponds to the Fourier transform of the aperture**.

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51.04.03
Fraunhofer Diffraction From a Single Narrow Slit

By single slit one means a narrow rectangular aperture very long, L, with respect to its width b. Let z be the coordinate normal to the plane, x the coordinate along the slit long side and y the coordinate along the slit short side. One has:

$$\boxed{r = R - y\sin[\chi]} \qquad \boxed{\beta \equiv \frac{kb\sin[\chi]}{2}} \quad ,$$

$$\psi[\mathbf{x},t] = CL\int_{-b/2}^{+b/2}\exp[\mathtt{i}kr]\;\mathtt{d}y = CL\int_{-b/2}^{+b/2}\exp[\mathtt{i}k\,(R - y\sin[\chi])]\;\mathtt{d}y = CL\exp[\mathtt{i}kR]\int_{-b/2}^{+b/2}\exp[-\mathtt{i}ky\sin[\chi]]\;\mathtt{d}y \quad ,$$

$$\boxed{\psi[\mathbf{x},t] \propto Lb\frac{\sin[\beta]}{\beta}} \quad .$$

Note that the amplitude of the field is proportional to the area of the slit.

Note that, strictly speaking if the slit were of infinite length, the transferred energy would be infinite and therefore the constant obtained by integrating along the infinite slit, would be infinite as well. In fact, one should use a very long slit, rather than an infinite one: $L \gg b$.

The intensity is given by:

$$\boxed{I = I_{\mathtt{M}}\left(\frac{\sin[\beta]}{\beta}\right)^2} \quad .$$

The quantity $I_{\mathtt{M}}$ is the intensity at the maximum of the intensity, $\beta = 0$.

The extrema of the function $\sin[\beta]/\beta$ are determined by the equation: $\tan[\beta] = \beta$. The absolute maximum is at $\beta = 0$, as the denominator causes the ratio to decrease at increasing β.

The zeros are at

$$\boxed{\beta = m\pi \qquad \Leftrightarrow \qquad \sin[\chi] = m\frac{\lambda}{b} \qquad (m \neq 0)} \quad .$$

This can be easily understood when observing that, when the path difference among the two edges of the slit is an integer number of wavelengths, the resulting field is a superposition of a continuous distributions of infinitesimal sources with phases uniformly distributed in an integer number of 2π intervals, so that they sum to zero.

The width of the diffraction peak is given by the angular with of the first zeros. We have assumed the hypothesis that $|\chi| \ll 1$ therefore:

$$|\sin[\chi_+] - \sin[\chi_-]| \simeq |\chi_+ - \chi_-| \equiv \boxed{\Delta\chi = 2\frac{\lambda}{b} \qquad (m = \pm 1)} \quad .$$

51.04.04
Fraunhofer Diffraction From a Single Rectangular Slit

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For a rectangular slit located in the xy plane the expression of the relative intensity is just a generalization of the expression for a linear slit. Assume a rectangular slit of sides b_x and b_y in the xy plane centered at the origin.

This gives, in terms of the angles χ_x and χ_y that the position vector from the origin to the observation point forms with the zy and zx planes:

$$\boxed{\beta_x \equiv \frac{kb_x\sin[\chi_x]}{2} \qquad \beta_y \equiv \frac{kb_y\sin[\chi_y]}{2}} \quad .$$

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The intensity is given by:

$$I = I_{\mathtt{M}} \left(\frac{\sin[\beta_x]}{\beta_x} \right)^2 \left(\frac{\sin[\beta_y]}{\beta_y} \right)^2 \quad .$$

The quantity $I_{\mathtt{M}}$ is the intensity at the maximum of the intensity.

51.04.05
Fraunhofer Diffraction From a Single Circular Slit

Consider a circular slit of radius R and diameter $D \equiv 2R$. The intensity is given in terms of:

$$\rho \equiv \frac{kD \sin[\chi]}{2} \quad , \tag{51.04.05}$$

by:

$$I = I_{\mathtt{M}} \left(\frac{2J_1[\rho]}{\rho} \right)^2 \quad , \tag{51.04.06}$$

where J_1 is a Bessel function of first kind [1] and $I_{\mathtt{M}}$ is the intensity at the maximum of the intensity (that is on the axis).

The first zero of the intensity is at:

$$\rho = 3.832 \qquad \sin[\chi] = 1.22 \left(\frac{\lambda}{D} \right) \quad .$$

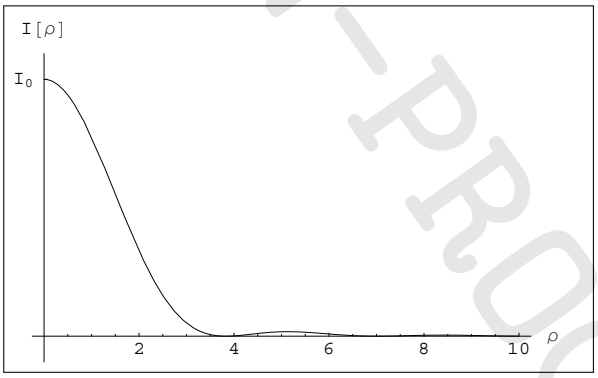

Figure 51.11: FIGURE

51.04.06
Resolving Power and Rayleigh Criterion

From the previous discussion it should be clear that nature puts an intrinsic limit to the resolving power of any optical instrument, due to the wave nature of light and diffraction.

The conventional definition of resolving power is the Rayleigh criterion. Two points are resolved if the maxima of the diffraction figures of the two points are at a distance larger that the half-width of the diffraction figure.

According to this criterion the resolving power is given by the relations in section § 51.04.

[1] Eric W. Weisstein. **Bessel Function of the First Kind**. From MathWorld–A Wolfram Web Resource. WEB - URL

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51.04.07
Fraunhofer Interference|Diffraction From a Interference|Diffraction Grating

When one needs to separate light of different wavelengths with high resolution a Interference|Diffraction Grating may be used, consisting in a large number of parallel, closely spaced slits. The Interference|Diffraction Grating leads to many applications for measurements.

One knows that a linear array of in-phase sources gives peaks whose position depends on the wavelength and on the inverse separation between sources (at a fixed order of the maximum, m). Moreover, the width of the peak decreases while increasing the number of sources. In order to separate different close wavelengths it is then necessary to use many sources with small spacing. This implies that the sources must be very thin slits (for instance). But this unavoidably introduce a significant diffraction.

Thanks to the large number of slits the intensity at the maximum is very sharp and the peak intensities are much higher while increasing the number of slits.

Let a Interference|Diffraction Grating be made of long, L, narrow rectangular slits having a width b ad a separation d.

$$\beta \equiv \frac{kb\sin[\chi]}{2} \qquad \delta \equiv \frac{kd\sin[\chi]}{2} \quad .$$

The intensity is given by:

$$I = I_{\mathtt{M}}\left(\frac{\sin[\beta]}{\beta}\right)^2\left(\frac{\sin[N\delta]}{N\sin[\delta]}\right)^2 \quad .$$

In order to keep the usual interpretation of the quantity $I_{\mathtt{M}}$ as the maximum intensity, the expression has been normalized dividing by N^2. The result is the result of a linear array whose amplitude is modulated by the diffraction term.

The smaller the slit the less evident is the effect of diffraction, consistently with the results obtained for the interference of a linear array of point sources. In fact, in this case, every single slit tends to become an isotropic emitter, thus realizing the linear array of isotropic antennas. The flatter the diffraction term the more and more maxima can be observed. If $b \ll d$ the modulation effect on the interference curve is small. In fact the limiting case $b = 0$ gives back the pure interference formula among N sources.

Some of the maxima of the interference figure might be missing if they coincide with zeros of the diffraction pattern.

The Interference|Diffraction Grating is normally simply called **diffraction grating** even if the name **interference grating** would be more appropriate, as it exploits interference among the slits.

Plots

51.04.07.01 Resolving Power of a Interference|Diffraction Grating

The maximum of order m of a Interference|Diffraction Grating with spacing d is given by:

$$\sin[\chi] = m\frac{\lambda}{d} \qquad m \text{ integer} \quad .$$

In the small angle approximation the angles where the maxima of order m for two different wavelengths are located are linked by:

$$\left|\sin\left[\chi_{\mathtt{M}|2}\right] - \sin\left[\chi_{\mathtt{M}|1}\right]\right| \simeq \left|\chi_{\mathtt{M}|2} - \chi_{\mathtt{M}|1}\right| = m\frac{|\lambda_2 - \lambda_1|}{d} \equiv m\frac{\Delta\lambda}{d} \quad .$$

The half-width of the interference peak, at the average wavelength, $\overline{\lambda} \equiv (\lambda_1 + \lambda_2)/2$, is, from equation (50.05.14),

$$\Delta\chi_{\mathtt{HW}} = \frac{\overline{\lambda}}{Nd} \quad .$$

In order to have separation of the two lines one then needs:

$$\frac{\overline{\lambda}}{Nd} = \Delta\chi_{\mathtt{HW}} < \left|\chi_{\mathtt{M}|2} - \chi_{\mathtt{M}|1}\right| = m\frac{\Delta\lambda}{d} \quad ,$$

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and the resolution is therefore:

$$\boxed{\frac{\Delta\lambda}{\overline{\lambda}} = \frac{1}{mN}} \quad .$$

Note that in order to have a large spread of different wavelengths it is necessary to have a small d and this implies a small b so that diffraction is not negligible.

It might be better called a interference grating.

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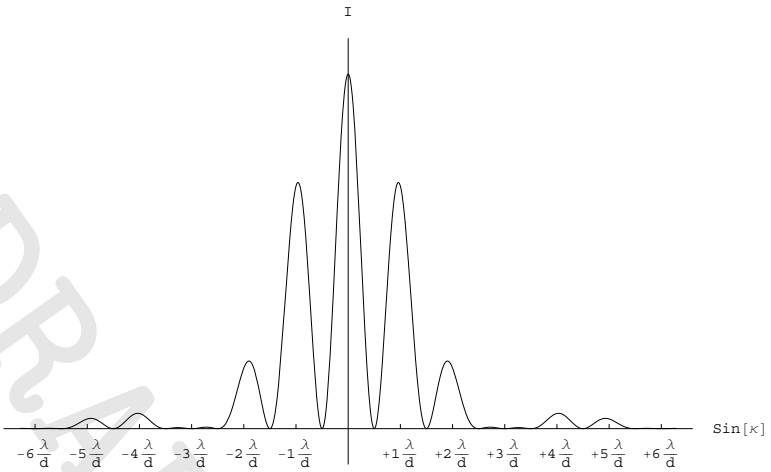
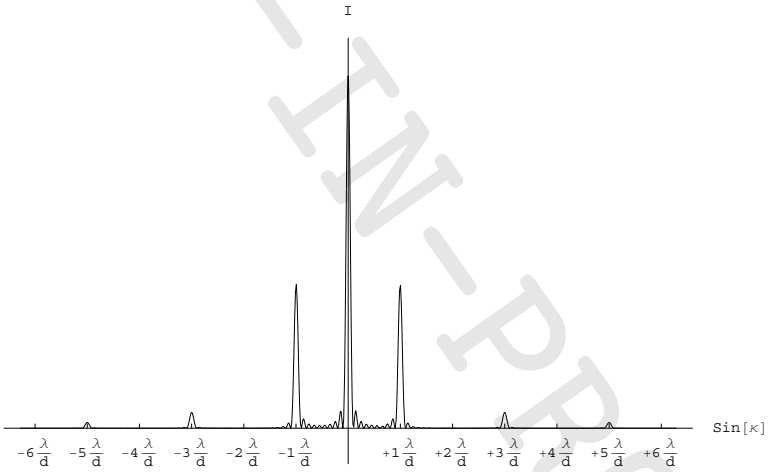


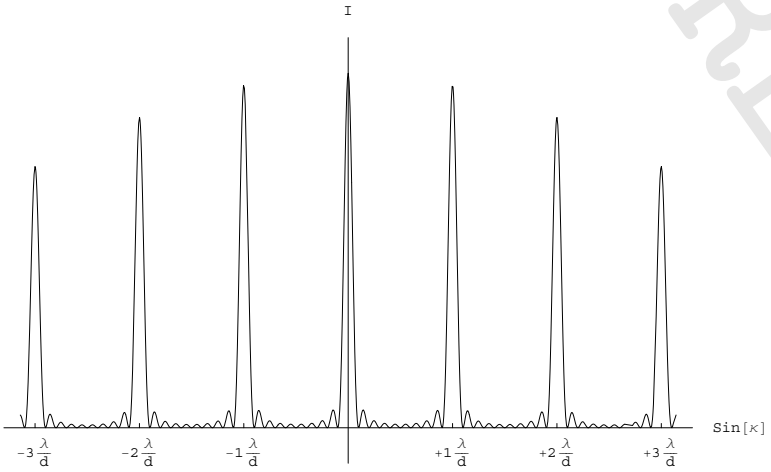
(a) $N = 2$, $d = 3b$ (and $\lambda = 3b$)

Figure 51.12: ....



(a) $N = 10$, $d = 2b$ (and $\lambda = 2b$)

Figure 51.13: ....



(a) $N = 10$, $d = 10b$ (and $\lambda = 10b$)

Figure 51.14: ....

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51.05 Fresnel Diffraction

51.06 Approximations In Diffraction and Realistic Diffraction situations

This § is referenced at pages:
[2282]

In the case of diffraction the single source must be coherent, that is, whatever the phase difference between any two infinitesimal sources, it remains constant in time, on average of time periods long with respect to the period of the monochromatic signal. Read § 50.09.

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51.07 Examples and Physical Applications

51.07.01 Diffraction From a Slit as a Limiting Case of the Interference of Infinite Sources

The diffraction from a single slit can be derived as a limiting case from the results of § 50.05.02.

Consider an increasing number of sources, N, in a fixed interval b with spacing d such that: $b = Nd$. Let the number of sources tend to infinity with b constant. In equation (50.05.11) I_0 is the intensity of a single source. In our limit we have an infinite number of infinitesimal sources. The limit shall be done by keeping a constant maximum intensity. It is then useful to normalize equation (50.05.11) to its maximum intensity, as it is normally done for diffraction:

$$I = \frac{I_{\text{M}}}{N^2}\left(\frac{\sin[N\delta]}{\sin[\delta]}\right)^2 = I_{\text{M}}\left(\frac{\sin[N\delta]}{N\sin[\delta]}\right)^2 \longrightarrow I_{\text{M}}\left(\frac{\sin[\beta]}{\beta}\right)^2 \quad .$$

51.07.02 Diffraction and Energy Conservation

This § is referenced at pages:
[2286]

Read § 50.10.07.

51.07.03 Resolving Power of a Telescope

In astronomy one likes optical instruments with large aperture both to collect more light from faint sources and to reduce the effects of diffraction.

Mount Palomar telescope: $D = 5$ m.

It follows, for $\lambda = 500$ nm, $\Delta\chi \simeq \lambda/D = 10^{-7}$ (diffraction limit).

The actual optical resolution is actually less than that due to the adverse effects of atmospheric turbulence which destroys the phase coherence.

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51.07.04
Angular Spread of a Laser-Pen

Assume a circular aperture with $D = 1$ mm.
It follows, for $\lambda = 500$ nm, $\Delta\chi \simeq \lambda/D = 5{\cdot}10^{-4}$.
At a distance $L = 100$ m this gives a spot size of ≈ 5 cm.

51.07.05
Resolving Power of the Human Eye (Rayleigh Criterion)

Assume a circular aperture with $D = 3$ mm.
The Rayleigh criterion gives, for $\lambda = 500$ nm, $\Delta\chi \simeq \lambda/D = 42''$.

51.07.06
Observation of Car-Lights (Rayleigh Criterion)

Find the maximum distance such that a typical human eye is capable to separate the two front lights of a car assuming their distance is $d = 1$ m.

51.07.07
A CD as a Grazing Interference|Diffraction Grating

©|WEB - URL ||*Many experiments with just a CD.*|

A CD is an example of a reflection Interference|Diffraction Grating. The typical distance between two neighboring tracks is $1.6\ \mu$m.

51.07.08
An Interference|Diffraction Grating

Assume: $\frac{\mathrm{d}N}{\mathrm{d}L} = 10^4$ lines/cm and $L = 1.0$ cm width. Therefore: $N = 10^4$ lines.

The first-order maximum ($m = 1$) gives: $\frac{\Delta\lambda}{\lambda} = 10^{-4}$. For $\lambda = 5{\cdot}10^{-7}$ m this implies: $\Delta\lambda = 0.5{\cdot}10^{-10}$ m. This is, roughly, the size of an hydrogen atom.

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51.08 Exercises Problems and Physical Applications

51-001 I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...

5.119, 5.120, 5.121, 5.122, 5.125, 5.126, 5.127, 5.136, 5.139, 5.140.

51-002 E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...

Read also sections from E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ..., § 10.

51-003 Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|§ 9|*A lot of problems.*|

51-004 Resolving Spectral Lines With an Interference Grating

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|5.139||

51-005 Resolving Power of a Telescope

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|5.146||

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|5.147||

51-006 Inclined Incident Wave in Fraunhofer regime

Generalize the expression for Fraunhofer regime, equation (51.04.01), to the case when the incident wave on the flat plane opaque screen forms an angle θ with respect to its normal.

51-007 Geometrical Relations and Fraunhofer Approximation (2)

©|E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|9.15||

Read § 50-010.

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CHAPTER 52

Interference Diffraction Wave-Fields and Coherence

52.01 Interference|Diffraction wrap-up 2310
52.02 Elementary Concepts About Coherence 2310
52.03 Interferometers 2315
52.04 Examples and Physical Applications 2315

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This § is referenced at pages:
[2257, 2292]

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52.01 Interference|Diffraction wrap-up

52.02 Elementary Concepts About Coherence

This § is referenced at pages:
[1165, 2143, 2259, 2259, 2261, 2316]

©|J.P.Pérez *et al.*, *Optique*, ..., DUNOD, ...Ed., WEB - URL.|excellent||

©|E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|excellent||

©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|||

In this section only **a few basic concepts** of the important theory of coherence will be given.

The waves from a distant, monochromatic point source is effectively a MPW and is said to be a **perfectly coherent wave**.

© - QUOTE

E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...
It is often convenient, even if artificial, to divide coherence effects into two classifications, temporal and spatial. The former relates directly to the finite bandwidth of the source, the latter to its finite extent in space.

It is convenient to classify coherence effects into two different classes: **temporal coherence and spatial coherence**.

Temporal coherence relates directly to the finite bandwidth, that is to the monochromaticity, of the source. For a single point source, the self-coherence (time-coherence) is essential for interference to occur and monochromaticity is implicit. It is also named **longitudinal coherence**.

Spatial coherence relates to the finite extent in space of the source. Real/Natural sources are typically made of a huge number of independent elementary sources at the atomic level and are said spatially incoherent sources, because, usually, there is no fixed relation among the waves emitted by the different elementary sources. Moreover, the intensities of the different sources may be different. For an extended source, the mutual coherence of the different component point sources is an essential point, even if each point source is self-coherent, in order to observe Interference | Diffraction. It is also named **lateral coherence**

© - QUOTE

R.Blandford & K.Thorne, *Applications Of Classical Physics*, ..., ..., ...Ed., WEB - URL, WEB - URL.
...
lateral or spatial coherence (coherence in the angular structure of the radiation field), and **temporal or longitudinal coherence** (coherence in the field's temporal structure, which clearly must imply something also about its frequency structure).
...
for both types of coherence there is a measurable quantity, called the **degree of coherence**, that is the **Fourier Transform** of either the radiation's angular intensity distribution (energy flux per unit angle or solid angle, as a function of direction) or its spectral energy flux (energy flux per unit angular frequency, as a function of angular frequency).

...

Independent sources may interfere as long as they are coherent sources; typical example are two different lasers, if they have a sufficient degree of mutual coherence.

52.02.01
Again on Necessary Conditions for Interference to Occur

In order to observe interference it is necessary to get rid of the randomness of the wave emission: interferometers may use only one source and apply either **division of front** or **division of amplitude**.

In order to have interference between two sources it is necessary that the two sources are, at least partially, coherent.

- One-dimensional case: the de-phasing at the observation point is only due to the two **phase constants** : there is no path-difference in the one-dimensional case, unless there are inversions of the velocity.
- Two/Three-dimensional case: the de-phasing at the observation point may also come from the difference in the two paths.

52.02.02
Natural Emission of ElectroMagnetic Waves

Two independent natural quasi-point sources do not normally interfere because of the randomness of natural emission, normally made of random wave-trains, independently emitted by quasi-point molecular|atomic|particle sources.

The wave-trains of an ElectroMagnetic MPW are emitted in a random way, which can be modeled in a simplified form as:

$$\boxed{\psi = A\cos\left[\omega_0 t + \phi\right]\widehat{\mathbf{e}}} \quad ,$$

with A **and** ϕ **real numbers, describing amplitude and phase,** $\widehat{\mathbf{e}}$ **a unit complex vector, describing the wave-Polarization state** (read § 46.03.03). The **parameters** A, phi **and** $\widehat{\mathbf{e}}$ **are assumed to vary in a random way** over times of the order of the **coherence time**, τ_c.

Moreover, as the MPW is not strictly monochromatic, the frequency has a spectral distribution around some central frequency, ω_0, instead of being a fixed value.

Some examples follow.

- $\tau_c \sim 10^{-13}$ s for an Hg lamp at $\lambda_0 = 546$ nm with spectral width $\Delta\lambda_{1/2} = 10$ nm: very short wave-trains of $\ell_c \sim 30$ μm, a few wavelengths long.
- $\tau_c \sim 0.7{\cdot}10^{-9}$ s for a He-Ne laser at $\lambda_0 = 632.8$ nm with spectral width $\Delta\nu_{1/2} = 1.4$ GHz: relatively log wave-trains of $\ell_c \sim 20$ cm, many wavelengths long.
- $\tau_c \sim 10^{-6}$ s for a stabilized He-Ne laser at $\lambda_0 = 632.8$ nm long wave-trains of $\ell_c \sim 400$ m, very many wavelengths long.

The period of the harmonic oscillations, for visible light, is of the order of $T \sim 10^{-14}$ s so that for the above cases $\tau_c \gg 1/\nu$.

Intensity is typically measured averaging over detection times, T_D, much longer than τ_c, for instance tenths of seconds by the human eye and milli-seconds by a CCD camera; the detection time is also typically longer than the period of the harmonic oscillation:

$$\boxed{T_D \gg \tau_c \qquad T_D \gg 1/\omega} \quad . \tag{52.02.01}$$

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52.02.03
Quasi-Monochromatic Point Source - Coherence Time and Lengths

This § is referenced at pages:
[Never referenced.]

Consider a quasi-monochromatic point source whose emission has the following characteristics: the wave is strictly monochromatic for a certain time τ_c, then the phase of the sinusoid | Cosinusoid changes abruptly in a random fashion, with uniform probability density for all possible values of the phase. The sequence repeats itself indefinitely. The time length τ_c is called the coherence time and it is clearly linked to the self-correlation function of the signal.

The ElectroMagnetic fields at any point can be expressed as:

$$\mathbf{F} = \mathbf{F}_0 \exp\left[-\mathtt{i}\omega_0 t\right] \exp\left[-2\pi\mathtt{i}\phi[t]\right] \quad ,$$

where $\phi[t]$ is a random step function with uniform probability distribution between zero and one. One can regard the above kind of field as an approximation for the radiation emitted by an atom.

Suppose the beam of light is divided into two beams that are subsequently brought together to interfere. It can be shown that the normalized auto-correlation function is given by:

$$\begin{cases} \gamma[\tau] = \left(1 - \dfrac{\tau}{\tau_c}\right) \exp\left[\mathtt{i}\omega_0 t\right] & \text{for } \tau < \tau_c \quad , \\ \gamma[\tau] = 0 & \text{for } \tau \geq \tau_c \quad , \end{cases} \quad .$$

The result means, in particular, that the path difference to observe interference must not exceed the coherence length:

$$\boxed{\ell_c \equiv c\tau_c = \frac{c}{\Delta\nu_{1/2}} = \frac{\lambda^2}{\Delta\lambda_{1/2}}} \quad ,$$

expressed in terms of the speed of the ElectroMagnetic wave, c.

Temporal coherence is a manifestation of spectral purity. If the waves were ideally monochromatic they would be a perfect sinusoids with an infinite coherence length. All real sources fall short of this, and **all real sources actually emit a range of frequencies**, albeit sometimes quite narrow.

52.02.04
Mutual Coherence of Two Different Monochromatic Point Sources

Two different point sources are said to be mutually coherent when the time-average of their phases is a non zero constant:

$$\boxed{\langle \cos\left[\Delta\Phi\right] \rangle = \text{constant} \neq 0} \quad ,$$

where the averaging is taken over averaging time $\mathcal{T}$ such that:

$$\boxed{\mathcal{T} \gg \tau_c \qquad \mathcal{T} \gg 1/\omega} \quad . \tag{52.02.02}$$

Two different point sources are said to be mutually incoherent when

$$\langle \cos\left[\Delta\Phi\right] \rangle = 0 \quad ,$$

where the averaging is taken over the detection time, T_D.

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52.02.05
Longitudinal|Transverse Spatial Coherence

Consider a single quasi-monochromatic quasi-point source.

52.02.05.01 Longitudinal Spatial Coherence

Consider two points aligned with the source, only differing in their distance from the source. The coherence between the fields at the two points will measure the **longitudinal spatial coherence** of the fields. It is clear that the longitudinal spatial coherence is just related to how large is the distance between the two points and the coherence length|time of the fields.

52.02.05.02 Transverse Spatial Coherence

Consider two points at the same distance from the source, only differing in their direction with respect to the source. The coherence between the fields at the two points will measure the transverse spatial coherence of the fields. In case the source is a strictly point source, the time dependence at the two points will be exactly the same, that is the fields at the two points will be totally coherent. In case the source is not a point source, but its size is small with respect to the distance between the two points where coherence is being studied, the time dependence at the two points will not be the same, in general.

52.02.06
Extended Source - Coherence at Different Points in Space - Transverse Coherence Length

This § is referenced at pages:
[Never referenced.]

This section deals with the problem of the coherence between two fields at different points in space. It depends on the size of the source.

Natural sources are typically made of independent almost-monochromatic almost-point sources, located at different positions. Each one must be treated as a single source and the combined effect of all the sources must be studied.

52.02.06.01 Extended Quasi-Monochromatic Source

Consider an extended quasi-monochromatic source. It can be considered made of a number of quasi-monochromatic quasi-point sources. The problem can be simplified by considering the extreme case of two point sources at the two edges of the extended source.

Assume that both the distance between the two point sources and the point where the fields are observed is small with respect to the distance between the sources and the observation point. It can be shown that the transverse coherence width is expressed, in terms of the wavelength, λ, and the angular separation of the sources as it is seen from the observation point, θ_s, as:

$$\boxed{\ell_t = \frac{\lambda}{\theta_s}} \quad .$$

It is called the **transverse spatial coherence width**. The transverse spatial coherence width gives the approximate dimensions of the width of the region of high mutual coherence of two different point sources. In fact, any two points at a distance D will have coherent wave-fields, that is wave-fields having a constant, on average, phase difference, as long as:

$$\boxed{D \ll \ell_t} \quad .$$

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Special case: demonstration from Young two-slits fenditure

Consider Young two-slits experiment with a source whose maximum dimension along the direction of the flat screen perpendicular to the slits is D, its distance from the screen is L. Two elementary quasi monochromatic quasi-point sources at the two extreme points will produce interference figures separated or not according to Rayleigh criterion depending on:

$$\theta_s \equiv \frac{D}{L} \leftrightarrow \frac{\lambda}{d} \quad .$$

The interference figure is observed if and only if:

$$\boxed{\theta_s \equiv \frac{D}{L} \ll \frac{\lambda}{d}} \quad .$$

Consider a slit experiment with a source whose maximum dimension along the direction of the flat screen perpendicular to the slits is D, its distance from the screen is L. Two elementary quasi monochromatic quasi-point sources at the two extreme points will produce diffraction figures separated or not according to Rayleigh criterion depending on:

$$\theta_s \equiv \frac{D}{L} \leftrightarrow \frac{\lambda}{b} \quad .$$

The diffraction figure is observed if and only if:

$$\boxed{\theta_s \equiv \frac{D}{L} \ll \frac{\lambda}{b}} \quad .$$

In both cases, the source has to be small enough to observe Interference | Diffraction.

In both cases, increasing the wavelength and | or decreasing $d \,|\, b$ helps to observe Interference | Diffraction, for a given source.

If one starts from a situation such that Interference | Diffraction is observed and $d \,|\, b$ is then increased, the Interference | Diffraction pattern, sooner or later, will smoothly disappear. Read § 52.04.03.

52.02.06.02 The Van Cittert-Zernicke Theorem

©|R.Blandford & K.Thorne, *Applications Of Classical Physics*, ..., ..., ...Ed., WEB - URL, WEB - URL.|§ 9.2.2||

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52.03 Interferometers

©|E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|§ 9.3 and 9.4||

Interferometers are a large important class of instruments useful for many purposes.

52.03.01 Amplitude Interferometry

Interferometers can also be built for particles, due to wave-particle dualism, in particular for electrons and neutrons.

52.03.01.01 Front Division - Rayleigh Interferometer

52.03.01.02 Amplitude Division - Michelson Interferometer

This § is referenced at pages:
[2398]

52.03.01.03 Amplitude Division - Jamin Interferometer

52.03.02 Intensity Interferometry

52.03.02.01 Handbury-Brown and Twiss Interferometry

52.04 Examples and Physical Applications

52.04.01 Coherence Width of Solar Light and Venus Light

Sun. Assume: $\theta = 32'$ and $\lambda = 550$ nm. One has: $\ell_t = 60$ μm.
Venus. Assume: $\theta = 1'$ and $\lambda = 550$ nm. One has: $\ell_t = 2$ mm.

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52.04.02
LASER

A laser is an artificial source which has a large coherence time and a large spatial coherence width, as compared to natural sources.

52.04.03
Michelson Stellar Interferometry

This § is referenced at pages:
[2314]

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|5.133||

Typical stellar diameters of large and nearby stars are in the range of 10^{-7} rad. This implies that the transverse coherence width at the Earth is of the order of a few meters.

A method to measure the angular diameter of stars was invented by Michelson, by observing the Young interference pattern with variable distance of the two slits. This is determined by just observing the disappearance of the interference fringes when the distance between the two slit is increased. Michelson thus measured the angular diameter of Betelgeuse to be $2.3{\cdot}10^{-7}$ rad.

It works by detecting the appearance and disappearance of the interference fringes when changing the distance between two parallel telescopes operating as an interferometer. The appearance and disappearance of the interference fringes is related to the size of the source, which can then be measured.

52.04.04
Radio-Telescope Interferometry

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|problem 9.57|*Interesting deep discussion*|

52.04.05
Source Extension

This section is a special case of the discussion of the coherence width in section § 52.02.

Suppose that one wants to look for diffraction from a circular slit of diameter $D = 0.2$ mm in a black screen with light with $\lambda = 500$ nm. We want to determine the maximum extension of the source to observe the diffraction figure if the source is located just behind the slit perpendicularly to the black screen at a distance L for the screen.

Two different point sources, constituents of the extended source and totally incoherent (actually independent), whose distance projected parallel to the black screen is ℓ (lateral distance), will give two central maxima of the diffraction figure separated, approximately, by

$$\chi \approx \frac{\ell}{L} \quad .$$

This must be much smaller than the width of the central peak itself in order that the two diffraction figures are not distinguishable:

$$\chi \approx \frac{\ell}{L} \ll \frac{\lambda}{D} \quad .$$

Therefore the maximum lateral extension of the source is:

$$\boxed{\ell \ll \frac{\lambda}{D} L} \quad .$$

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Example

Assume: $\lambda = 500$ nm, $D = 1$ mm and $L = 0.1$ m.
Then $\ell = 5{\cdot}10^{-5}$ m.

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CHAPTER 53

Microscopic Classical Model for Interaction of ElectroMagnetic Waves and Matter

53.01 Drude-Lorentz Dispersion Model for a Dilute non-dipolar and Non-Magnetic Medium 2321
53.02 Radiation Scattering From Atoms|Molecules in a Dilute Gas 2328
53.03 Coherent and Incoherent Scattering From Atoms|Molecules 2330
53.04 Examples and Physical Applications 2331
53.05 Exercises Problems and Physical Applications 2333

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This § is referenced at pages:
[2011, 2336, 2336, 2337, 2340, 2340, 2344, 2346, 2355]

A simple microscopic classical model, the Drude-Lorentz model, is elaborated, as an application of all the previously introduced elements, producing useful insight within a very simple framework.

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|6|*Good text.*|
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|7.5||

When ElectroMagnetic waves go cross matter, the electric charge of any obstacle, which might be electrons, ions, atoms, molecules, ..., are set into oscillatory motion by the ElectroMagnetic fields of the incident wave. Those accelerated charges re-radiate electromagnetic energy in all directions, a priori. This secondary radiation is called the **radiation scattering** by the obstacle. Scattering is given by excitation and subsequent re-radiation. In addition to re-radiating ElectroMagnetic energy, the excited elementary charges may transform part of the incident energy into other forms (thermal energy, for example): it is **radiation absorption**. Scattering and absorption are not mutually independent processes. The overall loss of energy is called **extinction: scattering plus absorption**. Read § 46.04.

In terms of photons, scattering is described by cross-section of processes as in figure **??**.

© - QUOTE

J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...
The scattering of ElectroMagnetic waves by systems whose individual dimensions are small when compared to the wavelength is a common case. It is convenient to think of the incident ElectroMagnetic fields as inducing electric and magnetic **Multipoles** oscillating with a definite phase relationship with respect to the incident ElectroMagnetic wave and radiate energy in directions other than the direction of incidence. The form of the angular distribution of radiated energy is governed by the coherent superposition of **Multipoles** induced by the incident ElectroMagnetic fields and in general also depends on the state of **wave-Polarization** of the incident ElectroMagnetic wave. If the wavelength of the radiation is long compared to the size of the scatterer, only the lowest **Multipoles** , usually **Electric Dipole** | **Magnetic Dipole** , are important and, in these circumstances, the induced **Dipoles** can be calculated from static or quasi-static boundary-value problems.

Electric and Magnetic **Dipoless** , oscillating with a definite phase relationship with respect to the incident ElectroMagnetic wave, are usually the dominant terms with respect to higher-order multipole fields.

When ElectroMagnetic waves interact with matter:

- the electric field excite Polarization in matter, and the induced **Electric Dipoles** re-radiate **Electric Dipole** radiation, typically.
- the magnetic field excite Magnetization in matter, and the induced **Magnetic Dipoles** re-radiate **Magnetic Dipole** radiation, typically.

Normally **Electric Dipole** radiation is more intense intense than **Magnetic Dipole** radiation due to relative weakness of **magnetic susceptibility** versus **electric susceptibility** for common LHI media (read § 49.07.05).

In the equation of motion of a non relativistic charge in ElectroMagnetic fields, the magnetic force is most often negligible with respect to the electric force: read § 46.05.02. This properties is assumed in Drude-Lorentz microscopic classical model of the ElectroMagnetic properties of matter.

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As the media which can be treated with simple models are basically linear media and because of the fact that linear magnetic media are restricted to paramagnetic and diamagnetic substances, whose **Magnetic-Permeability** is very close to one, normally, the **Magnetic-Permeability** will be assumed to be exactly one at any frequency in this section.

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53.01 Drude-Lorentz Dispersion Model for a Dilute non-dipolar and Non-Magnetic Medium

This § is referenced at pages:
[2328, 2341, 2344]

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|excellent!!!!!!!!!!!!!!||

The **simplified and approximate classical Drude-Lorentz model** for the Polarization of a medium when subject to external ElectroMagnetic waves is presented. In spite of its limitations the model clearly explains the physical bases of the fact that the **electric susceptibility is frequency dependent and it is a complex number**, implying **a phase shift between the applied (external) field** and the Polarization | Magnetization.

The relevant particles, in this context, are typically electrons in molecules, as will be clear at the end. Therefore it can be typically assumed that the molecules, thanks to their large mass, are subject to negligible motion when compared to electrons.

However, different sets of electrons must be considered as well, because different electrons might be subject to different restoring forces and | or different friction forces. The position will be typically either the position of the electron with respect to the atomic nucleus or the Center-Of-Mass of the molecule.

The dominant effect of electrons is confirmed by the results which will show an inverse dependence on the mass of the particles, thus consolidating the model based on electron oscillations.

The model, while not really suitable for realistic quantitative predictions, is definitely suitable to explain the **qualitative aspects** of the interaction between time-varying ElectroMagnetic fields and matter.

Consider a homogeneous, isotropic and linearly Polarizable medium with negligible magnetic properties:

$$\boxed{\mu_{\mathrm{R}} = 1} \quad ,$$

as justified in section § 49.07.05.

Consider different sets of particles with masses m_s and charges q_s whose position is $\mathbf{x}_s$ and displacement from the equilibrium position, $\mathbf{x}_{\mathrm{C}}$, is $\mathbf{r}_s \equiv \mathbf{x}_s - \mathbf{x}_{\mathrm{C}}$. The position $\mathbf{x}_{\mathrm{C}}$ is the generic position of the center for the restoring force acting on the particle. Assume each set of particles is subject to:

- a linear (elastic) restoring forces with proper frequency $\overline{\omega}_s$, $\mathbf{f}_{\mathrm{E}} = -m_s\overline{\omega}_s^2\mathbf{r}_s$, which is always applicable for small oscillations with respect to the equilibrium configuration;
- some kind of internal friction force, modeled by $\mathbf{f}_{\mathrm{F}} = -m_s\Gamma_s\dot{\mathbf{x}}_s = -m_s\Gamma_s\dot{\mathbf{r}}_s$; this is a rough way to model **energy losses**, such those from ElectroMagnetic radiation and losses of kinetic energies by collisions.

Other ElectroMagnetic properties (such as **Electric Dipole** | **Magnetic Dipole**) are assumed to have a negligible effect.

It is also assumed that the magnetic part of the Lorentz force has a negligible effect. Note that the latter assumption is not always acceptable, because the magnetic part of the Lorentz force is the one which determines the radiation pressure from ElectroMagnetic waves on matter: read § 46.05.02.

According to this model, the medium is a conducting medium if and only if there exist one set of particles, let us say $s = \widetilde{s} = 0$, with zero proper frequency, that corresponds to particles free to move with no restoring force:

$$\boxed{s = \widetilde{s} = 0 \qquad \Leftrightarrow \qquad \overline{\omega}_{\widetilde{s}} = 0} \quad .$$

Therefore, we assume here:

$$\boxed{\sigma = 0} \quad ,$$

and will interpret in section § 53.01.01 the results for the electrons with zero restoring force in terms of **conductivity** . **This approach, instead of separately considering free and bound electrons, is simpler for calculations as treats all electrons in the same way**.

Consider a ElectroMagnetic MPW, in vacuum, with a definite **wave-Polarization** , and a particular set of particles, possibly located near the origin of a given Coordinate System. In the general case superposition of fields might be used.

Assume that **the wavelength of the ElectroMagnetic wave is much larger than the dimensions of the system**,

$$\boxed{|\mathbf{x}_s - \mathbf{x}_C| \ll \lambda} \quad , \tag{53.01.01}$$

such that the ElectroMagnetic fields can be considered uniform on the set of charges.

Consider a **dilute medium**, so that **the difference between applied (external) field and the local field on the particles can be neglected**. If that is not the case the same arguments used for deriving Clausius-Mossotti relation may be used.

Assuming a **non relativistic motion** of the charges near the equilibrium position at $\mathbf{x}_C$, so that $\mathbf{x}_s \simeq \mathbf{x}_C$, the equation of motion is written in terms of the applied (external) field at $\mathbf{x}_s$:

$$\mathbf{r}_s \equiv \mathbf{x}_s - \mathbf{x}_C \quad ,$$
$$\overline{\mathbf{E}}_0 \exp\left[+\mathtt{i}\,\mathbf{k}\cdot\mathbf{x}\right] \exp\left[-\mathtt{i}\omega t\right] = \quad ,$$
$$\overline{\mathbf{E}}_0 \exp\left[+\mathtt{i}\,\mathbf{k}\cdot\mathbf{r}\right] \exp\left[+\mathtt{i}\,\mathbf{k}\cdot\mathbf{x}_C\right] \exp\left[-\mathtt{i}\omega t\right] \simeq \quad ,$$
$$\overline{\mathbf{E}}_0 \exp\left[+\mathtt{i}\,\mathbf{k}\cdot\mathbf{x}_C\right] \exp\left[-\mathtt{i}\omega t\right] \qquad \text{as } \mathbf{k}\cdot\mathbf{r} \ll 1 \text{ thanks to } |\mathbf{x}_s - \mathbf{x}_C| \ll \lambda \quad .$$

Therefore, for any charge of type s, with $\mathbf{r}_s \equiv \mathbf{x}_s - \mathbf{x}_C$:

$$\boxed{\ddot{\mathbf{r}}_s + \Gamma_s \dot{\mathbf{r}}_s + \overline{\omega}_s^2 \mathbf{r}_s = \frac{q_s}{m_s} \overline{\mathbf{E}}_0 \exp\left[+\mathtt{i}\,\mathbf{k}\cdot\mathbf{x}_C\right] \exp\left[-\mathtt{i}\omega t\right] \qquad \mathbf{r}_s \equiv \mathbf{x}_s - \mathbf{x}_C} \quad .$$

Any general **state of wave-Polarization is included in the description of the applied (external) field** $\overline{\mathbf{E}}_0$.

The **steady-state solution** for the position as a function of time for one specific specie of particle, s, near $\mathbf{x}_C$ is:

$$\boxed{\mathbf{r}_s[t] = \frac{\mathbf{p}_s[t]}{q_s} \equiv \frac{\mathbf{p}_{0s}[t]}{q_s} \exp\left[-\mathtt{i}\omega t\right] = \frac{q_s}{m_s} \left(\frac{\overline{\mathbf{E}}_0 \exp\left[+\mathtt{i}\,\mathbf{k}\cdot\mathbf{x}_C\right] \exp\left[-\mathtt{i}\omega t\right]}{\overline{\omega}_s^2 - \omega^2 - \mathtt{i}\omega\Gamma_s} \right)} \quad , \tag{53.01.02}$$

→ 2328

$$\boxed{\mathbf{p}_{0s}[\mathbf{x}_C] = \frac{q_s^2}{m_s} \left(\frac{\overline{\mathbf{E}}_0 \exp\left[+\mathtt{i}\,\mathbf{k}\cdot\mathbf{x}_C\right]}{\overline{\omega}_s^2 - \omega^2 - \mathtt{i}\omega\Gamma_s} \right)} \quad . \tag{53.01.03}$$

We now move from the description of the phenomena around some specific point, $\mathbf{x}_C$, to a continuum description as a function of a generic point $\mathbf{x}$. In the case of a dilute medium it is possible to neglect the distinction between local and average field and the matter Polarization can be written as:

$$\boxed{\mathbf{P}[\mathbf{x}, t] = \sum_s \eta_s q_s \mathbf{r}_s[t] = \sum_s \frac{\eta_s q_s^2}{m_s} \left(\frac{\overline{\mathbf{E}}_0 \exp\left[+\mathtt{i}\,\mathbf{k}\cdot\mathbf{x}\right] \exp\left[-\mathtt{i}\omega t\right]}{\overline{\omega}_s^2 - \omega^2 - \mathtt{i}\omega\Gamma_s} \right)} \quad , \tag{53.01.04}$$

→ 2322

where the number density of particles of the specie s, η_s, has been introduced. The resulting equation is simply the equation of a forced damped harmonic oscillator.

In the case the applied (external) field differs from the average local field the appropriate correction factors need to be introduced. Read § 53.01.02 for the case of a non-dilute, that is dense, medium.

The relation (53.01.04) shows the fundamental result that **a phase-shift is present between the applied (external) electric field and the Polarization if and only if there is a non-zero damping term**, $\Gamma_s \neq 0$. In order to better grasp the meaning of the relation (53.01.04) is it useful to make explicit the real and imaginary parts of the complex amplitude (the so-called absorptive and elastic amplitudes, read § 25.05.02):

$$\boxed{\frac{1}{\overline{\omega}^2 - \omega^2 - \mathtt{i}\omega\Gamma} = \left(\frac{\overline{\omega}^2 - \omega^2}{\left(\overline{\omega}^2 - \omega^2\right)^2 + \omega^2\Gamma^2} \right) + \mathtt{i} \left(\frac{\omega\Gamma}{\left(\overline{\omega}^2 - \omega^2\right)^2 + \omega^2\Gamma^2} \right) \equiv \mathcal{A}_{\mathtt{E}}[\omega] + \mathtt{i}\mathcal{A}_{\mathtt{A}}[\omega] \qquad \mathcal{A}_{\mathtt{E}} \gtrless 0 \text{ and } \mathcal{A}_{\mathtt{A}} \geq 0} \quad . \tag{53.01.05}$$

→ 2323 2333

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Note that in the frequency region far away from the resonance one has: Therefore **far away from the resonance the amplitude is essentially an elastic amplitude**, as

$$\left|\omega^2-\overline{\omega}^2\right|\gg\omega\Gamma\implies\begin{cases}\mathcal{A}_{\mathrm{E}}\approx\dfrac{1}{\overline{\omega}^2-\omega^2}\quad,\\ \mathcal{A}_{\mathrm{A}}\approx\dfrac{\omega\Gamma}{\left(\omega^2-\overline{\omega}^2\right)^2}\ll\dfrac{1}{\left|\omega^2-\overline{\omega}^2\right|}\ll\dfrac{1}{\omega\Gamma}\quad,\end{cases}\quad,\tag{53.01.06}$$

→ 2333

$$\left|\mathcal{A}_{\mathrm{E}}\right|\approx\frac{1}{\left|\overline{\omega}^2-\omega^2\right|}\gg\mathcal{A}_{\mathrm{A}}\quad.$$

When there are many resonance frequencies the curve become more complex if the different peaks are superimposed but easy quantitative conclusion can be drawn if the peaks are well separated.

It is worth remembering that the resonance curves in equation (53.01.05) have, mathematically, a very strong resemblance with the ones obtained from quantum-relativistic calculations, justifying their use regardless of the approximate derivation.

The **electric susceptibility** and **Electric-Permittivity** follow, where we denote them with a prime to make clear that they also **include the contribution from conduction charges**:

$$\varepsilon'_{\mathrm{R}}=1+\chi'_{\mathrm{E}}=1+\sum_s\frac{\eta_s q_s^2}{\varepsilon_0 m_s}\left(\frac{1}{\overline{\omega}_s^2-\omega^2-\mathrm{i}\omega\Gamma_s}\right)=1+\sum_s\frac{\eta_s q_s^2}{\varepsilon_0 m_s}\left(\frac{\overline{\omega}_s^2-\omega^2}{\left(\overline{\omega}_s^2-\omega^2\right)^2+\omega^2\Gamma_s^2}+\mathrm{i}\frac{\omega\Gamma_s}{\left(\overline{\omega}_s^2-\omega^2\right)^2+\omega^2\Gamma_s^2}\right)\quad.$$

(53.01.07)

→ 2323 2325 2344 2355

The separation of conduction from bound charges gives:

$$\varepsilon'_{\mathrm{R}}=1+\chi'_{\mathrm{E}}=1+\sum_s\frac{\eta_s q_s^2}{\varepsilon_0 m_s}\left(\frac{1}{\overline{\omega}_s^2-\omega^2-\mathrm{i}\omega\Gamma_s}\right)=1+\frac{\eta_0 q_0^2}{\varepsilon_0 m}\left(\frac{1}{-\omega^2-\mathrm{i}\omega\Gamma_0}\right)+\sum_{s>0}\frac{\eta_s q_s^2}{\varepsilon_0 m_s}\left(\frac{1}{\overline{\omega}_s^2-\omega^2-\mathrm{i}\omega\Gamma_s}\right)\quad.$$

(53.01.08)

→ 2323 2324

In the above expressions, equations (53.01.07), (53.01.08), the factor of the amplitudes is a squared frequency called **plasma frequency**, for charges of the specific type:

$$\left(\omega_p^2\right)_s\equiv\frac{\eta_s q_s^2}{\varepsilon_0 m_s}\quad,\tag{53.01.09}$$

see also equations (54.02.14).

Using the generic **Dispersion Relation** in equation (54.02.04) (read § 46.06):

$$k^2=\varepsilon_{\mathrm{R}}\mu_{\mathrm{R}}\varepsilon_0\mu_0\omega^2+\mathrm{i}\omega\mu_{\mathrm{R}}\mu_0\sigma\equiv\varepsilon'_{\mathrm{R}}\mu_{\mathrm{R}}\varepsilon_0\mu_0\omega^2\quad,$$

the relation between the **electric susceptibility/Electric-Permittivity** with/without inclusion of the conduction charges give:

$$\varepsilon'_{\mathrm{R}}=\varepsilon_{\mathrm{R}}+\mathrm{i}\frac{\sigma}{\omega\varepsilon_0}\qquad\chi'_{\mathrm{E}}=\chi_{\mathrm{E}}+\mathrm{i}\frac{\sigma}{\omega\varepsilon_0}\quad.\tag{53.01.10}$$

→ 2090 2324 2344 2351

The two equivalent ways to describe the problem consist therefore in using either of two sets of variables:

- free plus bound charges, separately:

$$\varepsilon_{\mathrm{R}}\qquad\mu_{\mathrm{R}}\qquad\sigma\neq0$$

- all charges:

$$\varepsilon'_{\mathrm{R}}\qquad\mu_{\mathrm{R}}\qquad\sigma\equiv0$$

Usually, which of the two approaches is used is implicit in the description.

Equation (53.01.07) shows that the imaginary part of the **Electric-Permittivity** is due to the damping factor Γ, that is it is due to some kind of internal friction. Moreover, equation (53.01.07) shows that the imaginary part is only significant near the resonance frequencies. **Away from the resonance frequencies the Electric-Permittivity can be considered, to a first approximation, as real and absorption is negligible**.

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In case of dilute media:

$$\eta\alpha_e \ll 1 \implies \boxed{n \equiv \sqrt{1+\chi_{\mathrm{E}}} \simeq 1+\chi_{\mathrm{E}}/2} \quad .$$

53.01.01
Result of the Drude Model for Conductors

This § is referenced at pages:
[2321, 2341]

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|ST9|*Detailed and complete discussion*|

In conductors, the bound charges give rise to a Polarization current as it is the case for insulators. However, **in non-static situations there is no real difference between free and bound charges**.

Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...- ST9

And finally the model, equation (53.01.08) and (53.01.10), gives the expression for the **conductivity** , essentially Drude model:

$$\boxed{\sigma = \frac{\eta_0 q_0^2}{m_0}\left(\frac{1}{\Gamma_0 - \mathrm{i}\omega}\right)} \quad .$$

The parameter Γ_0 is determined by the measured value of the **conductivity** at low frequency (the static **conductivity**).

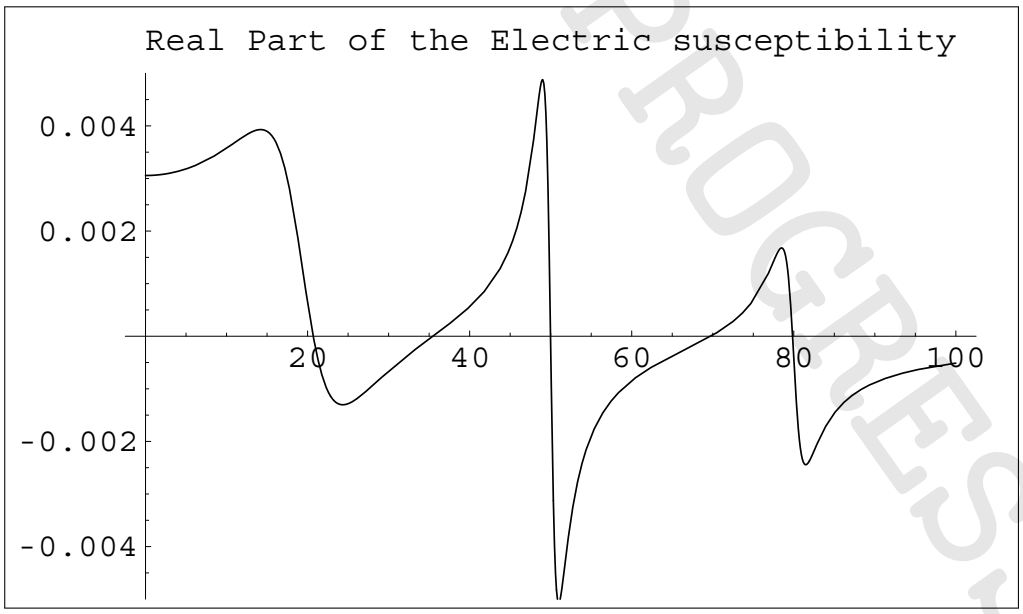

Figure 53.11: FIGURE

Note that, as the model is a resonant model, the real part of electric susceptibility becomes negative in narrow energy range around the resonance frequencies, the **anomalous dispersion region**.

WORK IN PROGRESS

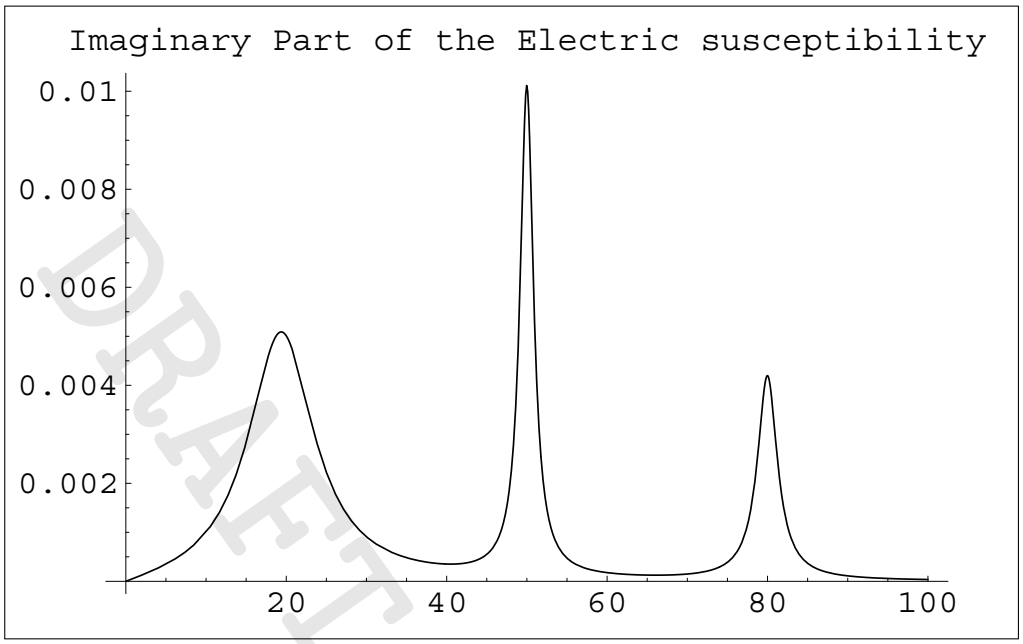

Figure 53.12: FIGURE

DRAFT IN PROGRESS

53.01.02
Dense Media

This § is referenced at pages:
[2322]

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If the density of the medium is not small enough the distinction between local and average field must be accounted for; read § 38.09, § 38.09. It will be assumed that this distinction only applies to bound charges, because, free charges typically feel the average fields, as long as they move on distances which are large on the microscopic scale; in this case they just feel what is, by definition, the average field. This is of course a reasonable assumption, but it is an assumption. In general, for bound electrons, the correction might be required.

Assume a non magnetic medium, $\mu_{\mathrm{R}} = 1$. One proceeds as in the case of Clausius-Mossotti, introducing the **Polarisability** , α, and writing the equation, for simplicity, for one single type of charges:

$$\mathbf{E}_{\mathrm{L}} = \mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0} \quad ,$$

$$\mathbf{P} = \eta_{\mathrm{N}} \alpha \varepsilon_0 \mathbf{E}_{\mathrm{L}} \quad ,$$

$$\mathbf{P} = \frac{3\eta_{\mathrm{N}} \alpha \varepsilon_0}{3 - \eta_{\mathrm{N}} \alpha \varepsilon_0} \mathbf{E} \quad ,$$

where the expression (53.01.07) was used.

Distinguishing between the local and average field one finds:

$$\frac{\varepsilon_{\mathrm{R}} - 1}{\varepsilon_{\mathrm{R}} + 2} = \frac{\eta \alpha_e}{3} = \frac{\eta q^2}{3\varepsilon_0 m} \left(\frac{1}{\overline{\omega}^2 - \omega^2 - \mathrm{i}\omega\Gamma} \right) \quad .$$

In the case of a dense medium with the different resonance frequencies widely spaced the use of **Clausius-Mossotti equation** with expression (53.01.07) is roughly equivalent to replace, in expression (53.01.07)

the proper resonance frequency according to:

$$\overline{\omega}^2 \to \overline{\omega}^2 - \frac{\eta q^2}{3m\varepsilon_0} \quad . \tag{53.01.11}$$

→ 2333

53.01.03
Take-away Lessons From the Microscopic Model

The simple microscopic model shows a few important facts.

- The **electric susceptibility depends on the inverse mass** of the oscillating particle and therefore it is justified to assume that the effect of electrons is dominant. One expects that the smaller the mass of the oscillating particle the larger its resonance frequencies, as this is typically inversely proportional to the mass of the particle, as for the harmonic oscillator.
- The **electric susceptibility depends on the frequency** of the applied EM fields.
- The **electric susceptibility is a complex number**, that is a phase shift is in general present between the applied ElectroMagnetic fields and the Polarization, if and only if there is some form of damping.
- In the **case of resonance frequencies widely spaced**, that is with peaks located at large distances with respect to the peak width, the **electric susceptibility** is almost a pure imaginary number, near the resonance frequencies, while far-away from the resonances the **electric susceptibility** is almost a real number.
- The **electric susceptibility tends to zero for very large frequencies**. The microscopic model allows to interpret this as due to the fact that the motions of the charges cannot follow the applied field for fields of very high frequency.

The relations between wave-vector, **index of refraction** and **phase-velocity** are all frequency dependent:

$$k^2[\omega] = \omega^2\varepsilon_0\mu_0\varepsilon'_{\mathrm{R}}[\omega]\mu_{\mathrm{R}}[\omega] = \frac{\omega^2}{c^2}\varepsilon'_{\mathrm{R}}[\omega]\mu_{\mathrm{R}}[\omega] \quad .$$

In the most common case of $\mu_{\mathrm{R}} = 1$ one has:

$$\mathbf{k} \equiv \mathbf{k}_{\mathrm{R}} + \mathrm{i}\mathbf{k}_{\mathrm{I}} \implies \mathbf{k}\cdot\mathbf{k} = k_{\mathrm{R}}^2 - k_{\mathrm{I}}^2 + 2\mathrm{i}\,\mathbf{k}_{\mathrm{R}}\cdot\mathbf{k}_{\mathrm{I}} = \frac{\omega^2}{c^2}\varepsilon'_{\mathrm{R}}[\omega] \quad .$$

It follows that the fact that the **Electric-Permittivity is a complex number requires an imaginary part of the wave-vector**, that is absorption if the EM waves. This is consistent with the fact that the imaginary part of the **Electric-Permittivity** is significantly different from zero only near the resonances when there is a damping factor, Γ_s, such that energy is transferred away from the ElectroMagnetic fields.

Also the **index of refraction** depends on the frequency:

$$n^2[\omega] = \varepsilon'_{\mathrm{R}}[\omega]\mu_{\mathrm{R}}[\omega] \qquad n[\omega] = \sqrt{\varepsilon'_{\mathrm{R}}[\omega]\mu_{\mathrm{R}}[\omega]} \quad .$$

The fact that the **index of refraction** depends on the frequency has no special consequence if one only considers monochromatic ElectroMagnetic waves at a fixed frequency, except that one needs to use the exact values for the **index of refraction** at the right frequency.

In the case of superposition of two or more frequencies there is no rigid propagation of the wave pulse any longer, as the different frequency components travel at different speeds. Moreover, the effect is also visible in different refraction angles for ElectroMagnetic waves of different frequencies.

UNDER THE PROGRESS

53.01.04
Normal and Anomalous Dispersion and Cauchy Relation

One speaks of **normal dispersion** in the frequency regions where the **index of refraction** is an increasing function of the frequency, which is seen to be everywhere except just above any resonance frequency. One speaks of **anomalous dispersion** in the regions where the **index of refraction** is a decreasing function of the frequency, which is seen to be just above any resonance frequency.

The name came from light dispersion: **normal dispersion** is the normal sequence of rainbow colors: shorter wavelengths (for instance blue with respect to red) are more refracted than longer wavelengths (blue more refracted than red) as the **index of refraction** of blue is larger than the **index of refraction** of red; **anomalous dispersion** is the inverted sequence of rainbow colors.

Many materials have proper resonance frequencies in the UV and therefore anomalous dispersion cannot be observed in visible light. At frequencies $\omega \ll \overline{\omega}$ and for small enough damping the **index of refraction** , considering only one term is the sum, that is the closest higher resonance frequency, can be written for sufficiently dilute media, as:

$$\left|\overline{\omega}^2 - \omega^2\right| >> \omega\Gamma \implies n \simeq 1 + \frac{\chi_{\mathrm{E}}}{2} \simeq \frac{\eta_{\mathrm{N}} q^2}{m\varepsilon_0} \frac{1}{\left(\overline{\omega}^2 - \omega^2\right)} = \frac{\eta_{\mathrm{N}} q^2}{m\varepsilon_0 \overline{\omega}^2} \frac{1}{1 - \omega^2/\overline{\omega}^2} \simeq A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \ldots \quad .$$

53.01.05
Phase-Velocity Greater Than Speed of Light in Vacuum

In the case of a **index of refraction** less than one the **phase-velocity** turns out to be larger that the **phase-velocity** of light in vacuum. This is not a real problem, however, because a MPW carries no information and therefore there is no violation of the relativity principles.

53.01.06
Extinction of ElectroMagnetic Waves in Matter

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©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|§ 2||

It can be shown that an external ElectroMagnetic disturbance traveling with the velocity of light in vacuum is exactly canceled out in matter and replaced by a secondary disturbance traveling with the velocity of ElectroMagnetic wave in matter. This is the so-called Ewald-Oseen extinction theorem.

One should remember that the total ElectroMagnetic field is always given by the sum of the EM fields produced by all the charges in the universe. Moreover, the radiation field produced by any charge, is the retarded field, with retardation speed always given by the speed of light in vacuum, c, even inside matter.

53.01.07
Link Between the Microscopic and the Macroscopic Descriptions

Read § 46.06.

DRAFT - IN - PROGRESS

53.02 Radiation Scattering From Atoms|Molecules in a Dilute Gas

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After having discussed the effect of ElectroMagnetic waves on matter at a microscopic level, we have to account for the fact that charges accelerated by the incoming ElectroMagnetic wave will radiate ElectroMagnetic radiation; this is called radiation scattering. Read § 46.04.01.

Only the case of a dilute gas is considered where the different Atoms|Molecules do not significantly interact each other.

Since the induced **Magnetic Dipole** moments of most gas molecules are negligible compared to the induced **Electric Dipole** moments, the scattering is purely **Electric Dipole** in character.

Consider the simplified model of § 53.01 for electric Dipole radiation from Atoms|Molecules under the action of an incoming monochromatic ElectroMagnetic wave. Consider the massive molecule fixed and the motion of the electrons in the molecule. Electrons are supposed to be elastically bound to the molecule with friction force. Assume the wavelength is much longer than the dimension of the molecule so that the electric field is basically constant over the extension of the molecule.

Equation (53.01.02) can be used, for the motion of the particle, together with equation (49.07.07), for the total radiated energy. Considering, for simplicity, only one proper frequency, one finds:

$$\boxed{\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}[\omega] = \frac{1}{4\pi\varepsilon_0}\frac{\omega^4|p_0|^2}{3c^3} = \frac{1}{4\pi\varepsilon_0}\left(\frac{e^4|E_0|^2}{3c^3m^2}\right)\left(\frac{\omega^4}{\left(\overline{\omega}^2-\omega^2\right)^2+\omega^2\Gamma^2}\right)} \quad . \tag{53.02.01}$$

The above formula is usually written introducing a length, called, for historical reasons, the **classical radius of the electron**:

$$\boxed{r_e \equiv \frac{1}{4\pi\varepsilon_0}\frac{e^2}{mc^2} = 3\cdot 10^{-15}\ \mathrm{m}} \quad . \tag{53.02.02}$$

$$\boxed{\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}[\omega] = \frac{c\varepsilon_0|E_0|^2}{2}\left(\frac{8\pi r_e^2}{3}\right)\left(\frac{\omega^4}{\left(\overline{\omega}^2-\omega^2\right)^2+\omega^2\Gamma^2}\right) \equiv \frac{c\varepsilon_0|E_0|^2}{2}\sigma[\omega]} \quad . \tag{53.02.03}$$

→ 2328

Equation (53.02.03) expresses the **scattered power** as a product of the incoming radiant intensity times a factor which is by definition the **scattering cross-section** (see section § 30.06):

$$\boxed{\sigma \equiv \frac{\text{Total Energy Scattered per unit time}}{\text{Energy incident per unit area per unit time}}} \quad .$$

The scattered light can be wave-Polarized, whatever the **wave-Polarization** of the incident wave. In fact, considering the Electric Dipole radiation one realizes that, for instance, looking at the incoming radiation, the scattered radiation has the same **wave-Polarization** as the incident radiation. However, when looking at a right angle with respect to the incoming radiation, the transversality of the incoming radiation produces linearly wave-Polarized scattered radiation.

Note that this **simplified purely classical model is inconsistent**, strictly speaking, when one realizes that the electrons moving around the nucleus should radiate continuously loosing energy. Only Quantum Physics can solve this issue.

By accounting for all species of electrons in matter one would write:

$$\boxed{\sigma[\omega] \equiv \frac{8\pi r_e^2}{3}\sum_s\left(\frac{\omega^4}{\left(\overline{\omega}_s^2-\omega^2\right)^2+\omega^2\Gamma_s^2}\right)} \quad .$$

- For electrons in molecules in air: $\nu_0 \approx 10^{17}$ Hz (UV).
- For visible light: $\nu_0 \approx 10^{14}$ Hz.

53.02.01
Validity of the No-Excitation Coherent-Scattering Regime

The above description is correct as long as the energy of the photons is not large enough to excite the molecules to higher energy levels, the **no-Excitation approximation**.

Moreover, the wavelength must be large with respect to the physical dimension of the molecules in order to have the coherent scattering, that is the whole molecule coherently oscillating due to the ElectroMagnetic fields, the **Coherent-Scattering approximation**.

53.02.02
Rayleigh Scattering

At low frequency:

$$\omega \longrightarrow 0 \qquad (\omega \ll \omega_s) \qquad \Longrightarrow \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}[\omega] \propto \omega^4 = \frac{1}{\lambda^4} \quad . \tag{53.02.04}$$

It is responsible for the blue of the sky and the red of Sun at sunset. It is also responsible for the visibility of a beam of light in a purified atmosphere.

53.02.03
Thomson Scattering

At high frequency:

$$\omega \longrightarrow +\infty \qquad (\omega \gg \omega_s) \qquad \Longrightarrow \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}[\omega] = \frac{c\varepsilon_0 |E_0|^2}{2} \frac{8\pi r_e^2}{3} \equiv \frac{c\varepsilon_0 |E_0|^2}{2} \sigma_T \quad , \tag{53.02.05}$$

$$\sigma_T = 0.67 \cdot 10^{-28}\ \mathrm{m}^2 \equiv 0.67\ \mathrm{barn} \quad . \tag{53.02.06}$$

It also applies to the **scattering from free charges**. The quantity σ_T is called the **Thomson scattering cross-section**: it is very large, as a cross-section, in fact, on a microscopic scale. It is, in fact, of the order of the barn, the unit of measure of cross-section.

Thomson scattering is responsible of the visibility of the Solar Corona during eclipses: the Solar Corona is made of a plasma and it is visible thanks to Thomson scattering on free electrons.

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53.03 Coherent and Incoherent Scattering From Atoms|Molecules

This § is referenced at pages:
[2277, 2278]

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|9.6.a; 9.6.d; 9.7||
©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...- ...
©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|22.9||
©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|||

Read § 50.05.04, § 50.05.05.

It can be shown that for a gas with regularly arranged molecules the scattering is zero. In fact, for any molecule scattering in one specific direction, one can always find another molecule scattering in the same direction but with opposite phase, so that total destructive interference happens, except for the forward direction. However, **density fluctuations in gases will produce a scattered intensity given by the sum of all the intensities**.

Scattering from liquids and solids is smaller, due to the smaller density fluctuations with respect to gases, even if the density is larger than for gases.

Moreover, the molecules are in motion so that, actually the scattered radiation will have a slightly **different frequency due to the Doppler effect** .

The scattering from N identical bodies is a three-dimensional interference problem and one has the following.

- Random positions: sum of the intensities.
- Regular arrangement: the scattered light is zero in almost all the directions, except for forward scattering, where there must be constructive interference, for sure. In other special directions there might be interference (for instance at $\pi/4$ for a cubic crystal in Bragg interference with X-rays). At $\pi/2$ there is destructive interference.

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53.04 Examples and Physical Applications

53.04.01
The index of refraction of Water

Values in the visible light range:

$$n[400\text{ nm} < \lambda < 700\text{ nm}] \approx 1.3 \qquad n[\omega \to 0] \approx 9 \quad .$$

53.04.02
Thomson Scattering in the Solar Corona

Estimation:

$$L \approx 7{\cdot}10^{8}\text{ m} \qquad \eta_{\mathbb{N}} = 10^{15}\text{ electrons/m}^3 \qquad \frac{1}{\lambda} = \eta_{\mathbb{N}}\sigma_T \implies \lambda = 1.5{\cdot}10^{13}\text{ m} \qquad \frac{\Delta I}{I} = 5{\cdot}10^{-5} \quad .$$

53.04.03
Blue Sky Light and Red Sun at Sunset

53.04.04
wave-Polarization of Blue Sky Light and Use for Orienteering

53.04.05
Clouds

Huge scattering from clouds is due to the fact that clouds are condensates of many particles. All the particles in a water drop scatters in phase thus leading to constructive interference. It does not increase ad infinitum because for too large dimensions (comparable with the incident radiation wavelength) the scatterers are not in phase any longer.

53.04.06
Microscopical Explanation of Brewster Angle

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53.04.07
Mie Scattering

The scattering from molecules and tiny particles is predominantly the Rayleigh scattering. For particle sizes larger than a wavelength the so-called Mie scattering predominates. This scattering produces a pattern like an antenna lobe with a sharper and more intense forward lobe for larger particles.

Mie scattering is not strongly wavelength dependent and produces the almost white glare around the sun when a lot of particulate material is present in the air. It also gives the white light from mist and fog.

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53.04.08
Tyndall Effect

The Tyndall effect, also known as Tyndall scattering, is the scattering of light by colloidal particles or particles in suspension.

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53.05 Exercises Problems and Physical Applications

53-001 I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...

5.0200, 5.0204, 5.0207, 5.0208, 5.0209, 5.0210.

53-002 Drude Lorentz In Dense Media

Demonstrate equation (53.01.11).

53-003 Elastic And Absorptive Amplitude and de-phasing

From equation (53.01.05) show that:

- for $\omega \to 0$, electric field and matter polarization are in-phase;
- for $\omega = \overline{\omega}$, electric field and matter polarization are in-quadrature;
- for $\omega \to +\infty$, electric field and matter polarization are opposite.

53-004 Elastic And Absorptive Amplitude Far From Resonance

Demonstrate equation (53.01.06).

53-005 Elastic and Absorptive Amplitude limits

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|3.2 pag 109||

Consider single resonance. Find the frequency range where

$$\left|\frac{\mathcal{A}_{\mathrm{E}}}{\mathcal{A}_{\mathrm{A}}}\right| \leq \alpha \quad .$$

WORKED SOLUTION

$$\left|\frac{\mathcal{A}_{\mathrm{E}}}{\mathcal{A}_{\mathrm{A}}}\right| = \frac{|\overline{\omega}^2 - \omega^2|}{\omega\Gamma} \leq \alpha \quad ,$$

$$-\frac{\alpha\Gamma}{2} + \sqrt{\overline{\omega}^2 + \frac{\alpha^2\Gamma^2}{4}} \leq \omega \leq +\frac{\alpha\Gamma}{2} + \sqrt{\overline{\omega}^2 + \frac{\alpha^2\Gamma^2}{4}}$$

The width of the frequency range turns out to be proportional to $\alpha\Gamma$ and it is centered on a location depending on $\alpha\Gamma$, at a frequency larger than $\overline{\omega}$.
For $\alpha = 1$ one finds: $\Delta\omega = \Gamma/2$.

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CHAPTER 54

ElectroMagnetic Waves in Matter

54.01 Work Done by the ElectroMagnetic Fields on the Matter 2336
54.02 EM Waves in a Generic LHI Material Medium 2338
54.03 Materials With Negative Electric-Permittivity and/or Magnetic-Permeability 2350
54.04 Examples and Physical Applications 2351
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This § is referenced at pages:
[2062, 2068, 2068, 2068, 2086, 2131, 2210]

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|6|*Good text.*|
©|M.V.Klein, *Optics*, 1970, J.Wiley & Sons, ...Ed., ...|||

When dealing with ElectroMagnetic waves in material media one should keep in mind that **ElectroMagnetic fields in matter always refer to average ElectroMagnetic fields** and **real matter is not really a continuum**. In particular this means that **the description of matter as a continuum with average ElectroMagnetic fields breaks down when the ElectroMagnetic fields change significantly over distances comparable with the inter-molecular distances** (for instance, in the case of harmonic fields, when the wavelength becomes so short to be comparable with inter-molecular distances). See the discussion in section § 37.03.01.

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54.01 Work Done by the ElectroMagnetic Fields on the Matter

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|||

The **Electric-Permittivity** , the **Magnetic-Permeability** and the **conductivity** are in general, complex quantities. In this case they represent, for a harmonic time dependence, a proportionality with a shift in time. See section § 53 for a modeling of these effects. This is further discussed in the next sections § 54.02.04.01 and § 54.02.02, for the **Electric-Permittivity** and the **conductivity** . As far as the **Magnetic-Permeability** is concerned this is, usually, negligibly close to one in the case of linear magnetic materials.

A non magnetic medium will be assumed: $\mu_{\mathrm{R}} = 1$. In this case the current is given by the free current plus the Polarization current, as the Magnetization current is zero for a non magnetic medium.

An harmonic time-dependence will be assumed.

The current is current from bound charges plus the current from free charges, that is one should coherently use, in the expression of the current in $\mathbf{j}\cdot\mathbf{E}$, either one of the two expressions:

$$\begin{cases} \boxed{\mathbf{j}^{\mathrm{P}} = -\mathrm{i}\omega\varepsilon_0\chi'_{\mathrm{E}}\mathbf{E}} \quad , \\ \boxed{\mathbf{j}^{\mathrm{F}} = \sigma\mathbf{E} \qquad \mathbf{j}^{\mathrm{P}} = -\mathrm{i}\omega\varepsilon_0\chi_{\mathrm{E}}\mathbf{E}} \end{cases} .$$

54.01.01 Via the Complex Electric-Permittivity Only

Use the description in terms of complex **Electric-Permittivity** only, such that free charges are just described as bound charges with zero restoring force and thus zero proper oscillation frequency.

The current is just the Polarization current as the free charges are described as weakly bound charges. The time-averaged work per unit volume done by the ElectroMagnetic fields on the matter is given by:

$$\langle \frac{\mathrm{d}W}{\mathrm{d}\tau} \rangle = \tag{54.01.01}$$

→ 2336

$$\frac{1}{2}\mathfrak{Re}\left[\overline{\mathbf{j}}\cdot\overline{\mathbf{E}}^{\star}\right] = \frac{1}{2}\mathfrak{Re}\left[\overline{\mathbf{j}^{\mathrm{P}}}\cdot\overline{\mathbf{E}}^{\star}\right] = \frac{1}{2}\mathfrak{Re}\left[-\mathrm{i}\omega\,\overline{\mathbf{P}}\cdot\overline{\mathbf{E}}^{\star}\right] = \frac{1}{2}\mathfrak{Re}\left[-\mathrm{i}\omega\varepsilon_0\overline{\chi'_{\mathrm{E}}}\,\overline{\mathbf{E}}\cdot\overline{\mathbf{E}}^{\star}\right] = \tag{54.01.02}$$

$$\boxed{\frac{1}{2}\omega\varepsilon_0\left|\overline{\mathbf{E}}\right|^2\mathfrak{Im}\left[\overline{\chi'_{\mathrm{E}}}\right] \geq 0} \quad . \tag{54.01.03}$$

The work done on matter, that is the energy transferred from the ElectroMagnetic wave to matter, that is the energy absorbed from the ElectroMagnetic fields is thus proportional to the imaginary part of the electric susceptibility.

The work done is normally positive, at least in the usual case of passive matter. This is formalized by the fact that the imaginary part of the complex susceptibility is positive. Read § 53.

54.01.02 Via the Complex Electric-Permittivity Plus Explicit conductivity for Free Charges

Use the description in terms of complex **Electric-Permittivity** plus explicit **conductivity** for free charges.

The part attributed to the Polarization current gives the same result as in equation (54.01.01). The part

DRAFT ENTREPRISE

attributed to the free current is calculated in a similar way:

$$\langle \frac{\mathrm{d}W}{\mathrm{d}\tau} \rangle = \tag{54.01.04}$$

$$\frac{1}{2}\omega\varepsilon_0 \left|\overline{\mathbf{E}}\right|^2 \mathfrak{Im}\left[\overline{\chi_{\mathrm{E}}}\right] + \frac{1}{2}\mathfrak{Re}\left[\overline{\mathbf{j}^{\mathrm{F}}}\cdot\overline{\mathbf{E}}^{\star}\right] = \frac{1}{2}\omega\varepsilon_0 \left|\overline{\mathbf{E}}\right|^2 \mathfrak{Im}\left[\overline{\chi_{\mathrm{E}}}\right] + \frac{1}{2}\mathfrak{Re}\left[\sigma\,\overline{\mathbf{E}}\cdot\overline{\mathbf{E}}^{\star}\right] = \tag{54.01.05}$$

$$\boxed{\frac{1}{2}\omega\varepsilon_0 \left|\overline{\mathbf{E}}\right|^2 \mathfrak{Im}\left[\overline{\chi_{\mathrm{E}}}\right] + \frac{1}{2}\left|\overline{\mathbf{E}}\right|^2 \mathfrak{Re}\left[\overline{\sigma}\right] \geq 0} \quad .$$

The work done on matter, that is the energy transferred from the ElectroMagnetic wave to matter, that is the energy absorbed from the ElectroMagnetic fields is thus proportional to the imaginary part of the electric susceptibility and to the real part of the **conductivity** .

The work done is normally positive, at least in the usual case of passive matter. This is formalized by the fact that the imaginary part of the complex susceptibility and the real part of the complex **conductivity** are positive. Read § 53.

54.01.03

Absorption of Energy: Energy Transfer From ElectroMagnetic Waves to Matter

From the previous results it follows that energy absorption comes from both the real part of the **conductivity** and the imaginary part of the **Electric-Permittivity** (or, equivalently, the **electric susceptibility**). This is consistent with relation (54.02.05).

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54.02 EM Waves in a Generic LHI Material Medium

This § is referenced at pages:
[2090, 2159]

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©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|ST9||
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|7||
©|M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|||

54.02.01 Introduction

In this section complex quantities will not be explicitly denoted as $\dddot{\ }$ for the sake of simplification of the notation, as it will be always clear when the variable is either a complex or a real variable.

In a insulator medium the free charge|current density were set to zero, as it is totally appropriate for both vacuum and globally neutral strictly insulator media. The case of a conducting medium involves those media where there are charges free to move, not bound to any particular site. Obviously the results obtained for insulator media must be obtained as limiting cases for conducting medium with limiting zero **conductivity** . However, in the case of conductors the flow of charge is not independently controlled and in general the free current density is not zero. Therefore in a conducting medium the **Maxwell Equations** with both free charges and currents must be employed, a priori.

The free charge density can be assumed to be zero in most conducting media, as it is shown in section § 36.20.21. The relaxation time for this process is

$$\tau = \frac{\varepsilon_0 \varepsilon_R}{\sigma} \quad . \tag{54.02.01}$$

→ 2338 2341

Therefore, at least in the case of ElectroMagnetic fields impinging on a globally neutral conductor from outside, the free charge density remains zero. From the expression for the relaxation time one can anticipate that the free charge in a conductor will have enough time to rearrange itself continuously for frequencies of the ElectroMagnetic fields such that:

$$\tau \lesssim \omega^{-1} \Longrightarrow \frac{\varepsilon_0 \varepsilon_R \omega}{\sigma} \lesssim 1 \quad .$$

Under these conditions the conducting medium will behave basically as in static conditions, that is it will continue to behave as a good conductor even in a non-static situation. The electric field inside the conductor will be basically zero. In summary we have the following.

- A perfect conductor is such that: $\sigma = \infty$ and $\tau = 0$.
- A good conductor is such that: $\varepsilon_R \varepsilon_0 \omega \ll \sigma$.
- A poor conductor is such that: $\varepsilon_R \varepsilon_0 \omega \gg \sigma$.

For perfect and good conductors it can be assumed that the free-charge is zero, as it is supposed that free-charge has enough time to rearrange itself during the relaxation time. In case there is some free charge density inside the medium one can still set $\rho^F = 0$ after a transient time given by equation (54.02.01). It turns out that, except for poor conductors the assumption $\rho^F = 0$ is justified.

For instance, $\sigma[Cu] = 59.6 \cdot 10^6$ SI and $\sigma[Ag] = 63.0 \cdot 10^6$ SI so that they are good conductors up to frequencies $\omega \approx 10^{18}$ Hz.

Obviously the free current density is not zero. One should remember that the zero free charge density implies that there are equal amounts of free positive and negative charges, balancing themselves, in

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each volume of the conductor. Therefore in a conductor at rest the free charges must include both the free conduction electrons and as many fixed positive charges as needed to balance the free conduction electrons. Similar arguments must be applied when dealing with other types of conductors, for instance electrolytic solutions and plasmas.

A conducting medium, in general, is expected to be non transparent to ElectroMagnetic waves, because currents normally imply Joule dissipation of energy, in the present case energy taken way from the ElectroMagnetic wave itself. In other words: conducting $\Longrightarrow$ dissipative.

54.02.02 Modeling of Free|Bound Charges|Currents

This § is referenced at pages:
[2336]

The **Dispersion Relation** is deduced by replacing a MPW into the equations (46.06.29), (46.06.04). The wave-vector is now necessarily complex

$$\boxed{\mathbf{k} = \mathbf{k}_{\mathrm{R}} + \mathrm{i}\mathbf{k}_{\mathrm{I}}} \quad . \tag{54.02.02}$$

→ 2339

For the present purposes one can assume that the wave-vector has a form less general than the one in equation (54.02.02), that is the real and imaginary parts are parallel:

$$\boxed{\mathbf{k} = k\widehat{\boldsymbol{\kappa}}} \quad , \tag{54.02.03}$$

where $\widehat{\boldsymbol{\kappa}}$ is a real unit vector: $\boldsymbol{\kappa}\cdot\boldsymbol{\kappa} = 1$. This is the case of homogeneous MPW. The more general case of non-homogeneous PW is much more complex and has less practical interest.

The general **Dispersion Relation** takes the form:

$$\boxed{k^2 = \varepsilon_{\mathrm{R}}\mu_{\mathrm{R}}\varepsilon_0\mu_0\omega^2 + \mathrm{i}\omega\mu_{\mathrm{R}}\mu_0\sigma} \quad . \tag{54.02.04}$$

→ 2011 2323 2339 2340 2341 2346 2355

Note that a general discussion of equation (54.02.04) is basically impossible, due to the many different parameters and properties of the materials; therefore only special cases under specific hypothesis/approximations are usually treated. Two main hypotheses are:

- almost real parameters, ε_{R}, μ_{R} and σ, as described in section § 54.02.03;
- behavior dominated by conduction charges, as described in section § 54.02.04.

54.02.02.01 Relation Between Electric-Permittivity and conductivity

It is clear from equations (54.02.04) that in the general case of complex ε_{R}, μ_{R} and σ the **Electric-Permittivity** and the **conductivity** are, to some extent, interchangeable for describing free charges. Actually **in presence of time varying fields the distinction between conduction current of free charges and Polarization current is, to a large extent, arbitrary**. From the physical point of view this reflects the fact that, for time varying harmonic fields, there is no qualitative difference between the motion of a free charge and the motion of a bound charge: both of them are harmonic motions. On the other hand there is a significant qualitative difference in the static case, as a bound charge will move until it reaches the equilibrium position, while a free charge will continue to move until the conditions to which it is subject change (for instance: boundaries, change of fields).

Let us denote, for the sake of clarity, with ε_{R} the **Electric-Permittivity** due to bound charges only and use explicitly σ to describe the free charges. On the other hand let us denote with $\varepsilon'_{\mathrm{R}}$ the **Electric-Permittivity** due to all charges, both bound and free charges.

From the formal point of view the relation among ε_{R}, $\varepsilon'_{\mathrm{R}}$ and σ can be understood by first writing equation (54.02.04) as:

$$\boxed{k^2 = \varepsilon_0\mu_0\mu_{\mathrm{R}}\omega^2\left(\varepsilon_{\mathrm{R}} + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\right)} \quad .$$

This equations confirms that actually, for non static situations, it is to some extent an arbitrary choice to express the properties of the material either in terms of a **conductivity** , σ, plus an **Electric-Permittivity** ,

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ε_R, describing only bound charges, or in terms of a **Electric-Permittivity** , ε'_R, describing all charges:

$$\varepsilon'_R \longleftrightarrow \left(\varepsilon_R + i\frac{\sigma}{\varepsilon_0\omega}\right) \quad . \tag{54.02.05}$$

→ 2337 2340

Read also § 54.04.01.

In general ε_R, ε'_R and σ may all be complex.

In view of the physical interpretation given above, equation (54.02.05) amounts to choose between two equivalent descriptions of the response of matter to ElectroMagnetic fields. In fact, one realizes that the concept of **Electric-Permittivity** can describe free charges as well by considering free charges as bound charges with an infinitely weak restoring force. Read § 53.

- Describe the free charges by means of a **conductivity** , σ, and the bound charge by means of the **Electric-Permittivity** , ε_R.
- Describe both free and bound charges by means of single parameter, the **Electric-Permittivity** , ε'_R.

In what follows the notation for the **Electric-Permittivity** will be normally ε_R as it will be clear from the context which one of the two approaches one is following, according to whether the **conductivity** is explicitly mentioned or not. The first approach (separate accounting for the **conductivity** and **Electric-Permittivity** of bound charges) will be normally employed, unless stated otherwise.

Actually, in the general case, one has to assume that the parameters in equation (54.02.04) (ε_R, μ_R and σ) are complex. In this case they represent, for a harmonic time dependence, a proportionality with a shift in time. Read § 53 for a modeling of these effects.

Following the approach to separately accounting for the **conductivity** and **Electric-Permittivity** of bound charges, the behavior of a material as conductor or insulator can be defined by the following limits:

- good conductor: $|\sigma| \gg \omega\varepsilon_0|\varepsilon_R|$, conduction current much larger than the displacement current, the medium behaves as a conductor characterized by its **conductivity** ;
- good insulator: $|\sigma| \ll \omega\varepsilon_0|\varepsilon_R|$, conduction current much smaller than the displacement current, the medium behaves as an insulator characterized by its **Electric-Permittivity** .

Most materials have an intermediate behavior with respect to these two extreme limits.

Note, however, that in general both **conductivity** and **Electric-Permittivity** are complex quantities. It should be anticipated that: the **Electric-Permittivity** is typically a real quantity except for the regions of frequency near the proper resonance frequencies of the charges in matter; the **conductivity** is a real parameter at low frequencies.

54.02.02.02 Relation among the three currents

Time-dependent currents in LHI matter may be described equivalently as polarization effects, magnetization effects, or any combination of the two.

Polarization effects can be made to include, for time-dependent fields, free charges|currents as for time-dependent field the two contributions can be hardly distinguished.

The distinction between polarization currents and magnetization currents density is similarly ambiguous. In fact,

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54.02.03
Almost Real Parameters

This § is referenced at pages:
[2339, 2341]

Assume that all the parameters in equation (54.02.04) (ε_{R}, μ_{R} and σ) are almost real, and a non magnetic medium. Therefore the conditions

$$\boxed{\mu_{\mathrm{R}} = 1 \qquad \Im\mathrm{m}\left[\varepsilon_{\mathrm{R}}\right] \simeq 0 \qquad \Im\mathrm{m}\left[\sigma\right] \simeq 0} \quad , \tag{54.02.06}$$

will be implicitly assumed except when specified otherwise. If that is not the case some of the results obtained in this section must be adapted.

It can be shown (read § 53.01, § 53.01.01) that this hypothesis is well satisfied for non magnetic media ($\mu_{\mathrm{R}} = 1$), far from proper resonance frequencies (ε_{R}) and at low enough frequencies (σ). In fact the assumptions for ε_{R} is well justified far-away from the resonance frequencies of the matter, where the **Electric-Permittivity** is essentially a real quantity. As far as the **Magnetic-Permeability** is concerned it is basically one for linear magnetic media in most cases. It is demonstrated in section § 54.02.04.01 that the **conductivity** of good conductors remains almost real up to large frequencies; it is anyway a real quantity at low frequencies, as it is well-known.

The above conditions are often satisfied in visible light as the resonance frequencies are often in the UV.

The real and imaginary parts of the wave-vector are then:

$$k_{\mathrm{R}}^2 - k_{\mathrm{I}}^2 = \frac{\omega^2 \varepsilon_{\mathrm{R}} \mu_{\mathrm{R}}}{c^2} \quad ,$$
$$2\,\mathbf{k}_{\mathrm{R}} \cdot \mathbf{k}_{\mathrm{I}} = \omega \mu_{\mathrm{R}} \mu_0 \sigma \qquad \Longrightarrow \qquad (2 k_{\mathrm{R}} k_{\mathrm{I}} = \omega \mu_{\mathrm{R}} \mu_0 \sigma \qquad \text{because } \mathbf{k} = k \widehat{\boldsymbol{\kappa}}\,) \quad ,$$

and the solution for the real and imaginary parts of the wave-vector is:

$$\boxed{k_{\mathrm{R}} = \frac{\omega}{c} \sqrt{\frac{\varepsilon_{\mathrm{R}} \mu_{\mathrm{R}}}{2}} \sqrt{\sqrt{1 + \left(\frac{\sigma}{\omega \varepsilon_0 \varepsilon_{\mathrm{R}}}\right)^2} + 1}} \quad ,$$

$$\boxed{k_{\mathrm{I}} = \frac{\omega}{c} \sqrt{\frac{\varepsilon_{\mathrm{R}} \mu_{\mathrm{R}}}{2}} \sqrt{\sqrt{1 + \left(\frac{\sigma}{\omega \varepsilon_0 \varepsilon_{\mathrm{R}}}\right)^2} - 1}} \quad .$$

Note that, from the algebraic point of view, the latter solutions are only valid assuming that all the parameters on the right-hand side are real.

The result shows that, as long as **all the parameters in equation (54.02.04) are real, a necessary and sufficient condition to have an imaginary part of the wave-vector is that the conductivity is different from zero**. In fact, in presence of conduction charge, one normally expects some absorption to be present, because the current excited by the external ElectroMagnetic fields will dissipate energy.

54.02.03.01 Good/Bad Conductors

Equation (54.02.04) is the basic equation describing the propagation of MPW inside matter. It can be re-written as the complex expression:

$$\boxed{k^2 \equiv \mathbf{k} \cdot \mathbf{k} = \varepsilon_{\mathrm{R}} \mu_{\mathrm{R}} \varepsilon_0 \mu_0 \omega^2 \left(1 + \mathrm{i} \frac{\sigma}{\varepsilon_{\mathrm{R}} \varepsilon_0 \omega}\right) = \varepsilon_{\mathrm{R}} \mu_{\mathrm{R}} \frac{\omega^2}{c^2} \left(1 + \mathrm{i} \frac{1}{\omega \tau}\right)} \quad , \tag{54.02.07}$$

→ 2342

where the first term corresponds to the displacement current while the second term corresponds to the conduction current and the relaxation time of equation (54.02.01) has been introduced.

Further progress can be done by assuming that ε_{R}, μ_{R} and σ are real quantities, possibly frequency dependent, as mentioned and explained in section § 54.02.03. In absence of a microscopic model, this is a kind of hypothesis which is needed to make further progress. The use of a microscopic model is described in section § 54.02.04.

PRELIMINARY DRAFT

The following notation is often introduced:

$$\boxed{k = \beta + \mathtt{i}\frac{\alpha}{2} \equiv k_{\mathtt{R}} + \mathtt{i}k_{\mathtt{I}}} \quad .$$

The Electric|Magnetic fields can be written, in general, as a function of the unit vector $\widehat{\boldsymbol{\kappa}}$ in the direction of the wave-vector $\mathbf{k}$:

$$\mathbf{E} = \mathbf{E}_0 \exp\left[-\alpha/2\, \boldsymbol{\kappa}\cdot\mathbf{x}\right] \exp\left[\mathtt{i}\beta\, \boldsymbol{\kappa}\cdot\mathbf{x} - \mathtt{i}\omega t\right] \quad ,$$
$$\mathbf{B} = \mathbf{B}_0 \exp\left[-\alpha/2\, \boldsymbol{\kappa}\cdot\mathbf{x}\right] \exp\left[\mathtt{i}\beta\, \boldsymbol{\kappa}\cdot\mathbf{x} - \mathtt{i}\omega t\right] \quad .$$

Admit therefore that $\varepsilon_{\mathtt{R}}$, $\mu_{\mathtt{R}}$ and σ are all real. Note that if this hypothesis is not satisfied one must start from equation (54.02.07) in any case.

Bad Conductors

This § is referenced at pages:
[2351]

A bad conductor, to first-order, gives an almost real index of refraction:

$$\left|\frac{\sigma}{\omega\varepsilon_0\varepsilon_{\mathtt{R}}}\right| \ll 1 \implies$$

$$k = \beta + \mathtt{i}\frac{\alpha}{2} \equiv k_{\mathtt{R}} + \mathtt{i}k_{\mathtt{I}} \simeq \frac{\omega}{c}\sqrt{\varepsilon_{\mathtt{R}}\mu_{\mathtt{R}}}\left(1 + \mathtt{i}\frac{\sigma}{2\omega\varepsilon_0\varepsilon_{\mathtt{R}}}\right) \qquad \text{to first-order in } \frac{\sigma}{\omega\varepsilon_0\varepsilon_{\mathtt{R}}} \quad .$$

Under this approximation the real part of the wave-vector is much larger than the imaginary part and the latter is independent of the frequency, except for possible dependencies induced by the **Electric-Permittivity** and **conductivity** . The **wave-vector is thus almost real**: there is absorption and the medium is, in the two extreme cases, either basically **transparent, in case the attenuation length is much longer than the dimension of the sample**, or basically **a black-body, in case the attenuation length is much shorter than the dimension of the sample**.

Note that actually the result is the same (almost real wave-vector) as long as the condition $|\sigma| \ll |\omega\varepsilon_0\varepsilon_{\mathtt{R}}|$ is satisfied irrespective of the reality or not of the **conductivity** σ.

The magnetic field is given by:

$$\mathbf{B}_0 = \frac{k}{\omega}\widehat{\boldsymbol{\kappa}}\wedge\mathbf{E}_0 \quad ,$$

but it should be kept in mind that the wave-vector is complex, and therefore the electric and magnetic fields are out-of-phase. However, thanks to the smallness of the imaginary part with respect to the real part of the wave-vector, the Electric|Magnetic fields are almost in phase.

The wave-impedance of a bad conductor, from equation (46.06.20), is given by:

$$\mathbf{E}\cdot\mathbf{E} \equiv Z^2\,\mathbf{H}\cdot\mathbf{H} \implies Z \equiv \sqrt{\frac{\mu_0\mu_{\mathtt{R}}}{\varepsilon_0\varepsilon'_{\mathtt{R}}}} = \sqrt{\frac{-\mathtt{i}\mu_0\mu_{\mathtt{R}}\omega}{\sigma - \mathtt{i}\varepsilon_0\varepsilon_{\mathtt{R}}\omega}} \quad . \tag{54.02.08}$$

In this case the wave-number is almost real and the **index of refraction** is given by:

$$n = \frac{ck}{\omega} = \sqrt{\varepsilon_{\mathtt{R}}\mu_{\mathtt{R}}}\left(1 + \mathtt{i}\frac{\sigma}{2\omega\varepsilon_0\varepsilon_{\mathtt{R}}}\right) \quad .$$

In case of a non-magnetic medium: $\mu_{\mathtt{R}} = 1$. If it can be assumed that the effect of the bound charges is negligible (free charges are accounted for by the **conductivity**) the **Electric-Permittivity** can be set to one: $\varepsilon_{\mathtt{R}} = 1$. In this case the **index of refraction** becomes:

$$n \simeq \left(1 + \mathtt{i}\frac{\sigma}{2\omega\varepsilon_0}\right) \quad .$$

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An ElectroMagnetic wave impinging at normal incidence onto such kind of material from vacuum is only weakly reflected because the reflectance is:

$$\varrho \simeq \left(\frac{\sigma/\left(2\omega\varepsilon_0\right)}{2}\right)^2 \ll 1 \quad .$$

The wave is thus almost completely transmitted/absorbed. If the thickness of the sample is large with respect to the attenuation length the wave will be totally absorbed inside the medium. Such a medium thus will appear as a black body. If the thickness of the sample is small with respect to the attenuation length the wave will be totally transmitted inside the medium. Such a medium thus will appear as a transparent one.

Good Conductors

This § is referenced at pages:
[2351]

A good conductor, to first-order, gives real and imaginary parts for the index of refraction which are almost equal:

$$\left|\frac{\omega\varepsilon_0\varepsilon_{\mathrm{R}}}{\sigma}\right| \ll 1 \implies$$

$$k = \beta + \mathtt{i}\frac{\alpha}{2} \equiv k_{\mathrm{R}} + \mathtt{i}k_{\mathrm{I}} = \sqrt{\mu_{\mathrm{R}}\mu_0\sigma\omega}\left(\frac{1+\mathtt{i}}{\sqrt{2}}\right) \qquad \text{to first-order in } \frac{\omega\varepsilon_0\varepsilon_{\mathrm{R}}}{\sigma} \quad .$$

The magnetic field is given by:

$$\mathbf{B}_0 = \frac{k}{\omega}\widehat{\boldsymbol{\kappa}}\wedge\mathbf{E}_0 \quad ,$$

but it should be kept in mind that the wave-vector is complex, and therefore the electric and magnetic fields are out-of-phase.

The wave-impedance of a good conductor, from equation (46.06.20), is given by:

$$\mathbf{E}\cdot\mathbf{E} \equiv Z^2\,\mathbf{H}\cdot\mathbf{H} \implies Z \equiv \sqrt{\frac{\mu_0\mu_{\mathrm{R}}}{\varepsilon_0\varepsilon'_{\mathrm{R}}}} = \sqrt{\frac{-\mathtt{i}\mu_0\mu_{\mathrm{R}}\omega}{\sigma}} \quad . \tag{54.02.09}$$

The real and imaginary parts of the wave-vector are equal in a good conductor. It follows that the Electric|Magnetic fields are out-of-phase by $\pi/4$. Moreover, the magnetic field is much larger than it would be in vacuum for the same electric field and therefore the energy of the wave is largely of magnetic type.

The imaginary part of the wave-vector gives the inverse attenuation length of the wave:

$$k_{\mathrm{I}} \equiv \frac{\alpha}{2} = \sqrt{\frac{\mu_{\mathrm{R}}\mu_0\sigma\omega}{2}} \equiv \delta^{-1} \quad ,$$

which is half the inverse attenuation length of the energy:

$$\boxed{2k_{\mathrm{I}} \equiv \alpha = \sqrt{2\mu_{\mathrm{R}}\mu_0\sigma\omega}} \quad . \tag{54.02.10}$$

The ElectroMagnetic fields thus penetrates the good conductor by a depth δ (the so-called skin-depth):

$$\boxed{\delta \equiv \sqrt{\frac{2}{\mu_{\mathrm{R}}\mu_0\sigma\omega}}} \quad . \tag{54.02.11}$$

→ 2352

The consequence of the latter result is the so-called skin effect such that ElectroMagnetic fields at high frequency do no enter inside conductors and currents mainly flow near the surface.

For instance: copper at $\omega \approx 10^{15}$ Hz has $\delta \simeq 10^{-8}$ m.

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54.02.04
Drude-Lorentz Model for Conducting Media

This § is referenced at pages:
[2339, 2341]

In this section it is assumed that **the behavior of the material is dominated by the free conduction charges**. This is well justified, among other cases, when all the proper resonance frequencies are far away from the wavelength region of interest. In fact in this case the contribution to the susceptibility from bound charges is very close to zero. Moreover, a non magnetic material is assumed.

Therefore the conditions

$$\boxed{\mu_{\text{R}} = 1 \qquad \varepsilon_{\text{R}} \simeq 1} \quad , \tag{54.02.12}$$

→ 2344 2346

will be implicitly assumed except when specified otherwise. If that is not the case some of the results obtained in this section must be adapted.

The simplified microscopic classical model for the dispersion described in section § 53 and equation (53.01.07) when applied to the case when one subset of charges, $k = 0$, has $\overline{\omega}_0 = 0$, that is they are free charges, gives, under the hypotheses in (54.02.12):

$$\varepsilon'_{\text{R}} = \varepsilon_{\text{R}} + \frac{\eta_0 q_0^2}{\varepsilon_0 m}\left(\frac{1}{-\omega^2 - \mathtt{i}\omega\Gamma_0}\right) = \varepsilon_{\text{R}} + \frac{\eta_0 q_0^2}{\varepsilon_0 m\omega}\left(\frac{\mathtt{i}}{\Gamma_0 - \mathtt{i}\omega}\right) \Longrightarrow \varepsilon'_{\text{R}} = 1 + \frac{\omega_p^2}{\omega}\left(\frac{\mathtt{i}}{\Gamma_0 - \mathtt{i}\omega}\right) \quad , \tag{54.02.13}$$

→ 2344

where the so-called plasma frequency, ω_p,

$$\boxed{\omega_p^2 \equiv \frac{\eta_0 q_0^2}{\varepsilon_0 m} \equiv \frac{\eta_0 q_e^2}{\varepsilon_0 m_e}} \quad , \tag{54.02.14}$$

→ 2323

was introduced (read § 54.02.04.02).

The comparison of equation (54.02.13) with equation (53.01.10) gives the following expression for the **conductivity** :

$$\boxed{\sigma[\omega] = \frac{\eta_0 q_0^2}{m_0}\left(\frac{1}{\Gamma_0 - \mathtt{i}\omega}\right) = \varepsilon_0\omega_p^2\left(\frac{1}{\Gamma_0 - \mathtt{i}\omega}\right) \qquad \sigma[\omega = 0] = \frac{\eta_0 q_0^2}{m_0\Gamma_0}} \quad . \tag{54.02.15}$$

→ 2344 2346 2355

This is essentially Drude model, read § 53.01. Equation (54.02.15) allows one to calculate the electric **conductivity** at various frequencies, after determining the damping constant, Γ_0, by using the values at low frequencies.

The result (54.02.15) shows that the **conductivity** is a real quantity, at very low frequencies, while it is a pure imaginary quantity, at very high frequencies.

54.02.04.01 Low Frequency Limit (Purely Resistive Regime)

This § is referenced at pages:
[2336, 2341]

At low frequencies, $\omega \ll \Gamma_0$, the **conductivity** is real and independent of the frequency, as in the static case. This is called the purely resistive regime, because the **conductivity** is basically the static **conductivity** . The charges, in fact, will be basically always at the terminal velocity appropriate to the instantaneous electric field. The contribution to the electric susceptibility is an imaginary one.

The square of the **index of refraction** is:

$$\boxed{n^2 = 1 + \mathtt{i}\frac{\omega_p^2}{\omega\Gamma_0}} \quad .$$

Two limiting cases arise for a resistive medium: dilute and dense.

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A Dense Resistive Medium

The condition $\omega \ll \Gamma_0$ is complemented by the additional conditions:

$$\boxed{\omega \ll \omega_p \qquad \frac{\omega_p^2}{\Gamma_0} \gg \omega} \quad .$$

It is called dense medium because ω_p^2 is proportional to the density.

The **index of refraction** is thus:

$$\boxed{n = n_{\mathrm{R}} + \mathrm{i}\, n_{\mathrm{I}} \simeq 1 + \mathrm{i}\frac{\omega_p^2}{2\omega\Gamma_0} \qquad \text{with } |n_{\mathrm{I}}| \gg n_{\mathrm{R}} = 1} \quad .$$

The **index of refraction** is thus essentially a imaginary number:

$$\boxed{n \simeq \sqrt{\mathrm{i}\frac{\omega_p^2}{\omega\Gamma_0}} = \sqrt{\mathrm{i}\frac{\sigma}{\omega\varepsilon_0}} \qquad \text{with } |n_{\mathrm{I}}| \gg n_{\mathrm{R}} = 1} \quad .$$

The real and imaginary parts of the wave-vector are thus equal and large with respect to the vacuum value, ω/c.

The mean penetration distance of the wave is small with respect to the vacuum wavelength. A wave is then reflected with negligible absorption, because there is so little penetration that a little energy is absorbed.

The reflectivity is than almost equal to one.

A dense resistive medium gives almost complete reflection.

A Dilute Resistive Medium

The condition $\omega \ll \Gamma_0$ is complemented by the additional conditions:

$$\boxed{\omega_p \ll \Gamma_0 \qquad \frac{\omega_p^2}{\Gamma_0} \ll \omega} \quad .$$

It is called dilute medium because ω_p^2 is proportional to the density.

The **index of refraction** is thus:

$$\boxed{n = n_{\mathrm{R}} + \mathrm{i}\, n_{\mathrm{I}} \simeq 1 + \mathrm{i}\frac{\omega_p^2}{2\omega\Gamma_0} \qquad \text{with } |n_{\mathrm{I}}| \ll n_{\mathrm{R}} = 1} \quad .$$

The wave-vector is:

$$\boxed{k = n\frac{\omega}{c} = k_{\mathrm{R}} + \mathrm{i}\, k_{\mathrm{I}} \simeq \frac{\omega}{c} + \mathrm{i}\frac{\sigma_0}{2\varepsilon_0 c} \qquad \text{with } |k_{\mathrm{I}}| \ll k_{\mathrm{R}}} \quad .$$

The real part of the wave-vector is the same as in vacuum. The fact that the imaginary part of the wave-vector is much smaller than the real one implies that the attenuation length is much larger than the vacuum wavelength.

The reflectivity is much smaller than one.

The wave-vector is thus almost real: there is absorption and the medium is, in the two extreme cases, either basically transparent, in case the attenuation length is much longer than the dimension of the sample, or basically a black-body, in case the attenuation length is much shorter than the dimension of the sample.

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54.02.04.02 High Frequency Limit (Purely Elastic Regime)

This § is referenced at pages:
[2344]

At sufficiently high frequencies the **Electric-Permittivity** of any substance tends to one (see section § 53). This happens when the frequency is well above all the resonance frequencies of the system: $\omega \gg \overline{\omega}_{\mathtt{MAX}}$, in terms of the highest resonance frequency of the system, $\overline{\omega}_{\mathtt{MAX}}$. Therefore, in this case, the hypothesis, $\varepsilon_{\mathtt{R}} = 1$, invoked in (54.02.12), is certainly satisfied.

At high frequencies, $\omega \gg \Gamma_0$, the **conductivity** is imaginary and dependent of the frequency as ω^{-1}.

Therefore at sufficiently high frequencies the conclusions of this section are of rather general applicability.

The **conductivity** at very high frequencies, $\omega \gg \Gamma_0$, according to equation (54.02.15), gives:

$$\sigma[\omega] = \mathtt{i}\frac{\eta_0 q_0^2}{m\omega} \quad .$$

By using this expression in the **Dispersion Relation** (54.02.04) one finds:

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2} \quad .$$

Below the Plasma Frequency (Reactive Frequency Range)

The conditions $\omega \gg \Gamma_0$ and $\omega_p \gg \Gamma_0$, in this case, are complemented by the additional condition:

$$\omega < \omega_p \implies -\frac{\omega_p^2}{\Gamma_0} \ll n^2 \leq 0 \quad .$$

Below the plasma frequency, if the conditions $\omega \gg \Gamma_0$ (and $\omega \gg \overline{\omega}_{\mathtt{MAX}}$) are satisfied, the **index of refraction** and wave-vector are imaginary. The wave decays exponentially with attenuation length given by ${k_{\mathtt{I}}}^{-1}$. The wave is totally reflected without absorption.

Above the Plasma Frequency (Dispersive Frequency Range)

The conditions $\omega \gg \Gamma_0$ and $\omega_p \gg \Gamma_0$, in this case, are complemented by the additional condition:

$$\omega > \omega_p \quad .$$

Above the plasma frequency, if the conditions $\omega \gg \Gamma_0$ (and $\omega \gg \overline{\omega}_{\mathtt{MAX}}$) are satisfied, the wave-number becomes real: the wave is transmitted without any absorption. The **index of refraction** is:

$$n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \leq 1 \quad .$$

It follows that the **index of refraction** is less than one and the **phase-velocity** is larger than the **phase-velocity** of ElectroMagnetic waves in vacuum:

$$v[\omega] \equiv \frac{\omega}{k} = \frac{\omega c}{\sqrt{\omega^2 - \omega_p^2}} \geq c \quad .$$

Actually the **phase-velocity** tends to infinity when the frequency approaches the plasma frequency.

As the **index of refraction** is real, positive and less than one it follows that the medium is transparent and there is no absorption.

This regime applies to X-rays in good conductors.

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54.02.05
Summary of Regimes for a Conducting Medium

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low enough frequency	$\omega \ll \Gamma$ and $\omega \ll \omega_p$ and $\omega\Gamma \ll \omega_p^2$	dense resistive medium	n^2 is pure imaginary (high reflection)
high enough frequency	$\Gamma \ll \omega_p \ll \omega$	dispersive medium	$0 \leq n^2 \leq 1$ is pure real (totally transparent, no absorption)
medium such that: $\Gamma \ll \omega_p$	for $\omega \ll \Gamma$ and $\omega_p^2 \ll \omega\Gamma$	dilute resistive medium	$n \simeq 1 + \mathrm{i}\dfrac{\omega_p^2}{2\omega\Gamma}$ (wave absorption without reflection; the mean attenuation length long with respect to wavelength)
medium such that: $\omega_p \ll \Gamma$	for $\Gamma \ll \omega$ and $\omega \ll \omega_p$	purely reactive medium	$-\omega_p^2 \ll n^2\Gamma^2 \leq 0$; n is purely imaginary (total reflection without absorption)
medium such that: $\omega_p \approx \Gamma$		no frequency range for which it is dilute resistive or purely reactive	at low frequency: dense resistive medium; at high frequency: dispersive medium

The properties of any conductor may be summarized as follows.

- At low enough frequency any conductor is a dense resistive medium. It gives almost total reflection with very little absorption.
- At high enough frequency any conductor is a dispersive medium. It is then transparent. Conductors can then be divided into three classes, according to the relative values of Γ_0 and ω_p.
  - If $\Gamma_0 \gg \omega_p$ has a frequency range where it is a dilute resistive medium. In that range i can absorb a wave without reflection. Such a conductor cannot have a purely reactive frequency range and therefore it cannot give total reflection at any frequency.
  - If $\Gamma_0 \ll \omega_p$ has a frequency range where it is a purely reactive medium. In that range it can give total reflection without absorption. Such a conductor cannot have a dilute resistive frequency range and therefore it cannot give total absorption without reflection.
  - If $\Gamma_0 \approx \omega_p$ it has no frequency range where it is purely reactive nor dilute resistive medium.

54.02.05.01 Summary Results

Consider ElectroMagnetic MPW at a certain fixed frequency reaching a non-magnetic material ($\mu_R = 1$) and coming from a material with real refractive index n_0 (typically $n_0 \simeq 1$) at normal incidence for simplicity.

$$\boxed{k = n\frac{\omega}{c}} \quad .$$

Recall the expressions of **reflectivity** and **transmissivity** at normal incidence for non-magnetic media:

$$\varrho = \left(\frac{n_0 - n}{n_0 + n}\right)^2 \qquad \tau = 1 - \varrho \quad .$$

- If n is purely real, then we have a perfectly transparent material and the energy which enters, determined by ϱ and τ, is transmitted without absorption:

$$\alpha = 0 \quad . \tag{54.02.16}$$

- If n is pure imaginary, and the index of refraction n_0 of the first medium is real, there is no penetration

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of the wave but only exponential attenuation in the material. Moreover: $\varrho = 1$ and $\tau = 0$ and there is no energy entering the material.

- If $\mathfrak{Re}[n] \simeq \mathfrak{Im}[n]$ we have a penetration by the skin-depth

$$k_{\mathrm{I}} = \alpha/2 = \delta^{-1} = \sqrt{\frac{\mu_0 \mu_{\mathrm{R}} \sigma \omega}{2}} \quad , \tag{54.02.17}$$

and there is penetration of a harmonic damped wave.

- If $n = A + \mathrm{i}\epsilon$ with $|\epsilon| \ll A$, A real, and n_0 is the real index of refraction of the first medium, the reflectivity can be very small if $A \simeq$, in fact:

$$\varrho = \left| \frac{A - n_0}{A + n_0} \right|^2 \quad , \tag{54.02.18}$$

the wave is absorbed in the material or not depending on the relationship between the attenuation length and the dimension of the material (either a black-body or a transparent medium).

- If $n = 1 + \epsilon + \mathrm{i}B$ with $|\epsilon| \ll |B|$ there is high reflection and high absorption of the energy which enters the medium, but little energy enters to be absorbed, depending on the refractive index n_0 of the first medium.

54.02.05.02 Plots

See the summary plots 54.11, 54.12, 54.13, 54.14 and 54.15.

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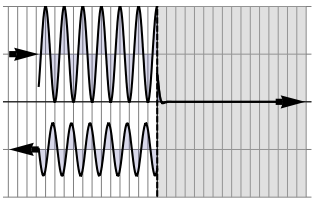
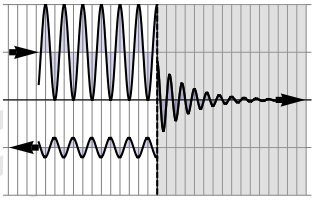


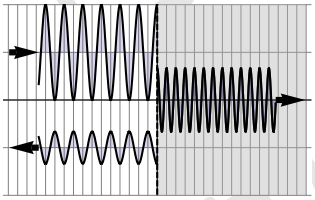
(a) index of refraction: $\mathfrak{Re}[n] = 1.5$ and $\mathfrak{Im}[n] = 1.5$

Figure 54.11: ...



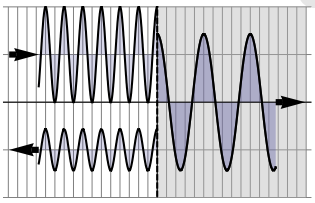
(a) index of refraction: $\mathfrak{Re}[n] = 1.5$ and $\mathfrak{Im}[n] = 0.1$

Figure 54.12: ...



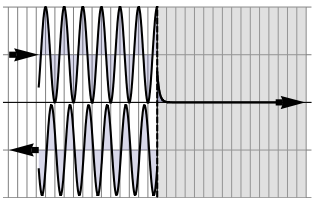
(a) index of refraction: $\mathfrak{Re}[n] = 2.0$ and $\mathfrak{Im}[n] = 0$

Figure 54.13: ...



(a) index of refraction: $\mathfrak{Re}[n] = 0.4$ and $\mathfrak{Im}[n] = 0$

Figure 54.14: ...



(a) index of refraction: $\mathfrak{Re}[n] = 0.1$ and $\mathfrak{Im}[n] = 1.5$

Figure 54.15: ...

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54.03 Materials With Negative Electric-Permittivity and/or Magnetic-Permeability

This § is referenced at pages:
[2131]

Read § 46.08 for general properties.
Most materials have positive **Electric-Permittivity** |**Magnetic-Permeability** .
Artificial materials exist with both negative **Electric-Permittivity** and **Magnetic-Permeability** . These materials feature unusual properties.

© - QUOTE

Wave Propagation From Electrons to Photonic Crystals and Left-Handed Materials Peter Markos Costas M. Soukoulis
It is important to note that we are not aware of the existence of such materials in nature. However, recently man-made composites were prepared, which exhibit, in a certain frequency region, negative effective permittivity and permeability. 'Effective' means that these parameters describe the response to an external electromagnetic field with a frequency ν such that the wavelength of the field, $k = 2\pi c/\nu$, is much larger than any spatial in-homogeneity of the material.

See WEB - URL.

PREVIEW-PROGRESS

54.04 Examples and Physical Applications

54.04.01 Conductivity and Electric-Permittivity With/Without Bound Charges

This § is referenced at pages:
[2340]

Consider a conducting LHI medium ($\mathbf{j}^{\mathrm{F}} = \sigma\mathbf{E}$) with a normal **Electric-Permittivity** , ε_{R}, describing the behavior of bound charges only. The Maxwell-Ampère relation, in presence of an harmonic time dependence, $\exp[-\mathrm{i}\omega t]$, becomes:

$$\mathbf{rot}\mathbf{B} = \mu_0\mu_{\mathrm{R}}\sigma\mathbf{E} + \varepsilon_{\mathrm{R}}\varepsilon_0\mu_0\mu_{\mathrm{R}}\partial_t\mathbf{E} \Longrightarrow \mathrm{i}\,\mathbf{k}\wedge\mathbf{B} = \mu_0\mu_{\mathrm{R}}\sigma\mathbf{E} - \mathrm{i}\omega\varepsilon_{\mathrm{R}}\varepsilon_0\mu_0\mu_{\mathrm{R}}\mathbf{E} = \mu_0\mu_{\mathrm{R}}\left(\sigma - \mathrm{i}\omega\varepsilon_{\mathrm{R}}\varepsilon_0\right)\mathbf{E} \quad ,$$

showing that, under the specified hypotheses, a conducting medium can described by the same equation as an insulator medium by using an **electric susceptibility/Electric-Permittivity** , $\varepsilon'_{\mathrm{R}}$ and χ'_{E}, given by (equation (53.01.10)):

$$\varepsilon'_{\mathrm{R}} \equiv \varepsilon_{\mathrm{R}} + \mathrm{i}\frac{\sigma}{\omega\varepsilon_0} \qquad \chi'_{\mathrm{E}} \equiv \chi_{\mathrm{E}} + \mathrm{i}\frac{\sigma}{\omega\varepsilon_0} \quad .$$

If one does not introduce the relation $\mathbf{j}^{\mathrm{F}} = \sigma\mathbf{E}$ and all the behavior of the medium is attributed to its Polarizable material properties the latter must be described by the **electric susceptibility/Electric-Permittivity** , ε_{R} and χ_{E}, only.

These considerations show that the distinction between insulators and conductors is largely artificial, at least for non-zero frequencies.

54.04.02 Radio-Wave Propagation in the Ionosphere

Radio-Communication: $\nu \approx 3$ GHz (short waves).

54.04.03 Radio-Communication Black-Out With a Spaceship at Reenter Into the Atmosphere

If the plasma created around the spaceship by the atmospheric drag is such that the plasma frequency around the spaceship becomes larger (high enough density) than the radio-communication frequency the radio waves to the spaceship are reflected before reaching it.

54.04.04 MicroWave Cooking

©|G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|||

Typical MicroWave oven frequency: assume $\nu = 3$ GHz; actually the resonance frequency of water is at $\nu = 2.45$ GHz but one should not stay too close to the resonance frequency otherwise there is little energy penetration. Assume the following parameters for fat animal tissues at the given frequency: $\varepsilon_{\mathrm{R}} = 5.5$, $\sigma = 0.15\ (\Omega\cdot\mathrm{m})^{-1}$. In this case

$$\left|\frac{\sigma}{\omega\varepsilon_0\varepsilon_{\mathrm{R}}}\right| \simeq 0.16 \Longrightarrow k \simeq \omega\sqrt{\varepsilon_0\mu_0\varepsilon_{\mathrm{R}}\mu_{\mathrm{R}}}\left(1 + \frac{\mathrm{i}\sigma}{2\omega\varepsilon_0\varepsilon_{\mathrm{R}}}\right) = (1 + 0.08\mathrm{i})\,\omega\sqrt{\varepsilon_0\mu_0\varepsilon_{\mathrm{R}}\mu_{\mathrm{R}}} \quad ,$$

and therefore neither the bad conductor nor the good conductor approximations are suitable (Read § 54.02.03.01, § 54.02.03.01). The full calculation is required. ■

WORK-IN-PROGRESS

One then finds:

$$\lambda_0 = c/\nu = 10 \text{ cm} \quad ,$$
$$k_{\text{R}} = 1.5{\cdot}10^2 \text{ m}^{-1} \quad ,$$
$$\lambda = \frac{2\pi}{k_{\text{R}}} = 4.2 \text{ cm} \quad ,$$
$$k_{\text{I}} = 1.2{\cdot}10^1 \text{ m}^{-1} \quad ,$$
$$\delta \equiv {k_{\text{I}}}^{-1} = 8.3 \text{ cm} \quad .$$

Note that:

$$\frac{k_{\text{I}}}{k_{\text{R}}} \simeq 0.08 \simeq \frac{1}{2}\left|\frac{\sigma}{\omega\varepsilon_0\varepsilon_{\text{R}}}\right| \quad ,$$

as it should.

The process involved is the oscillation of the permanent **Dipole** moments of water molecules.

Note that one should not stay at the resonance frequency because in that case the microwave is reflected. One should not stay too much far away from it, because in this case the amount of absorption is too small. The real frequency is a compromise.

54.04.05
Reflection From Conducting Materials

54.04.06
Selective Reflectance From Conductors

54.04.07
Skin Effect

The skin effect, that is the fact that high-frequency ElectroMagnetic fields penetrates into a conductor by a typically small amount given by the skin depth (equation (54.02.11)), implies, as a consequence, that high-frequency currents only flow at the surface of a conductor. As a consequence the effective resistivity of a conductor is much larger, at high-frequency, than expected from considerations based on the geometry of the wire itself.

This explains, among other things, the use of copper braids as ground connections for high-frequency currents.

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54.04.08
Color

When looking at any object one sees either the light emitted by the object, for light emitting sources, or the light, coming from some source, reflected by the object. In the latter case the light might come from both surface reflection (surface color) and body reflection (body color). A third way to **see** an object is by observing the absorption by the object of the light coming from a source behind it.

Extinction of an ElectroMagnetic wave inside matter is caused by both absorption (energy transfer from ElectroMagnetic energy into internal energy) and scattering (energy spread around as Electro-Magnetic energy).

54.04.08.01 Surface Color

Typical for high reflectivity bodies.

It is determined by the properties of the surface. Selective reflection: when some wavelength is reflected more than others the body, when seen in reflection, will appear colored of the colors best reflected. When observed in transmission it will appear colored of the complementary color.

54.04.08.02 Body Color

Typical for low reflectivity bodies.

It is determined by the properties of the body. The wave penetrates and some frequencies are more absorbed than others, which are scattered. Selective scattering will give rise to the color, and scattering is in all directions. In this case the color is the same in reflection and transmission.

A typical example is the blue of the sky produced by Rayleigh scattering of the solar light: the radiation received is not coming from the surface of the medium but from all the medium itself.

Other examples are the color of water (sea-water, for instance) and colored glasses.

54.04.09
Reflection From a Good Conductor

Consider, for simplicity, ElectroMagnetic waves at normal incidence on a medium from vacuum. A non magnetic medium will be assumed: $\mu_{\mathrm{R}} = 1$.

In general terms the presence of absorption, at fixed real part of the **index of refraction** , always increases the reflection. In fact:

$$n \equiv n_{\mathrm{R}} + \mathrm{i}\, n_{\mathrm{I}} \quad ,$$

$$n^2 \equiv (n_{\mathrm{R}} + \mathrm{i}\, n_{\mathrm{I}})^2 \equiv \varepsilon'_{\mathrm{R}} \mu_{\mathrm{R}} \simeq \varepsilon'_{\mathrm{R}} = \Re\mathfrak{e}\left[\varepsilon'_{\mathrm{R}}\right] + \mathrm{i}\, \Im\mathfrak{m}\left[\varepsilon'_{\mathrm{R}}\right] = (n_{\mathrm{R}})^2 - (n_{\mathrm{I}})^2 + 2\mathrm{i}\, n_{\mathrm{R}} n_{\mathrm{I}} \quad ,$$

$$\Im\mathfrak{m}\left[\varepsilon'_{\mathrm{R}}\right] \neq 0 \qquad \Leftrightarrow \qquad n_{\mathrm{R}} n_{\mathrm{I}} \neq 0 \quad ,$$

$$\varrho \equiv \left| \frac{n-1}{n+1} \right|^2 = \frac{(n_{\mathrm{R}}-1)^2 + {n_{\mathrm{I}}}^2}{(n_{\mathrm{R}}+1)^2 + {n_{\mathrm{I}}}^2} \geq \frac{(n_{\mathrm{R}}-1)^2}{(n_{\mathrm{R}}+1)^2} \quad .$$

The above formula makes clear the fact that the amplitude reflection coefficient is large in either of the two cases:

$n_{\mathrm{I}} \gg 1 \implies \varrho \simeq 1$ strong absorption: total reflection with little energy penetrating to be absorbed ,

$n_{\mathrm{R}} = 0 \implies \varrho \simeq 1$ it applies to good conductors at frequencies: $\Gamma_0 \ll \omega \ll \omega_p$.

In the case of incidence on a conductor a phase shift is normally present, because the reflection and amplitude transmission coefficients are complex number, as long as the **index of refraction** is a complex number. Therefore in presence of absorption there is always a phase shift in the reflected and refracted wave.

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54.04.10
Relationship Between Absorption and Surface Reflection

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|||

High absorption is caused by a high value of the imaginary part of the **index of refraction** , that is from a large imaginary part of the wave-vector. This implies a amplitude reflection coefficient close to one. Then almost all the energy is reflected and little energy remains to be absorbed inside the medium.

As an example, a red body, if red is caused by surface (that is interface) reflection, appears blue in transparency because red is reflected and absorbed while other frequencies are preferentially transmitted.

Reflection From Gold and Silver

Silver has an almost uniform and high (> 0.9) reflectivity in the visible wavelength range. Therefore it reflects strongly and uniformly all the visible light. Then it appears metallic white.

Gold has an almost uniform and high (> 0.9) reflectivity in the visible wavelength range down to about $0.5\ \mu\mathrm{m}$ (yellow). Below that value the reflectivity drops to a smaller value. Therefore white light is selectively reflected by gold, with preferential reflection of the long wavelengths, giving to light reflected by gold the typical yellowish aspect. For the same reasons, white light transmitted by a thin sheet of gold has the complementary greenish color, because light below about $0.5\ \mu\mathrm{m}$ is well transmitted.

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54.05 Exercises Problems and Physical Applications

54-001 Electric-Permittivity of a Conductor

Replace equation (54.02.15) in equation (54.02.04) and show that the result for the resulting **Electric-Permittivity** is consistent with the microscopic model presented in section § 53 (equation (53.01.07) with $\overline{\omega} = 0$).

54-002 Skin Depth for Copper

Calculate the skin depth for copper from visible light knowing that: $\sigma = 6{\cdot}10^{7}$ SI.

WORKED SOLUTION

Visible light has $\nu \approx 10^{15}$ SI. It follows: $\delta \approx 10^{-8}$ m.

54-003 Conductivity, Damping Factor and Plasma Frequency for Copper

This § is referenced at pages:
[Never referenced.]

Copper has valence one, density $\rho = 8920\ \mathrm{kg/m^3}$, molar mass $\mathcal{M}_0 = 63.55{\cdot}10^{-3}\ \mathrm{kg/mol}$, and the low frequency **conductivity** is $\sigma^{-1}[\omega = 0] = 1.7{\cdot}10^{-8}\ \Omega \cdot \mathrm{m}$. Calculate the damping factor, Γ_0, and the plasma frequency, ω_p.

54-004 Conductivity Damping Factor and Plasma Frequency for Silver

This § is referenced at pages:
[Never referenced.]

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|ST9||

Silver has density $\rho = 10490\ \mathrm{kg/m^3}$, molar mass $\mathcal{M}_0 = 107.9{\cdot}10^{-3}\ \mathrm{kg/mol}$, low frequency **conductivity** is $\sigma^{-1}[\omega = 0] = 1.6{\cdot}10^{-8}\ \Omega \cdot \mathrm{m}$; it has one conduction electron per atom. Calculate the damping factor, Γ_0, and the plasma frequency, ω_p.

54-005 Conductivity, Damping Factor and Plasma Frequency for Graphite

This § is referenced at pages:
[Never referenced.]

©|Berkeley Physics Course, Vol. 3, *Waves*, 1968, McGraw-Hill, ...Ed., ...|ST9||

Graphite has valence four, density $\rho = 2.0{\cdot}10^{3}\ \mathrm{kg/m^3}$, molar mass $\mathcal{M}_0 = 12.0{\cdot}10^{-3}\ \mathrm{kg/mol}$, and the low frequency **conductivity** is $\sigma^{-1}[\omega = 0] = 3.0{\cdot}10^{-5}\ \Omega \cdot \mathrm{m}$. Calculate the damping factor, Γ_0, and the plasma frequency, ω_p.

DRAFT - DO NOT REPRODUCE

CHAPTER 55

Geometrical|Ray Optics

55.01 Introduction 2358
55.02 Fermat Principle 2359
55.03 Propagation of Light Rays in Linear Isotropic piece-wise homogeneous media 2361
55.04 General Properties of Optical Systems 2366
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This § is referenced at pages:
[2264, 2264]

©||WEB - URL |*Nice advenced introductory overview.*|

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55.01 Introduction

55.01.01
Applicability of Geometrical|Ray Optics

Geometrical Optics | Ray Optics is a model for the description of the propagation of electromagnetic waves with small wavelength compared to the relevant dimensions ($\lambda \longrightarrow 0$). In this case it is possible to neglect, as a first approximation, the wave effects and consider the propagation of ElectroMagnetic waves as a propagation of rays of light.

The modeling of rays in geometrical optics is, in a first approximation, just based on parametric curves in space. Refined models include, wavelength, intensity and polarization.

55.01.02
index of refraction

Dal punto di vista dell'ottica geometrica le caratteristiche ottiche di un mezzo sono completamente caratterizzate dal suo indice di rifrazione, definito in ottica geometrica come il rapporto tra la velocità (di fase) della luce nel vuoto, c, e la velocità (di fase) della luce nel mezzo in questione, v:

$$n \equiv \frac{c}{v} \quad .$$

Si noti che l'indice di rifrazione può anche essere minore di uno in quanto la velocità di fase della luce in un mezzo può anche essere maggiore della velocità di fase della luce nel vuoto. In generale l'indice di rifrazione dipende dalla lunghezza d'onda e dallo stato di Polarizzazione della luce ed è complesso.

Un mezzo otticamente omogeneo è un mezzo in cui la velocità di fase della luce non dipende dal punto. Un mezzo otticamente isotropo è un mezzo in cui la velocità di fase della luce in ogni punto non dipende dalla direzione di propagazione né dalla Polarizzazione della luce.

DRAFT! WORK IN PROGRESS

55.02 Fermat Principle

This § is referenced at pages:
[2185, 2263]

©|A.J.deWitte|Equivalence of Huygens' Principle and Fermat's Principle in Ray Geometry| *WEB - URL* |

©|E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|4.5|*Excellent*|

Fermat Principle: a light ray, moving through any set of linear and isotropic (non necessarily homogeneous) media, in going from any fixed point S to any fixed point P must traverse an optical path

$$\Delta t \propto \int_S^P n[\mathbf{x}]\, \mathrm{d}s \propto \int_S^P \mathrm{d}t \quad , \tag{55.02.01}$$

with length stationary with respect to small variations of the optical path; that is, such as to render its optical path length (nearly) equal to the length of other paths closely adjacent to the actual one. In short, the optical path length is a stationary functional of path (not necessarily a minimum). In this case, there is at P constructive interference, as the phases at arrival are synchronized to within some tolerance.

The key to Fermat principle is constructive Interference|Diffraction among nearby optical paths with almost equal path-lengths. Read § 50.02.

The formalism required to handle such kind of problems is variational calculus:

$$\delta \int_S^P n[\mathbf{x}]\, \mathrm{d}s = \delta \int_S^P \frac{c}{v[\mathbf{x}]}\, \mathrm{d}s = 0 \quad . \tag{55.02.02}$$

For a sequence of linear and isotropic media, k:

$$\sum_k \Delta t_k = 0 \qquad \text{for a sequence of linear and isotropic media} \quad .$$

© - QUOTE

E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...
Suppose that a beam of light advances through a homogeneous isotropic medium so that a ray passes from points S to P. Atoms within the material are driven by the incident disturbance, and they re-radiate in all directions. Wavelets progressing along paths in the immediate vicinity of a stationary straight-line path will reach P by routes that differ only slightly in OPL. They will therefore arrive nearly in-phase and reinforce each other. Think of each wavelet represented by a tiny phasor that rotates once around as the wave advances one wavelength along any ray path. Because the OPLs are all about the same, the phasors at P all point in more or less the same direction, and even though they're all small they combine to make the dominant contribution. Wavelets taking other paths far from the stationary one will arrive at P appreciably out-of-phase with each other and will therefore tend to cancel. In other words, there will be large angles between the little phasors; placed tip-to-tail they'll spiral around producing only a tiny net contribution. Keep in mind that we've just drawn three ray paths-the argument would be better made with millions of them in each group. We can conclude that energy will effectively propagate along the ray from S to P that satisfies Fermat's Principle. And this is true whether we're talking about interfering electromagnetic waves or photon probability amplitudes (p. 138). We can expect that this same logic holds for all propagation processes.

55.02.01
Principle of Reversibility of the Optical Paths

The principle of reversibility of the optical paths follows from Fermat principle: **a light beam going from B to A will travel in the opposite direction along the same trajectory traveled by a light beam going from A to B.**

55.02.02
Consequences of Fermat Principle

55.02.02.01 Reflection

Reflection from a plane surface: minimum optical path.

55.02.02.02 Transmission|Refraction

Read § 23-012.

55.02.02.03 Example of Maximum Optical Length Path

Spherical internally reflecting mirror with point source at one point on the surface: the effective optical path, source-sphere-source, is the one with reflection at the point diametrically opposite to the source; among all possible paths, source-sphere-source, this is the longest optical path one.

55.02.02.04 Example of Stationary Optical Length Path

An elliptical mirror with one point source at one focus and point detector at the other one: all paths with reflections have identical paths lengths, and therefore there is a stationary point, but longer than the shortest one, going directly from one focus to the other. There exist other paths, not stationary paths, but shorter than the one-reflection path.

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55.03 Propagation of Light Rays in Linear Isotropic piece-wise homogeneous media

55.03.01 Rectilinear Propagation in Homogeneous Isotropic Media

...

55.03.02 Reflection at a Plane Interface

At the interface between two isotropic media, a light ray is (partially) reflected; the reflected ray lies in the plane of incidence determined by the incident light ray and the surface normal at the point of incidence; the angles of incidence and reflection, θ and θ'', are equal:

$$\boxed{\theta = \theta''} \quad .$$

Let $\hat{\mathbf{u}}$ be the unit vector at the incidence point; the reflected ray direction, $\mathbf{k}''$, is given by:

$$\boxed{\mathbf{k}'' = \mathbf{k} - 2\hat{\mathbf{u}}\,(\mathbf{k}\cdot\hat{\mathbf{u}})} \quad . \tag{55.03.01}$$

→ 1252

Note that no information is provided by Fermat principle about the reflection or transmission coefficients: just a **geometrical information** is provided. Moreover, Fermat principle applies to the case of specular reflection, that is, to one perfectly smooth interface surface. Basically, it means that the imperfections of the surface (roughness) are much smaller than the wavelength. In the case of a surface with significant roughness, also **diffuse reflection** occurs.

55.03.03 Transmission|Refraction at a Plane Interface

At the interface between two isotropic media, a light ray is (partially) transmitted; the transmitted ray lies in the plane of incidence determined by the incident light ray and the surface normal at the point of incidence; the angles of incidence and transmission, θ and θ', are linked by:

$$\boxed{n\sin\theta = n'\sin\theta'} \quad .$$

Let $\hat{\mathbf{u}}$ be the unit vector at the incidence point; the transmitted ray direction, $\mathbf{k}'$, is given by:

$$n'\mathbf{k}' = n\mathbf{k} + \hat{\mathbf{u}}\left(n\,(\mathbf{k}\cdot\hat{\mathbf{u}}) - \sqrt{n'^2 - n^2 + n^2\,(\mathbf{k}\cdot\hat{\mathbf{u}})^2}\right) = n\left(\mathbf{k} + \hat{\mathbf{u}}\left((\mathbf{k}\cdot\hat{\mathbf{u}}) - \sqrt{\left(\frac{n'}{n}\right)^2 - 1 + (\mathbf{k}\cdot\hat{\mathbf{u}})^2}\right)\right) \quad , \tag{55.03.02}$$

→ 1252 2364

which only depends on the ratio between the two index of refractiones.

Note that no information is provided by Fermat principle about the reflection or transmission coefficients: just a **geometrical information** is provided. Moreover, Fermat principle applies to the case of one perfectly smooth interface surface. Basically, it means that the imperfections of the surface (roughness) are much smaller than the wavelength. In the case of a surface with significant roughness, also **diffuse transmission** occurs.

DRAFT-IN-PROGRESS

55.03.04
Paraxial Reflection and Transmission|Refraction at a Spherical Surface

Beware that **different conventions** exist in the literature concerning the signs of the different quantities entering equations.

55.03.04.01 Paraxial Reflection at a Spherical Mirror

This § is referenced at pages:
[2362]

Both Transmission|Refraction and reflection in general do exist (read § 55.03.04.02) at a surface, even if there are cases when one of the two can be negligible.

Small angles approximation.

$$\boxed{\frac{1}{p}+\frac{1}{q}=-\frac{2}{R}\equiv\frac{1}{f}} \quad . \tag{55.03.03}$$

→ 2362 2377

$$\boxed{f_1=-\frac{2}{R} \qquad \text{for } q\longrightarrow+\infty} \quad ,$$

$$\boxed{f_2=-\frac{2}{R} \qquad \text{for } p\longrightarrow+\infty} \quad .$$

The equation (55.03.03) shows **focusing**: the distance q does not depend on the angle of the ray leaving the object.

Sign Conventions for Reflecting Surfaces:

- Light-side, incoming light, along the z axis, from left to right (the optical axis);
- Objects on incident-light side: p is positive;
- Images on reflected-light side: q is positive, along $-z$;
- Center of curvature right of V: R is positive;
- ρ positive above the optical axis, erect object and/or image;
- Parameters are negative if they don't meet the above criteria for being positive.

55.03.04.02 Paraxial Transmission|Refraction at a Spherical Refracting Interface

This § is referenced at pages:
[2362, 2363]

Both Transmission|Refraction and reflection in general do exist (read § 55.03.04.01) at a surface, even if there are cases when one of the two can be negligible.

Small angles approximation.

$$\boxed{\frac{n_1}{p}+\frac{n_2}{q}=\frac{n_2-n_1}{R}} \quad . \tag{55.03.04}$$

→ 2362

$$\boxed{f_1=\frac{n_1R}{n_2-n_1} \qquad \text{for } q\longrightarrow+\infty} \quad ,$$

$$\boxed{f_2=\frac{n_2R}{n_2-n_1} \qquad \text{for } p\longrightarrow+\infty} \quad .$$

The equation (55.03.04) shows **focusing**: the distance q does not depend on the angle of the ray leaving the object.

Sign Conventions for Refracting Surfaces:

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- Light-side, incoming light, along the z axis, from left to right (the optical axis);
- Objects on incident-light side: p is positive;
- Images on refracted-light side: q is positive, along $+z$;
- Center of curvature right of V: R is positive;
- ρ positive above the optical axis, erect object and/or image;
- Parameters are negative if they don't meet the above criteria for being positive.

55.03.04.03 Paraxial Transmission|Refraction at Two|More Close Spherical Interfaces

This § is referenced at pages:
[Never referenced.]

Consider the surfaces at the same position, that is with zero distance between them. Consider a sequence such that the image of any surface is the object of the next one, accounting for their signs:

$$p \equiv p_1 \qquad q_1 = -p_2 \ldots q_{N-1} = p_N \qquad q \equiv q_N \tag{55.03.05}$$

→ 2363

55.03.04.04 Paraxial Transmission|Refraction at a Thin Lens

This § is referenced at pages:
[Never referenced.]

Using equation (55.03.05) the equation for the thin lens is found.
Small angles approximation.

$$\boxed{\frac{1}{p} + \frac{1}{q} = \frac{n_L - n_0}{n_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \equiv \frac{1}{f}} \quad . \tag{55.03.06}$$

→ 2377

55.03.05
Paraxial Transmission|Refraction at a Generic Surface

Small angles approximation.
It goes as in section § 55.03.04.02 with the local properties of the surface.
In fact, Reflection and Transmission|Refraction depend on local properties of the surface only. Therefore, as every surface can be locally approximated, most often, with a quadratic surface, the understanding of spherical surfaces is very important. In fact, within every normal plane, the surface intersection with the normal plane can be approximated with a circumference, so that inside each normal plane ray tracing can benefit from the discussion of spherical surfaces.

55.03.06
Special Paraxial Rays

Consider a cylindrically symmetrical optical system.
Special Rays for Refracting|Reflecting Surfaces are defined as follows, with reference to either the ray or its extension, forward or backward as required.

- The principle of reversibility of optical paths applies.
- Any ray that passes **through the focal point**: the Refracting|Reflecting ray is parallel to the central axis, as it is focused to infinity.
- A ray that is initially **parallel to the central axis**: the Refracting|Reflecting ray passes through the focal point, also form reversibility from the previous point.
- A ray that passes **through the center of curvature**: the Refracting|Reflecting ray propagates along the same direction as the incident ray.
- A ray that arrives at the vertex of a thin lens passes with the same angle with respect to the axis, as

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the two surfaces are parallel each other and perpendicular to the axis.

- A ray that arrives at the vertex of a mirror is reflected with the same angle with respect to the axis, as the surface is perpendicular to the axis.

55.03.07
General Transmission|Refraction at a Generic Interface

Equation (55.03.02) must be used, with the local properties of the surface.

55.03.08
Type of lenses|mirrors

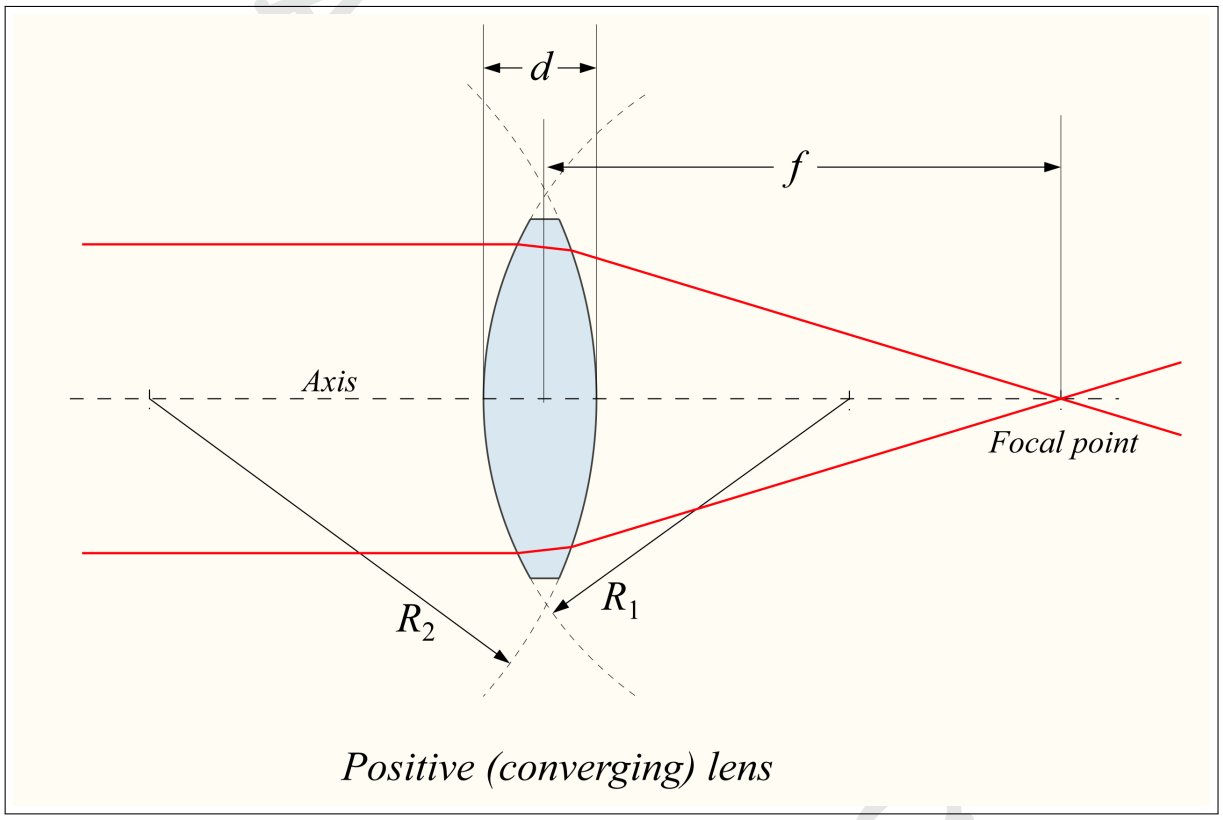

Figure 55.11: Biconvex Lens - WEB - URL.

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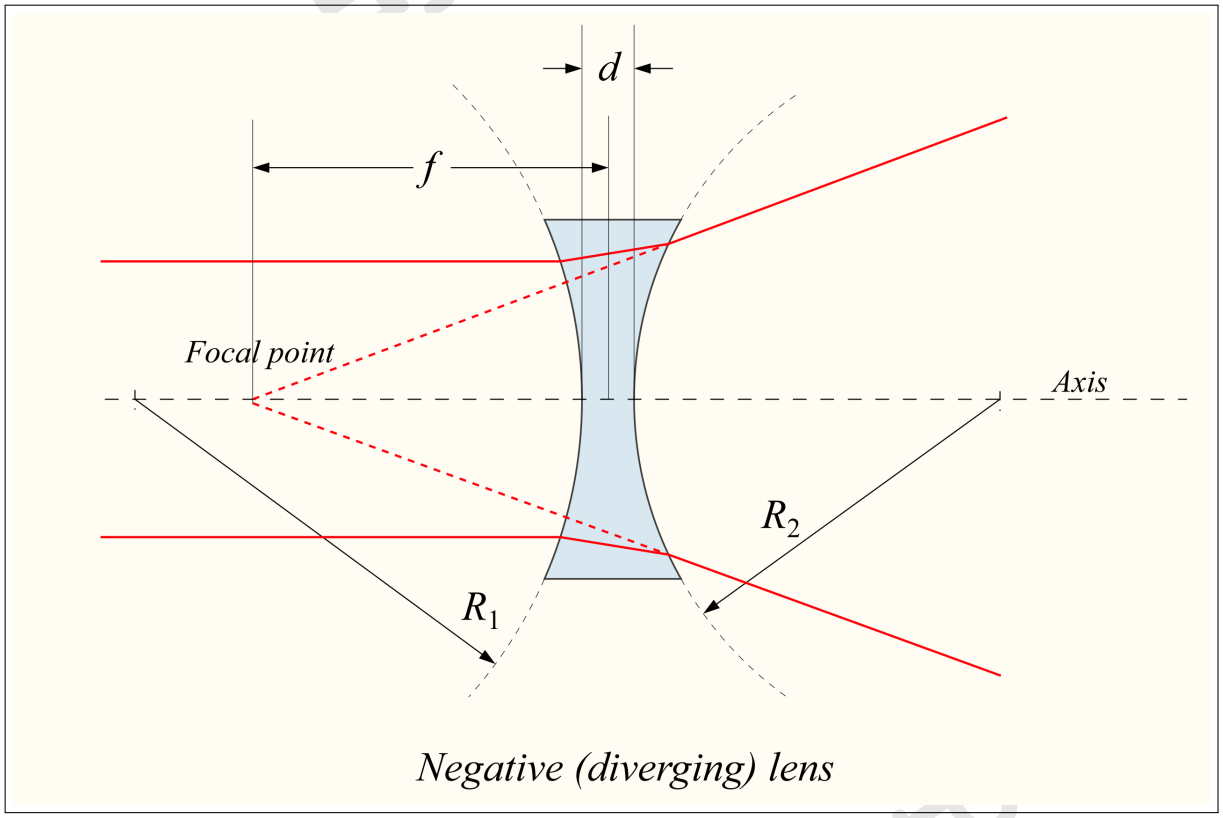

Figure 55.12: Biconcave Lens - WEB - URL.

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55.04 General Properties of Optical Systems

- A real object is one such that light rays physically emanate from the object.
- A real image is one such that light rays physically intersect at the image point.
- A virtual object is one from which light rays appear to emanate, but physically do not.
- A virtual image is one in which light rays do not physically intersect at the image point, but appear to diverge from that point.
- Real images may be displayed on a screen while virtual images may not.

Un sistema ottico è detto stigmatico per una coppia di punti P e P' se per un cono di raggi luminosi che escono dal punto P esiste un corrispondente cono di raggi luminosi che passano per il punto P'. In tal caso P' è detto immagine perfetta di P.

55.04.01 Object and Image

Un punto oggetto è un punto che emette raggi luminosi in tutte le direzioni. L'immagine di un punto è un punto in cui convergono i raggi luminosi emessi da un punto oggetto. I due punti sono detti coniugati.

55.04.02 Real and Virtual

Un raggio reale è un raggio luminoso realmente esistente ed accessibile. Un raggio virtuale si riferisce all'estrapolazione ad una zona non fisica di un raggio reale, non accessibile.

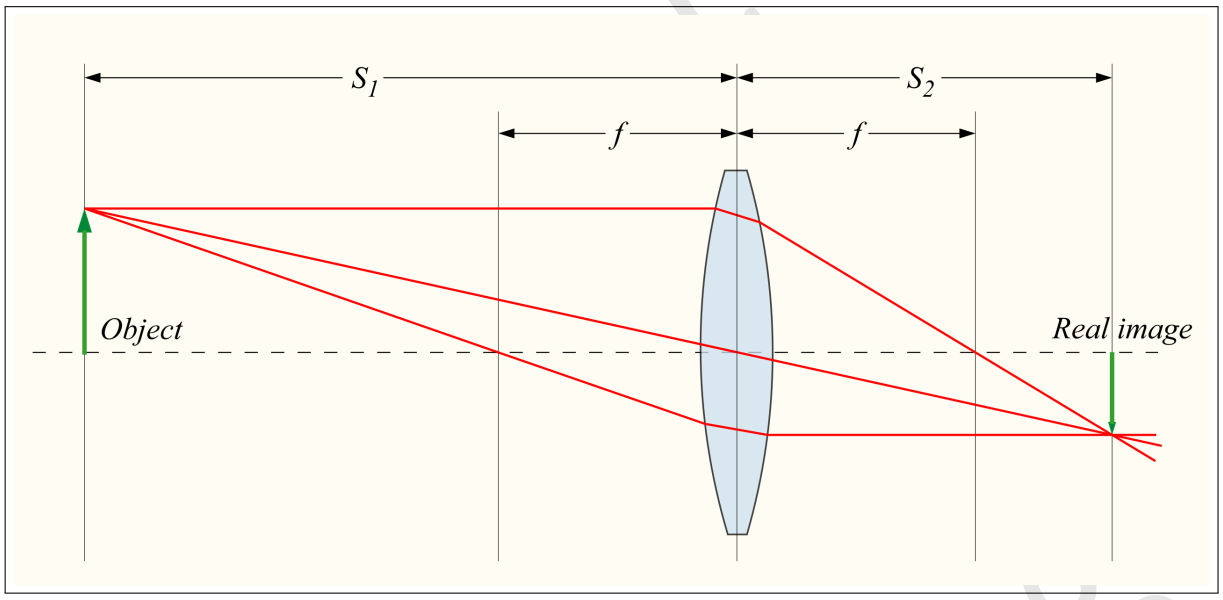

Figure 55.13: Real Image - WEB - URL.

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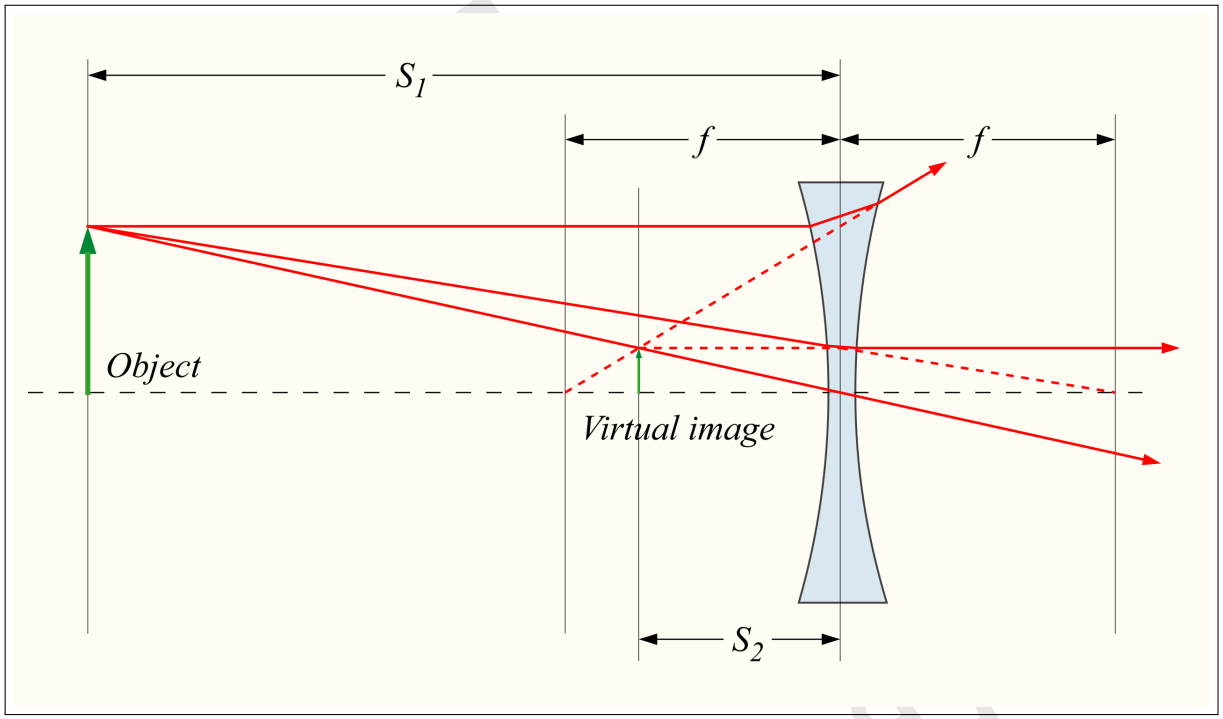

Figure 55.14: Virtual Image - WEB - URL.

55.05 Centered Optical Systems

Un sistema ottico si dice centrato quando il sistema ha simmetria di rotazione attorno ad un asse, detto asse ottico del sistema. I sistemi ottici centrati coprono una gran parte dei sistemi ottici di uso pratico.

55.05.01 Paraxial Approximation

Dato un sistema ottico centrato nell'approssimazione parassiale si considerano solo i raggi che raggiungono la superficie in un punto a distanza dall'asse ottico piccola rispetto ai raggi di curvatura delle superfici e con angolo piccolo rispetto all'asse ottico.

55.05.02 Magnification

Il caso di gran lunga più interessante in pratica è quello di un oggetto piano e di una superficie ricevente piana, entrambi perpendicolari all'asse ottico.

L'ingrandimento trasversale è il rapporto tra l'altezza (distanza dall'asse), con segno, dell'immagine, h', e quella dell'oggetto h, con segno:

$$\boxed{m_T \equiv \frac{h'}{h}} \quad ,$$

è una grandezza che può assumere qualunque valore reale.

L'ingrandimento longitudinale o assiale, lungo l'asse ottico z, è il rapporto tra la lunghezza (con segno) dell'immagine e quella dell'oggetto (con segno) nel limite di piccole lunghezze:

$$\boxed{m_L \equiv \frac{\Delta z'}{\Delta z}} \quad ,$$

è una grandezza che può assume qualunque valore reale.

L'ingrandimento angolare è il rapporto tra l'angolo (con segno) tra due punti dell'immagine e l'angolo (con segno) tra gli stessi due punti dell'oggetto, nel limite di piccoli angoli:

$$\boxed{m_\alpha \equiv \frac{\Delta \alpha'}{\Delta \alpha}} \quad ,$$

è una grandezza che può assume qualunque valore reale.

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55.05.02.01 Visual Magnification

Con strumenti ottici la percezione visiva delle dimensioni dell'oggetto dipende dalla sua estensione angolare. L'ingrandimento visuale è il rapporto tra l'estensione angolare dell'immagine relativa a quella dell'oggetto visto direttamente. L'estensione angolare dell'oggetto visto direttamente dipende dalla distanza. Si considera l'oggetto posizionato al punto più vicino che consente visione distinta, convenzionalmente scelta come $\mathcal{D} = 25\ \mathrm{cm}$.

55.05.03
Cardinal and Principal Planes nodal points and planes

Un qualunque sistema ottico centrato è caratterizzato completamente, nell'approssimazione parassiale, dalla posizione dei suoi piani cardinali.

Un fascio di raggi paralleli all'asse ottico viene focalizzato in un punto dell'asse ottico (secondo fuoco). Il piano passante per il secondo fuoco e perpendicolare all'asse ottico è il secondo piano focale. Un fascio di raggi che escono parallelamente all'asse ottico ha come punto oggetto un punto dell'asse ottico detto primo fuoco. Il piano passante per il primo fuoco e perpendicolare all'asse ottico è il primo piano focale.

Esiste una coppia di piani coniugati (piano oggetto e piano immagine) che sono l'uno l'immagine dell'altro con ingrandimento trasversale unitario. Tali piani sono detti i piani principali.

Esistono due punti dell'asse ottico detti punti nodali: points of an optical system for which an incoming ray, directed at a nodal point, leaves the system with the same direction. I piani corrispondenti perpendicolari all'asse ottico sono detti piani nodali e sono caratterizzati da ingrandimento angolare unitario.

In the frequently encountered situation where the index of refraction is the same in front of and behind the optical system, the nodal points coincide with the principal points.

55.05.04
Thin Lenses

La lunghezza focale di una lente sottile in aria (indice di rifrazione uguale ad uno) è

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

dove R_1 ed R_2 sono i raggi di curvatura rispettivamente della prima e della seconda superficie, adottando la convenzione descritta sopra.

Per una lente sottile si ha l'equazione nella forma gaussiana

$$\frac{1}{o} + \frac{1}{i} = \frac{1}{f} \quad . \tag{55.05.01}$$

→ 2370

Definendo $z_i \equiv i - f$ e $z_o \equiv o - f$ (distanze rispetto ai fuochi) si ha l'equazione nella forma di Newton

$$z_i z_o = f^2 \quad .$$

L'ingrandimento lineare vale

$$M = -\frac{i}{o}$$

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55.05.04.01 Combinazione Di Lenti Sottili

La distanza focale di una combinazione di due lenti sottili è

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \quad ,$$

dove d è la distanza tra le due lenti.

55.05.05
Thick Lenses

La lunghezza focale di una lente spessa è data da

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2} - \frac{t\,(n-1)}{n R_1 R_2}\right) \quad ,$$

dove t è lo spessore della lente.

Le posizioni dei piani principali sono date da

$$d_1 = ft\left(\frac{1-n}{nR_2}\right) \quad ,$$

$$d_2 = ft\left(\frac{1-n}{nR_1}\right) \quad .$$

In una lente spessa tutte le distanze nello spazio oggetto e nello spazio immagine sono misurate a partire dai piani principali e si applica la relazione (55.05.01).

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55.06 Paraxial Optics of Axially Symmetric Systems via Transfer Matrices

Un raggio che giace in un piano passante per l'asse ottico di un sistema ottico centrato (raggio meridionale) è descritto completamente dalla coordinata di un punto del raggio (la sua distanza dall'asse ottico, ρ) e dall'angolo che in quel punto forma con l'asse ottico (α).

In approssimazione parassiale il passaggio del raggio attraverso il sistema ottico centrato può essere descritto da una successione di trasformazioni lineari nella coppia di variabili ρ e α:

$$\begin{pmatrix} \rho' \\ \alpha' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \rho \\ \alpha \end{pmatrix} = \mathcal{M} \begin{pmatrix} \rho \\ \alpha \end{pmatrix} \quad , \tag{55.06.01}$$

→ 2372

descritte da una opportuna matrice $\mathcal{M}$. I successivi passaggi (traslazioni, rifrazioni, riflessioni) possono essere dunque descritti da una successione di matrici e il risultato finale sarà il prodotto delle varie matrici di trasformazione, scritte da destra verso sinistra, in ordine cioè inverso rispetto a quello in cui sono attraversate.

55.06.01 Matrici Di Trasformazione

La matrice di traslazione di una distanza d è

$$\mathcal{T} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \quad . \tag{55.06.02}$$

→ 2371

La matrice della rifrazione ad una superficie sferica con raggio di curvatura r che separa due mezzi con indice di rifrazione n ed n' è

$$\mathcal{R} = \begin{pmatrix} 1 & 0 \\ -\dfrac{P}{n'} & \dfrac{n}{n'} \end{pmatrix} \quad , \tag{55.06.03}$$

→ 2371

dove si è indicata con P la potenza della superficie rifrangente

$$P \equiv \frac{n' - n}{r} \quad , \tag{55.06.04}$$

positiva per una superficie che faccia convergere i raggi e negativa per una superficie che faccia divergere i raggi, con la convenzione che il raggio di curvatura sia positivo per una superficie convessa e negativo per una superficie concava.

La matrice per la riflessione ad una superficie sferica si può scrivere adottando la convenzione che in presenza di una superficie riflettente il verso dell'asse ottico si inverta alla riflessione:

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ \dfrac{2}{R} & 1 \end{pmatrix} \quad , \tag{55.06.05}$$

Dalle (55.06.02), (55.06.03) si deduce che per un sistema ottico centrato una qualunque successione di traslazioni e rifrazioni su superfici sferiche è rappresentato da una matrice il cui determinante è il rapporto tra l'indice di rifrazione del primo mezzo con quello dell'ultimo mezzo:

$$\det[\mathcal{M}] = \frac{n}{n'} \quad . \tag{55.06.06}$$

→ 2372

La matrice di trasformazione per una lente, tra i piani definiti dai vertici delle superfici sferiche, si ottiene dalla composizione delle (55.06.02) e (55.06.03),

$$\begin{pmatrix} 1 - t\dfrac{P_1}{n_L} & t\dfrac{n}{n_L} \\ t\dfrac{P_1 P_2}{n' n_L} - \dfrac{P_1}{n'} - \dfrac{P_2}{n'} & \dfrac{n}{n'}\left(1 - t\dfrac{P_2}{n_L}\right) \end{pmatrix} \quad , \tag{55.06.07}$$

→ 2374

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dove t è la distanza tra i vertici delle due superfici ed n_L l'indice di rifrazione del materiale della lente.

Per una lente sottile si riduce a

$$\begin{pmatrix} 1 & 0 \\ -\dfrac{P}{n'} & \dfrac{n}{n'} \end{pmatrix} \quad , \qquad \rightarrow 2373$$

dove la potenza della lente, in termini dei raggi di curvatura, è data da

$$P = P_1 + P_2 = \frac{n_L - n}{r_1} + \frac{n' - n_L}{r_2} \quad .$$

ed ha dunque la stessa forma di quella della rifrazione semplice.

55.06.02
Punti e Piani Cardinali

I punti nodali, i punti focali e i punti principali costituiscono i punti cardinali di un sistema ottico centrato.

Nel caso generale si ha come sola condizione sulla matrice $\mathcal{M}$ ((55.06.01)) la condizione sul determinante (55.06.06). Tale condizione implica che al massimo due elementi della matrice possono essere nulli, e se due sono nulli non possono essere né sulla stessa riga né sulla stessa colonna.

Il più generale sistema che forma immagini è caratterizzato dalla proprietà

$$M_{12} = 0 \qquad \Leftrightarrow \qquad \rho' = \beta\rho$$

con β l'ingrandimento lineare. In tal caso i due piani sono caratterizzati dall'essere due piani coniugati.

Dato un generico sistema si possono sempre determinare due piani coniugati (piani principali) che sono l'uno l'immagine dell'altro con ingrandimento lineare trasversale unitario. La coppia di piani coniugati cui corrisponde $\beta = 1$ definisce i piani principali.

Il più generale sistema afocale, o telescopico, in cui ogni fascio di raggi paralleli esce come un fascio di raggi paralleli, è caratterizzato dalla proprietà

$$M_{21} = 0 \qquad \Leftrightarrow \qquad \alpha' = \gamma\alpha$$

con γ l'ingrandimento angolare.

La coppia di piani coniugati cui corrisponde $\gamma = 1$ definisce i piani nodali.

La condizione $M_{11} = 0$ implica che raggi che entrano nel sistema paralleli all'asse ottico ($\alpha = 0$) vengono focalizzati ad $\rho' = 0$ qualunque sia ρ. Tale condizione corrisponde al fatto che il secondo piano è il secondo piano focale del sistema.

La condizione $M_{22} = 0$ implica che se $\rho = 0$ allora $\alpha' = 0$, qualunque sia α. Quindi in tal caso il primo piano è il primo piano focale del sistema.

55.06.03
Determinazione Dei Punti/Piani Principali

Two planes perpendicular to the axis are said principal planes when they are conjugate planes with unit linear transverse magnification.

Dato un generico sistema descritto dalla matrice $\mathcal{M}$, che trasforma i punti del piano B nei punti del piano B' (entrambi ortogonali all'asse ottico), si considerino due traslazioni, $\mathcal{T}$ da un piano H al piano B e $\mathcal{T}'$ dal piano B' ad un piano H'. La trasformazione complessiva è descritta dalla matrice $\mathcal{M}' = \mathcal{T}'\mathcal{M}\mathcal{T}$. Tale trasformazione lascia invariato l'elemento M_{21} ($M'_{21} = M_{21}$) ed $\mathcal{M}'$ risulta essere

$$\mathcal{M}' = \begin{pmatrix} M_{11} + D'M_{21} & DM_{11} + D'M_{22} + M_{12} + DD'M_{21} \\ M_{21} & DM_{21} + M_{22} \end{pmatrix} \quad . \qquad \rightarrow 2373$$

Richiedendo che i due piani H ed H' siano coniugati con ingrandimento unitario (i piani principali) si hanno le condizioni $M'_{11} = \beta = 1$ e $M'_{12} = 0$. Inoltre in generale $M'_{22} = n/n'$. Da queste si ricavano

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le distanze dei piani principali dai piani B e B'

$$D = \frac{(n/n') - M_{22}}{M_{21}} \qquad (55.06.07)$$
$$D' = \frac{1 - M_{11}}{M_{21}}$$

→ 2373 2374

con la condizione $M_{21} \neq 0$, che rappresenta la condizione necessaria e sufficiente all'esistenza dei piani principali ed esprime la richiesta che il sistema abbia potenza non nulla. Le due condizioni (55.06.07) implicano $M'_{12} = 0$ per ogni matrice a determinante unitario.

La matrice di trasformazione di un generico sistema ottico tra i due piani principali si scrive allora

$$\begin{pmatrix} 1 & 0 \\ -\frac{P}{n'} & \frac{n}{n'} \end{pmatrix} \qquad \text{con} \qquad P \equiv -n' M_{12} \quad . \qquad (55.06.08)$$

Quindi nell'approssimazione parassiale tutti i sistemi ottici che formano immagini hanno lo stesso comportamento formale della lente sottile e la matrice di trasformazione tra i piani principali è data dalla (55.06.01).

55.06.04
Determinazione Dei Punti/Piani Focali

Dalla relazione (55.06.03) si può determinare la posizione dei piani focali di un generico sistema ottico. Dalle condizioni $M_{11} = 0$ e $M_{22} = 0$ si ricava

$$F' = -\frac{M_{11}}{M_{21}} \qquad (55.06.09)$$
$$F = -\frac{M_{22}}{M_{21}}$$

55.06.05
Determinazione Dei Punti/Piani Nodali

Two planes perpendicular to the axis are said nodal planes when they are conjugate planes with unit angular magnification.

Dalla relazione (55.06.06) imponendo la condizione $M_{22} = 0$ si determina la posizione dei piani nodali di un sistema. La distanza L di cui il primo piano nodale precede il primo piano principale e la distanza L' di cui il secondo piano nodale segue il secondo piano principale sono date da

$$L = -L' = \frac{n - n'}{P} \quad . \qquad (55.06.10)$$

Poiché $L = -L'$ la distanza tra i piani nodali è la stessa che la distanza tra i piani principali e le due coppie di piani sono coincidenti nel caso $n = n'$.

55.06.06
La Formazione Delle Immagini

Si possono trovare le relazioni tra i piani principali e due qualunque piani coniugati di un sistema di potenza P effettuando due generiche traslazioni, S ed S', a partire dai piani principali. La matrice della trasformazione risultante è

$$\begin{pmatrix} 1 - S'\frac{P}{n'} & S - SS'\frac{P}{n'} + S'\frac{n}{n'} \\ -\frac{P}{n'} & \frac{n}{n'}\left(1 - S\frac{P}{n}\right) \end{pmatrix} \quad .$$

→ 2373

Si trova

$$\frac{n}{S} + \frac{n'}{S'} = P = \frac{n}{f} = \frac{n'}{f'} \quad ,$$

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avendo introdotto le distanze focali

$$S \longrightarrow +\infty \implies S' = \frac{n'}{P} \equiv f' \quad ,$$

$$S' \longrightarrow +\infty \implies S = \frac{n}{P} \equiv f \quad .$$

Equivalentemente, indicando con z ed z' le distanze misurate a partire dai fuochi, si ha la forma newtoniana:

$$zz' = ff' \quad .$$

Quando si ha formazione di immagine la matrice prende la forma

$$\begin{pmatrix} \beta & 0 \\ -\dfrac{P}{n'} & \gamma \end{pmatrix} \quad ,$$

dove gli ingrandimenti lineare ed angolare valgono

$$\beta = -\frac{nS'}{n'S} \quad ,$$

$$\gamma = -\frac{S}{S'} \quad .$$

Dalla condizione sul determinante si ottiene

$$\beta\gamma = \frac{n}{n'} \quad ,$$

che in base alla definizioni di β e γ

$$\beta \equiv \frac{\rho'}{\rho} \qquad \gamma \equiv \frac{\Delta\alpha'}{\Delta\alpha} \quad ,$$

fornisce l'invariante ottico di Lagrange o Smith-Helmholtz:

$$n\rho\alpha = n'\rho'\alpha' \quad .$$

55.06.07

Combinazione Di Due Sistemi Che Formano Immagini

Si considerino due generici sistemi che formano immagini, descritti dalle matrici $calM_1$ ed $calM_2$ rispetto ai loro piani principali, ed aventi potenza, rispettivamente

$$P_1 = \frac{n_1}{f_1} = \frac{n_1'}{f_1'} \qquad \text{e} \qquad P_2 = \frac{n_2}{f_2} = \frac{n_2'}{f_2'} \quad .$$

Nel caso specifico si ha $n_1' = n_2$, mentre le proprietà dei due sistemi ottici sono condensate nelle due potenze, P_1 e P_2.

La matrice del sistema formato ponendo a distanza d il secondo piano principale del primo sistema dal primo piano principale del secondo sistema ha la stessa struttura della (55.06.01) ed è data da

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ -\dfrac{P_2}{n_2'} & \dfrac{n_2}{n_2'} \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\dfrac{P_1}{n_1'} & \dfrac{n_1}{n_1'} \end{pmatrix} = \begin{pmatrix} 1 - d\dfrac{P_1}{n_2} & d\dfrac{n}{n_2} \\ d\dfrac{P_1P_2}{n'n_2} - \dfrac{P_1}{n'} - \dfrac{P_2}{n'} & \dfrac{n}{n'}\left(1 - d\dfrac{P_2}{n_2}\right) \end{pmatrix} \quad . \tag{55.06.11}$$

Dalla relazione derivata si possono poi ricavare le posizioni dei piani principali del sistema complessivo tramite le (55.06.07).

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55.06.08
Sistemi Telescopici

A telescopic system converts a bundle of parallel rays making an angle with the optical axis to a second parallel bundle of rays making another angle. The ratio of the two angles is called the angular magnification.

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55.07 Optical Aberrations

Le deviazioni dall'ottica parassiale sono dette aberrazioni che si dividono in aberrazioni monocromatiche, esistenti anche per luce di lunghezza d'onda definita, e cromatiche che dipendono dal fatto che l'indice di rifrazione è in generale funzione della lunghezza d'onda.

L'ottica parassiale si ottiene considerando tutte le relazioni esatte approssimate al primo ordine. Nell'approssimazione successiva, al terzo ordine, le aberrazioni si possono classificare in cinque tipi: aberrazione sferica, coma e astigmatismo, che deteriorano la qualità dell'immagine, distorsione e curvatura di campo che deformano l'immagine senza deteriorarla.

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55.08 Examples and Physical Applications

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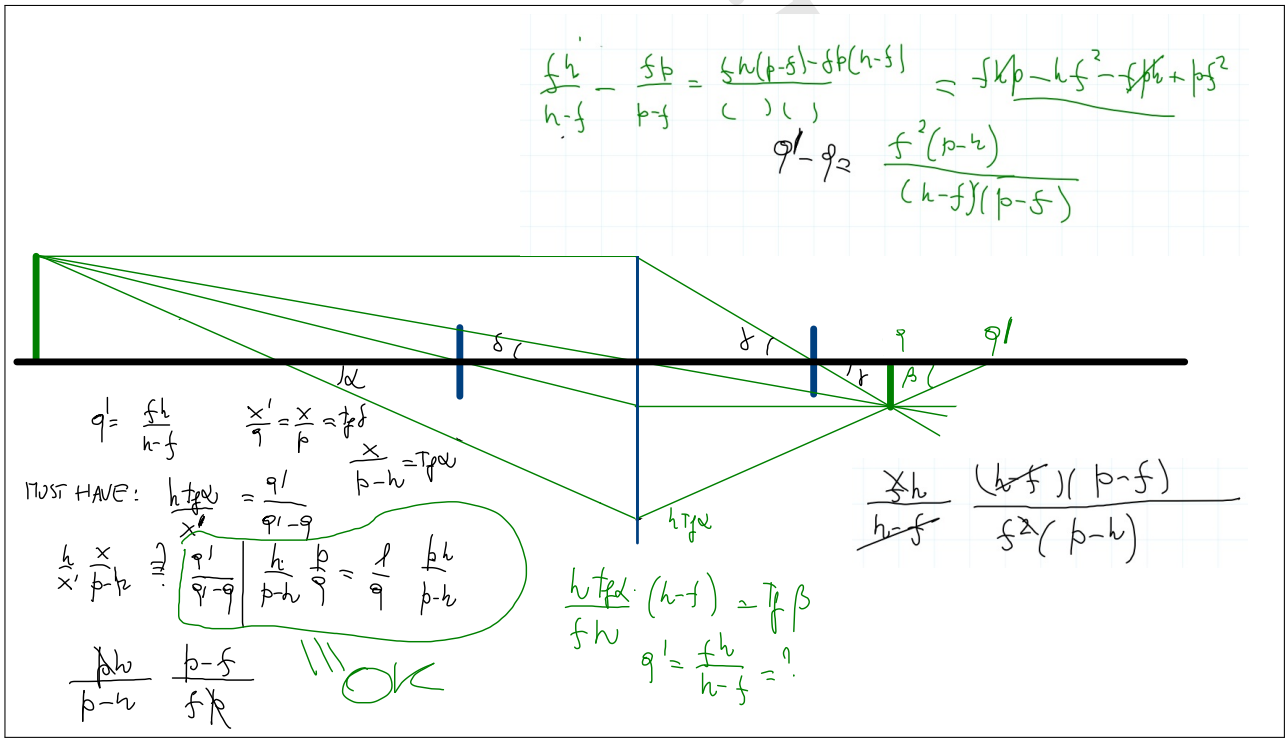
55.08.01
PinHole camera

1. How does a pinhole camera work?
2. How does it generate an image?
3. Why a large aperture does not focus?

55.08.02
ParaxialOffAxisRayTracing

From the fundamental paraxial relations, equations (55.03.03) and (55.03.06), it is easy to trace rays crossing the axis and|or passing by the focii.

Demonstrate, from geometric considerations, that all rays emitted by an off-axis point at an arbitrary angle with the axis in the paraxial approximation are always all focused at the same point.



55.08.03
Ray Diagram for spherical lenses

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55.08.04
Ray Diagram for spherical mirrors

©|WEB - URL |||

55.08.05
A single Thin lens

©|WEB - URL |||

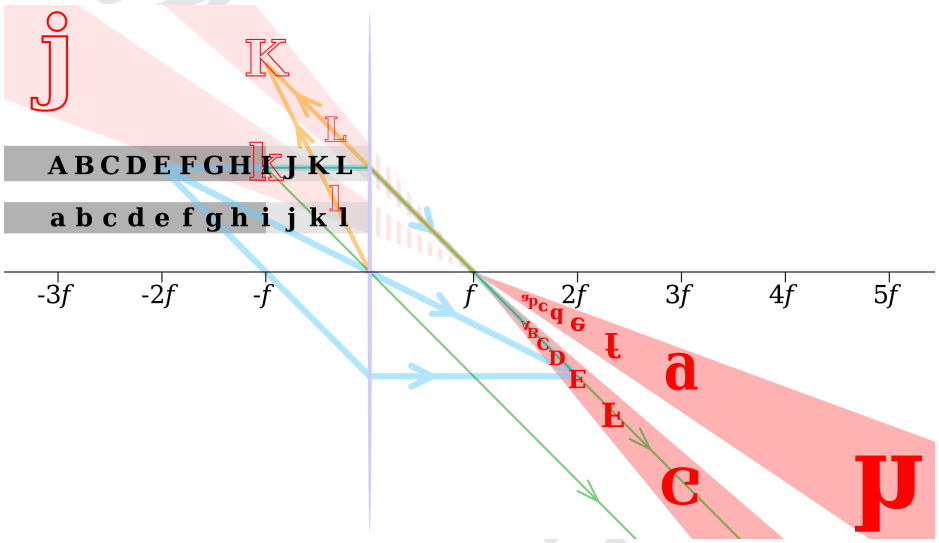

Figure 55.15: Images of the black letters in a thin convex lens of focal length f are shown in red. Selected rays are shown for letters E, I and K in blue, green and orange, respectively. Note that E (at $2f$) has an equal-size, real and inverted image; I (at $-f$) has its image at infinity; K (at $-f/2$) has a double-size, virtual and upright image.

55.08.06
The Camera

- The pinhole camera.
- The camera.

55.08.07
The simple Magnifier

It is the basis of many optical instruments: the ocular.
The object has to be close; the image is virtual.

DRAFT IN PROGRESS

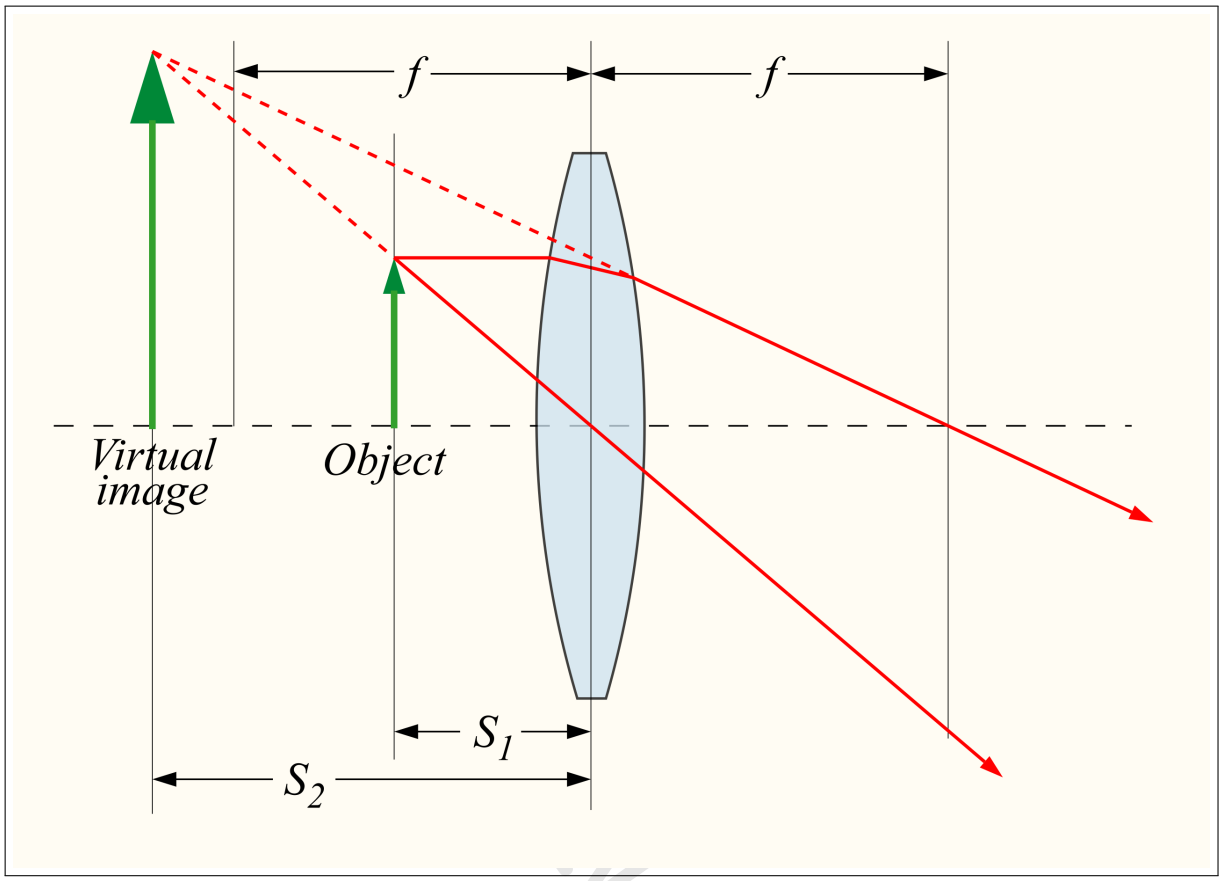

Figure 55.16: Magnifying glass - WEB - URL.

WORK IN PROGRESS

55.08.08
Ray Diagrams For Microscope And Telescope

©|https://demonstrations.wolfram.com/RayDiagramsForMicroscopeAndTelescope/||*EXCELLENT*|

55.08.09
The Telescope

55.08.09.01 The Astronomical Telescope

Two positive lenses with second focus of the objective coincident with the first focus of the ocular.
It works on objects at infinity, angular magnification is the key.
The ocular is a magnifying glass using as object the image of the objective.

55.08.09.02 The Galilean Telescope

One positive objective lens, and one negative ocular lens with the second foci of the two lenses coincident.
It works on objects at infinity, angular magnification is the key.
The ocular is a magnifying glass using as object the image of the objective.

55.08.10
The Microscope

Two positive lenses with second focus of the objective and first focus of the ocular separated by a fixed distance.
It works on close objects.
The ocular is a magnifying glass using as object the image of the objective.

55.08.11
Field Lenses

55.08.12
Human Eye

55.08.13
A Perfect Lens

©||WEB - URL |*Applet*|

DRAFT - IN-PROGRESS

55.08.14
Depth Of Field
WEB - URL

55.08.15
Atmospheric Optics
©|J.P.Pérez *et al.*, *Optique*, ..., DUNOD, ...Ed., WEB - URL.|17.3||

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55.09 Exercises Problems and Physical Applications

55-001 Problems on Geometrical Optics

©|E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|§ 5, 6||

See plenty of problems in reference.

55-002 A parabolic mirror

Show that a parabolic mirror (built from a paraboloid, $z = \alpha r^2$, in cylindrical coordinates) focuses rays parallel to its axis exactly into its geometrical focus and determine its position as a function of α.

55-003 A Hyperbolic Lens

Show that a plano-convex lens, with the curved surface shaped as a hyperboloid of revolution, and axis perpendicular to the plane face, exactly focuses rays parallel to the axis, incident on the plane face onto a point and determine its position as a function of the index of refraction of the lens. Determine the position of the other focus. Does it suffer chromatic aberrations?

55-004 Hemispherical Lens

Also named immersion lens.

DRAFT-IN-PROGRESS

CHAPTER 56

Other Topics

DRAFT-IN-PROGRESS

Part F

Basics of Relativity

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62 Other Topics 2518

DRAFT-IN-PROGRESS

CHAPTER 57

General References

DRAFT-IN-PROGRESS

Pagina Web del Corso

Informazioni sul corso, testi di studio, problemi ed esercizi, fonti di dati, materiale vario di studio e multimediale:
WEB - URL.
Many classical texts are excellent texts, sometimes considered to be a *Bible* in their field, as they often are. However, physics progresses very very fast, so that excellent texts written many years ago remain a milestone, but may become old.

Useful General Mathematical Resources

©||S.Wolfram Mathematics Encyclopedia|*WEB - URL*|

©||J.Wevers Mathematics Formulary|*WEB - URL*|

©||WEB - URL|*Nice summary website.*|

Some Useful General Physics Resources/References

©||D.Halliday & R.Resnick & K.S.Krane, *Physics*, 2001, J.Wiley & Sons, 5thEd., ...|*Excellent, basic, modern; but do not ...*

©||Berkeley Physics Course, in 5 Vol., , ..., McGraw-Hill, ...Ed., ...|*Excellent text, but do not forget it was written many ...*

©||D.V.Sivuchin, , ..., ..., ...Ed., ...|*Excellent text, but do not forget it was written many years ago, and physics progresse- ...*

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©||W.M.Haynes *et al.*, *CRC Handbook of Chemistry and Physics*, 2016, CRC press, 97thEd., WEB - URL.|*Source of Data*|

©||R.Blandford & K.Thorne, *Applications Of Classical Physics*, ..., ..., ...Ed., WEB - URL, WEB - URL.|*Excellent, ...*

Some Useful Physics Specific Resources/References

©||G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|*Excellent text, ...*

©||Barger & Olsson, , ..., ..., ...Ed., ...||

©||M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|*Excellent, advanced text; but do not forget it was ...*

©||H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|*Excellent, advanced text; but do not forget it was ...*

©|M.dePodesta, *Understanding The Properties Of Matter*, 2002, CRC Press, 2ndEd., ...|Now freely available: ...

©||H.Goldstein *et al.*, *Classical Mechanics*, 2014, Pearson Education, 3rdEd., ...|*Excellent, advanced text; but do not ...*

©||D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|*Excellent, basic, modern.*|

©||E.Hecht, *Optics*, 2002, Addison Wesley Longman, 4thEd., ...|*Excellent, basic, modern.*|

©||J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|*Excellent, advanced text; but do not forget ...*

©||W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|*Excellent, ...*

©||F.Reif, *Fundamentals Of Statistical And Thermal Physics*, 2009, Waveland Press, ...Ed., ...|*Excellent, basic text; but ...*

©||A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|*Excellent, intermediate, modern.*|

©||M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|*Excellent, basic text; but ...*

Some Useful Resources/References on Physics and the Real World

©||L.Bloomfield, *How Things Work*, ..., ..., ...Ed., WEB - URL.||

Other references

©|K.McDonald|WEB - URL|*A real treasure of tens of detailed and critical analyses of key physics problems, showing how ...*

The SI of Units of Measure

Read § B.

DRAFT-IN-PROGRESS

References for this section

Some References for this section follow.

- ...

Vedere Appunti Di Fisica Delle Particelle per varie applicaizoni

Exercises and problems for this part

DRAFT-IN-PROGRESS

CHAPTER 58

From Galilei Transformation to Lorentz Transformation

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This § is referenced at pages:
[2481]

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©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|||
©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...|||
©|R.Resnick, *Introduction To Special Relativity*, 1968, J.Wiley & Sons, ...Ed., ...||*Clear and detailed introduction*|
©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||
©|H.Goldstein *et al.*, *Classical Mechanics*, 2014, Pearson Education, 3rdEd., ...|||
©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|||
©|S.Weinberg, *The Quantum Theory of Fields, Vol. 1,2,3*, 1995, CUP, ...Ed., WEB - URL.|||
©|C.W.Misner & K.S.Thorne & J.A.Wheeler, *Gravitation*, 1973, W.H.Freeman and Co., ...Ed., WEB - URL.|||
©|MFayngold - SpecialRelativityAndMotionsFasterThanLight|§ 5|*Discussion of many paradoxes*|
©|G.L.Naber|The Geometry Of Minkowski SpaceTime|*Excellent*|

WORK - IN - PROGRESS

58.01 Introduction

This § is referenced at pages:
[Never referenced.]

While Special Relativity is often presented starting from mechanical experiments and measurements of times and distances, the real historical development originates from analysis classical electromagnetism, as A.Einstein first paper confirms: *On the electrodynamics of moving bodies*.

Classical ElectroMagnetism turned out to be fully consistent with the Theory of Relativity, while **Galilei-Newton Mechanics** did not. Therefore, **Galilei-Newton Mechanics** had to be modified in order to make it consistent with the Theory of Relativity. ElectroMagnetism did not require any fundamental modification, but just required to be re-written using the relativistic formalism as well as understood and interpreted in the proper context of the Theory of Relativity, shedding more light on ElectroMagnetic phenomena.

The basic element of the Theory of Relativity is **Lorentz Transformation** for time and space coordinates, affecting the time-space coordinates of the world. **Lorentz Transformation** is therefore the basic tool required for physics, before any other development and deserves a very detailed discussion.

After defining the **Lorentz Transformation** for time and space, the Lorentz group will be used to define the general scalars, vectors and tensors (for the Lorentz group) for any other physical quantity.

WORK-IN-PROGRESS

58.02

Experimental Basis of Special Relativity

©|T.Roberts & S.Schleif & J.M.Dlugosz|The experimental basis of Special Relativity |*Excellent, extensive and detailed*|

© - QUOTE

©|T.Roberts & S.Schleif & J.M.Dlugosz|**WEB - URL** ||

Physics is an experimental science, and as such the experimental basis for any physical theory is extremely important. The relationship between theory and experiments in modern science is a multi-edged sword.

- It is required that the theory not be refuted by any undisputed experiment within the theory's domain of applicability.
- It is expected that the theory be confirmed by a number of experiments that:
  - cover a significant fraction of the theory's domain of applicability;
  - examine a significant fraction of the theory's predictions.

OPEN-IN-PROGRESS

58.03 Phenomenology

58.03.01 Asymmetries in the Description of ElectroMagnetic Phenomena

This § is referenced at pages:
[2481, 2496, 2496]

The example of the bar moving in a constant and uniform magnetic field (section § 36.20.23 , § 36.20.24) shows that the observer moving with the bar must see an electric field in order to explain the induced currents. This is perfectly coherent with **Galilei Transformation** (58.04.01).

However, even if this explanation is perfectly correct, the observer moving with the bar seems to be unable to attribute the electric field to any electric charge nor to a time-varying magnetic field, and therefore the appearance of the electric field seems to be a sort of a mystery to him. Apparently he does not know the origin of the electric field: one will discover later on that there are indeed electric charges producing the observed electric field, in its judgment.

Moreover, even if the two observers make the same predictions, they explain the situation in two different ways: what is seen as a magnetic field by one observer is seen as an electric plus magnetic field by another observer in relative motion: while $\mathbf{B}$ does not change from one observer to the other, $\mathbf{E}$ does change.

Therefore:

- either the two observers are not equivalent, as they describe in a different way the same phenomena;
- or Electric|Magnetic fields are strictly connected and there is a common underlying framework for both observers.

Note that **the asymmetry is only in the description of ElectroMagnetic induction phenomena, not in the phenomena themselves, nor in the predictions that the two observers make**. Actually the two observers give the same predictions while describing the phenomena in a different way. Moreover, **the observer moving with the bar is troubled by his lack of identification of real charges producing the electric field he is observing**. Is this a violation of the Principle of Relativity? Is this just an accident or is it the result of the fact that the two observers are in fact equivalent but this equivalence is a more subtle one?

58.03.02 Internal Forces of a Pair of Moving Charges

Consider the forces exchanged between two identical charged point particles moving at the same constant velocity $\mathbf{v}$. Any observer at rest with respect to the particles attributes the force to the Coulomb law only. Other observers must also consider the magnetic field produced by each particle, which acts on the other moving electric charges according to $\mathbf{v} \wedge \mathbf{B}$.

Consider now a particular case of two particles such that their velocities are orthogonal to their relative position (they somewhat run side-by-side). The magnetic field force term is opposite to the force term due to the electric field repulsion, thus reducing the repulsive force with respect to the Coulomb force, as the two particles are actually two identical parallel currents. This leads to different relative accelerations observed in the two Rest Frame, which is in contrast with the Galilean relativity principle. Actually **the two observers give different predictions and describe the phenomena in different ways**; this is an apparent violation of the Principle of Relativity.

How is this explained? Actually forces transform from one Reference Frame to another, in relativistic mechanics, at variance from the Galilei-Newton world, so that the comparison in this example requires a different analysis.

Read § 15-026.

PRELIMINARY PROOFS

58.03.03
Relative Motion Between a Point Charge and a Small Magnet

This § is referenced at pages:
[2506]

Consider the following situation in the light of sections § 61.09.08.

A point charge moves along the positive direction of the x axis. A small magnet is fixed in the xy plane with its **Magnetic Dipole** moment along the positive direction of the y axis and it is located at some $y > 0$.

For the observer fixed with the magnet, the charge produces a magnetic field at the location of the magnet and the magnet therefore experiences a torque whose direction is along the positive direction of the x axis.

For an observer in the Rest Frame of the point charge, the magnet is seen moving along the negative direction of the x axis. It is not immersed in any magnetic field but only in the electric field of the point charge. Therefore the magnet does not experience any torque.

Actually **the two observers give different predictions and describe the phenomena in different ways**: this is an apparent violation of the Principle of Relativity.

The explanation for this issue has to be found in the fact that, according to Relativity, a moving **Magnetic Dipole** also possesses an **Electric Dipole** moment. As the predictions are different, this example is even more troublesome than that in section § 36.20.23, § 36.20.24.

WORKED SOLUTION

It turns out that the moving magnet will acquire an **Electric Dipole** moment and will experience a torque in the electric field of the stationary charge, according to § 61.09.08.

58.03.04
Action-Reaction Principle (Apparent) Violation in EM

This § is referenced at pages:
[1335]

A charge in motion at constant velocity in a Inertial Reference Frame , S, the source charge, sets up a magnetic field. Another charge, the test charge, moves through this ElectroMagnetic field with a velocity **u**: the test charge experiences a magnetic force in the S Inertial Reference Frame . Consider an observer in another Inertial Reference Frame , S', which moves relative to S, either with the velocity of the test charge in S or with the velocity of the source charge in S. In either one of these Rest Frame there will be no magnetic force, for either the velocity of the test charge is zero, or the source charge is at rest and there is no magnetic field: is there or isn't there a magnetic force?

PREPRINT - In Progress

58.04 Galilei Transformation of Space-Time Coordinates and Galilei-Newton Mechanics

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Read § 24.10.02.

58.04.01 ElectroMagnetism and Galilei-Invarian*

This § is referenced at pages:
[1315, 2488]

©|J.A.Heras|The Galilean limits of Maxwell's equations|*Am. J. Phys., 78, 1048 (2010);* WEB - URL.|

See § 24.10.03, § 61.03.05.

The non-relativistic limit of ElectroMagnetism is tricky. This might be expected because ElectroMagnetism is a truly relativistic theory: there are zero mass photons, with **zero-mass particles only making sense in relativistic theories**, see § 24.03.02.01.

Moreover, **charge conservation is a local one, as required by any relativistic theory**. On the other hand, **in Galilei-Newton Mechanics , charge conservation can be non-local, because simultaneity is an absolute concept**: two equal and opposite charges can simultaneously and instantaneously appear at large distances.

Note that in the SI (read § B), the units for the electric field and magnetic field are different. Therefore it is sometimes useful to reduce them to the same units, to better show the relativistic symmetry between the two fields. This can be done by using, instead of the magnetic field, the **reduced magnetic field**:

$$\mathbf{B} \to c\mathbf{B} \equiv \mathbf{B}' \quad ,$$

as suggested by the Lorentz force, by the relation between the fields in ElectroMagnetic waves in vacuum and by the expressions of the energy densities. This convention makes the transformation formulas for the field a symmetrical one, as for the time-space **Lorentz Transformation** .

The **Galilei-Newton Mechanics** transformation law for ElectroMagnetic fields, the so-called instantaneous limit, can be determined as follows.

Galilei Transformation imply that the accelerations in the two inertial Reference Frame are the same but velocities are different. Therefore, when dealing with a velocity dependent force, such as the Lorentz force, one should be careful. In **Galilei-Newton Mechanics** , the accelerations are the same, so that the Lorentz forces on one point charge are also the same in both Reference Frames. Therefore:

$$\mathbf{v} = \mathbf{v}' + \mathbf{u} \quad ,$$
$$\mathbf{E}' + \mathbf{v}' \wedge \mathbf{B}' = \mathbf{E} + (\mathbf{v}' + \mathbf{u}) \wedge \mathbf{B} \qquad \text{for any } \mathbf{u} \text{ and any } \mathbf{v}' \quad ,$$
$$\mathbf{v}' = 0 \implies \mathbf{E}' = \mathbf{E} + \mathbf{u} \wedge \mathbf{B} \quad ,$$
$$\implies \mathbf{v}' \wedge (\mathbf{B} - \mathbf{B}') \qquad \text{for any } \mathbf{v}' \implies \mathbf{B} = \mathbf{B}' \quad .$$

Therefore, **Galilei-Invarian*** implies:

$$\boxed{\mathbf{E}' = \mathbf{E} + \mathbf{u} \wedge \mathbf{B} \qquad \mathbf{B}' = \mathbf{B} \qquad \rho' = \rho \qquad \mathbf{j}' = \mathbf{j} - \rho\mathbf{u}} \quad . \tag{58.04.01}$$

→ 1769 2394 2488

In fact, many of the equations of ElectroMagnetism, starting with the Lorentz force law, make explicit reference to the velocity of the charge, which presupposes the choice of a Reference Frame with respect to which velocities are to be measured.

However, the above relations, technically making the so-called **instantaneous** limit, are only a part of a much more complex history, see § 61.03.05, and different non-relativistic limits do exist.

DRAFT - IN PROGRESS

58.05 The Speed of Light

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Speed of light should always be understood to mean the speed of light in free vacuum space unless it is explicitly stated otherwise. The speed of light in a material medium is, in general, different from c.

It is an experimental fact that no faster method of sending signals has ever been discovered.

The speed of light, in the theory of Relativity, was raised to a fundamental constant of nature, in the sense that it is postulated that there exists a limiting speed, c, that ElectroMagnetic waves or photons in vacuum travel at the limiting speed c and that this speed is independent of the motion of both the source and the receiver.

It is believed that ElectroMagnetic waves or photons are not unique in this respect but, actually, any massless particle/field would travel, in vacuum, at the same speed as the speed of light, independently of the motion of both the source and the receiver. So, in more general terms, one should say that there exist one and only one invariant speed in vacuum, including, as the most prominent example, the speed of light [1].

If there is something mysterious in special relativity this is the invariance of the speed of light. Once the invariance of the speed of light is granted, then everything else follows directly and fairly simply, even if many results may appear paradoxical and it is often difficult to feel comfortable with them. Every new situation must be analyzed carefully, without prejudices.

It will be shown, when discussing relativistic dynamics, that an infinite energy is required to accelerate a particle with non-zero mass to the speed of light. Therefore no massive particle can reach the speed of light. On the other hand relativistic dynamics shows that massless particles always travel at the speed of light. The speed of light, therefore, stands out as a limiting, maximum possible, speed. Read § 58.12.01.

Relativistic effects are negligible in real life because the speed of light is so large with respect to every-day speeds. In fact it will be shown that all relativistic effects disappear when the speed of light tends to infinity.

58.05.01 Measurements of the Speed of Light

See any textbook for the description of different methods for measuring the speed of light.

It was understood since a long time that light had some finite, even if very large, speed. One should in fact remember the experiments carried on by Galilei with a lamp and an distant friend.

The following basic methods had been employed, historically, to measure the speed of light [2].

- Prediction of the eclipses times of the main satellite of Jupiter by Romër.
- Measurements of the stellar aberration by Bradley.
- Rotating mirrors and toothed wheels.
- Cavity resonator.
- Electro-optic switch via the Kerr cell.

[1] In fact, following common usage, *speed of light* will be used, in short

[2] Read Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...

DRAFT INTEGRITAS

58.05.02
Michelson-Morley Experiment

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|15||

The Michelson-Morley experiment was designed to try to detect the **ether wind**. In fact, it was realized that any effect of the **ether wind** was indeed very small and therefore only very sophisticated experiments might be able to detect it.

The idea was to measure the speed of light in different directions in the ether frame, to try to detect the changes in the speed of light due to the Earth moving with respect to the ether frame, by also exploiting the annual motion of the Earth. In fact, one would expect different speeds in different directions.

The experiment clearly showed that the ether does not exist, as it did not detect any **ether wind**, because the measured speed was the same in all directions, while the experiment was in principle sensitive to measure the predicted ether wind.

The null result of Michelson-Morley experiment could only be understood if the speed of light is infinite, but it was known that this is not the case.

On the basis of experimental evidence Einstein then postulated that Galilei Transformation are not correct and must be replaced with new linear transformation law, Lorentz Transformation . As a consequence ElectroMagnetism was all right while **Galilei-Newton Mechanics** had to be updated.

The new postulate changed drastically the way space and time are conceived.

The invariance of the laws of ElectroMagnetism (**Lorentz-Invarian***) was taken to apply to all physical phenomena.

The new transformation law must be such that the speed of light is invariant when passing from one inertial Reference Frame to another. Moreover, the new transformation law must transform a motion at uniform velocity into another with uniform velocity and therefore they have to be linear.

58.05.02.01 Analysis of the Michelson-Morley Experiment

See any textbook for the description of the Michelson-Morley experiment.

Read also § 52.03.01.02.

Let the speed of light be have module c in the unknown ether frame.

Let the Earth travel at a speed $\mathbf{v}$ with respect to the ether frame.

The Michelson-Morley experiment, using the Michelson interferometer, basically compares, with high-precision, the time for light to travel forth and back a segment, one of the arms of the Michelson interferometer, comparing the travel times along two orthogonal arms.

Assume, for simplicity, that the two orthogonal arms are one parallel and the other one perpendicular to the speed of the Earth, $\mathbf{v}$, with respect to the ether frame.

$$v \approx 30 \text{ km/s} \quad ,$$

$$\Delta t_{\parallel} = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2L}{c\left(1-v^2/c^2\right)} \quad ,$$

$$\Delta t_{\perp} = \frac{2L}{c\sqrt{1-v^2/c^2}} \quad ,$$

$$\left|\Delta t_{\perp} - \Delta t_{\parallel}\right| \simeq \frac{2L}{c}\left|\left(1+\frac{v^2}{2c^2}\right) - \left(1+\frac{v^2}{c^2}\right) + \mathcal{O}\left(\frac{v}{c}\right)^4\right| = \frac{Lv^2}{c^3} \approx 0.3{\cdot}10^{-16} \text{ s} \quad ,$$

a very small time interval but measurable with a good interferometric technique.

PREPRINT PROGRESS

58.06 Postulates of Special Relativity

All the consequences of the special theory of relativity follow from the following two postulates:

1. The Principle of (Einsteinian) Relativity, § 58.06.01;
2. The Principle of Invariance of the Speed of Light, § 58.06.02.

58.06.01 Principle of (Einsteinian) Relativity

This § is referenced at pages:
[2399]

The relativity principle is assumed to be a principle valid for all the laws of physics, not only for mechanics.

The principle of relativity: space is isotropic and uniform; the fundamental laws of physics are identical for any two inertial observers in uniform relative motion.

Corollary: there exist at least one inertial Reference Frame (and therefore infinite).

58.06.02 Principle of Invariance of the Speed of Light

This § is referenced at pages:
[2399]

The null result of the Michelson-Morley experiment to detect the drift of the Earth through an ether (as well as other results) can be understood by assuming the following new principle.

The principle of invariance of the speed of light: the speed of light in empty space is independent of both the motion of the light source and the receiver; light propagates in free empty space with a definite velocity c, a universal constant of nature.

58.06.03 Clock Hypothesis/Postulate

This § is referenced at pages:
[2415]

The clock hypothesis states that **the tick rate of a clock when measured in an inertial frame depends only upon its velocity relative to that frame and it is independent of its acceleration or higher derivatives of position**.

It relies on the experiments, such those described in section § 58.12.02.

© - QUOTE

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On Accelerated Clocks.
The special theory of relativity describes and relates measurements which are independent of the detailed structure of real bodies. It makes no prediction about the dynamical effects of acceleration, such as the stresses induced by acceleration. If such stresses are absent or may be ignored, the

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theory does give us an unambiguous description of the effect of acceleration on clock rates. The result is as if at each instant an accelerated clock had a different velocity, with a rate to be calculated using in Eq. (11.11) the appropriate instantaneous velocity. If this prediction is correct, two consequences follow.

- If the speed is constant but the direction varies, Eq. (11.11) holds without change. The frame of the clock is non-inertial. If the speed is constant except for brief moments of acceleration or deceleration (moments negligibly short in comparison with the total time), then Eq. (11.11) will still describe accurately the relation between the proper time and the stationary laboratory time. A fast charged particle in a constant magnetic field experiences an acceleration perpendicular to its motion, but the speed never changes.
- If the particle is unstable, the measured half-life should be exactly the same as if it moved with the same speed in a straight line with no magnetic field present. This forecast is confirmed by experiments on the J/ψ-meson, which decays with a proper mean life of $2.2 \cdot 10^{-6}$ s into an electron and neutrinos. The same proper lifetime is observed for J/ψ-meson which are free or spiraling in a magnetic field or allowed to come to rest. It is believed that the special theory of relativity gives a good description of the circular (accelerated) motion of particles in a magnetic field.

58.06.04
Ultimate Speed

© - QUOTE

Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...
... Many other experiments suggest, as this one does, that c is the upper limit to the velocity of particles. Thus we believe firmly that c is the maximum signaling speed with either particles or ElectroMagnetic waves: c is the ultimate speed.

It is, in the end, an experimental fact.

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58.07 Events and Reference Frames

58.07.01 Synchronization of Clocks

© - QUOTE

©|R.Resnick, *Introduction To Special Relativity*, 1968, J.Wiley & Sons, ...Ed., ...|||

Some **obvious** methods of synchronizing clocks turn out to be erroneous. For example, we can set the two clocks so that they always read the same time as seen by observer A. This means that whenever A looks at the B clock it reads the same to him as his clock. The defect here is that if observer B uses the same criterion (that is, that the clocks are synchronized if they always read the same time to him), he will find that the clocks are not synchronized if A says that they are. For this method neglects the fact that it takes time for light to travel from B to A and vice versa. The student should be able to show that, if the distance between the clocks is L, one observer will see the other clock lag his by $2L/c$ when the other observer claims that they are synchronous. We certainly cannot have observers in the same Rest Frame disagree on whether clocks are synchronized or not, so we reject this method. An apparent way out of this difficulty is simply to set the two clocks to read the same time and then move them to the positions where the events occur. In principle, we need clocks everywhere in our Rest Frame to record the time of occurrence of events, but once we know how to synchronize two clocks we can, one by one, synchronize all the clocks. The difficulty here is that we do not know ahead of time, and therefore cannot assume, that the motion of the clocks (which may have different velocities, accelerations, and path lengths in being moved into position) will not affect their readings or time-keeping ability. Even in classical physics, the motion can affect the rate at which clocks run.

58.07.02 Reference Frame- Rigid Rulers and Synchronized Clocks

A Reference Frame is defined as a set of rigid rulers and synchronized clocks.

- Rigid bodies are incompatible with the finite speed of propagation of light. Therefore, starting from a fixed origin and **one single clock**, one defines the coordinate position of any point from the known speed of light and measurement of the round-trip propagation time: origin-point-origin. In this way, **a position label** can be attached to every point in space. Note that measuring time on a closed path allows to avoid any problem with synchronization of clocks: only one clock is used.
- Secondly, one puts **identical clocks at every point in space**; the clocks are then synchronized from the known distance from the origin and the known speed of light. In this way, all clocks can be synchronized with the one at the origin. It is then assumed transitivity of the *clocks are synchronized* relation.

58.07.03
Events and Lorentz Intervals

An event is a well defined and unambiguously identifiable circumstance happening at a well defined time and at a well defined place, in any Reference Frame. It is thus characterized by a time and space coordinate:

$$\boxed{\mathcal{E} \equiv \{t, \mathbf{x}\}} \quad .$$

Given two events, $\mathcal{E}_1 \equiv \{t_1, \mathbf{x}_1\}$ and $\mathcal{E}_2 \equiv \{t_2, \mathbf{x}_2\}$ their Lorentz invariant interval is defined as:

$$\boxed{(\Delta s)^2 \equiv c^2\,(\Delta t)^2 - (\Delta \mathbf{x})^2 = c^2\,(t_2 - t_1)^2 - (\mathbf{x}_2 - \mathbf{x}_1)^2} \quad . \tag{58.07.01}$$

The quantity Δs has the same value for any inertial observer when calculated between the same to events.

The invariance of the Lorentz interval can be checked for the special cases discussed above.

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58.08 Consequences of the Postulates and Thought Experiments

This § is referenced at pages:
[2407]

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|||

Whenever a inertial Reference Frame moves with respect to another Reference Frame with relative velocity $\mathbf{u}$ it is common practice to introduce the shorthand notations:

$$\beta_u \equiv \beta[u] \equiv \frac{u}{c} \qquad \gamma_u \equiv \gamma[u] \equiv \frac{1}{\sqrt{1-u^2/c^2}} \quad .$$

Whenever there is no possibility of confusion about which velocity beta and gamma refer to, the notation is usually simplified to: $\beta_u \equiv \beta$, $\gamma_u \equiv \gamma$.

58.08.01 Relativity of Simultaneity

Consider a light source at the center of a freight-car moving with constant velocity $\mathbf{u}$.

The light source emits a lamp at a certain time and the arrival of the light signals at the two extremes of the freight-car (of rest length L') are recorded (events $\mathcal{E}_+$ and $\mathcal{E}_-$).

The observer inside the freight-car says that the two events are simultaneous, and that the light travel lasts

$$\Delta t' = L'/(2c) \quad .$$

The observer on the ground measures two different time durations, Δt_+ and Δt_-, as the freight-car is moving during the light travel:

$$c\Delta t_+ = L/2 + u\Delta t_+ \qquad c\Delta t_- = L/2 - u\Delta t_- \quad ,$$
$$\Delta t_+ = L/(2(c-u)) \qquad \Delta t_- = L/(2(c+u)) \quad .$$

Here we are looking at both the moving photon and the two moving ends of the freight-car and calculate the **mutual velocity** between the photon and the leading/trailing side of the freight-car, read § 59.01.05. There is no problem here: the same observer is looking at two different objects and finds mutual velocities $u-c$ and $u+c$, for the leading/trailing side of the freight-car.

Therefore simultaneity is a relative concept: two events which happen at the same time, but at different places, for one observer may be not simultaneous for another observer.

Note that in the case of infinite speed of propagation of light, $c \gg u$, the simultaneity is recovered as $\Delta t_+ = \Delta t_-$. In fact the freight-car has to be going very fast in order the discrepancy becomes easily detectable.

Note that L and L', **the two lengths, are not the same, a priori**, but this does not affect the conclusion that the two events $\mathcal{E}_+$ and $\mathcal{E}_-$ are simultaneous in one Reference Frame and are not in the other.

Two events that are simultaneous at different places in one Inertial Reference Frame may be not simultaneous, in general, in another Inertial Reference Frame .

Note that **the relativity of simultaneity makes problematic the application of the Action-Reaction Principle** . Consider in fact the case of time-dependent forces: action-reaction must be equal and opposite at the same time but, as simultaneity is a relative concept, another observer may not agree on the fact that action and reaction are equal and opposite. Only in the case of point-contact interactions, with the two forces applied at the same physical point (as well as in the trivial case where the forces are constant), the **Action-Reaction Principle** can be retained.

Note also that the effect is a real one, depending on how time floes, not to be confused with spurious effects such as the time delay between lighting and thunder.

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58.08.02
Time Dilatation (Time Interval Between the Same Two Events)

Consider a light source at the center of the ceiling of a freight-car moving with constant velocity **u**.

Consider a light ray which strikes the floor just below the light source for the observer inside the freight-car. Let H' be the rest height of the lamp from the floor.

For the observer inside the freight-car the time between light emission and light reaching the floor is:

$$\Delta t' = H'/c \quad .$$

For the observer on the ground the time between light emission and light reaching the floor is:

$$c^2\Delta t^2 = H^2 + u^2\Delta t^2 \qquad \Delta t = H/\sqrt{c^2-u^2} \quad ,$$

with the same same speed c, but for this observer the light makes a longer travel as the freight-car is moving.

Note that the height of the freight-car is the same for both observers, as demonstrated in section § 58.08.03:

$$\boxed{H' = H} \quad .$$

Therefore one finds the time dilatation formula:

$$\boxed{\Delta t = \gamma \Delta t'} \quad . \qquad (58.08.01)$$

→ 2405

the time elapsed between the same two events (light leaves light source and light strikes the floor just below the light source) is different for the two observers.

Running clocks, thus, run slow.

The **proper time** is the time in the Reference Frame where the two events happen at the same place ($\mathbf{x}_1 = \mathbf{x}_2$) or, as a weaker condition applying in this problem, at the same place only in the coordinate along the velocity **u**.

Note that **the result concerns time itself, not the way clocks work, as it is experimentally demonstrated by experiments on many systems**.

Note that in the case of infinite speed of propagation of light, $c \gg u$, the time dilatation disappears. In fact the freight-car has to be going very fast in order the discrepancy becomes easily detectable.

It might appear that the time dilatation, in particular the fact that running clocks run slow, violates the principle of relativity, but it does not. In fact consider how two different observers measure the rate at which a moving clock is running: they are measuring two different things.

58.08.03
Length Contraction (Distance Between the Same Two Events at the Same Time)

This § is referenced at pages:
[2404, 2407]

Note that **every measure of length must be carried on by taking the position of all the points at the same time**: as simultaneity is a concept relative to the observer, length measurements may depend on the observer.

58.08.03.01 Length Contraction Along the Direction of Motion

As the speed of light in free space is, by hypothesis, a universal constant, it can be used to measure lengths.

Consider a light source at one end of a freight-car moving with constant velocity **u**. There is reflecting mirror on the other side.

For the observer inside the freight-car the travel time is related to the length, $\Delta L'$, of the freight-car by:

$$\Delta t' = \frac{2\Delta L'}{c} \quad .$$

For the observer on the ground the travel time is related to the length, ΔL, of the freight-car by:

$$\Delta t = \Delta t_+ + \Delta t_- = \frac{\Delta L}{(c-u)} + \frac{\Delta L}{(c+u)} \Longrightarrow \Delta t = \frac{2\Delta L c}{(c^2-u^2)}$$

as given by the sum of two travel times Δt_+ and Δt_+ with mutual velocity $c+u$ and $c-u$ between the light and the freight-car.

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Here, again, the mutual velocity as seen by the ground observed, between light and the freight-car, is used.

These two time intervals are related by the time dilatation formula (58.08.01), so that we obtain the length contraction formula:

$$\boxed{\Delta L = \Delta L'/\gamma} \quad . \tag{58.08.02}$$

Moving objects are thus shortened.

Note that in the case of infinite speed of propagation of light, $c \gg u$, the length contraction disappear. In fact the freight-car has to be going very fast in order the discrepancy becomes easily detectable.

Note that it does not matter that the path is asymmetrical for the moving observer one just uses the correct formula relating time and space.

58.08.03.02 Length (Non)-Contraction Perpendicularly to the Direction of Motion

It should be emphasized that moving objects are shortened only along the direction of motion (that is parallel to the velocity **u**).

There is no shortening in the direction perpendicular to motion.

The following argument applies [1]. Suppose the train is running near a wall where a horizontal blue line is painted at one meter height from the ground. A woman inside the train leans out the window and paints a horizontal red line one meter above the ground. Which of the two lines is higher? If, for instance, lengths in direction perpendicular to the velocity contracts, then the man on ground would predict the red line is lower than the blue line. On the other hand the woman on the train would predict the red line is higher than the blue line (as the wall is moving for the woman). The principle of relativity allows us to conclude that the two lines must coincide:

$$\boxed{x' = x \qquad y' = y \qquad \text{for } \mathbf{u} \parallel \widehat{\mathbf{e}}_3} \quad .$$

Note that no subtlety of synchronization enters this reasoning.

Of course, there is no direct experimental measurement of Lorentz contraction, because it is not practically possible to get an object of measurable size going near the speed of light.

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[1] See D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ..., and references therein.

58.09 Lorentz Transformation of Space-Time Coordinates

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 12.1.3|*Alternative derivation.*|

58.09.01 Special Transformations of Space-Time coordinates

This § is referenced at pages:
[2406]

©|Berkeley Physics Course, Vol. 1, *Mechanics*, 1965, McGraw-Hill, ...Ed., ...|11|*A more general Ansatz.*|

This special case contains all the essential ingredients of Relativity without too much non essential mathematical complexity.

Consider the special case when the Cartesian axes of both Coordinate System are parallel and with the same orientation, that the two origins coincide at times $t = t' = 0$ and that the velocity of Reference Frame $\mathcal{I}'$ with respect to Reference Frame $\mathcal{I}$ is parallel to both z and z' axes and let its component along z be $\mathbf{u} \equiv u\widehat{\mathbf{e}}_3$.

Note that the transformation must be a linear one, as the inertia principle requires that a uniform rectilinear motion is transformed into a uniform rectilinear motion.

Assume that at $t = t' = 0$ a short light lamp is emitted at the coincident origins of the two Reference Frames: $\mathbf{r} = \mathbf{r}' = 0$.

The equation of motion of the light front is a spherical one, in both Reference Frames, thanks to the isotropy of space and the speed of light is the same, for both Reference Frames:

$$x^2 + y^2 + z^2 = c^2t^2 \qquad x'^2 + y'^2 + z'^2 = c^2t'^2 \quad .$$

In fact every inertial observer must see a spherical light front, independently on the velocity of the light source.

The correct time-space transformations must transform the first expression into the second one:

$$x^2 + y^2 + z^2 = c^2t^2 \longrightarrow x'^2 + y'^2 + z'^2 = c^2t'^2 \quad .$$

58.09.01.01 Special Galilei Transformation Are Not Compatible With the New Postulates

This § is referenced at pages:
[2407]

When using **Galilei Transformation** (24.10.02) in the second equation one finds:

$$x^2 + y^2 + z^2 + \left(u^2t^2 - 2zut\right) = c^2t^2 \quad .$$

It is therefore clear that **Galilei Transformation** cannot manage the equality of the speed of light in both Reference Frames, due to the Galilean law of addition of the velocities.

Moreover, in order to cancel the undesired time dependence $u^2t^2 - 2zut$ one needs to invoke a dependence on time of the space component z. In fact, since the distances perpendicular to the speed $\mathbf{u}$ cannot change, it is assumed that t' only depends on z'.

As a result of the previous reasoning we are led to § 58.09.01.

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58.09.01.02 Derivation of the Special Lorentz Transformation

Continue as in section § 58.09.01.01.

The relativity of time intervals, derived in section § 58.08, unavoidably compels us to assume a transformation which includes and mixes space and time coordinates.

In order to be consistent with the principle of inertia the space-time transformations between any two inertial Reference Frames must be a linear one, as any uniform motion must be mapped into a uniform motion.

The principle of reciprocity is also assumed: whenever the velocity of Reference Frame $\mathcal{I}'$ with respect to Reference Frame $\mathcal{I}$ is $\mathbf{u}$, the velocity of Reference Frame $\mathcal{I}$ with respect to Reference Frame $\mathcal{I}'$ is $-\mathbf{u}$.

In order to find the one-dimensional **Lorentz Transformation** (and its inverse) one can try the following educated guess, in terms of the so far unknown parameter γ:

$$u \equiv \mathbf{u}\cdot\hat{\mathbf{e}}_3 \gtreqless 0 \quad ,$$

$$\boxed{z' = \gamma\,(z - ut)} \qquad \Leftrightarrow \qquad \boxed{z = \gamma\,(z' + ut')} \quad , \qquad \rightarrow 2407$$

by inverting the second one above to find t' and using the first one above to replace z' one finds:

$$\boxed{t' = \gamma t + \frac{z}{u}\left(\frac{1-\gamma^2}{\gamma}\right)} \qquad \Leftrightarrow \qquad \boxed{t = \gamma t' - \frac{z}{u}\left(\frac{1-\gamma^2}{\gamma}\right)} \quad . \qquad \rightarrow 2407$$

The, so far unknown, factor γ is necessarily a function of the module of the velocity $\mathbf{u}$, thanks to the isotropy of space.

In order to preserve the orientation of the flow of time we must impose: $\gamma \geq 0$.

Note that the principle of reciprocity has been used, so that the double implication applies in equations (58.09.01.02), (58.09.01.02) as the observer $\mathcal{I}'$ sees the inertial Reference Frame $\mathcal{I}$ moving with velocity $-\mathbf{u}$.

The constancy of the speed of light implies

$$\{z = ct \qquad \Leftrightarrow \qquad z' = ct'\} \quad , \qquad (58.09.01) \qquad \rightarrow 2407$$

and using equations (58.09.01.02), (58.09.01.02) in the first relation above and comparing the result with the second relation above, one has

$$\boxed{\gamma^2 = \frac{1}{1 - u^2/c^2} \geq 1} \quad . \qquad (58.09.02)$$

Note that the above relation (58.09.01) cannot be satisfied by any **Galilei Transformation** (except for $c \to \infty$).

It follows that:

$$t' = \gamma\left(t - \frac{uz}{c^2}\right) \qquad \Leftrightarrow \qquad t = \gamma\left(t' + \frac{uz'}{c^2}\right) \quad .$$

Moreover, we already know (read § 58.08.03) that:

$$x' = x \qquad y' = y \quad .$$

Let

$$\boxed{\beta \equiv \frac{u}{c} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}} \quad .$$

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58.09.01.03 The Special Lorentz Transformation

All in all we have the special **Lorentz Transformation** :

$$\boxed{t' = \gamma\left(t - uz/c^2\right)} \quad , \tag{58.09.03}$$ → 2429

$$\boxed{x' = x} \quad , \tag{58.09.04}$$

$$\boxed{y' = y} \quad , \tag{58.09.05}$$

$$\boxed{z' = \gamma\left(z - ut\right)} \quad . \tag{58.09.06}$$ → 2415

And the inverse **Lorentz Transformation** :

$$\boxed{t = \gamma\left(t' + uz'/c^2\right)} \quad , \tag{58.09.07}$$ → 2414

$$\boxed{x = x'} \quad , \tag{58.09.08}$$

$$\boxed{y = y'} \quad , \tag{58.09.09}$$

$$\boxed{z = \gamma\left(z' + ut'\right)} \quad . \tag{58.09.10}$$

In the instantaneous non-relativistic limit the Lorentz Transformation become the Galilei Transformation .

58.09.01.04 Special Lorentz Transformation in Matrix Form

The special **Lorentz Transformation** read, in matrix form:

$$\boxed{r \equiv \{ct; x, y, z\} \equiv \{ct; \mathbf{r}\} \qquad r' = \mathbb{L}r \qquad \text{with} \qquad \mathbb{L} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}} \quad .$$

58.09.01.05 Inverse Special Lorentz Transformation

Inverse special **Lorentz Transformation** can be obtained by the replacement $\boldsymbol{\beta} \to -\boldsymbol{\beta}$. It can be shown by direct calculation that the inverse **Lorentz Transformation** , given replacing $\boldsymbol{\beta} \to -\boldsymbol{\beta}$, is described by the matrix inverse of the **Lorentz Transformation** one (according to (58.09.17)):

$$\boxed{\{\boldsymbol{\beta} \to -\boldsymbol{\beta}\} \qquad \Leftrightarrow \qquad \{\mathbb{L} \to \mathbb{L}^{-1}\}} \quad .$$

58.09.01.06 Composition of Two Special Lorentz Transformation Along the Same Direction

Imagine to apply two different **Lorentz Transformation** along the same z direction, from the Reference Frame $\mathcal{I}_1$ to $\mathcal{I}_2$ (with velocity v_{12} along z) and from the Reference Frame $\mathcal{I}_2$ to $\mathcal{I}_3$ (with velocity v_{23} along z).

Determine the Lorentz matrix corresponding to the composite Lorentz transformation from the Reference Frame $\mathcal{I}_1$ to $\mathcal{I}_3$ (with velocity along z).

By explicit calculation of the product of the two matrices, $\mathbb{L}_{23}[v_{23}]$ and $\mathbb{L}_{12}[v_{12}]$, and remembering that the matrix of a special **Lorentz Transformation** has the main diagonal $\{\gamma, 1, 1, \gamma\}$ one finds:

$$\mathbb{L}_{13} = \mathbb{L}_{23}[v_{23}]\mathbb{L}_{12}[v_{12}] = \mathbb{L}_{13}[\frac{v_{12} + v_{23}}{1 + v_{12}v_{23}/c^2}] \quad .$$

The following algebraic identity must be used:

$$\gamma[v] \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \boxed{\gamma[v_{12}]\gamma[v_{23}]\left(1 + v_{12}v_{23}/c^2\right) = \gamma[\frac{v_{12} + v_{23}}{1 + v_{12}v_{23}/c^2}]} \quad .$$

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58.09.01.07 Special Lorentz Transformation and Rapidity

©|Am.J.Phys, , ..., ..., ...Ed., ... 47 (12) 1979|Additivity, rapidity, relativity||

The special **Lorentz Transformation** can be written in the alternative form in terms of hyperbolic functions (read § 14.03.02):

$$r' = \mathbb{L} r \qquad \text{with} \qquad \mathbb{L} = \begin{pmatrix} +\operatorname{coh}\chi & 0 & 0 & -\operatorname{sih}\chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\operatorname{sih}\chi & 0 & 0 & +\operatorname{coh}\chi \end{pmatrix} ,$$

where the χ parameter, defined by relations:

$$\gamma \equiv \operatorname{coh}\chi \equiv \frac{\exp[+\chi] + \exp[-\chi]}{2} ,$$ → 1048

$$\beta\gamma \equiv \operatorname{sih}\chi \equiv \frac{\exp[+\chi] - \exp[-\chi]}{2} ,$$

$$\beta \equiv \tanh[\chi] \equiv \frac{\exp[+\chi] - \exp[-\chi]}{\exp[+\chi] + \exp[-\chi]} ,$$

is called **rapidity** . **Rapidity is an alternative, commonly used quantity, to measure a relativistic velocity**.

Special **Lorentz Transformation** expressed in terms of **rapidity** have the nice property that the special **Lorentz Transformation** corresponding to two special Lorentz transformations in series, both with the velocities parallel to the same $z = z' = z''$ axes, has a **rapidity** given by the algebraic sum of the rapidities of the two transformations in series (**hyperbolic functions**).

This is analogous to the fact, from elementary geometry, that when composing two rotations in series in the same plane the overall rotation has a rotation angle given by the algebraic sum of the angles of the two rotations. This is manifest when using the rotation angle as a parameter for the rotation (**trigonometric functions**). It is not at all obvious if one would use, for instance, the cosinus or the sinus of the angle as the rotation parameter: in this case the overall rotation would not be described by the sum of the parameters of the two rotations.

Therefore, note the similarity between the Lorentz matrix in terms of **rapidity** and the one describing ordinary rotations in euclidean space.

58.09.02 Boosts

This § is referenced at pages:
[Never referenced.]

As a second particular case of **Lorentz Transformation** , consider the case when the Cartesian axes of both Coordinate System are parallel to each other and with the same orientation, that the two origins coincide at times $t = t' = 0$ and the velocity of Reference Frame $\mathcal{I}'$ with respect to Reference Frame $\mathcal{I}$ is the arbitrary, $\mathbf{u}$. These kind of transformation is called **boost** or pure **Lorentz Transformation** .

The transformation can be obtained by simply decomposing the spatial portion of the vectors into vectors parallel and perpendicular to the velocity and read:

$$t' = \gamma\,(t - \boldsymbol{\beta}\cdot\mathbf{r}/c) , \tag{58.09.11}$$ → 2410 2435 2440 2460

$$\mathbf{r}' = \mathbf{r} + \boldsymbol{\beta}\left(\frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{r}) - \gamma c t\right) = \mathbf{r} + \boldsymbol{\beta}\left(\frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{r}) - \gamma c t\right) . \tag{58.09.12}$$ → 2410 2435 2460

The following identity is often useful to deal with the above expressions:

$$\frac{\gamma^2}{\gamma+1} = \frac{\gamma-1}{\beta^2} \quad .$$

The **Lorentz Transformation** matrix has the following form for a boost along a direction indicated by the unit vector:

$$\mathbb{L} = \begin{pmatrix} \mathbb{L}^{0\cdot}_{\cdot 0} & \mathbb{L}^{0\cdot}_{\cdot j} \\ \mathbb{L}^{k\cdot}_{\cdot 0} & \mathbb{L}^{k\cdot}_{\cdot j} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma\widehat{n}^{\cdot}_{j} \\ -\beta\gamma\widehat{n}^{k}_{\cdot} & (\gamma-1)\,\widehat{n}^{k}_{\cdot}\widehat{n}^{\cdot}_{j} + \delta^{k\cdot}_{\cdot j} \end{pmatrix} \qquad \widehat{\mathbf{n}} \equiv \frac{\mathbf{u}}{u} \quad . \tag{58.09.13}$$

Note that, for three-dimensional vectors, the position of the index, either high or low, is not relevant:

$$\widehat{\mathbf{n}} \equiv \left\{\widehat{n}^{k}_{\cdot}\right\} = \{\widehat{n}^{\cdot}_{k}\} \equiv \widehat{\mathbf{n}} \quad .$$

This expression is coherent with equation (58.09.17): to take the inverse matrix one must change $+\mathbf{n} \longrightarrow -\mathbf{n}$

One should not forget that **the 3-vectors** in equations (58.09.11) and (58.09.12) **must be considered in a pure algebraic sense**, that is as triples of real numbers as they actually belong to different Euclidean spaces. In other words, they do not transform as the 3-vectors under spatial rotations, SO(3).

58.09.02.01 Transformation of the Parallel and Transverse Components of the Time-Space Four-Vector

From the equations (58.09.11), (58.09.12) one can derive the following relations, for the **Lorentz Transformation** of the space components:

$$\mathbf{r}'_{\parallel} = \gamma\left(\mathbf{r}_{\parallel} - c\boldsymbol{\beta}t\right) \qquad \mathbf{r}'_{\perp} = \mathbf{r}_{\perp} \quad . \tag{58.09.14}$$

They are consistent with the special case of the special **Lorentz Transformation** .

58.09.03
Rotations - sub-group of Lorentz Transformation

58.09.04
General Lorentz Transformation

Consider two Inertial Reference Frames, $\mathcal{I}$ and $\mathcal{I}'$. Assume that $\mathcal{I}'$ is moving, with respect to $\mathcal{I}$, with a speed $\mathbf{u}$. Assume that a orthonormal Cartesian Coordinates Coordinate System and a time coordinate are defined in both Inertial Reference Frames.

Lorentz Transformation give the relation between the time-space coordinates of the Inertial Reference Frame $\mathcal{I}$, the four-tuple of real numbers $x^{\mu} \equiv \{ct, x_1, x_2, x_3\}$, and the time-space coordinates of the Inertial Reference Frame $\mathcal{I}'$, the four-tuple of real numbers $x^{\mu\prime} \equiv \{ct', x'_1, x'_2, x'_3\}$. Note that the Greek four-dimensional indexes $\{...\}^{\mu}$ are placed high.

Define the real 4×4 matrix of the linear transformation between $x^{\mu} \equiv \{ct, x_1, x_2, x_3\}$ and $x^{\mu\prime} \equiv \{ct', x'_1, x'_2, x'_3\}$ as $\mathbb{L}$:

$$x \longrightarrow x' \qquad x' = \mathbb{L}x \quad .$$

Let:

$$x^0 \equiv ct \qquad x'^0 \equiv ct' \quad .$$

The **Lorentz Transformation** can be written, with explicit indexes, as:

$$x'^{\mu}_{\cdot} = \mathbb{L}^{\mu\cdot}_{\cdot\nu} x^{\nu}_{\cdot} \quad .$$

The general **Lorentz Transformation** between $x^{\mu} \equiv \{ct, x, y, z\}$ and $x'^{\mu} \equiv \{ct', x', y', z'\}$ are derived from the basic postulates of the relativity theory as the set of linear transformations leaving invariant

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the matrix η (the metric tensor):

$$\eta^{\cdot\cdot}_{\mu\nu} \equiv \eta^{\mu\nu}_{\cdot\cdot} \equiv \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad .$$

As a matrix η has the following properties (by direct calculation):

$$\eta = \eta^{-1} = \eta^{\mathsf{T}} \qquad \eta\eta = \mathbb{I} \quad .$$

The matrix η with high indexes is defined as the inverse of the η with low indexes which therefore equals η with high indexes as $\eta = \eta^{-1}$:

$$\eta^{\mu\nu}_{\cdot\cdot} \equiv \left(\eta^{-1}\right)^{\cdot\cdot}_{\mu\nu} \qquad \text{no index convention} \quad .$$

In components one finds:

$$\eta^{\alpha\rho}_{\cdot\cdot}\eta^{\cdot\cdot}_{\rho\beta} = \delta^{\alpha}_{\cdot\beta} \equiv \eta^{\alpha}_{\cdot\beta} \quad .$$

In fact the η matrix, the metric tensor, is used to raise and lower the indexes (read § 20).

Lorentz Transformation are defined by the set of real 4×4 matrices, $\mathbb{L}$, satisfying the Lorentz condition:

$$\eta = \mathbb{L}^{\mathsf{T}}\eta\mathbb{L} \tag{58.09.15}$$

→ 2411

In terms of explicit indexes condition (58.09.15) becomes:

$$\eta = \mathbb{L}^{\mathsf{T}}\eta\mathbb{L} \implies \eta^{\cdot\cdot}_{\mu\nu} = \mathbb{L}^{\mathsf{T}}{}^{\cdot\alpha}_{\mu\cdot}\eta^{\cdot\cdot}_{\alpha\beta}\mathbb{L}^{\beta\cdot}_{\cdot\nu} = \mathbb{L}^{\alpha\cdot}_{\cdot\mu}\mathbb{L}^{\beta\cdot}_{\cdot\nu}\eta^{\cdot\cdot}_{\alpha\beta} \quad . \tag{58.09.16}$$

→ 2412

The inverse **Lorentz Transformation** matrix can be derived in a straightforward way from the condition (58.09.15) by right multiplying by $\mathbb{L}^{-1}$ and left multiplying by η:

$$\mathbb{L}^{-1} = \eta\mathbb{L}^{\mathsf{T}}\eta \quad . \tag{58.09.17}$$

→ 2408 2410 2411 2412 2430

The Lorentz matrix inversion thus amounts to just transposing and changing the sign of the elements on the first row and first column, except for the $L^{0\cdot}_{\cdot 0}$ element as implied by equation (58.09.17).

From equation (58.09.17), after left multiplication by $\mathbb{L}$ and right multiplication by η, one also obtains:

$$\eta = \mathbb{L}\eta\mathbb{L}^{\mathsf{T}} \implies \eta^{\cdot\cdot}_{\mu\nu} = \mathbb{L}^{\cdot\alpha}_{\mu\cdot}\eta^{\cdot\cdot}_{\alpha\beta}\mathbb{L}^{\mathsf{T}}{}^{\beta\cdot}_{\cdot\nu} = \mathbb{L}^{\cdot\alpha}_{\mu\cdot}\mathbb{L}^{\cdot\beta}_{\nu\cdot}\eta^{\cdot\cdot}_{\alpha\beta} \quad . \tag{58.09.18}$$

Finally, from equation (58.09.17), one obtains:

$$\left(\mathbb{L}^{-1}\right)^{\mathsf{T}} = \left(\mathbb{L}^{\mathsf{T}}\right)^{-1} = \eta\mathbb{L}\eta \quad . \tag{58.09.19}$$

When handling Lorentz matrices and indexes one should carefully remember the rules/conventions stated in section § 20.02.03.

The following additional conditions define the orthochronous proper **Lorentz Transformation** :

$$\det[\mathbb{L}] = +1 \qquad \mathbb{L}^{0\cdot}_{\cdot 0} \geq 1 \quad , \tag{58.09.20}$$

→ 2412

which corresponds to exclude from **Lorentz Transformation** both Space-Inversion parity inversion and time inversion.

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58.09.04.01 Algebra of Lorentz Matrices

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The **Lorentz Transformation** form a group, the Lorentz group, defined as the group of all the 4×4 real matrices such that the Lorentz condition (58.09.16) is satisfied:

$$\eta = \mathbb{L}^{\mathsf{T}} \eta \mathbb{L} \quad .$$

The additional restrictions (58.09.20)

$$\det[\mathbb{L}] = +1 \qquad \mathbb{L}^{0}_{\cdot 0} \geq 1 \quad ,$$

correspond to exclude both the Space-Inversion parity inversion and the Time-Reversal, thus defining a sub-group of the Lorentz group, the orthochronous and proper Lorentz group.

The general **Lorentz Transformation** matrix can be written, in block form, as:

$$\boxed{\mathbb{L} = \begin{pmatrix} \gamma & \mathbb{R} \\ \mathbb{C} & \mathbb{M} \end{pmatrix}} \quad ,$$

where:

- $\mathbb{R}$ is a 1×3 matrix, and R is a mnemonics for **row**;
- $\mathbb{C}$ is a 3×1 matrix, and C is a mnemonics for **column**;
- $\mathbb{M}$ is a 3×3 matrix, and M is a mnemonics for **matrix**.

We have:

$$\boxed{\mathbb{L}^{\mathsf{T}} = \begin{pmatrix} \gamma & \mathbb{C}^{\mathsf{T}} \\ \mathbb{R}^{\mathsf{T}} & \mathbb{M}^{\mathsf{T}} \end{pmatrix}} \quad .$$

Moreover, equations (58.09.17) implies:

$$\boxed{\mathbb{L}^{-1} = \begin{pmatrix} \gamma & -\mathbb{C}^{\mathsf{T}} \\ -\mathbb{R}^{\mathsf{T}} & \mathbb{M}^{\mathsf{T}} \end{pmatrix}} \quad .$$

The constraints imposed by the Lorentz condition can be derived from:

$$\mathbb{L}^{-1}\mathbb{L} = \mathbb{L}\mathbb{L}^{-1} = \mathbb{I} \quad .$$

One finds the two set of relations:

$$\mathbb{C}^{\mathsf{T}}\mathbb{C} = \gamma^2 - 1 \quad , \tag{58.09.21}$$

$$\gamma\mathbb{R} = \mathbb{C}^{\mathsf{T}}\mathbb{M} \quad , \tag{58.09.22}$$

$$\gamma\mathbb{R}^{\mathsf{T}} = \mathbb{M}^{\mathsf{T}}\mathbb{C} \quad , \tag{58.09.23}$$

$$\mathbb{I} = \mathbb{M}^{\mathsf{T}}\mathbb{M} - \mathbb{R}^{\mathsf{T}}\mathbb{R} \quad , \tag{58.09.24}$$

→ 2413 2428

and

$$\mathbb{R}\mathbb{R}^{\mathsf{T}} = \gamma^2 - 1 \quad , \tag{58.09.25}$$

$$\gamma\mathbb{C}^{\mathsf{T}} = \mathbb{R}\mathbb{M}^{\mathsf{T}} \quad , \tag{58.09.26}$$

$$\gamma\mathbb{C} = \mathbb{M}\mathbb{R}^{\mathsf{T}} \quad , \tag{58.09.27}$$

$$\mathbb{I} = \mathbb{M}\mathbb{M}^{\mathsf{T}} - \mathbb{C}\mathbb{C}^{\mathsf{T}} \quad . \tag{58.09.28}$$

→ 2413 2428

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58.09.04.02 Rotations

It can be shown that the matrix

$$\mathbb{D} = \mathbb{M} - \frac{\mathbb{C}\mathbb{R}}{\gamma+1} \qquad \mathbb{D}^{\mathsf{T}} = \mathbb{M}^{\mathsf{T}} - \frac{\mathbb{R}^{\mathsf{T}}\mathbb{C}^{\mathsf{T}}}{\gamma+1} \quad , \tag{58.09.29}$$

→ 2428

is a rotation matrix, that is an orthogonal matrix in $\mathsf{O}(3)$:

$$\mathbb{D}^{\mathsf{T}}\mathbb{D} = \mathbb{D}\mathbb{D}^{\mathsf{T}} = \mathbb{I} \quad .$$

It can be shown that a necessary a sufficient condition for a Lorentz matrix to be a rotation matrix is that its inverse is equal to its transpose:

$$\boxed{\mathbb{L}^{-1} = \mathbb{L}^{\mathsf{T}}} \quad . \tag{58.09.30}$$

58.09.04.03 Boosts

A **Lorentz Transformation** without rotation is defined by $\mathbb{D} = \mathbb{I}$, that is by the condition:

$$\mathbb{M} = \mathbb{I} + \frac{\mathbb{C}\mathbb{R}}{\gamma+1} \qquad \mathbb{M}^{\mathsf{T}} = \mathbb{I} + \frac{\mathbb{R}^{\mathsf{T}}\mathbb{C}^{\mathsf{T}}}{\gamma+1} \quad .$$

By using the relations (58.09.21), (58.09.25) one can show that this implies:

$$\mathbb{R} = \mathbb{C}^{\mathsf{T}} \qquad \mathbb{R}^{\mathsf{T}} = \mathbb{C} \quad .$$

All in all a boost, a transformation without rotation, can be written as:

$$\mathbb{L} = \begin{pmatrix} \gamma & \mathbb{R} \\ \mathbb{R}^{\mathsf{T}} & \mathbb{I} + \dfrac{\mathbb{R}^{\mathsf{T}}\mathbb{R}}{\gamma+1} \end{pmatrix} = \begin{pmatrix} \gamma & \mathbb{C}^{\mathsf{T}} \\ \mathbb{C} & \mathbb{I} + \dfrac{\mathbb{C}\mathbb{C}^{\mathsf{T}}}{\gamma+1} \end{pmatrix} \quad .$$

That is **the Lorentz pure transformation matrix, in addition to satisfying the Lorentz condition, is also symmetrical**:

$$\boxed{\mathbb{L} = \mathbb{L}^{\mathsf{T}}} \quad . \tag{58.09.31}$$

Note that **the boost do not form a sub-group of the Lorentz group**, while rotations do, because the product of two symmetrical matrices, in general, is not symmetrical.

It can be shown that **the composition of two pure transformations is a pure transformation if and only if the velocities are parallel to each other**. If the two velocities are not parallel to each other the resulting transformation is not a pure one: the resulting rotation is called Thomas rotation (or precession).

58.09.04.04 General Lorentz Transformation

The following results can be demonstrated. A general **Lorentz Transformation** can be factorized in different ways.

- Consider one specific rotation. The general **Lorentz Transformation** can be factorized into:
  - one pure transformation followed by the rotation;
  - the rotation followed by one pure transformation.

  The two pure transformations, applied before/after the same rotation, are in general different. this corresponds to the physical fact that the two pure transformations have the same velocity but expressed in components with respect to different Coordinate System.
- The general **Lorentz Transformation** can be also factorized into: one rotation to align the z axis along the velocity followed by one special transformation followed by one rotation to align the z axis along the desired direction.

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58.10 Consequences of the Lorentz Transformation

Consider special **Lorentz Transformation** along the z axis for simplicity.

58.10.01 Time Dilatation and Proper Time (Special Lorentz Transformation)

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Consider an object fixed in $\mathcal{I}'$: $\mathbf{r}'_1 = \mathbf{r}'_2$.
The time interval between any two events happening at the same place is so-called **proper time**.
More generally: the **proper time** is the time between two events whenever:

$$\Delta\mathbf{r}' = 0 \qquad \text{same position along the direction of } \mathbf{u} \text{ is enough} \quad .$$

Therefore one needs to relate: $\Delta t'$, Δt and $\Delta\mathbf{r}' = 0$. From the appropriate expressions of the **Lorentz Transformation** (equation (58.09.07)) one finds:

$$\Delta\tau \equiv \Delta t' = \Delta t/\gamma \quad .$$

Any clock in motion runs slower. However, the effect is only noticeable when the clock travels at relativistic speeds.

Moreover: $\Delta z = u\Delta t$, as it is obvious from the definition of velocity for a point fixed in $\mathcal{I}'$.

The observer who sees the clock moving, $\mathcal{I}$, will not agree that the positions have been recorded at the same place but it will measure a difference of position $\Delta z = u\Delta t$, as it is obvious from the definition of velocity for a point fixed in $\mathcal{I}'$.

Note that $\mathbf{r}'$ is held fixed, because we are watching one specific clock moving. If one would keep $\mathbf{r}$ fixed then one would watch a whole series of different clocks in $\mathcal{I}'$ passing by and this would not tell us whether one of them is running slow or not.

Of course no physical effect has happened to the clock: it is only the process of measurement in the moving Reference Frame which has given a different result.

It can be shown that the time dilatation is a reciprocal effect among the two observers, such that any of the two sees the clocks of the other observer running slower [1].

Note that moving clocks are not synchronized among each other as synchronization is not an invariant concept so that if the clocks are synchronized in the frame where they are at rest they are not in other systems; therefore it is essential when checking time dilation to focus attention on a single moving clock.

Proper time is normally denoted as $\Delta\tau$ or $đ\tau$. Proper time of an object, that is time measured by a clock at rest with the object, is clearly an invariant, by definition. In many cases it is a quantity more useful than time.

[1] D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ..., § 12.1.2

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58.10.01.01 Proper Time and Proper Rest Frame

The definition of proper time is not limited to objects moving at constant velocity.

Consider any Reference Frame, $\mathcal{I}'$, moving arbitrarily, without rotations, with respect to the observer $\mathcal{I}$. That is assume that the two Reference Frames are always linked by a pure **Lorentz Transformation** , at any time, a different boost at any time.

At any instant the relations between the times measured in the two Reference Frame is given by:

$$\boxed{\mathrm{d}t = \gamma\, \mathrm{d}t' \equiv \gamma\, \mathrm{d}\tau} \quad .$$

The invariant $\mathrm{d}\tau$ is the proper time that is the time measured in the proper Rest Frame, because that gives the time interval when $\mathrm{d}x = 0$, that is the time measured by an observer at rest with respect to the system.

In fact it is a **basic postulate of Relativity that time only depends on the speed of the clock, not on acceleration nor any other higher-order derivative**, read § 58.06.03.

58.10.02
Length Contraction and Proper/Rest Length (Special Lorentz Transformation)

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It is necessary to understand precisely what the length represents: **length is the distance between two points taken at the same instant of time as judged in the Reference Frame of the observer**. However, the criterion of simultaneity is different for any other observer, and hence the result of measuring the length can be different, too.

In order to measure the length of a moving segment one needs to pick-up the position of the two ends of the segment at the same time: one looks at the position labels of the points coincident with the two extremes of the moving segment at the same time, as measured by the two clock at the two points.

Proper/Rest Length is defined as the length measured when the object is at rest.

Consider an object at rest in $\mathcal{I}'$, whose ends are fixed at $\mathbf{r}'_1$ and $\mathbf{r}'_2$.

When observing from the Reference Frame $\mathcal{I}$ the observer must take the position of the two ends at the same time in order to measure its length, so that:

$$\boxed{\Delta t = 0} \quad .$$

Therefore one needs to relate: $\Delta\mathbf{r}$, $\Delta\mathbf{r}'$ and $\Delta t = 0$. From the appropriate expressions of the **Lorentz Transformation** (equation (58.09.06)) one finds:

$$\boxed{\Delta z' = \gamma \Delta z} \quad .$$

On the other hand we don't need to care about times in $\mathcal{I}'$ because the segment is at rest in $\mathcal{I}'$. In fact for the $\mathcal{I}$ observer the positions of the two ends are a function of its time, while for the $\mathcal{I}'$ observer the positions of the two ends are time-independent and therefore the time in $\mathcal{I}'$ does not enter.

The observer at rest with the object, $\mathcal{I}'$, will not agree that the positions have been recorded at the same time but it will measure a difference of time $\Delta t' = -u\Delta z'/c^2$.

Any object in motion appears to be shorter along the direction of motion. However, the effect is only noticeable when the objects travels at relativistic speeds.

Of course no physical effect has happened to the rod: it is only the process of measurement in the moving Reference Frame which has given a different result.

It can be shown that the length contraction is a reciprocal effect among the two observers, such that any of the two sees length of the objects at rest with respect the other observer being shorter [1].

[1] D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ..., § 12.1.2

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A.P.French, *Vibrations And Waves*, 1971, M.I.T. Introductory Physics Series, ...Ed., ...
Despite the fact that the relativistic transformations are founded on the complete equivalence of inertial frames, there may still seem to be something paradoxical about the length- contraction and time-dilation phenomena. Suppose we have two identical meter sticks in relative motion along the direction of their length. If I am riding in the rest frame of one of them I shall measure the length of the other to be less than 1 m. How, then, can I avoid the conclusion that my meter stick appears more than I m long to an observer in the rest frame of the other stick? You need never be perplexed by such a question if you once realize that each inertial frame must speak for itself, as it were. Observations made in one frame provide a description of events from the standpoint of that frame alone. The observations that lead A to say that B's meter stick is shrunk are A's observations; the conclusions drawn from those observations are valid for A but not for B. B can make his own observations-similar to A's, but not the same-and his conclusion is that A's meter stick is contracted. Both are correct. There is no contradiction, because it is physically impossible for both sets of observations to refer to one and the same Rest Frame.

58.10.03
Transformation Law for the Volume

Consider a parallelepiped, fixed with respect to the moving Reference Frame $\mathcal{I}'$, with its sides either parallel or perpendicular to the speed $\mathbf{u}$. The sides perpendicular to the speed $\mathbf{u}$ do not contract. The side along the speed $\mathbf{u}$ contracts by a factor γ. Therefore volumes change as:

$$\boxed{\Delta V = \Delta V'/\gamma} \quad .$$

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58.11 Physical Laws and Four-Dimensional Formalism

This § is referenced at pages:
[Never referenced.]

Read § 20.03.07, § 20.03.06 for the general **Quotient Law** and **Principle of Covariance** , which are applied in this section in the context of **Lorentz Transformation** of Physical Quantities and Laws.

58.11.01 Notations and Conventions

Note that the system of units such that $c = 1$ will be often used except for some of the basic formulas.

A four-vector has one index (always a Greek letter) which can have four values: $0, 1, 2, 3$. The index zero denotes the so-called time-component of a four-vector. The indexes 1, 2, 3, denote the so-called space-component of a four-vector. More general tensors have more indexes with the same characteristics as the four-vectors. Note that the position of the vector/tensor indexes (high/low) are important and high/low indexes are not equivalent. Therefore, any meaningful equality must carry indexes in the same position. Whenever a pair of indexes are summed one of them must be high while the other one must be low. As **Lorentz Transformation** mixes the time and space components, it is necessary that the four components have the same physical dimensions; in fact the Lorentz matrix is dimensionless. When grouping together two different physical quantities to form a four-vector we will use the convention of retaining the physical dimensions of the tri-vector and modifying with suitable constants the physical dimensions of the zero-component.

Three dimensional space position vectors will be denoted either as

$$\mathbf{r} \equiv r_k \equiv r^k = \{x, y, z\} \qquad \text{or} \qquad \mathbf{x} \equiv x_k \equiv x^k = \{x_1, x_2, x_3\} \quad .$$

The position of the vector/tensor indexes (high/low) is not relevant.

Four dimensional time-space four-vectors will be denoted either as

$$r \equiv r^\mu = \{ct, \mathbf{r}\} = \{ct, x, y, z\} \qquad \text{or} \qquad x \equiv x^\mu = \{ct, \mathbf{x}\} = \{ct, x_1, x_2, x_3\} \quad .$$

The position of the vector/tensor indexes (high/low) are important and high/low indexes are not equivalent.

58.11.02 Scalars Under Lorentz Transformation

By definition a Lorentz scalar is any physical quantity which has the same value in any inertial Reference Frame, that is any physical quantity which is invariant under **Lorentz Transformation** :

$$\boxed{S' = S} \quad .$$

Examples of Lorentz scalar quantities are: the Lorentz interval between any two events, the mass of any system and the electric charge.

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58.11.03
Four-Vectors Under Lorentz Transformation

A four-vector is any set of four physical quantities transforming, under Lorentz transformation:

- either as the differentials of the time-space coordinates (a **contravariant vector**);
- or as the partial derivatives with respect to the time-space coordinates (a **covariant vector**);

Read also § 20.

In fact two different possible transformations laws exists for four-vectors under **Lorentz Transformation** , as it is the case for general coordinate transformations, read § 20. The two kinds of four-vectors are distinguished by:

- upper (high) indexes, contravariant vectors;
- lower (low) indexes, covariant vectors.

58.11.03.01 Contravariant and Covariant Form of Four-Vectors

The following heuristic considerations can be carried on.

It can be shown that the η matrix can be used to raise and lower the indexes (read § 20), so that **contravariant and covariant form of a four-vectors are just different ways to represent the same four-vector**.

The relation between contravariant, V^α, and covariant, V_α, form of a four-vector is:

$$V_\alpha = \eta_{\alpha\beta} V^\beta \qquad V^\alpha = \eta^{\alpha\beta} V_\beta \quad .$$

By definition the time-space four-vector is (contravariant/covariant forms):

$$x^\alpha \equiv \{ct, +\mathbf{x}\} \quad \Leftrightarrow \quad x_\alpha \equiv \{ct, -\mathbf{x}\} \quad .$$

By definition the time-space four-gradient is (contravariant/covariant forms)::

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} \equiv \left\{ \frac{1}{c}\partial_t, +\frac{\partial}{\partial \mathbf{x}} \right\} \equiv \left\{ \frac{1}{c}\partial_t, +\boldsymbol{\nabla} \right\} \quad \Leftrightarrow \quad \partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} \equiv \left\{ \frac{1}{c}\partial_t, -\frac{\partial}{\partial \mathbf{x}} \right\} \equiv \left\{ \frac{1}{c}\partial_t, -\boldsymbol{\nabla} \right\} \quad .$$

The direct and inverse **Lorentz Transformation** in matrix form read:

$$\boxed{x' = \mathbb{L} x \quad \Leftrightarrow \quad x = \mathbb{L}^{-1} x'} \quad .$$

The direct and inverse **Lorentz Transformation** with explicit indexes read:

$$\boxed{x'^\alpha_{\cdot} = \mathbb{L}^\alpha_{\cdot\,\beta} x^\beta_{\cdot} \quad \Leftrightarrow \quad x^\alpha_{\cdot} = \left(\mathbb{L}^{-1}\right)^\alpha_{\cdot\,\beta} x'^\beta_{\cdot}} \quad .$$

The Jacobian matrices of the two transformations are:

$$\boxed{\frac{\partial x'^\alpha_{\cdot}}{\partial x^\beta_{\cdot}} = \mathbb{L}^\alpha_{\cdot\,\beta} \quad \Leftrightarrow \quad \frac{\partial x^\alpha_{\cdot}}{\partial x'^\beta_{\cdot}} = \left(\mathbb{L}^{-1}\right)^\alpha_{\cdot\,\beta}} \quad .$$

The differentials of the space-time coordinates transform, in matrix form, as:

$$\boxed{\mathrm{d}x' = \mathbb{L}\, \mathrm{d}x} \quad .$$

Introduce the useful notation:

$$\boxed{\frac{\partial}{\partial x^\mu_{\cdot}} \equiv \partial^{\cdot}_\mu \qquad \frac{\partial}{\partial x'^\mu_{\cdot}} \equiv \partial'^{\cdot}_\mu} \quad .$$

The partial derivative of the space-time coordinates transform as:

$$\boxed{\frac{\partial}{\partial x'^\beta_{\cdot}} = \frac{\partial}{\partial x^\alpha_{\cdot}} \frac{\partial x^\alpha_{\cdot}}{\partial x'^\beta_{\cdot}} = \frac{\partial}{\partial x^\alpha_{\cdot}} \left(\mathbb{L}^{-1}\right)^\alpha_{\cdot\,\beta} \implies \partial'^{\cdot}_\beta = \partial^{\cdot}_\alpha \left(\mathbb{L}^{-1}\right)^\alpha_{\cdot\,\beta}} \quad .$$

It is therefore clear that for a given **Lorentz Transformation** the partial derivatives transform in a different way from the differentials: in matrix form the row four-gradient transformed vector is found from the product between the row four-gradient vector and the inverse Lorentz matrix $\mathbb{L}^{-1}$.

The partial derivative of the space-time coordinates transform, in matrix form, as:

$$\boxed{\partial'^{\cdot}_\beta = \partial^{\cdot}_\alpha \left(\mathbb{L}^{-1}\right)^\alpha_{\cdot\,\beta} \quad \Leftrightarrow \quad \partial'^{\mathsf{T}} = \partial^{\mathsf{T}} \mathbb{L}^{-1}} \quad .$$

In order to distinguish between the two different way of transformations of four-vectors under the same **Lorentz Transformation** :

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1. the first type of four-vectors, those transforming as the differentials of the coordinates, have their Greek indexes in the high position and called contravariant four-vectors;
2. the second type of four-vectors, those transforming as the gradient of the coordinates, have their Greek indexes in the low position and called covariant four-vectors.

58.11.03.02 Properties of Four-Vectors

It can be shown that the relations between the contravariant and covariant components of a four vector are:

$$\boxed{V_{\mu}^{\cdot} = \eta_{\mu\nu}^{\cdot\cdot} V_{\cdot}^{\nu} \qquad V_{\cdot}^{\mu} = \eta_{\cdot\cdot}^{\mu\nu} V_{\nu}^{\cdot}} \quad . \tag{58.11.01}$$

→ 2430

Contravariant vectors, by definition, transforms as the differentials of the coordinates, that is:

$$\boxed{V'^{\mu}_{\cdot} = \mathbb{L}^{\mu}_{\cdot\nu} V_{\cdot}^{\nu} \qquad V' = \mathbb{L}V \qquad \text{in matrix form}} \quad . \tag{58.11.02}$$

→ 2419

Covariant vectors, by definition, transforms as the gradient of the coordinates, that is:

$$\boxed{V'_{\mu}^{\cdot} = \mathbb{L}_{\mu}^{\cdot\nu} V_{\nu}^{\cdot} \qquad V'^{\mathsf{T}} = V^{\mathsf{T}} \mathbb{L}^{-1} \qquad \text{in matrix form}} \quad . \tag{58.11.03}$$

→ 2419 2430

Note that the above result just comes from lowering-rising the indexes. It is a particular case of the more general treatment is section § 20 which would be valid for any change between Coordinate System.

Read § 58-018.

58.11.03.03 Scalar Product of Four-Vectors

It can be shown that in order to produce a scalar as product of two four-vectors one needs to use the contravariant form of one four-vector and the covariant form of the other one:

$$\boxed{A \cdot B \equiv A^{\mu} B_{\mu} = A_{\mu} B^{\mu}} \quad .$$

The invariance of the scalar product is clear from the above relations (58.11.02), (58.11.03). In fact by considering the contravariant form as a column vector and the covariant form as a row vector the scalar product can be written in matrix form as:

$$A_{\mu} B^{\mu} = A^{\mathsf{T}} \cdot B = B^{\mathsf{T}} \cdot A = B_{\mu} A^{\mu} \quad ,$$

as the transpose of a 1×1 matrix is the same number.

It is then clear from equations (58.11.02), (58.11.03) that applying a **Lorentz Transformation** the result does not change as the Lorentz matrix and its inverse cancel out.

It immediately follows that in order to have a scalar one needs to sum one high and one low index.

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58.11.03.04 Contravariant and Covariant Form in General Coordinates

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58.11.04 Higher Order Tensors Under Lorentz Transformation

This § is referenced at pages:
[Never referenced.]

They are defined according to the general procedure outlined in section § 20.

By definition, a Lorentz tensor of rank $m = h + l$ is a physical quantity with an ordered set of h contravariant indexes and l covariant indexes:

$$\left(\mathbb{T}'\right)^{r_1 \cdot r_2 \cdots r_h \cdot \cdot \cdot}_{\cdot a_1 \cdot \cdot \cdot a_2 \cdots a_l} = \mathbb{L}^{r_1 \cdot}_{\cdot s_1} \mathbb{L}^{\cdot b_1}_{a_1 \cdot} \mathbb{L}^{r_2 \cdot}_{\cdot s_2} \dots \mathbb{L}^{r_h \cdot}_{\cdot s_h} \mathbb{L}^{\cdot b_2}_{a_2 \cdot} \mathbb{L}^{\cdot b_l}_{a_l \cdot} \left(\mathbb{T}\right)^{s_1 \cdot s_2 \cdots s_h \cdot \cdot \cdot}_{\cdot b_1 \cdot \cdot \cdot b_2 \cdots b_l} \quad . \tag{58.11.04}$$

Some examples follow.

1. Rank two constant tensors: $\eta_{\mu\nu}$ and $\delta^{\alpha}_{\cdot\beta}$.
2. Rank three tensor: the totally anti-symmetrical**Levi-Civita tensor** $\epsilon_{\mu\nu\sigma\rho}$ in Minkowski space.
3. The orbital Boost-AngMomentum Tensor, with respect to the origin of the Coordinate System, is defined, as a function of the Energy-LinMomentum four-vector, as:
$$\gamma^{\alpha\beta}_{\cdot\cdot} \equiv x^{\alpha} p^{\beta} - x^{\beta} p^{\alpha} \quad .$$

58.11.04.01 Relativistic Angular Momentum

In relativity, angular momentum is a part of a second-order anti-symmetric tensor. Therefore it is wrong to treat the relativistic transformation of angular momentum as those of a vector. It is a second-order anti-symmetric tensor. Its transformation law, is such that the vector angular momentum transforms the other way round with respect to the space part of real four-vector: its longitudinal component does not change, while its transverse component does change. This is the same as for the Electric|Magnetic field, not surprisingly, as they are also part of a second-order anti-symmetric tensor, read section § 61.05.02.

Read also § **??**.

58.11.04.02 Examples of Lorentz Tensors

Important scalars (invariants) include:

- the mass;
- the electric charge.

Important four-vectors include:

- the time-space four-vector;
- the Energy-LinMomentum (Four-Momentum) four-vector (proportional to the four-velocity);
- the frequency-wave-vector (four-wave-vector) four-vector;
- the charge-current (Four-Current) four-vector;
- the ElectroMagnetic four-potential;
- the spin four-vector.

Usually, the physical dimensions of the zero component is adapted to the physical dimension of the 3-vector to get a four-vector with all components having the same dimensions.

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Similarly, usually, the name of the four-vector is taken from the name of the 3-vector quantity (four-position, four-momentum, four-current, four-wave-vector).

Important tensors of second order include:

- the ElectroMagnetic field tensor;
- the energy momentum tensor.

58.11.05
Spinors and Spinorial Representations

©|Gerrit Coddens. Spinors for everyone. 2017. cea-01572342|WEB - URL |*Full critical discussion.*|

58.11.06
Transformation of Fields Under Lorentz Transformation

Read § **??**.

58.11.07
Invariant Conserved Constant Physical Quantities

One has to be careful to avoid confusion between the concept of a scalar (invariant) quantity and the concept of a conserved quantity.

- A quantity that is constant does not change with time.
- A Lorentz scalar (Lorentz invariant) is a quantity which has the same numerical value in any inertial Reference Frame.
- A conserved quantity is a quantity which does not change before and after some process/encounter.

The speed of light in a vacuum is constant and invariant. These are entirely different and independent statements.

The rest mass is a scalar. However, the rest mass of a complex system (also called the invariant mass) is not, in general, conserved. The rest mass (just called mass) of an elementary entity (particle) is an intrinsic properties of the entity: if it changes we assume that the identity of the elementary entity has changed.

Energy-LinMomentum of an isolated system is a conserved quantity, but it is also conserved in collisions/interactions internal to the system.

Electric charge is both a scalar and a conserved quantity.

Velocity is neither conserved nor invariant.

58.11.08
Tensors Under Lorentz Versus Tensors Under Galilei

Rotation transformations (leave $\mathbb{I}$ invariant):

$$\mathbb{I} = \mathbb{R}^{\mathsf{T}}\mathbb{R} = \mathbb{R}\mathbb{R}^{\mathsf{T}} \implies \mathbb{R}^{-1} = \mathbb{I}\mathbb{R}^{\mathsf{T}}\mathbb{I} \quad ,$$

$$\boxed{\delta^{\cdot\cdot}_{pq} = R^{\cdot a}_{p\cdot} R^{\cdot b}_{q\cdot} \delta^{\cdot\cdot}_{ab}} \quad . \qquad \rightarrow 2421$$

Note that equation (58.11.08) actually means that δ is a second order tensor for the rotations group, actually an invariant one, that has the same components in any Coordinate System.

Lorentz Transformation (leave η invariant):

$$\eta = \mathbb{L}^{\mathsf{T}}\eta\mathbb{L} = \mathbb{L}\eta\mathbb{L}^{\mathsf{T}} \implies \mathbb{L}^{-1} = \eta\mathbb{L}^{\mathsf{T}}\eta \quad ,$$

$$\boxed{\eta^{\cdot\cdot}_{\mu\nu} = \mathbb{L}^{\cdot\alpha}_{\mu\cdot} \mathbb{L}^{\cdot\beta}_{\nu\cdot} \eta^{\cdot\cdot}_{\alpha\beta}} \quad . \qquad \rightarrow 2421$$

Note that equation (58.11.08) actually means that η is a second order tensor for the Lorentz group, actually an invariant one, that has the same components in any Reference Frame.

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58.11.09
Covariance of Equations of Physical Laws

This § is referenced at pages:
[1187]

Read § 20.03.06.

© - QUOTE

R.Hagedorn, *Relativistic Kinematics*, 1963, W.A.Benjamin,, ...Ed., WEB - URL.
I wish to stress once more the following point: scalars, vectors, tensor operators and such things are defined abstractly as physical or geometrical quantities. As soon as a system of coordinates is introduced, these quantities will be represented by components with respect to the coordinate axes. This representation will depend on the choice of the coordinates whereas the abstract quantities are independent: their existence logically precedes the existence of coordinates. If in the following we shall find different representations of, for instance, a four vector by covariant and contravariant components, and if these representations are different in different metrics they nevertheless represent one and the same physical quantity.

© - QUOTE

R.Hagedorn, *Relativistic Kinematics*, 1963, W.A.Benjamin,, ...Ed., WEB - URL.
If a question is of such a nature that its answer will always be the same, no matter in which Lorentz system one starts, it must be possible to formulate the answer entirely with the help of those invariants which one can build with the available four-vectors. One then finds the answer in a particular Lorentz system which one can choose freely and in such a way that the answer is there obvious or most easy. One looks then how the invariants appear in this particular system, expresses the answer to the problem by these invariants, and one has found at the same time the general answer. This looks sometimes like Hocus-Pocus, as you will see in the following examples. It is worthwhile to devote some thinking to this method of calculation until one has completely understood that there is really no jugglery or guesswork in it and that it is absolutely safe.

© - QUOTE

R.Hagedorn, *Relativistic Kinematics*, 1963, W.A.Benjamin,, ...Ed., WEB - URL.
If an equation given in any particular Lorentz Reference Frame can be written in a manifestly covariant form (that is, both sides have the same transformation properties under **Lorentz Transformation**) which in the particular Lorentz frame reduces to the equation given originally, which is a known exact equation, this covariant form is the unique generalization of the equation given.

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58.12 Examples and Physical Applications

58.12.01 Speed of Light as a Limiting Speed

This § is referenced at pages:
[2397, 2427, 2451]

©|W.Rindler|Special relativity: kinematics|*WEB - URL* |

© - QUOTE

W.Rindler, *Relativity*, 2006, Oxford, 2ndEd., ...
The γ factor of **Lorentz Transformation** becomes infinity, for $u = c$, and imaginary, for $u > c$. Therefore the relative velocity between two different Reference Frames must be less than the speed of light, if finite real-valued coordinates in one Reference Frame must correspond to finite real-valued coordinates in any other Reference Frame. Based on this line of reasoning, it follows that no particle can move super-luminally relative to any Reference Frame, because any set of such particles moving parallel among each other would constitute a Reference Frame moving super-luminally relative to the first Reference Frame.
Other indications show that the speed of particles and of all physical signals, is upper-limited by c, such as violation of causality in presence of super-luminal signal transmission. Consider, in fact, any signal or process whereby a first event causes a second event (or whereby information is sent from the first event to the second event) at super-luminal speed, U, relative to some inertial Reference Frame. By applying **Lorentz Transformation** it is then possible to find an inertial Reference Frame, moving at speed u, such that $c^2/U < u < c$, where the time-ordering of the two events is inverted. Therefore there would exist some inertial Reference Frame in which the second event precedes the first one, so that cause and effect are reversed and the signal is considered to travel in the opposite spatial direction.
The whole concept of super-luminal signaling is seen to produce paradoxes. We are thus led to accept the axiom that no super-luminal signals can exist, so that c is an upper-limit to the speed of macroscopic information-conveying signals. In particular, this speed limit must apply to particles, since they can convey messages.
Relativistic mechanics provides a speed limit by having the mass of particles increase beyond all bounds as their speed approaches the speed of light. Any massive particle that constantly accelerates, no matter how strongly, approaches but never attains the speed of light.
One consequence of the upper relativistic speed limit is that such things as rigid-bodies and incompressible fluids have become impossible objects, even as idealizations or limits: in fact, by definition, they would transmit signals instantaneously.
Note, However, that arbitrarily large velocities are possible for moving points that carry no information. The sweep of a light spot from the Earth on the Moon surface, where a movable laser beam from Earth impinges, or the intersection point of two rulers that cross each other at an arbitrarily small angle are two such examples.

58.12.02
Experimental Tests of Special Relativity

This § is referenced at pages:
[2399]

©|T.Roberts & S.Schleif & J.M.Dlugosz|WEB - URL |*Excellent, extensive and detailed*|

Experimental tests of special relativity are particularly important because either special relativity has many counter-intuitive predictions and, in order to appreciate relativistic results, it is necessary to have objects moving close to the speed of light, which is difficult to attain.

Experiments on the contraction of lengths are clearly impossible, as this would require to run physical objects close to the speed of light.

Experiments on the dilatation of time are easier.

Read also § 59.02.05 for wave-related experiments.

58.12.02.01 Atmospheric Muons Lifetime Paradox

This § is referenced at pages:
[2427, 2477, 2477]

The lifetime increase of a high-energy particle is often used in High-Energy Physics to observe the decay point of a short-lived particle away from the production point.

Muons, for instance, have a lifetime $\tau_0 = 2.2{\cdot}10^{-6}$ s and mass $m = 0.105\ \mathrm{GeV/c^2}$.

1. What energy is necessary to a muon to reach the Earth surface once it was produced in the high part of the atmosphere at $H = 10$ km while running radially?
2. How is the kinematics described by the observer at the Earth and by an observer moving with the muon?

WORKED SOLUTION

$$H = \tau_0\gamma\beta c = \frac{\tau_0 p}{m} = \frac{\tau_0\beta\varepsilon}{mc} \longrightarrow \varepsilon = 1.6\ \mathrm{GeV} \qquad \text{assuming } \beta \simeq 1 \quad .$$

Description of the observer at the Earth: the muon lives $\tau_0\gamma$ and travels $H = \tau_0\gamma\beta c$ (time dilatation).
Description of the muon: the muon lives τ_0 and si standing still. The atmosphere, of length H, is traveling towards the muon, with velocity βc (length contraction). The muon feels a shorter life, by a factor γ, but it sees distances contracted, by a factor γ, so that the atmosphere she is seeing is only H/γ long.
Therefore the two descriptions, the one given by the muon and the one given by the ground observer, are totally coherent.

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58.12.02.02 Atomic Clocks on Commercial Airplanes

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©|J.Hafele & R.Keating||*WEB - URL* |

They flew cesium atomic clocks on commercial airliners around the world in both directions and compared the time elapsed on the airborne clocks with the time elapsed on an earthbound clock.

The effect includes both an effect of special relativity and an effect of general relativity.

58.12.02.03 Lifetime of Muons in Storage Rings

Bailey *et al.*, Measurements of relativistic time dilation for positive and negative muons in a circular orbit, Nature 268 (July 28, 1977) Pag. 301.

They stored muons in a storage ring and measured their lifetime. When compared with measurements of the muon lifetime at rest this is a highly relativistic twin scenario as $v \simeq 0.9994c$, for which the stored muons are the traveling twin and return to a given point in the lab every few μs.

The experiment confirms the clock hypothesis stating that the tick rate of a clock when measured in an inertial frame depends only upon its velocity relative to that frame, and it is independent of its acceleration or higher derivatives with respect to time. In fact muons in the storage ring are accelerated and their lifetime could be measured to only depend on the velocity.

58.12.02.04 Time Dilatation and the Clock Hypotesis and the Twin Paradox

The so-called **twin paradox** occurs when two clocks are synchronized, separated, one of them takes a long journey, rejoined and their readings are finally compared.

If one clock remains in an inertial frame then the other must be accelerated sometimes during its journey and it will display less elapsed proper time than the inertial clock at the end of the journey. Therefore the situation is not symmetrical between the two clocks.

This is a paradox only in that it appears to be inconsistent, but it is not.

This has been tested using muons stored in a storage ring and measured their lifetime. When combined with measurements of the lifetime of muons at rest (the twin at rest) this becomes a **twin paradox** experiment.

58.12.02.05 Experimental Test of the Second Postulate

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||

Experiment performed at CERN, Geneva, Switzerland, in 1964: the speed of photons with $\varepsilon = 6$ GeV (speed $0.99975c$), produced in the decay of very energetic neutral pions, was measured by time of flight over paths up to 80 meters. Within the experimental errors it was found that the speed of the photons emitted by the extremely rapidly moving source was equal to c.

58.12.03 Charged Particle in Uniform Magnetic Field - Larmor Radius

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 12.2.4||

Read § 15.05.09, § 15.05.10.

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58.12.04
Barn and the Ladder Paradox

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 12.1.2, 12.1.3||

It is all about rigid-body in relativity and the meaning of measuring a length.

58.12.05
Ehrenfest Paradox

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 12.1.3||

It is about the space geometry in relativity.

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58.13 Exercises Problems and Physical Applications

58-001 Super-Luminal Motion

Read § 58.12.01 for the explanation in the context of Relativity.

The value of c is the limiting speed for real material bodies and for processes that can be used for the transmission of a signal, that is the transmission of a certain amount of energy carrying some information.

Super-luminal motion refers to apparent speeds larger than the speed of light. It is only a visual effect, nothing is running faster than light, nothing carrying information.

Super-luminal motion also refers to **motion** which is faster than light but not for real objects nor for transfer of information.

As an example, imagine a machine-gun sending projectiles, at a rate of n Hz while rotating in the horizontal plane with a fixed angular speed ω. Consider a very far circular wall, centered on the machine-gun, at distance D, and consider the motion of the track of the impinging projectiles. Determine the **speed** of the track left by the projectiles on the wall. Projectiles arrive at the rate they leave the machine-gun, separated by an angular distance $\Delta\theta = \omega\Delta t = \omega/n$ so that the distance on the wall is $D\omega/n$ and the **speed** of the track is $D\omega$, which can be as large as one desires provided D is large enough.

58-002 Lifetime of Unstable Particles

The lifetime of short-lived particles is increased with their speed, see section § 58.12.02.01.

58-003 Simultaneity Synchronization and Time Dilation

Suppose that at time $t = 0$ the observer $\mathcal{I}$ decides to examine all the clocks in $\mathcal{I}'$. He will find that they all read different times, depending on their location: those at negative z are ahead, and those at positive z are behind, by an amount that increases in proportion to their distance. Only the master clock at the origin reads $t' = 0$.

How does the observer examines the clocks at rest in $\mathcal{I}'$? He needs to compare at any time the moving clock with its own clock at rest at the position of the moving clock.

58-004 Invariant Quantity of Lorentz Transformation

Show, by direct calculation, that, for a special **Lorentz Transformation** :

$$\boxed{c^2\,(\Delta t)^2 - (\Delta z)^2 = c^2\,(\Delta t')^2 - (\Delta z')^2} \quad .$$

Any general **Lorentz Transformation** leaves invariant the quantity:

$$\boxed{c^2\,(\Delta t)^2 - (\Delta \mathbf{x})^2 = c^2\,(\Delta t')^2 - (\Delta \mathbf{x}')^2} \quad .$$

58-005 An Algebraic Identity

Show that:

$$\boxed{\frac{\gamma^2 - 1}{\gamma^2} = \beta^2} \quad .$$

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58-006 Non-Relativistic Limit

Show that **Lorentz Transformation** reduce to **Galilei Transformation** in the limit: $c \to \infty$.

58-007 Composition of Lorentz Transformation

©|E.Byckling & K.Kajantie, *Particle Kinematics*, 1993, J.Wiley & Sons, ...Ed., WEB - URL.|problem 2.19||

Consider three Inertial Reference Frames, $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$. Let $\mathbf{u}_{21}$ be the velocity of $\mathcal{I}_2$ with respect to $\mathcal{I}_1$, let $\mathbf{u}_{32}$ be the velocity of $\mathcal{I}_3$ with respect to $\mathcal{I}_2$ and let $\mathbf{u}_{31}$ be the velocity of $\mathcal{I}_3$ with respect to $\mathcal{I}_1$. Choose the Coordinate Systems in such a way that both transformations $(1) \to (2)$ and $(2) \to (3)$ are pure transformations.

Show that:

$$\gamma_{\mathbf{u}_{31}} = \gamma_{\mathbf{u}_{21}} \gamma_{\mathbf{u}_{32}} \left(1 + \frac{\mathbf{u}_{21} \cdot \mathbf{u}_{32}}{c^2} \right) \quad .$$

58-008 Group Properties of the Lorentz Transformation

Demonstrate that the **Lorentz Transformation** form a group.

58-009 Rotations in Lorentz Transformation

Demonstrate that the matrix (58.09.29) is a rotation matrix.

WORKED SOLUTION

By direct computation of $\mathbb{D}^{\mathsf{T}}\mathbb{D} = \mathbb{D}\mathbb{D}^{\mathsf{T}} = \mathbb{I}$, using relations (58.09.21), (58.09.25). It is long but straightforward.

58-010 Boosts in Lorentz Transformation

Show that the composition of two boost is a boost if and only if the two velocities are parallel.

58-011 Infinitesimal Lorentz Transformation

Show that for an infinitesimal **Lorentz Transformation** ,

$$\mathbb{L} = \mathbb{I} + \mathbb{B}\, \mathrm{d}\theta + \mathcal{O}\left(\mathrm{d}\theta \right)^2 \quad ,$$

the matrix $\mathbb{B}$ satisfies:

$$\eta\mathbb{B} = -\eta\mathbb{B}^{\mathsf{T}} \quad ,$$

that is the matrix $\eta\mathbb{B}$ is anti-symmetrical.

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58-012 Thomas Precession

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||

58-013 Change of Angles

The Reference Frame $\mathcal{I}'$ is related to the Reference Frame to $\mathcal{I}$ by the special **Lorentz Transformation** (58.09.03).

Calculate in $\mathcal{I}'$ the length and the angle with the z' axis of a rod of length L at an angle θ with the z axis in $\mathcal{I}$.

58-014 Change of Direction in a Lorentz Transformation

Show that, because of the length contraction in the direction of motion, only oriented segments either parallel or perpendicular to the velocity of the **Lorentz Transformation** keep the same direction after a **Lorentz Transformation** .

WORKED SOLUTION

The formula of transformation of directions is:

$$\boldsymbol{\lambda} = \frac{\boldsymbol{\lambda}' + \mathbf{u}\left(\left(\frac{1}{\gamma} - 1\right)\frac{\mathbf{u}\cdot\boldsymbol{\lambda}'}{u^2}\right)}{\sqrt{1 - \left(\mathbf{u}\cdot\boldsymbol{\lambda}'/c\right)^2}} \quad .$$

58-015 Change of Angles in a Lorentz Transformation

Show that the concept of parallelism is invariant under **Lorentz Transformation** , but the concept of perpendicularity it is not.

Show that the angle does not change when both directions are either perpendicular or parallel to the velocity.

WORKED SOLUTION

The formula of transformation of angles is:

$$\cos\left[\alpha\right] = \frac{\cos\left[\alpha'\right] - \frac{\left(\mathbf{u}\cdot\boldsymbol{\lambda}_1'\right)\left(\mathbf{u}\cdot\boldsymbol{\lambda}_2'\right)}{c^2}}{\left(1 - \left(\mathbf{u}\cdot\boldsymbol{\lambda}_1'/c\right)^2\right)\left(1 - \left(\mathbf{u}\cdot\boldsymbol{\lambda}_2'/c\right)^2\right)} \quad .$$

58-016 Check of the Reciprocity Relation

Use equation (59.01.01) and consider its inverse equation. Replace the inverse equation into (59.01.01) and check that the result is an identity.

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58-017 Raising/Lowering Indexes

Show that raising a previously lowered indexes as well as lowering a previously raised index recovers the original four-vector.

58-018 Relation Between Contravariant and Covariant Transformations

This § is referenced at pages:
[2419]

Find the transformation law for covariant vectors from the one for contravariant vectors.

WORKED SOLUTION

$$V_{\mu}^{\cdot} \equiv \eta_{\mu\beta}^{\cdot\cdot} V_{\cdot}^{\beta} \implies V'^{\cdot}_{\mu} = \eta_{\mu\beta}^{\cdot\cdot} V'^{\beta}_{\cdot} = \eta_{\mu\beta}^{\cdot\cdot} \mathbb{L}_{\cdot\alpha}^{\beta\cdot} V_{\cdot}^{\alpha} = \eta_{\mu\beta}^{\cdot\cdot} \mathbb{L}_{\cdot\alpha}^{\beta\cdot} \eta_{\cdot\cdot}^{\alpha\nu} V_{\nu}^{\cdot} \equiv \mathbb{L}_{\mu\cdot}^{\cdot\nu} V_{\nu}^{\cdot} \quad . \tag{58.13.01}$$

→ 2430

Note that the transformation matrix is exactly obtained by $\mathbb{L}_{\cdot\nu}^{\mu\cdot}$ by lowering the first index and raising the second one.

58-019 Direct Check of Invariance

Check by direct calculation for a special **Lorentz Transformation** that the scalar product of any two four-vectors is invariant.

58-020 Non-Invariance of d'Alembert Operator for Galilei Transformation

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|problem 11.1||

58-021 Matrix Form of the Covariant Transformation

The above relation (58.11.03), describing the transformation of covariant vectors as a function of the transformation of contravariant vectors, can be written in matrix form as:

$$\boxed{V' = (\boldsymbol{\eta}\mathbb{L}\boldsymbol{\eta})\, V} \quad .$$

Beware that the vectors in the above equation are covariant vectors, obtained from the contravariant vectors from equation (58.11.01).

WORKED SOLUTION

It is implicit in the last step of equation (58.13.01) when writing in matrix form the expression: $\eta_{\mu\beta}^{\cdot\cdot} \mathbb{L}_{\cdot\alpha}^{\beta\cdot} \eta_{\cdot\cdot}^{\alpha\nu}$.

In order both to find the relation between the matrix $\mathbb{L}_{\mu\cdot}^{\cdot\nu}$ and the matrix $\mathbb{L}_{\cdot\nu}^{\mu\cdot}$ and to make this result consistent with the general discussion of § 20 it is better to rewrite the above relation in a different way.

Moreover, using the symmetry of η and equation (58.09.17), one has:

$$\mathbb{L}_{\mu\cdot}^{\cdot\nu} = \eta_{\mu\beta}^{\cdot\cdot} \mathbb{L}_{\cdot\alpha}^{\beta\cdot} \eta_{\cdot\cdot}^{\alpha\nu} = \eta_{\mu\beta}^{\cdot\cdot} \mathbb{L}^{\mathsf{T}}{}_{\alpha\cdot}^{\cdot\beta} \eta_{\cdot\cdot}^{\alpha\nu} = \eta_{\cdot\cdot}^{\nu\alpha} \mathbb{L}^{\mathsf{T}}{}_{\alpha\cdot}^{\cdot\beta} \eta_{\beta\mu}^{\cdot\cdot} = \left(\eta \mathbb{L}^{\mathsf{T}} \eta\right)_{\cdot\mu}^{\nu\cdot} = \left(\mathbb{L}^{-1}\right)_{\cdot\mu}^{\nu\cdot} = \left(\left(\mathbb{L}^{-1}\right)^{\mathsf{T}}\right)_{\mu\cdot}^{\cdot\nu} = \left(\left(\mathbb{L}^{\mathsf{T}}\right)^{-1}\right)_{\mu\cdot}^{\cdot\nu} \quad .$$

Therefore:

$$\boxed{V'^{\cdot}_{\mu} = \left(\mathbb{L}^{-1}\right)_{\cdot\mu}^{\nu\cdot} V_{\nu}^{\cdot}} \quad .$$

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58-022 Sum of Time-Like Four-Vectors

This § is referenced at pages:
[2459]

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CHAPTER 59

Relativistic Kinematics

59.01 Transformation Laws of Velocity and Acceleration 2434
59.02 Transformation of Frequency and wave-vector for MPW 2438
59.03 Examples and Physical Applications 2442
59.04 Exercises Problems and Physical Applications 2446

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This § is referenced at pages:
[Never referenced.]

Note that $c = 1$ will be often used except for some of the basic formulas.

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59.01 Transformation Laws of Velocity and Acceleration

The transformation law of velocities is a rather complex one, and it is a non-linear one. In fact the transformation law of the velocities requires to transform both the numerator and the denominator of the derivative and this makes the law complex.

59.01.01 Special Lorentz Transformation for the Velocity

This § is referenced at pages:
[2435, 2442, 2446, 2446, 2446]

Consider two Inertial Reference Frames, $\mathcal{I}$ and $\mathcal{I}'$. Assume the two Reference Frames are connected by a special **Lorentz Transformation** and that $\mathcal{I}'$ is moving, with respect to $\mathcal{I}$, with a speed $\mathbf{u} \equiv \boldsymbol{\beta} c$ along the z axis ($\mathbf{u} = u\widehat{\mathbf{e}}_3$):

$$v'_z \equiv \lim_{\Delta t' \to 0} \frac{\Delta z'}{\Delta t'} \quad ,$$

$$v'_{(x|y)} \equiv \lim_{\Delta t' \to 0} \frac{\Delta (x'|y')}{\Delta t'} \quad .$$

When $\Delta t' \to 0$ then also $\Delta x' \to 0$, $\Delta y' \to 0$ and $\Delta z' \to 0$.

Therefore **Lorentz Transformation** imply that $\Delta t \to 0$ and also $\Delta x \to 0$, $\Delta y \to 0$ and $\Delta z \to 0$.

$$v'_z \equiv \lim_{\Delta t' \to 0} \frac{\Delta z'}{\Delta t'} = \lim_{\Delta t \to 0} \frac{\gamma\,(\Delta z - u\Delta t)}{\gamma\,(\Delta t - u\Delta z/c^2)} = \frac{v_z - u}{1 - uv_z/c^2} \quad ,$$ → 2429

$$v'_{(x|y)} \equiv \frac{\Delta (x'|y')}{\Delta t'} = \lim_{\Delta t' \to 0} \frac{\Delta (x|y)}{\gamma\,(\Delta t - u\Delta z/c^2)} = \frac{v_{(x|y)}}{\gamma\,(1 - uv_z/c^2)} \quad .$$

The non-relativistic limit applies when all velocities (on the right-hand side of the above equations) are non relativistic: $v_z \ll c$, $v_{(x|y)} \ll c$ and $|u| \ll c$. In this limit the above relations reduce to the one predicted by **Galilei Transformation** and the resulting velocities on the left-hand side turns out to be non-relativistic.

These equations implement, among other things, the fact that the maximum possible velocity is the speed of light, so that no composition of velocities can lead to a speed greater than the speed of light. Consider, for instance, the special case $v_z = c$, which implies $v_x = v_y = 0$, one has $v'_z = c$, which implies $v'_x = v'_y = 0$, whatever u. The general case, for arbitrary velocity, is discussed in section § 59-002.

In summary, in the case of a special **Lorentz Transformation** along the z axis ($\mathbf{u} = u\widehat{\mathbf{e}}_3$):

$$\boxed{v'_x = \frac{v_x}{\gamma\,(1 - uv_z/c^2)}} \quad ,$$ → 2435

$$\boxed{v'_y = \frac{v_y}{\gamma\,(1 - uv_z/c^2)}} \quad ,$$ → 2435

$$\boxed{v'_z = \frac{v_z - u}{1 - uv_z/c^2}} \quad .$$ → 2435

Note that the transverse components of the velocity transform, because even if the transverse components of the position vector do not transform, one has to take into account the transformation law for the time at the denominator.

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59.01.02
Boost Lorentz Transformation for the Velocity

This § is referenced at pages:
[2442, 2446, 2446, 2446]

Consider two Inertial Reference Frames, $\mathcal{I}$ and $\mathcal{I}'$. Assume the two Reference Frames are connected by a boost and that $\mathcal{I}'$ is moving, with respect to $\mathcal{I}$, with a speed $\mathbf{u} \equiv \boldsymbol{\beta} c$.

It can be straightforwardly derived, for a boost, from the transformation laws (58.09.11), (58.09.12):

$$\mathbf{v}' \equiv \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t'} = \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{\mathbf{v} + \boldsymbol{\beta}\left(\frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{v}) - \gamma c\right)}{\gamma\left(1 - \frac{\boldsymbol{\beta}\cdot\mathbf{v}}{c}\right)} \quad , \tag{59.01.01}$$

→ 2435

and the inverse:

$$\mathbf{v} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t'}\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{\mathbf{v}' + \boldsymbol{\beta}\left(\frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{v}') + \gamma c\right)}{\gamma\left(1 + \frac{\boldsymbol{\beta}\cdot\mathbf{v}'}{c}\right)} \quad . \tag{59.01.02}$$

The expressions in the case the velocity $\mathbf{v}$ is either parallel or orthogonal to the relative velocity $\boldsymbol{\beta}$ can be derived by direct computation from equation (59.01.01):

$$\mathbf{v} \parallel \mathbf{u} \implies \mathbf{v}' = \frac{\mathbf{v} - \mathbf{u}}{1 - uv/c^2} \implies \mathbf{v}' \parallel \mathbf{u} \quad ,$$

→ 2435

$$\mathbf{v} \perp \mathbf{u} \implies \mathbf{v}' = \frac{\mathbf{v}}{\gamma} - \mathbf{u} \implies \mathbf{v}' \text{ is not perpendicular to } \mathbf{u}, \text{ differently than } \textbf{Galilei Transformation} \quad .$$

→ 2435

These equations are consistent with equations coming from a special **Lorentz Transformation** § 59.01.01.

Equation (59.01.02), in fact, is the same as equation (59.01.01).

Equation (59.01.02), when $v_z = 0$, in fact, implies that:

- the component of $\mathbf{v}'$ parallel to $\mathbf{u}$ is $-\mathbf{u}$, which is the same result coming from equation (59.01.01) when $v_z = 0$;
- the component of $\mathbf{v}'$ perpendicular to $\mathbf{u}$ is $\mathbf{v}/\gamma$, which is the same result coming from equations (59.01.01), (59.01.01) when $v_z = 0$.

The addition law of relativistic velocities is neither commutative nor associative.

59.01.03
Ordinary Velocity and Proper Velocity

Ordinary velocity is defined as: $\mathbf{v} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$; from now on it will be simply called **velocity**. Both quantities, $\mathrm{d}\mathbf{x}$ and $\mathrm{d}t$ are measured with respect to the inertial Reference Frame of the observer, for a moving object.

Sometimes it might be useful to consider the distance per unit proper time, which defines a hybrid quantity, the so-called **proper-velocity**: $\mathbf{V} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}$. In this case $\mathrm{d}\mathbf{x}$ is measured with respect to the inertial Reference Frame of the observer but the proper time is measured in the Reference Frame at rest with the moving object.

In fact, while remembering the relation between time and proper time

$$\mathrm{d}t = \gamma\, \mathrm{d}\tau \quad .$$

PROGRESS

one can define, alongside with the velocity, the proper velocity, $\mathbf{V}$:

$$\mathbf{v} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \qquad \mathbf{V} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} \quad ,$$
$$\mathbf{V} = \gamma_v \mathbf{v} \qquad |u| \ll c \implies \mathbf{V} = \mathbf{v} \quad .$$

The definition can be extended to the four-dimensional quantity:

$$V \equiv \frac{\mathrm{d}x}{\mathrm{d}\tau} \qquad \begin{cases} \mathbf{V} \equiv \dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \gamma_v \mathbf{v} \\ V^0 \equiv \dfrac{\mathrm{d}x^0}{\mathrm{d}\tau} = \gamma_v c \end{cases} ,$$

which is called the proper velocity four-vector.

The proper velocity, V, will transform according to **Lorentz Transformation** , that is a much easier transformation law than the transformation law for velocity.

It immediately follows from its definition that:

$$\boxed{V \cdot V = c^2} \quad .$$

Note that the proper velocity is a hybrid quantity: it is the ratio of a displacement as measured by the Reference Frame of the observer, $\mathcal{I}$, and the time, as measured by an observer at rest with the moving object, that is moving with respect to Reference Frame of the observer, $\mathcal{I}$.

59.01.04
Transformation Laws for the Acceleration

The transformation law for ordinary acceleration is rather complex. Therefore proper acceleration is most often used, which transforms as a four-vector:

$$\boxed{A \equiv \frac{\mathrm{d}V}{\mathrm{d}\tau} \equiv \frac{d^2x}{\mathrm{d}\tau^2}} \quad .$$

It turns out that:

$$\boxed{V \cdot A = 0} \quad .$$

59.01.05
Mutual and Relative Position Velocity Acceleration

This § is referenced at pages:
[2403, 2446]

- The concept of **mutual** vector involves **one single observer and two different observed points**: mutual velocity between two points, A and B, as seen by the observer. Consequently, the concept of mutual vector cannot involve the Galilei/Lorentz transformations, because there is only one observer.
- The concept of **relative** vector, based on the **composition of vectors, involves two different observers and one observed vector**: I am observing one and ask for it as seen by another observer. Consequently, the concept of relative vector involves the Galilei/Lorentz transformations.

The most interesting case is that of the velocity vector.
These are, therefore, two different concepts:

- **mutual velocity** between two points, as seen by one external observer: $\mathbf{u}_2 - \mathbf{u}_1$ (one observer but two different observed points);
- **relative velocity** of one point with respect to another Reference Frame, moving with respect to the observer, that is the velocity of one point as measured by another Reference Frame, moving with respect to the observer (two different observers but one observed point).

The two concepts coincide in **Galilei-Newton Mechanics** due to **GINV** .

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While the expression $\mathbf{v}_2 - \mathbf{v}_1$ is invariant under Galilei transformations it is not invariant under **Lorentz Transformation** .

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59.02 Transformation of Frequency and wave-vector for MPW

This § is referenced at pages:
[2136]

59.02.01 Lorentz Invariance of the Phase of MPW

Consider a MPW (a scalar one, for simplicity), with angular frequency ω and wave-vector $\mathbf{k}$:

$$\boxed{\psi[t,\mathbf{x}] = \psi^0 \cos\left[\mathbf{k}\cdot\mathbf{x} - \omega t - \phi\right] \qquad \Phi[\mathbf{x},t] \equiv \mathbf{k}\cdot\mathbf{x} - \omega t - \phi} \quad , \tag{59.02.01}$$

→ 2439

where ψ^0 is a scalar, giving the scalar character to the MPW.

Thanks to the linearity of **Lorentz Transformation** the phase-factor, $\cos\left[\mathbf{k}\cdot\mathbf{x} - \omega t - \phi\right]$, will transform into a similar phase factor in any other Reference Frame, and therefore the MPW will appear as a MPW.

As the phase of the MPW must be a Lorentz invariant, as demonstrated below, the transformed MPW will have the same scalar/vector/tensor character in any Reference Frame, the one given by the multiplicative factor to the phase-factor (the scalar ψ^0 in our case).

The **phase of the wave must be a scalar** (that is Lorentz invariant) because the measurement of the difference of the phase at two different space-time events can be reduced to the number of counts of the wave-crests in between the two space-time events, which must give the same result in any Reference Frame.

As a first example consider the difference of the phase at two different space points at the same time:

$$\Delta\Phi = \mathbf{k}\cdot(\mathbf{x}_2 - \mathbf{x}_1) = k|\mathbf{x}_2 - \mathbf{x}_1|\cos[\theta] \quad ,$$

where θ is the angle between the vectors $\mathbf{k}$ and $(\mathbf{x}_2 - \mathbf{x}_1)$. The difference of the phase is just the fractional number of wavelengths (space periods) between the two space points (times 2π).

As a second example consider the difference of the phase at the same space point between two different times:

$$\Delta\Phi = -\omega\,(t_2 - t_1) \quad .$$

The difference of the phase is just the fractional number of time periods passing away at the fixed space point between the times t_1 and t_2 (times 2π).

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59.02.02
Lorentz Transformation of Frequency and wave-vector

If one defines, for a wave with **phase-velocity** v, the object with four-components k^μ:

$$k^\mu \equiv \{\omega/c, \mathbf{k}\} \equiv \left\{k^0, \mathbf{k}\right\} \qquad v|\mathbf{k}| = \omega \quad ,$$

The fact that the phase is invariant implies that k^μ is a four-vector (thanks to the **Quotient Law** § 20.03.07): the **frequency wave-vector four-vector**. Note that the time-like component is always ω/c and not ω/v, as it follows from the expression of the phase in equations (59.02.01). Its transformation laws give the **Doppler effect** and the aberration of fronts.

59.02.03
Motion of the Source

59.02.04
EM Radiation

In the case of ElectroMagnetic radiation:

$$c|\mathbf{k}| = \omega \implies k^\mu k_\mu = 0 \quad .$$

In the ElectroMagnetic case the **phase-velocity** is always c in any Reference Frame.

Consider the Reference Frame $\mathcal{I}$, where a MPW has four-vector $k \equiv \{\omega/c, \mathbf{k}\}$, and an observer moving with a velocity $\mathbf{u}$ with its Reference Frame $\mathcal{I}'$, seing the MPW with four-vector $k' \equiv \left\{\omega'/c, \mathbf{k}'\right\}$. A light ray is described by the unit vector $\widehat{\mathbf{n}}$, in Reference Frame $\mathcal{I}$, and by the unit vector $\widehat{\mathbf{n}}'$, in $\mathcal{I}'$.

Relativistic optics is rather simple as there is no privileged Reference Frame. Light rays in vacuum follow straight paths with speed c in any Inertial Reference Frame.

Let us consider the boost between the two inertial Reference Frames, $\mathcal{I}$ and $\mathcal{I}'$, the latter moving with velocity $\mathbf{u}$ with respect to the former, and denote the velocities of the same light ray in the two Reference Frames with $\mathbf{v} \equiv c\widehat{\mathbf{n}}$ and $\mathbf{v}' \equiv c\widehat{\mathbf{n}}'$.

59.02.04.01 Aberration of Light

The law of addition of the velocities gives:

$$\frac{\mathbf{v}'}{c} \equiv \widehat{\mathbf{n}}' = \frac{\dfrac{\widehat{\mathbf{n}}}{\gamma_u} + \mathbf{u}\left(\left(1 - \dfrac{1}{\gamma_u}\right)\dfrac{\mathbf{u}\cdot\widehat{\mathbf{n}}}{u^2} - \dfrac{1}{c}\right)}{1 - \dfrac{\mathbf{u}\cdot\widehat{\mathbf{n}}}{c^2}} \quad , \qquad (59.02.02) \quad \rightarrow 2439\ 2440$$

with inverse law:

$$\frac{\mathbf{v}}{c} \equiv \widehat{\mathbf{n}} = \frac{\dfrac{\widehat{\mathbf{n}}'}{\gamma_u} + \mathbf{u}\left(\left(1 - \dfrac{1}{\gamma_u}\right)\dfrac{\mathbf{u}\cdot\widehat{\mathbf{n}}'}{u^2} + \dfrac{1}{c}\right)}{1 + \dfrac{\mathbf{u}\cdot\widehat{\mathbf{n}}'}{c^2}} \quad . \qquad (59.02.03) \quad \rightarrow 2439\ 2440$$

Alternatively the aberration can be described via the law of transformation of the wave-vector, $\mathbf{k}$, such that for a pure transformation one has:

$$\mathbf{k}' = \mathbf{k} + \boldsymbol{\beta}\left(\frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{k}) - \gamma\frac{\omega}{c}\right) = \mathbf{k} + \boldsymbol{\beta}\left(\frac{\gamma^2}{\gamma + 1}(\boldsymbol{\beta}\cdot\mathbf{k}) - \gamma\frac{\omega}{c}\right) \quad .$$

The relation between $\widehat{\mathbf{n}}$ and $\widehat{\mathbf{n}}'$ can be deduced from equations (59.02.02), (59.02.03), describing light aberration.

Let us define:

$$\widehat{\mathbf{n}}\cdot\mathbf{u} \equiv -u\cos\left[\alpha\right] \quad ,$$
$$\widehat{\mathbf{n}}'\cdot\mathbf{u} \equiv -u\cos\left[\alpha'\right] \quad .$$

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Using equations (59.02.02), (59.02.03) one finds:

$$\cos\left[\alpha'\right] = \frac{\cos\left[\alpha\right] + \dfrac{u}{c}}{1 + \dfrac{u}{c}\cos\left[\alpha\right]} \quad .$$

59.02.04.02 Doppler Effect

$$\omega' = \gamma\left(\omega - \mathbf{u}\cdot\mathbf{k}\right) \quad .$$

Particular Case of Source at Rest in the Moving System

If the source is at rest in $\mathcal{I}'$, so that the $k' \equiv \left\{\omega'/c, \mathbf{k}'\right\}$ is the proper frequency-wave-vector of the source, the observer will see the source moving with velocity $\mathbf{u}$ with respect to $\mathcal{I}$. Assume that the incoming light wave-vector, $\mathbf{k}$, is perpendicular with respect to the velocity $\mathbf{u}$.

Therefore:

$$\mathbf{k}\cdot\mathbf{u} = 0 \implies \omega' = \gamma\omega \geq \omega \quad ,$$

so that the observer sees a smaller frequency than the proper frequency of the source, that is a red-shift with respect to the proper frequency of the source.

Therefore the transformation law for the frequency gives the so-called transverse **Doppler effect** when the source is moving perpendicularly the line of sight in the opinion of the observer, which is a consequence of time dilatation.

This is a purely relativistic effect, for classically one would not expect any frequency shift from a source that moves by right angles with respect to the line of sight.

Transverse Doppler Effect

The transverse **Doppler effect** comes in when a light ray is reaching the observer Reference Frame, $\mathcal{I}$, perpendicularly to the velocity, $\mathbf{u}$, of $\mathcal{I}'$: $k\cdot u = 0$.

Consider in fact the boost, equation (58.09.11), with $\mathbf{k}\cdot\mathbf{u} = 0$, one can write:

$$\mathbf{k}\cdot\mathbf{u} = 0 \implies \begin{cases} \omega' = \gamma\omega \quad , \\ \mathbf{k}' = \mathbf{k} - \boldsymbol{\beta}\gamma\omega/c \end{cases} .$$

As a consistency check one can use the previous result to evaluate:

$$\omega = \gamma\left(\omega' + c\,\boldsymbol{\beta}\cdot\mathbf{k}'\right) = \gamma\left(\omega' + \beta^2\gamma\omega\right) \implies \omega = \omega'/\gamma \quad .$$

Note that the presence of the source is, so-far, inessential.

59.02.05
Experimental Tests of Special Relativity With Waves

This § is referenced at pages:
[2424]

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59.02.05.01 Experimental Test of Special Relativity via the Stellar Redshift

59.02.05.02 Experimental Test of Special Relativity via the Ives-Stilwell Experiment

In 1938 Ives and Stilwell observed the spectral lines of a high-velocity beam of hydrogen atoms. The spectral lines are blue shifted when looking in front of the incoming beam while they are red-shifted when looking at the beam from behind.

Show that the average forward-backward shift is blue shifted by a factor γ.

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59.03 Examples and Physical Applications

59.03.01 Airplane and Speeds

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|12.2||

In order to appreciate the different use of the ordinary and proper velocity consider somebody flying on a airplane.

When the pilot is speaking about the velocity is speaking about the distance measured on ground divided by the time measured on ground, presumably. This is what you are concerned if you have an appointment after arrival.

On the other hand if you are interested in whether you will be hungry or not after arrival you are dealing with the proper time for you, and therefore you would like to know the hybrid proper velocity.

59.03.02 Relativity and Absolute Quantities

Note that there actually exist absolute quantities in relativity, indeed: the rest length of a rod is an absolute quantity, the same for all inertial observers; the proper time of a clock, which might better have been called **the local time**, is an invariant quantity.

Also, in non-relativistic physics, it is a common notion that measured physical quantities depend on the inertial Reference Frame: velocity, kinetic energy, frequency....

© - QUOTE

R.Resnick, *Introduction To Special Relativity*, 1968, J.Wiley & Sons, ...Ed., ...
Moreover, there are absolute lengths and times in relativity. The rest length of a rod is an absolute quantity, the same for all inertial observers: if a given rod is measured by different inertial observers by bringing the rod to rest in their respective frames, each will measure the same length. Similarly for clocks, the proper time (which might better have been called *local time*) is an invariant quantity: the frequency of oscillation of an ammonia molecule, for instance, would be measured to be the same by different inertial observers who bring the molecule to rest in their respective frames.

59.03.03 Lorentz Transformation of the Module of Speed

Read § 59-002.

Consider the situation of section § 59.01.01 and/or section § 59.01.02.

The Lorentz transform of any speed less than c is less than c.

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59.03.04
Laser Cooling

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Laser cooling refers to the number of techniques in which atomic and molecular samples are cooled through the interaction with one or more laser light fields. The first example of laser cooling, and also still the most common method of laser cooling (so much so that it is still often referred to as laser cooling) is **Doppler** cooling.
Doppler cooling, which is usually accompanied by a magnetic trapping force to give a magneto-optical trap, is by far the most common method of laser cooling. It is used to cool low density gasses down to the **Doppler** cooling limit, which for Rubidium 85 is around 150 microkelvin. As **Doppler** cooling requires a very particular energy level structure, known as a closed optical loop, the method is limited to a small handful of elements.
In **Doppler** cooling, the frequency of light is tuned slightly below an electronic transition in the atom. Because the light is detuned to the "red" (i.e. at lower frequency) of the transition, the atoms will absorb more photons if they move towards the light source, due to the **Doppler effect** . Thus if one applies light from two opposite directions, the atoms will always scatter more photons from the laser beam pointing opposite to their direction of motion. In each scattering event the atom loses a momentum equal to the momentum of the photon. If the atom, which is now in the excited state, emits a photon spontaneously, it will be kicked by the same amount of momentum but in a random direction. The result of the absorption and emission process is to reduce the speed of the atom, provided its initial speed is larger than the recoil velocity from scattering a single photon. If the absorption and emission are repeated many times, the mean velocity, and therefore the kinetic energy of the atom will be reduced. Since the temperature of an ensemble of atoms is a measure of the random internal kinetic energy, this is equivalent to cooling the atoms.

59.03.05
Astronomical Apparently Super-Luminal Objects

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|Problem 12.6||

Gli astronomi hanno scoperto oggetti che emettono getti di materia ad altissima velocità. Se la distanza dell'oggetto dalla Terra è nota (sia d) la misura della velocità angolare alla quale si vede il getto muoversi sulla sfera celeste permette di determinare la velocità con cui il getto di materia è emesso dal nucleo centrale. In alcuni casi sono stati osservati getti in moto con velocità apparente superiore alla velocità della luce, fatto in apparente contraddizione con la teoria della relatività speciale. Mostrare come sia possibile ottenere velocità apparenti superiori a quella della luce e come ciò non contrasti con la relatività speciale considerando l'emissione del getto ad un angolo θ rispetto alla linea di vista dell'osservatore terrestre.

1. Determinare la relazione tra velocità apparente e quella reale del getto di materia.
2. Determinare la velocità apparente di un sistema in cui la direzione di emissione del getto forma un angolo $\theta = 9.2°$ con la direzione di vista supponendo la velocità di emissione pari a $v = 0.987c$ (questa si può determinare dallo studio dello spostamento **Doppler** delle linee spettrali).
3. Determinare il valore dell'angolo θ per cui la velocità apparente del getto sulla sfera celeste è massima e il valore di tale velocità massima.

PROGETTO

WORKED SOLUTION

Sia t_0 l'istante di emissione del getto di materia in direzione θ rispetto alla linea di vista. L'emissione del getto viene osservata da Terra al tempo t_1:

$$t_1 = t_0 + d/c$$

Il getto viene osservato da Terra, dopo che ha percorso la distanza $v\Delta t$, all'istante

$$t_2 = t_0 + \Delta t + \frac{d - v\Delta t \cos[\theta]}{c}$$

L'intervallo di tempo Δt tra le due osservazioni da Terra è

$$t_2 - t_1 = \Delta t(1 - \frac{v\cos[\theta]}{c}) < \Delta t \qquad (\text{se } \cos[\theta] > 0)$$

L'astronomo conosce la distanza d dell'oggetto e misura uno spostamento angolare $\Delta\theta$ sulla sfera celeste da cui deduce uno spostamento apparente Δx del getto. Lo spostamento apparente Δx sulla sfera celeste vale

$$\Delta x \equiv \Delta\theta d = \frac{v\Delta t \sin[\theta]}{d} d = v\Delta t \sin[\theta]$$

è da notare il fatto che la relazione precedente è esatta essendo $d \gg v\Delta t$ per cui non è necessario distinguere fra la distanza dell'oggetto che emette il getto, d, e la distanza del getto al tempo t_2 che vale $\simeq d - v\Delta t\cos[\theta]$. La velocità apparente osservata risulta dunque:

$$v_{\text{app}}(\theta) = \frac{\Delta x}{t_2 - t_1} = \frac{d\Delta\theta}{t_2 - t_1} = \frac{v\sin\theta}{1 - (v/c)\cos[\theta]}$$

che può essere maggiore di c.

Il massimo della velocità al variare dell'angolo di emissione θ si trova derivando rispetto a θ e ponendo la derivata uguale a zero:

$$\frac{dv_{\text{app}}(\theta)}{d\theta} = \frac{v\cos[\theta] - v^2/c}{(1 - v/c\cos[\theta])^2} = 0$$

da cui segue che l'angolo a cui si ottiene la velocità apparente massima e la velocità apparente massima valgono

$$\cos[\theta_{\text{max}}] = v/c \qquad v_{\text{app}}\Big|_{\text{max}} = \frac{v}{\sqrt{1 - v^2/c^2}}$$

Con i dati del problema si ottiene $v_{\text{app}} = 6.14c$.

WORKED SOLUTION

1. $v_{\text{app}}(\theta) = v\sin\theta/(1 - (v/c)\cos[\theta])$;
2. $v_{\text{app}} = 6.14c$;
3. $\cos[\theta_{\text{max}}] = v/c$.

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59.03.06
Relativistic Twins

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|11.6||

A spacecraft leaves the Earth in the year 2000. One of a couple of twins born in 1970 remains on Earth while the other one travels in the spacecraft. The spacecraft has an acceleration g in its own Rest Frame in order to make the occupants feel at home. It accelerates in a straight line path for five years (by its own clocks); it decelerates at the same rate for 5 more years; it turns around, it accelerates for 5 years, it decelerates for 5 years, and it finally lands on the Earth. The twin in the rocket is 40 years old. What year is it on Earth? How far away from the Earth did the rocket ship travel?

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59.04 Exercises Problems and Physical Applications

59-001 Composition of Two Velocities Close to the Speed of Light

Consider the situation of section § 59.01.01 and/or section § 59.01.02.
Assume $|u| \leq c$ and $u \simeq c$ as well as $|v'_z| \leq c$ and $v'_z \simeq c$. Show that $v_z \simeq c$.

59-002 Modules of Relativistic Velocities

This § is referenced at pages:
[2434, 2442]

©|H.M.Schwartz, *Introduction To Special Relativity*, 1968, McGraw-Hill, ...Ed., ...|||

Consider the situation of section § 59.01.01 and/or section § 59.01.02.

1. Show that the modules of the velocities are related by:

$$c^2 - v'^2 = c^2 \frac{\left(c^2 - v^2\right)\left(c^2 - u^2\right)}{\left(c^2 - \mathbf{u}\cdot\mathbf{v}\right)^2} \quad . \tag{59.04.01}$$

→ 2446

   Note that the expression is symmetrical in $\mathbf{u}$ and $\mathbf{v}$.
2. Show that equation (59.04.01) can be also written as:

$$v'^2 = c^2 \frac{c^2\left(\mathbf{v} - \mathbf{u}\right)^2 - \left(\mathbf{v}\wedge\mathbf{u}\right)^2}{\left(c^2 - \mathbf{u}\cdot\mathbf{v}\right)^2} \quad .$$

3. Use equation (59.04.01) to show that the module of the velocity v'^2 will always be smaller than c as long as both modules of $\mathbf{v}$ and $\mathbf{u}$ are smaller than c.

59-003 Relation Between the gamma factors

©|H.M.Schwartz, *Introduction To Special Relativity*, 1968, McGraw-Hill, ...Ed., ...|problem 3.3||

Consider the situation of section § 59.01.01 and/or section § 59.01.02.
Show that:

$$\gamma_{v'} = \left(1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right)\gamma_v\gamma_u \quad .$$

59-004 Relative Velocity in Relativistic kinematics

Read § 59.01.05.

If one observer measures two objects to be traveling at velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively what is the relative speed between the two objects, that is what speed observer (1) would measure observer (2) to be traveling at, or vice versa.

In Galilean mechanics the relative velocity is just:

$$\mathbf{v}_{21} = \mathbf{v}_2 - \mathbf{v}_1 \quad .$$

Show that in relativistic mechanics the relative velocity is:

$$\mathbf{v}_{21} \equiv \frac{\sqrt{\left(\mathbf{v}_2 - \mathbf{v}_1\right)^2 - \left(\mathbf{v}_1\wedge\mathbf{v}_2\right)^2/c^2}}{\left(1 - \mathbf{v}_1\cdot\mathbf{v}_2/c^2\right)} \quad .$$

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59-005 Properties of the Four-Acceleration

Consider the four-vector four-acceleration.

1. Show that:

$$A{\cdot}A = A^\alpha A_\alpha = \gamma_v^4 \left(\mathbf{a}{\cdot}\mathbf{a} + \gamma_v^2 \left(\mathbf{v}{\cdot}\mathbf{a} \right)^2 / c^2 \right) \quad .$$

2. Show that in the particle rest-frame the module of the four-acceleration is the square of the proper acceleration, $\mathbf{a}_0$.
3. Show that:

$$A{\cdot}A = A^\alpha A_\alpha = |\mathbf{a}_0|^2 = \gamma_v^4 \left(a_\perp^2 + \gamma_v^2 a_\parallel^2 \right) \quad .$$

59-006 The three-Velocity and the Four-Velocity

For events along the world-line of a particle traveling with three-velocity $\mathbf{u}$ the link between the three-velocity and the four-velocity is:

$$V^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = \gamma_v \frac{\mathrm{d}x^\alpha}{\mathrm{d}t} \equiv \gamma_v v^\alpha = \{\gamma_v c, \gamma_v \mathbf{v}\} \quad . \tag{59.04.02}$$

1. Show that:

$$V{\cdot}V = V^\alpha V_\alpha = V_\alpha V^\alpha = c^2 \quad .$$

59-007 The three-Acceleration and the Four-Acceleration

For events along the world-line of a particle traveling with three-acceleration $\mathbf{a}$ the link between the three-acceleration and the four-acceleration is:

$$A^\alpha \equiv \frac{d^2x^\alpha}{\mathrm{d}\tau^2} = \frac{\mathrm{d}V^\alpha}{\mathrm{d}\tau} = \gamma_v \frac{\mathrm{d}V^\alpha}{\mathrm{d}t} = \gamma_v \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_v v^\alpha) = \gamma_v \frac{\mathrm{d}}{\mathrm{d}t}\{\gamma_v c, \gamma_v \mathbf{v}\} \quad . \tag{59.04.03}$$

In fact, the relationship between the three-acceleration and the four-acceleration is less straightforward the corresponding one among velocities.

1. Show that:

$$A^\alpha = \gamma_v \left(c\frac{\mathrm{d}\gamma_v}{\mathrm{d}t}, \gamma_v \mathbf{a} + \mathbf{v}\frac{\mathrm{d}\gamma_v}{\mathrm{d}t} \right) \qquad \text{where} \qquad \mathbf{a} \equiv \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \quad .$$

2. Show that in the Rest Frame of the particle one has:

$$V^\alpha = \{c, \mathbf{0}\} \qquad A^\alpha = \{0, \mathbf{a}\} \quad .$$

3. Show that:

$$V{\cdot}A = 0 \quad .$$

WORKED SOLUTION

The Rest Frame moves with the particle. Therefore in that system one has:

$$\mathbf{v} = \mathbf{0} \qquad \text{constant} \qquad \gamma_v = 1 \qquad \text{constant} \quad .$$

It therefore follows:

$$\frac{\mathrm{d}\gamma_v}{\mathrm{d}t} = 0 \quad ,$$

and then

$$V^\alpha = \{c, \mathbf{0}\} \qquad A^\alpha = \{0, 1 \cdot \mathbf{a} + \mathbf{v} \cdot 0\} = \{0, \mathbf{a}\} \quad .$$

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59-008 Law of Transformation of the Acceleration

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|Problem 11.3||

Derive the law of Transformation of the Acceleration.

59-009 Transformation of the module of the Acceleration From/to the Rest Frame

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|Problem 16.3||

Show that the transformation of $\dot{u}^2$ from the Rest Frame to a general frame is:

$$\dot{u}^2 \longrightarrow \gamma^6 \left(\dot{u}^2 - (\dot{\mathbf{u}} \wedge \boldsymbol{\beta})^2 \right) \quad .$$

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CHAPTER 60

Relativistic Dynamics

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[Never referenced.]

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Alternative introduction to relativistic dynamics can be found in the standard textbooks.

Note that $c = 1$ will be often used without mention, except for some of the basic formulas.

WORK IN PROGRESS

60.01 Relativistic Energy/Linear Momentum and Power/Force

This § is referenced at pages:
[2456]

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©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|11.5, 3rd ed.|*Energy and linear momentum in ...*

The laws of **Galilei-Newton Mechanics** are not compatible with the postulates of relativity, because, among other things, a constant force can make the speed of any mass point arbitrarily large, if acting long enough. In fact, constant force means that the linear momentum and energy go to infinity: this is incompatible with the speed of light being **a limiting speed**, read § 58.12.01.

Moreover, for speeds approaching the speed of light, total energy and total linear momentum of an isolated system, as Newton defined linear momentum and energy, are not conserved; this is an **experimental fact**.

60.01.01 Relativistic Four-Momentum

In **Galilei-Newton Mechanics** , linear momentum is a vector, defined as the product of mass times the velocity, $\mathbf{p} = m\mathbf{v}$. Therefore it transforms as the velocity, not in the simple way of a **Lorentz Transformation** : it is not, in fact, the space part of a four-vector. Under **Lorentz Transformation** , the transformation law of the linear momentum of **Galilei-Newton Mechanics** , is the same as the transformation law of velocity: cumbersome and non linear.

Therefore when trying to extend the definition of linear momentum into relativistic physics one wonders whether ordinary or proper velocity needs to be used. Any extension of the concept of linear momentum must fulfill what we usually think a linear momentum is:

- reduce to the **Galilei-Newton Mechanics** expression at low velocities;
- being additive, that is the total linear momentum of a systems is defined as the sum of the linear momenta;
- ensure total linear momentum conservation in isolated systems, regardless of the velocities of the particles relative to Reference Frame;
- linear momentum conservation in isolated systems must be an invariant concept;
- linear momentum changes must be linked to the total external forces for non-isolated systems.

If one attempts to extend the concept of linear momentum using the four-velocity, instead of the ordinary velocity, one finds an expression whose space part transforms as the space part of a four-vector and reduces to the **Galilei-Newton Mechanics** expression of the ordinary velocity in the low velocity limit. Moreover, the time component of the four-velocity is proportional to γ: it is not clear, so far, what it is.

One can then try the following definitions, waiting for **experimental confirmation** that the resulting quantity is useful to describe the dynamics, that is it satisfies the above specifications.

See § 60.01.03 and § 60.01.02.

Let us define the tentative Four-Momentum as the contravariant four-vector:

$$\boxed{p^\mu \equiv \{\varepsilon/c, \mathbf{p}\} \equiv mV^\mu \qquad \text{whatever } p^\mu \text{ is}} \quad ,$$

$$V \cdot V = c^2 \implies \boxed{p \cdot p = m^2c^2} \quad .$$

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Moreover:

$$\boxed{p^0 \equiv \varepsilon/c = m\gamma c} \quad , \qquad \rightarrow 2452$$

$$\boxed{\mathbf{p} \equiv m\mathbf{V} = m\gamma\mathbf{v}} \quad , \qquad \rightarrow 2452$$

$$\boxed{\gamma \equiv \gamma_v \equiv \left(1 - v^2/c^2\right)^{-1/2}} \quad .$$

While the space-part looks like a sensible extension of the linear momentum of **Galilei-Newton Mechanics** , it is not clear what ε is, yet. It is anticipated that it is the energy, see § 60.01.03. Linear momentum and energy are different concepts but in relativity they are combined and put on equal footing as for space and time.

All in all we have the special **Lorentz Transformation** for the Four-Momentum is:

$$\boxed{\varepsilon' = \gamma_u \left(\varepsilon - up_z\right)} \quad , \tag{60.01.01}$$

$$\boxed{p'_x = p_x} \quad , \tag{60.01.02}$$

$$\boxed{p'_y = p_y} \quad , \tag{60.01.03}$$

$$\boxed{p'_z = \gamma_u \left(p_z - u\varepsilon/c^2\right)} \quad . \tag{60.01.04}$$

And the inverse **Lorentz Transformation** :

$$\boxed{\varepsilon = \gamma_u \left(\varepsilon' + up'_z\right)} \quad , \tag{60.01.05}$$

$$\boxed{p_x = p'_x} \quad , \tag{60.01.06}$$

$$\boxed{p_y = p'_y} \quad , \tag{60.01.07}$$

$$\boxed{p_z = \gamma_u \left(p'_z + u\varepsilon'/c^2\right)} \quad . \tag{60.01.08}$$

One needs to understand what is the meaning of p^0, the time component of the four-vector defined above. It is what was called **relativistic mass** in the past, but this term is misleading, as of today and should not be used. Nowadays it is usually called just the **energy**.

Squaring equation (60.01.01), (60.01.01) and subtracting side by side one finds:

$$\boxed{\varepsilon = \sqrt{p^2c^2 + m^2c^4} > 0} \quad .$$

However, at this stage, it is not clear what ε is, yet; read § 60.01.03. Dividing side by side equation (60.01.01) by (60.01.01) one finds:

$$\boxed{c^2\mathbf{p} = \varepsilon\mathbf{v} \qquad \Leftrightarrow \qquad \mathbf{v} = c^2\frac{\mathbf{p}}{\varepsilon} \qquad |\mathbf{v}| \leq c} \quad .$$

60.01.01.01 Contravariant and Covariant Forms of Four-Momentum

By definition the Energy-LinMomentum four-vector of any system is defined in terms of the energy and linear momentum as:

$$\boxed{p^\alpha \equiv \{\varepsilon/c, +\mathbf{p}\} \qquad \Leftrightarrow \qquad p_\alpha \equiv \{\varepsilon/c, -\mathbf{p}\}} \quad ,$$

with

$$\boxed{p{\cdot}p = p^\alpha p_\alpha = p_\alpha p^\alpha = \varepsilon^2/c^2 - \mathbf{p}{\cdot}\mathbf{p} = m^2c^2 \qquad m \text{ is the } \textbf{mass}} \quad .$$

The mass m is what was called **rest mass** in the past, but this term is redundant, as of today. Nowadays it is usually called just the **mass** an intrinsic property of any system. Note that the **mass**, $\varepsilon_0 = mc^2$, **is a Lorentz invariant but not conserved** quantity.

DRAFT IN PROGRESS

60.01.02
Invariance and Four-Momentum Conservation for Isolated Systems

This § is referenced at pages:
[2451, 2455]

Consider a collision problem among any number of particles in the initial state and any other number of particles in the final state, a priori different particles from the ones in the initial state.

For simplicity only, consider a one-dimensional collision problem with just two particles in the initial state and just two particles in the final state:

$$\boxed{a+b \to A+B \qquad (a+b+c+... \to A+B+C+...)} \quad . \tag{60.01.09}$$

60.01.02.01 Galilei Transformation and Classical Linear Momentum Conservation

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Using Galilei Transformation , if classical linear momentum, $\mathbf{p} = m\mathbf{v}$, is conserved in one inertial Reference Frame it is conserved in another inertial Reference Frame if and only if the sum of the masses (the total mass) is conserved:

$$(m_A + m_B + \ldots) = (m_C + m_D + \ldots) \quad . \tag{60.01.10}$$

Note that there is no assumption that the number of incoming particles is equal to the number of outgoing particles, only that the total mass is conserved.

Even if it is known that **Galilei Transformation** are non correct in general this result shows that **Galilei-Newton Mechanics** is self-consistent when it is valid, that is for velocities small with respect to the speed of light.

WORKED SOLUTION

$$\begin{aligned}
(p_A + p_B) - (p_C + p_D) &= 0 \quad , \\
(m_A v_A + m_B v_B) - (m_C v_C + m_D v_D) &= 0 \quad , \\
\left(m_A \left(v'_A + u\right) + m_B \left(v'_B + u\right)\right) - \left(m_C \left(v'_C + u\right) + m_D \left(v'_D + u\right)\right) &= 0 \quad , \\
\left(\left(p'_A + p'_B\right) - \left(p'_C + p'_D\right)\right) + u\left(\left(m_A + m_B\right) - \left(m_C + m_D\right)\right) &= 0 \quad .
\end{aligned}$$

60.01.02.02 Lorentz Transformation and Classical Linear Momentum Non-Conservation

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It can be shown that the classical linear momentum, $\mathbf{p} = m\mathbf{v}$ is incompatible with linear momentum conservation and **Lorentz Transformation** . **It is not true that, using Lorentz Transformation , if classical linear momentum, $\mathbf{p} = m\mathbf{v}$, is conserved in one inertial Reference Frame it is conserved in another inertial Reference Frame**.

As it is known that **Lorentz Transformation** are the correct transformation rules among different Reference Frame, not **Galilei Transformation** , this implies that the notion of classical linear momentum of **Galilei-Newton Mechanics** must be abandoned.

PRELIMINARY

60.01.02.03 Lorentz Transformation and Relativistic Linear Momentum Conservation

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Using Lorentz Transformation , if relativistic linear momentum, $\mathbf{p} = m\gamma\mathbf{v}$, is conserved in one inertial Reference Frame it is conserved in another inertial Reference Frame if and only if the sum of the relativistic energies (the total energy) is conserved.

As we know that **Lorentz Transformation** are the correct transformations in general, this result shows that in order to generalize the definition of classical linear momentum of **Galilei-Newton Mechanics** to something meaningful one needs to use the relativistic linear momentum defined as the product of mass times the proper velocity. This is necessary in order to have an invariant conservation of linear momentum.

WORKED SOLUTION

Direct Calculation via the Velocity Transformations

The direct calculation is long and tedious, because the transformation law for the velocity is complex. The following identity can be used:

$$\boxed{\gamma_v v = \gamma_u \gamma_{v'} \left(u + v'\right)} \quad .$$

Therefore:

$$v_X = \frac{v'_x + u}{1 + uv'_x/c^2} \quad ,$$

$$(p_A + p_B) - (p_C + p_D) = 0 \quad ,$$

$$(m_A\gamma_{v_A} v_A + m_B\gamma_{v_B} v_B) - (m_C\gamma_{v_C} v_C + m_D\gamma_{v_D} v_D) = 0 \quad ,$$

$$\left(m_A\gamma_{v_A}\left(\frac{v'_A + u}{1 + uv'_A/c^2}\right) + m_B\gamma_{v_B}\left(\frac{v'_B + u}{1 + uv'_B/c^2}\right)\right) - \left(m_C\gamma_{v_C}\left(\frac{v'_C + u}{1 + uv'_C/c^2}\right) + m_D\gamma_{v_D}\left(\frac{v'_D + u}{1 + uv'_D/c^2}\right)\right) = 0 \quad ,$$

... by developing the calculations one is led to use: $\gamma_v v = \gamma_u \gamma_{v'} \left(u + v'\right)$,

$$(m_A\gamma_{v_A} v_A + m_B\gamma_{v_B} v_B) - (m_C\gamma_{v_C} v_C + m_D\gamma_{v_D} v_D) = 0 \quad ,$$

$$\left(m_A\gamma_u\gamma_{v'_A}\left(u + v'_A\right) + m_B\gamma_u\gamma_{v'_B}\left(u + v'_B\right)\right) - \left(m_C\gamma_u\gamma_{v'_C}\left(u + v'_C\right) + m_D\gamma_u\gamma_{v'_D}\left(u + v'_D\right)\right) = 0 \quad ,$$

$$\gamma_u\left(\left(m_A\gamma_{v'_A} v'_A + m_B\gamma_{v'_B} v'_B\right) - \left(m_C\gamma_{v'_C} v'_C + m_D\gamma_{v'_D} v'_D\right)\right) + u\gamma_u\left(\left(m_A\gamma_{v'_A} + m_B\gamma_{v'_B}\right) - \left(m_C\gamma_{v'_C} + m_D\gamma_{v'_D}\right)\right) = 0 \quad .$$

Using the Four-Momentum Transformation Law

By using the transformation law for the Four-Momentum the calculation is simpler:

$$\boxed{p = \gamma_u\left(p' + u\varepsilon'/c^2\right)} \quad .$$

The immediate result is:

$$(p_A + p_B) - (p_C + p_D) = 0 \quad ,$$

$$\left(\left(p'_A + p'_B\right) - \left(p'_C + p'_D\right)\right) + \frac{u}{c^2}\left(\left(\varepsilon'_A + \varepsilon'_B\right) - \left(\varepsilon'_C + \varepsilon'_D\right)\right) = 0 \quad .$$

60.01.02.04 Galilei Transformation and Relativistic Linear Momentum Non-Conservation

One might also show, by direct calculation, that **it is false, in general, that: using Galilei Transformation , if relativistic linear momentum, $\mathbf{p} = m\gamma\mathbf{v}$, is conserved in one inertial Reference Frame it is conserved in another inertial Reference Frame**.

Obviously, the statement will be true in the **Galilei-Newton Mechanics** low-velocity limit, but it is not true in general.

60.01.03 Relativistic Energy

This § is referenced at pages:
[2451, 2452, 2452]

Let us now try to figure out what the meaning of ε is.

- First of all, recall the strict link between the space and time components of the Four-Momentum induced by Lorentz invariance, as discussed in section § 60.01.02.
- Consider the low-energy limit. When the object is at rest we obtain the **rest energy**, which is non-zero, for $m > 0$:

$$\boxed{\varepsilon_0 = mc^2 \qquad \Leftrightarrow \qquad \mathbf{p} = 0 \qquad \Leftrightarrow \qquad \mathbf{v} = 0} \quad .$$

  The remainder of the total energy when the rest energy is subtracted is defined as the kinetic energy, as it reduces to the kinetic energy of classical mechanics at low velocities:

$$\boxed{\mathcal{K} \equiv \varepsilon - mc^2 \qquad \mathcal{K} \simeq \frac{1}{2}mv^2 + \frac{3mv^4}{8c^2} + \dots \qquad \text{for low velocities } |\mathbf{v}| \ll c} \quad . \tag{60.01.11}$$

All in all, it makes sense to consider the zero component of the Four-Momentum as the relativistic extension of the **total energy** of a point-mass.

60.01.03.01 Massless Particles

Note that in **Galilei-Newton Mechanics** there is no such thing as a massless particle. In fact for any massless system **Galilei-Newton Mechanics** predicts that the resultant of the external force must be zero and therefore the force applied by the massless system onto any other system is zero as well. Moreover, it has zero linear momentum and zero kinetic energy. Therefore a massless system in **Galilei-Newton Mechanics** has no interactions with nothing else.

Conversely in relativistic mechanics one has a meaningful:

$$\boxed{m = 0 \implies \{ \qquad \varepsilon = pc \qquad |\mathbf{v}| = c \qquad \}} \quad .$$

The question arises what distinguishes two photons with different energies. Relativity provides no answer. Quantum physics shows that they differ in the frequency of the associated wave, which, in the case of photons, relates to the wavelength of the ElectroMagnetic radiation.

60.01.03.02 (Non/Ultra)-Relativistic Limits

The non-relativistic and extreme relativistic limits are:

$$\begin{cases} \boxed{p \ll mc \implies \varepsilon \simeq mc^2 + \dfrac{p^2}{2m}} \\ \boxed{p \gg mc \implies \varepsilon \simeq pc + \dfrac{m^2c^3}{2p}} \end{cases} \quad , \qquad .$$

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60.01.03.03 On Relativistic Energy

This § is referenced at pages:
[1315]

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So far what has been described in section § 60.01 may be just a useful notation, and nothing more with any physics content.

If all the velocities are so small that the non-relativistic approximation is valid, then the classical laws of conservation of mass and linear momentum are equivalent to the conservation of the time and space parts of Four-Momentum.

This suggests the use of Four-Momentum as an extension of the classical linear momentum.

60.01.03.04 Conservation of Energy

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The total energy of an isolated system is conserved. Total energy is conserved in all interactions: elastic and inelastic collisions as well as creations, transformations, decays and annihilations of particles. In contrast the total kinetic energy of a system calculated using **Galilei-Newton Mechanics** for low-speed is conserved only for elastic collisions, which are in fact defined as collisions in which kinetic energy is conserved. In inelastic collisions kinetic energy transforms into other forms of energy, such as internal energy, chemical energy, potential energy or other forms of energy. In **Galilei-Newton Mechanics** each of these forms of energy must be treated separately and conservation of energy must be invoked as a separate principle, as something beyond the **Galilei-Newton Mechanics** analysis of mechanical energy.

60.01.04
Non Isolated Systems of point particles only

The non invariance of simultaneity poses problems, except for point particles interacting, essentially, by contact interactions only. In fact, in this case, the interactions are applied essentially at the same point in space and simultaneity is not an issue. The other case which can be treated is that of constant interactions. Read § 60.04.12.

In other cases, field theory must be used.

60.01.04.01 First and second Cardinal Equation for Relativistic Point particles

The First **Cardinal Equations** are still valid for point masses, provided the relativistic expressions of linear momentum and energy are used:

$$\pi \equiv \mathbf{v}\cdot\mathbf{f} = \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \quad , \tag{60.01.12}$$

→ 2475

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \quad , \tag{60.01.13}$$

→ 2475

$$\{\pi/c, \mathbf{f}\} = \frac{\mathrm{d}}{\mathrm{d}t}\{\epsilon/c, \mathbf{p}\} = \frac{\mathrm{d}p}{\mathrm{d}t} \quad . \tag{60.01.14}$$

The Second **Cardinal Equations** are still valid for point masses and pole at the point itself, provided the relativistic expressions are used:

$$\gamma = \frac{\mathrm{d}s}{\mathrm{d}t} \quad . \tag{60.01.15}$$

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60.01.04.02 Relation Between Force and Acceleration

This § is referenced at pages:
[2471]

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\left(\gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{v}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) = m\left(\gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{v}\frac{\gamma^3 v}{c^2}\frac{\mathrm{d}v}{\mathrm{d}t}\right) = m\left(\gamma\mathbf{a} + \mathbf{v}\frac{\gamma^3 v}{c^2}\frac{\mathrm{d}v}{\mathrm{d}t}\right) \quad . \qquad (60.01.16)$$

→ 2457

The component perpendicular to the velocity, which has contributions from the first term only in (60.01.16), is:

$$\mathbf{f}_\perp = m\gamma\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)_\perp = m\gamma\mathbf{a}_\perp \quad .$$

The component parallel to the velocity, which has contributions from both terms in (60.01.16), is (read § 60.04.08):

$$\mathbf{f}_\parallel = m\gamma^3\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)_\parallel = m\gamma^3\mathbf{a}_\parallel \quad .$$

Therefore:

$$\mathbf{f}_\parallel = \left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_\parallel = m\gamma^3\mathbf{a}_\parallel \quad ,$$

$$\mathbf{f}_\perp = \left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_\perp = m\gamma\mathbf{a}_\perp \quad .$$

→ 2475

Note that the inertia to forces parallel/perpendicular to the velocity are different.

The relation can also be written as:

$$\mathbf{f} = m\gamma\left(\mathbf{a} + \gamma^2\frac{\mathbf{v}\,(\mathbf{v}\cdot\mathbf{a})}{c^2}\right) \quad .$$

The relation between force and acceleration is not that simple as it is in non-relativistic mechanics. Note that, even though it is caused by a force, **acceleration does not generally point in the direction of the force**: at high speeds the force is not, in general, parallel to the acceleration.

Non-relativistic limit

For slow motions, the second term in equation (60.01.16) is negligible with respect to the first one and $\gamma \to 1$, so the expression reduces to the non-relativistic one.

60.01.04.03 Transformation Laws for Force and Power

Let $\mathbf{v}$ be the velocity of the particle with respect to the Reference Frame $\mathcal{I}$ and $\mathbf{v}'$ the velocity of the particle with respect to the Reference Frame $\mathcal{I}'$. Let and $\mathbf{u}$ the velocity of the **Lorentz Transformation** from the Reference Frame $\mathcal{I}$ to the Reference Frame $\mathcal{I}'$.

Let:

$$\mathbf{u} = c\boldsymbol{\beta} \qquad \gamma \equiv \gamma_u \quad .$$

The transformation law for force and power can be derived in the same way as the transformation law for the velocities. In case $\beta \parallel \hat{\mathbf{e}}_3$ one finds:

$$\pi' = \frac{\pi - u_z f_z}{1 - u_z v_z/c^2} \quad ,$$

$$f'_z = \frac{f_z - \beta\pi/c}{1 - \beta v_z/c} \quad ,$$

$$f'_{x|y} = \frac{f_{x|y}}{\gamma\,(1 - \beta v_z/c)} \quad .$$

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In the general case:

$$\boxed{\pi' = \frac{\pi - c\,\boldsymbol{\beta}\cdot\mathbf{f}}{1 - \boldsymbol{\beta}\cdot\mathbf{v}/c}} \quad , \tag{60.01.17}$$

$$\boxed{\mathbf{f}' = \frac{\mathbf{f} + \boldsymbol{\beta}\left(\dfrac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{f}) - \gamma\dfrac{\pi}{c}\right)}{\gamma\,(1 - \boldsymbol{\beta}\cdot\mathbf{v}/c)}} \quad . \tag{60.01.18}$$

60.01.04.04 Minkowski Four-Force

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Similarly to what is done when passing from ordinary velocity to four-velocity one can formally introduce the quantities:

$$\boxed{\mathbf{f}_{\mathtt{M}} \equiv \gamma_v \mathbf{f} \qquad \pi_{\mathtt{M}} \equiv \gamma_v \pi = \mathbf{v}\cdot\mathbf{f}_{\mathtt{M}}} \quad .$$

The so-called Minkowski power and force, $\pi_{\mathtt{M}}$ and $\mathbf{f}_{\mathtt{M}}$,

$$\boxed{f^{\mu}_{\mathtt{M}} \equiv \{\pi_{\mathtt{M}}/c, \mathbf{f}_{\mathtt{M}}\}} = \frac{\mathrm{d}p}{\mathrm{d}\tau} \quad ,$$

is a four-vector.

60.01.04.05 On the Practical Use

The Minkowski four-force is easier to handle than ordinary power-force when doing **Lorentz Transformation** .

However, the Minkowski four-force is a hybrid quantity, relating Four-Momentum, measured in the Reference Frame of the observer, to the proper time, measured in the Rest Frame of the system. Therefore, it is often less useful than ordinary power-force, in practice, because the latter relates quantities measured in the Reference Frame of the observer.

60.01.05
Four-Momentum for Isolated and Non-Isolated System

Physics lies in the experimental fact that Four-Momentum is conserved for every isolated system, that this is a Lorentz-Invariant property and that changes of Four-Momentum are related to the total external force.

60.01.05.01 Dynamical Invariants

Relativistic invariants often lead to surprisingly simple calculations, avoiding long, tedious and error prone algebra.

If any question is of such a nature that its answer will be the same in any Inertial Reference Frame , then it must be possible to formulate its answer entirely using the invariants quantities which can be formed with the available Lorentz tensors.

One can then find the answer in a particular Inertial Reference Frame , freely chosen, in such a way that the answer is a simple one. One then looks at how suitable invariants quantities appear in that system and expresses the answer in term of invariants quantities. This gives the unique general answer. In fact if a Lorentz Invariant expression assumes the correct value in any specific Inertial Reference Frame , then it assumes that same value in any other Inertial Reference Frame , because it is invariant.

Refer to a single particle with Four-Momentum $p = (\varepsilon, \mathbf{p})$. The following are invariants quantities.

$$p^2 = \varepsilon^2 - \mathbf{p}\cdot\mathbf{p} = m^2 \qquad \text{the mass of a particle} \quad ,$$

$$\mathrm{d}\mathbf{p}/\varepsilon \qquad \text{the phase-space factor} \quad .$$

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For any given system of particles, $k = 1, 2, \ldots n$, having four-momenta p_k, the invariant mass of the system is defined as:

$$M_0^2 \equiv \left(\sum_k p_k \right)^2 \equiv s \quad , \tag{60.01.19}$$

where the commonly used Mandelstam variable s has been introduced. Read § 58-022 to check that the definition well-posed, as the square of the sum of time-like four-vectors is a time-like four-vector.

The invariant mass of a system of particles corresponds to the total energy in the [glo:CMRF]Center-of-Momenta Reference It is also the total energy available to produce new particles.

60.01.06
Energy and Linear Momentum and Relativistic Four-Momentum

In **Galilei-Newton Mechanics** energy is a scalar: invariant under change of the Coordinate System but not invariant under changes of Reference Frame; linear momentum is a vector. They are entirely different quantities. Energy remains energy and linear momentum remains linear momentum under **Galilei Transformation** . In relativistic physics energy and linear momentum are mixed by Lorentz transformations, the same way as time and space are mixed.

In **Galilei-Newton Mechanics** the total mass is always conserved and therefore there is no such thing as energy conversion into/from mass. However, in relativistic physics there is no such thing as mass conservation, but only energy conservation. This opens the way to energy conversion into/from rest mass.

60.01.06.01 Energy Versus Relativistic Mass

In the old times, in an attempt to keep the classical definition of linear momentum as the product of mass times velocity, it was said that mass changes with velocity (the **Relativistic Mass**).

Nowadays it is preferred to consider mass as a (relativistic invariant) intrinsic property of any specific system and assume that the expression of linear momentum as a function of velocity is different from the classical one. This approach perfectly fits the concept of linear momentum as a four-vector.

The name **Relativistic Mass** shall not be used.

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60.02 Zero-Momentum Rest Frame

The relativistic extension of the classical concept of the [glo:CMRF]Center-of-Momenta Reference Frame is the concept of **Zero-Momentum frame** (**Center of Momentum frame**).

60.02.01 Zero-Momentum Rest Frame- the General Case

Let us consider a pair of particles having, in a certain frame of reference, the four-momenta $(\varepsilon_1, \mathbf{p}_1)$ and $(\varepsilon_2, \mathbf{p}_2)$. The derivation will show that the extension to the general case of an arbitrary number of particles is straightforward.

The [glo:CMRF]Center-of-Momenta Reference Frame is defined as that frame where the total 3-momentum of the system of particles is zero. Let us call $\mathbf{u}$ the velocity of [glo:CMRF]Center-of-Momenta Reference Frame with respect to the current frame and let us denote with * the quantities in the [glo:CMRF]Center-of-Momenta Reference Fram
Let: $E \equiv \varepsilon_1 + \varepsilon_2$ and $\mathbf{P} \equiv \mathbf{p}_1 + \mathbf{p}_2$. The equation to solve in order to find $\mathbf{u}$ is thus:

$$\mathbf{p}_1^\star + \mathbf{p}_2^\star = \mathbf{0} \quad .$$

From equation (58.09.11) and equation (58.09.12) one has:

$$(\mathbf{p}_1 + \mathbf{p}_2) + \gamma \mathbf{u} \left(\frac{\gamma}{\gamma + 1} (\mathbf{u} \cdot (\mathbf{p}_1 + \mathbf{p}_2)) - (\varepsilon_1 + \varepsilon_2) \right) = \mathbf{0} \quad , \qquad \rightarrow 2460$$

$$\mathbf{P} + \gamma \mathbf{u} \left(\frac{\gamma}{\gamma + 1} (\mathbf{u} \cdot \mathbf{P} - E) \right) = \mathbf{0}$$

The equation can be solved for $\mathbf{u}$ in the following way. First take the scalar product of the equation with $\mathbf{u}$ and use the identity

$$\boxed{\gamma_u = 1 + \frac{u^2 \gamma_u^2}{\gamma_u + 1}}$$

to obtain

$$u \cdot P = u^2 E \quad .$$

Secondly use the latter equation in (60.02.01) to obtain the final result:

$$\boxed{\mathbf{u} = \frac{\mathbf{P}}{E}} \quad . \qquad (60.02.01) \quad \rightarrow 2460$$

One then immediately finds that:

$$\boxed{\gamma_u \equiv \left(1 - u^2\right)^{-1/2} = \left(1 - \left(\frac{\mathbf{P}}{E}\right)^2\right)^{-1/2} = \frac{E}{\sqrt{E^2 - \mathbf{P}^2}} = \frac{E}{\sqrt{s}}} \quad . \qquad (60.02.02) \quad \rightarrow 2460$$

Note that equations (60.02.01), (60.02.02) are consistent with the single particle case because $\sqrt{s}$ is actually the invariant mass of the particle system.

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60.03 Energy-LinMomentum Tensor and Boost-AngMomentum Tensor

This § is referenced at pages:
[1653, 1653, 2493]

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60.03.01 Balance of Extensive Quantities in Relativistic Form

This § is referenced at pages:
[1922, 2493]

Read § 21.03.

Laws describing the balance of some extensive quantity (read § 21.03) are expressed, in relativistic covariant form, by an equation of type:

$$\boxed{\partial_\alpha T^{\alpha\beta\cdots}_{\cdot\cdot\cdot} = G^{\beta\cdots}_{\cdot\cdot} \qquad \text{balance law}} \quad .$$

Conservation laws, in particular, have the above form with $G = 0$:

$$\boxed{\partial_\alpha T^{\alpha\beta\cdots}_{\cdot\cdot\cdot} = 0 \qquad \text{conservation law}} \quad .$$

60.03.02 Energy-LinMomentum Tensor

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The discussion in this section is just an heuristic introduction.

The covariant Energy-LinMomentum tensor is derived from field-theoretical considerations. Only an heuristic discussion and a few important facts are summarised here. The references are required for a more serious approach.

Relativistic physics shows that energy and linear momentum goes together as the component of a four-vector. Therefore either the conservation of energy/linear momentum (in the case of an isolated system) and the balance law of energy/linear momentum (in the case of a non-isolated system) must be packed together for a relativistically covariant description.

The covariant Energy-LinMomentum tensor, is built from:

- the energy volume density, u;
- the energy flux, $\mathbf{S}$;
- the linear momentum volume density, $\mathbf{G}$;
- the three-dimensional stress-tensor, $\Xi^{\cdot}_{kj}$, representing the flux of linear momentum: flux along the direction k of the component j of linear momentum.

For each component of the Four-Momentum one can write a balance (conservation) law in term of

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their volume density and their flux:

$$\partial_t u + \mathbf{div}\mathbf{S} = \ldots (= 0) \quad ,$$
$$\partial_t(\,\mathbf{G}\cdot\mathbf{e}_x) + \frac{\partial \Xi^{\cdot\cdot}_{kx}}{\partial x^k} = \ldots (= 0) \quad ,$$
$$\partial_t\big(\,\mathbf{G}\cdot\mathbf{e}_y\big) + \frac{\partial \Xi^{\cdot\cdot}_{ky}}{\partial x^k} = \ldots (= 0) \quad ,$$
$$\partial_t(\,\mathbf{G}\cdot\mathbf{e}_z) + \frac{\partial \Xi^{\cdot\cdot}_{kz}}{\partial x^k} = \ldots (= 0) \quad .$$

This is a parallel development to the one for the ElectroMagnetic fields only (see section § 40).

One therefore writes:

$$T^{\alpha\beta}_{\cdot\cdot} \equiv \begin{pmatrix} u & \mathbf{S}/c \\ c\mathbf{G} & \Xi^{\cdot\cdot}_{kj} \end{pmatrix} \quad ,$$

which can be shown to be a Lorentz tensor.

Note the usual dimensional convention has been used to take as it is the space-space part of the tensor and modify the other components accordingly. In fact:

$$[\Xi] = [\mathrm{u}] = \mathrm{J/m^3} \qquad [\mathbf{G}] = (\mathrm{J/m^3})/\mathrm{c} \qquad [\mathbf{S}] = (\mathrm{J/m^3})\mathrm{c} \quad .$$

60.03.02.01 Symmetry of the Energy-LinMomentum Tensor

In general: whenever there is a flow of energy of any kind, with speed of the energy flow equal to v, the energy flowing per unit time across a unit surface perpendicular to the flow direction, when multiplied by c^{-2} gives the linear momentum per unit volume in space.

In fact the relativistic relation between energy and linear momentum is:

$$\mathbf{v} = \frac{c^2\mathbf{p}}{\varepsilon} \quad .$$

First Demonstration

The energy flux in direction k and density of the k component of the linear momentum are, respectively,

$$\frac{\Delta\varepsilon}{\Delta V} v_k \qquad \frac{\Delta p_k}{\Delta V} \quad .$$

The equality, to within a constant factor c^2, comes from:

$$\frac{\Delta\varepsilon}{\Delta V} v_k = \frac{\Delta\varepsilon}{\Delta V}\frac{c^2\Delta p_k}{\Delta\varepsilon} = \frac{\Delta p_k}{\Delta V} c^2 \quad ,$$

that is:

$$\boxed{\mathbf{S} = c^2\mathbf{G}} \quad .$$

This allows to write the covariant Energy-LinMomentum tensor as:

$$T^{\alpha\beta}_{\cdot\cdot} \equiv \begin{pmatrix} u & \mathbf{S}/c \\ \mathbf{S}/c & \Xi^{\cdot\cdot}_{kj} \end{pmatrix} = T^{\alpha\beta}_{\cdot\cdot} \equiv \begin{pmatrix} u & c\mathbf{G} \\ c\mathbf{G} & \Xi^{\cdot\cdot}_{kj} \end{pmatrix} \quad .$$

The covariant Energy-LinMomentum tensor is thus symmetrical, or, better to say, the Energy-LinMomentum tensor of an isolated system can always be cast in a symmetrical form. ∎

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Second Demonstration

A slightly different derivation is the following.

Assume that energy and linear momentum are carried by particles having energy ε and linear momentum $\mathbf{p}$ while moving with speed $\mathbf{v}$.

The energy flux, in terms of the number density, $\rho_{\mathcal{N}}$, is given by:

$$\mathbf{S} = \rho_{\mathcal{N}} \varepsilon \mathbf{v} \quad .$$

The linear momentum density, in terms of the number density, $\rho_{\mathcal{N}}$, is given by:

$$\mathbf{G} = \rho_{\mathcal{N}} \mathbf{p} \quad .$$

The relativistic relation between energy and linear momentum

$$\mathbf{v} = \frac{c^2 \mathbf{p}}{\varepsilon} \quad ,$$

then implies the desired relation that is:

$$\boxed{\mathbf{S} = c^2 \mathbf{G}} \quad .$$

60.03.02.02 Balance Equation for the Energy-LinMomentum Tensor

As in the case of the ElectroMagnetic fields, section § 40), the balance equation for a system with symmetrical Energy-LinMomentum tensor $\mathbb{T}$ is given by:

$$\partial_\alpha T^{\alpha\beta}_{\cdot\cdot} = S^\beta \quad ,$$

where the four-vector S^β is the generic source to changes in the Four-Momentum of the system, and originating externally of the system.

The Energy-LinMomentum tensor is defined only to within a divergence-less tensor of second rank. This is similar to the non-uniqueness of the Poynting vector.

The non-uniqueness of the Energy-LinMomentum tensor allows to symmetrise a non symmetrical tensor.

60.03.02.03 Conservation of the Energy-LinMomentum Tensor of an Isolated System

An isolated system has $S^\beta = 0$ so that the total Energy-LinMomentum tensor of an isolated system is always conserved:

$$\partial_\alpha T^{\alpha\beta}_{\cdot\cdot} = 0 \quad .$$

The conserved quantities that can be derived from the Energy-LinMomentum tensor are:

$$\boxed{P^\mu \equiv \iiint T^{\mu 0} \, \mathrm{d}V_{\mathbf{x}} \qquad \text{total Four-Momentum: four independent components}} \quad ,$$

$$\boxed{\mathcal{E} \equiv P^0 \equiv \iiint T^{00} \, \mathrm{d}V_{\mathbf{x}}} \quad ,$$

$$\boxed{P^k \equiv \iiint T^{k0} \, \mathrm{d}V_{\mathbf{x}}} \quad ,$$

which obviously are energy and linear momentum.

60.03.02.04 Mass Four-Velocity and Center-Of-Mass of the System

This § is referenced at pages:
[2473]

Define the invariant mass of the system as:

$$\boxed{M \equiv \sqrt{P^\mu P_\mu} \geq 0} \quad .$$

Define the four-velocity of the system (the Center-Of-Mass velocity) as:

$$\boxed{U^\mu \equiv \frac{P^\mu}{M} \qquad \text{for } M > 0} \quad .$$

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Define the Center-Of-Mass (aka **Center of Energy**) of the system as:

$$\boxed{\mathbf{X}_{\text{E}} \equiv \mathbf{X}_{\text{CM}} \equiv \frac{1}{\mathcal{E}} \iiint \mathbf{x} T^{00}_{\cdot\cdot} \, \mathrm{d}x} \quad .$$

The basic relativistic relation is thus:

$$\boxed{\frac{\mathrm{d}\mathbf{X}_{\text{E}}}{\mathrm{d}t} = c^2 \frac{\mathbf{P}}{\mathcal{E}}} \quad .$$

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60.03.03 Boost-AngMomentum Tensor

The Boost-AngMomentum Tensor is defined as:

$$\boxed{M^{\gamma\alpha\beta} \equiv x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma} = -M^{\gamma\beta\alpha} \qquad \text{for } T^{\alpha\beta}_{\cdot\cdot} = T^{\beta\alpha}_{\cdot\cdot}} \quad ,$$

and it is by definition anti-symmetric in the second-third indexes.

The definition can be generalized by considering the Angular Momentum with respect to an arbitrary origin, the time-space event a^μ:

$$\boxed{M^{\gamma\alpha\beta}[a] \equiv (x^\alpha - a^\alpha) T^{\beta\gamma} - \left(x^\beta - a^\beta\right) T^{\alpha\gamma} = -M^{\gamma\beta\alpha} \qquad \text{for } T^{\alpha\beta}_{\cdot\cdot} = T^{\beta\alpha}_{\cdot\cdot}} \quad .$$

If the Energy-LinMomentum tensor is conserved and symmetrical has:

$$\partial_\gamma M^{\gamma\alpha\beta} = 0 \quad .$$

The conserved quantities that can be derived from the Boost-AngMomentum Tensor are:

$$J^{\alpha\beta}_{\cdot\cdot} \equiv \iiint M^{0\alpha\beta} \, \mathrm{d}V_{\mathbf{x}} = -J^{\beta\alpha}_{\cdot\cdot} \qquad \text{total Angular Momentum: six independent components} \quad (60.03.01)$$

→ 2464

$$J^{0i}_{\cdot\cdot} \equiv tP^i - \iiint x^i T^{00}_{\cdot\cdot} \, \mathrm{d}V_{\mathbf{x}} = -J^{i0}_{\cdot\cdot} \qquad \text{boost-related part: three independent components} \quad (60.03.02)$$

$$J^{ij}_{\cdot\cdot} \equiv \iiint \left(x^i T^{j0}_{\cdot\cdot} - x^j T^{i0}_{\cdot\cdot}\right) \, \mathrm{d}V_{\mathbf{x}} = -J^{ji}_{\cdot\cdot} \qquad \text{true Angular Momentum: three independent components}$$

The Angular Momentum defined by equation (60.03.01) transforms, under translation of the time-space coordinates by the amount a^α, as:

$$x^\gamma \longrightarrow \hat{x}^\gamma \equiv x^\gamma + a^\gamma \implies J^{\alpha\beta}_{\cdot\cdot} \longrightarrow \hat{J}^{\alpha\beta}_{\cdot\cdot} \equiv J^{\alpha\beta}_{\cdot\cdot} + a^\alpha P^\beta - a^\beta P^\alpha \quad .$$

60.03.03.01 Intrinsic Boost-AngMomentum Tensor (Spin)

The intrinsic Angular Momentum is defined as the angular momentum about any event on the world line of the Center-Of-Mass.

The part of the Angular Momentum which is independent of translation of the time-space coordinates is extracted via the spin four-vector as follows.

The spin four-vector is defined as the four-vector whose components in the Rest Frame is

$$\{0, \mathbf{S}\} \qquad \mathbf{S} \equiv \iiint (\mathbf{x} - \mathbf{x}_{\text{CM}}) \wedge \mathbf{G} \, \mathrm{d}x \quad .$$

It turns out that:

$$S_\alpha \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} J^{\beta\gamma} U^\delta \qquad \text{for } M > 0 \quad ,$$

$$S_\alpha \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} J^{\beta\gamma} P^\delta \qquad \text{for } M = 0 \quad .$$

The spin four-vector has only three independent components because:

$$S_\mu U^\mu = 0 \quad .$$

→ 2478

In the system Rest Frame (CM frame) one has:

$$U^\mu = \{1; \mathbf{0}\} \qquad \text{in the Rest Frame} \quad ,$$

and therefore

$$S_0 = 0 \quad ,$$
$$S_1 = J^{23} \quad ,$$
$$S_2 = J^{31} \quad ,$$
$$S_3 = J^{12} \quad .$$

This shows that the spin four-vector represents the internal/intrinsic angular momentum of a system.

Decomposition of Angular Momentum Into Intrinsic and Orbital Parts

If one picks an arbitrary event, a, whose perpendicular displacement from the Center-Of-Mass world line is Y^α, so that

$$Y^\alpha U_\alpha = 0 \quad ,$$

→ 2478

then the angular momentum with respect to a is:

$$J^{\mu\nu}_{\cdot\cdot} = U_\alpha S_\beta \epsilon^{\alpha\beta\mu\nu}_{\cdot\cdot\cdot\cdot} + Y^\mu P^\nu - Y^\nu P^\mu \quad .$$

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60.04 Examples and Physical Applications

60.04.01

✓ Kinematics of Ultra-Relativistic Particles at LEP

The Large Electron-Positron (LEP) Collider operated at CERN from year 1989 to 2000. In the first phase of operation of the Collider electrons and positrons were colliding head-on, with opposite momenta, at a Center-Of-Mass-energy of $\sqrt{s} \simeq 92$ GeV. Determine the velocity of the electron/positron. If the energy is known with an uncertainty of $\Delta\varepsilon$ determine the uncertainty in the velocity.

WORKED SOLUTION

The relation between energy and velocity is given by the γ:

$$v = c\sqrt{1-\frac{1}{\gamma^2}} \simeq c\left(1-\frac{1}{2\gamma^2}\right) \quad .$$

In this problem one has:

$$\gamma = \frac{\varepsilon}{m_e c^2} = \frac{\sqrt{s}}{2m_e c^2} = 9.0{\cdot}10^4 \quad ,$$

and therefore:

$$1-\frac{v}{c} = 6.2{\cdot}10^{-11} \quad .$$

The error propagation gives:

$$\beta[\gamma] = \sqrt{1-\frac{1}{\gamma^2}} \Longrightarrow |\Delta\beta| = \frac{1}{\gamma^3\sqrt{1-\frac{1}{\gamma^2}}}|\Delta\gamma| = \frac{1}{\gamma^3\beta}|\Delta\gamma| \quad .$$

In the current case $\beta \simeq 1$ and therefore:

$$|\Delta\beta| = \frac{1}{m_e c^2\gamma^3}|\Delta\varepsilon| = 2.7{\cdot}10^{-15}|\Delta\varepsilon| \simeq \frac{|\Delta\beta|}{\beta} \quad ,$$

the latter equation coming from the fact that $\beta \simeq 1$.

60.04.02

✓ LHC Hadron Collider Parameters

The Large Hadron Collider (LHC) at CERN will collide head-on two proton beams at $\varepsilon = 7$ TeV $\oplus$ 7 TeV. Every beam is made of $n_b = 2808$ bunches of protons ($m = 938$ GeV/c^2) and every bunch of protons has $n_p = 1.15{\cdot}10^{11}$ protons. The total length of the collider, which is approximately a circular one, is $L = 26.659$ km.

1. Calculate the gamma factor and velocity of the protons.

WORKED SOLUTION

$$\gamma = \frac{\varepsilon}{mc^2} = 7.5{\cdot}10^3 \gg 1 \qquad \text{the protons have a very high } \gamma\text{: they are ultra-relativistic} \quad ,$$

$$\frac{1}{\gamma} = \sqrt{1-\beta^2} = \sqrt{1-\beta}\sqrt{1+\beta} \simeq \sqrt{2\,(1-\beta)} \Longrightarrow \beta = 1-\frac{1}{2\gamma^2} = 1-9.0{\cdot}10^{-9} \quad .$$

2. Determine the current.

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WORKED SOLUTION

$$\nu_0 = \frac{v}{L} \simeq \frac{c}{L} = 1.1253{\cdot}10^4 \text{ Hz} \quad ,$$
$$I = q\nu_0 n_p n_b = 0.58 \text{ A} \quad .$$

3. Determine the average magnetic field required to keep the protons in the orbit.

WORKED SOLUTION

$$p = q\rho B \quad ,$$
$$\rho = \frac{L}{2\pi} = 4.24 \text{ km} \quad ,$$
$$\gamma \gg 1 \implies \varepsilon \simeq cp \quad ,$$
$$B = \frac{7{\cdot}10^{12} \text{ eV}/c}{1.6{\cdot}10^{-19} 4.24{\cdot}10^3 \text{ C}\cdot\text{m}} = \frac{7{\cdot}10^{12} \langle\!\langle e \rangle\!\rangle SI/c \text{ J}}{\langle\!\langle e \rangle\!\rangle SI \langle\!\langle \rho \rangle\!\rangle SI \text{ Cm}} = 5.5 \text{ T} \quad .$$

4. Determine the total energy stored in the proton beams.

WORKED SOLUTION

$$E = 2n_p n_b \varepsilon = 0.7 \text{ GJ} \quad .$$

The stored energy can melt half-ton of Copper and corresponds to the kinetic energy of a Jumbo (bare mass equal to 181 tons) moving at 314 km/h.

5. Compare the available energy in the Center-Of-Mass ($\sqrt{s} = 2\varepsilon = 14$ TeV) with the one that would be available by sending a proton beam at $\varepsilon' = 14$ TeV against a fixed target.

WORKED SOLUTION

$$\sqrt{s} = \sqrt{2m_p^2 + 2m_p\varepsilon'} \simeq \sqrt{2m_p\varepsilon'} = 0.16 \text{ TeV} \quad .$$

60.04.03

☒ A Relativistic Totally Inelastic Collision

Two objects, each one with mass m, collide head-on with equal and opposite speeds and stick together.

1. If the speed is a typical speed of a gun projectile, $v = 300$ m/s $= 1{\cdot}10^{-6}c$, what is the relative increase of mass of the composite lump with respect to the sum of the masses of the two projectiles?
   Use:
$$\gamma - 1 \simeq \frac{1}{2}\left(\frac{v}{c}\right)^2 \qquad \text{for } v \ll c \quad .$$
2. If the speed is $v = 0.6c$, what is the mass of the composite lump?

WORKED SOLUTION

Conservation of linear momentum is trivial.
Conservation of energy gives:

$$2m\gamma c^2 = Mc^2 \quad .$$

The final mass is larger than the sum of the initial masses: mass is not conserved; energy is conserved. Kinetic energy was converted into rest energy, so mass increased.

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In a classical analysis one says that kinetic energy has been converted into internal energy: the composite object is hotter than the two original colliding objects. This is true also in the relativistic analyses. Internal energy is the sum of the random kinetic and potential energies as well as rest masses of all the particles, atoms and molecules of the objects. Relativity tells us that these microscopic energies are represented in the mass of the object: a hot object is heavier than a cold object as well as a compressed spring is heavier than a relaxed spring. This is normally not a large macroscopic quantity as internal energy, $\mathcal{U}$, contributes an amount $\mathcal{U}/c^2$ to the mass, and c^2 is a very large number when compared to our typical units of measure. Therefore such kind of relativistic effects are hardly observable in the macroscopic world. However, in the realm of elementary particles the effects can be very striking. Nuclear energy production, either by fission or fusion, rely on these kind of phenomena.

1. $$\frac{M-2m}{2m} = 0.5\cdot 10^{-12} \quad .$$
2. $$2m\gamma c^2 = Mc^2 \implies M = 2.5m > (m+m) \quad .$$

60.04.04

☒ Creation of a composite particle by fusion

It is an example of conversion of energy into mass.

Si abbia una particella di massa $m_0 = 1$ GeV/c^2 ed energia cinetica $k = 2m_0c^2$. La particella interagisce con una particella fissa nel Reference Frame del laboratorio di massa $2m_0$.

- Le due particelle formano una stato legato: derminarne la massa e l'impulso.
- What the process looks like rolling time backwards? It would be an example of conversion of mass into energy.

WORKED SOLUTION

Occorre anzitutto notare che la massa della particella finale non è $3m_0$, cioè la somma delle masse di riposo, perchè la massa tiene conto anche dell'energia di legame dello stato legato (come nei nuclei e negli atomi).

La conservazione dell'energia fornisce per l'energia dello stato finale:

$$\varepsilon/c^2 = (m_0 + 2m_0) + 2m_0 = 5m_0 \quad .$$

L'impulso del proiettile iniziale vale invece:

$$|\mathbf{p}_0| = c\sqrt{(3m_0)^2 - m_0^2} = 2.8m_0c = 2.8 \text{ GeV/c} \quad .$$

Per la conservazione dell'impulso questo è anche il valore dell'impulso dello stato finale. La massa dello stato finale vale dunque:

$$M = \sqrt{(5m_0)^2 - 8m_0^2} = 4.1m_0 = 4.1 \text{ GeV/c}^2 \quad .$$

This kind of process can be seen, for instance, as the time-reversed nuclear alpha decay.

60.04.05

☒ Neutral Pion at Rest Decaying Into Two Photons

It is an example of conversion of mass into energy.

A neutral pion at rest (mass $M_{\pi^0} = 135\ \mathrm{GeV/c^2}$) decays into two massless photons. Determine the energy and linear momentum of the outgoing photons.

In this process pure mass is transformed into pure energy.

60.04.06

Einstein Box and the Inertia of Energy

The equivalence of energy and mass is such an important and striking result of Relativity that Einstein himself, after his derivation of the result, sought and found an alternative elementary physical line of reasoning that leads to the same conclusion. It is called Einstein box. The purpose of this thought experiment is to suggest that energy must have associated with it a certain inertial mass equivalent.

Consider the following thought experiment. One photon leaves one side of a freight-car, which is free to move on a rail, directed towards the other side. Due to linear momentum conservation, after the photon has started the freight-car recoils. The freight-car stops again when the photon is absorbed at the other side. However, the Center-Of-Mass seems to have moved for a massless photon. Assume that the freight-car moves at low velocity and therefore its dynamics can be described by the **Galilei-Newton Mechanics** .

A Simplified Solution

Assume the velocity of the freight-car is $|V| \ll c$ and the photon **mass-equivalent**, if any, is $m \ll M$, much less that the freight-car mass M.

Assume the photon has zero mass-equivalent. It follows:

$$p_\gamma = \frac{\varepsilon}{c} = -MV = M|V| > 0 \quad ,$$

$$\Delta Z = V\Delta t = \frac{VL}{c} = -\frac{p_\gamma L}{Mc} < 0 \quad .$$

The Center-Of-Mass has moved.

Now imagine that the photon carries some kind of **mass-equivalent**, m. It follows that in order to keep the Center-Of-Mass fixed one needs to have:

$$M\Delta Z + mL = 0 \quad ,$$

$$m = -\frac{M\Delta Z}{L} = \frac{p_\gamma}{c} = \frac{\varepsilon}{c^2} \quad .$$

The **mass-equivalent**, m, associated to the photon is therefore, $\frac{\varepsilon}{c^2}$.

A More Precise Solution

Assume the photon has zero mass-equivalent. It follows:

$$p_\gamma = \frac{\varepsilon}{c} = -MV = M|V| > 0 \quad ,$$

$$z_{\mathrm{CM}} = \frac{L}{2} \quad ,$$

$$\Delta t = \frac{L}{c - V} = \frac{L}{c + |V|} \quad ,$$

$$\Delta Z = V\frac{L}{c + |V|} < 0 \quad ,$$

$$z_{\mathrm{CM}}{}' = \frac{L}{2} + \Delta Z \quad .$$

Therefore the Center-Of-Mass seems to have moved.

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Now imagine that the photon carries some kind of **mass-equivalent**, m. It follows:

$$p_\gamma = \frac{\varepsilon}{c} = -MV = M|V| > 0 \quad ,$$

$$z_{\text{CM}} = \frac{ML/2}{m+M} \quad ,$$

$$\Delta t = \frac{L}{c-V} = \frac{L}{c+|V|} \quad ,$$

$$\Delta Z = V\frac{L}{c+|V|} < 0 \quad ,$$

$$z_{\text{CM}}{}' = \frac{M\left(\frac{L}{2}+\Delta Z\right)+m\,(L+\Delta Z)}{m+M} = z_{\text{CM}} + \Delta Z + L\frac{m}{m+M} \quad ,$$

$$z_{\text{CM}}{}' = z_{\text{CM}} \implies \Delta Z = -L\frac{m}{m+M} < 0 \quad ,$$

$$\implies \varepsilon = mc^2 \quad .$$

Therefore consistency is ensured by assuming the photon carries a mass equivalent to its energy (divided by c^2). In other words: even if the rest mass of the photon is zero its energy has the role of a mass for the calculation of the Center-Of-Mass position.

A simplified derivation assuming $m \ll M$ and $|V| \ll c$ gives the same result.

The transport of mass by the photon can be understood only as a new feature of the raaiation itself.

60.04.07

Charged Pion at Rest Decaying Into a Muon and a Neutrino

A charged pion at rest (mass M) decays into a muon (mass m) and a (massless) neutrino. Determine the energy of the outgoing muon in terms of the two masses.

Find the velocity of the muon.

WORKED SOLUTION

$$\varepsilon = c^2\frac{M^2+m^2}{2M} \quad .$$

60.04.08

✓ Component Parallel to the Velocity of the Force

This § is referenced at pages:
[2457]

$$va_\parallel \equiv \mathbf{v}\cdot\mathbf{a} = \mathbf{v}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}t} \quad ,$$

$a_\parallel = \dfrac{\mathrm{d}v}{\mathrm{d}t}$ as it is well-known from the representation in intrinsic coordinates of the acceleration ,

$$\mathbf{f}_\parallel \equiv m\left(\gamma\mathbf{a}+\mathbf{v}\frac{\gamma^3 v}{c^2}\frac{\mathrm{d}v}{\mathrm{d}t}\right)\cdot\frac{\mathbf{v}}{v} = m\left(\gamma a_\parallel + \frac{v^2\gamma^3}{c^2}a_\parallel\right) = m\gamma a_\parallel\left(1+\frac{v^2}{c^2}\gamma^2\right) = m\gamma^3 a_\parallel \quad ,$$

$$m\gamma^3\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)_\parallel = m\gamma^3\mathbf{a}_\parallel \quad .$$

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60.04.09

☒ Relativistic Motion From Rest Subject to a Constant Force

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©|J.G.Pereira *et al.*|Relativistic motion under constant force: velocity and acceleration behavior|*WEB - URL*|

An object of mass m is subject to a constant force $\mathbf{F} = F\hat{\mathbf{e}}_3$. Determine the law of motion:

1. when starting from rest at time $t = 0$;
2. when starting with $v_z[t = t_0] = v_0$.

WORKED SOLUTION

It can be solved by using the relations in section § 60.01.04.02.
Use:

$$\int \gamma^3 \, \mathrm{d}v = \gamma v \quad .$$

1. The law of motion is:

$$\alpha \equiv \frac{mc}{F} \qquad z[t] = c\sqrt{t^2 + \alpha^2} - \alpha c \implies v_z[t] = \frac{ct}{\sqrt{t^2 + \alpha^2}} \leq c \quad .$$

While in **Galilei-Newton Mechanics** a constant force gives a parabolic dependence of the position on time, in relativistic mechanics the position as a function of time is hyperbolic. The asymptote of the hyperbola $z[t] \approx c\,(t - \alpha)$ gives the limiting speed:

$$z[t] \approx ct \qquad \text{for very large times } t \gg \alpha \equiv \frac{mc}{F} \quad .$$

The short-time behavior reproduces the classical result, as it is a low-speed result:

$$z[t] \approx \frac{1}{2}\frac{c}{\alpha}t^2 = \frac{1}{2}\frac{F}{m}t^2 \qquad \text{for very short times } t \ll \alpha \equiv \frac{mc}{F} \quad .$$

Note that both energy and linear momentum increase indefinitely, but the speed never reaches the speed of light. In fact the linear momentum increase at a constant rate, as from the equation of motion, while the energy is always greater or equal to the linear momentum times the speed of light.

2. The solution is:

$$v[t] = \frac{c\,(\gamma_0 v_0 + (F\Delta t)/m)}{\sqrt{c^2 + (\gamma_0 v_0 + (F\Delta t)/m)^2}} \quad .$$

60.04.10

Motion Subject to a Constant Force Perpendicular to the Initial Velocity

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60.04.11
Motion Subject to a Force Always Perpendicular to the Velocity

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Read § 60-008.

60.04.12
Action-Reaction Principle in Relativistic Physics

This § is referenced at pages:
[2456]

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... Newton's third law does not, in general, extend to the relativistic domain. Indeed, if the two objects in question are separated in space, the third law is incompatible with the relativity of simultaneity. For suppose the force of A on B at some instant t is $+\mathbf{F}$, and the force of B on A at the same instant is $-\mathbf{F}$; then the third law applies in this Rest Frame.
But a moving observer will report that these equal and opposite forces occurred at different times; in his system, therefore, the third law is violated.
Only in the case of contact interactions, where the two forces are applied at the same physical point (and in the trivial case where the forces are constant) can the third law be retained.
...

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60.04.13
Hidden ElectroMagnetic LinMometum and ElectroMagnetic AngMometum

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 12.2.4||

Read also § 40.02.02.

It refers to microscopical mechanical momentum of moving particles, which must be included in the momentum balance together with the ones of ElectroMagnetic origin, also in case of static fields.

60.04.14
Relativistic Center of Mass in Field Theory

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|§ 12.2.4|*Center-Of-Energy*|

Read § 60.03.02.04 for the relativistic extension of the concept of Center-Of-Mass to Center-Of-Energy and the relation of the Center-Of-Energy with the total momentum, including the energy and momentum carried by the fields, in addition to those carried by particles.

60.04.15
Relativistic Center of Mass in Classical Field Theory With Spin

©|C.Lorcé|The relativistic center of mass in field theory with spin|*Eur. Phys. J. C (2018) 78:785* - *WEB - URL* |

60.04.16
Neutrino mass

Suppose that the neutrinos emitted by the supernova SN1987A were all generated and left the core during a period of one second. A large fraction of the twenty neutrinos detected $T = 160000$ years later were all observed in a period of $\Delta t = 2$ s, with a mean energy of about $\epsilon = 8$ MeV Make a rough estimate of an upper limit on the neutrino mass.

DRAFT - IN - PROGRESS

60.05 Exercises Problems and Physical Applications

60-001 ✓ Kinetic Energy and Galilei Transformation in Galilei-Newton Mechanics

Show that in **Galilei-Newton Mechanics** the transformation law for the kinetic energy of a system of point particles, assuming that the mass of a point particle is Galilei invariant, is:

$$\mathcal{K} = \mathcal{K}' + \frac{1}{2}Mu^2 + \mathbf{u}\cdot\mathbf{P} \quad .$$

Show, using the above relation, that a collision is elastic if and only if the total mass and total linear momentum are conserved.

WORKED SOLUTION

Denote with a prime the quantities after the collision:

$$\mathcal{K} = \mathcal{K}' + \frac{1}{2}Mu^2 + \mathbf{u}\cdot\mathbf{P} = \mathcal{K}'' + \frac{1}{2}M'u^2 + \mathbf{u}\cdot\mathbf{P}' = \mathcal{K}' \quad .$$

60-002 ✓ Non-Relativistic and Ultra-Relativistic Approximation

What is the maximum value of v and γ if the non-relativistic approximation for energy, $\varepsilon \approx mc^2 + p^2/(2m)$, is to be used with an error less than α, that is with an absolute error on the energy equal to $\alpha p^2/(2m)$. Hence show that for an error of $\alpha = 0.01$ the linear momentum satisfies $p \approx 0.2mc$.

What is the minimum value of v and γ if the ultra-relativistic approximation for energy, $\varepsilon \approx pc$, is to be used with an error less than α, that is with an absolute error on the energy equal to αpc. Hence show that for an error of $\alpha = 0.01$ the linear momentum satisfies $p \approx 7mc$.

60-003 ✓ Relation Between Kinetic Energy and Linear Momentum

Determine the relativistic relation between kinetic energy and linear momentum.
L'approssimazione non relativistica vale per $\mathcal{K} \ll mc^2$, equivalente a $p \ll mc$:

$$\mathcal{K} = \varepsilon - mc^2 = mc^2(\gamma - 1) = \sqrt{p^2c^2 + m^2c^4} - mc^2 \simeq \frac{p^2}{2m}$$

L'approssimazione non relativistica corrisponde a $v \ll c$, o equivalentemente, $\varepsilon \ll mc^2$.

60-004 ☒ Outrun a Light Ray

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 12.37||

Suppose you are subject to a constant force and start a race with a light ray.
How much head start do you need in order that the light ray will never reach you?

WORK IN PROGRESS

60-005 ✓ Invariant Mass of a Pair of Particles

Show that for any pair of particles having four-momenta p_1 and p_2 and masses m_1 and m_2, one has the identity:

$$p_1 \cdot p_2 = \left(\frac{s - (m_1^2 + m_2^2)}{2} \right) \quad ,$$

in terms of the invariant mass of the pair, $M_0 = \sqrt{s}$.

60-006 ✓ On the Mass of a Isolated System

1. Does conservation of the Four-Momentum of an isolated system implies that collisions and interactions within an isolated system cannot change the total system mass?
2. Does conservation of the Four-Momentum of an isolated system implies that the constituents that enter a collision are necessarily the same in individual mass and in number as the constituents that leave that collision?
3. Is the mass of an isolated system composed of freely-moving objects obtained by adding the masses of the individual objects?

60-007 Relation Between Force and Power

Show that equations (60.01.12), (60.01.13) imply:

$$\pi = c^2 \frac{\mathrm{d}}{\mathrm{d}t}(m\gamma) \quad .$$

Show that equations (60.01.12), (60.01.13) imply:

$$m\gamma \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{f} - \frac{\pi}{c^2}\mathbf{v} \quad .$$

The latter equation shows that in the relativistic case, in general, force and acceleration are not parallel, at variance with respect to the classical case. Force and acceleration are parallel if and only if either $\mathbf{f} \parallel \mathbf{v}$ or $\mathbf{f} \perp \mathbf{v}$.

60-008 ✓ A Charged Particle in a Uniform and Constant Magnetic Field

This § is referenced at pages:
[2472, 2477]

Read also section § 15.05.09, § 15.05.10.

A charged particle is moving in a constant and uniform magnetic field. At a certain instant its velocity is perpendicular to the magnetic field.

The Lorentz force does no work and therefore the energy, velocity as well as the gamma factor, are constant:

$$a_{\parallel} = \frac{\mathrm{d}v}{\mathrm{d}t} = 0 \quad .$$

The motion lies in plane as at the initial instant its velocity is perpendicular to the magnetic field.

The Lorentz force is perpendicular to the velocity, therefore equation (60.01.04.02) can be used at once as

$$\mathbf{f}_{\perp} = \left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \right)_{\perp} = m\gamma \mathbf{a}_{\perp} = m\gamma \frac{v^2}{\rho} \mathbf{e}_N = q\, \mathbf{v} \wedge \mathbf{B} \implies \boxed{\rho = \frac{m\gamma v}{qB} = \frac{p}{qB} = \text{constant}} \quad .$$

Note that, for a uniform circular motion:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = p \frac{\mathrm{d}\theta}{\mathrm{d}t} \widehat{\mathbf{e}}_{\theta} = p \frac{v}{R} \widehat{\mathbf{e}}_{\theta} \quad .$$

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60-009 Dynamic Properties of a Particle as Seen From the Rest System of Another Particle

Two particles have four-momenta p_1 and p_2. Determine the energy, module of linear momentum and velocity of particle 2 as seen from the rest system of particle 1: E_{21}, $|\mathbf{p}_{21}|$ and $|\mathbf{v}_{21}|$.

WORKED SOLUTION

The answer is the same no matter in which Lorentz system we work. The answer must be therefore expressible in terms of invariants. The available invariants are:

$$p_1^2 = m_1^2 \qquad p_2^2 = m_2^2 \quad ,$$

$$p_1 \cdot p_2 \qquad \text{or} \qquad (p_1 + p_2)^2 \qquad \text{or} \qquad (p_1 - p_2)^2 \quad .$$

The energy of particle 2 as seen from the rest system of particle 1 is:

$$E_{21} = \varepsilon_2 \qquad \text{in the system where } \mathbf{p}_1 = \mathbf{0} \quad .$$

It must be written in terms of invariants. In the rest system of particle 1 one has:

$$p_1 \cdot p_2 = m_1 \varepsilon_2 \qquad \text{in the system where } \mathbf{p}_1 = \mathbf{0} \quad .$$

In a general system the latter becomes:

$$\boxed{E_{21} = \frac{p_1 \cdot p_2}{m_1} \neq E_{12}} \quad .$$

One can then calculate:

$$\boxed{|\mathbf{p}_{21}|^2 = \frac{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{m_1^2} \neq |\mathbf{p}_{12}|^2} \quad .$$

One can finally calculate:

$$\boxed{|\mathbf{v}_{21}|^2 = \frac{|\mathbf{p}_{21}|^2}{E_{21}^2} = \frac{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{(p_1 \cdot p_2)^2} = |\mathbf{v}_{12}|^2} \quad .$$

60-010 An Electron-Positron Annihilation

The positron is the anti-particle of the electron, in all identical to the electron except for having its charges of the opposite sign.

A positron having kinetic energy equal to its mass strikes an electron at rest. They annihilate, creating two high-energy photons. One photon enters a detector placed at right angle with respet to the direction of the initial positron with respect to the interaction point.

Determine the energies of both photons and the direction of motion of the second photon.

60-011 Energy Production in the Sun

This § is referenced at pages:
[2614]

Read § 65-004.

Energy from the Sun reaches the outer atmosphere of the Earth at a rate of about $1.4\ \mathrm{kW/m^2}$ (this is called the solar constant).

1. How much of the mass of the Sun is converted into energy every second to supply the energy that reaches the Earth.
2. What total mass is converted to energy every second in Sun?
3. Most of the energy comes from burning hydrogen nuclei (mostly protons) into helium nuclei (mostly a two-proton-two-neutron combination). How much hydrogen must be converted into helium per second?
4. Estimate how long Sun will continue to warm Earth, neglecting all other processes in Sun and emissions from Sun.

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60-012 An Isolated Photon

Demonstrate that an isolated photon cannot split into two photons going into two different directions other than going both into the original direction.

60-013 Particle-antiParticle Pair Production

A high-energy photon may carry an energy larger than the rest energy of an electron-positron pair. Nevertheless it cannot give rise to an electron-positron pair in the absence of other particles

1. Prove, in the [glo:LRF]Laboratory Reference Frame, that this process is incompatible with the laws of conservation of linear momentum and energy as employed.
2. Repeat the demonstration in the [glo:CMRF]Center-of-Momenta Reference Frame of the electron-positron pair.

60-014 ✓ Muon in the Atmosphere

Refer to § 58.12.02.01 and section § 60-008.

Assume, as a gross approximation, that the Earth magnetic field is 0.5 Gauss radially directed.

Determine how much the horizontal component of the velocity of the muon changes in direction if its energy is $\alpha = 1/100$ more than the one determined in section § 58.12.02.01, just enough to reach the Earth surface and the $\alpha\varepsilon$ additional energy is in carried by the horizontal component of the velocity.

60-015 Maximum Kinetic Energy in a Decay

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If a system of mass M decays or transforms at rest into a number of final state particles, the sum of whose masses is less than M by an amount ΔM, show that the maximum kinetic energy of the i-th particle (with mass m_i) is:

$$\mathcal{K}_i \leq \Delta M \left(1 - \frac{m_i}{M} - \frac{\Delta M}{2M}\right) .$$

60-016 Exact Velocity Distribution in the Head-Light Effect

This § is referenced at pages:
[Never referenced.]

Determine the exact velocity distribution of the photons as it is seen by the $\mathcal{I}$ Reference Frame observer in problem § **??**.

60-017 Angular Momentum for Non-Symmetrical Energy-LinMomentum Tensors

Write the conservation of Boost-AngMomentum Tensors in the case the Energy-LinMomentum tensor is non-symmetrical.

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60-018 Spin Four-Vector

Show that the spin four-vector, defined in (60.03.03.01), is invariant under translations of the time-space coordinates.

60-019 Properties of the Spin Four-Vector

Demonstrate that:

$$S_\mu U^\mu = 0 \quad .$$

This shows that the spin four-vector has only three independent components.

60-020 Properties of the Spin Four-Vector of a Free Particle or Isolated System

Consider a free particle or an isolated system, such that Four-Momentum and angular momentum are conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t} P^\mu = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t} J^{\alpha\beta}_{\cdot\cdot} = 0 \quad .$$

Demonstrate that:

$$\frac{\mathrm{d}}{\mathrm{d}t} S_\mu = 0 \quad .$$

60-021 Spin Four-Vector in the System Rest Frame

Demonstrate equation (60.03.03.01).

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CHAPTER 61

ElectroMagnetism and Relativity

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This § is referenced at pages:
[1727, 1954]

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LN-PROGRESS

61.01 Introduction

Read § 58 for introductory remarks.
Read § 21.03 for recap on continuity equations.
Note that $c = 1$ will be often used except for some of the basic formulas.

61.01.01 EM and Relativity

ElectroMagnetism is consistent with the principle of relativity from the beginning: all inertial observers describe all physical phenomena, including ElectroMagnetic ones, in the same way.

However, the interpretations may be different and what appears to be an Electric|Magnetic phenomenon to one observer may appear as a magnetic/electric phenomenon to another observer. Moreover, the understanding of ElectroMagnetic is greatly increased by the relativistic framework, in particular in all cases when moving observers are present: read § 61.09, § 58.03.01, § 61.09.08, § 61.09.03.

Relativistic invariance of ElectroMagnetism is not demonstrated but, on the other hand as for any physical law, it is an assumption arrived at via a process of induction from the experiments.

61.01.02 Comments on the Relativistic Formulation of EM

It is easy to be distracted by the mathematical elegance of the relativistic formulation of ElectroMagnetism; However, one should not loose the physical meaning of the equations and the complex physical relations underlying the equations and their relativistic invariance.

The importance of the ordinary three-dimensional representation of physical quantities must not be neglected even if complex and compact relativistic invariant relations might embrace different physical concepts. For instance the **Maxwell Equations** without Charge|Current will be compacted into a single four-dimensional equation, but this single equation embraces, at the same time, the lack of magnetic Monopoles and the Faraday-Neumann-Lenz law of ElectroMagnetic induction.

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61.02 Charge and Current

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61.02.01 Invariance Conservation Quantization of the Electric Charge

This § is referenced at pages:
[2496]

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The electric charge, in addition to be a Lorentz invariant and a conserved quantity, comes in multiples of the elementary charge of the electron and the proton.

The absolute values of the electric charge of proton and electron are the same as far as we know today, within the current experimental sensitivity.

The electric charge is an intrinsic, **Lorentz-Invarian*** , property of elementary particles, together with the mass, the spin and any other **charges** (such as baryon charge and lepton charge) which one discovers in fundamental physics.

61.02.01.01 Some Experimental Evidence about charge invariance

The equality of the absolute values of the electric charge of the electron and the proton is an experimental fact. These experimental facts [1] can be also used to support the fact that the electric charge is a Lorentz invariant quantity, or, in other words, the invariance of the charge of the particle from its velocity. In fact neutral atoms at rest with different atomic number have electrons moving with different velocities, depending on the energy levels, which in turn strongly depend on the atomic number. The fact the atoms are neutral provide support for the **independence of the electric charge of the electrons from their velocity**.

Invariance of the electric charge means that every observer will measure the same value for the flux of the electric field:

$$\oiint_{\Sigma[t]} \mathbf{E}\cdot \mathbb{d}\mathbf{S} = \oiint_{\Sigma'[t']} \mathbf{E}'\cdot \mathbb{d}\mathbf{S}' \quad .$$

One must be very careful that the electric fields, surfaces and times must be taken for the two different observers. In particular the distinction between t and t' must not be overlooked because one knows that events that are simultaneous in $\mathcal{I}$ need not be simultaneous in $\mathcal{I}'$. Each of the surface integrals in must be evaluated at one precise instant in its Reference Frame. If charges lie on the boundary one has to be rather careful about ascertaining that the charges within $\mathcal{I}$ at t are the same as those within $\mathcal{I}'$ at t'. If the charges are well away from the boundary there is no problem in this respect.

[1] J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...

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61.02.02
Four-Current

61.02.02.01 Transformation Law of the Charge Density

As the electric charge is a relativistic invariant the transformation of charge volume density is dictated by the law of transformation of volumes. In particular the relation between the charge volume density in the proper Rest Frame, ρ_0, and in any other Reference Frame, ρ, is given by:

$$\mathrm{d}V = \frac{\mathrm{d}V_0}{\gamma} \Longrightarrow \frac{\mathrm{d}q}{\mathrm{d}V} \equiv \rho = \rho_0\gamma \equiv \frac{\mathrm{d}q}{\mathrm{d}V_0}\gamma \quad . \tag{61.02.01}$$

→ 2483

61.02.02.02 Transformation Law of the Current Density

As the electric charge is a relativistic invariant the transformation of current volume density is dictated by the law of transformation of volumes. In particular the relation between the current volume density in the proper Rest Frame, ρ_0, and in any other Reference Frame, ρ, is given by:

$$\mathrm{d}V = \frac{\mathrm{d}V_0}{\gamma} \Longrightarrow \mathbf{j} \equiv \rho\mathbf{v} = \frac{\mathrm{d}q}{\mathrm{d}V}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathrm{d}q}{\mathrm{d}V}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \frac{\mathrm{d}q}{\mathrm{d}V}\frac{1}{\gamma}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \rho_0\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} \equiv \rho_0\mathbf{V} \quad . \tag{61.02.02}$$

→ 2483

61.02.02.03 Current Four-Vector

The two relations (61.02.01), (61.02.02) allows one to define the current four-vector.

Consider, to start with, a set of all identical particles moving with the same velocity.

The current is a four-vector defined as

$$j^\alpha \equiv \rho_0\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \rho_0\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = \rho_0\gamma\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = \rho\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = \begin{cases} j^0 \equiv \rho c = \rho_0\gamma c & , \\ \mathbf{j} \equiv \rho\mathbf{v} = \rho_0\gamma\mathbf{v} = \rho_0\mathbf{V} \end{cases} ,$$

showing explicitly the four-vector character of the four-current, in terms of the Lorentz-invariant volume charge density in the Rest Frame of the charge, ρ_0, and the Lorentz-invariant proper time, τ.

Note that:

$$j^\alpha \equiv \rho_0\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \qquad \text{expresses } j^\alpha \text{ in terms of quantities in the proper Rest Frame only} \quad ,$$

$$j^\alpha \equiv \rho\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} \qquad \text{expresses } j^\alpha \text{ in terms of quantities in the Laboratory only} \quad .$$

In the proper Rest Frame the definition reduces to:

$$j^\alpha = \{c\rho_0, \mathbf{0}\} \qquad \text{in the proper Rest Frame} \quad .$$

The current four-vector so defined is time-like:

$$j^\mu j_\mu = c^2\rho_0^2 = \text{invariant} \geq 0 \quad .$$

In the general case where the current is due to either the motion of different species of charges and|or a different sets of charges with different kinematics the Four-Current must be defined as the algebraic sum of the different components, all of them being convective currents:

$$j^\alpha \equiv \sum_k j_k^\alpha \equiv \sum_k \rho_k\frac{\mathrm{d}x_k^\alpha}{\mathrm{d}t} \quad , \tag{61.02.03}$$

possibly generating non convective currents, that is conduction currents.

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61.02.02.04 Conduction and Convection Currents

This § is referenced at pages:
[1213]

The motion of charges produces a current.

- The convection current is the one due to the motion of the Center-Of-Mass of the element of matter. In the simplest case that is a number of all identical particles moving with the same velocity.
- The conduction current is the one due to the motion relative to the Center-Of-Mass. As an example one can quote a plane circular loop of current whose Center-Of-Mass is at rest (no convection current) while it has a conduction current due to the motion of the free charges with respect to the Center-Of-Mass.

Read also § 21.03.03.03.

61.02.02.05 Some Comments on the Current Four-Vector

Note that the definition of j^μ is totally analogous to the definition of the Four-Momentum, p^μ, of a particle, with the volume charge density, ρ, replacing the energy and the volume charge density in the Rest Frame of the particle, ρ_0, replacing the mass of the particle.

One can also write explicitly the definition of current as:

$$j^\mu \equiv \{c\rho, \mathbf{j}\} \equiv \{c\rho, \rho\mathbf{v}\} \qquad \text{with} \qquad \rho = \gamma_{\mathbf{u}}\rho_0 \quad ,$$

where ρ_0 is the volume charge density in the rest frame of the particle.

A stationary charge in the $\mathcal{I}$ Reference Frame is described by a time-like four-vector. As it is expected when the charge is seen from the $\mathcal{I}'$ Reference Frame, moving with velocity $\mathbf{V}$ with respect to $\mathcal{I}$, the four-vector has both a time component and a space component. This is not a surprise as we know that a moving charge possesses both a charge and a current. What is less obvious is the behavior of a space-like current four-vector and the fact that a globally neutral system of currents will show a total net charge when it is seen from a Reference Frame moving with respect to the other system. In general one can say that, as **volume charge density and current density are components of a four-vector, they can mix** in a **Lorentz Transformation** .

One consequence, as discussed in section § 61.09, is that a moving **Magnetic Dipole** appears as having an **Electric Dipole** and vice-versa.

61.02.02.06 Charge Conservation and the Current Four-Vector

The charge continuity equation itself becomes, in terms of the Four-Current:

$$\partial_\mu j^\mu = 0 \quad .$$

DRAFT WORK IN PROGRESS

61.03 Transformation of ElectroMagnetic Fields

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It might be thought that the Electric|Magnetic fields are the space part of a four-vector but the situation is actually more complex than that.

It is not unexpected that a stationary charge distribution only produces an electric field while **a moving charge distribution produces an electric field plus a magnetic field**.

It is less obvious that **an electricly neutral steady current, only producing a magnetic field, is seen by a moving observer as a magnetic field plus a electric field**. This applies, for instance, to an observer moving with respect to a real magnet.

It is important to stress that when transforming fields, one need to transform:

- both the field quantity;
- and the space-time coordinates describing the event where the filed is calculated and applied.

In fact, a physical field, $\psi_k[x]$, with k components transforms, under **Lorentz Transformation** , as:

$$\psi_k[x'] = \mathcal{D}_{kj}\psi_j[x] \qquad x' = \mathbb{L}x + a \qquad \text{with } \mathcal{D} \text{ any representation of the Poincaré group} \quad .$$

Read § **??**.

Several arguments in this section are adapted from the references [1].

61.03.01 Elementary Considerations on the Transformations of Electric Fields

This § is referenced at pages:
[2514]

61.03.01.01 An Infinite Ideal Capacitor

Consider two infinite parallel sheets of charge, perpendicular to the z axis at $z = \pm d/2$, and charged with surface charge density $\pm\rho_S$ ($\rho_S > 0$) at rest for the inertial observer $\mathcal{I}$.

The electric field is zero outside the plates and uniform along the z axis in between the two plates and the magnetic field is zero:

$$\mathbf{E} = -\frac{\rho_S}{\varepsilon_0}\widehat{\mathbf{e}}_3 \qquad \mathbf{B} = 0 \quad .$$

An inertial observer $\mathcal{I}'$ moving with velocity $\mathbf{u}$ perpendicularly to the z axis will see the surface charge density increased by a factor γ, as charge is invariant and the distances are contracted by γ along the direction of motion. Any inertial observer moving perpendicularly to the z axis with velocity $\mathbf{u}$ will reach the same conclusion.

Therefore, if the electric field is still perpendicular to the xy plane:

$$\left\{\widehat{\mathbf{e}}_3 = \widehat{\mathbf{e}}'_3 \qquad \rho'_S = \gamma\rho_S\right\} \implies \mathbf{E}_\perp{}' = \gamma\mathbf{E}_\perp \quad .$$

Moreover, the moving observer will also see a surface current density $\propto -\mathbf{u}$. In fact he will see two infinite parallel sheets of currents with surface current:

$$\mathbf{j}_S^{\pm\prime} = \mp\rho'_S\mathbf{u} = \mp\gamma\rho_S\mathbf{u} \quad .$$

[1] Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...
D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...

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He will then see a magnetic field in between the two sheets, given by the sum of the magnetic fields produced by the two sheets:

$$\mathbf{B}' = \mu_0 \frac{\left(\mathbf{j}_S^{+\prime}\right) \wedge \left(-\hat{\mathbf{e}}_3{}'\right)}{2} + \mu_0 \frac{\left(\mathbf{j}_S^{-\prime}\right) \wedge \left(+\hat{\mathbf{e}}_3{}'\right)}{2} = \mu_0 \rho_S' \, \mathbf{u} \wedge \hat{\mathbf{e}}_3' \quad .$$

Note the appropriate signs for $\hat{\mathbf{e}}_3$ in order to pick up the region internal to the two sheets and the factor two in the formula for the magnetic field of one sheet of current removed by summing up the magnetic fields produced by the two sheets. All in all:

$$\mathbf{j}_S^{\pm\prime} = \mp \gamma \rho_S \mathbf{u} \qquad \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3{}' \qquad \mathbf{B}' = \mu_0 \rho_S \gamma \left(\mathbf{u} \wedge \hat{\mathbf{e}}_3\right) = -\varepsilon_0 \mu_0 \gamma \left(\mathbf{u} \wedge \mathbf{E}\right) = -\frac{\gamma}{c^2} \mathbf{u} \wedge \mathbf{E} \quad .$$

Therefore:

$$\boxed{\mathbf{B}'_\perp = -\frac{\gamma}{c^2} \mathbf{u} \wedge \mathbf{E} = -\frac{\gamma}{c^2} \mathbf{u} \wedge \mathbf{E}_\perp} \quad .$$

It is not strange that charges in motion produce a current and therefore a magnetic field.

An inertial observer $\mathcal{I}'$ moving along the z axis will see the surface charge density unchanged. She would see the distance between the two plates contracted, but this has no effect on the electric field.

Therefore, if the field is still seen as a field perpendicular to the plane:

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel \quad .$$

61.03.02
Elementary Considerations on the Transformations of Magnetic Fields

This § is referenced at pages:
[Never referenced.]

61.03.02.01 An Infinite Sheet of Current

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|2-18.4||

Consider one plane infinite neutral sheet of current, perpendicular to the z axis, located at $z = 0$, and carrying a surface current density $\mathbf{j}_S$ along the x axis at rest for the inertial observer $\mathcal{I}$.

The magnetic field is uniform and directed along the y axis:

$$\mathbf{j}_S = |\mathbf{j}_S| \hat{\mathbf{e}}_1 \qquad \mathbf{B} = \pm \frac{\mu_0 |\mathbf{j}_S|}{2} \hat{\mathbf{e}}_2 = \begin{cases} z > 0 \implies \mathbf{B} \cdot \hat{\mathbf{e}}_2 < 0 \\ z < 0 \implies \mathbf{B} \cdot \hat{\mathbf{e}}_2 > 0 \end{cases} .$$

An inertial observer $\mathcal{I}'$ moving along the y axis will see the current volume density and|or the surface current density unchanged. Moreover, the sheet was electricly neutral and it is still seen as a neutral sheet, as the observer is moving perpendicularly to the current density. In this case the positive ions at rest do not enter as we are moving perpendicularly to the current.

Therefore, if the field is still seen as a field along the y axis:

$$\mathbf{B}'_\parallel = \mathbf{B}_\parallel \quad .$$

An inertial observer $\mathcal{I}'$, moving along the x axis, will see a different surface current density, increased by a factor γ, while moving along the current. Moreover, the sheet will also appear electricly charged. In this case the positive ions at rest do enter as we are moving along the current, as for a wire carrying current.

The transformation can be more simply derived from **Lorentz Transformation** :

$$j'_x = \gamma \left(j_x - u\rho\right) = \gamma j_x \quad ,$$
$$\rho' = \gamma \left(\rho - u j_x / c^2\right) \quad .$$

Therefore, if the field is still seen as a field along the y axis:

$$\mathbf{B}'_\perp = \gamma \mathbf{B}_\perp \quad .$$

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An inertial observer $\mathcal{J}'$, moving along the z axis, will see the thickness of the sheet of current to decrease. Moreover, the time dilatation enters.

61.03.02.02 An Infinite Ideal Solenoid

Another simple example can be taken from a ideal infinite solenoid, with axis the z axis, whose n_l (helical windings per unit length) carry a constant current I. The constant and uniform magnetic field at the interior of the solenoid is: $B_z = \mu_0 n_l I$.

An observer in motion along the z axis, who sees the solonoid moving along the negative z axis, will see the number of windings per unit length increase by a factor γ, as the number of windings does not change but the length of the solenoid decreases by a factor γ. Therefore: $n'_l = \gamma n_l$.

The current I, assumed to be running exactly perpendicularly to the z axis, is given by the product between the volume current density, which does not change as it runs perpendicular to the relative velocity of the two Reference Frame, and the cross-section of the wire, which shrinks by a factor γ, when the solenoid is moving. Therefore the current is reduced, in the judgement of the observer who sees the solenoid moving: $I' = I/\gamma$.

Alternatively, one can reach the same conclusion on the current by observing that the moving observer will see the same charge across any section of the wire in a time longer than the time measured by the stationary observer (who measures the proepr time). Therefore the current measured by the moving observer is reduced by a factor γ. Therefore, again: $I' = I/\gamma$.

Therefore, all in all, the magnetic field does not change, as the two factors cancel.

61.03.03
Transformations Mixing-Up Electric|Magnetic Fields

Read § 61.09.02 for one example of magnetic field transofrming into an electric field.

61.03.04
Relativistic Transformation Laws of ElectroMagnetic Fields

This § is referenced at pages:
[2514]

In relativity Electric|Magnetic fields have no separate meaning: the single concept of an ElectroMagnetic field exists.

However, the Electric|Magnetic field vectors are not the space component of any four-vector. In fact they are members of a four-tensors with two indexes.

The full transformation laws can be derived from the study of other simple situations, as the one described above, as well as from a more general treatment (see section § 61.05.02). The results are:

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \tag{61.03.01}$$

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \tag{61.03.02}$$

$$\mathbf{E}'_{\perp} = \gamma \left(\mathbf{E} + c^{+1} \boldsymbol{\beta} \wedge \mathbf{B} \right)_{\perp} \tag{61.03.03}$$

$$\mathbf{B}'_{\perp} = \gamma \left(\mathbf{B} - c^{-1} \boldsymbol{\beta} \wedge \mathbf{E} \right)_{\perp} \quad . \tag{61.03.04}$$

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61.03.05
Non-Relativistic Transformation Laws of ElectroMagnetic Fields

This § is referenced at pages:
[1315, 2396, 2396]

©|G.Manfredi|Non-relativistic limits of Maxwell's equations| *Eur. J. Phys. 34 (2013) 859-871; WEB - URL* |
©|G.Rousseaux|Forty years of Galilean Electromagnetism (1973-2013)|*Eur. Phys. J. Plus 128, 81 (2013); WEB - URL* |

Read § 58.04.01, § 24.10.03.

It is immediate, from equation (61.03.01), (61.03.02), (61.03.03), (61.03.04), that the limit for $c \to \infty$ provides the non relativistic transformation laws, equation (58.04.01).

However, the non-relativistic limit is tricky. The limit for $c \to \infty$ is called the **instantaneous limit**, and other limits are possible.

Note that taking $\mathbf{u} = 0$ is not a sensible way to take the non-relativistic limit, because $\mathbf{u} = 0$ is just the identity transformation, that is no transformation at all, so they have to agree.

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61.04 Transformation of ElectroMagnetic Potentials

This § is referenced at pages:
[1731]

As discussed in section § 36.12 the Electric|Magnetic fields do not change under the gauge transformations (equations (36.12.02), (36.12.03)):

$$\boxed{\Phi' = \Phi - \partial_t \Lambda} \quad ,$$

$$\boxed{\mathbf{A}' = \mathbf{A} + \mathbf{grad}\Lambda} \quad .$$

The arbitrariness can be fixed by fixing a gauge. The covariant **Lorenz Gauge** :

$$\boxed{\partial_\mu A^\mu = \frac{1}{c^2}\partial_t \Phi + \mathbf{div}\,\mathbf{A} = 0} \quad ,$$

is normally used in relativistic physics.

Define the four-potential of the ElectroMagnetic field, whose transformation properties are not yet known:

$$\boxed{A^\mu \equiv \{\Phi/c, \mathbf{A}\}} \quad .$$

One can show that the four-potential of the ElectroMagnetic field, $A^\mu \equiv \{\Phi/c, \mathbf{A}\}$, is a four-vector. In fact, in the **Lorenz Gauge** , it is the solution to the d'Alembert equation:

$$\Box \ldots \equiv \partial\partial\,(\ldots) \equiv \partial_\mu \partial^\mu\,(\ldots) \quad ,$$

$$\Box\Phi = \frac{\rho}{\varepsilon_0} = \mu_0 c^2 \rho \quad ,$$

$$\Box\mathbf{A} = \mu_0 \mathbf{j} = \frac{1}{\varepsilon_0 c^2}\mathbf{j}$$

or:

$$\boxed{\Box A^\mu = \frac{j^\mu}{\varepsilon_0 c^2} = \mu_0 j^\mu \qquad \text{with} \qquad \partial_\mu A^\mu = 0} \quad .$$

As the four-potential is solution to the covariant d'Alembert equation it must be a four-vector, thanks to the **Quotient Law** § 20.03.07 and considering the equation in Fourier-transformed space, with arbitrary PW: $(k_\mu k^\mu)\, A^\nu = k^2 A^\nu \propto j^\nu$.

In four dimensional notation the gauge transformation (equations (36.12.02), (36.12.03)) becomes:

$$\boxed{A^\mu \longrightarrow A^\mu - \partial^\mu \Lambda} \quad . \qquad (61.04.01)$$

ElectroMagnetic potentials in the **Lorenz Gauge** are not uniquely defined by the **Lorenz Gauge** condition any potential connected to a **Lorenz Gauge** potential by a gauge transformation (61.04.01) with $\Box\Lambda = 0$ is still a **Lorenz Gauge** potential. The arbitrariness can be removed, however, by introducing suitable boundary conditions on the potentials.

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61.05 ElectroMagnetism in Relativistic Notation

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||

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61.05.01
ElectroMagnetic Field Tensor

The Electric|Magnetic fields are built from space and time derivatives of the potentials. In order to build a covariant description time and space coordinates (and derivatives) must mix. In fact the magnetic field is given in terms of the spatial derivatives only. Under a **Lorentz Transformation** of the spatial derivatives the temporal derivative appears and the temporal derivatives uniquely appear in the expression of the Electric field.

Moreover, we know already how to build a four-vector from the potentials. Therefore one might try to build a second-rank tensor from A^μ and its space and time derivatives, to try to sort out the expressions for the Electric|Magnetic field.

Let us define the (gauge invariant) anti-symmetric ElectroMagnetic field tensor, $\mathbb{F}$, as the generalisation to four dimensions of the three dimensional rotor: $\mathbb{F}$ is defined as the four-dimensional rotor of the four-potential (read § **??**).

The anti-symmetric ElectroMagnetic field tensor is thus defined as:

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ +E_x/c & 0 & -B_z & +B_y \\ +E_y/c & +B_z & 0 & -B_x \\ +E_z/c & -B_y & +B_x & 0 \end{pmatrix} \quad , \quad \Longrightarrow \quad F^{\mu\nu} + F^{\nu\mu} = 0 \quad . \qquad \rightarrow 2511$$

One can verify that:

$$F_{\mu\nu} \equiv \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}_{\cdot\cdot} \quad \Leftrightarrow \quad \mathbb{F}^{\cdot\cdot}_{\cdot\cdot} = \boldsymbol{\eta}\mathbb{F}^{\cdot\cdot}_{\cdot\cdot}\boldsymbol{\eta} \quad ,$$

can be obtained by changing the sign of the electric field vector:

$$\{\mathbf{B} \longrightarrow +\mathbf{B} \quad \mathbf{E} \longrightarrow -\mathbf{E}\} \quad \Leftrightarrow \quad F^{\mu\nu} \longrightarrow F_{\mu\nu} \quad .$$

By explicit calculation (read § 20.05.04) one can easily verify that the components of the tri-vector fields are:

$$E^k = E_k = -cF^{0k}_{\cdot\cdot} = +cF^{k0}_{\cdot\cdot} \quad ,$$

$$B^k = B_k = \frac{1}{2}\epsilon^{\cdot\cdot\cdot}_{krs}F^{sr}_{\cdot\cdot} \quad .$$

By its very definition, from two four-vectors, $\mathbb{F}$ is a rank-two Lorentz tensor. The Electric|Magnetic fields, therefore, have no independent existence, but they are parts of the same ElectroMagnetic Field, and transform into each other by **Lorentz Transformation** .

The dual ElectroMagnetic field tensor is defined, in terms of the totally anti-symmetrical **Levi-Civita tensor** (read § **??** and comments therein about different conventions)

$$G^{\cdot\cdot}_{\mu\nu} \equiv \frac{1}{2}\epsilon^{\cdot\cdot\cdot\cdot}_{\mu\nu\alpha\beta}F^{\alpha\beta}_{\cdot\cdot} = \begin{pmatrix} 0 & +B_x & +B_y & +B_z \\ -B_x & 0 & -E_z/c & +E_y/c \\ -B_y & +E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & +E_x/c & 0 \end{pmatrix} \quad , \quad G^{\cdot\cdot}_{\mu\nu} + G^{\cdot\cdot}_{\nu\mu} = 0 \quad . \qquad \rightarrow 2491\ 2511$$

The elements of $\mathbb{G}$ are obtained from the elements of $\mathbb{F}$ by applying:

$$\{\mathbf{E} \longrightarrow -\mathbf{B} \quad \mathbf{B} \longrightarrow +\mathbf{E}\} \quad \Leftrightarrow \quad F^{\mu\nu} \longrightarrow G^{\mu\nu} \quad . \qquad (61.05.01) \qquad \rightarrow 2491$$

Warning: equation (61.05.01), (61.05.01) have the opposite sign with respect to J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...! due his different convention on the **Levi-Civita tensor** (read **§ ??**).

61.05.02
Relativistic Transformations of the ElectroMagnetic Fields

This § is referenced at pages:
[2420, 2487, 2497, 2501]

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|18.3||

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|26||

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|11||

In order to calculate how the Electric|Magnetic fields transform one just need to do the calculation with the **Lorentz Transformation** equations. It is just long and tedious. The calculations are done more easily if the transformation law is written in matrix form from $F^{\mu\nu}_{\cdot\cdot}$:

$$\mathbb{F}' = \mathbb{L}\mathbb{F}\mathbb{L}^{\mathsf{T}} \quad .$$

One finds:

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \quad ,$$

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \quad ,$$

$$\mathbf{E}'_{\perp} = \gamma\left(\mathbf{E} + c^{+1}\,\boldsymbol{\beta}\wedge\mathbf{B}\right)_{\perp} \quad ,$$

$$\mathbf{B}'_{\perp} = \gamma\left(\mathbf{B} - c^{-1}\,\boldsymbol{\beta}\wedge\mathbf{E}\right)_{\perp} \quad .$$

61.05.03
Maxwell Equations in Relativistic Notation

The two homogeneous **Maxwell Equations** , equations (36.08.07), (36.08.08), those describing the intrinsic properties of the ElectroMagnetic fields independently from Charge|Current, become, in covariant form:

$$\partial^{\alpha}F^{\beta\gamma}_{\cdot\cdot} + \partial^{\beta}F^{\gamma\alpha}_{\cdot\cdot} + \partial^{\gamma}F^{\alpha\beta}_{\cdot\cdot} = 0 \quad . \qquad (61.05.02)$$

→ 2512

The two homogeneous **Maxwell Equations** are written most compactly using the dual tensor of the field tensor:

$$\partial_{\alpha}G^{\alpha\beta}_{\cdot\cdot} = 0 \quad . \qquad (61.05.03)$$

The two non-homogeneous **Maxwell Equations** , equations (36.08.01), (36.08.04), those describing how the properties of the ElectroMagnetic fields are related to Charge|Current, become, in covariant form:

$$\partial_{\alpha}F^{\alpha\beta}_{\cdot\cdot} = \frac{j^{\beta}}{\varepsilon_0 c^2} = \mu_0 j^{\beta} \quad , \qquad (61.05.04)$$

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valid in the **Lorenz Gauge** and representing the just the wave equation in the **Lorenz Gauge** .

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61.06 Dynamics of Particles in ElectroMagnetic Field

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|11.9||
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See the very important comment in section § 36.05.03 about the modeling of matter with point charges, **Dipoles** and higher-order **Multipoles** .

The equations of motion of a charged point particle in an ElectroMagnetic field are:

$$\boxed{\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = q\,\mathbf{u}\cdot\mathbf{E}} \quad ,$$

$$\boxed{\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\,(\mathbf{E} + \mathbf{u}\wedge\mathbf{B})} \quad .$$

Moreover, if the particle has an intrinsic **Electric Dipole** | **Magnetic Dipole** moment, in the particle Rest Frame one can write the second **Cardinal Equation** as:

$$\boxed{\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \mathbf{p}\wedge\mathbf{E} + \mathbf{m}\wedge\mathbf{B} \qquad \mathbf{j} = \mathbf{s}} \quad .$$

In fact, for a mass-point, the orbital angular momentum dynamics is dictated by the first cardinal equation:

$$\boxed{\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\,(\mathbf{E} + \mathbf{u}\wedge\mathbf{B}) + \boldsymbol{\nabla}\,(\mathbf{p}\cdot\mathbf{E}) + \boldsymbol{\nabla}\,(\mathbf{m}\cdot\mathbf{B}) \qquad \text{fixed } \textbf{Dipoles}} \quad .$$

In this case the, in presence of non-uniform Electric|Magnetic fields, the first **Cardinal Equation** is also modified to account for the force on a **Dipole** , depending on the gradient of the field (read § 36.05.01).

Quantum Physics, however, requires that any intrinsic **Electric Dipole** | **Magnetic Dipole** moment is parallel to the spin of the particle: $\mathbf{p} = g_E\mathbf{s}$ and $\mathbf{m} = g_B\mathbf{s}$.

Note, however, that one might limit to consider charged particles only, avoiding introducing explicitly **Dipoles** and|or higher order **Multipoles** , when one realizes that any higher order charge or current **Multipole** can be always described by a suitable combination of charge with a suitable dynamics. Therefore, in principle, one would not need to introduce explicitly the equations for **Dipoles** and|or higher order **Multipoles** . However, the explicit use of **Dipoles** and|or higher order **Multipoles** is often a shortcut allowing a simpler description.

Also note that when one consider a generic continuous distribution of Charge|Current using it for calculations of integrated properties, such as forces/couples, one can always model point charges with a delta charge distribution, **Electric Dipoles** | **Magnetic Dipoles** with a suitable first derivative of a delta charge distribution, and so on.

In terms of the four-velocity and Four-Momentum:

$$U^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = \gamma_{\mathbf{u}}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} \equiv \gamma_{\mathbf{u}}u^\alpha \qquad p^\alpha = mU^\alpha = m\gamma_{\mathbf{u}}u^\alpha \quad ,$$

one can write:

$$\frac{\mathrm{d}p_0}{\mathrm{d}\tau} = q\,\mathbf{U}\cdot\mathbf{E} \quad ,$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = q\,(U_0\mathbf{E} + \mathbf{U}\wedge\mathbf{B}) \quad .$$

The equation of motion can be written in covariant form:

$$\boxed{\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} \equiv m\frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} = qF^{\alpha\beta}U_\beta} \quad . \qquad (61.06.01)$$

→ 2492

Note that it would be possible to deduce the transformation properties of the EM fields from the equation of motion (61.06.01), as the transformation properties of all other quantities are known.

61.07 Energy-LinMomentum and Boost-AngMomentum Tensors of ElectroMagnetic Fields

This § is referenced at pages:
[Never referenced.]

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Read § 21.03 and § 60.03.01.
Start from equations § 40.01.02, § 40.01.03, § 40.01.04.
In order to build a relativistic covariant expression for the energy and linear momentum, according to section § 60.03, one needs to build a suitable covariant expression quadratic in the field tensor $\mathbb{F}$ as we know that energy density, energy flux and linear momentum density are quadratic in the fields.
Possible components are therefore:

$$\eta^{\cdot\cdot}_{\alpha\beta} F^{\mu\alpha}_{\cdot\cdot} F^{\beta\nu}_{\cdot\cdot} = F^{\mu\cdot}_{\cdot\delta} F^{\delta\nu}_{\cdot\cdot} \qquad \eta^{\mu\nu}_{\cdot\cdot} F^{\alpha\beta}_{\cdot\cdot} F^{\cdot\cdot}_{\alpha\beta} \quad .$$

Comparing with the known expressions in section § 40.01 one finds the correct expression:

$$T^{\mu\nu}_{\cdot\cdot} = \boxed{\varepsilon_0 c^2 \left(\eta^{\cdot\cdot}_{\alpha\beta} F^{\mu\alpha}_{\cdot\cdot} F^{\beta\nu}_{\cdot\cdot} + \frac{1}{4} \eta^{\mu\nu}_{\cdot\cdot} F^{\alpha\beta}_{\cdot\cdot} F^{\cdot\cdot}_{\alpha\beta} \right)} \quad . \tag{61.07.01}$$

→ 2512

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61.08 Transformation Properties of the Polarization and Magnetization Fields

©|W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|18||

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61.09 Examples and Physical Applications

This § is referenced at pages:
[2481, 2484]

61.09.01
Four-Current of a Ohmic Conductor Wire

Consider the Four-Current in a typical globally neutral Ohmic metallic rectilinear conductor with conduction current four-vector equal to:

$$j = \{c\rho_+, \mathbf{0}\} + \{c\rho_-, \rho_-\mathbf{v}_-\} = \{0, \rho_-\mathbf{v}_-\} \quad ,$$

in terms of the charge density, ρ_-, and drift velocity, $\mathbf{v}_-$, of the electrons and of the charge density, ρ_+, of the positive ions. All quantities are measured in the Laboratory Rest Frame.

Show that the Current four-vector, j, is space like with

$$j^\mu j_\mu = -(\rho_-)^2 {\mathbf{v}_-}^2 = -\left(\rho_+^0\right)^2 {\mathbf{v}_-}^2$$

in terms of the charge density of the positive ions at rest in the conductor, ρ_+^0.

WORKED SOLUTION

$$\left|\rho_+^0\right| = |\rho_-| \quad ,$$
$$j = \{0, \rho_-\mathbf{v}_-\} = \left\{c\rho_+^0, \mathbf{0}\right\} + \{c\rho_-, \rho_-\mathbf{v}_-\} \quad ,$$
$$\rho_- = \rho_-^0\gamma \qquad \rho_+ = \rho_+^0 \quad ,$$
$$j^\mu j_\mu = -\left(\rho_+^0\right)^2 {\mathbf{v}_-}^2 = -(\rho_-)^2 {\mathbf{v}_-}^2 < 0 \quad .$$

61.09.02
EM Fields for an Observer Moving With Respect to Straight Infinite Neutral Current Wire (1)

This § is referenced at pages:
[2487]

©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...- ...

Read also § 61.09.03, for a different solution.

Consider a very long rectilinear electricly neutral current-carrying wire and consider it as an infinite rectilinear electricly neutral current-carrying wire (aligned with the z axis). Suppose the current $I > 0$ is generated by electrons moving with velocity $u < 0$ inside a lattice of fixed positive ions. Let $\gamma \equiv \gamma_u$.

Let us also consider an observer, $\mathcal{I}'$, moving with the negative electrons.

The wire is neutral, in the Laboratory Reference Frame, with positive ions at rest and negative electrons moving in the negative z direction. The positive ions are at rest and the free electrons move at constant speed. The system is electricly neutral bacause the measured separation beetween positive ions and electrons are identical.

The Laboratory Reference Frame is the Rest Frame for the positive ions, but it not the Rest Frame for the electrons and therefore:

$$\rho^+ = \rho_0^+ \qquad \rho^- = \gamma\rho_0^- > \rho_0^- \quad ,$$
$$\rho^+ + \rho^- = 0 \qquad \text{as the wire is electrically neutral in the Laboratory Reference Frame} \quad ,$$
$$\rho \equiv \rho_0^+ = \rho^+ = -\rho^- > 0 \qquad \text{just a definition} \quad .$$

The wire will generate a tangential magnetic field and no electric field:

$$\mathbf{B} = B[r]\widehat{\mathbf{e}}_\theta \qquad B[r] = \frac{\mu_0 I}{2\pi r} \qquad \text{and} \qquad \mathbf{E} = 0 \quad .$$

Note that the cross-section of the wire as well as the radial distance do not change, when passing from the Laboratory Reference Frame to $\mathcal{J}'$:

$$S' = S \qquad r' = r \quad .$$

We have, in the Laboratory Reference Frame:

$$I \equiv -\rho u S = -\rho c \beta S > 0 \qquad j_z = \rho^- u = -\rho u > 0 \quad .$$

Lorentz Transformation imply

$$\rho' = -\gamma\beta j_z/c = \gamma\rho\beta^2 > 0 \quad ,$$
$$j'_z = \gamma j_z \qquad I' = \gamma I \quad .$$

The law of transformation of the charge density can be also derived, with more physical insight, directly as follows. An observer moving with the electrons will see a charge density:

$$\rho' = \rho^{+\prime} + \rho^{-\prime} = \gamma\rho_0^+ + \rho_0^- = \gamma\rho_0^+ + \rho^-/\gamma = \rho\,(\gamma - 1/\gamma) = \rho\beta^2\gamma > 0 \quad .$$

Therefore he will see a radial electric field (to within the sign, which will be determined later on):

$$E'[r] = \frac{\rho'\mathbf{S}'}{2\pi\varepsilon_0 r} = \frac{\rho\beta^2\gamma S}{2\pi\varepsilon_0 r} = \gamma\beta c \frac{1}{\varepsilon_0\mu_0 c^2}\frac{\mu_0\rho c\beta S}{2\pi r} = \gamma\beta c B \quad .$$

It turns out, introducing vectors (all the fields are perpendicular to $\mathbf{u}$):

$$\mathbf{E}'_\perp = \gamma\,\mathbf{u}\wedge\mathbf{B} = \gamma\,\mathbf{u}\wedge\mathbf{B}_\perp \implies \left|\mathbf{E}'\right| = \left|\mathbf{E}'_\perp\right| = \frac{\rho\gamma\beta^2 S}{2\pi\varepsilon_0 r} \quad .$$

The moving observer will then Read also an electric force on a point charge, not only a magnetic force.

It is important to stress that the electric field seen by the moving observer is due to the appearance of a real charge density from the point of view of this observer. Compare with the discussion of § 58.03.01.

The magnetic field is:

$$\mathbf{B}'_\perp = \gamma\mathbf{B}_\perp = \frac{\mu_0 I'}{2\pi r} \quad .$$

Note that the apparent non-invariance of electric charge, due to the fact that the wire appears neutral to one observer and charged to the other one, is an artifact of the idealisation of infinite wire with infinite charge. It might seem to be paradoxical, in view of the invariance of the total electric charge, that the wire appears to be charged to one observed and neutral to another one. Actually there is no problem because the rectilinear infinite wire is an idealisation with charges entering from infinity and exiting to infinity, such that it is not clear what is the total charge (which is, in any case, infinite). Read § 61.09.08 for a more realistic situation.

One should consider a finite loop of current to avoid such kind of problems (read § 61.09.08). In the real case there will be parts of the wire positively charged and parts of the wire negatively charged, thus retaining the overall neutrality of the wire.

However, when considering a finite piece of wire inside a closed finite surface, one might still wonder what happens with the total internal charge, which should be an invariant quantity. In this case one should remember the discussion in section § 61.02.01, about the relativity of simultaneity.

Note that it as been shown not only that an observer in motion with respect to a stationary neutral current distribution will see an electric field but also that the sources of the electric field are real charges (as any real electric field must be traced to specific source charges). This fact was not clear from the discussion in section § 58.03.01.

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61.09.03
EM Fields for an Observer Moving With Respect to Straight Infinite Neutral Current Wire (2)

This § is referenced at pages:
[2481, 2495, 2505, 2505, 2505, 2513, 2513]

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©|Berkeley Physics Course, Vol. 2, E.M.Purcell & D.J.Morin, *Electricity And Magnetism*, 2013, CUP, ...Ed., ...- ...

Consider the central part of a long rectilinear and uniform wire of radius r, section S, carrying a constant current I and electricly neutral, in global.

Assume that the conduction current is provided by electrons moving inside a lattice of fixed positive ions.

Calculate the ElectroMagnetic fields at a distance R from the wire according to an observer at rest with respect to the wire (an observer in the Laboratory Frame).

Calculate the ElectroMagnetic fields at a distance R from the wire according to an observer moving with the same velocity as the average velocity of the (current producing) electrons inside the wire.

Verify that the ElectroMagnetic fields according to the moving observer are produced by the charge density and current as seen from that system.

Verify that the two invariants of the ElectroMagnetic fields (section § 61-004) are actually invariant in the transformation.

Introduce a cylindrical Coordinate System with the wire on its axis and choose for the z axis the same orientation as the current flow.

For the stationary observer the electric field is zero, because the wire is electricly neutral. The magnetic field is tangential. It is:

$$\mathbf{B}[R] = \frac{\mu_0 I}{2\pi R}\widehat{\mathbf{e}}_\theta \qquad \mathbf{j}\cdot\mathbf{e}_z > 0 \quad .$$

Consider an observer moving with the same velocity as the average velocity of the (current producing) electrons inside the wire. Use the transformation formulas in section § 61.05.02, use the fact that for this Lorentz transformation we have

$$\mathbf{B}[R] = \mathbf{B}_\perp = \frac{\mu_0 I}{2\pi R}\widehat{\mathbf{e}}_\theta \quad ,$$

use the fact that the observer is moving like the electrons, that is with a negative component of the velocity, $V_z < 0$, to find:

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel = 0 \tag{61.09.01}$$ → 2513

$$\mathbf{B}'_\parallel = \mathbf{B}_\parallel = 0 \tag{61.09.02}$$ → 2513

$$\mathbf{E}'_\perp = \gamma_{\mathbf{V}}\left(\mathbf{E} + c^{+1}\,\boldsymbol{\beta}\wedge\mathbf{B}\right)_\perp = \gamma_{\mathbf{V}} c\,\boldsymbol{\beta}_{\mathbf{V}}\wedge\mathbf{B}_\perp \tag{61.09.03}$$ → 2513

$$\mathbf{B}'_\perp = \gamma_{\mathbf{V}}\left(\mathbf{B} - c^{-1}\,\boldsymbol{\beta}\wedge\mathbf{E}\right)_\perp = \gamma_{\mathbf{V}}\mathbf{B}_\perp \quad . \tag{61.09.04}$$ → 2498 2513

A radial electric field has appeared while the perpendicular magnetic field has been modified.

The relation between the charge volume density in the Rest Frame, ρ_0, and the one in an arbitrary frame, moving with velocity $-\mathbf{V}$ with respect to the Rest Frame, is given by:

$$\boxed{\rho = \gamma_{\mathbf{V}}\rho_0} \quad . \tag{61.09.05}$$ → 2498

In the Rest Frame the volume of an element of charge is maximum and the charge density is thus minimum, due to the invariance of the charge.

Let ρ_+ and ρ_- be the charge volume density in the Laboratory Frame. Let ρ'_+ and ρ'_- be the charge volume density in the Reference Frame moving with the same velocity as the average velocity of the electrons in the wire. Let ρ^0_+ and ρ^0_- be the charge volume density in the Rest Frame of the corresponding charge.

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We have:

$$\begin{aligned}
\rho_+ + \rho_- &= 0 \qquad \text{globally neutral in the Laboratory system} \quad , \\
|\rho_+| = |\rho_-| &\equiv \rho > 0 \qquad \text{definition} \quad , \\
\rho_- &= \gamma \rho_-^0 \qquad \text{from the transformation of the charge density} \quad , \\
\rho_+ &= \rho_+^0 \qquad \text{from the transformation of the charge density} \quad , \\
\rho'_- &= \rho_-^0 = \rho_- / \gamma \quad , \\
\rho'_+ &= \gamma \rho_+^0 = \gamma \rho_+ \quad , \\
0 \leq I = \rho_- V_z S &= -\rho_- |V_z| S = \rho |V_z| S \equiv \rho V_z S \quad ,
\end{aligned}$$

having used equation (61.09.05).

61.09.03.01 First Method

From equation (61.09.05) the following relations can be derived:

$$\rho' \equiv \rho'_+ + \rho'_- = \gamma \rho_+ + \rho_- / \gamma = \rho \left(\gamma - 1/\gamma \right) = \rho \left(\frac{\gamma^2 - 1}{\gamma} \right) = \rho \gamma \beta^2 > 0 \quad . \tag{61.09.06}$$

→ 2498

Note that ρ_- decreases in the transformation, as we are moving onto the Rest Frame, while ρ_+ increases, as we are moving away from the Rest Frame. Therefore we expect to see the wire electricly charged with positive charge.

The moving observer will see a positive charge density and thus a radial electric field. Its calculation of the (radial) electric field gives:

$$E'_r = \frac{\rho' S}{2\pi\varepsilon_0 R} = \frac{\rho \gamma \beta^2 S}{2\pi\varepsilon_0 R} \quad , \tag{61.09.07}$$

→ 2498

At the same time equation (61.09.03) gives:

$$\mathbf{E}'_\perp = \gamma_\mathbf{V} c \boldsymbol{\beta}_\mathbf{V} \wedge \mathbf{B}_\perp = \gamma \beta c \left(\frac{\mu_0 I}{2\pi R} \right) \left(-\hat{\mathbf{e}}_3 \wedge \hat{\mathbf{e}}_\theta \right) = \frac{\mu_0 c^2 \rho \gamma \beta^2}{2\pi R} \hat{\mathbf{e}}_r = \frac{\rho \gamma \beta^2 S}{2\pi\varepsilon_0 R} \hat{\mathbf{e}}_r \quad ,$$

in agreement with (61.09.07). The two methods provide then the same value for the electric field, as it should.

61.09.03.02 Second Method

It is obvious that the wire appears charged because the current transforms in the Lorentz transformation.

Write the special Lorentz transform, along the z axis, from the Laboratory Frame to the frame of an observer moving with the same velocity as the average velocity of the (current producing) electrons inside the wire:

$$\begin{aligned}
\beta &= -V < 0 \quad , \\
\rho'_+ &= \gamma \left(\rho_+ - \beta j_+ \right) = \gamma \rho_+ \quad , \\
\rho'_- &= \gamma \left(\rho_- - \beta j_- \right) \quad , \\
j'_+ &= \gamma \left(j_+ - \beta \rho_+ \right) \quad , \\
j'_- &= \gamma \left(j_- - \beta \rho_- \right) = 0 \quad ,
\end{aligned}$$

where the last passages come from the fact that in this problem:

$$\begin{aligned}
j_+ &= 0 \quad , \\
j'_- = 0 &\Longrightarrow j_- = \beta \rho_- \quad .
\end{aligned}$$

It follows:

$$\begin{aligned}
\rho'_+ &= \gamma \rho_+ \quad , \\
\rho'_- &= \gamma \left(\rho_- - \beta^2 \rho_- \right) = \gamma \rho_- \left(1 - \beta^2 \right) = \rho_- / \gamma \quad , \\
\rho' = \rho'_+ + \rho'_- &= \gamma \rho_+ + \gamma \rho_- = \gamma \rho_+ + \rho_- / \gamma = \rho \gamma \beta^2 > 0 \quad ,
\end{aligned} \tag{61.09.08}$$

where the last passages are the same as in equation (61.09.06).

This second method is actually more general than the first one, as it can be applied to any transformation, not only to the specific one considered in this problem, moving to a Reference Frame where the electrons

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are at rest. In this case the generalisation requires to distinguish between the velocity of the **Lorentz Transformation** and the drift velocity of the charges. It is than better to stick to the more general relation ($\rho = 0$):

$$\rho' = -\gamma\beta j \quad . \tag{61.09.09}$$

It might seem to be paradoxical, in view of the invariance of the total electric charge, that the wire appears to be charged to one observed and neutral to another one. Actually there is no problem because the rectilinear infinite wire is an idealisation with charges entering from infinity and exiting to infinity, such that it is not clear what is the total charge (which is, in any case, infinite).

Read § 61.09.08 for a more realistic situation.

61.09.04
A Beam of Identical Charged Particles

This § is referenced at pages:
[2511]

Consider a cylindrical beam of identical charged particles moving at constant velocity, $\mathbf{u}$, on straight line with uniform charge density and a circular section of radius R. This is often a good approximation for a beam of particles in a particle accelerator.

Show that the repulsive force between particles is damped by a factor γ^2, as a consequence of the attractive effect of the magnetic field, with respect to the case when the particles are at rest.

Alternatively show that the same effect can be explained by the relativistic time dilatation.

WORKED SOLUTION

In the Laboratory Frame let the beam density be ρ. Let the z axis be on the direction of the beam and let the current density along z be $\mathbf{j}_z = \rho v$. Therefore in the Laboratory Frame:

$$E_r[r] = \frac{\rho r}{2\varepsilon_0} \qquad \text{at an point internal to the beam: } r \le R \quad , \quad ,$$

$$B_\phi[r] = \frac{\mu_0 j_z r}{2} = \frac{\mu_0 \rho v r}{2} \quad , \quad ,$$

$$\mathbf{f} = q\left(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}\right) = q\left(E_r[r]\widehat{\mathbf{e}}_r + v\widehat{\mathbf{e}}_3\wedge B_\phi[r]\widehat{\mathbf{e}}_\psi\right) \quad , \quad ,$$

$$f_r[r] = q\left(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}\right)\cdot\widehat{\mathbf{e}}_r = q\frac{\rho r}{2\varepsilon_0}\frac{1}{\gamma^2} \quad .$$

The repulsive force in the Laboratory Frame is trying to scatter away the particles in the beam but in the Laboratory Frame the time evolution is slowed down.
let us look at the situation as it is seen by the Reference Frame where the charged particles are at rest, $\mathcal{I}'$. In this system the force is of electric nature only. The charge density is:

$$\rho' = \rho/\gamma \le \rho \quad .$$

The perpendicular component of the electric field is:

$$\mathbf{E}'_\perp = \gamma\left(\mathbf{E} + c\boldsymbol{\beta}\wedge\mathbf{B}\right)_\perp = \gamma\left(\frac{\rho r}{2\varepsilon_0}\widehat{\mathbf{e}}_r + \widehat{\mathbf{e}}_3\wedge\widehat{\mathbf{e}}_\psi\frac{\mu_0\rho v^2 r}{2}\right) = \frac{\gamma\rho r}{2\varepsilon_0}\left(1 - \frac{v^2}{c^2}\right)\widehat{\mathbf{e}}_r = \frac{\rho r}{2\varepsilon_0\gamma}\widehat{\mathbf{e}}_r = \frac{\rho' r}{2\varepsilon_0}\widehat{\mathbf{e}}_r \quad .$$

As the electric charge is invariant the force on one particle in Reference Frame $\mathcal{I}'$ is thus:

$$\mathbf{f}' = \frac{q\rho' r}{2\varepsilon_0}\widehat{\mathbf{e}}_r = \frac{q\rho r}{2\varepsilon_0\gamma}\widehat{\mathbf{e}}_r = \gamma\mathbf{f} \quad .$$

This can be written as the Minkowski force as:

$$\mathbf{f}'_{\text{M}} \equiv \mathbf{f}' = \frac{q\rho' r}{2\varepsilon_0}\widehat{\mathbf{e}}_r \quad ,$$

where $\gamma = 1$ in the definition of the Minkowski force because particles are at rest in the $\mathcal{I}'$ Reference Frame.

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On the other hand the force on one particle in Reference Frame $\mathcal{J}$ is:

$$\mathbf{f} = \frac{q\rho r}{2\varepsilon_0\gamma^2}\widehat{\mathbf{e}}_r = \frac{q\rho' r}{2\varepsilon_0\gamma}\widehat{\mathbf{e}}_r \quad .$$

This can be written as the Minkowski force as:

$$\mathbf{f}_{\mathrm{M}} \equiv \gamma\mathbf{f} = \frac{q\rho' r}{2\varepsilon_0}\widehat{\mathbf{e}}_r \quad .$$

It is thus evident that:

$$\mathbf{f}_{\mathrm{M}} = \mathbf{f'}_{\mathrm{M}} \quad ,$$

as it is required for the transformation of the transverse component of the spatial part of a four-vector as the Minkowski force is.

61.09.05

Force on a Charge Near a Straight Infinite Neutral Current Wire

Consider the central part of a long rectilinear and uniform cylindrical wire of radius R, cross-sectional area S, carrying a constant current I and electricly neutral globally.

Consider a positive point charge moving parallel to the wire with speed $\mathbf{v}$ and determine the force it feels.

Compare the description of the observatory in the Laboratory Frame ($\mathcal{J}$) and the observatory moving with the charge ($\mathcal{J}'$).

WORKED SOLUTION

Let the wire be along the z axis.

The current, charge and current density in the Laboratory Frame are given by:

$$\rho = 0 \qquad \mathbf{j} \equiv j\widehat{\mathbf{e}}_3 \qquad I = j\pi R^2 \quad .$$

The charge and current density in the $\mathcal{J}'$ Reference Frame are given by:

$$\rho' = -\gamma\,\boldsymbol{\beta}\cdot\mathbf{j}/c = -\gamma v j/c^2 \qquad \mathbf{j}' = \gamma\mathbf{j} \quad .$$

A positive charge moving equiverse with the current is attracted by the wire by the Lorentz force.

In the Laboratory Frame one has:

$$\mathbf{E}_{\parallel} = 0 \quad ,$$
$$\mathbf{E}_{\perp} = 0 \quad ,$$
$$\mathbf{B}_{\parallel} = 0 \quad ,$$
$$\mathbf{B}_{\perp} = \frac{\mu_0 I}{2\pi r}\widehat{\mathbf{e}}_{\psi} \quad .$$

Actually not that $\mathbf{E}_{\parallel} = 0$ is only approximatively valid at large distance from the central part of a long wire. In fact the conservation of the tangential component of the electric field implies that the electric field just outside the wire is the same as inside it. Moreover, a radial component of the field exists in a real finite wire.

In the $\mathcal{J}'$ Reference Frame one has the transformed fields:

$$\mathbf{E'}_{\parallel} = \mathbf{E}_{\parallel} = 0 \quad ,$$
$$\mathbf{E'}_{\perp} = \gamma\,\mathbf{v}\wedge\mathbf{B} = \gamma\,\mathbf{v}\wedge\mathbf{B}_{\perp} \qquad \text{it appears} \quad ,$$
$$\mathbf{B'}_{\parallel} = \mathbf{B}_{\parallel} = 0 \quad ,$$
$$\mathbf{B'}_{\perp} = \gamma\mathbf{B}_{\perp} = \gamma\frac{\mu_0 I}{2\pi r}\widehat{\mathbf{e}}_{\psi} \qquad \text{it is modified} \quad .$$

The electric field which appears in the $\mathcal{J}'$ Reference Frame requires that the wire appears as charged. In fact:

$$E'_{\perp} = \gamma v B = \gamma v\frac{\mu_0 I}{2\pi r} = \frac{\gamma v\mu_0\pi R^2 j}{2\pi r} = -\frac{c^2\rho\mu_0 R^2}{2r} = -\frac{\rho R^2}{2\varepsilon_0 r} \quad ,$$

as it should be for an attractive electric field due to a circular cylindrical wire of charges.

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The Lorentz force in the Laboratory Frame is:

$$\mathbf{B} = \mathbf{B}_{\perp} \qquad \mathbf{f} = q\,\mathbf{v}\wedge\mathbf{B} = \frac{\mu_0 q v I}{2\pi r}\widehat{\mathbf{e}}_3\wedge\widehat{\mathbf{e}}_\psi = -\frac{\mu_0 q v I}{2\pi r}\widehat{\mathbf{e}}_r \qquad \text{attractive} \quad .$$

The Lorentz force in the $\mathcal{I}'$ Reference Frame (the velocity of the charge is zero) is:

$$\mathbf{E} = \mathbf{E}_{\perp} \qquad \mathbf{f}' = q\left(\mathbf{E}'\right) = q\gamma\,\mathbf{v}\wedge\mathbf{B} = \gamma\mathbf{f} \quad .$$

As the **Lorentz Transformation** is perpendicular to the force the Minkowski force should be unchanged by the transformation. In fact:

$$\mathbf{f}_{\mathtt{M}} = \gamma\mathbf{f} \quad ,$$
$$\mathbf{f}'_{\mathtt{M}} = \mathbf{f}' \quad ,$$

which are actually identical, as demonstrated by the previous equations.

61.09.06
ElectroMagnetic Fields of a Point Charge Moving at Constant Velocity

This § is referenced at pages:
[2236, 2236, 2253, 2504, 2512, 2512, 2513, 2513, 2514]

We are going to use **Lorentz Transformation** and the transformation laws for the ElectroMagnetic fields (section § 61.05.02) to determine the ElectroMagnetic fields produced by a point charge q moving at constant velocity $\mathbf{V}$ (with $\boldsymbol{\beta} \equiv \mathbf{V}/c$) in a certain Reference Frame, $\mathcal{I}$.

Let $\mathcal{I}$ be the inertial Reference Frame where the charge is seen to move at constant velocity $\mathbf{V}$. Let $\mathcal{I}'$ be the inertial Reference Frame where the charge is at rest.

Let the fixed point P, a fixed point in $\mathcal{I}$, be the point where the ElectroMagnetic fields have to be calculated.

Let $\mathbf{r}_{\mathtt{P}}$ and $\mathbf{r}_{\mathtt{C}}$ be the positions, in the Reference Frame $\mathcal{I}$, respectively, of the point P and of the moving point charge C.

Let $\mathbf{r}[t]$ be the vector from the position of the moving point charge to the point P, so that the kinematics of the motion of the point charge is:

$$\mathbf{r}[t] \equiv \mathbf{r}_{\mathtt{P}} - \mathbf{r}_{\mathtt{C}}[t] = \mathbf{r}_{\mathtt{P}} - \mathbf{V}\,(t - t_0) - \mathbf{r}_{\mathtt{C}}[t = t_0] \quad .$$

Note that the relation relating the position of P to the origin of the Coordinate System in $\mathcal{I}$ and to C are the same in terms of the coordinates of both $\mathcal{I}$ and $\mathcal{I}'$:

$$\mathbf{r}_{\mathtt{P}} = \mathbf{r} + \mathbf{r}_{\mathtt{C}} \qquad \mathbf{r}'_{\mathtt{P}} = \mathbf{r}' + \mathbf{r}'_{\mathtt{C}} \quad ,$$

as all the coordinates are related by the linear **Lorentz Transformation** :

$$\mathbf{x}' = \mathbb{L}\mathbf{x} \quad .$$

Let $\theta[t]$ be the angle between the velocity vector $\mathbf{V}$ and the position vector $\mathbf{r}$:

$$\cos\left[\theta[t]\right] = \frac{\mathbf{V}\cdot\mathbf{r}}{|\mathbf{r}||\mathbf{V}|} \quad .$$

We can show that the (time dependent) electric field is:

$$\boxed{\mathbf{E}[\mathbf{r}_{\mathtt{P}}] = \frac{1}{4\pi\varepsilon_0}\frac{q\left(1-\beta^2\right)}{\left(1-\beta^2\sin^2\left[\theta\right]\right)^{3/2}}\frac{\mathbf{r}}{r^3}} \quad , \tag{61.09.10}$$

while the (time dependent) magnetic field is:

$$\boxed{\mathbf{B} = \frac{1}{c}\boldsymbol{\beta}\wedge\mathbf{E}_{\perp} = \frac{1}{c}\boldsymbol{\beta}\wedge\mathbf{E} = c\frac{\mu_0}{4\pi}\frac{q\left(1-\beta^2\right)}{\left(1-\beta^2\sin^2\left[\theta\right]\right)^{3/2}}\frac{\boldsymbol{\beta}\wedge\mathbf{r}}{r^3}} \quad . \tag{61.09.11}$$

→ 2504 2514

Note that the fields are expressed in terms of the instantaneous position of the point charge, that is they are radial with respect to the instantaneous position of the point charge. They are not expressed in terms

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of the **retarded position** of the point charge (read § 49 for the precise definition of the **retarded position** of the point charge, that is its apparent position).

Choose the z and z' axes parallel to $\mathbf{V}$ and denote as $\widehat{\mathbf{n}} \equiv \widehat{\mathbf{e}}_3$ the unit vector associated with this direction.

Let the moving charge be at the origin of the Coordinate System of $\mathcal{I}'$ and let the two origins of the two Coordinate System of $\mathcal{I}$ and $\mathcal{I}'$ be coincident at the times: $t = t' = 0$.

We are going to calculate the ElectroMagnetic fields as a function of the instantaneous position of the point charge, not of the retarded one (read § 49 for the precise definition of the **retarded position** of the point charge, that is its apparent position). We will therefore calculate, at a generic time t, the ElectroMagnetic fields at a point P, with coordinates $\mathbf{r}_\mathrm{P}$ and fixed in $\mathcal{I}$. The point P is located at a distance H from the velocity vector $\mathbf{V}$ and has a distance d from the origin along the direction of $\mathbf{V}$, that is:

$$\widehat{\mathbf{n}} \equiv \widehat{\mathbf{e}}_3 \equiv \mathbf{V}/V \qquad \mathbf{r}_\mathrm{P} = \widehat{\mathbf{n}} d + (H/|\mathbf{r}|)\, \mathbf{n} \wedge (\mathbf{r} \wedge \mathbf{n}) \quad .$$

The distance H is the so-called impact parameter of the point particle with respect to the point where the field is being calculated.

Moreover, with our conventions of $\mathcal{I}$ and $\mathcal{I}'$, we have:

$$\mathbf{r}_\mathrm{C} = \mathbf{V}t \qquad \mathbf{r}'_\mathrm{C} = \mathbf{0} \quad ,$$

and

$$\mathbf{r}_\mathrm{P} = \{x_P, y_P, z_P = d\} \qquad H \equiv \sqrt{x_P^2 + y_P^2} \quad .$$

Let us then transform to the Rest Frame of the moving point charge, the inertial Reference Frame $\mathcal{I}'$ whose z' axis is parallel to the z axis (and to $\mathbf{V}$).

The **Lorentz Transformation** of the time-space coordinates of the moving point charge, C, and of the point P are:

$$\begin{aligned} &t' = \gamma\left(t - Vz/c^2\right) && && \\ &z' = \gamma\left(z - Vt\right) && z'_C = 0 && z'_P[t] = \gamma\left(z_P - Vt\right) = \gamma\left(d - Vt\right) \equiv z'_C + z' = z' \quad , \\ &r'_\perp = r_\perp = H && && \end{aligned}$$

We can now calculate, first, and transform to the $\mathcal{I}$ Reference Frame, later, the ElectroMagnetic fields produced at the point P in the rest Reference Frame of the point charge, $\mathcal{I}'$.

It must be emphasized that this transformation involves both transformation of the ElectroMagnetic fields and transformation of the time-space coordinates where the fields are to be calculated. In particular equations (61.03.01), (61.03.03), for the electric field, as well as equations (61.03.02), (61.03.04), are valid provided the ElectroMagnetic fields are those at the same time-space point which, however, has different coordinates in the two different Reference Frame.

The components of the ElectroMagnetic fields parallel to $\mathbf{V}$ do not change.

Electric Field

The components of the Electric field parallel to $\mathbf{V}$ do not change.

The transformation of component of the Electric field perpendicular to $\mathbf{V}$ can be calculated by using the inverse transform of equation (61.03.03):

$$\mathbf{E}_\perp = \gamma\left(\mathbf{E}' - c^{+1}\,\boldsymbol{\beta} \wedge \mathbf{B}'\right)_\perp \quad ,$$

and the fact that $\mathbf{B}' = 0$ one finds:

$$\mathbf{E}'[\mathbf{r}'_\mathrm{P}, t'] = \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{r}'}{r'^3} = \begin{cases} E'_z[\mathbf{r}'_\mathrm{P}, t'] = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r'^3}(z') = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r'^3}(\gamma\,(d - Vt)) \quad , \\ \Longrightarrow\; E_z[\mathbf{r}_\mathrm{P}, t] = E'_z[\mathbf{r}'_\mathrm{P}, t'] = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r'^3}(\gamma\,(d - Vt)) \quad , \\ E'_\perp[\mathbf{r}'_\mathrm{P}, t'] = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r'^3}(r'_\perp)\,\dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r'^3}(H) \quad , \\ \Longrightarrow\; \mathbf{E}_\perp[\mathbf{r}_\mathrm{P}, t] = \gamma\mathbf{E}'_\perp[\mathbf{r}'_\mathrm{P}, t'] = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r'^3}(\gamma H) \end{cases} .$$

→ 2504

Note that in both components a multiplicative factor γ appears, in one case due to the transformation of abscissa in the other case due to the transformation of the electric field.

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Moreover, one needs to transform r':

$$r' \equiv \sqrt{(x')^2 + (y')^2 + (z')^2} = \sqrt{x^2 + y^2 + (\gamma\,(d - Vt))^2} \quad .$$

We observe that all the dependence on d and t just happens in the combined variable $d - Vt$. Therefore the ElectroMagnetic fields, for the observer $\mathcal{J}$, translate rigidly with time. We can therefore limit to calculate the fields at $t = 0$ and generalise the final result for any time by replacing, in the final result, $d \to d - Vt$.

The expression for r', at $t = 0$, can be written as:

$$r' = \left(H^2 + \gamma^2 d^2\right)^{1/2} = \gamma \left(d^2 + \frac{H^2}{\gamma^2}\right)^{1/2} = \gamma \left(d^2 + H^2 + H^2 \left(\frac{1}{\gamma^2} - 1\right)\right)^{1/2} = \quad ,$$

$$\gamma \left(d^2 + H^2\right)^{1/2} \left(1 - \beta^2 \frac{H^2}{d^2 + H^2}\right)^{1/2} \equiv \gamma \left(d^2 + H^2\right)^{1/2} \left(1 - \beta^2 \sin^2[\theta]\right)^{1/2} \quad .$$

Finally, letting θ be the angle between $\mathbf{r}$ and $\mathbf{V}$ in the stationary Rest Frame, one finds, at $t = 0$:

$$E_z[\mathbf{r}_P, t = 0] = \frac{1}{4\pi\varepsilon_0} \frac{q}{r'^3} (\gamma d) = \quad ,$$

$$\frac{1}{4\pi\varepsilon_0} \frac{q\gamma d}{\gamma^3 \left(d^2 + H^2\right)^{3/2} \left(1 - \beta^2 \sin^2[\theta]\right)^{3/2}} = \frac{1}{4\pi\varepsilon_0} \frac{qd\left(1 - \beta^2\right)}{\left(d^2 + H^2\right)^{3/2} \left(1 - \beta^2 \sin^2[\theta]\right)^{3/2}} \quad ,$$

$$E_\perp[\mathbf{r}_P, t = 0] = \frac{1}{4\pi\varepsilon_0} \frac{q}{r'^3} (\gamma H) = \quad ,$$

$$\frac{1}{4\pi\varepsilon_0} \frac{q\gamma H}{\gamma^3 \left(d^2 + H^2\right)^{3/2} \left(1 - \beta^2 \sin^2[\theta]\right)^{3/2}} = \frac{1}{4\pi\varepsilon_0} \frac{qH\left(1 - \beta^2\right)}{\left(d^2 + H^2\right)^{3/2} \left(1 - \beta^2 \sin^2[\theta]\right)^{3/2}} \quad .$$

Putting together the above equations and replacing $d \to d - Vt$ one finds:

$$\mathbf{E}[\mathbf{r}_P, t] = \frac{1}{4\pi\varepsilon_0} \frac{q\left(1 - \beta^2\right)}{\left(1 - \beta^2 \sin^2[\theta]\right)^{3/2}} \frac{\mathbf{r}}{r^3} \qquad \mathbf{r}[t] \equiv \mathbf{r}_P - \mathbf{r}_C[t] = \mathbf{r}_P - \mathbf{V}t \qquad \theta = \theta[t] \quad . \tag{61.09.12}$$

→ 2513

Note the electric field is not conservative, because it is a radial field whose module depends on the angle. However, as the situation is not a static one this is not a problem and the rotor of the electric field will be compensated by a changing magnetic field.

The result shows that the electric field, when one considers the moving charge at the present position, is still radial, but its module depends on θ. This is not a violation of the finite velocity of propagation of signals because the point charge has been known to move at constant velocity since an infinite time.

At fixed distance from the charge the θ dependence shows that the field is weaker in the forward and backward directions. This anisotropy increases at increasing velocities.

In ultra-relativistic limit, $\beta \longrightarrow 1$, the electric field becomes basically a field transverse to the velocity of the charge. In fact:

$$\begin{cases} \theta = 0 \longrightarrow E \sim \left(1 - \beta^2\right) & E \longrightarrow 0 \quad \text{for } \beta \longrightarrow 1 \\ \theta = \pi/2 \longrightarrow E \sim \left(1 - \beta^2\right)^{-1/2} & E \longrightarrow \infty \quad \text{for } \beta \longrightarrow 1 \end{cases} \quad .$$

This behavior is not incompatible with Gauss law, because the field increases in the plane perpendicular to the velocity but the region in solid angle where the field increases is decreasing at high speeds.

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Magnetic Field

The components of the Magnetic field parallel to $\mathbf{V}$ do not change.

It is therefore zero in each Rest Frame. The transformation of component of the Magnetic field perpendicular to $\mathbf{V}$ can be calculated by using the inverse transform of equation (61.03.04):

$$\mathbf{B}_\perp = \gamma\left(\mathbf{B}' + c^{-1}\boldsymbol{\beta}\wedge\mathbf{E}'\right)_\perp \quad . \tag{61.09.13}$$

By using equation (61.09.06) to express $\mathbf{E}'_\perp$ and observing that $\mathbf{E}'_\perp$ can be replaced with $\mathbf{E}$ without changing the result of the vector product one finds:

$$\boxed{\mathbf{B}[\mathbf{r}_\mathrm{P},t] = \mathbf{B}_\perp = \frac{1}{c}\boldsymbol{\beta}\wedge\mathbf{E}_\perp = \frac{1}{c}\boldsymbol{\beta}\wedge\mathbf{E} = c\frac{\mu_0}{4\pi}\frac{q\left(1-\beta^2\right)}{\left(1-\beta^2\sin^2[\theta]\right)^{3/2}}\frac{\boldsymbol{\beta}\wedge\mathbf{r}}{r^3} \qquad \mathbf{r}[t] \equiv \mathbf{r}_\mathrm{P} - \mathbf{r}_\mathrm{C}[t] = \mathbf{r}_\mathrm{P} - \mathbf{V}t \qquad \theta = \theta[t]} \quad . \tag{61.09.14}$$

The result shows that the magnetic field, when one considers the moving charge at the present position, is still radial, but its module depends on θ. This is not a violation of the finite velocity of propagation of signals because the point charge has been known to move at constant velocity since an infinite time.

At fixed distance from the charge the θ dependence shows that the field is weaker in the forward and backward directions. This anisotropy increases at increasing velocities.

In ultra-relativistic limit, $\beta \longrightarrow 1$, the magnetic field has the same behavior than the electric field, and becomes basically a field transverse to the velocity of the charge. In fact:

$$\begin{cases} \theta = 0 \longrightarrow B \sim \left(1-\beta^2\right) \qquad B \longrightarrow 0 \qquad \text{for } \beta \longrightarrow 1 \quad , \\ \theta = \pi/2 \longrightarrow B \sim \left(1-\beta^2\right)^{-1/2} \qquad B \longrightarrow \infty \qquad \text{for } \beta \longrightarrow 1 \end{cases} \quad .$$

ElectroMagnetic Fields

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The Electric|Magnetic fields are always perpendicular: $\mathbf{E}\cdot\mathbf{B}$.

In ultra-relativistic limit, $\beta \longrightarrow 1$, the modules tend to become identical, in a system of units where $c = 1$.

The ElectroMagnetic fields produced by a ultra-relativistic charge resembles a impulse of a PW.

Note that the uniformly moving charge is not radiating ElectroMagnetic waves.

The ElectroMagnetic fields move as rigid fields because the charge is supposed to be moving with constant velocity since an infinite time so that any effect of past accelerations has already reached an infinite distance.

61.09.07

Magnetic Field Produced by a Rectilinear Infinite Line of Charges Moving at Constant Velocity

Use the results of § 61.09.06 to calculate the magnetic field produced by a rectilinear line of charges moving at constant velocity, like the moving charges in a infinite rectilinear current wire of section A carrying a current I.

Introduce a cylindrical Coordinate System having as the z axis the current wire, oriented as the current. Use equation (61.09.11) for an infinitesimal element $\mathrm{d}z$ of the current wire. Let R be the distance of the point P from the wire. Then:

$$\mathrm{d}\mathbf{B} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,\mathrm{d}z\beta\left(1-\beta^2\right)}{\left(1-\beta^2\sin^2[\theta]\right)^{3/2}}\frac{\sin^3[\theta]}{R^2}\widehat{\mathbf{e}}_\psi \quad ,$$

$$z = -R\cot\theta \qquad \Longrightarrow \qquad \mathrm{d}z = \frac{R}{\sin^2[\theta]}\mathrm{d}\theta \quad ,$$

$$\mathbf{B} = \int_0^\pi \mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\lambda\beta}{R}\widehat{\mathbf{e}}_\psi \quad ,$$

$$\lambda = \rho A \qquad \Longrightarrow \qquad \mathbf{B} = \frac{\mu_0}{4\pi}\frac{\beta\rho A}{R}\widehat{\mathbf{e}}_\psi = \frac{\mu_0}{4\pi}\frac{\beta\rho A}{R}\widehat{\mathbf{e}}_\psi = \frac{\mu_0}{4\pi}\frac{jA}{R}\widehat{\mathbf{e}}_\psi = \frac{\mu_0}{4\pi}\frac{I}{R}\widehat{\mathbf{e}}_\psi \quad .$$

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The indefinite integral:

$$\int \frac{\sin[\theta]}{\left(1-\beta^2 \sin[\theta]^2\right)^{3/2}} \mathrm{d}\theta = \frac{\sqrt{2}\cos[\theta]}{(-1+\beta^2)\sqrt{2-\beta^2+\beta^2\cos[2\theta]}} \quad ,$$

was used.

61.09.08
A Simplified Model for a Moving Magnetic Dipole

This § is referenced at pages:
[2395, 2395, 2481, 2496, 2496, 2499]

According to § 61.09.03 a current carrying long rectilinear stationary wire, globally neutral, will appear as charged to an observer moving along the wire. This is an apparent violation of the Lorentz invariance of the charge, as the two different observers will measure a different total charge. The problem lies in the unphysical infinite wire.

In this section a real (possibly super-conducting) closed current loop is modeled. The use of a super-conducting loop avoids nuisances such as batteries, external wires and geometrical asymmetries.

Use the results of problem § 61.09.03 to cross-check the results of problem § 61.09.10 in the simple case of a real **Magnetic Dipole** made of a plane rectangular current loop, with sides a and b and negligible dimensions of its cross-section (of area S), where a current I passes in its own Rest Frame.

Assume that the loop is moving, with respect to an inertial Reference Frame, with constant velocity $\mathbf{u}$ parallel to its sides of length a along the axes $z \equiv z'$. Let the sides with length a be parallel to the axes $z \equiv z'$ and the sides with length b be perpendicular to the axes $z \equiv z'$. Look at the loop so that the current runs in counterclock-wise direction. Therefore the **Magnetic Dipole** moment of the loop is directed toward the observer.

The four sides of the rectangle appear neutral in its own Rest Frame: $\rho' = 0$. On the other hand: $\mathbf{j}' \neq 0$.

In the upper side of length a the current goes in the negative $z \equiv z'$ direction, so that it appears negatively charged to the observer who sees the loop moving.

In the lower side of length a the current goes in the positive $z \equiv z'$ direction, so that it appears positively charged to the observer who sees the loop moving.

Whatever the velocity $\mathbf{u}$, the two sides of length a will appear charged to the observer who sees the loop moving. The **Lorentz Transformation** of the current four-vector allows one to calculate the charge density seen by the moving observer, as discussed in problem § 61.09.03.

In fact:

$$\rho = \gamma\left(\rho' + \beta {j'}_z/c\right) = \gamma\beta {j'}_z/c < 0 \qquad \text{in the upper side with } {j'}_z < 0 \quad ,$$
$$\rho = \gamma\left(\rho' + \beta {j'}_z/c\right) = \gamma\beta {j'}_z/c > 0 \qquad \text{in the lower side with } {j'}_z > 0 \quad .$$

Therefore the observer who sees the loop moving will see on the two sides of length a the charge densities:

$$\rho = \pm\gamma|\beta|\left|j'\right|/c \quad ,$$

the $\pm$ sign, different for the two sides, being decided by the current being parallel or anti-parallel to $\mathbf{u}$, according to the above equations.

Note that there is no effect on the charge and currents the other two sides, of length b, because for those sides $\mathbf{j}\cdot\mathbf{u} = 0$, that is the current density is transverse to the velocity of the **Lorentz Transformation** and there is therefore no transformation at all (it remains neutral). However, the cross-section of the other two sides, of length b, is shrinked by a factor γ due to the contraction of the length in the direction of motion.

The loop appears now charged, in its two sides of length a, which appear contracted by a factor γ, generating an **Electric Dipole** moment with absolute value:

$$|p| = \left(\gamma|\beta|\left|j'\right|/c\right)(Sa/\gamma)(b) = \left(|\beta|\left(\left|I'\right|/S\right)/c\right)Sab = |\beta|\left|I'\right|ab/c = |\beta|\left|m'\right|/c \quad .$$

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61.09.09
Relative Motion Between a Point Charge and a Small Magnet

Read § 58.03.03.

61.09.10
Potentials for a Moving Magnetic Dipole

This § is referenced at pages:
[2505, 2513]

©|V.Namias|Am.J.Phys, , ..., ..., ...Ed., ... 56, 898 (1988) - doi: 10.1119/1.15406|*Definitive!*|
©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|||

Consider an ideal point **Magnetic Dipole** , with **Magnetic Dipole** moment, $\mathbf{m}'$, which is moving with respect to the observer with constant velocity $\mathbf{V}$ while keeping a constant magnetic Dipole moment vector, $\mathbf{m}'$.

We want to determine the **scalar electric potential** as seen by the observer and determine its non relativistic limit. Let $\mathbf{r}$ be the position of the point where we want to calculate the **ElectroMagnetic potentials** and the position of the **Dipole** in the **Dipole** Rest Frame.

Use $c = 1$. We know the potentials of the **Magnetic Dipole** in its rest system, $\mathcal{I}'$, (read § 16.12.12):

$$\Phi'[\mathbf{r}', t'] = 0 \qquad \mathbf{A}'[\mathbf{r}', t'] = \frac{\mu_0}{4\pi}\frac{\mathbf{m}' \wedge \mathbf{r}'}{r'^3} \quad ,$$

and we want to calculate the potentials in the $\mathcal{I}$ Reference Frame of the observer who sees the **Magnetic Dipole** moving with constant velocity $\mathbf{V}$ while keeping a constant **Magnetic Dipole** moment, $\mathbf{m}'$.

Choose a special **Lorentz Transformation** with $z = z'$ and $\mathbf{V}$ along the $z = z'$ axis:

$$\begin{aligned} z' &= \gamma\,(z - Vt) \quad , \\ t' &= \gamma\,(t - Vz) \quad , \\ x' &= x \quad , \\ y' &= y \quad . \end{aligned}$$

We can write ($\mathbf{V}$ is the velocity of $\mathcal{I}'$ with respect to $\mathcal{I}$):

$$\Phi[\mathbf{r}, t] = \gamma\left(\Phi'[\mathbf{r}', t'] + \mathbf{V}\cdot A'[\mathbf{r}', t']\right) = \gamma V A'_z[\mathbf{r}', t'] \tag{61.09.15}$$

→ 2507

$$\Phi[\mathbf{r}, t] = \frac{\mu_0}{4\pi}\gamma\frac{\mathbf{V}\cdot(\mathbf{m}' \wedge \mathbf{r}')}{r'^3} = \frac{\mu_0}{4\pi}\gamma\frac{\mathbf{r}'\cdot(\mathbf{V} \wedge \mathbf{m}')}{r'^3} = \tag{61.09.16}$$

→ 2507

$$\frac{\mu_0}{4\pi}\gamma\frac{\mathbf{r}'\cdot(V\widehat{\mathbf{e}}_3 \wedge \mathbf{m}')}{r'^3} = \frac{\mu_0}{4\pi}\gamma V\frac{y'\mathbf{m}'_1 - x'\mathbf{m}'_2}{\left((x')^2 + (y')^2 + (z')^2\right)^{3/2}} = \quad ,$$

$$\frac{\mu_0}{4\pi}\gamma V\frac{y\mathbf{m}'_1 - x\mathbf{m}'_2}{\left(x^2 + y^2 + (\gamma\,(z - Vt))^2\right)^{3/2}} \quad .$$

As the potential is not a function of z and t separately but only of the combination $z - Vt$, the function moves rigidly with time with speed V. Therefore we can limit to calculate the potential at a specified time

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$t=0$, when the **Dipole** is located at $\mathbf{r}=\mathbf{0}$ in the $\mathcal{I}$ Reference Frame. One thus finds:

$$\Phi[\mathbf{r},t=0]=\frac{\mu_0}{4\pi}\gamma V\frac{y\mathbf{m}'_1-x\mathbf{m}'_2}{\left(x^2+y^2+(\gamma z)^2\right)^{3/2}}= \quad ,$$

$$\frac{\mu_0}{4\pi}\gamma V\frac{y\mathbf{m}'_1-x\mathbf{m}'_2}{\left(r^2+z^2\left(\gamma^2-1\right)\right)^{3/2}}= \quad ,$$

$$\frac{\mu_0}{4\pi}\gamma V\frac{y\mathbf{m}'_1-x\mathbf{m}'_2}{r^3\left(1+\cos^2[\theta]\left(\gamma^2-1\right)\right)^{3/2}}= \quad ,$$

$$\frac{\mu_0}{4\pi}\gamma V\frac{y\mathbf{m}'_1-x\mathbf{m}'_2}{\gamma^3 r^3\left(1-\beta^2\sin^2[\theta]\right)^{3/2}} \quad .$$

In order to recover the time dependence in equation (61.09.17) one needs to do the replacement $z\longrightarrow z-Vt$. Finally:

$$\Phi[\mathbf{r},t=0]=\frac{\mu_0}{4\pi}\frac{\mathbf{r}\cdot(\mathbf{V}\wedge\mathbf{m}')}{r^3\left(1-\beta^2\sin^2[\theta]\right)^{3/2}}\left(1-\beta^2\right) \qquad \mathbf{r}[t]\equiv\mathbf{r}_{\mathrm{P}}-\mathbf{r}_{\mathrm{C}}[t]=\mathbf{r}_{\mathrm{P}}-\mathbf{V}\,(t-t_0)-\mathbf{r}_{\mathrm{C}}[t=t_0] \quad .$$

(61.09.17) → 2507

The identity

$$\gamma^2-1=v^2\gamma^2 \quad ,$$

was used in the derivation.

In the last passage the vector form has been resurrected. That form would have been obtained in case a general boost would be used.

Let us develop to lowest order in the velocity to determine the non-relativistic limit. To do that, it is enough to take the lowest order expression of all the terms. One finds:

$$\Phi[\mathbf{r},t=0]=\frac{\mu_0}{4\pi}\gamma\frac{\mathbf{r}\cdot(\mathbf{V}\wedge\mathbf{m}')}{\gamma^3 r^3\left(1-\beta^2\sin^2[\theta]\right)^{3/2}}\simeq\frac{\mu_0}{4\pi}\frac{\mathbf{r}\cdot(\mathbf{V}\wedge\mathbf{m}')}{r^3}\equiv\frac{\mu_0}{4\pi}\frac{\mathbf{r}\cdot\mathbf{p}_{\mathrm{E}}}{r^3} \quad ,$$

where the equivalent **Electric Dipole** moment has been introduced:

$$\boxed{\mathbf{p}_{\mathrm{E}}\equiv\frac{\mathbf{V}\wedge\mathbf{m}'}{c^2}} \quad .$$

Note that the observer will also see a **vector magnetic potential** which can be calculated in a similar way. The observer, therefore, will not observe a pure **Electric Dipole** field but also an additional ElectroMagnetic field.

Note that the identification with an **Electric Dipole** field is not limited to the non-relativistic limit. In fact inspection of equation (61.09.15) shows that a pure electric Dipole field moving with velocity $\mathbf{V}$ would give exactly the transformed **scalar electric potential** of equation (61.09.16).

Note that the factor $\mathbf{r}'/r'^3$ in equation (61.09.16) which need to be transformed, is exactly the same appearing in the ElectroMagnetic fields of the charge moving at constant velocity, and this explains the very similar results and intermediate passages.

Note that when one passes to describe the induced Polarization (that is induced **Electric Dipole** moment per unit volume) an additional factor γ appears due to the transformation of the volume.

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61.09.11
Moving Electric|Magnetic Dipoles

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|problem 11.28, 11.29, 11.30||

61.09.12
Poynting-Robertson Effect (Poynting-Robertson Drag)

This § is referenced at pages:
[1628]

©|B.W.Carroll & D.A.Ostlie, *An Introduction To Modern Astrophysics*, ..., ..., 1stEd., ...|4.3, P20.13, P20.14, P21.3||

©|T.Padmanabhan, *Theoretical Astrophysics, Vol. 1,2,3*, 2000/2001/2002, CUP, ...Ed., ...|||

The Poynting-Robertson effect is a consequence of the Head-Light effect, read **§ ??**. Consider a grain of dust orbiting the Sun in a circular orbit. Assume it is in thermal equilibrium within the solar radiation field, for instance by assuming that it is spinning very fast. Assume, in order to simplify the problem, that for any photon absorbed another photon of the same energy is emitted, in order to maintain thermal equilibrium, within the kinematic constraints imposed by the Head-Light effect.

Neglect the **Doppler shift** of the emitted radiation on the basis of the fact that the velocity of the grain of dust is assumed to be non relativistic.

Determine the linear momentum loss of the grain of dust by using both the description in the Reference Frame fixed to the Sun (heliocentric Rest Frame) and in the Reference Frame moving with the grain of dust. Verify the compatibility of the two results.

Determine the equation describing the change in the angular momentum of the grain of dust as a function of time and as a function of the photon absorption cross-section, Σ_g.

WORKED SOLUTION

Describe the problem in the heliocentric Rest Frame.
The conservation of linear momentum in the radial direction provides the effect of the radiation pressure, which counteracts the effect of the gravitational field of the Sun, see problem § 46.05.02. On the other hand the tangential linear momentum balance give the following equation, where all the components of the momenta must be interpreted as the tangential component of the linear momentum,

$$p_\gamma + P = \hat{p}_\gamma + \hat{P} \Longrightarrow P = \hat{p}_\gamma + \hat{P} \Longrightarrow \hat{P} = P - \hat{p}_\gamma = P - \beta\varepsilon \Longrightarrow \Delta P \equiv \hat{P} - P = -\beta\varepsilon \quad ,$$

in terms of the tangential components of the momenta of the grain of dust before and after the absorption-emission process, P and $\hat{P}$, the corresponding tangential components of the momenta of the photons, p_γ and $\hat{p}_\gamma$ and the last passage follows from equation (**??**). Note that the energy ε appearing in the last equation is the energy of the emitted photon, which is, according to the hypotheses, equal to the energy of a corresponding absorbed photon.
The β factor to use in the previous equation needs some attention. If one considers the real process where a small but macroscopic grain of dust is immersed, in thermal equilibrium, in the Sun radiation field one actually has a great many absorption and re-emission processes which change almost continuously the speed of the grain of dust. Using the results of **§ ??** one can then conclude that the speed to use is the current speed of the grain of dust. On the other hand if one considers a single absorption and re-emission process, possibly not an instantaneous one, and one notes that, according to the results of **§ ??** one must use the speed of the grain of dust before the emission, the speed of the grain of dust at the re-emission is slightly changed due to the change in energy following the photon absorption. However, the change of speed can be calculated, from the

conservation of the tangential components of the momenta, as

$$V_1 = \frac{\mathcal{E}}{\mathcal{E}+\varepsilon} V_0 \simeq V_0 \left(1 - \frac{\varepsilon}{\mathcal{E}}\right) \quad ,$$

in terms of the total energy of the grain of dust, $\mathcal{E}$, and the total energy of the photon ε. It is then clear that the use of the exact velocity at emission, V_1, would give a correction of the order of $\varepsilon|\mathcal{E}$, which is small under our hypotheses.
When photons traveling in the radial direction are absorbed by the grain of dust the tangential component of the linear momentum of the grain of dust does not change, while the radial component does change. When photons are emitted with a isotropic distribution in the grain of dust rest Rest Frame the grain of dust has a recoil. By hypothesis the energy of the emitted photon, on average, is equal to the energy of the incoming photon.
Every absorption-emission event thus implies a linear momentum loss of the grain of dust approximately equal to $\Delta P = -\beta\varepsilon/c$ (re-introducing c for future reference).
Consider all the photons to have the same energy ε, equal to the average photon energy. The following equations can be derived as a function of the intensity I_0 of the Sun light, the number of incoming photons, N_0, the Sun Luminosity $\mathcal{L}_\odot$ and the cross section for photon absorption by the dust of grain, Σ_g, the distance form the Sun, r.

$$\frac{\mathrm{d}N}{\mathrm{d}t} \equiv \Sigma_g \frac{d^2 N_0}{\mathrm{d}t\ \mathrm{d}A} \quad ,$$

$$I_0 = \varepsilon \frac{d^2 N_0}{\mathrm{d}t\ \mathrm{d}A} \quad ,$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \Sigma_g \frac{I_0}{\varepsilon} = \frac{\Sigma_g}{\varepsilon} \frac{\mathcal{L}_\odot}{4\pi r^2} \quad .$$

The orbital angular momentum loss of the grain of dust, as a function of the orbital angular momentum loss corresponding to a single absorption-emission event, ΔL, is thus:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \Delta L \frac{\mathrm{d}N}{\mathrm{d}t} = r\Delta P \frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\Sigma_g}{\varepsilon} \frac{\mathcal{L}_\odot}{4\pi r^2} \frac{-\beta r\varepsilon}{c} = -\Sigma_g \frac{\mathcal{L}_\odot}{4\pi r^2 mc^2} L \quad .$$

Orbital angular momentum will not be conserved, due to the non-central character of the Poynting-Robertson force, actually also named Poynting-Robertson drag, for this very reason. The linear momentum loss per unit time represents the Poynting-Robertson drag.
In the grain of dust Rest Frame there is no recoil due to the emission of the photons but there is a recoil due to the absorption of the photons, because of the aberration of the light seen by the Rest Frame.

61.09.13
Yarkovsky Effect

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The Yarkovsky effect is similar to the Poynting-Robertson effect but it comes into play when the emission of thermal photons is anisotropic in the Rest Frame of the grain of dust, for instance because there is no thermalization of the grain of dust due, for instance, to slow rotation.

If often causes small irregularities on the orbits of small asteroids due to their irregular shape.

The Yarkovsky effect is small in case of very small and very large bodies. In fact the force is approximately proportional to the surface area and the mass to the volume; therefore the resulting acceleration is inversely proportional to the dimensions of the body. On the other hand the conduction of heat across the whole small body effectively diminishes temperature differences on the surface and the corresponding emission is then almost isotropic.

61.09.14
Transformation Law for the Angular Intensity of Light

©|W.Greiner *et al.*, , ..., ..., ...Ed., ...|||

Consider an isotropic light source at rest in the Reference Frame $\mathcal{I}$. An observer in the Reference Frame $\mathcal{I}'$, moving with velocity $\mathbf{u}$ with respect to $\mathcal{I}$, sees a non isotropic light distribution, $f'[\theta']$.

Determine $f'[\theta']$.

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61.10 Exercises Problems and Physical Applications

61-001 Beams of Charged Particles in Particle Accelerators

The reduction of the repulsive force among identical charged particles forming a straight beam is well-known in the physics of particle accelerators.

The explanation for this issue has to be found by employing the correct Lorentz transformations for the ElectroMagnetic fields and dynamical quantities, see section § 61.09.04.

61-002 Invariants of the ElectroMagnetic Field

Show that the two quantities

$$\left(c^2B^2 - E^2\right) \qquad \mathbf{E}\cdot\mathbf{B} \quad , \tag{61.10.01}$$

are Lorentz invariant by direct calculation from equations (61.03.01), (61.03.02), (61.03.03), (61.03.04).

61-003 Potentials for a Uniform and Constant ElectroMagnetic Field

This § is referenced at pages:
[Never referenced.]

©|J.P.Pérez *et al.*, *Électromagnétisme*, ..., DUNOD, ...Ed., WEB - URL.|P16.3||

Read also § 36-042.

Consider a uniform and constant ElectroMagnetic field:

$$\mathbf{E} = E_0\widehat{\mathbf{e}}_2 \qquad \mathbf{B} = B_0\widehat{\mathbf{e}}_3 \quad .$$

- Show that two possible choices of potentials are the following:

$$\Phi_1 = -E_0 y \qquad \mathbf{A}_1 = \frac{1}{2}\mathbf{B}\wedge\mathbf{r} \quad ,$$

$$\Phi_2 = 0 \qquad \mathbf{A}_2 = (xB_0 - tE_0)\,\widehat{\mathbf{e}}_2 \quad .$$

- Determine the scalar field allowing to pass from one set to the other.

61-004 Some Properties of the ElectroMagnetic Field Tensor

This § is referenced at pages:
[2497]

- Verify that the ElectroMagnetic field tensor $F^{\mu\nu}$ is gauge invariant and that its expression in terms of the Electric|Magnetic fields is the one of equation (61.05.01).
- Verify that the expression of $G^{\mu\nu}$ in equation (61.05.01) follows from the definition in the same equation (61.05.01) as a function of $F^{\mu\nu}$.
- Show that the two quantities

$$\boxed{F_{\mu\nu}F^{\mu\nu} = 2\left(B^2 - E^2/c^2\right)} \qquad \boxed{G_{\mu\nu}F^{\mu\nu} = 4\,\mathbf{E}\cdot\mathbf{B}/c} \quad , \tag{61.10.02}$$

  are Lorentz invariant.
- Show that:

$$\boxed{\det[\mathbb{F}] = \left(\mathbf{E}\cdot\mathbf{B}\right)^2/c^2} \quad .$$

BRAFT - W - PROGRESS

61-005 Field Equations in Covariant Form

Verify the equivalence between the four **Maxwell Equations** ((36.08.01), (36.08.02), (36.08.03), (36.08.04)), and their expression in covariant form ((61.05.02), (61.05.04)).

61-006 Consistency

Show that equation (61.05.04) is consistent with charge conservation.

61-007 Trace of the ElectroMagnetic Energy-LinMomentum Tensor

Show that the trace of the Energy-LinMomentum tensor in equation (61.07.01) is zero.

61-008 Properties of the ElectroMagnetic Energy-LinMomentum Tensor

Show that equation (61.07.01) gives: $T^{00}_{\cdot\cdot} = \dfrac{\varepsilon_0 \mathbf{E}^2}{2} + \dfrac{\mathbf{B}^2}{2\mu_0}$.

61-009 EM Energy-LinMomentum Tensor for MPW

Consider ElectroMagnetic MPW and show, from the general expression of the EM Energy-LinMomentum tensor in equation (61.07.01), that the EM Energy-LinMomentum tensor is:

$$T^{\mu\nu}_{\cdot\cdot} = u\frac{c^2}{\omega^2}k^\mu k^\nu \qquad u \equiv \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) = \varepsilon_0 E^2 = \frac{B^2}{\mu_0} \quad .$$

This result shows that

$$\frac{E\sqrt{\varepsilon_0}}{\omega} = \frac{B}{\omega\sqrt{\mu_0}} = \quad \text{Lorentz invariant} \quad .$$

61-010 Conduction Current

Determine the conduction current in the loop seen by the stationary observer.

Verify that the stationary observer sees the same current crossing each section of each side of the loop, coherently with the fact that he is seing a steady situation.

Determine the convection current seen by the stationary observer.

61-011 Convection and Conduction Currents - Ohm Law

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|Problem 11.16||

61-012 ElectroMagnetic Fields of a Charge Moving at Constant Velocity

Consider a point charge moving at constant velocity in an inertial Reference Frame. Use the explicit expression of the electric field (section § 61.09.06), with its time dependence, to show that the electric field is divergence-less at any point outside the charge. Use the explicit expression of the magnetic field (section § 61.09.06), with its time dependence, to show that the magnetic field is divergence-less at any point outside the charge.

DRAFT-EM-PROBLEMS

61-013 Charge Invariance

Consider a point charge moving at constant velocity in an inertial Reference Frame. Use the explicit expression of the electric field (section § 61.09.06), with its time dependence, to show that Gauss law is satisfied by calculating the flux on a sphere centered on the charge and with arbitrary radius.

Note that if the sphere is moving along with the charge Gauss law must still apply.

61-014 Internal Forces of a Pair of Charges Moving at Constant Velocity

Use the results of this section to calculate the force between two identical charges, such that one charge fixed at the origin of an inertial Reference Frame while the second one is moving with constant velocity $\mathbf{V}$ with respect to the same inertial Reference Frame with impact parameter $b > 0$ with respect to the fixed charge. Calculate the sum of the internal forces of the two charges.

61-015 Transformation Laws for Polarization and Magnetization

Use the results of section § 61.09.10 as well as charge invariance and volume contraction to write down the relativistic transformation laws of the Polarization and Magnetization vector fields, analogous to equations (61.03.01), (61.03.02), (61.03.03), (61.03.04).

61-016 Unipolar Generator

61-017 A Cross-Check

Verify that equations (61.09.01), (61.09.02), (61.09.03), (61.09.04) are consistent with the invariants of the ElectroMagnetic field.

61-018 A Variation

Solve the problem § 61.09.03 or the case of arbitrary velocity of the observer, parallel to the field, interpreting it in terms of relativistic transformations of charges and currents. Solve the problem using both methods used in section § 61.09.03.

61-019 Properties of the ElectroMagnetic Fields Produced by a Charge in Motion at Constant Velocity

Refer to § 61.09.06.

- Show that Gauss law is satisfied by equation (61.09.12) at fixed time on an arbitrary sphere centered on the charge.
- Show that equation (61.09.12) gives zero flux of the magnetic field at fixed time on an arbitrary sphere centered on the charge.
- Show that for $\beta \longrightarrow 1$ one finds: $\boldsymbol{\beta}\cdot\mathbf{E} = \boldsymbol{\beta}\cdot\mathbf{B} = 0$ and the ElectroMagnetic fields tend to zero at any point $\theta \neq \pi/2$.
- Consider a fixed point and the moving charge. Show that an estimate of the length of time during which a significant field is seen at the fixed point is of the order of:

$$\Delta t \approx \frac{H}{\beta\gamma} \quad .$$

DRAFT - IN - PROGRESS

61-020 Biot-Savart Law

Refer to § 61.09.06.
Show that equation (61.09.11) gives, in the ultra-relativistic limit, the Biot-Savart law.

61-021 Poynting Vector for a Charge Moving at Constant Velocity

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Find the Poynting vector for a point charge in uniform motion.

61-022 Force Between the Armatures of a Plane Capacitor

Refer to the plane infinite capacitor described in section § 61.03.01.
The observer at rest with respect to the capacitor measures an attractive force between the two armatures. The observer in motion sees a larger electric field, so that he measures a larger attractive force. However, he also measures a repulsive force due to the magnetic field. Show that the overall forces is consistent with the force measured by the observer at rest taking into account the law of transformation of the forces.

61-023 Lorentz Transformation of the ElectroMagnetic Field

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|11.10||

Use equations (61.03.01), (61.03.03), for the electric field, and equations (61.03.02), (61.03.04), for the magnetic field and combine each of the two pairs to write in a single vector equation the transformation law for both the electric and the magnetic field.

WORKED SOLUTION

Use $c = 1$ to simplify the formulae. One finds:

$$\mathbf{E}' = \gamma\,(\mathbf{E} + \boldsymbol{\beta}\wedge\mathbf{B}) - \frac{\gamma^2\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\mathbf{E})}{1+\gamma} \quad ,$$

$$\mathbf{B}' = \gamma\,(\mathbf{B} - \boldsymbol{\beta}\wedge\mathbf{E}) - \frac{\gamma^2\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\mathbf{B})}{1+\gamma} \quad .$$

61-024 Invariants of the ElectroMagnetic Fields

Using the relations in section § 61.03.04 show that the two quantities

$$\mathbf{E}\cdot\mathbf{B} \qquad c^2B^2 - E^2 \quad ,$$

are invariant under **Lorentz Transformation** .

DRAFT - IN PROGRESS

61-025 Transformation Laws for a Purely Electric or Purely Magnetic Field

Use the Lorentz invariants

$$\mathbf{E}\cdot\mathbf{B} \qquad c^2B^2 - E^2 \quad ,$$

to show that a purely electric (magnetic) field in one Reference Frame cannot transform to a purely magnetic (electric) field in another Reference Frame.

61-026 Transformation Laws for an ElectroMagnetic Field

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|12.64||

In a certain inertial Reference Frame the electric field and the magnetic field are neither parallel nor perpendicular at a particular space-time point.

1. Show that in a different inertial Reference Frame moving velocity $\mathbf{u}$ given by
$$\mathbf{u} = \left(1 + \frac{u^2}{c^2}\right) \frac{\mathbf{E}\wedge\mathbf{B}}{B^2 + E^2/c^2} \quad ,$$
2. Is there a Reference Frame in which the two fields are perpendicular?

61-027 Electric|Magnetic Fields Around a Straight Current Wire

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|12.66||

Start from a point charge at rest in a Reference Frame which moves with velocity $\mathbf{u}$ with respect to the laboratory Reference Frame and where there is a pure electric field.

Transform to fields to the Laboratory Reference Frame to derive the Lorentz force law, by also transforming the force from one Reference Frame to the other.

61-028 Larmor Radiation Formula

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|12.69||

Use the **Larmor Radiation Formula** and special relativity to derive the Lienard formula.

61-029 D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...- problem 12.69

©|D.J.Griffiths, *Introduction To Electrodynamics*, 1999, Prentice Hall, 3rdEd., ...|problem 12.69||

61-030 Electric|Magnetic Fields Around a Straight Current Wire

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[Never referenced.]

©|G.S.Smith, *An Introduction To Classical Electromagnetic Radiation*, 1997, CUP, ...Ed., ...|||

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|papero||

WORKED SOLUTION

DRAFT-IN-PROGRESS

61-031 Inverse Compton Scattering

©|T.Padmanabhan, *Theoretical Astrophysics, Vol. 1,2,3*, 2000/2001/2002, CUP, ...Ed., ...|2.6.5||

When a particle (even an elementary charged particle) moves inside a isotropic radiation field it scatters radiation, and loses energy to the radiation field: this is called inverse Compton scattering.

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CHAPTER 62

Other Topics

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Part G

Basics of ThermoDynamics

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64 Framework of Thermodynamics and Temperature 2526
65 Work Energy Heat and the First Principle 2568
66 Entropy and the Second Principle 2618
67 Thermodynamics in Action 2662
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CHAPTER 63

General References

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Pagina Web del Corso

Informazioni sul corso, testi di studio, problemi ed esercizi, fonti di dati, materiale vario di studio e multimediale:
WEB - URL.
Many classical texts are excellent texts, sometimes considered to be a *Bible* in their field, as they often are. However, physics progresses very very fast, so that excellent texts written many years ago remain a milestone, but may become old.

Useful General Mathematical Resources

©||S.Wolfram Mathematics Encyclopedia|*WEB - URL*|

©||J.Wevers Mathematics Formulary|*WEB - URL*|

©||WEB - URL|*Nice summary website.*|

Some Useful General Physics Resources/References

©||D.Halliday & R.Resnick & K.S.Krane, *Physics*, 2001, J.Wiley & Sons, 5thEd., ...|*Excellent, basic, modern; but do not ...*

©||Berkeley Physics Course, in 5 Vol., , ..., McGraw-Hill, ...Ed., ...|*Excellent text, but do not forget it was written many ...*

©||D.V.Sivuchin, , ..., ..., ...Ed., ...|*Excellent text, but do not forget it was written many years ago, and physics progresse- ...*

©||J.P.Pérez *et al.*, 5 Vol., , ..., ..., ...Ed., WEB - URL.|*Excellent, basic, modern.*|

©||R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|*Excellent text, but do not ...*

©||L.D.Landau *et al.*, , ..., ..., ...Ed., ...|*Excellent text, but do not forget it was written many years ago, and physics ...*

©||W.M.Haynes *et al.*, *CRC Handbook of Chemistry and Physics*, 2016, CRC press, 97thEd., WEB - URL.|*Source of Data*|

©||R.Blandford & K.Thorne, *Applications Of Classical Physics*, ..., ..., ...Ed., WEB - URL, WEB - URL.|*Excellent, ...*

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©||G.Bekefi & A.H.Barrett, *Vibrazioni Elettromagnetiche Onde E Radiazione*, 1981, Zanichelli, ...Ed., ...|*Excellent text, ...*

©||Barger & Olsson, , ..., ..., ...Ed., ...||

©||M.Born & E.Wolf, *Principles Of Optics*, 1986, CUP, 6thEd., ...|*Excellent, advanced text; but do not forget it was ...*

©||H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|*Excellent, advanced text; but do not forget it was ...*

©|M.dePodesta, *Understanding The Properties Of Matter*, 2002, CRC Press, 2ndEd., ...|Now freely available: ...

©||H.Goldstein *et al.*, *Classical Mechanics*, 2014, Pearson Education, 3rdEd., ...|*Excellent, advanced text; but do not ...*

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©||W.K.H.Panoffsky & M.Phillips, *Classical Electricity And Magnetism*, 1962, Addison-Wesley, 2ndEd., ...|*Excellent, ...*

©||F.Reif, *Fundamentals Of Statistical And Thermal Physics*, 2009, Waveland Press, ...Ed., ...|*Excellent, basic text; but ...*

©||A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...|*Excellent, intermediate, modern.*|

©||M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|*Excellent, basic text; but ...*

Some Useful Resources/References on Physics and the Real World

©||L.Bloomfield, *How Things Work*, ..., ..., ...Ed., WEB - URL.||

Other references

©|K.McDonald|WEB - URL|*A real treasure of tens of detailed and critical analyses of key physics problems, showing how ...*

The SI of Units of Measure

Read § B.

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  the reference, with problems;
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  excellent summary book with many many solved problems;
- Berkeley Physics Course, Vol. 5, *Statistical Physics*, 1965, McGraw-Hill, ...Ed., ...
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  two excellent books with many clear explanations and applications;
- H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...
  the bible;
- M.dePodesta, *Understanding The Properties Of Matter*, 2002, CRC Press, 2ndEd., ...
  excellent book with phenomenological material and applications;
- I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...
  the standard book for problems.
- Nicholas Wheeler Physics Lectures:
  - ©|WEB - URL |Nicholas Wheeler Physics Lectures|*advanced, part 0*|
  - ©|WEB - URL |Nicholas Wheeler Physics Lectures|*advanced, part 1*|
  - ©|WEB - URL |Nicholas Wheeler Physics Lectures|*advanced, part 2*|
  - ©|WEB - URL |Nicholas Wheeler Physics Lectures|*advanced, part 3*|

- Daniel Arovas - Lecture Notes on Thermodynamics and Statistical Mechanics (A Work in Progress) 10/12/2019 - Excellent!
- John Denker - Modern Thermodynamics
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Exercises and problems for this part

List of Exercises and Problems

- M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...
  - § 1
    1.1, 1.2, 1.3, 1.4, 1.5, 1.7, 1.9.
  - § 2
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  - § 9
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    10.1, 10.3, 10.4, 10.6, 10.9, 10.10, 10.12, 10.13, 10.16, 10.17, 10.22.
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    Esercizi
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    1, 2, 15
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    Esercizi
    11, 13, 19, 21, 24, 25, 26, 27, 28, 29, 30
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    14, 21
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CHAPTER 64

Framework of Thermodynamics and Temperature

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64.01 Aim and Scope of Thermodynamics

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©|PIppard||*Similar approach to M.Zemansky*|

©|H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|||

The views and opinions expressed in any referenced document do not necessarily state or reflect those of AP.

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©|E.H. Lieb, J. Yngvason|Phys. Rep. 310 (1999) 1||

The second law of thermodynamics is, without a doubt, one of the most perfect laws in physics. Any reproducible violation of it, However, small, would bring the discoverer great riches as well as a trip to Stockholm. The world's energy problems would be solved at one stroke. It is not possible to find any other law (except, perhaps, for super selection rules such as charge conservation) for which a proposed violation would bring more skepticism than this one. Not even Maxwell laws of electricity or Newton law of gravitation are so sacrosanct, for each has measurable corrections coming from quantum effects or general relativity.
Classical thermodynamics of equilibrium states. Classical means that there is no mention of statistical mechanics here and equilibrium means that we deal only with states of systems in equilibrium and do not attempt to define quantities such as entropy and temperature for systems not in equilibrium. This is not to say that we are concerned only with thermostatics because, as will be explained more fully later, arbitrarily violent processes are allowed to occur in the passage from one equilibrium state to another.

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True to his tradition, the primary emphasis is placed on the thermodynamic (macroscopic) study of temperature, energy and entropy, while recognizing that equations of state, temperature variations of specific heats, and valuable insight come from the statistical physics (microscopic) approach.

Thermodynamics, that is **classical thermodynamics**, should be more properly called **thermostatics**, as it deals with equilibriums states only.

Thermodynamics (Thermostatics) is a science of the macroscopic world, with, in principle, no reference to the microscopic world.

However, it is out of question that the microscopic view is enlightening and complementary and, in soma cases, necessary to Thermodynamics (Thermostatics).

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Thermodynamics (...) neither claims a unique domain of systems over which it asserts primacy, nor does it introduce a new fundamental law analogue to Newton or **Maxwell Equations** . In contrast to the speciality of mechanics and electromagnetism, the hallmark of thermodynamics is generality. Generality first in the sense that thermodynamics applies to all types of systems in macroscopic aggregation, and second in the sense that thermodynamics does not predict specific numerical values for observable quantities. Instead thermodynamics sets limits (inequalities) on permissible physical processes, and it establishes relationships among apparently unrelated properties.

Whereas thermodynamics is not based on a new and particular law of nature, it instead reflects a commonality of universal feature of all laws. In brief, thermodynamics is the *study of restrictions on the possible properties of matter that follows from the symmetry properties of the fundamental laws of physics*.

Thermodynamics (Thermostatics) is the study of restrictions on the possible properties of matter that follows from the symmetry properties of the fundamental laws of physics.

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64.02 Microscopic Versus Macroscopic

64.02.01 Systems and Surroundings

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...||*Adapted from.*|

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The study of any special branch of natural science starts with a separation of a restricted region of space or a finite portion of matter from its surroundings by means of a closed surface called the boundary.
The region within the arbitrary boundary and on which the attention is focused is called the system, and everything outside the system that has a direct bearing on the system behavior is known as the surroundings, which could be another system.
If no matter crosses the boundary, then the system is closed; but if there is an exchange of matter between system and surroundings, then the system is open.
When a system has been chosen, the next step is to *describe it in terms of quantities related to the behavior of the system or its interactions with the surroundings, or both*.
There are, in general, two points of view that may be adopted: the *macroscopic point of view* and the *microscopic point of view*.

- The macroscopic point of view considers variables or characteristics of a system at approximately the human scale, or larger.
- The microscopic point of view considers variables or characteristics of a system at approximately the molecular scale, or smaller.

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Vladislav Čápek and Daniel P. Sheehan
There are no truly isolated systems in nature; all are connected by long-range gravitational and perhaps ElectroMagnetic forces; all are likely affected by other uncontrollable interactions, such as by neutrinos, dark matter, dark energy and perhaps local cosmological expansion; and all are inevitably coupled thermally to their surroundings to some degree. Straightforward calculations show, for instance, that the gravitational influence of a minor asteroid in the Asteroid Belt is sufficient to instigate chaotic trajectories of molecules in a parcel of air on Earth in less than a microsecond. Since gravity cannot be screened, the exact molecular dynamics of all realistic systems are constantly affected in essentially unknown and uncontrollable ways. Unless one is able to model the entire universe, one probably cannot exactly model any subset of it. Fortunately, statistical arguments (e.g., molecular chaos, ergodicity) allow thermodynamics to proceed quite well in most cases.

There are no truly isolated systems in nature: not different from modeling in other branches of physics...

64.02.02
Macroscopic Point of View

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...||*Adapted from.*|

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Vladislav Čápek and Daniel P. Sheehan
Macroscopic properties (e.g., temperature, number density, pressure) are ones that exhibit statistically smooth behavior at equilibrium.
Scale lengths are critical; for example, one expects macroscopic properties for typical liquids at scale lengths greater than about μm. At shorter scale lengths statistical fluctuations become important and can undermine the second law.

REMEMBER probability and statistics.

© - QUOTE

Let us take as a system the contents in a cylinder of an automobile engine. A chemical analysis would show a mixture of hydrocarbons and air before being ignited, and after the mixture has been ignited there would be combustion products describable in terms of new chemical compounds. A statement of the amounts of these chemicals describes both the mass and the composition of the system.
At any moment, the system can be described further by specifying the volume, which varies as the piston moves in the cylinder. The volume can be easily measured and, in the laboratory, is recorded automatically by means of a device coupled to the piston.
Another quantity that is indispensable in the description of our system is the pressure of the gases in the cylinder. After ignition of the mixture, the pressure is large; after the expulsion of the combustion products, the pressure is small. In the laboratory, a pressure gauge may be used to measure and record the changes of pressure as the engine operates.
Finally, there is one more quantity without which we should have no adequate description of the operation of the engine. This quantity is the temperature, and, as we shall see, in many instances it can be measured just as simply as the other quantities.
We have described the contents in a cylinder of an automobile engine by specifying the quantities of mass, composition, volume, pressure, and temperature. These quantities refer to the large-scale characteristics, or aggregate properties, of the system and provide a macroscopic description. The quantities are, therefore, called macroscopic coordinates.
For a system other than a gas, such as a paramagnetic salt, the different quantities must be specified to provide a macroscopic description of the system; but macroscopic coordinates, in general, have the following properties in common:

- They involve no special assumptions concerning the structure of matter, fields, or radiation.
- They are few in number needed to describe the system.
- They are fundamental, as suggested more or less directly by our sensory perceptions.
- They can, in general, be directly measured.

In short, a macroscopic description of a system involves the specification of a few fundamental measurable properties of a system.
Thermodynamics, then, is the branch of natural science that deals with the macroscopic properties or characteristics of nature and always includes the macroscopic coordinate of temperature for every system. The presence of temperature distinguishes thermodynamics from other macroscopic branches of science, such as geometrical optics, mechanics, or electricity and magnetism.

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In short, a macroscopic description of a system involves the specification of a few fundamental measurable properties of a system.

64.02.03
Microscopic Point of View

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The microscopic point of view is the result of the tremendous progress of molecular, atomic, and nuclear science during the past hundred years. From this point of view, a system is considered to consist of an enormous number of particles, each of which is capable of existing in a set of states defined by their energy, linear momentum, angular momentum, etc.... The particles are assumed to interact with one another by means of fields. The system of particles may be imagined to be isolated or, in some cases, may be considered to be embedded in a set of similar systems, or ensemble of systems. The mathematics of probability is applied, and the equilibrium state of the system is assumed to be the state of highest probability.
The fundamental problem is to find the number of particles in each of the microscopic states (known as the populations of the states) when equilibrium is reached.
Statistical Physics, then, is the branch of natural science that deals with the microscopic characteristics of nature.
A microscopic description of a system involves the following properties:

- Assumptions are made concerning the structure of matter, fields, or radiation.
- Many quantities must be specified to describe the system.
- These quantities specified are not usually suggested by our sensory perceptions, but rather by our mathematical models.
- They cannot be directly measured, but must be calculated.

In short, a microscopic description of a system involves various assumptions about the internal structure of the system and then calculations of system-wide characteristics.

In short, a microscopic description of a system involves various assumptions about the internal structure of the system and then calculations of system-wide characteristics.

64.02.04
Microscopic Versus Macroscopic

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Although it might seem that the two points of view are hopelessly different and incompatible, both points of view, applied to the same system, must lead to the same conclusion.
The two points of view are reconciled because the few directly measurable properties whose specification constitutes the macroscopic description are really averages, over a period of time, of a large number of microscopic characteristics. For example, the macroscopic quantity, pressure, is the average rate of change of linear momentum due to the large number of molecular collisions made on a unit of area. Pressure, however, is a property that is perceived by our senses. We feel the effects of pressure. Pressure was experienced, measured, and used long before there was reason to believe in the existence of molecular impacts. If molecular theory is changed, for example, by

incorporating the results of chaos, the concept of pressure will still remain and be understood by all normal human beings.
The few measurable macroscopic properties are as sure as our senses. They will remain unchanged as long as our senses remain the same and are not deceived. Herein lies an important distinction between the macroscopic and microscopic points of view.
The microscopic point of view, however, goes much further than our senses and many direct experiments. It assumes the structure of microscopic particles, their motion, their energy states, their interactions, etc., and then calculates measurable quantities. The microscopic point of view has changed several times, and we can never be sure that the assumptions are justified until we have compared some deduction made on the basis of these assumptions with a similar deduction based on the experimentally proven macroscopic point of view. In other words, when we seek to understand the physical reality of a result of a microscopic calculation, we look to the macroscopic point of view for guidance.
Throughout its history, the study of thermodynamics has always looked for general laws, relationships, and procedures for understanding macroscopic temperature-dependent phenomena. Because it makes no assumptions about the microscopic structure of matter, thermodynamics has not been disproved as the various specific microscopic classical and quantum models of matter have been incorporated into statistical mechanics.

Macroscopic variables are typically averages of microscopic quantities.

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In thermodynamics, however, the attention is directed to the interior of a system. A macroscopic point of view is adopted, and emphasis is placed on those macroscopic quantities which have a bearing on the internal state of a system.
It is the function of experiment to determine the quantities that are appropriate for a description of such an internal state.
Macroscopic quantities, including temperature, having a bearing on the internal state of a system are called thermodynamic coordinates. Such coordinates serve to determine the internal energy of a system.
It is the purpose of thermodynamics to find, among the thermodynamic coordinates, general relations that are consistent with the fundamental laws of thermodynamics.

PROGRESS

64.03 Thermodynamic State

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M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...
It has been emphasized that a description of the large-scale characteristics of a system by means of a few of its measurable properties, suggested more or less directly by our sensory perceptions, constitutes a macroscopic description.
Such descriptions are the historic starting point of all investigations in all branches of natural science. For example, in dealing with the mechanics of a rigid body, we adopt the macroscopic point of view in that only the external aspects of the rigid body are considered. The position of its center of mass is specified with reference to coordinate axes at a particular time. Position and time and a combination of both, such as velocity, constitute some of the macroscopic quantities used in classical mechanics and are called mechanical coordinates. The mechanical coordinates serve to determine the potential and the kinetic energy of the rigid body with reference to the coordinate axes, namely, the kinetic and the potential energy of the body as a whole. These two types of energy constitute the external, or mechanical, energy of the rigid body. It is the purpose of mechanics to find such relations between the position coordinates and the time as are consistent with Newton's laws of motion.
In thermodynamics, however, the attention is directed to the interior of a system. A macroscopic point of view is adopted, and emphasis is placed on those macroscopic quantities which have a bearing on the internal state of a system.
It is the function of experiment to determine the quantities that are appropriate for a description of such an internal state.
Macroscopic quantities, including temperature, having a bearing on the internal state of a system are called thermodynamic coordinates. Such coordinates serve to determine the internal energy of a system.
It is the purpose of thermodynamics to find, among the thermodynamic coordinates, general relations that are consistent with the fundamental laws of thermodynamics.
A system that may be described in terms of thermodynamic coordinates is called a thermodynamic system.
In engineering, the important thermodynamic systems are a gas, such as air; a vapor, such as steam; a mixture, such as gasoline vapor and air; and a vapor in contact with its liquid, such as liquid and vaporized freon. Chemical thermodynamics deals with these systems and, in addition, with reactions, surface films, and electric cells. Physical thermodynamics includes, in addition to the above, such systems as wire resistors, electric capacitors, and magnetic substances.

64.03.01 Macroscopic Variables of State

The choice of the appropriate state variables depends on the system under consideration and on the degree of accuracy that one wishes to achieve. It is ultimately decided by experiment only.

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Les systèmes physiques considérés en thermodynamique contiennent généralement un très grand nombre de particules (atomes, molécules, etc.), de l'ordre du nombre de Avogadro. L'expérience courante montre que de tels systèmes présentent un comportement que l'on peut décrire et prédire, notre échelle macroscopique, pourvu que l'on puisse maitriser quelques variables caractéristiques

de ces systèmes, appelés variables d'état
L'état thermodynamique d'un système est stationnaire si les paramétrés macroscopiques qui définissent son état (volume, pression, nombre de particules par unit de volume, température, etc.) n'évoluent pas au cours du temps.
Un système est en équilibre thermodynamique lorsque son état est stationnaire, en l'absence d'exchange avec le milieu extérieur

© - QUOTE

Robert Floyd Sekerka - **Thermal Physics: Thermodynamics and Statistical Mechanics**
Much of our treatment will be concerned with homogeneous bulk systems in a state of equilibrium. By bulk systems, we refer to large systems for which surfaces, either external or internal, make negligible contributions
In defining bulk samples, we must be careful to exclude samples such as thin films or thin rods for which one or more dimension is small compared to others. Thus, a thin film ..... which will not be negligible for a sufficiently thin film. We must also exclude samples that are finely subdivided, such as those containing many internal cavities.
From the considerations of the preceding section, atoms of bulk samples can be regarded as being equivalent to one another, independent of location. It follows that certain state variables needed to describe such systems are proportional to the number of atoms.

64.03.02
Intensive and Extensive Macroscopic Properties

Imagine a system in equilibrium to be divided into two equal parts, each with equal mass. Those properties of each half of the system that remain the same are said to be intensive; those that are halved are called extensive.

The intensive coordinates of a system, such as temperature and pressure, are independent of the mass; the extensive properties are proportional to the mass. Since the intensive variables are ratios or derivatives involving extensive variables the total number of independent intensive variables is one less than the total number of independent extensive variables.

To every extensive variable its intensive variable can be associated. The reverse is not true; temperature and pressure, for instance,have no extensive counterpart.

One has to be careful that examples of properties of already known systems do not necessarily satisfy this property, tat is either intensive or extensive, which is far from trivial. Consider, for instance, a set of n (an even number) identical point masses: the gravitational interaction energy, as a function of the mass, m, and average distance between all pairs, $\langle r\rangle$, goes like $\mathcal{U} \sim -\,\mathrm{Bin}[n,2]\mathcal{G}m^2/\langle r\rangle$. Therefore, if the system is divided into two equal sets, with the same average distance, the gravitational interaction energy is not halved, because it depends on the quadratic function $\mathrm{Bin}[n,2]$. How to deal with it, it will be mentioned later.

At least one extensive variable must be used, else one cannot determine the overall size of the system.

IN-PROGRESS

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64.03.03
Additive and Extensive Properties

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Additive quantities: a property is additive whenever, under combination of two independent subsystems, the property of the resulting system is the sum of the the properties of the original systems.

Extensive quantities: a function f of the variables Z_i is extensive if it satisfies the scaling law

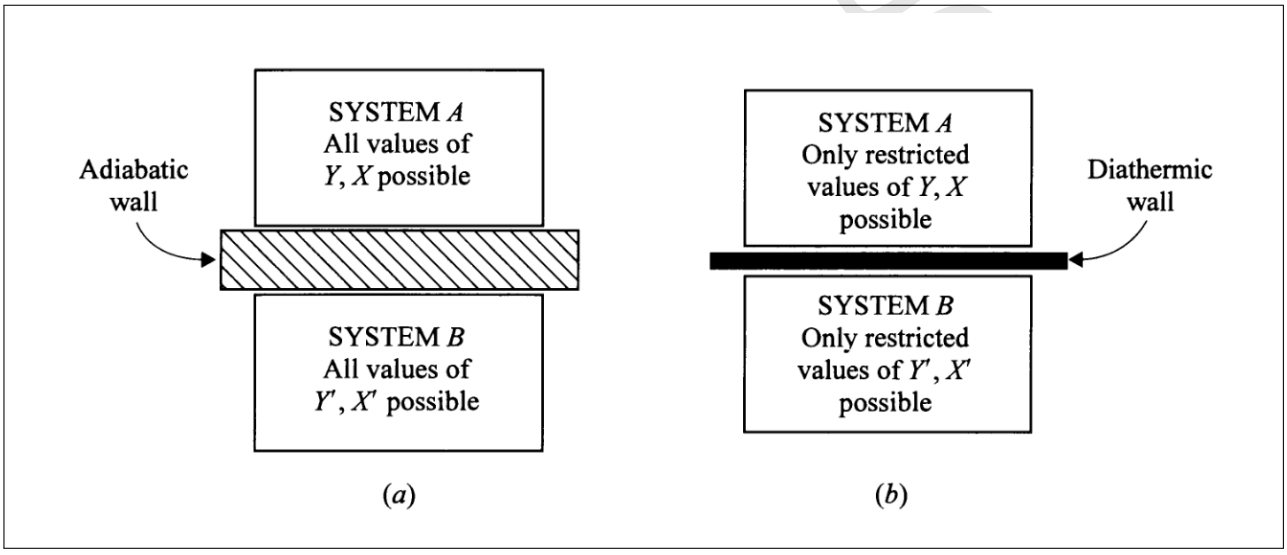
$$f(qZ_1, qZ_2, ...) = qf(Z_1, Z_2, ...) \qquad q > 0 \quad .$$

That is, f is homogeneous of degree one, and the theorem of Euler applies.

© - QUOTE

GRANDY
While additivity can readily be shown to imply extensivity, at least for continuous f, the converse is not necessarily true if systems differing in some way are being brought together; for example, even though the two separate systems are extensive, there may be a chemical reaction upon mixing. In any event, the two notions are logically independent.

Additivity implies extensivity; the reverse is not true. For instance, gravitational potential energy is extensive, as it is proportional to the total mass, but it is not additive, as the gravitational potential energy of a system composed of two subsystems, is not the sum of the gravitational potential energies of the subsystems because there is also the interaction energy term (the gravitational force, in fact, is not short-ranged).



64.03.04
State Functions and Path Functions

Any quantity which, in equilibrium, depends only on the state variables is called a state function.

64.04 Thermal Equilibrium and Temperature

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...||*Adapted from.*|

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For any pure substance, pressure and volume are always state variables, as well as number and type of particles and temperature.
Experiment shows that, for a given composition and for a constant mass and temperature, many different values of pressure and volume are possible for a gas. If the pressure is kept constant, the volume may vary over a wide range of values, and vice versa. In other words, the pressure and the volume are independent coordinates but are related in a simple equation, namely, Boyle law. More recently, experiment has shown that, for a wire of constant mass, the tension and the length are independent coordinates, whereas, in the case of a surface film, the surface tension and the area may be varied independently. Some systems that, at first sight, seem quite complicated, such as an electric cell with two different electrodes and an electrolyte, may still be described with the aid of only two independent coordinates. On the other hand, some thermodynamic systems composed of a number of homogeneous parts require the specification of two independent coordinates for each homogeneous part.
For the present, to simplify our discussion, we shall deal only with systems of constant mass and composition, each requiring only one pair of independent coordinates for its description. This involves no essential loss of generality and results in a considerable saving of words. In referring to any unspecified system, we shall use the symbols X and Y for the pair of independent coordinates, where the symbol X refers to a generalized force (for instance, the pressure of a gas) and Y refers to a generalized displacement (for instance, the volume of a gas).
A state of a system in which the coordinates X and Y have definite values that remain constant so long as the external conditions are unchanged is called an equilibrium state. Experiment shows that the existence of an equilibrium state in one system depends on the proximity of other systems and on the nature of the boundary or wall separating the different systems. Walls are said to be either adiabatic or diathermic in ideal cases. If a wall is adiabatic, an equilibrium state for system A may coexist with any equilibrium state of system B for all attainable values of the four quantities, X, Y and X', Y' provided only that the wall is able to withstand the stress associated with the difference between the two sets of coordinates. Thick layers of wood, concrete, asbestos, felt, or polystyrene, as well as dewars, are, in this order, increasingly better experimental approximations to ideal adiabatic walls.
If the two systems are separated by a diathermic wall, the values of X, Y and X', Y' will change spontaneously until an equilibrium state of the combined system is attained. The two systems are then said to be in thermal equilibrium with each other. The most common experimental diathermic wall is a thin metallic sheet. Thermal equilibrium is the state achieved by two (or more) systems, characterized by restricted values of the coordinates of the systems, after they have been in communication with each other through a diathermic wall. Unlike the diathermic wall, an adiabatic wall prevents two systems from communicating with each other and coming to thermal equilibrium with each other. Although we have not yet defined the concept of heat, it may be said that a diathermic wall is a boundary through which heat is communicated from one system to another system, yet remains closed to the transport of matter. An ideal adiabatic wall does not communicate heat.

A state of a system in which the coordinates X and Y have definite values that remain constant so long as the external conditions are unchanged is called an equilibrium state.

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64.04.01
Principle Zero

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|||

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M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...
These experimental facts may then be stated concisely in the following transitive relation: Two systems in thermal equilibrium with a third are in thermal equilibrium with each other. As suggested by Ralph Fowler, this postulate of transitive thermal equilibrium has been numbered the zeroth law of thermodynamics, which establishes the basis for the concept of temperature and for the use of thermometers.

Two systems in thermal equilibrium with a third system are in thermal equilibrium.

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All states of corresponding isotherms of all systems have something in common, namely, that they are in thermal equilibrium with one another. The systems themselves, in these states, may be said to possess a property that ensures their being in thermal equilibrium with one another. We call this property temperature.
The temperature of a system is a property that determines whether or not a system is in thermal equilibrium with other systems.

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Questa legge può apparire elementare, ma non è così ovvia come sembra. Possono esistere altre situazioni in cui un sistema C può interagire equivalentemente con due sistemi A e B, senza che A e B presentino uguale interazione tra di loro. Immaginiamo per esempio che A e B siano due chiodi di ferro non Magnetizzati e che C sia una calamita; orbene, C attrae sia A sia B, ma questi ultimi non si attraggono tra di loro.

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Un altro enunciato della legge zero che fa riferimento alla temperatura è il seguente. Esiste una grandezza scalare chiamata temperatura, che è una proprietà di tutti i sistemi termodinamici in equilibrio. Due sistemi sono in equilibrio termico se, e solo se, sono uguali le loro temperature. Questa legge definisce dunque la nozione di temperatura, grazie alla quale abbiamo tratto l'idea di costruire e utilizzare i termometri.

The principle zero defines an equivalence relations partitioning all possible systems into subsets characterized by having the same temperature.

Temperature is the property of a system at thermal equilibrium defining which other systems the system is in equilibrium with.

Note that, it is not necessary to define a temperature scale to define thermal equilibrium.

The time required for a system to come to thermal equilibrium is called the relaxation time. How long the systems need to come into thermal equilibrium is of no concern for thermodynamics (thermostatics).

Time is not a variable in thermostatics.

Isotherm: the locus of states X, Y which are in thermal equilibrium with a third reference system and among them.

The locii of states X, Y which are in thermal equilibrium when changing the temperature, provide a generic relation:

$$\boxed{f[X,Y,T]=0} \quad ,$$

which is the equation of state.

The equation of state is assumed to:

- be locally (and only locally) differentiable, as much as required;
- be locally (and only locally) invertible in any of the X, Y, T variables:

$$X = f_X[Y,T] \quad ,$$
$$Y = f_Y[T,X] \quad ,$$
$$T = f_T[X,Y] \quad ,$$

- depend on pairs of conjugate variables, X, Y, X', Y', X'', Y'', ..., plus the temperature (more on this in section § 65.02.05).

DRAFT

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WHEELER
Select two systems, $\mathcal{S}_1$ and $\mathcal{S}_2$, and arrange (see the figure) that they be permitted to interact by heat exchange, but in no other way. What happens? One finds that

- either the states of both $\mathcal{S}_1$ and $\mathcal{S}_2$ remain unaltered, which we will record by writing $\mathcal{S}_1 \sim \mathcal{S}_2$;
- both states change, which we record by writing $\mathcal{S}_1 \sim \mathcal{S}_2$. One never observes a change in the state of one system which is unaccompanied by a change in the state of the other . . . though "thermostats" ("heat reservoirs"), created by making one system very much more massive than the other, are designed to approximate such behavior. One finds, moreover, that
- if $\mathcal{S}_1 \sim \mathcal{S}_2$ and $\mathcal{S}_1 \sim \mathcal{S}_3$ then $\mathcal{S}_2 \sim \mathcal{S}_3$.

The physical relationship that we have denoted ..... is therefore an equivalence relation. The "diathermal wall test" has served to resolve the set of all systems in all states into disjoint equivalence classes, just as "congruence" serves to resolve into equivalence classes the set of all triangles. There is a lesson to be learned from looking more closely to the set of all triangles inscribed on the Euclidean plane. For it is geometrically evident that

- 3 numbers are required to identify an equivalence class if congruence is the defining relation (angle/side/angle or side/angle/side);
- only 2 numbers are required to identify an equivalence class if similarity is the defining relation (angle/angle);
- only one number is required to identify an equivalence class if isometry is the defining relation (area).

Evidently the resolution of a set into disjoint equivalence classes conveys-in and of itself-no information concerning how many numbers must be specified to identify an equivalence class (the "dimension" of the resolution). The question now before us: What is the dimension of the resolution achieved by the diathermal wall test? How many numbers must be presented to identify one of the resulting equivalence classes, to distinguish it from all others? It is a lesson of simple sense experience that systems which are equivalent in the above sense are invariably adjudged to have, in the most naive tactile sense, the same "temperature." And conversely.
The important implication of the experience thus summarized is formulated in the so-called Zeroth Law of Thermodynamics: The diathermal wall test achieves a one-dimensional resolution of the set {states of systems} into disjoint equivalence classes. Every such class (to say the same thing another way) can be distinguished from every other such class by specification of a single real number, which we agree to call the "empirical temperature."
However, natural it may be to assign larger values to "hotter" system-states, smaller values to cooler ... the 0th law does not by itself obligate us to do so, nor does it provide guidance as to how we might/should do so: it permits a natural ordering of system-states, but places no constraint on

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how we might elect to graduate that ordering.

64.04.02
Empirical Concept of Temperature

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...||*Adapted from.*|

Useful to have a handy instrument to compare temperatures of different systems.

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M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...
Since temperature is a scalar quantity, the temperature of all systems in thermal equilibrium may be represented by a number. The establishment of a temperature scale is merely the adoption of a set of rules for assigning one number to a set of corresponding isotherms, and a different number to a different set of corresponding isotherms. Once this is done, the necessary and sufficient condition for thermal equilibrium between two systems is that they have the same temperature. Also, when the temperatures are different, we may be sure that the systems are not in thermal equilibrium.

Subjective temperature: common concept of **hotness**.
Dependence of various physical quantities on temperature: size, electric and **Magnetic-Permeability** , resistivity, Young/Poisson modules and relatives, ...
Use the dependence on temperature of various physical quantities (X) to build an **empirical thermometer**, a transducer, for instance mercury/alcohol thermometer or a platinum resistance thermometer.
Define a measurement method for temperature, for instance:

$$\theta[X] \equiv \alpha_X X \qquad \text{or} \qquad \theta[Y] \equiv \alpha_Y Y \qquad \text{or} \qquad \theta[X,Y] \equiv g[X,Y] \quad .$$

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It should be noted further that different empirical temperature scales usually result when this arbitrary relation is applied to different kinds of thermometers and even when it is applied to different systems of the same kind, such as constant volume hydrogen or nitrogen thermometers.

Therefore the detailed knowledge of this law is essential to calibrate the instrument. For instance, if two different thermometers are calibrated at zero and 100 °C, they do not necessarily measure the same temperature at other temperatures.
Concept of thermal equilibrium: two bodies in thermal contact reach the same temperature, as measured by an empirical thermometer, if the wall is diathermic; the other macroscopic parameters will change as well, in general.
Two bodies in contact via an adiabatic wall will not change their temperature, as measured by an empirical thermometer, nor the other macroscopic parameters.
Thermalization: exchange of internal energy, on the microscopic scale. No macroscopic work is involved.

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In linea di principio qualsiasi proprietà di una sostanza che dipenda dalla temperatura, detta per questo proprietà termometrica, può servire allo scopo; ne sono esempi il volume di un liquido, come nei termometri a mercurio e ad alcool, la pressione di un gas confinato, la resistenza elettrica di un filo, la lunghezza di una striscia metallica, o il colore di un filamento da lampadina; tutte grandezze che variano al variare della temperatura e tutte utilizzate in comuni termometri. Considerato però che la dipendenza di queste grandezze dalla temperatura non è in generale espressa da una

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legge lineare, ne consegue che la conoscenza dettagliata di questa legge di dipendenza diventa un elemento cruciale per la taratura e per l'accuratezza del termometro basato su ognuna di queste grandezze. L'adozione di una semplice dipendenza lineare tra la grandezza misurata e la temperatura da misurare fa sì che ognuno dei citati procedimenti misuri una propria temperatura empirica, che sarà diversa a seconda del tipo di termometro utilizzato. Così facendo, anche ammesso che tutti i tipi siano concordi quando misurano il punto triplo dell'acqua generale non lo saranno più ad altre temperature, sia maggiori sia minori. come illustra il Problema svolto 21.1. Ciononostante i termometri empirici, se tarati opportunamente per confronto con campioni riconosciuti, possono servire utilmente come campioni secondari di misura della temperatura. E difatti quasi tutti i termometri esistenti sono campioni secondari siffatti.

64.04.03
Type of Thermometers and Measurement of Temperature

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- Liquid mercury thermometer.
- Platinum resistance thermometer.
- Bi-metallic lamina thermometer.
- Constant-volume gas thermometer.
- .....

64.04.03.01 Temperature Scales

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64.04.03.02 Ideal-Gas Temperature Thermometer

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64.04.04
Preliminary Microscopic Interpretation of Heat and Temperature

64.04.04.01 Kinetic Temperature

This § is referenced at pages:
[2731]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 1, § 39||

Consider a system made of a large number of molecules.

Temperature is a measure of the average internal energy of particles; the kinetic temperature is defined from the random motion of the Center-Of-Mass of the molecule. Also rotations and vibrations, for poly-atomic molecules, are related to temperature, via the equipartition theorem, read § 69.02.02.01.

Consider a small part of the large system, large on a microscopic scale and small on a macroscopic scale. The part might have an overall motion, described by the motion of the Center-Of-Mass of all its molecules, with overall linear momentum $\mathbf{P} = 0$. **Koënig theorem** for kinetic energy may be applied.

DRAFT - IN PROGRESS

Kinetic temperature is defined, locally, in terms of the average squared velocity of the Center-Of-Mass of the single molecules in the Rest Frame of the Center-Of-Mass of the part, $\mathbf{v}_{\text{CM}}$, as:

$$\frac{1}{2}m\langle v_{\text{CM}}^2\rangle \equiv \frac{3}{2}k_{\text{B}}T[\mathbf{r},t] \quad \Leftrightarrow \quad \mathbf{P}=0 \quad \Leftrightarrow \quad \langle \mathbf{p}_i\rangle = 0 \quad ,$$

defining the temperature field, $T[\mathbf{r},t]$, whenever there is no net macroscopic motion, $\mathbf{P}=0$.

64.04.04.02 Collisions and Kinetic Temperature

This § is referenced at pages:
[2726]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 1|*§ 39.4*|

©|Am.J.Phys, , ..., ..., ...Ed., ... 52 (1), January 1984|one-dimensional case||

From the kinematics of classical elastic interactions between any species of molecules, whenever the velocity distribution is isotropic in the laboratory frame, and the angular distribution of the decay products in the [glo:CMRF]Center-of-Momenta Reference Frame of the pair of particles is isotropic (as it is the case due to the total randomness),

$$\langle \mathbf{v}_1\rangle = \langle \mathbf{v}_2\rangle = 0 \qquad \langle \mathbf{v}_1{\cdot}\mathbf{v}_2\rangle = 0 \qquad \langle \mathbf{v}_1^{\star\prime}{\cdot}\mathbf{V}_{\text{CM}}\rangle = \langle \mathbf{v}_2^{\star\prime}{\cdot}\mathbf{V}_{\text{CM}}\rangle = 0 \quad .$$

For any two-body collision $\mathbf{V}_{\text{CM}}$ is the Center-Of-Mass velocity of the pair.ions are derived, for quantities before and after the collision:

$$\Delta\mathcal{K}_1 \equiv \mathcal{K}_1' - \mathcal{K}_1 = \frac{2\mu}{M}(\mathcal{K}_2 - \mathcal{K}_1) + \mu\left(\frac{m_1-m_2}{M}\right)(\mathbf{v}_1{\cdot}\mathbf{v}_2) + m_1\left(\mathbf{v}_1^{\star\prime}{\cdot}\mathbf{V}_{\text{CM}}\right) \quad ,$$

$$V_{\text{CM}}^2 = \frac{2}{M^2}(m_1\mathcal{K}_1 + m_2\mathcal{K}_2 + m_1m_2(\mathbf{v}_1{\cdot}\mathbf{v}_2)) \quad .$$

Therefore, on average:

$$\Longrightarrow \langle\Delta\mathcal{K}_1\rangle = \frac{2\mu}{M}(\langle\mathcal{K}_2\rangle - \langle\mathcal{K}_1\rangle) \qquad \langle\Delta\mathcal{K}_2\rangle = \frac{2\mu}{M}(\langle\mathcal{K}_1\rangle - \langle\mathcal{K}_2\rangle) \quad ,$$

→ 2540

such that, on average, the kinetic energies tend to become equal (thermalize).

From equation (64.04.04.02) it can be easily shown that, for a fixed particle mass, the maximum average kinetic energy change happens whenever the other particle has the same mass, when μ/M is maximum. This is exploited, for instance, for neutron thermalization in nuclear reactors.

Note that, the above results are valid for poly-atomic molecules provided the velocity above of the molecules are taken as the Center-Of-Mass velocities of the single molecules.

64.04.04.03 Heat

Heat is exchange of energy at the microscopic level, without changing macroscopic parameters.

64.04.04.04 Microscopic Inter-Particle Interactions

Inter-particle interactions can be described to a first approximation by the Lennard-Jones potential.

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The Lennard-Jones. potential (also referred to as the LJ potential, 6-12 potential, or 12-6 potential) is a mathematically simple model that approximates the interaction between a pair of neutral atoms or molecules.

DRAFT - PRELIMINARY

The form of this inter-atomic potential was first proposed in 1924 by John Lennard-Jones. The most common expressions of the LJ potential are

$$V_{\text{LJ}} = 4\varepsilon \left[\left(\frac{\sigma}{r} \right)^{12} - \left(\frac{\sigma}{r} \right)^{6} \right] = \varepsilon \left[\left(\frac{r_m}{r} \right)^{12} - 2 \left(\frac{r_m}{r} \right)^{6} \right]$$

where ε is the depth of the potential well, σ is the finite distance at which the inter-particle potential is zero, r is the distance between the particles, and r_m is the distance at which the potential reaches its minimum. At r_m, the potential function has the value $-\varepsilon$. The distances are related as $r_m = 2^{1/6}\sigma \approx 1.122\sigma$.

These parameters can be fitted to reproduce experimental data or accurate quantum chemistry calculations. Due to its computational simplicity, the Lennard-Jones. potential is used extensively in computer simulations even though more accurate potentials exist.

The r^{-12} term, which is the repulsive term, describes Pauli repulsion at short ranges due to overlapping electron orbitals and the r^{-6} term, which is the attractive long-range term, describes attraction at long ranges (van der Waals force, or dispersion force).

Average energy increase implies, usually, average distance increase, that is thermal dilatation. In some cases the shape of the interaction potential can be such that an average energy increase causes an average distance decrease.

WORK-IN-PROGRESS

64.05 Thermodynamic Equilibrium

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A system in thermodynamic equilibrium is one for which the macroscopic variables T, (X, Y), (X', Y'), (X'', Y''),... are well defined (meaning that all intensive variables have the same value throughout the whole system) and remain constant over time (provided that external factors do not change); and the same is true for every subsystem.

64.05.01 Thermodynamic Equilibrium States

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If the state of a thermodynamic system can be fully characterized by the values of the thermodynamic variables, and if these values are invariant over time, one says that it is in a state of thermodynamic equilibrium. Thermodynamic equilibrium, as R. Feynman says, occurs when all fast processes have already occurred, while the slow ones have yet to take place. Clearly the distinction between fast and slow processes is dependent on the observation time that is being considered.

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When there is no unbalanced force or torque in the interior of a system and also none between the system and its surroundings, the system is said to be in a state of mechanical equilibrium. When these conditions are not satisfied, either the system alone or both the system and its surroundings will undergo a change of state, which will cease only when mechanical equilibrium is restored.
When a system in mechanical equilibrium does not tend to undergo a spontaneous change of internal structure, such as a chemical reaction, or a transfer of matter from one part of the system to another, such as diffusion or solution, However, slow, then it is said to be in a state of chemical equilibrium. A system not in chemical equilibrium undergoes a change of state that, in some cases, is exceedingly slow. The change ceases when chemical equilibrium is reached.
Thermal equilibrium exists when there is no spontaneous change in the coordinates of a system in mechanical and chemical equilibrium when it is separated from its surroundings by diathermic walls. In other words, there is no exchange of heat between the system and its surroundings. In thermal equilibrium, all parts of a system are at the same temperature, and this temperature is the same as that of the surroundings. When these conditions are not satisfied, a change of state will take place until thermal equilibrium is reached.
When the conditions for all three types of equilibrium are satisfied, the system is said to be in a state of thermodynamic equilibrium; in this condition, it is apparent that there will be no tendency whatever for any change of state, either of the system or of the surroundings, to occur. States of thermodynamic equilibrium can be described in terms of macroscopic coordinates that do not involve the time, that is, in terms of thermodynamic coordinates. Thermodynamics does not attempt to deal with any problem involving the rate at which a process takes place. The investigation of problems involving the time dependence of changes of state is carried out in other branches of science, as in the kinetic theory of gases, hydrodynamics, and chemical kinetics.
When the conditions for anyone of the three types of equilibrium that constitute thermodynamic equilibrium are not satisfied, the system is said to be in a non-equilibrium state. Thus, when

there is an unbalanced force or torque in the interior of a system, or between a system and its surroundings, the following phenomena may take place: acceleration, turbulence, eddies, waves, etc. While such phenomena are in progress, a system passes through non-equilibrium states. If an attempt is made to give a macroscopic description of anyone of these non-equilibrium states, it is found that the pressure varies from one part of a system to another. There is no single pressure that refers to the system as a whole. Similarly, in the case of a system at a different temperature from its surroundings, a non-uniform temperature distribution is set up and there is no single temperature that refers to the system as a whole.
Therefore, we conclude that when the conditions for mechanical and thermal equilibrium are not satisfied, the states traversed by a system cannot be described in terms of thermodynamic coordinates referring to the system as a whole.
It must not be concluded, however, that we are entirely helpless in dealing with such non-equilibrium states. If we divide the system into a large number of small mass elements, then thermodynamic coordinates may be found in terms of which a macroscopic description of each mass element may be approximated. There are also special methods for dealing with systems in mechanical and thermal equilibrium but not in chemical equilibrium. All these special methods will be considered later. At present, we shall deal exclusively with systems in thermodynamic equilibrium.

Thermodynamic equilibrium is established by the exchange of energy, volume or particle number between different systems or subsystems:

- energy exchange $\rightarrow T =$ constant and uniform $\rightarrow$ thermal equilibrium, driven by ΔT;
- volume exchange $\rightarrow p =$ constant and uniform $\rightarrow$ mechanical equilibrium, driven by Δp;
- particle exchange $\rightarrow \mu =$ constant and uniform $\rightarrow$ chemical equilibrium, driven by $\Delta \mu$.

64.05.02
Thermodynamic Transformations

A system is in a state of thermodynamic equilibrium at a certain time, and is in a different state of thermodynamic equilibrium at a later time. The act of passing from the first to the second state is a transformation. How long it takes is not a problem of thermostatics.

64.05.02.01 Trasformazioni Quasi Statiche

Sono trasformazioni ideali che avvengono in modo così graduale che il sistema può essere sempre considerato in equilibrio termodinamico in ogni momento della trasformazione.

64.05.02.02 Types of Thermodynamic Transformations

Include:

- iso-thermal: quasi-static, with constant temperature;
- iso-bare: quasi-static, with constant pressure;
- iso-core: quasi-static, with constant volume.

Generic, non quasi-static, transformations:

- mono-thermal: non necessarily quasi-static, with equal temperature in the initial and final states;
- mono-bare: non necessarily quasi-static, with equal pressure in the initial and final states;
- mono-core: non necessarily quasi-static, with equal volume in the initial and final states.

WORK-IN-PROGRESS

64.06 Simple Thermodynamical Systems

This § is referenced at pages:
[1289]

64.06.01 pVTN Systems

A phase is a homogeneous region of matter, separated from other homogeneous regions by a phase boundary with a well defined and smooth geometrical shape enclosing a well-defined volume; the different regions (phases) will have different physical and|or chemical properties; when more than one phase exist, the system is heterogeneous, but the length scale of heterogeneity can vary from nano-meter to macroscopic dimensions.

Polymorphism is the ability of a solid material to exist in more than one form or crystal structure. Fluid polyamorphism is the existence of different condensed amorphous states in a single-component fluid. Polyamorphism of amorphous ice exists!

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WEB - URL

Fluid polyamorphism is the existence of different condensed amorphous states in a single-component fluid. It is either found or predicted, usually at extreme conditions, for a broad group of very different substances, including helium, carbon, silicon, phosphorous, sulphur, tellurium, cerium, hydrogen, and tin tetra-iodide. This phenomenon is also hypothesized for metastable and deeply supercooled water, presumably located a few degrees below the experimental limit of homogeneous ice formation. We present a generic phenomenological approach to describe polyamorphism in a single-component fluid, which is completely independent of the molecular origin of the phenomenon. We show that fluid polyamorphism may occur either in the presence or in the absence of fluid phase separation depending on the symmetry of the order parameter. In the latter case, it is associated with a second-order transition, such as in liquid helium or liquid sulphur. To specify the phenomenology, we consider a fluid with thermodynamic equilibrium between two distinct interconvertible states or molecular structures. A fundamental signature of this concept is the identification of the equilibrium fraction of molecules involved in each of these alternative states. However, the existence of the alternative structures may result in polyamorphic fluid phase separation only if mixing of these structures is not ideal. The two-state thermodynamics unifies all the debated scenarios of fluid polyamorphism in different areas of condensed-matter physics, with or without phase separation, and even goes beyond the phenomenon of polyamorphism by generically describing the anomalous properties of fluids exhibiting interconversion of alternative molecular states.

It is a matter of experience that the thermodynamic properties of a homogeneous phase depend on temperature, pressure, volume, mass and composition. It is found, experimentally, that many thermodynamic systems of fixed composition can be described in terms of pressure and volume only as macroscopic thermodynamical variables (plus composition).

For closed systems, N is fixed parameter. For open systems N becomes a thermodynamic variable.

Moreover, pressure and volume, a priori, are macroscopic variables which, a priori, should enter the description of any thermodynamical system, even if in some cases they can be neglected as non relevant with respect to the interesting aspects of the problem.

In all cases temperature is a macroscopic thermodynamical variable.

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Similarly the number of molecules (or moles) of every kind of molecule is always a relevant macroscopic variable (for open systems) or fixed parameter (for closed systems).

Examples include:

- a pure substance (gas, liquid, solid);
- a homogeneous mixture of pure substances;
- a heterogeneous mixture of homogeneous phases;
- an ideal solution (read § 67.11.06);
- any composite system made of a number of the previous subsystems.

Note that negative pressures are not forbidden in themselves: they simply characterize a system under extension and not under compression. By pulling a solid, one subjects it to such a negative pressure.

Pressure, by convention in thermodynamics, is defied as the pressure exerted by the system on the walls, as opposed to the stresses on a elastic medium, which is exerted and positive in the opposite direction. This is the origin of the negative sign that accompanies hydrostatic work.

64.06.01.01 Thermo-Elastic Coefficients

Coefficienti Di Dilatazione Termica Lineare E Volumico

Il coefficiente di dilatazione termica lineare a pressione costante è definito come

$$\alpha \equiv \frac{1}{L}\left(\frac{\partial L}{\partial T}\right)_p = \alpha[T,p] \quad . \tag{64.06.01}$$

→ 2561

See table 7.7 and 7.8 of WEB - URL for values.

Il coefficiente di espansione termica volumetrica a pressione costante è definito come

$$\beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = \beta[T,p] \quad . \tag{64.06.02}$$

→ 2562

Nelle definizioni si assume che le eventuali altre variabili, da cui dipende lo stato del sistema, oltre alla temperatura, siano tenute fisse. In generale i coefficienti così definiti dipendono quindi anche da tali altre variabili, se significative.

Per una sostanza isotropa il coefficiente di dilatazione termica lineare non dipende dalla direzione. In tal caso si ha

$$\beta = 3\alpha \quad .$$

In generale per una sostanza non isotropa, detti α_1, α_2 ed α_3 i coefficienti di dilatazione lineare lungo tre assi ortogonali si ha:

$$\beta = \alpha_1 + \alpha_2 + \alpha_3 \quad . \tag{64.06.03}$$

All the thermal dilatation coefficients are often positive, but it is not forbidden for them to be negative (for instance, liquid water below 4 °C at atmospheric pressure).

Coefficiente Di Compressibilità Isoterma

This § is referenced at pages:
[Never referenced.]

Il coefficiente di compressibilità isoterma è definito come

$$\kappa_{\mathrm{T}} \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \equiv \kappa_{\mathrm{T}}[T,p] \quad . \tag{64.06.04}$$

See table 7.4, 7.5 and 7.6 of WEB - URL for values.

Si può dimostrare, come i metodi della termodinamica, che per un sistema in equilibrio stabile, il modulo di compressibilità isoterma è sempre maggiore o uguale di zero:

$$\kappa_{\mathrm{T}} \geq 0 \quad . \tag{64.06.05}$$

UNDER CONSTRUCTION

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La plausibilità si può dedurre dal seguente ragionamento. Supponiamo che un oggetto abbia $\kappa_T < 0$. Si consideri l'oggetto rinchiuso in un recipiente rigido. Si aumenti la pressione entro il recipiente (ad esempio iniettando del gas). Essendo $\kappa_T < 0$ il volume dell'oggetto aumenta. Ma questo causerebbe un ulteriore aumento della pressione circostante. Qualunque aumento della pressione causerebbe dunque un ulteriore aumento della stessa. Analogamente si deduce che qualunque diminuzione di pressione causerebbe un'ulteriore diminuzione della stessa. Il sistema sarebbe instabile per fluttuazioni di pressione.

Isochore Pressure Variation

Equation (64.08.02) shows the, for a general pVT system:

$$\boxed{\left(\frac{\partial p}{\partial T}\right)_V = \frac{\beta}{\kappa_T}} \quad .$$

64.06.01.02 Pure Substances as pVTN Systems

A pure substance viewed as a $pVTN$ system is described by the equation of state, $f[p, V, T] = 0$, which can be represented as a 3D graph as well as via its three 2D projections: pV, pT and TV.

---

pV Diagram

It is useful for visualizing work. In fact work is the signed integral of $p[V, T]$. Read § 65.02.05.01.

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pT Diagram

It is useful to discuss phase transitions and equilibrium among phases. Read § 67.08.

Triple Points of Water

Triple points is any combination of (p, V, T) values such that three different phases can co-exist.

The gas-liquid-solid triple point of water corresponds to the minimum pressure at which liquid water can exist. At pressures below the triple point (as in outer space), solid ice when heated at constant pressure is converted directly into water vapor in a process known as sublimation.

Super-cooling is the process of lowering the temperature of either a liquid or a gas below its freezing point without solidification; this can happen in the absence of a seed crystals or nuclei around which a crystal structure can form.

The term sub-cooling refers to a liquid existing at a temperature below its normal boiling point.

Triple point of gaseous-liquid-solid water: $T_{tp} = 273.16$ K and $p_{tp} = 0.00604$ atm.

WEB - URL

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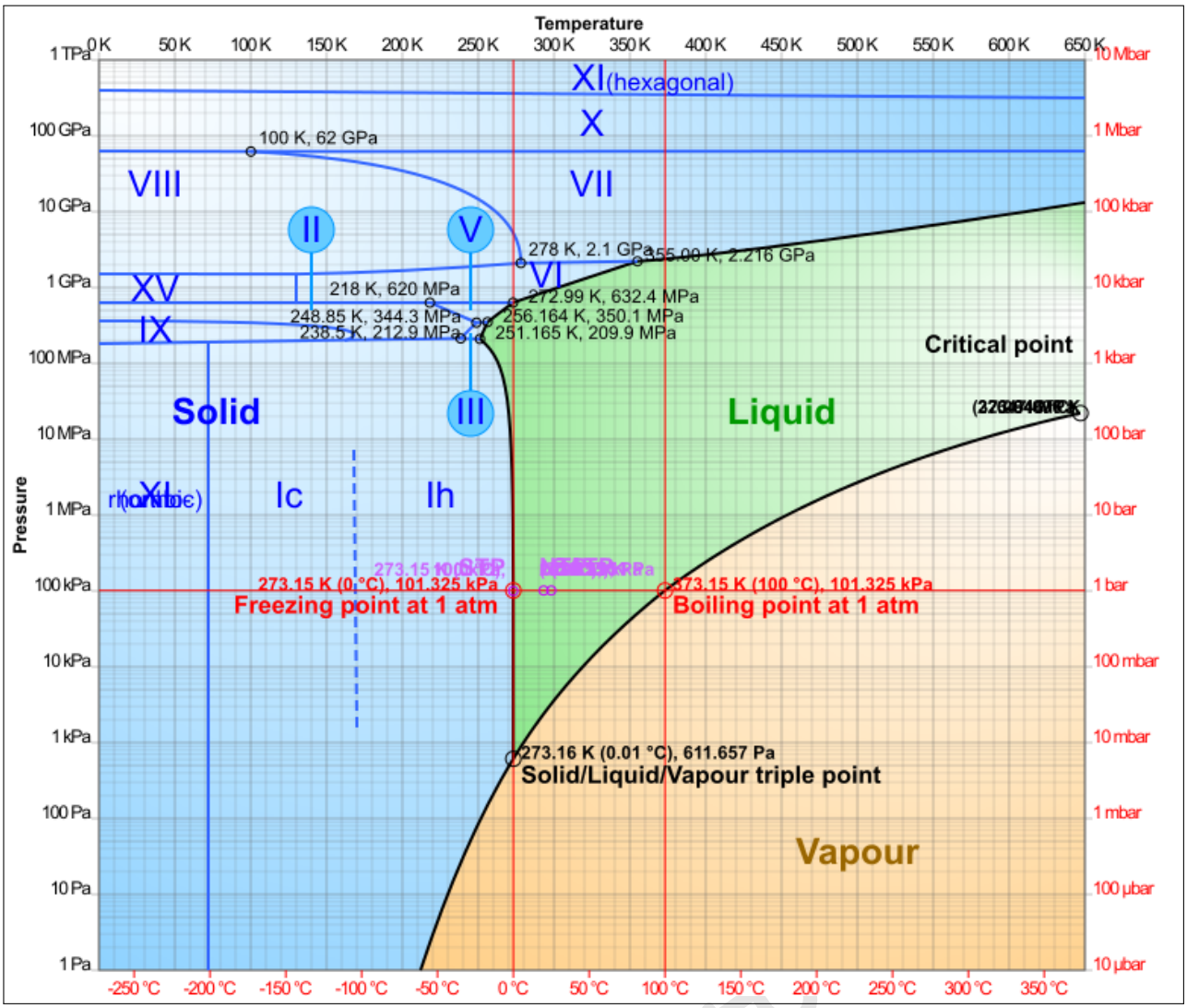

Phase Diagrams of Water

©|WEB - URL |Water|*Excellent interactive 3D models*|

Phase Diagrams of Carbon Dioxide

©|WEB - URL |Carbon Dioxide|*Excellent interactive 3D models*|

PROGRESS

Phase Diagrams of Ammonia

©|WEB - URL |Ammonia|*Excellent interactive 3D models*|

64.06.02
Wires Under Tension as 1D Thermodynamical Systems

Consider an elastic wire under tension, that is a rectilinear wire, whose cross-sectional area, A, is much smaller than its length, L, and such that the changes of its cross-sectional area are negligible. Note that, when dealing with Hooke law in mechanics, all processes are usually considered to be iso-thermal processes; in particular, Young modulus is the iso-thermal one.

Let the stretched elastic wire with fixed cross-sectional area A, be defined by its thermodynamic variables:

- length, L; it is an extensive macroscopic variable.
- tension per unit area, $\tau \equiv F_\perp / A$, the normal stress, where $F_\perp$ is the normal force at the two ends of the wire, positive when attempting to extend the wire; it is an intensive macroscopic variable.

It is, therefore, considered as a one-dimensional system. Define:

$$\alpha \equiv \frac{1}{L}\left(\frac{\partial L}{\partial T}\right)_\tau \equiv \alpha_\tau \quad ,$$

$$Y \equiv L\left(\frac{\partial \tau}{\partial L}\right)_T \equiv Y_T \quad ,$$

$$\left(\frac{\partial \tau}{\partial T}\right)_L = -\alpha_\tau Y_T \qquad \text{from the mathematical relations on mixed derivatives} \quad ,$$

having used equation (64.08.02). The Young modulus, as defined in elementary mechanics, is the isothermal Young modulus.

The linear expansivity at fixed tension is most often a positive number, with rubber as a notable exception. In general, for any change of temperature, at fixed length, the change of tension will be opposite to the sign of the linear expansivity coefficient.

It is assumed, as for every thermodynamic system, that there exists an equation of state, relating L, τ and T:

$$G[L,\tau,T]=0 \implies \mathrm{d}L = \left(\frac{\partial L}{\partial T}\right)_\tau \mathrm{d}T + \left(\frac{\partial L}{\partial \tau}\right)_T \mathrm{d}\tau = L\alpha_\tau\,\mathrm{d}T + \frac{L}{Y_T}\,\mathrm{d}\tau \implies \frac{\mathrm{d}L}{L} = \alpha_\tau\,\mathrm{d}T + \frac{1}{Y_T}\,\mathrm{d}\tau \quad .$$

At constant tension, whenever α_τ does not depend on temperature, the equation can be integrated to get:

$$L[T] = L_0 \exp\left[\alpha_\tau[\tau_0]\,(T-T_0)\right] \qquad \text{at constant } \tau=\tau_0 \quad .$$

At constant temperature, whenever Y_T does not depend on tension, the equation can be integrated to get:

$$L[\tau] = L_0 \exp\left[(\tau-\tau_0)\,/Y_T[T_0]\right] \qquad \text{at constant } T=T_0 \quad .$$

Examples include:

- a Hooke spring;
- an ideal elastic wire;
- a non-elastic wire;
- a rubber band.

PRELIMINARY-PROGRESS

64.06.03
Surfaces Under Surface Tension as 2D Thermodynamical Systems

64.06.04
Non-Isotropic Solids

The mechanical behavior of a anisotropic region, such as a volume element of a non-cubic crystal, cannot be described by two variables only, such as volume and pressure.

In general the state of deformation is given by a set of six variables, the strain components, and the state of stress is given by another set of six variables, the stress components.

The quantities analogous to the volume of an isotropic region are the six extensive properties giving the volume of the reference state. The stress components correspond to the negative pressure and are intensive properties.

64.06.05
Polarizable Materials in External Electric Field

It is made complex by the fact that ElectroMagnetic interactions are long-range interaction, and therefore the problem does not directly fit the framework of elementary classical thermodynamics.

Read § 67.12.02.

64.06.06
Magnetizable Materials in External Magnetic Field

It is made complex by the fact that ElectroMagnetic interactions are long-range interaction, and therefore the problem does not directly fit the framework of elementary classical thermodynamics.

Read § 67.12.03

64.06.07
ElectroMagnetic Radiation

It is actually a $pVTN$ system, but in this case N is not fixed.

Read § 65.09.01.

DRAFT – TAU – PROGRESS

64.07 Equations of State of Various Systems

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...||*Adapted from.*|

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An equation of state expresses the individual peculiarities of one system as compared with another system and must, therefore, be determined either by experiment or by molecular theory. A general theory like thermodynamics, based on general laws of nature, is incapable of generating an equation of state of any system. An equation of state, therefore, is not a theoretical deduction from thermodynamics, but is usually an experimental addition to thermodynamics. It expresses the results of experiments in which the thermodynamic
It is obvious that no equation of state exists for the states traversed by a system that is not in mechanical and thermal equilibrium, since such states cannot be described in terms of thermodynamic coordinates referring to the system as a whole. For example, if a gas in a cylinder were to expand and to impart to a piston an accelerated motion, the gas might have, at any moment, a definite volume and temperature, but the corresponding pressure, calculated from an equation of state, would not apply to the system as a whole. The pressure would not be a thermodynamic coordinate, because it would depend not only on the velocity and the acceleration of the piston but would also vary from point to point.

Macroscopic thermodynamic variables always come in pairs, the two variables being the conjugate variables appearing in the expression of the work done on the system.

Moreover, from the expression of the first principle for an infinitesimal quasi-static transformation, as the internal energy is an extensive variable, the work term must be extensive as well, and therefore one of the two conjugate variables must be intensive, while the second one must be extensive.

An equation of state is the (either empirical of theoretical) relation among one pair of conjugate variables plus temperature. For each pair of conjugate variables, $\{Y, X\}$ there exist one equation of state: $G[X, Y, T] = 0$. The number of equations of state is therefore equal to the number of pairs of conjugate variables.

The equation of state so defined, is often called the **thermal equation of state**, or just **equation of state**, in short.

64.07.01 pVTN Systems

64.07.01.01 Ideal Gas

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|§ 5||

The **molecular ideal gas** - **ideal gas** in short - is the ideal gas of massive particles (molecules, ...).

The ideal gas is made of non-interacting particles; thermalization is assumed to happen thanks to the interactions with the walls.

1. Boyle law:

$$pV = \text{constant} \quad .$$

IN PROGRESS

2. Dalton law: In a mixture of non-reacting ideal gases, the total pressure is equal to the sum of the partial pressures of the individual gases:

$$p_i = p x_i \qquad p = \sum_i^n p_i \quad ,$$

where x_i is the mole fraction of the i-th component in the total mixture of n components.

3. Charles law (also known as the law of volumes): At constant pressure on gas, the temperature and the volume are proportional.

$$V \propto T \qquad \text{at constant pressure} \quad .$$

4. Ideal gas state equation:

$$pV = nRT = Nk_{\mathrm{B}}T \quad .$$

Clearly, the ideal gas state equation summarizes Boyle, Dalton and Charles laws.

64.07.01.02 Non Ideal Gases

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 5.6||

1. Clausius state equation (first type) for a hard core gas:

$$p\left(V - nb\right) = nRT \quad . \tag{64.07.01}$$

2. Van der Waals state equation:

$$\left(p + \frac{an^2}{V^2}\right)\left(V - nb\right) = nRT \quad . \tag{64.07.02}$$

3. The empirical algebraic Redlich-Kwong equation of state:

$$p = \frac{nRT}{v - nb} - \frac{an^2}{\sqrt{T}v\left(v + nb\right)} \quad . \tag{64.07.03}$$

4. Dieterici equation of state:

$$p = \left(\frac{RT}{v - b}\right)\exp\left[-\frac{a}{RTv}\right] \quad . \tag{64.07.04}$$

→ 2703

At low pressures the Dieterici equation becomes identical to Van der Waals equation.

5. Berthelot equation of state:

$$\left(p + \frac{an^2}{TV^2}\right)\left(V - nb\right) = nRT \quad . \tag{64.07.05}$$

6. Clausius state equation (second type):

$$\left(p + \frac{an^2}{T\left(c + V\right)^2}\right)\left(V - nb\right) = nRT \quad . \tag{64.07.06}$$

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64.07.01.03 Incompressible Solid/liquid

Dal punto di vista della termodinamica un liquido o un solido ideale è un sistema pVT caratterizzato dalla proprietà di incompressibilità assoluta. L'equazione di stato di una liquido o un solido ideale è dunque

$$\boxed{V = V_0} \quad . \tag{64.07.07}$$

Clearly, the above relation is nothing but the zero-order development of the state function $V(T, p)$.

For many phenomena liquids and solids can essentially be regarded as incompressible under changes of pressure, but still changing when temperature changes, so that their equation of state reduces to

$$\boxed{V = V[T]} \quad . \tag{64.07.08}$$

64.07.01.04 Elastic Solid/liquid

Una approssimazione migliore rispetto all'equazione di stato di solido e liquido incomprimibile, consiste nello sviluppare il volume al primo ordine della serie di Taylor in pressione e temperatura:

$$V[p, T] \simeq V[p_0, T_0] + \left(\frac{\partial V}{\partial p}\right)_T \Bigg|_0 (p - p_0) + \left(\frac{\partial V}{\partial T}\right)_p \Bigg|_0 (T - T_0) \quad ,$$

where the derivatives must be evaluated at the starting point of the Taylor series development.

Dal punto di vista della termodinamica un liquido oppure un solido reale possono essere caratterizzati come sostanze elastiche ideali. stato di un sistema elastico ideale fornisce per il volume una variazione lineare in funzione della pressione e temperatura attorno ad un punto a temperatura T_0 e pressione p_0

$$\boxed{V = V_0 + \beta V_0 (T - T_0) - \kappa_T V_0 (p - p_0)} \quad , \tag{64.07.09}$$

→ 2583

dove V_0 è il volume alla pressione p_0 e temperatura T_0 mentre β e κ_T sono considerati costanti. Quest'ultima assunzione è ben verificata normalmente per un liquido o un solido pur di considerare variazioni di temperatura e pressione non troppo estese. Si noti che non è necessario che il solido sia isotropo.

Clearly, the above relation is nothing but the first-order development of the state function $V(T, p)$.

64.07.01.05 Carnahan-Starling Equation of State

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The Carnahan-Starling equation of state is a good equation of state for a fluid made of hard spheres:

$$\boxed{\frac{pV}{Nk_BT} = \frac{1 + \eta + \eta^2 - \eta^3}{(1 - \eta)^3} \qquad \eta \equiv \frac{\pi N d^3}{6V} \qquad d \text{ is the hard sphere diameter}} \quad . \tag{64.07.10}$$

DRAFT - In Progress

64.07.01.06 ElectroMagnetic Radiation - the Photon Gas

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 1|*§ 39.3*|

The photon gas is the ideal gas of massless particles (photons).
Read § 65.09.01.06 for the discussion.

64.07.01.07 Stiffened Equation of State

©|WEB - URL |||

When considering water under very high pressures (typical applications are underwater nuclear explosions, sonic shock lithotripsy, and sonoluminescence) the stiffened equation of state is often used:

$$p = \rho(\gamma - 1)e - \gamma p^0$$

where e is the internal energy per unit mass, γ is an empirically determined constant typically taken to be about 6.1, and p^0 is another constant, representing the molecular attraction between water molecules. The magnitude of the correction is about 2 GPa. For a ideal gas, $p_0 = 0$.

64.07.02 Wires Under Tension

64.07.02.01 Hooke Ideal Wire/spring

It satisfies Hooke law:

$$\boxed{\tau = (a - bT)\,(L - L_0)} \quad ,$$

where a and b are constants.

64.07.02.02 Ideal Elastic Wire

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 2.7, 3.8, 3.9||

Read § 64.11.13.

64.07.02.03 Rubber Band

Read § 67.11.03.

64.07.03 Surfaces of a Pure Liquid in Equilibrium With Their Vapor

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|§ 2.6|*Adapted from*|

The surface of a pure liquid in equilibrium with its vapor has a particularly simple equation of state. Experiment shows that the surface tension of such a surface does not depend on the area but is a function of the temperature only. For most pure liquids in equilibrium with their vapor phase, the surface tension has an equation of state of the form originated by Van der Waals, but further developed by Guggenheim:

$$\boxed{\gamma[T] = \gamma_0 \left(1 - \frac{T}{T_c}\right)^{\alpha}} \quad ,$$

where γ_0 is the surface tension at a standard temperature, T_c is the critical temperature, that is the temperature above which no amount of pressure will condense vapor into liquid, and α is a constant that lies between 1 and 2. It is clear from this equation that the surface tension decreases as T increases, becoming zero when $T = T_c$.

DRAFT - IN PROGRESS

64.07.04
Materials in External ElectroMagnetic Fields

64.07.04.01 Curie and Curie-Weiss Laws for Linear Polarizable|Magnetizable Substances

A linear Polarizable|Magnetizable substance material in an external Electric|Magnetic field assumes a Polarization/Magnetization approximately directly proportional to the applied field.

If the temperature of the material is changed, this proportionality is changed as well: for a fixed value of the field, the Polarization/Magnetization is (approximately) inversely proportional to temperature (Curie law). For a simple homogeneous, isotropic and linear material:

$$\chi_{\mathtt{E}} \propto C_e/T \qquad \chi_{\mathtt{M}} \propto C_m/T \quad .$$

A more general law is the Curie-Weiss law: For a simple homogeneous, isotropic and linear material:

$$\chi_{\mathtt{E}} \propto C_e/\left(T+\theta_e\right) \qquad \chi_{\mathtt{M}} \propto C_m/\left(T+\theta_m\right) \quad .$$

The Curie constants C_e, C_m, θ_e, θ_m depend on the material.

64.07.04.02 Other Equations of State for Polarizable|Magnetizable Substances

Include, non homogeneous and|or non isotropic materials, ferro-magnetic and ferro-electric, general non linear equation of state.

64.07.05
Non-Isotropic Solids

The full stress-tensor and strain tensor must be used to write the expression of work.

There exists a stress-strain relation which is the equation of state. The simplest relation of this kind is the generalized Hooke law which holds for small deformations.

DRAFT - IN-PROGRESS

64.08 Some Relevant Mathematical Theorems

64.08.01 Relations Among Partial Derivatives

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|§ 2.4||

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Let $f[x,y,z]=0$ that is, only two out of the three variables x, y, z are independent. Let w a fourth variable, with $x=x[w]$, $y=y[w]$, $z=z[w]$. Then:

$$\left(\frac{\partial y}{\partial x}\right)_z\left(\frac{\partial x}{\partial y}\right)_z=+1 \qquad \text{reciprocal rule} \quad , \tag{64.08.01}$$

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y=-1 \Longrightarrow \left(\frac{\partial x}{\partial y}\right)_z=-\left(\frac{\partial x}{\partial z}\right)_y\left(\frac{\partial z}{\partial y}\right)_x \qquad \text{cyclic rule} \quad , \tag{64.08.02}$$

→ 2546 2548 2584

$$\left(\frac{\partial x}{\partial y}\right)_w\left(\frac{\partial y}{\partial z}\right)_w\left(\frac{\partial z}{\partial x}\right)_w=+1 \qquad \text{chain rule} \quad , \tag{64.08.03}$$

$$\left(\frac{\partial x}{\partial w}\right)_z=\left(\frac{\partial x}{\partial w}\right)_y+\left(\frac{\partial x}{\partial y}\right)_w\left(\frac{\partial y}{\partial w}\right)_z \qquad \text{Jacobian rule} \quad , \tag{64.08.04}$$

64.08.02 Second Mean Value Theorem for Integrals

If f is continuous on $[a,\ b]$, $a<b$, show that there exists $c\in[a,b]$ such that we have $\int_a^b f = f(c)(b-a)$. This result is sometimes called the *Mean Value Theorem for Integrals.*

If f and g are continuous on $[a,\ b]$ and $g(x)>0$ for all $x\in[a,b]$, show that there exists $c\in[a,b]$ such that $\int_a^b fg = f(c)\int_a^b g$. Show that this conclusion fails if we do not have $g(x)>0$. (Note that this result is an extension of the preceding exercise.)

WORK-IN-PROGRESS

64.09 Miscellaneous Topics

64.09.01 Legge Del Raffreddamento Di Newton

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|22.9 22.10||

È esperienza comune che gli oggetti caldi tendono a raffreddarsi e quelli freddi a riscaldarsi alla temperatura dell'ambiente circostante. Se la differenza di temperatura tra l'oggetto e l'ambiente, $\Delta T \equiv T_{\text{ogg}} - T_{\text{amb}}$, non è troppo elevata la velocità di raffreddamento o riscaldamento è proporzionale, in prima approssimazione, alla differenza di temperatura

$$\frac{\mathrm{d}(\Delta T)}{\mathrm{d}t} = -A\,(\Delta T)$$

dove A è una costante.

1. Da quali fattori dipende la costante A e quali sono le sue dimensioni?
2. Se all'istante $t = 0$ si ha $\Delta T = \Delta T_0$ mostrare che l'andamento previsto della differenza di temperatura è

$$\Delta T = \Delta T_0 \exp -At$$

UNDER-IN-PROGRESS

64.10 Summary of the Framework of Thermodynamics

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Definitions

- System: the portion of universe one is interested in.
- Surroundings: the complement to the system; also, often, known as **ambient**.
- Universe: system plus surroundings.
- Boundary: the frontier between the system and surroundings, a closed surface; usually a well-defined geometrical shape enclosing a definite volume.
- Isolated System: no exchange of matter with surroundings; no exchange of energy with surroundings; no exchange of any other dynamical quantities with surroundings (linear momentum, angular momentum,...); no exchange of work nor heat.
- Closed system: no exchange of matter with surroundings; can exchange energy and|or any other dynamical quantities with surroundings, including work and heat.
- Open system: can exchange matter with surroundings as well as energy and|or other dynamical quantities, including work and heat.
- Adiabatic wall: two systems put in contact across an adiabatic wall will keep the same macroscopic parameters; when an adiabatic wall is used to separate two systems, the macroscopic properties of one system are found to be uninfluenced by those of the other.
- Diathermic wall: two systems put in contact across a diathermic wall will change their macroscopic parameters until thermodynamic equilibrium is reached; the two systems will undergo changes until thermal equilibrium is reached.
- An isotherm is the locus of all points representing states in which a system is in thermal equilibrium with one state of another reference system; we make no assumption as to the continuity of the isotherm, although experiments on simple systems indicate usually that at least a portion of an isotherm is a continuous curve.
- A phase is a homogeneous region of matter, separated from other homogeneous regions by a phase boundary with a well defined and smooth geometrical shape enclosing a well-defined volume; the different regions (phases) will have different physical and|or chemical properties; when more than one phase exist, the system is heterogeneous, but the length scale of heterogeneity can vary from nano-meter to macroscopic dimensions. Usually, three kind of phases are identified: solid, liquid, gas.
- A fluid phase: liquid and gas.
- A condensed phase: solid and liquid.
- System in steady state: the state variables do not change with time.
- System in thermodynamic equilibrium: given a system in steady state, if furthermore there are no macroscopic currents of energy (work and|or heat) or particle number flowing through the system, the system is said to be in equilibrium.
- A thermodynamic path: a continuous succession of equilibrium states; it can be represented as a smooth curve in a multidimensional space whose axes are labeled by the state variables.
- A thermodynamic process/transformation: any change or succession of changes which results in a change of the state variables.

  - A cyclic process: the initial and final states are the same.
  - A quasi-static process: the system passes through a continuous succession of equilibria.
  - A reversible process: the external conditions and the thermodynamic path of the system can be time-reversed and the initial state ba be reached again leaving both the system and the environment in exactly the same state as before. It is both quasi-static and non-dissipative. To reverse the process, infinitesimal changes in the opposite direction must be applied.
  - A quasi-static process which is not reversible: slow drag a mass on a floor; slow leak of air from a tire; slow thermal transfer between two objects at different temperatures across a wall with very little thermal **conductivity** .
  - Irreversible processes, as a rule, are dissipative.
- Thermal contact: heat conduction by direct physical contact, heat convection via some moving heated fluid, heat exchange via radiation.

Conceptual Bases

- Objective of thermodynamics.
  - In thermodynamics, systems of **many-many bodies** are studied (many-many: number of Avogadro).
  - **Macroscopic approach**.
  - Here, we limit to thermostatics.
- Define the system.
  - Define with precision what belongs to the **system** and what does not: in thermodynamics, it may be tricky.
  - Define the **boundary** between system and surroundings (assume it is a geometrically smooth surface).
  - Define the **surroundings** (the complement of the system).
  - **Universe**: the system plus the surroundings.
- Define how the system interacts with the surroundings.
  - As in mechanics, but with emphasis on exchanges of energy.
  - **Exchanges of energy**.
  - But also: exchanges of linear and|or angular momentum, and|or whatever else
  - Here, we only study exchanges of energy (**work** and|or **heat**).
- Define the macroscopic variables.

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  - Select the **macroscopic variables relevant to the proper description of the system**, that is relevant to the scope, within the empirical/experimental/phenomenological accuracy desired/required and within the known constraints, limits, capabilities.
  - The selection of the macroscopic variables is an **empirical/experimental/phenomenological** step.
  - Do not forget anything relevant to the problem
  - Do not overdo, by including less relevant quantities
  - You decide what is relevant to the scope and what is not relevant to the scope: to do this you must know what you want to achieve.
  - You should know the physics, to decide what you can/must ignore and you will not ignore.
  - The macroscopic variables must be able to describe the exchanges of energy: pairs of macroscopic variables, **a generalized force** and a **generalized displacement**, to describe macroscopic work.
  - The generalized force and generalized displacement depend on the type of system.
  - Work is energy exchange due to some change of the macroscopic parameters.
  - Work is opposed to heat: energy exchange without changes in macroscopic parameters (that is at a microscopic level).
  - The pairs of macroscopic variables shall be independent variables; name them (X, Y), (X', Y'), (X'', Y''),...
  - If the set of pairs of macroscopic variables have been chosen in a correct way, that is all and only the variables relevant to the scope, then they describe in a unambiguous way the **state** of the system.
  - Here we will mainly describe the formalism for systems defined by one single pair of macroscopic variables, (X, Y), to avoid a cumbersome notation; the extension of the formalism to systems described by more than one pair of macroscopic variables is straightforward.

- Walls.
  - Experience shows that there exist materials such that two systems, A and B, put in contact via a material wall will keep the same values of their macroscopic variables (the same state) for a very long time;
    $$(X_A, Y_A) \to (X'_A, Y'_A) = (X_A, Y_A) \quad ,$$
    and
    $$(X_B, Y_B) \to (X'_B, Y'_B) = (X_B, Y_B) \quad ,$$
    this is named an **adiabatic wall**.
  - Experience shows that there exist materials such that two systems, A and B, put in contact via a material wall will change the values of their macroscopic variables (they change their state) in a short time, reaching new values which will be then kept for very long times:
    $$(X_A, Y_A) \to (X'_A, Y'_A) \neq (X_A, Y_A) \quad ,$$
    and
    $$(X_B, Y_B) \to (X'_B, Y'_B) \neq (X_B, Y_B) \quad ,$$
    this is named a **diathermic wall**.
  - In the case of contact via a diathermic wall, the two systems (better, the states of the two systems), at the end, are said to be in **thermal equilibrium**.
  - Whenever two systems are in thermal equilibrium, not all the values of the macroscopic variables (X_A, Y_A) and (X_B, Y_B) are allowed but only a restricted subset, namely those corresponding to thermal equilibrium between the two systems.
  - Adiabatic wall and diathermic walls are limiting concepts (limiting models).
  - Note that **contact** does not necessarily mean physical contact; one speaks of **thermal contact**, which means to let heat flow, that is to let energy exchanges on a microscopic scale, which can happen via physical contact (conduction), via mass in contact flow (convection) or via radiation.

- Thermal equilibrium.

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  - **Principle zero**: if the states of two systems are in thermal equilibrium with the state of a third system, then they are in thermal equilibrium.
  - The locus of states, in the XY plane, of any one system which are in thermal equilibrium with a well-defined state of a third (reference) system, is called an **isothermal curve**; the reference system is named **thermometer**.
  - Any isothermal curve is assume to be, locally, differentiable just enough.

- Empirical temperature scale.
  - An **empirical temperature scale** can be defined by associating a numerical value and unit of measure to any function of the macroscopic variables of the thermometer, for instance: $\Theta[X,Y]=g[X,Y]$; it is often used in a simplified form of $\Theta_X[X]=g_X[X]$ or $\Theta_Y[Y]=g_Y[Y]$.

- Equations of state.
  - As the pair of macroscopic variable is assumed to describe unambiguously the macroscopic state of the system, the empirical temperature is uniquely defined, and there is therefore a well-defined relations among X,Y,T which can be written in the most general way in implicit form as $G[X,Y,T]=0$ and it is called **equation for state**.
  - It is assumed that, locally, the equation of state is invertible in any of the variables:
$$X=f_X[Y,T] \qquad Y=f_Y[T,X] \qquad T=f_T[X,Y] \quad .$$

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64.11 Examples and Physical Applications

64.11.01 Applications of Thermal Expansion

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Per aprire il coperchio troppo stretto di un vaso di conserva si può sovente riuscire nell'intento ponendolo sotto un getto di acqua calda. Al crescere della temperatura il coperchio metallico si espande un po' di più rispetto al vetro.
La dilatazione termica non è però sempre desiderabile. Conosciamo tutti le fessure che devono essere lasciate nei giunti delle rotaie e dei ponti per consentirne la dilatazione.
Nell'industria aerea i bulloni e gli altri agganci sono raffreddati con ghiaccio secco prima dell'inserzione e quindi lasciati espandere una volta che si sono correttamente adattati.
Esistono termometri e termostati che fondano il loro funzionamento sulla differenza di dilatazione tra le componenti di un bimetallo.

64.11.02 Lo Specchio Del Telescopio Di Monte Palomar

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Lo specchio in vetro Pyrex del telescopio di Monte Palomar (il telescopio Hale) misura $D = 200$ pollici di diametro. Le temperature più estreme misurate a Monte Palomar sono $T_{\mathtt{min}} = -10°\mathrm{C}$ e $T_{\mathtt{max}} = +50°\mathrm{C}$. Determinare la massima variazione possibile del diametro D e della superficie A dello specchio causata dalle variazioni di temperatura. Si assuma per il coefficiente di dilatazione lineare medio del Pyrex, nell'intervallo di temperatura considerato, il valore $\alpha = 3.2{\cdot}10^{-6}{}^{\circ}\mathrm{C}^{-1}$.

WORKED SOLUTION

Dalla relazione (64.06.01), valida per variazioni a tensione costante in approssimazione di solido elastico e per piccoli intervalli di temperatura cosicché α possa essere considerato costante, si ha

$$\Delta L = L_0 \alpha \Delta T$$

Nel caso di solidi isotropi (o liquidi) il coefficiente di espansione lineare non dipende dalla direzione per cui per una variazione piccola di temperatura si può valutare la variazione di una piccola superficie rettangolare A come

$$A \equiv XY = X_0 Y_0 \left(1 + \alpha\Delta T\right)\left(1 + \alpha\Delta T\right) = A_0\left(1 + 2\alpha\Delta T\right) + \mathcal{O}\left(\Delta T^2\right)$$

fa cui si deduce che il coefficiente di dilatazione superficiale vale 2α. Il coefficiente di espansione volumetrica è definito dalla

$$\beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

mantenendo costante la pressione (nel caso di un sistema pVT) oppure lo sforzo normale (nel caso di un sistema elastico $\tau\mathfrak{X}\mathfrak{T}$). Analogamente si trova che vale $\beta = 3\alpha$ nel caso di sistemi isotropi oppure in generale $\beta = \alpha_1 + \alpha_2 + \alpha_3$ nel caso di un sistema anisotropo con coefficienti di dilatazione lineare α_1, α_2 ed α_3 lungo i tre assi. Infatti dalla definizione di β e ponendo $V = xyz$ e derivando si ha $\beta = 3\alpha$.
Con i dati del problema si ottiene la variazione del diametro $\Delta D = 0.98\ \mathrm{mm}$ e una variazione di superficie $\Delta \mathbf{S} = 78\ \mathrm{cm}^2$. In questo caso la variazione della superficie dello specchio può essere

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ottenuta anche dalla relazione tra $\Delta\mathbf{S}$ ed il diametro d

$$\Delta\mathbf{S} = \frac{\pi d}{2}\Delta d = \frac{\pi d^2}{2}\alpha\Delta T = A_0 2\alpha\Delta T$$

valida per piccole variazioni di diametro.

WORKED SOLUTION

$\Delta D = 0.98$ mm; $\Delta\mathbf{S} = 78$ cm^2.

64.11.03
Effetto Delle Variazioni Di Temperatura Su Un Barometro

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Dimostrare che quando la temperatura del liquido di un barometro varia delle quantità ΔT e la pressione resta costante, l'altezza del liquido nel barometro varia di $\Delta h = \beta h \Delta T$ dove β è il coefficiente di espansione volumetrica del liquido definito dalla (64.06.02). Si supponga di poter trascurare la variazione di dimensioni del bulbo di vetro del barometro.

64.11.04
Espansione Termica in Un Termometro a Mercurio

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Si consideri un termometro a mercurio con bulbo e capillare di vetro. Si supponga che la sezione del capillare sia A costante e che V sia il volume del bulbo a temperatura $T_0 = 0\,^\circ$C. Si supponga che il mercurio riempia esattamente il bulbo alla temperatura $T_0 = 0\,^\circ$C.

Dimostrare che la lunghezza L della colonna di mercurio nel capillare a temperatura T (espressa in gradi centigradi) vale $L = V\,(\beta - 3\alpha)\,T/A$ ed è dunque proporzionale alla temperatura con β il coefficiente di espansione volumica del mercurio e α il coefficiente di espansione lineare del vetro.

64.11.05
Movimento a Micro-Velocità

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In un certo esperimento occorre muovere un sorgente radioattiva ad un velocità assegnata estremamente piccola. Questo può essere ottenuto fissando la sorgente radioattiva ad un estremo di una sbarra di alluminio e riscaldando la parte centrale della sbarra in modo controllato tramite un avvolgimento elettrico. Se la sezione della sbarra effettivamente riscaldata è lunga $z = 1.8$ cm a quale ritmo costante deve essere variata la temperatura della sbarra se la sorgente radioattiva deve muoversi alla velocità costante $v_0 = 96{\cdot}10^{-9}$ ms

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64.11.06
Coefficiente Di Dilatazione Termica Dipendente Dalla Temperatura

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Mostrare che, se il coefficiente di dilatazione termica

$$\alpha \equiv \frac{1}{L}\frac{\partial L}{\partial T}$$

di una sostanza dipende dalla temperatura, la lunghezza L alla temperatura T è legata alla lunghezza L_0 alla temperatura T_0 dalla relazione

$$L = L_0 \exp\left[\int_{T_0}^{T} \alpha(T)\, \mathrm{d}T\right] \quad .$$

64.11.07
Dipendenza Della Densità Dalla Temperatura

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Dimostrare che, se il volume di un oggetto dipende dalla temperatura, la densità varia con la temperatura secondo la relazione

$$\Delta\rho = -\beta\rho\Delta T \tag{64.11.01}$$

64.11.08
Coefficiente Di Espansione Termica Di Una Sbarra Composta

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Una sbarra cilindrica di lunghezza $L = L_1 + L_2$ è composta di due parti di lunghezza L_1 ed L_2, di materiali differenti, attaccate ad un estremo. I coefficienti di espansione lineare dei due materiali sono, rispettivamente, α_1 ed α_2.

1. Determinare il coefficiente di espansione lineare effettivo della sbarra.
2. Utilizzando acciaio ed ottone progettare una sbarra composta di lunghezza $L = 52.4\ \mathrm{cm}$ avente un coefficiente effettivo di espansione lineare pari a $\alpha = 13{\cdot}10^{-6}{}^{\circ}\mathrm{C}^{-1}$

WORKED SOLUTION

1. $\alpha = (\alpha_1 L_1 + \alpha_2 L_2)\,/L$

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64.11.09
Variazione Del Momento Di Inerzia Al Variare Della Temperatura

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Dimostrare che la variazione del momento di inerzia rispetto ad un asse di un solido, al variare della temperatura, è data da

$$\Delta I = 2\alpha I \Delta T$$

64.11.10
Sbarra Composta Di Lunghezza Fissata

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This § is referenced at pages:
[Never referenced.]

Una sbarra cilindrica è composta di due parti di uguale lunghezza L, fatte di due diversi materiali aventi coefficiente di espansione termica e modulo di Young pari rispettivamente ad α_1, α_2 ed E_1, E_2. Le due parti sono collegate ad un estremo, hanno sezione A e la sbarra è posta tra due supporti rigidi e fissi, a distanza fissa $2L$. La temperatura iniziale del sistema è T e, a questa temperatura, la sbarra non è in tensione. La sbarra è riscaldata uniformemente e la temperatura cresce di ΔT. Si ignori la variazione della sezione della sbarra.

1. Dimostrare che la superficie di congiunzione tra le due parti si sposta della quantità
$$\Delta L = \left(\frac{\alpha_1 E_1 - \alpha_2 E_2}{E_1 + E_2} \right) L \Delta T$$
2. Determinare lo sforzo normale all'interfaccia tra le due parti.
3. Generalizzare e rispondere alle domande precedenti assumendo che le due parti abbiano lunghezza diversa, L_1 e L_2.
4. Verificare che i risultati alla domanda precedente si riducano correttamente nei casi limite che si possono considerare.

64.11.11
I Cavi Del Golden Gate Bridge

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La distanza tra le torri dell'arcata principale del Golden Gate Bridge a San Francisco vale $d = 4200$ ft. La freccia del cavo, cioè la differenza di altezza tra il punto in cui il cavo è attaccato presso la cima della torre e il punto più basso, vale $z = 470$ ft ad una temperatura di $50\ ^\circ\mathrm{F}$. Si assuma un coefficiente di espansione termica del cavo pari a $\alpha = 6.5 \cdot 10^{-6}\ ^\circ\mathrm{F}^{-1}$, che il cavo sospeso assuma una forma parabolica e che non vari la distanza tra le due torri. Si supponga che la temperatura vari da $T_1 = 10\ ^\circ\mathrm{F}$ a $T_2 = 90\ ^\circ\mathrm{F}$.

1. Calcolare la variazione di lunghezza del cavo.
2. Calcolare la variazione della freccia del cavo.

DRAFT - IN PROGRESS

64.11.12
L'aria in Una Stanza

Una stanza di volume $V = 60\ \mathrm{m}^3$ è mantenuta a temperatura costante pari a $T_0 = 27°\mathrm{C}$. Se il sistema di riscaldamente viene spento la temperatura della stanza scende lentamente a $T_1 = -3°\mathrm{C}$. Determinare la massa di aria che entra nella stanza attraverso gli spifferi di porte e finestre.

WORKED SOLUTION

$\approx 7\ \mathrm{kg}$

64.11.13
Elastic Wire

This § is referenced at pages:
[2553]

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 2.7, 3.8, 3.9||

Read § 67.11.04
The equation of state of an ideal elastic wire of cross-section A is:

$$\tau = \frac{Y_0 A}{3}\left(\frac{L}{L_0} - \frac{L_0^2}{L^2}\right) \quad . \tag{64.11.02}$$

Consider an ideal elastic wire.
1. Calculate the isothermal Young modulus.
2. Show that the isothermal Young modulus at zero tension is Y_0.
3. Calculate the linear thermal expansivity.

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64.12 Exercises Problems and Physical Applications

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CHAPTER 65

Work Energy Heat and the First Principle

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DRAFT-IN-PROGRESS

65.01 Introduction

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If a system as a whole exerts a force on its surroundings and a displacement takes place, the work that is done, either by the system or on the system, is called external work. Thus, a gas, confined in a cylinder and at uniform pressure, while expanding and imparting motion to a piston does external work on its surroundings. The work done, however, by one part of a system on another part is called internal work. The interactions of molecules, atoms, or electrons on one another constitute internal work.

Internal work cannot be discussed in macroscopic thermodynamics. Only the work that involves an interaction between a system and its surroundings is analysed. When a system does external work, the changes that take place can be described by means of macroscopic quantities referring to the system as a whole, in which case the changes may be imagined to accompany the raising or lowering of a suspended weight, the winding or unwinding of a spring, or, in general, the alteration of the position or configuration of some external mechanical device. This may be regarded as the ultimate criterion as to whether external work is done or not. It will often be found convenient throughout the remainder of this book to describe the performance of external work in terms of, or in conjunction with, the operation of a mechanical device, such as a suspended weight or deformed spring. Unless otherwise indicated, the word work, unmodified by any adjective, will mean external work.

In mechanics, we are concerned with the behavior of systems acted on by external forces. When the total force exerted on a mechanical system is in the same direction as the displacement of the system, the work of the force is positive, work is said to be done on the system, and the energy of the system increases. For thermodynamics to be consistent with mechanics, we adopt the same sign convention for work. Thus, when work is done on the system, the work is regarded as positive. Conversely, when work is done by the system, the work is regarded as negative. This sign convention disagrees with engineering practice, in which positive work is done by the system on an external object.

A process performed in this ideal way is said to be quasi-static. During a quasi-static process, the system is at all times infinitesimally near a state of thermodynamic equilibrium, and all the states through which the system passes can be described by means of thermodynamic coordinates referring to the system as a whole. An equation of state is valid, therefore, for all these states. A quasi-static process is an idealization that is applicable to all thermodynamic systems, including Electric|Magnetic systems. The conditions for such a process can never be rigorously satisfied in the laboratory, but they can be approached with almost any degree of accuracy.

The expression

$$\delta \mathcal{W} \equiv -p \, đV$$

is an infinitesimal amount of work and has been. There is, however, an important distinction between an infinitesimal amount of work and the other infinitesimals, such as $đp$ or $đV$. An infinitesimal amount of hydrostatic work is an inexact differential; that is, it is a quantity but it is not the differential, that is the infinitesimal change, of an actual function of the thermodynamic coordinates. To indicate that an infinitesimal amount of work is not a mathematical differential of a function and to emphasize at all times that it is an inexact differential, the symbol used is $\delta \mathcal{W}$. Another explanation of the symbolism is that $đV$ is a very small change in the value of the volume, whereas $\delta \mathcal{W}$ is a very small value of work, not a change of something.

There are various interpretations of the fact that thermodynamic work is an inexact differential. Already it has been shown that work is a path-dependent integral; in other words, work is not a state function. There is no function of the thermodynamic coordinates representing the work in a body. The physical interpretation of thermodynamic work is that the performance of work is an external activity or process that leads to a change in the state of the body, namely, to increase or decrease the energy. The phrase **work in a body** has no meaning. Notice that work is not a quantity that itself moves, but is a process that moves another quantity, namely, energy of some type.

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65.02 Work

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DRAFT WORK IN PROGRESS

65.02.01 Thermodynamic Work

Work is always defined as the work done by the surroundings on the system (positive work).

Unless otherwise indicated, the word work, unmodified by any adjective, will mean external work.

It is postulated that work done on a system can be always calculated, at least in principle, in terms of physical quantities of the universe.

Consider a composite system made by two subsystems A and B.

By definition, the total work done on the composite system is the sum of the works done on the two sub-systems:

$$\mathcal{W}_{A+B} \equiv \mathcal{W}_A + \mathcal{W}_B \quad .$$

It is postulated that contact Force | Torque (that is short-range Force | Torque) are only allowed. This implies:

$$\mathcal{W}_{A\to B} + \mathcal{W}_{B\to A} = 0 \quad .$$

If this does not apply, some of the conclusions of classical thermodynamics might not work and the appropriate thermodynamics framework must be re-built from the start.

Clearly, the postulate is false, in general, but it applies straightforwardly to many common thermodynamic systems, whenever only contact forces, that is short-range forces, are responsible for the work, in such a way that two small parts of the two systems are in contact and therefore always have the same displacement.

Notable exceptions are long-range gravitational and ElectroMagnetic interactions.

However, gravitational forces are very weak on the every-day scale, and negligible. On astrophysical scales, the thermodynamic treatment of gravitating systems must take into account its long-range character.

On the every-day scale, physics is dominated by ElectroMagnetic interactions. However, free charges tend to redistribute in such a way to self-shield, so that neutral systems are obtained, which only exert Van-der-Walls type, short ranges, forces.

Read § 29.06.23 for the characterization of long-range interactions.

65.02.02 Quasi Static Work

If, and only if, the transformation is a quasi-static one, that is it proceeds very slowly and gradually so that system and surroundings are always in thermodynamic equilibrium, the work can be calculated in terms of the state variables of the system, thanks to the thermodynamic equilibrium.

Consider a system defined by the set of pairs of macroscopic variables:

$$(X, Y) \qquad (X', Y') \qquad (X'', Y'') \qquad ...$$

the expression of work, for a quasi-static transformation, in terms of variables of the system:

$$\delta\mathcal{W} = Y\, \mathrm{d}X + Y'\, \mathrm{d}X' + Y''\, \mathrm{d}X'' + ...$$

As quasi-static transformations are idealized transformations, calculating the work in terms of the state variables of the system must be considered, conceptually, the exception to the general rule.

65.02.03
Work Depends on the Transformation

Work done on a system between two arbitrary states depends, in general, on the path followed, for a quasi-static transformation, and in general on the transformation details.

Consider, for instance, two transformations of a pVT systems from state A and A', with the same initial and final temperatures, T_0, with the same initial, V_1, and final volumes, V_2, assuming, for the sake of clarity, $V_2 < V_1$, and the same initial and final pressures, p_1 and p_2.

1. For the (unique) iso-thermal transformation, the work is: $\mathcal{W} = -\int_{V_1}^{V_2} p[V, T_0]\, \mathrm{d}V$.
2. For the unique sequence of isocore followed by isobare transformations, via the intermediate state $B = (p_2, V_1, T')$: $\mathcal{W} \equiv \mathcal{W}_{AB} + \mathcal{W}_{BA'} = 0 - p_2\,(V_2 - V_1) > 0$
3. For the unique sequence of isobare followed by isocore transformations, via the intermediate state $C = (p_1, V_2, T'')$: $\mathcal{W} \equiv \mathcal{W}_{AC} + \mathcal{W}_{CA'} = -p_1\,(V_2 - V_1) + 0 > 0$

The microscopic interpretation helps to understand the difference between slow quasi-static transformations and non quasi-static transformations, such as a sudden expansion/compression of the gas. Imagine the gas inside the typical piston-cylinder, with the piston initially clamped in a fixed position with a weight resting on it exerting a pressure on the piston. Depending on the weight, the pressure on the piston may be either smaller or larger than the initial gas pressure. If friction is negligible and the volume of gas decreases then the work done by the weight on the system is larger than it would be for a quasi-static transformation, because the external pressure is larger. The local compression of the gas near the piston generates a larger resistance which must be overcome by the external force. The reverse happens in case of a sudden expansion.

65.02.04
Adiabatic Work

If a closed system is caused to change from an initial state to a final state by adiabatic means only, then the work done on the system is the same over all adiabatic paths connecting the two states: work done adiabatically on a system is a state function.

© - QUOTE

Consider a system enclosed by a thermally isolated (adiabatic), impermeable wall, so that the sole interaction with the external world will appear under the form of a mechanical work $\mathcal{W}$, for instance by expansion of its volume or by stirring. Referring to the famous experience of Joule, the work can be measured by the decrease in potential energy of a slowly falling weight. During the evolution of the system between the two given equilibrium states A and B, it is checked experimentally that the work $\mathcal{W}$ is determined exclusively by the initial and the final states A and B, independently of the transformation paths. This observation allows us to identify $\mathcal{W}$ with the difference $\Delta\mathcal{U} = \mathcal{U}(B) - \mathcal{U}(A)$ of a state variable $\mathcal{U}$ which will be given the name of internal energy.

65.02.05
Work on Various Systems

This § is referenced at pages:
[2537]

Un sistema termodinamico è descritto da un insieme di variabili macroscopiche. Tali variabili intervengono sempre a coppie (variabili coniugate) nell'espressione del lavoro fatto sul sistema

$$\delta\mathcal{W} = \sum_i Y_i\, \mathrm{d}X_i$$

dove Y_i è una forza generalizzata (ad esempio la pressione per un sistema pVT) ed $\mathrm{d}X_i$ è uno spostamento generalizzato (ad esempio il volume per un sistema pVT).

65.02.05.01 Work on the Generalized Force-Displacement Diagram

This § is referenced at pages:
[2546, 2572]

It is useful for visualizing work. In fact work is the signed integral of $p[V,T]$. Read § 65.02.05.01.

For a system described by the variables X and Y, with work $\delta\mathcal{W} = Y\,\mathrm{d}X$, the work in a quasi-static transformation from A to B is the signed definite integral

$$\mathcal{W} = \int_{X_A}^{X_B} Y[X,T]\,\mathrm{d}X > 0 \qquad \Leftrightarrow \qquad Y > 0 \qquad \&\& \qquad X_B > X_A \quad .$$

65.02.05.02 Sistema pVT

65.02.05.03 Solido Elastico

Il lavoro fatto su una sbarra elastica di sezione A, definita dalle variabili termodinamiche lunghezza, L, e sforzo normale, tensione per unità di area, τ, vale

$$\delta\mathcal{W} = +\tau A\,\mathrm{d}L \quad .$$

Si adotta la convenzione che lo sforzo τ è positivo per una sbarra in tensione e negativo per una sbarra in compressione.

65.02.05.04 Superfici

Il lavoro fatto per variare una superficie, dovuto alla presenza della tensione superficiale γ (Read § 28.06), vale

$$\delta\mathcal{W} = +\gamma\,\mathrm{d}\mathbf{S} \quad .$$

65.02.05.05 Polarizable Materials in External Electric Field

© - QUOTE

S.Bobbio, *Electrodynamics of Materials*, 2000, Academic Press, 1stEd., ...
The definition of the energy of an ElectroStatic field is borrowed from that of potential energy in mechanics, and is grounded upon the irrotational character of the ElectroStatic field. Basing ourselves on this important property, we are able to introduce the ElectroStatic energy of a continuous distribution of charges: it is, by definition, the work "we" should spend in order to "build" the given distribution (in an arbitrary way), by counterbalancing only the forces due to the long-range ElectroStatic interactions between charges.
...
This work is independent of the way in which the given distribution of charge is built (as the ElectroStatic field is irrotational), we conclude that the ElectroStatic energy of the distribution is expressed, by definition, as...

It is made complex by the fact that ElectroMagnetic interactions are long-range interaction, and therefore the problem does not directly fit the framework of elementary classical thermodynamics.

Read § 67.12.02.

For the sake of illustration, it is anticipated that the work done when changing the external electric field, with no work done by the generators, and considering as a system an infinitesimal volume of matter, $\mathrm{d}V$, including the ElectroMagnetic field interaction energy, is:

$$\delta\mathcal{W} = -(\mathbf{P}\cdot\Delta\mathbf{E}_0) \qquad \text{for a path in the plane } \mathbf{P}\text{-}\mathbf{E}_0 \quad .$$

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65.02.05.06 Magnetizable Materials in External Magnetic Field

© - QUOTE

S.Bobbio, *Electrodynamics of Materials*, 2000, Academic Press, 1stEd., ...
In this section we intend to evaluate the work that we have to spend on "building" a steady, continuous current distribution.
...

It is made complex by the fact that ElectroMagnetic interactions are long-range interaction, and therefore the problem does not directly fit the framework of elementary classical thermodynamics.

Read § 67.12.03

For the sake of illustration, it is anticipated that the work done when changing the external magnetic field, with no work done by the generators, and considering as a system an infinitesimal volume of matter, $\mathrm{d}V$, including the ElectroMagnetic field interaction energy, is:

$$\delta \mathcal{W} = -(\mathbf{M}\cdot\Delta\mathbf{B}_0) \qquad \text{for a path in the plane } \mathbf{M}\text{-}\mathbf{B}_0 \quad .$$

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65.03 Internal Energy

For every system, there is a scalar extensive state function called energy. When the system is isolated, the energy is conserved.

The change in internal energy of a system when changing from state 1 to state 2, is defined as the adiabatic work done:

$$\boxed{\Delta \mathcal{U}_{1\to 2} \equiv \int_1^2 \delta \mathcal{W}^{\mathcal{A}} \equiv \mathcal{W}^{\mathcal{A}}_{1\to 2}} \quad , \tag{65.03.01}$$

Internal energy is assumed to be additive over subsystems:

$$\boxed{\mathcal{U}_{A+B} = \mathcal{U}_A + \mathcal{U}_B} \quad ,$$

and extensive:

$$\boxed{\mathcal{U} = M\mathrm{u}} \quad .$$

In fact, the former implies the latter.

Additivity follows from the hypothesis of weak-coupling of the systems.

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65.04 Heat

©|Mark W. Zemansky|**The Use and Misuse of the Word "Heat" in Physics Teaching** |*The Physics Teacher 8, 295 ...*

The heat system when changing from state A to state B, is defined as:

$$\mathcal{Q}_{1\to 2} \equiv \Delta\mathcal{U}_{1\to 2} - \mathcal{W}^{\mathcal{A}}_{1\to 2} \quad . \tag{65.04.01}$$

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Microscopically, mechanical work is related to coherent correlated motions of the particles while heat represents that part of motion, which is uncorrelated, say incoherent.

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65.05 First Principle of ThermoDynamics

This § is referenced at pages:
[1299, 2584]

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65.05.01 Statement of the First Principle of ThermoDynamics for Closed Systems

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The first principle of ThermoDynamics states the energy balance for a generic system and the conservation of energy for isolated systems (energy is never created nor destroyed).

In a very general form it states that, for any closed system, having, possibly, an overall linear momentum $\mathbf{P} \neq 0$ but an overall angular momentum $\mathbf{J} = 0$ [1], the change of total energy, $\mathcal{E}$, due to energy transfers across the boundaries, is:

$$\boxed{\Delta\mathcal{E} \equiv \Delta\mathcal{K} + \Delta\Phi + \Delta\mathcal{U} = \mathcal{W} + \mathcal{Q} \equiv \mathcal{W} + (\mathcal{Q}_{\mathtt{cnd}} + \mathcal{Q}_{\mathtt{cnv}} + \mathcal{Q}_{\mathtt{rad}})} \quad , \qquad (65.05.01)$$

→ 2580 2581 2605

where:

- $\mathcal{K}$ is the macroscopic kinetic energy of the system, that is the kinetic energy associated with the motion of the Center-Of-Mass, according to **Koënig theorem** for kinetic energy (section § 24.11.03); for complex continuous systems this must be thought as the macroscopic kinetic energy of any macroscopically small but microscopically large part of the system [a];
- Φ is the potential energy associated to external forces/fields (the macroscopic potential energy); for complex continuous systems this must be thought as the macroscopic potential energy of any macroscopically small but microscopically large part of the system;
- the sum $\mathcal{K} + \Phi$ is normally called the macroscopic energy of any part of the system;
- $\mathcal{U}$ is the internal energy of the system (see section § 65.05.04); it includes the potential energy of all internal forces and normally it cannot be attributed to individual particles, while the kinetic and intrinsic energy can; internal energy is invariant for **Galilei Transformation** ;
- $\mathcal{W}$ is the work done by the surroundings on the system, counted as positive if the work done on the system; it shall include the work of all Force|Torque which is not accounted for as potential energy; whenever any force is conservative one might choose between including it into the potential energy or counting it as work in the right-hand side of the expression; it is assumed that any contribution to the work $\mathcal{W}$ can be always be calculated from macroscopic parameters;
- $\mathcal{Q} \equiv (\mathcal{Q}_{\mathtt{cnd}} + \mathcal{Q}_{\mathtt{cnv}} + \mathcal{Q}_{\mathtt{rad}})$ is the heat exchanged between the system and the surroundings, counted as positive if it is given to the system; heat is a way to transfer energy from/to a system without changing the macroscopic parameters of the system;
- $\mathcal{Q}_{\mathtt{cnd}}$ is the heat exchanged between the system and the surroundings via thermal conduction, counted as positive if it is given to the system; heat is a way to transfer energy from/to a system without changing the macroscopic parameters of the system; $\mathcal{Q}_{\mathtt{cnd}}$ is an exchange of energy with

[1] The case of $\mathbf{J} \neq 0$ is slightly more complex and has some subtleties, see the references.

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physical contact;
- $\mathcal{Q}_{\mathtt{cnv}}$ is the heat exchanged between the system and the surroundings via thermal convection, counted as positive if it is given to the system; heat is a way to transfer energy from/to a system without changing the macroscopic parameters of the system; $\mathcal{Q}_{\mathtt{cnv}}$ is an exchange of energy with physical contact;
- $\mathcal{Q}_{\mathtt{rad}}$ is the energy given by radiation to the system from the surroundings, counted as positive if it is given to the system; $\mathcal{Q}_{\mathtt{rad}}$ refers to exchange of energy without physical contact.

[a] In case $\mathbf{J} \neq 0$ the overall macroscopic kinetic rotational energy must be included in the macroscopic kinetic energy. However, when the volume tends to zero, the kinetic rotational energy goes to zero faster than the translational kinetic energy.

The right-hand terms (work, heat and radiation), are actually energy in transfer: they are the three possible ways any system can exchange energy with the external world.

It is assumed that one always knows how to calculate the work done by the surroundings on the system, by means of parameters external to the system. In case of quasi-static transformations, that is transformations so slow that the system is always in equilibrium, and only in this case, the work can be also calculated via parameters of the system.

The work term, $\mathcal{W}$, must include all and only the interactions with the surroundings which are not included in the external macroscopic potential energy term $\Delta\Phi$: in fact one should make sure both to include all the interactions and avoid double-counting them.

The work of internal forces is usually included in the internal energy, ans internal forces are normally conservative.

It is worth emphasizing once more the following. In classical elementary thermodynamics it is assumed that, given two arbitrary systems, A and B, the work A does on B is equal and opposite to the work that B does on A:

$$\mathcal{W}_{A\to B} + \mathcal{W}_{B\to A} = 0 \qquad \text{basic assumption of classical elementary thermodynamics} \quad . \tag{65.05.02}$$

→ 2577

This results, which is obviously totally false in general, comes from the assumption of classical elementary thermodynamics that the external forces included in $\mathcal{W}$ are contact forces. Therefore the **Action-Reaction Principle** , coupled to the fact that for contact forces the application points of the action-reaction pair move by the same amount gives the desired result. This hypothesis ignores the systems where long-range forces are acting, as for these systems the additivity of energies do not apply (this is the case of gravitational and ElectroMagnetic interactions).

In addition to the assumption that internal energy is additive,

$$\mathcal{U}_{A+B} = \mathcal{U}_A + \mathcal{U}_B \quad ,$$

equation (65.05.02) immediately leads to the conclusion that, if the system $A + B$ is isolated, then

$$\mathcal{Q}_{A\to B} + \mathcal{Q}_{B\to A} = 0 \qquad \textbf{heat conservation} \text{ for isolated systems} \quad .$$

Isolated System means: no work, that is no change in the macroscopic parameters of the system; no heat nor radiation exchange, as enforced by adiabatic walls [1]. Adiabatic walls therefore refers to walls which do not allow passage to heat nor radiation, that is totally radiation absorbing walls.

When, as it is often the case, there is no changes in macroscopic mechanical energies, $\mathcal{K} + \Phi$, one has the simplified form of the first principle of thermodynamics:

$$\Delta\mathcal{U} = \mathcal{W} + \mathcal{Q} \equiv \mathcal{W} + (\mathcal{Q}_{\mathtt{cnd}} + \mathcal{Q}_{\mathtt{cnv}} + \mathcal{Q}_{\mathtt{rad}}) \quad . \tag{65.05.03}$$

→ 2580

It is important to emphasize that, in general, the internal energy, $\mathcal{U}$, of an isolated system is not necessarily conserved; only the total energy is.

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Thermodynamics of Natural Systems
G. M. Anderson
Note that we have not "proved" the first law. It is a principle that has been deduced from the way

[1] see M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...

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things work in our experience, but the fact that it has never been known to fail does not constitute a proof. Neither does the fact that the sun has never failed to rise in the east constitute a proof that it will rise in the east tomorrow, but I wouldn't bet against it.

65.05.02
Isolated Systems and Energy Conservation

The first principle of thermodynamics is a balance law for energy accounting for the change of energy due to heat and work.

For an isolated system, there is no heat and no work, and therefore the balance law becomes a conservation law, the law of energy conservation. As in other branches of physics, because the strength of interactions decrease with distance and therefore any system sufficiently far-away from sources of interactions can be considered as an isolated system.

65.05.03
Local Balance and Conservation of the Total Energy

The principle of conservation of energy is a local conservation law (see section § 21.03), that is a balance equation for a quantity, energy, which is not created nor destroyed. The total energy of a system can change if and only if energy flows across the boundaries as work, heat or radiation. The change of energy in side an arbitrary volume of space, that is, can only be due to energy flow through the boundary of the volume. Two separate regions of space cannot exchange energy except through a flow of energy through a region of space that connects them. This is required by the theory of relativity.

See, for instance § 40.01 for the ElectroMagnetic field.

65.05.04
Internal Energy and Conservation of the Energy

This § is referenced at pages:
[1293, 2576]

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Physical systems we observe in nature (such as gases, liquids, solids and plasmas as well as molecules, atoms, nuclei hadrons and other particles), are typically composed of interacting particles.

Particles are the recognisable units composing the system at any time. Particles might have an internal structure, that is, the particles themselves may be systems and are not always strictly geometrical points (an erroneous idea that often appears in some textbooks). Some particles may be stable, but others may change in time as a result of processes occurring in the system (such as chemical and particle reactions) or related to the internal structure of the particles themselves (such as decays).

The internal kinetic energy of the system is defined as the sum of the kinetic energies of the particles, k_i, measured relative to the Center-Of-Mass of the system, by using **Koënig theorem** . The internal kinetic energy of a system does not depend on the Reference Frame used to measure the velocities of the particles: it is a property of the system, independent of its motion relative to the observer.

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In classical physics the translational part of the internal kinetic energy can be expressed in terms of the translational velocity of the Center-Of-Mass of each particle, $\mathbf{v}_i$, thanks to the classical theorem of equipartition of energy, as:

$$\sum_i \frac{1}{2} m_i v_i^2 = \frac{3}{2} N k_{\mathrm{B}} T \quad .$$

The absolute temperature T in the equation above is called the kinetic temperature.

Note that the theorem of equipartition of energy implies that the total internal kinetic energy is shared among the internal degrees of freedom in such a way that to any quadratic term in the Hamiltonian is associated a mean kinetic energy of $k_{\mathrm{B}}T/2$. That is to say: the theorem of equipartition of energy implies that the system thermalizes among all its internal degrees of freedom. In Quantum Physics thermalization still holds but to a different extent such that the different degrees of freedom do not always have the same energy but share it according to well-defined rules. In fact in Quantum Physics the equipartition of energy holds among the so called **accessible** degrees of freedom.

Internal forces can do work on the particles as a consequence of their relative motion, and therefore one may associate an interaction energy with these internal forces. The internal interaction energy of the system, associated with the internal forces, is defined as the sum of the interaction energies, of the particles, which, in many cases, may be assumed to be expressible as interactions between pairs of particles, $u_{ij}^{\cdot\cdot}$. In many cases the interaction energy can be expressed as a potential energy depending on the relative dynamical coordinates of the particles:

$$\delta \mathcal{W}_{\mathtt{I}} = - \mathrm{d}\Phi_{\mathtt{I}} \quad .$$

In this case one speaks about an internal potential energy. When that is not possible, other descriptions of the interaction might be necessary.

In case three-body or n -body forces are significant the extension is straightforward.

Since the particles in a system may have an internal structure, which can be modified either as a result of inelastic collisions and other interactions among the particles or by external agents, one must take into account the energy associated with that structure. This energy is called the intrinsic particle energy, ε_i.

Therefore, the internal energy $\mathcal{U}$ of a system is defined as the sum of the three energies we have just considered, kinetic, interaction and intrinsic particle energies:

$$\mathcal{U} = \sum_i \mathtt{k}_i + \frac{1}{2} \sum_{a \neq b} \Phi_{ab} + \sum_i \varepsilon_i \quad . \tag{65.05.04}$$

→ 2579 2580

Note that the interaction energy $u_{ij}^{\cdot\cdot}$ corresponds to a coupling between particles, individual particle energies are not well defined and therefore the internal energy of a system is not the sum of individual particle energies. Only the internal kinetic energies and the intrinsic energies are additive. Only the total internal energy of the system is well defined.

The internal structure of particles may sometimes be ignored, as long as it does not affect the overall physical properties and one is not interested in the internal details. Moreover, Quantum Physics ensures that only energy exchanges larger than a minimum quantum jump in energy are required to change the quantum state of a particles/system. When the involved energies are smaller than the minimum possible energy changes of the internal state of the particle/system, energy changes are not actually possible. Of course, some particles are assumed to be really structure-less, such as, as far as we know today, the leptons and quarks, and thus have only the mass-energy $m\gamma c^2$ contributing to their intrinsic energy.

The way one treats the particles of a system depends on the relative values of the three terms in equation (65.05.04) and the type of process involved. That is: models must be considered as flexible tools and must be adapted to each specific situation.

One should note that a system may be composed of several kinds of different particles, each kind with a different internal structure, and the number and kind of particles do not always remain constant [1].

[1] For instance for a gas of photons.

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There may also be more than one kind of interaction, Most often only one interaction is dominant and the others can be neglected to a first approximation: the dominant interaction determines the physics of the system.

Also note that that we consider particles in a system may themselves be systems composed of other particles. For instance molecules can be considered as the particles in certain problems but the must be considered as composite systems, made of atoms, in other problems: the intrinsic energy of the molecules then becomes the internal energy of the molecule, with atoms having kinetic, interaction and intrinsic energies.

Changes in the internal energy of a system, which generally consist in changes of the three terms in equation (65.05.04), must be related to energy exchanges with other systems, such as the environment.

When the system is just one particle, so that there is no interaction energy with external fields (because external fields do not exist), and the notion of heat does not apply, equation (65.05.03) reduces to a simplified form which is the logical generalisation of the standard kinetic energy theorem for the dynamics of a single particle:

$$\boxed{\Delta\mathcal{E} \equiv \Delta\mathcal{K} + \Delta\mathcal{U} = \mathcal{W}} \quad .$$

For an isolated system, that is, a system not subject to any external action/interaction, the three terms on the right-hand side of equation (65.05.01) are zero, so that the total energy of the system remains constant.

One may point out that the real and interesting physics is contained in the interaction energy and in the intrinsic particle energy. These two terms are essential to explain those processes that may occur or what structures may appear in a physical system.

In most of the processes traditionally considered in classical thermodynamics one may ignore the intrinsic energy, because thermodynamics deals only with relatively low energy processes during which the intrinsic energy stays, most often, constant.

65.05.05
From Mechanics to ThermoDynamics

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The work-energy theorem for a general system in mechanics states that the change in total kinetic energy of any system is equal to the total work done on the system, by both external and internal interactions:

$$\boxed{\Delta\mathcal{K}_{\mathtt{tot}} = \mathcal{W}' \equiv \mathcal{W}'_{\mathtt{ext}} + \mathcal{W}'_{\mathtt{int}}} \quad .$$

When some of the interactions are conservative one can account for them either as work or as potential energy, obtaining the balance (not conservation, in the general case) of mechanical energy:

$$\Delta E_{\mathtt{mech}} \equiv \Delta\mathcal{K}_{\mathtt{tot}} + \Delta\Phi_{\mathtt{int}} + \Delta\Phi_{\mathtt{ext}} = \mathcal{W} \quad ,$$

describing some (or all) conservative interactions as potential energy, and letting the other conservative interactions non described as potential energy as $\mathcal{W}$. Obviously non conservative interactions cannot be described as potential energy and are necessarily described by the work $\mathcal{W}$ term.

The mechanical energy $E_{\mathtt{mech}}$ is made of the sum of the total kinetic energy and the total potential energy, both of internal and external interactions.

In the most general case mechanical energy is not a conserved quantity, unless non conservative interactions are absent, as it is well known from mechanics.

It is However, possible to extend the principle of energy conservation in such a way to obtain a conserved quantity for isolated systems: the total energy.

Consider an isolated system, that is a system which cannot exchange energy nor matter with its surroundings. The general case of an open system can be deduced later from this case by the usual procedure to consider a larger global system made of the actual non isolated system plus the matter and energy exchanged so that the larger system is an isolated one.

The total kinetic energy can be separated in two contributions according to **Koënig theorem** .

PRELIMINARY - IN PROGRESS

For any isolated system it is possible to define a state function $\mathcal{U}$, the internal energy, a function of the extensive variables, such that the total energy is conserved, that is the term of energy creation is zero:

$$\Delta\mathcal{E} = 0 \quad \Leftrightarrow \quad \mathcal{E} = \mathcal{K} + \Phi + \mathcal{U} \quad \text{with } \Phi \equiv \Phi_{\mathtt{ext}} \quad \text{and } \mathcal{K}_{\mathtt{tot}} \equiv \mathcal{K} + \mathcal{K}_{\mathtt{int}} \quad . \tag{65.05.05}$$

→ 2581

For a general system, not isolated, one has the first principle of thermodynamics, equation (65.05.01).

Note that the extensivity character is a working hypothesis, satisfied in many cases, but not necessarily in all cases (for instance in systems with long-range interactions).

It follows that any change of the total energy of a system must come from exchanges of energy with the surroundings (energy balance).

The potential energy is separated into its internal and external parts according to whether the interaction is either an internal or an external interaction.

The internal energy is made of the sum of the microscopic kinetic energy (that is the kinetic energy with respect to the Center-Of-Mass of the system, according to **Koënig theorem**) plus the potential energy due to all internal interactions plus any other possible intrinsic contribution (intrinsic particle energy) such as the rest energy of the particles.

The separation between macroscopic and microscopic kinetic energy is easily made by means of the **Koënig theorem** . Consider, in fact, for any macroscopically small but microscopically large piece of the system its Center-Of-Mass and decompose the kinetic energy accordingly. Note that as the internal energy is defined in terms of the internal kinetic energy it does not depend on the Reference Frame.

The internal energy is basically defined by equation (65.05.05), as the total energy minus the macroscopic kinetic energy and minus the potential energy due to external fields.

The intrinsic particle energy depends on the internal structure of the particles.

65.05.06

Summary - the First Principle of ThermoDynamics for Closed Systems

Il primo principio della termodinamica per un generico sistema si scrive

$$\boxed{\Delta\mathcal{K} + \Delta\Phi + \Delta\mathcal{U} = \Delta E^{\mathtt{macro}} + \Delta\mathcal{U} = \Delta\mathcal{E} = \mathcal{Q} + \mathcal{W}} \tag{65.05.06}$$

→ 2603 2634

dove $\mathcal{K}$ indica l'energia cinetica macroscopica del sistema, Φ l'energia potenziale associata a campi di forza macroscopici esterni al sistema, $E^{\mathtt{macro}}$ l'energia totale macroscopica del sistema, $\mathcal{E}$ l'energia totale del sistema, $\mathcal{U}$ l'energia interna del sistema, $\mathcal{Q}$ il calore assorbito dal sistema (contatto, convezione o irraggiamnto), $\mathcal{W}$ il lavoro fatto sul sistema dalle forze esterne che non sono descritte dal termine di energia potenziale Φ.

Nel caso l'energia cinetica e potenziale macroscopica non cambino il primo principio si riduce alla forma semplificata

$$\boxed{\Delta\mathcal{U} = \mathcal{W} + \mathcal{Q} \equiv \mathcal{W} + (\mathcal{Q}_{\mathtt{cnd}} + \mathcal{Q}_{\mathtt{cnv}} + \mathcal{Q}_{\mathtt{rad}})} \quad . \tag{65.05.07}$$

Si assume che il lavoro fatto su un sistema possa essere sempre espresso in termini dei parametri esterni al sistema. Solo in caso di trasformazioni quasi statiche il lavoro può anche essere espresso in termini delle variabili di stato del sistema. In tal caso infatti il sistema è sempre in equilibrio termodinamico e quindi le variabili di stato del sistema sono ben definite.

65.05.07

First Principle of ThermoDynamics for Open Steady Systems

L'equazione del bilancio energetico per un sistema aperto in moto stazionario è

$$\boxed{\Delta\mathtt{k} + \Delta\phi + \Delta\mathtt{h} = \mathtt{w}' + \mathtt{q}} \tag{65.05.08}$$

→ 2606

dove $\mathtt{h}$ è l'entalpia per unità di massa, $\mathtt{k}$ l'energia cinetica macroscopica per unità di massa, ϕ l'energia potenziale macroscopica per unità di massa, $\mathtt{q}$ il calore ricevuto per unità di massa e $\mathtt{w}'$ il lavoro utile effettivo per unità di massa fatto sul sistema, ad esclusione cioè del lavoro fatto dalle forze di pressione del fluido, che è quello che realmente interessa in problemi di ingegneria.

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65.05.08
First Principle of ThermoDynamics for Open Generic Systems

This § is referenced at pages:
[1327]

In general, energy can change due to transfer of matter across the boundary of the system. In this case:

$$\boxed{\mathrm{d}\mathcal{K} + \mathrm{d}\Phi + \mathrm{d}\mathcal{U} = \delta\mathcal{Q} + \delta\mathcal{W} + \mu\,\mathrm{d}N} \quad , \tag{65.05.09}$$

in terms of the change in number of particles, N, and the chemical potential.

65.05.09
First Principle in Terms of Power for Generic Open Systems

$$\boxed{\dot{\mathcal{K}} + \dot{\Phi} + \dot{\mathcal{U}} = \dot{\mathcal{Q}} + \dot{\mathcal{W}} + \mu\dot{N}} \quad . \tag{65.05.10}$$

65.05.10
Dissipative Phenomena

Dissipative phenomena convert some form of energy into some other form of energy such that the potential of the final form to do mechanical work is less than that of the initial form.

One such example is when macroscopic energy is transformed into internal energy.

Another example is heat transfer which is dissipative because it is a transfer of internal energy from a hotter body to a colder one.

Dissipative phenomena include: mechanical solid friction, viscosity, turbulence, heat transport, electric resistance, free expansion, particle diffusion, hysteresis, chemical reactions....

DRAFT - IN-PROGRESS

65.06 Internal Energy for Some Thermodynamic Systems

The equation providing the internal energy for a particular system is often called the **caloric equation of state**.

Some of the results of this section require the use of the equations of internal energy, read § 67.04.01

65.06.01 Incompressible Solid/Liquid

The equation of state implies:

$$V = V_0 \qquad \left(\frac{\partial V}{\partial p}\right)_T = 0 \qquad \left(\frac{\partial V}{\partial T}\right)_p = 0 \qquad \left(\frac{\partial p}{\partial T}\right)_V = \text{undetermined} \quad .$$

The internal energy can be considered as a function of any pair of the three variables pVT, for instance V and T. But, as $V = V_0$ it is only a function of temperature.

Alternatively and more formally, equation (67.04.01) implies:

$$\left(\frac{\partial \mathcal{U}}{\partial p}\right)_T = 0$$

Therefore, l'energia interna dipende solo dalla temperatura (as volume does not change):

$$\mathrm{d}\mathcal{U} = C_{\mathrm{V}}[T]\,\mathrm{d}T \quad .$$

In fact, equation (67.04.01) implies that the internal energy does not depend on the pressure whenever the volume is constant.

65.06.02 Elastic Solid/Liquid

La capacità termica a volume costante e quella a pressione costante di un liquido o solido descritti dall'equazione di stato (64.07.09) non dipendono, rispettivamente, dal volume e dalla pressione, come si deduce dalla relazione (67.03.05), (67.03.05). Equation (67.03.02) implies

$$C_{\mathrm{p}} - C_{\mathrm{V}} = \frac{\beta^2 V_0 T}{\kappa_{\mathrm{T}}} \quad ,$$

and therefore, as C_{V} does not depend on V and $C_{\mathrm{p}} - C_{\mathrm{V}}$ only depends on T, and as C_{p} does not depend on p and $C_{\mathrm{p}} - C_{\mathrm{V}}$ only depends on T, both C_{p} and C_{V} only depend on temperature.

Si ha:

$$\left(\frac{\partial \mathcal{U}}{\partial V}\right)_T = \left(\frac{V - V_0}{\kappa_{\mathrm{T}} V_0} - p_0 + \frac{\beta T_0}{\kappa_{\mathrm{T}}}\right) \qquad \left(\frac{\partial \mathcal{U}}{\partial p}\right)_T = V_0\,(p\kappa_{\mathrm{T}} - T\beta) \quad .$$

L'energia interna di un liquido o solido descritti dall'equazione di stato (64.07.09) vale dunque

$$\mathcal{U}[T, V] = \left(\frac{V - V_0}{2\kappa_{\mathrm{T}} V_0} - p_0 + \frac{\beta}{\kappa_{\mathrm{T}}} T_0\right)(V - V_0) + \int_{T_0}^{T} C_{\mathrm{V}}[T]\,\mathrm{d}T \quad .$$

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65.06.03
Ideal Gas

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Experimentally, it is found, via experiments of free expansion inside a rigid and adiabatic container, $\mathcal{W} = \mathcal{Q} = 0$, that all gases at sufficiently low densities, do not change their temperature.

Considering $\mathcal{U} = \mathcal{U}[T, V]$, if no temperature change takes place in a free expansion, $\Delta\mathcal{U} = 0$ then

$$\left(\frac{\partial \mathcal{U}}{\partial V}\right)_T = 0 \quad ,$$

that is, $\mathcal{U}$ does not depend on V.

Considering $\mathcal{U} = \mathcal{U}[T, p]$, if no temperature change takes place in a free expansion, $\Delta\mathcal{U} = 0$ then

$$\left(\frac{\partial \mathcal{U}}{\partial p}\right)_T = 0 \quad ,$$

that is, $\mathcal{U}$ does not depend on p.

Therefore, $\mathcal{U}$ is a function of temperature only.

An ideal (molecular) gas is defined as a system following the equation of state of ideal gases and whose internal energy only depends on temperature:

$$\boxed{pV = nRT \qquad \mathcal{U} = \mathcal{U}[T]} \quad .$$

Note that, given equation (64.08.02), and the equation of state of a ideal gas, the equality to zero of the partial derivatives of the internal energy with respect to volume and pressure, at constant temperature, are equivalent:

$$\left(\frac{\partial \mathcal{U}}{\partial V}\right)_T = 0 \qquad \Leftrightarrow \qquad \left(\frac{\partial \mathcal{U}}{\partial p}\right)_T = 0 \qquad \text{for an ideal molecular gas} \quad .$$

Note that, the specific heats at constant volume/pressure can depend on temperature, in general.

65.06.04
ElectroMagnetic Field

This § is referenced at pages:
[2588, 2595]

Poynting theorem is the first principle (energy balance) for the ElectroMagnetic field.

Poynting theorem is the energy balance applied to the system made of the EM fields (the system). When comparing with the first principle of thermodynamics § 65.05 one identifies the expression (40.01.02) of the energy density $u[\mathbf{x}]$ with the total energy density of the ElectroMagnetic fields, the term $\mathbf{j}\cdot\mathbf{E}$ as the work per unit volume done by the ElectroMagnetic fields on matter and the flux of Poynting vector as a radiation term, for exchanging ElectroMagnetic energy across the boundaries of the system.

Note that when writing the first principle of thermodynamics to some piece of matter (now the system) the term $\mathbf{j}\cdot\mathbf{E}$ becomes the work per unit volume done by the ElectroMagnetic fields on the system.

Heat is radiation, here, given by the Poynting vector.

In fact, Heat only exists in a macroscopic approach.

DRAFT - PRE-PRESS

65.07 Enthalpy

©|D.V.Schroeder, *Thermal Physics*, 1999, Addison-Wesley, ...Ed., ...|Introduction to thermal physics|*§ 1*|

By definition:

$$\boxed{\mathcal{H} \equiv \mathcal{U} + pV} \quad .$$

Enthalpy, therefore, can increase for two reasons: either because the energy increases or because the system expands doing work on the external pressure to make room for it.

During constant pressure transformations, any change of the enthalpy of a system is caused only by heat and, possibly, other forms of work except compression-expansion work. That is, compression-expansion work can be ignored, because it is automatically accounted for by enthalpy. In fact, the symbols $\mathcal{H}$ is used, because, whenever no other types of work are being done, except compression-expansion work the change in enthalpy provides the heat added to the system.

Clearly, enthalpy is very important for chemical reactions, which very often happen at constant external pressure.

DRAFT – IN-PROGRESS

65.08 Capacità Termica Calore Specifico/Molare E Calore Latente

This § is referenced at pages:
[2667]

La definizione generale di capacità termica di un sistema lungo una trasformazione quasi-statica, arbitraria ma completamente definita, Γ è definita da

$$C_C \equiv \left(\frac{\delta \mathcal{Q}}{\delta T}\right)_C \quad .$$

Occorre specificare la trasformazione in quanto il calore non è una funzione di stato e il calore scambiato dipende quindi dalla trasformazione.

Per un sistema pVT si utilizzano frequentemente le capacità termiche a volume e pressione costante

$$C_{\mathtt{V}} \equiv \left(\frac{\delta \mathcal{Q}}{\delta T}\right)_V \quad , \quad ,$$

$$C_{\mathtt{p}} \equiv \left(\frac{\delta \mathcal{Q}}{\delta T}\right)_p \quad .$$

Il calore specifico è la capacità termica per unità di massa.

Il calore molare è la capacità termica per unità di mole.

$$c_C^{(m)} \equiv \frac{1}{m}\left(\frac{\delta \mathcal{Q}}{\delta T}\right)_C \quad , c_C^{(n)} \equiv \frac{1}{n}\left(\frac{\delta \mathcal{Q}}{\delta T}\right)_C \quad .$$

65.08.01 Capacità Termica a Volume Costante

This § is referenced at pages:
[Never referenced.]

Si consideri un sistema pVT chiuso a composizione costante.

Data una qualunque trasformazione a volume invariato, isocora, cioè una qualunque trasformazione tra due stati di equilibrio termodinamico, tale che il volume del sistema, è uguale all'inizio e alla fine della trasformazione, il lavoro fatto sul sistema è nullo. Allora si ha:

$$\Delta \mathcal{U} = \mathcal{Q} = \text{funzione di stato} \quad \Leftrightarrow \quad V = \text{constant} \quad .$$

Si noti che il volume deve restare invariato, isocora e non monocora, e non è in generale sufficiente che il volume iniziale sia uguale a quello finale per garantire che il lavoro totale fatto sul sistema sia nullo, ad esempio nel caso siano presenti forze di attrito.

Nel caso il lavoro sia nullo comunque il calore risulta essere una funzione di stato.

$C_{\mathtt{V}}$ might be called **energy capacity**. Moreover, there doesn't have to be any heat involved at all, since the energy could just as well enter as other work.

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65.08.02
Capacità Termica a Pressione Costante

This § is referenced at pages:
[2684]

Si consideri un sistema pVT chiuso a composizione costante.

Data una qualunque trasformazione tra due stati di equilibrio termodinamico, tale che la pressione del sistema, p, è uguale all'inizio e alla fine della trasformazione, e quindi ugulae alla presisone esterna, si ha:

$$\Delta\mathcal{H} = \Delta\mathcal{U} + \Delta(pV) = \Delta\mathcal{U} + p\,(V_{\mathtt{F}} - V_{\mathtt{I}}) = \Delta\mathcal{U} + p^{\mathtt{ext}}\,(V_{\mathtt{F}} - V_{\mathtt{I}})$$

Negli stati iniziale e finale, essendo in equilibrio termodinamico, la pressione esterna coincide con quella del sistema: $p = p^{\mathtt{ext}}$.

Se la pressione esterna, che per ipotesi è la stessa all'inizio e alla fine della trasformazione, è costante durante la trasformazione, cioè la trasformazione è isobara non monobara con riferimento alla pressione esterna, in modo che il lavoro è:

$$\mathcal{W} = -p^{\mathtt{ext}}\,(V_{\mathtt{F}} - V_{\mathtt{I}}) \quad ,$$

allora si ha

$$\boxed{\Delta\mathcal{H} = \mathcal{Q} = \text{funzione di stato} \quad \Leftrightarrow \quad p^{\mathtt{ext}} = \text{constant}} \quad .$$

$C_{\mathtt{p}}$ might be called **enthalpy capacity**. Moreover, there doesn't have to be any heat involved at all, since the enthalpy could just as well enter as other work.

65.08.03
Latent Heats

In certain situations it is possible to put heat into a system without increasing its temperature. This happens at a phase transformation. Technically, the **Heat-Capacity** is infinite in these conditions.

It is an experimental fact that phase transitions happen at constant pressure and temperature.

The latent heat of transformation is the amount of heat necessary to induce, reversibley, the phase transformation of a certain amount of matter. However, this simple definition is ambiguous, since any amount of work could also be done during the process. By convention, it is assumed that pressure and temperature are constant, and that no other work is done. Therefore lantent heats are entalpies.

The specific/molar latent heat of transformation is a characteristic of the substance:

$$\boxed{\mathcal{Q} \equiv m\lambda^{(spe)} = n\lambda^{(mol)} \implies \lambda^{(mol)} = \mathcal{M}_0\lambda^{(spe)}} \quad .$$

Whether the latent heat of transformation is either specific or molar is clear from the units of measure.

65.08.04
Heat Reservoirs

Heat reservoirs (aka: heat baths, heat sources) are bodies which such a large mass that that can exchange any amount of heat without changing their temperature. That is, ideally:

$$\boxed{C \to +\infty} \quad .$$

Clearly, it is an abstraction, well approximated by bodies with large thermal capacity.

On practical grounds a mixture of two (or more) phases of a pure substance, represent a system, such that its temperature does not change, until more than one phase is present.

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65.09 Energy in Transit as Heat

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©|F.Reif, *Fundamentals Of Statistical And Thermal Physics*, 2009, Waveland Press, ...Ed., ...|||

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|I 43, II 2.6, 3.4, 12.2||

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|II 12.4||

65.09.01 EM Thermal Radiation

This § is referenced at pages:
[2549]

©|M.W.Zemansky & M.M.Abbot & H.C.VanNess, *Fondamenti Di Termodinamica per Ingegneri*, ..., ..., ...Ed., ...|11.31||

Read § 65.06.04.

Thermal (heat) radiation, is the ElectroMagnetic radiation emitted by any body of temperature $T > 0$ K.

Whenever a surface is mentioned, the surface can be either real or imaginary (that is, a mathematical surface, not a real interface between different things).

65.09.01.01 Kinematics of Particle/Radiation Transport

This § is referenced at pages:
[1033, 1213, 1213]

Consider the transport of a generic physical quantity, Q, such as mass, energy, linear momentum, particles, electric charge...

Two cases can be envisaged:

- sources: the transported quantity is emitted by some source and we are describing the properties of the matter;
- receiver: the transported quantity is directed onto some detector of any kind, describing the properties of the incoming flux.

One-Dimensional Flow

All particles have velocity along a fixed direction in space: $\mathbf{v} = v_n \widehat{\mathbf{n}}$.

To quantify the flow of any quantity it is useful to introduce the flux, defined as the amount of quantity crossing a fixed surface, perpendicular to the velocity, per unit surface perpendicular area per unit time:

$$\frac{d^2 Q}{\mathrm{d}A_\perp \, \mathrm{d}t} \equiv \frac{\mathrm{d}Q}{\mathrm{d}V} v_n \equiv \mathcal{I}[\mathbf{x}] \quad .$$

The quantity $\mathcal{I}[\mathbf{x}]$ is often called irradiance.

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Omni-Dimensional General Flow

Particles have velocities along any direction in space.

In this case, to quantify the flow, one needs also to take into account the direction. The differential flux in solid angle is defined as the amount of quantity crossing a fixed surface, perpendicular to the velocity, per unit surface perpendicular area per unit time per unit solid angle around the normal to the surface:

$$\boxed{\frac{d^3Q}{\mathrm{d}A_\perp\ \mathrm{d}t\ \mathrm{d}\Omega} \equiv \frac{1}{\cos[\theta]}\frac{d^3Q}{\mathrm{d}A\ \mathrm{d}t\ \mathrm{d}\Omega} \equiv \mathcal{R}[\mathbf{x},\theta,\phi]} \quad .$$

The quantity $\mathcal{R}[\mathbf{x},\theta,\phi]$ is often called radiance.

The flux across a fixed surface, regardless of the angle inside the solid angle Ω_0, is:

$$\boxed{\frac{d^2Q}{\mathrm{d}A\ \mathrm{d}t}[\mathbf{x}] = \mathcal{I}[\mathbf{x}] = \iint_{\Omega_0} \mathcal{R}[\mathbf{x},\theta,\phi]\cos[\theta]\ \mathrm{d}\Omega} \quad .$$

In case the $\mathcal{R}$ of a source is independent from the angles, the source is called a **Lambertian source** (read § 46-028).

In the important case of fixed $\mathcal{R}$, that is homogeneous and isotropic flow:

$$\frac{d^2Q}{\mathrm{d}A\ \mathrm{d}t} = \pi\mathcal{R} \quad .$$

Radiance Differential in Energy

It is often useful, for instance in the case of ElectroMagnetic energy transport, to introduce the spectral quantities, in terms of λ and|or ν:

$$\boxed{\mathcal{R}_\lambda \equiv \frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\lambda}[\mathbf{x},\theta,\phi,\lambda] \equiv \frac{d^4Q}{\mathrm{d}A_\perp\ \mathrm{d}t\ \mathrm{d}\Omega\ \mathrm{d}\lambda} \qquad \mathcal{R}_\nu \equiv \frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\nu}[\mathbf{x},\theta,\phi,\nu] \equiv \frac{d^4Q}{\mathrm{d}A_\perp\ \mathrm{d}t\ \mathrm{d}\Omega\ \mathrm{d}\nu}} \quad .$$

65.09.01.02 Deposition of Radiation

When ElectroMagnetic radiation impinges upon the surface of a substance, three processes may occur:

- Absorption: the ElectroMagnetic power is deposited into the substance and converted to another type of energy, typically internal energy.
- Transmission: the ElectroMagnetic radiation passes through the substance unhindered.
- Reflection: the radiation is reflected at the surface of the substance.

In general, the three processes do not occur separately but simultaneously:

$$\boxed{\alpha+\tau+\varrho=1} \qquad \boxed{\text{absorbance} \quad + \quad \text{transmittance} \quad + \quad \text{reflectance} \quad = \quad 1} \quad .$$

65.09.01.03 Radiometry - Transport of ElectroMagnetic Energy

Is the science of measurement of radiant energy (including light) in terms of absolute power.

Classical radiometric nomenclature is as follows [1].

- Radiant energy, measured in joules.
- Radiant power, measured in watts.
- Radiant flux: Energy per unit Time per unit normal Area (also called Intensity); it is the flux of energy. The flux can leave/reach the surface in any direction.
  There are two possible cases:
  - the flux can be arriving at the surface, in which case the radiant flux is referred to as irradiance, $\mathcal{I}$;
  - the flux can be leaving the surface, due to emission and|or reflection, in which case the radiant flux is referred to as radiant exitance, $\mathcal{E}$.

  The importance of **either real or imaginary** surface cannot be overstated: the radiant flux can be measured anywhere including on the surface of physical objects, in the space between them and

[1] Beware that radiometric nomenclature is quite confusing and varying; here we try to avoid names which are misleading with respect to their use in other branches of physics.

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inside transparent media.
Radiant flux is measured in watts per square meter.

- Spectral radiant flux: Energy per unit Time per unit normal Area per unit frequency/wavelength; it is the radiant flux differential in frequency/wavelength.
  There are two possible cases:
  - the flux can be arriving at the surface, in which case the spectral radiant flux is referred to as spectral irradiance;
  - the flux can be leaving the surface, due to emission and|or reflection, in which case the spectral radiant flux is referred to as spectral radiant exitance.
- Radiant flux differential in solid angle: Energy per unit Time per unit normal Area per unit solid angle; it is the irradiance differential in solid angle.
  Unlike the radiant flux, the definition of radiance does not distinguish between flux arriving at or leaving a surface. In fact, the formal definition of radiance (ANSI/IES 1986) states that it can be **leaving, passing through or arriving at the surface**.
  Radiance is measured in watts per square meter per steradian.
- Spectral Radiance: radiance differential in frequency/wavelength.

65.09.01.04 Photometry - ElectroMagnetic Radiation as Perceived by the Human Eye

Photometry is the science of the measurement of light, in terms of its perceived brightness to the human eye.

To pass from radiometric to photometric units: replace the prefix **rad-** with the **lum-** prefix.

65.09.01.05 Thermal Radiation

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- Consider an enclosure which has an arbitrary shape and which may contain several bodies within it. Its walls, which may consist of any material, are maintained at a fixed absolute temperature. The enclosure thus acts as a heat reservoir.
- All the enclosure is assumed to be at thermodynamic equilibrium.
- Let $f_\alpha[\mathbf{x}, \mathbf{k}]$ be the mean number of photons per unit volume at the point $\mathbf{x}$, with wave-vector $\mathbf{k}$ and with **wave-Polarization** specified by the index α.
- It turns out that $f_\alpha[\mathbf{x}, \mathbf{k}]$ is independent of $\mathbf{x}$: the radiation field is homogeneous. Consider, in fact, what would happen if two identical small bodies at temperature T were placed at two different positions inside the cavity. Imagine these to be surrounded by filters which transmit only frequencies in a specified range and which transmit only radiation of a specified **wave-Polarization** . Then, if the values of $f_\alpha[\mathbf{x}, \mathbf{k}]$ were different at the different points, the temperature would increase, at one place, and would decrease at the other place, contrary to thermodynamics equilibrium.
- It turns out that $f_\alpha[\mathbf{k}]$ is independent of the direction of $\mathbf{k}$, but depends only on $|\mathbf{k}|$: the radiation field is isotropic. Suppose, in fact, that $f_\alpha[\mathbf{k}]$ did depend on the direction $\mathbf{k}$, for instance that is greater if $\mathbf{k}$ points north than if it points east. We could again imagine that two identical small bodies at temperature T (and surrounded by the same filters as before) are introduced into the enclosure. Then the body on the north side would have more radiation incident on it and thus absorb more power than the body on the east side. This would again lead to a non permissible temperature difference being produced between these bodies.
- It turns out that $f_\alpha[|\mathbf{k}|]$ is independent of the **wave-Polarization** of the radiation, that is: the radiation field is non wave-Polarized. Suppose, in fact, that $f_\alpha[|\mathbf{k}|]$ did depend on the **wave-Polarization** . Then we could imagine that two identical small bodies at the temperature T are introduced side by side into the enclosure and are surrounded by filters which transmit different directions of **wave-Polarization** . Hence different amounts of radiation would be incident upon these bodies, and a temperature difference would be developed between them in contradiction to the equilibrium condition.
- It turns out that $f[|\mathbf{k}|]$ does not depend on the shape nor volume of the enclosure, nor on the material of which it is made, nor on the bodies it may contain. Consider, in fact, two different enclosures, both at the temperature T, and suppose that the two functions $f[|\mathbf{k}|]$ describing their radiation fields were different. Imagine that we connect the two enclosures through a small hole (containing a filter which transmits only radiation in a narrow frequency range about and of the specified **wave-Polarization**). This would represent an equilibrium situation if both enclosures are at the same temperature. If the two functions $f[|\mathbf{k}|]$ are different more radiation per unit time would pass from one enclosure to the other than in the opposite direction. A temperature difference would then develop between the two enclosures, in contradiction to the equilibrium condition of uniform temperature.
- At thermodynamic equilibrium, consider a closed cavity with a small body inside. The body emits radiation and radiation is continually incident upon the body which absorbs a certain fraction of it. To keep equilibrium, the absorbed power must be equal to the emitted power: Kirchhoff law.
- Imagine that the body is surrounded by a filter which absorbs completely all radiation except that, in one small element of area, it is completely transparent to radiation of one specific **wave-Polarization** and of one narrow frequency range. The presence of this shield cannot affect such intrinsic parameters of the body as its emissivity or absorptivity. Nor can it, by the arguments of the preceding section, affect the nature of the radiation in the enclosure. Since the equilibrium situation can equally well exist in the presence of the shield, it follows that the energy balance must hold for this particular element of area, **wave-Polarization** , and frequency range.

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Thermal radiation is important in thermometry, since it is the basis of precisely measured high temperatures above the range of gas thermometers. As an example of the effects of thermal radiation, consider placing your hand near a hot object and experiencing the warming that occurs before the object is touched. Evidently, heat is transmitted across the space between the object and the hand. That this can occur without the intervention of matter is proved by the fact that heat comes to us from the Sun across 150 million km of empty space. By conduction and convection, the transfer of heat is accomplished through the medium of matter, and heat can be transferred in this way only as far as matter extends or can itself be transported. By the process of radiation, however, heat is separated from its association with matter and can travel as radiation as far as empty space extends.

A substance may be stimulated to emit ElectroMagnetic radiation in a number of ways, including:

1. An electric conductor carrying a high-frequency alternating current emits radio waves.
2. Electrons oscillating in a magnetron-type tube emit microwaves.
3. A hot solid or liquid emits thermal radiation, that is, infrared radiation.
4. A gas experiencing an electric discharge may emit visible or ultraviolet radiation.
5. A substance exposed to ultraviolet radiation from an external source may emit fluorescent light.
6. A metal target bombarded by high-speed electrons emits x-rays.
7. A substance whose atoms are radioactive may emit gamma rays.

All these radiations are ElectroMagnetic waves, differing only in frequency (or wavelength) in vacuum.

We shall be concerned in this section only with thermal radiation, namely, the radiation emitted by a solid or a liquid by virtue of its temperature. In these cases, the close-packing of the atoms generate a continuous ElectroMagnetic spectrum.

The radiation characteristics of gases require special treatment. In fact, for gases, the spectrum is typically the emission/absorption discrete spectrum of the gas molecules. In this section, we shall assume that gases are transparent to thermal radiation.

When thermal radiation is dispersed by a prism or Interference|Diffraction Grating, one obtains a continuous spectrum of the visible spectrum and the invisible infrared radiation. The distribution of energy among the various wavelengths is such that, at temperatures below about $500°C$, the energy is infrared radiation; at higher temperatures, increasingly more visible light is emitted. In general, the higher the temperature of a body, the greater the total energy emitted. The loss of energy due to the emission of thermal radiation may be compensated in a variety of ways. The emitting body may be a source of energy itself, such as the sun; or, there may be a constant supply of electric energy from the surroundings, as in the case of a lamp. Energy may be supplied also by heat conduction or by the performance of work on the emitting body. In the absence of these sources of supply, the only other way in which a body may receive energy is by the absorption of radiation from its surroundings. In the case of a body that is immersed in radiation, the internal energy of the body will remain constant when the rate at which radiant energy is emitted equals the rate at which radiant energy is absorbed; in other words, the exiting radiant power equals the incident radiant power.

Experiment shows that the radiant power leaving a body as thermal radiation depends on the temperature, on the area of the surface, and on the nature of the surface of the body. The total radiant power exiting an infinitesimal element of surface, divided by the area of that surface, is called the radiant exitance, $\mathcal{E}$, of the body. For example, the radiant exitance of tungsten at $1000\ \mathrm{K}$ is $6.46\ \mathrm{kW/m^2}$, at $2000\ \mathrm{K}$ is $236\ \mathrm{kW/m^2}$, and at $3000\ \mathrm{K}$ is $1534\ \mathrm{kW/m^2}$.

In general, some of the radiation may be absorbed, some reflected, and some transmitted.

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In general, the incident isotropic radiation of all wavelengths that is absorbed depends on the temperature and the nature of the surface of the absorbing body.
The fraction of the total incident radiant power that is absorbed is called the absorptivity.
In thermal equilibrium, the processes of absorption and emission of radiant power are equal and opposite. So, the emissivity, equal to the absorptivity, is defined as the fraction of the power provided to a real body that is emitted through a material surface as thermal radiation, where the word total includes all wavelengths of ElectroMagnetic radiation. As a practical matter, it is easier to measure emissivity than absorptivity.
To summarize:

- radiant exitance $\mathcal{E}$: total radiant power emitted per unit area;
- emissivity ε: fraction of the total radiant power that is emitted as thermal radiation, equal to the absorptivity.

The emissivity depends on both the temperature and the nature of the emitting surface. The emissive nature of surfaces is revealed by comparing emissivities at the same temperature. At 300 K, when all bodies emit only infrared radiation, the emissivity of polished steel is 0.09, rough oxidized steel is 0.81, and ocean water is 0.96. There are some substances, such as lampblack or carbon soot, whose emissivity is very nearly unity, that is, almost an ideal emitter.
For theoretical purposes, it is useful to conceive of an ideal substance capable either of absorbing all the thermal radiation falling on it or of emitting all the energy provided to it in the form of thermal radiation. Such a substance is called a blackbody. If a blackbody is indicated by the subscript bb, we have, for the emissivity:

$$\varepsilon_{bb} = 1 \quad .$$

A very good experimental approximation to a blackbody is provided by a cavity enclosed by high-temperature opaque walls. The interior walls, which are maintained at a uniform temperature, permit thermal radiation to pass through a hole small in comparison with the dimensions of the cavity. Any radiation entering the hole is completely absorbed by the walls after repeated reflections at the walls, with only a negligible amount eventually finding its way out the hole. This is true regardless of the composition of the materials of the interior walls. The radiation emitted by the interior walls is similarly absorbed or diffusely reflected a large number of times, so that the cavity is filled with isotropic blackbody radiation.
Let us define the irradiance, $\mathcal{I}$, as the radiant power per unit area incident upon a surface within the cavity.
Suppose a blackbody whose temperature is the same as that of the walls is introduced into the cavity. Since the temperature of the blackbody remains constant, the radiant power per unit area that is absorbed must equal the radiant power per unit area that is leaving; whence the irradiance within a cavity whose walls are at the temperature T is equal to the radiant exitance of a blackbody at the same temperature. For this reason, the radiation enclosed within a cavity is called blackbody radiation.
Such radiation, which provides standard radiation in terms of wavelengths and intensities, is a function only of temperature and is studied by allowing a small amount to escape from a small hole in the cavity. Since $\mathcal{I}$ is independent of the materials of which the interior walls are composed, it follows that the radiant exitance of a blackbody is a function of the temperature only.
The radiant exitance of a non-blackbody depends as much on the nature of the surface as on the temperature, according to a simple law that we may derive as follows.
Suppose that a non-blackbody at the temperature T, with radiant exitance $\mathcal{E}$, and emissivity ε, is introduced into a cavity whose interior walls are at the same temperature and where the irradiance is $\mathcal{I}$:

$$\mathcal{E} = \varepsilon \mathcal{E}_{bb} \quad .$$

The radiant exitance of any body at any temperature is equal to a fraction of the radiant exitance of a blackbody at that temperature, this fraction being the emissivity at that temperature. This

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equation, known as Kirchhoff law and named after the German physicist, shows that the emissivity of a body may be determined experimentally by measuring the radiant exitance of the body and dividing it by the radiant exitance of a blackbody at the same temperature.
It should be emphasized that the tabulated values of emissivity refer to the thermal radiation appropriate to the temperature listed in the temperature- range column. Thus, the emissivity of ice is 0.97 not for visible radiation, but for the long infrared waves associated with matter at $0°C$ $(273\ K)$. It should be noticed that the word "heat" has not appeared as yet. If there is a temperature difference between a body and its surroundings, then, in a given interval of time, the body loses an amount of internal energy equal to the energy radiated minus the energy absorbed, whereas the surroundings gain an amount of internal energy equal to the energy absorbed minus the energy radiated. The gain of the surroundings equals the loss of the body. The gain or loss of internal energy of the body, equal to the difference between the energy of the thermal radiation which is absorbed and that which is radiated, is called heat. This statement is in agreement with the original definition of heat, since a gain or loss of internal energy by radiation and absorption will take place only if there is a difference in temperature between a body and its surroundings. If the two temperatures are the same, there is no net gain or loss of internal energy of either the body or its surroundings, and there is, therefore, no transfer of heat.
The rate at which heat is transferred by radiation is proportional to the difference between the radiant exitance of a black-body at the two temperatures in question.
The first measurements of the heat transferred by radiation between a body and its surroundings were made by Tyndall. On the basis of these experiments, it was concluded by Stefan in 1879 that the heat radiated was proportional to the difference of the fourth powers of the absolute temperatures. This purely experimental result was later derived thermodynamically by Boltzmann, who showed that the radiant exitance of a black-body at any temperature T is equal to:

$$\mathcal{E}_{bb}[T] = \sigma_{\text{SB}} T^4$$

This law is now known as the Stefan-Boltzmann law, and σ_{SB} is called the Stefan-Boltzmann constant.

$$\sigma_{\text{SB}} = 5.67 \cdot 10^{-8}\ \text{Wm}^2\text{K}^{-4}$$

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©|Y.A.Cengel, *Introduction To Thermodynamics And Heat Transfer*, 2008, McGraw-Hill, 2ndEd., ...|||

A blackbody is defined as a perfect emitter and absorber of radiation. At a specified temperature and wavelength, no surface can emit more energy than a blackbody. A blackbody absorbs all incident radiation, regardless of wavelength and direction. Also, a blackbody emits radiation energy uniformly in all directions per unit area normal to direction of emission. That is, a blackbody is a diffuse emitter. The term diffuse means **independent of direction**.
For a blackbody: $\varrho = \tau = 0$ and $\alpha = 1$.
The emissivity of a surface represents the ratio of the radiation emitted by the surface at a given temperature to the radiation emitted by a black-body at the same temperature.
The fraction of irradiation absorbed by the surface is called the absorptivity; the fraction reflected by the surface is called the reflectivity; and the fraction transmitted is called the transmissivity.
For opaque surfaces, $\tau = 0$. This is an important property relation since it enables us to determine both the absorptivity and reflectivity of an opaque surface by measuring either of these properties.
The total hemispherical emissivity of a surface at temperature T is equal to its total hemispherical absorptivity for radiation coming from a blackbody at the same temperature. This relation, which greatly simplifies the radiation analysis, was first developed by Gustav Kirchhoff in 1860 and is now called Kirchhoff law. Note that this relation is derived under the condition that the surface temperature is equal to the temperature of the source of irradiation, and the reader is cautioned

against using it when considerable difference (more than a few hundred degrees) exists between the surface temperature and the temperature of the source of irradiation.
The form of Kirchhoff law that involves no restrictions is the spectral directional form, that is, the emissivity of a surface at a specified wavelength, direction, and temperature is always equal to its absorptivity at the same wavelength, direction, and temperature.
It is very tempting to use Kirchhoff law in radiation analysis since it enables us to determine all three properties of an opaque surface from a knowledge of only one property.
Surfaces emit radiation as well as reflect it, and thus the radiation leaving a surface consists of emitted and reflected parts. The calculation of radiation heat transfer between surfaces involves the total radiation energy streaming away from a surface, with no regard for its origin. The total radiation energy leaving a surface per unit time and per unit area is called radiosity and is denoted by J.
For an opaque surface:

$$\alpha = \varepsilon \qquad \alpha + \varrho = 1 \qquad J = \varepsilon \mathcal{E}_{bb} + \varrho \mathcal{J} = \varepsilon \mathcal{E}_{bb} + (1 - \varepsilon)\, \mathcal{J}$$

where $\mathcal{E}_{bb}$ is the blackbody radiant exitance of the surface and $\mathcal{J}$ is the irradiation, the radiation energy incident on the surface per unit time per unit area.
During a radiation interaction, a surface loses energy by emitting radiation and gains energy by absorbing radiation emitted by other surfaces. A surface experiences a net gain or a net loss of energy, depending on which quantity is larger. The net rate of radiation heat transfer from a surface of surface area A is expressed as:

$$\dot{\mathcal{Q}} = A\,(\mathcal{J} - J)$$

Consider two diffuse, grey, and opaque surfaces of arbitrary shape maintained at uniform temperatures. Recognizing that the radiosity J_k represents the rate of radiation leaving surface k per unit surface area and that the view factor $F_{ij}^{\cdot\cdot}$ represents the fraction of radiation leaving surface i that strikes surface j, the net rate of radiation heat transfer from surface i to surface j can be expressed as

$$\dot{\mathcal{Q}}_{ij}^{\cdot\cdot} = A_i F_{ij}^{\cdot\cdot} J_i - A_j F_{ji}^{\cdot\cdot} J_j = A_i F_{ij}^{\cdot\cdot} (J_i - J_j) \qquad A_i F_{ij}^{\cdot\cdot} = A_j F_{ji}^{\cdot\cdot}$$

65.09.01.06 Ideal Photon Gas

This § is referenced at pages:
[2552, 2684]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 1|*§ 39.3*|
©|S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...|§ 23||
©|H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|3.6||
©|Am.J.Phys, , ..., ..., ...Ed., ... 70 (8) August 2002|TeachingThePhotonGasInIntroductoryPhysics||

Recall Poynting theorem in section § 65.06.04.

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Adapted from.... ?
Consider a container of volume V, whose walls are maintained at temperature T. The walls of the cavity, are diathermic, meaning they are in thermal contact with their surroundings, so that the temperature within the cavity may be controlled.
Suppose it has been emptied of matter by a vacuum pump. It cannot be entirely empty because the walls radiate photons into the container. Some photons scatter off the walls, with some being absorbed and new ones being emitted continually. A dynamic equilibrium exists when the average

absorption and emission rates are equal. Thus, **an apparently empty container actually is filled with a photon gas**.
Consider an open molecular ideal gas system: in this case N is not a fixed parameter, but an additional variable: one has three independent variable and one equation of state. Unlike the molecular ideal gas, for which there are three independent variables among p, T, V and N, if the number of particles is not fixed, the photon gas has just two independent, controllable variables, among p, T, V and N; that is, there is a second relation linking the four variables, and N is not a parameter, but a variable.
We can envisage building a photon gas from energy stored in the container walls. Consider a container. Imagine purging it of all atoms with a vacuum pump, and then moving the piston to the left until it touches the left wall. The volume is then zero and the walls, including the piston, have temperature T. Now slowly move the piston to the right, keeping the wall temperature constant using a reservoir. Photons will pour out of the walls as the volume increases, until the dynamic equilibrium described above occurs.
In this way, we mentally construct a photon gas of volume V and temperature T, with average photon number $N[T, V]$. Building the photon gas using this thought experiment can help develop an understanding of the nature of the variable particle photon gas.

- Either ElectroMagnetic waves or photons.
- EM radiation in thermal equilibrium inside a closed cavity with rigid walls kept at a fixed temperature T. The photons inside the cavity are in thermal equilibrium with the cavity walls, and form ElectroMagnetic standing waves.
- The ElectroMagnetic energy density is known from Maxwell. This is the internal energy of the ElectroMagnetic field.
- Photons are continuously absorbed and emitted by/from the walls. It is by virtue of these mechanisms that the ElectroMagnetic radiation inside the container depends on the temperature of the walls.
- A key feature of the photon gas is that it has a variable particle number, N. For photons, N is not conserved! The total number of photons inside the enclosure is not fixed, but depends on the temperature T of the walls.

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This observation calls our attention to the fact that in the empty cavity there exist no conserved particles to be counted by a parameter N, but N is a variable.
The ElectroMagnetic radiation within the cavity is governed by a fundamental equation of the form $\mathcal{S} = \mathcal{S}[\mathcal{U}, V, N] \to \mathcal{S}[..., ...]$ in which there are only two rather than three independent extensive parameters!

The equation of state depends on the energy momentum relation for the gas particle. It is different

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for non-relativistic particles / relativistic particles / photons.

$$m = 0 \implies \varepsilon = pc \qquad \varepsilon = \hbar\omega \equiv h\nu \qquad , \quad ,$$

$$u = \frac{\mathcal{U}}{V} = \eta_{\mathrm{N}}\varepsilon \quad , \quad ,$$

$$pV = \frac{\mathcal{U}}{3} \qquad \text{internal energy for a photon gas} \quad , \quad , \qquad \rightarrow 2706\ 2754$$

$$pV = \frac{2\mathcal{U}}{3} \qquad \text{internal energy for a molecular mono-atomic gas} \quad , \quad , \qquad \rightarrow 2754$$

$$u \equiv \frac{\mathcal{U}}{V} \propto T^4 \implies \mathcal{U} = \alpha V T^4 \qquad \text{internal energy (photons)} \quad , \quad , \qquad \rightarrow 2706$$

$$\eta_{\mathrm{N}} \propto T^3 \implies N = \beta V T^3 \qquad \text{number of particles (photons)} \quad , \quad ,$$

$$pV = \frac{\alpha}{3\beta} N[V,T]T \quad , \quad ,$$

$$p = \frac{\alpha T^4}{3} \qquad \text{independent variables are: } V \text{ and (} p \text{ or } T \text{)} \quad , \quad ,$$

$$\text{Energy Flux} = uc = \sigma_{\mathrm{SB}} T^4 \qquad \text{jet-like, collinear distribution} \quad , \quad , \qquad \rightarrow 2599\ 2735$$

$$\text{Energy Flux} = \frac{1}{4}uc = \sigma_{\mathrm{SB}} T^4 \qquad \text{isotropic distribution} \quad . \qquad \rightarrow 2736$$

The two equations of state are:

$$\boxed{\mathcal{U} = \frac{6\zeta[4]\hbar c}{\pi^2} V \left(\frac{k_{\mathrm{B}}T}{\hbar c}\right)^4 = \frac{\pi^2 \hbar c}{15} V \left(\frac{k_{\mathrm{B}}T}{\hbar c}\right)^4} \quad ,$$

and:

$$\boxed{N = \frac{2\zeta[3]}{\pi^2} V \left(\frac{k_{\mathrm{B}}T}{\hbar c}\right)^3} \quad ,$$

where:

- $\zeta[3] \simeq 1.202$,
- $\zeta[4] = \dfrac{\pi^4}{90}$,

are values of the zeta function.

The mean photon energy is:

$$\boxed{\frac{\mathcal{U}}{N} = \frac{\pi^4}{30\zeta[3]} k_{\mathrm{B}}T \simeq 2.70118 k_{\mathrm{B}}T} \quad . \qquad (65.09.01)$$

Notice that N is strictly a function of T and V and must not be considered an independent parameter. In fact the above relations show that everything is known as soon as one knows, for instance, V and T, which are perhaps the most natural variables for an ElectroMagnetic cavity, or any other pair of variables. In other words, N cannot be chosen independently from V and T. While for a molecular ideal gas you can choose N as you like, for photons, given T and V, then N is given.

That is, **we're dealing with an open system where the number of particles is not conserved**.

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©|?|?|*Adapted*|

The ElectroMagnetic energy density is known from Maxwell. This is the internal energy of the ElectroMagnetic field.
A priori it depends on the position inside the cavity and linear momentum of the photons: **x**

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and $\mathbf{p} = \hbar\mathbf{k}$. The cavity dimension is assumed to be much larger than the longest wavelength of significance in the discussion.
The energy density

$$u \equiv u[\mathbf{x}, \mathbf{k}, \lambda, \alpha, \text{MAT}, \text{shapes}, ...]$$

of ElectroMagnetic radiation is a quantity that tells you how many Joules are stored in a cubic meter of cavity. What we want to do now is to specify in which frequency ranges that energy is stored.
... but we want to continue to apply a classical thermodynamic treatment to see how far we can get. To do this, consider two containers, each in contact with thermal reservoirs at temperature T and joined to one another by a tube... Two cavities at temperature T: one is lined with soot and the other with a mirror coating. The system is allowed to come to equilibrium.
The thermal reservoirs are at the same temperature T and so we know from the second law of thermodynamics that there can be no net heat flow from either one of the bodies to the other.

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- Therefore there can be no net energy flux along the tube, so that the energy flux from the soot-lined cavity along the tube from left to right must be balanced by the energy flux from the mirror-lined cavity along the tube from right to left. Equation (65.09.01.06) thus tells us that each cavity must have the same energy density. This argument can be repeated for cavities of different shape and size as well as different coatings. Hence we conclude that u **is independent of shape, size, or material of the cavity**.
- But maybe one cavity might have more energy density than the other at certain wavelengths, even if it has to have the same energy density overall? This is not the case, as we shall now prove.
  The spectral energy density u_λ is defined as follows:
  $$\mathrm{d}E \equiv u_\lambda \, \mathrm{d}\lambda$$
  is the energy density due to those photons which have wavelengths between λ and $\lambda + \mathrm{d}\lambda$. Now imagine that a filter, which only allows a narrow band of radiation at wavelength to pass, is inserted at ..... and the system is left to come to equilibrium. The same arguments listed above apply in this case: there is no net energy flux from one cavity to the other and hence the specific internal energy within a narrow wavelength range is the same for each case. This demonstrates that **the spectral internal energy density has no dependence on the material, shape, size, or nature of a cavity**.
- Suppose that u were different at two positions in the enclosure, in equilibrium at temperature T. Consider what would happen if two identical small bodies at temperature T were placed at these positions. Imagine these to be surrounded by filters which transmit only frequencies in the specified range and which transmit only radiation of the specified **wave-Polarization** . Since different amounts of radiation would be incident on the two bodies, they would absorb different amounts of energy per unit time and their temperatures would therefore become different. This would contradict the equilibrium condition.
- The energy density is independent of the direction of **k**, but depends only its module, that is wavelength; the radiation field is therefore isotropic. Suppose that u did depend on the direction of **k**, for instance that is greater if **k** points north than if it points east. We could again imagine that two identical small bodies at temperature T (and surrounded by the same filters as before) are introduced into the enclosure. Then the body on the north side would have more radiation incident on it and would become hotter, contradicting the equilibrium condition.
- The energy density is independent of the direction of **wave-Polarization** of the radiation, that is the radiation field in the enclosure is non wave-Polarized. Suppose that u did depend on the direction of **wave-Polarization** . Then we could imagine that two identical small bodies at the temperature T are introduced side by side into the enclosure and are surrounded by filters which transmit different directions of **wave-Polarization** . Hence different amounts of radiation would be incident upon these bodies, and a temperature difference would be developed between them in contradiction to the equilibrium condition. Hence, u is independent of the **wave-Polarization** .

There exists, therefore, a very close connection between the emissivity and the absorptivity a of a body. A good emitter of radiation is also a good absorber of radiation, and vice versa. This is a qualitative statement of Kirchhoff law.
Note that this statement **refers only to properties of the body and is thus generally valid, even if the body is not in equilibrium**; but we arrived at this conclusion by investigating the conditions which must be fulfilled to make the properties of the body consistent with a possible equilibrium situation.

The ElectroMagnetic energy density inside the cavity only depends on T and λ.
The spectral energy density inside the cavity is thus a universal function of λ and T only.

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Kirchhoff law,

$\alpha_\lambda[T]$ spectral absorpitity, dimensionless; ratio between the absorbed enery and the incident energy , ,

$e_\lambda[T]$ spectral emissive power, $\mathrm{Wm^{-2}m^{-1}}$; power emitted per unit area per unit wavelenght ,

$$\boxed{\frac{e_\lambda[T]}{\alpha_\lambda[T]} = \frac{c}{4\pi} u_\lambda[T]} \quad ,$$

states that the ratio e_λ/α_λ is a universal function of λ and T.

Therefore, if you fix λ and T, the ratio e_λ/α_λ is fixed and hence

$$\boxed{e_\lambda \propto \alpha_\lambda} \quad .$$

In other words **good absorbers are good emitters** and **bad absorbers are bad emitters**.

A better statement of Kirchhoff laws would be **good absorbers at one wavelength are good emitters at the same wavelength**.

A black body is an object that is defined to have $\alpha_\lambda = 1$ for all λ. Kirchhoff law tells us that for this maximum value, a black body is the best possible emitter.

65.09.01.07 Black-Body Radiation

This § is referenced at pages:
[Never referenced.]

Planck Law

The spectral radiance and spectral energy density of Black-Body radiation are given by the Planck function.

Spectral radiance:

$$\boxed{\frac{d^4E}{\mathrm{d}A_\perp\ \mathrm{d}t\ \mathrm{d}\Omega\ \mathrm{d}\lambda} \equiv \mathcal{R}^{bb}_\lambda[T] = \frac{2hc^2}{\lambda^5}\left(\frac{1}{(\exp[(hc)/(\lambda k_\mathrm{B}T)]-1)}\right) \quad \text{spectral radiance; physical dimensions } \mathrm{W\cdot m^{-2}m^{-1}sr^{-1}}} \quad ,$$

$$\boxed{\frac{d^4E}{\mathrm{d}A_\perp\ \mathrm{d}t\ \mathrm{d}\Omega\ \mathrm{d}\nu} \equiv \mathcal{R}^{bb}_\nu[T] = \frac{2h\nu^3}{c^2}\left(\frac{1}{(\exp[(h\nu)/(k_\mathrm{B}T)]-1)}\right) \quad \text{spectral radiance; } \mathrm{W\cdot m^{-2}Hz^{-1}sr^{-1}}} \quad .$$

The spectral energy density is:

$$\boxed{u^{bb}_\lambda[T] = \frac{4\pi}{c}\mathcal{R}^{bb}_\lambda[T] \quad \text{spectral energy (volume) density; } \mathrm{J\cdot m^{-3}m^{-1}}} \quad ,$$

$$\boxed{u^{bb}_\nu[T] = \frac{4\pi}{c}\mathcal{R}^{bb}_\nu[T] \quad \text{spectral energy (volume) density; physical dimensions } \mathrm{J\cdot m^{-3}Hz^{-1}}} \quad .$$

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Wien Displacement Law

Wien displacement law gives the maximum of the Planck distribution:

$$\boxed{\lambda_{\max} T = 2.9{\cdot}10^{-3}\ \mathrm{m}\cdot\mathrm{K} \qquad \text{for } \mathcal{R}^{bb}_{\lambda}[T]} \quad ,$$

$$\boxed{\nu_{\max}/T = 5.9{\cdot}10^{+10}\ \mathrm{Hz/K} \qquad \text{for } \mathcal{R}^{bb}_{\nu}[T]} \quad .$$

Stefan-Boltzmann Law

Stefan-Boltzmann law provides the total energy per unit area per unit time emitted by a piece of plane surface in all directions:

$$\boxed{\frac{d^2E}{\mathrm{d}A\,\mathrm{d}t} = \mathcal{I}_{bb}[T] = \pi \int_0^{+\infty} \mathcal{R}^{bb}_{\lambda}[T]\,\mathrm{d}\lambda = \pi \int_0^{+\infty} \mathcal{R}^{bb}_{\nu}[T]\,\mathrm{d}\nu \equiv \sigma_{\text{SB}} T^4} \quad , \quad ,$$

$$\boxed{\sigma_{\text{SB}} = \frac{2\pi^5 k_{\text{B}}^4}{15c^2h^3} = 5.67{\cdot}10^{-8}\ \mathrm{W}\cdot\mathrm{m}^{-2}\mathrm{K}^{-4}} \qquad \text{Stefan-Boltzmann constant} \quad .$$

One Directional Flow

$$cu = \sigma_{\text{SB}} T^4 \qquad \text{power per unit area} \quad ,$$

$$u = \frac{\sigma_{\text{SB}}}{c} T^4 \qquad \text{energy density} \quad ,$$

$$p = u \qquad \text{pressure on the cavity walls} \quad .$$

Omni-Directional Flow

$$\frac{1}{4} cu = \sigma_{\text{SB}} T^4 \qquad \text{power per unit area} \quad ,$$

$$u = \frac{4\sigma_{\text{SB}}}{c} T^4 \qquad \text{energy density} \quad ,$$

$$p = \frac{u}{3} \qquad \text{pressure on the cavity walls} \quad .$$

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Radiant Energy Emitted/received From/to a Plane Surface

$$E = \iiint \mathcal{R}[\mathbf{x}, \theta, \phi] \cos[\theta] \; \mathrm{d}\Omega \, \mathrm{d}\mathbf{S} \, \mathrm{d}t \qquad \text{energy} \quad .$$

In case $\mathcal{R}$ is independent of direction: $\Delta E = \pi \mathcal{R}[\mathbf{x}] \Delta \mathbf{S} \Delta t$.

Note that, in all definitions above, the area is the area perpendicular to the velocity, that is to the specific direction considered; it is the area projected along the direction, that is:

$$\mathrm{d}\mathbf{S}_{\perp} \equiv \mathrm{d}\mathbf{S} \cos[\theta]$$ θ:= angle between the direction and the normal to the surface .

65.09.02
Heat Conduction

This § is referenced at pages:
[2742]

Read § 68.05.06

65.09.03
Heat Convection

This § is referenced at pages:
[Never referenced.]

Convection is energy exchange by means of exchange of matter. Actually, while conduction and radiation can be considered fundamental processes, convection is made by conduction and radiation with moving matter as transport medium.

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A flow of liquid or gas that absorbs heat at one place and then moves to another place, where it mixes with a cooler portion of the fluid and rejects heat, is called a convection current. If the motion of the fluid is caused by a difference in density that accompanies a temperature difference, the phenomenon is called natural convection. If the fluid is made to move by the action of a pump or a fan, it is called forced convection.
Consider a fluid in contact with a flat or curved wall whose temperature is higher than that of the main body of the fluid. Although the fluid may be in notion, there is a relatively thin layer of stagnant fluid next to the wall, the thickness of the layer depending upon the character of the motion of the main body of fluid. The more turbulent the motion, the thinner the layer. Heat is transferred from the wall to the fluid by a combination of conduction through the layer and convection in the fluid.

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65.10 Examples and Physical Applications

65.10.01 Compression Work on a Solid/liquid

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|§ 3.6||

65.10.02 Urto Anaelastico Di Un Contenitore Con Gas

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|2.27||

Un recipiente a pareti rigide e adiabatiche contiene n moli di un gas ideale di capacità termica $c_{\mathtt{V}}$ e massa molecolare $\mathcal{M}_0$ ed è in moto a velocità costante $\mathbf{v}_0$, orizzontale. Il recipiente urta in modo totalmente anaelastico un muro rigido fermandosi.

- Determinare l'aumento di temperatura del gas contenuto nel recipiente. Si trascuri la capacità termica del recipiente.
- Determinare l'aumento di temperatura del gas contenuto nel recipiente se la capacotà termica del recipiente vale C_0.

WORKED SOLUTION

Essendo le pareti del recipiente rigide si ha $\mathcal{W} = 0$, ed essendo adiabatiche si ha $\mathcal{Q} = 0$. Inoltre $\Delta\Phi = 0$. Ne segue

$$\Delta\mathcal{U} = -\Delta\mathcal{K} = nc_{\mathtt{V}}\Delta T = \frac{1}{2}n\mathcal{M}_0 v_0^2$$

Da cui si ricava

$$\Delta T = \frac{\mathcal{M}_0 v_0^2}{2c_{\mathtt{V}}}$$

65.10.03 Il Proiettile Che Si Conficca Nel Muro

Un proiettile di piombo di massa m e temperatura $T_0 = 30°\mathrm{C}$ che viaggia orizzontalmente a velocità $v_0 = 346\,\mathrm{ms}$ incontra una parete in cui penetra fermandosi. Supponendo che tutta l'energia meccanica dissipata vada in energia interna del proiettile determinare la temperatura finale del proiettile, $T_{\mathtt{F}}$, e l'eventuale frazione della massa del proiettile, x, che fonde in conseguenza dell'urto. Per il piombo si usino i seguenti dati: $T_{\mathtt{fus}} = 601\ \mathrm{K}$, $c_{\mathtt{V}}^{\mathtt{sol}} = 0.129\cdot 10^3\ \mathrm{J\ kg^{-1}\ K^{-1}}$, $c_{\mathtt{V}}^{\mathtt{liq}} = xx\cdot 10^{yy}\ \mathrm{J\ kg^{-1}\ K^{-1}}$ e $\lambda_{\mathtt{fus}} = 24.7\cdot 10^3\ \mathrm{J\ kg^{-1}}$. Si consideri il proiettile puntiforme.

WORKED SOLUTION

Si consideri come sistema l'insieme formato dal proiettile e dal muro. Il primo principio della termodinamica, equazione (65.05.06), si scrive

$$\Delta\mathcal{K} + \Delta\Phi + \Delta\mathcal{U} = \mathcal{W} + \mathcal{Q} \equiv \mathcal{W} + (\mathcal{Q}_{\mathtt{cnd}} + \mathcal{Q}_{\mathtt{cnv}} + \mathcal{Q}_{\mathtt{rad}})$$

Considerando il sistema proiettile-muro isolato termicamente ($\mathcal{Q} = 0$) ed energeticamente ($\mathcal{W} = 0$), si ha

$$\Delta\mathcal{K} + \Delta\mathcal{U}_p + \Delta\mathcal{U}_m = 0 \qquad (65.10.01)$$

Si denoti con T la temperatura finale del piombo incognita e con x la eventuale frazione della massa del proiettile che liquefa ($0 \le x \le 1$). Assumendo che tutta l'energia meccanica (in questo caso l'energia cinetica macroscopica) vada tutta in energia interna del proiettile, cioè $\Delta\mathcal{U}_m = 0$, si ha, a seconda del valore di $\Delta\mathcal{U}_p$, l'equazione

$$\begin{aligned}
\Delta\mathcal{U}_p &= -\Delta\mathcal{K} = mc_{\mathtt{V}}^{\mathtt{sol}}(T - T_0) \qquad \text{se } T \le T_{\mathtt{fus}} \quad ,\\
\Delta\mathcal{U}_p &= -\Delta\mathcal{K} = mc_{\mathtt{V}}^{\mathtt{sol}}(T_{\mathtt{fus}} - T_0) + mx\lambda_{\mathtt{fus}} \qquad \text{se } 0 \le x \le 1 \quad ,\\
\Delta\mathcal{U}_p &= -\Delta\mathcal{K} = mc_{\mathtt{V}}^{\mathtt{sol}}(T_{\mathtt{fus}} - T_0) + m\lambda_{\mathtt{fus}} + mc_{\mathtt{V}}^{\mathtt{liq}}(T - T_{\mathtt{fus}}) \qquad \text{se } T \le T_{\mathtt{ebol}} \quad ,\\
&\ldots\ldots \quad .
\end{aligned}$$

Nel primo caso si determina la temperatura finale del proiettile nel caso l'energia meccanica dissipata non sia sufficiente a far salire la temperatura fino alla temperatura di fusione. Nel secondo caso si determina la frazione della massa del proiettile che fonde, nel caso l'energia meccanica dissipata non sia sufficiente a farlo fondere tutto. Nel terzo caso si determina la temperatura finale del piombo liquido. Con i dati del problema si ha, usando la prima equazione,

$$T - T_0 = -\frac{\Delta\mathcal{K}}{mc_{\mathtt{V}}^{\mathtt{sol}}} = 464°\mathrm{C}$$

cui corrisponde una temperatura superiore a quella di fusione del piombo per cui occorre usare la seconda equazione. Dalla seconda equazione si ricava allora

$$x = \frac{-\Delta\mathcal{K} - mc_{\mathtt{V}}^{\mathtt{sol}}(T_{\mathtt{fus}} - T_0)}{m\lambda_{\mathtt{fus}}} = 0.86$$

che fornisce la frazione della massa del proiettile che fonde.
L'assunzione che tutta l'energia cinetica vada in energia interna del solo proiettile fornisce in ogni caso un lmiite superiiore allatemperatura finale del proiettile.

65.10.04
Blocco Trascinato a Velocità Costante Su Un Piano Scabro

Un blocco rigido è trascinato a velocità costante su un piano rigido e scabro da una forza costante F parallela al piano. Determinare il bilancio energetico del blocco, del piano e del sistema complessivo. Si consideri il sistema blocco più piano isolato termicamente e che non scambi lavoro con l'esterno (eccetto quello relativo alla forza F).

65.10.05
Bilancio Energetico Di Un'auto in Discesa

©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|1.15||

©|H.C.Ohanian, , ..., ..., ...Ed., ...|21.34||

Un'auto di massa m scende a motore spento lungo una discesa a pendenza costante.

1. Scrivere il bilancio energetico dell'auto se il guidatore aziona i freni in modo che la velocità dell'auto sia costante e uguale a v_0 e supponendo che si sia raggiunta una situazione stazionaria in cui l'energia interna dell'auto è costante.
2. Scrivere il bilancio energetico dell'auto se il guidatore, partendo da uan situazione in cui l'auto scende in discesa libera, vede un semaforo in fondo alla discesa, e frena fino a fermarsi, iniziando a frenare dalla velocità v_0 alla quota z_0.

Si supponga che in ogni caso le ruote ruotino senza strisciare sull'asfalto.

PROGETTO

WORKED SOLUTION

Consideriamo come sistema l'auto nella sua intrerezza.

1. L'energia cinetica e interna sono costanti per ipotesi. Inoltre le forze esterne che agiscono sull'auto, la reazione normale della strada e la forza di attrito statico del terreno, non compiono lavoro mentre l'effetto della forza peso viene incluso nel termini di energia potenziale del sistema nei campi esterni. L'attrito dell'aria è trascurato.
   Indicando con z la quota dell'auto, il primo principio della termodinamica (65.05.01) per l'auto, in condizioni stazionarie, si scrive allora:
   $$\Delta\mathcal{U} = 0 \implies \Delta\mathcal{E} = mg\Delta z = \mathcal{Q} < 0 \quad .$$
   L'energia potenziale gravitazionale dell'auto viene in questo caso ceduta all'ambiente esterno sotto forma di calore in quanto, per ipotesi, l'energia interna dell'auto resta costante. Infatti il calore $\mathcal{Q}$ risulta negativo.
2. Nel caso in cui il guidatore frena rallentando non è più accettabile l'ipotesi di costanza dell'energia interna. Il primo principio della termodinamica (65.05.01) per l'auto tra l'istante di inizio della frenata e lo stop completo si scrive allora
   $$\Delta\mathcal{E} \equiv \Delta\mathcal{K} + \Delta\mathcal{U} + \Delta\mathcal{U} = -\frac{mv_0^2}{2} + mg\Delta z + \Delta\mathcal{U} = \mathcal{Q} \implies \Delta\mathcal{U} - \mathcal{Q} = -\Delta\mathcal{K} - \Delta\mathcal{U} = \frac{mv_0^2}{2} + mgz_0 \quad ,$$
   da cui si deduce che la variazione di energia interna dell'auto più il calore $-\mathcal{Q}$ che l'auto cede all'ambiente eguagliano l'energia meccanica totale che l'auto aveva all'inizio della frenata e che viene dunque dissipata, nel linguaggio della meccanica, in parte sotto forma di calore ceduto all'ambiente e in parte come aumento di energia interna. Non è possibile sulla sola base del primo principio della termodinamica determinare come si ripartisce tra energia interna e calore ceduto all'ambiente l'energia meccanica dissipiata. Qualitativamente si può però affermare quanto segue. Inizialmente, a freni freddi, non può esserci cessione di calore all'ambiente in quanto l temperatura è uniforme. Allora inizialmente l'energia meccanica si trasforma in energia interna dell'auto (di fatto dei freni dell'auto). Questo ha per conseguenza un aumento della temperatura dei freni che rende possibile una cessione di calore verso l'esterno. La temperatura aumenta finché la cessione di calore diventa sufficiente a dissipare l'energia meccanica che va dissipata. A questo punto si instaura una situazione stazionaria in cui l'energia interna dell'auto non varia. Si osservi che la forza di attrito statico, pur essendo quella che effettivamente decelera l'auto non compie però lavoro.
   Notare che il segno di $\mathcal{Q}$, in generale, dipende dalle condizioni: se siamo all'equatore e l'auto è appena uscita da un frigorifero inizialmente si ha comunque $\mathcal{Q} > 0$. In generale il motore di un'auto è tipicamente a temperature maggiore di quella ambiente, per cui si ha sempre una cessione di calore dal motore all'ambiente.

PRESS

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65.10.06
Bilancio Energetico Di Un Paracadutista

Scrivere il bilancio energetico di un paracadutista in discesa.

65.10.07
Bilancio Energetico Di Un Fluido in Moto Stazionario

©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|§???||

Si deduca l'espressione del primo principio della termodinamica per il moto stazionario di un fluido (sistema aperto in moto stazionario). Si consideri un volume di controllo e il fluido che entra ed esce dal volume.

WORKED SOLUTION

L'equazione del bilancio energetico per un fluido in moto stazionario (sistema aperto) è la (65.05.08):

$$\Delta \mathtt{k} + \Delta\phi + \Delta \mathtt{h} = \mathtt{w}' + \mathtt{q}$$

dove $\mathtt{h}$ è l'entalpia per unità di massa, $\mathtt{k}$ l'energia cinetica macroscopica per unità di massa, ϕ l'energia potenziale macroscopica per unità di massa, $\mathtt{q}$ il calore ricevuto per unità di massa ed $\mathtt{w}'$ il lavoro utile effettivo per unità di massa fatto sul sistema, ad esclusione cioè del lavoro fatto dalle forze di pressione del fluido, che è quello che realmente interessa in problemi di ingegneria. La variazione riguarda le proprietà del fluido tra due punti del condotto di flusso mentre per l'ipotesi di stazionarietà le proprietà del fluido in ogni punto non cambiano.
L'equazione fondamentale del bilancio energetico si ricava considerando il moto stazionario di un fluido in un volume di controllo fissato in cui entra il calore $\mathcal{Q}$ su cui viene compiuto il lavoro totale $\mathcal{W}$, e con variazione, tra l'uscita e l'ingresso del volume di controllo, di energia cinetica $\Delta\mathcal{K}$, di energia potenziale associata ai campi esterni $\Delta\Phi$, di energia interna $\Delta\mathcal{U}$. Si consideri la massa di fluido contenuta nel volume di controllo più la massa di fluido che entrerà nel volume di controllo in un certo intervallo di tempo arbitrario fissato Δt. Nello stesso intevallo di tempo un'uguale massa di fluido uscirà dal volume di controllo. Essendo il moto stazionario tutte le proprietà del fluido entro il volume di controllo non cambiano per cui il bilancio energetico coinvolgerà solo la differenza delle proprietà della massa di fluido che esce rispetto alla massa di fluido che entra nel volume di controllo, per la conservazione della massa. Dal primo principio della termodinamica si ha dunque

$$\Delta\mathcal{K} + \Delta\Phi + \Delta\mathcal{U} = \mathcal{W} + \mathcal{Q} \equiv \mathcal{W} + (\mathcal{Q}_{\mathtt{cnd}} + \mathcal{Q}_{\mathtt{cnv}} + \mathcal{Q}_{\mathtt{rad}}) \implies \mathcal{K}_2 - \mathcal{K}_1 + \Phi_2 - \Phi_1 + \mathcal{U}_2 - \mathcal{U}_1 = \mathcal{Q} + p_1V_1 - p_2V_2 + \mathcal{W}'$$

avendo indicato con l'indice 1 le grandezze relative alla massa di fluido che entra nel volume di controllo in Δt e con l'indice 2 le grandezze relative alla massa di fluido che esce dal volume di controllo in Δt. Si è poi indicato con V_1 e V_2, rispettivamente, il volume di fluido che entra e quello che esce dal volume di controllo nell'intervallo di tempo Δt e si è separato il lavoro fatto dalle forze di pressione sul fluido del volume di controllo dal restante lavoro $\mathcal{W}'$. Si ha poi

$$\mathcal{W}' - \mathcal{W} = p_2V_2 - p_1V_1 = \Delta m\,(p_2v_2 - p_1v_1)$$

dove si è tenuto conto che, per la conservazione della massa, la massa che entra nel volume di controllo è pari a quella che esce e avendo indicato con v i volumi specifici del fluido. Riarrangiando i termini e semplificnado la massa si perviene alla (65.05.08).

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65.10.08
Ebollizione Dell'acqua per Agitazione

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Un fisico si sveglia un mattino e trova la stufa guasta. Decide allora di far bollire l'acqua per il caffè agitandola dentro un thermos. Si supponga che usi $560\ \mathrm{cm}^3$ di acqua alla temperatura di $59\ ^\circ\mathrm{F}$, che capovolga ripetutamente il thermos, che ad ogni capovolgimento l'acqua cada di $35\ \mathrm{cm}$ e che il fisico capovolga il thermos 30 volte al minuto.

1. Trascurando ogni perdita di energia si stimi per quanto tempo il fisico deve agitare il thermos per portare ad ebollizione l'acqua.

WORKED SOLUTION

1. 2.4 giorni

65.10.09
Riscaldatore Ad Energia Solare

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In un riscaldatore ad energia solare l'energia solare è raccolta da un collettore sul tetto di un edificio entro il quale si trovano i tubi in cui circola l'acqua. La radiazione solare entra nel collettore tramite una finestra trasparente e riscalda l'acqua nei tubi. L'acqua è poi pompata entro un serbatoio. Assumendo che l'efficienza globale del sistema sia 20%, cioè che 80% dell'energia solare non sia utilizzata per riscaldare l'acqua, quale superficie deve avere il collettore per riscaldare $200\ \mathrm{l}$ di acqua da $20°\mathrm{C}$ a $40°\mathrm{C}$ in un'ora? L'intensità della luce incidente è $700\ \mathrm{W}/\ \mathrm{m}^2$.

65.10.10
Il Calorimetro a Flusso

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Un calorimetro a flusso è utilizzato per misurare il calore specifico di un liquido. Si cede calore al liquido ad un tasso noto e costante mentre questo passa entro il calorimetro e si misura la differenza di temperatura del liquido tra l'uscita e l'ingresso. Un liquido di densità $\rho = 0.85\ \mathrm{g/cm}^3$ scorre nel calorimetro ad un flusso di $8.2\ \mathrm{cm}^3/\ \mathrm{s}$, si cedono $250\ \mathrm{W}$ al fluido e la differenza di temperatura che si stabilisce in situazione stazionaria tra uscita ed ingresso vale $15\ °\mathrm{C}$.

1. Determinare il calore specifico del liquido.

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65.10.11
Calore Specifico Dell'argento

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La capacità termica molare dell'argento, misurata a pressione atmosferica, varia con la temperatura, nell'intervallo tra $50\ \mathrm{K}$ e $100\ \mathrm{K}$, secondo la relazione empirica

$$C = 0.318T - 0.00109T^2 - 0.628$$

dove C è espressa in $\mathrm{J/\ mol\ K}$ e la temperatura T in K. Calcolare la quantità di calore necessaria ad alzare la temperatura di $316\ \mathrm{g}$ di argento da $50\ \mathrm{K}$ a $90\ \mathrm{K}$. La massa molare dell'argento vale $M = 107.87\ \mathrm{g/\ mol}$.

65.10.12
Adiabatic Transformations and Thermal Capacities for Ideal Gas

Consider a quasi-static adiabatic transformation of an ideal gas. One can show that:

$$\frac{\mathrm{d}p}{p} + \gamma[T]\frac{\mathrm{d}V}{V} = \text{constant} \qquad \gamma[T] \equiv \frac{C_{\mathrm{p}}[T]}{C_{\mathrm{V}}[T]} \quad .$$

If $\gamma[T]$ does not depend on temperature, the equation can be integrated:

$$\boxed{pV^{\gamma} = \text{constant}} \quad .$$

In fact, for a ideal gas (and only for a ideal gas):

$$\mathcal{U} = \mathcal{U}[T] \qquad \mathcal{H} \equiv \mathcal{U}[T] + pV = \mathcal{U}[T] + nRT \quad ,$$
$$\mathrm{d}\mathcal{U} = C_{\mathrm{V}}[T]\ \mathrm{d}T \qquad \mathrm{d}\mathcal{H} = C_{\mathrm{p}}[T]\ \mathrm{d}T \quad ,$$
$$C_{\mathrm{p}}[T] = C_{\mathrm{V}}[T] + nR \quad .$$

65.10.13
Heat-Capacity for a Generic Polytropic Transformation for Ideal Gas

Consider a generic polytropic transformation, Γ, for an ideal gas.

$$\delta\mathcal{Q} = C_{\mathrm{V}}[T]\ \mathrm{d}T + p\ \mathrm{d}V \quad ,$$

$$pV^{\alpha} = \text{constant} \implies TV^{\alpha-1} = \text{constant} \implies \mathrm{d}TV^{\alpha-1} + (\alpha-1)\,TV^{\alpha-2}\,\mathrm{d}V = 0 \implies -\frac{\mathrm{d}TV}{T\,(\alpha-1)} = \mathrm{d}V \quad ,$$

$$\delta\mathcal{Q} = C_{\mathrm{V}}[T]\ \mathrm{d}T - pV\ \mathrm{d}T\frac{1}{T\,(\alpha-1)} = C_{\mathrm{V}}[T]\ \mathrm{d}T - nR\,\mathrm{d}T\frac{1}{(\alpha-1)} \quad ,$$

$$C_C[T] = C_{\mathrm{V}}[T] - \frac{nR}{(\alpha-1)} = nR\left(\frac{1}{(\gamma[T])} - \frac{1}{(\alpha-1)}\right) \quad .$$

Note that, for some transformations ($0 < \alpha < \gamma$) the thermal capacity is negative. This means that the change of temperature due to work overcomes the one due to heat.

DRAFT-IN-PROGRESS

65.10.14
Ciclo Termodinamico Di Un Gas Ideale

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|2.32||

Un cilindro contiene $n = 2$ moli di un gas ideale a temperatura $T_0 = 300$ K e pressione p_0. Il sistema viene sottoposto ad una compressione quasi-statica che riduce la pressione a $p = p_0/\alpha$, con $\alpha > 1$, mantenendo il volume costante. Successivamente il sistema è sottoposto ad una espansione quasi-statica in cui la pressione esterna resta costante mentre la temperatura viene riportata al valore iniziale T_0. Determinare il calore complessivo ceduto al sistema. Determinare il calore ceduto al sistema in ciascuna delle due trasformazioni. Discutere cosa cambia se le due trasformazioni non sono quasi-statiche

65.10.15
Free Expansion of Different Model Gases

A gas, n moles, is initially in equilibrium at temperature T_0 and volume V_0. The gas undergoes a free expansion to a total final volume V, while it is contained in a rigid and adiabatic container.

Determine the final temperature for:

1. an ideal gas;
2. a Clausius gas, explaining the energy changes;
3. a Van der Waals gas, explaining the energy changes;
4. a Dieterici gas (can you answer?);
5. a Berthelot gas(can you answer?).

If the equation for the internal energy is not known, try to assume $\mathcal{U}[T, V, N] \propto V^\alpha T^\beta N^\gamma$ For the expression to be extensive one needs $\alpha + \gamma = 1$. Therefore:

$$\mathcal{U}[T, V, N] \propto V^\alpha T^\beta N^\gamma \qquad \alpha + \gamma = 1 \quad ,$$
$$\Longrightarrow \Delta\mathcal{U} = 0 \Longrightarrow V_1^\alpha T_1^\beta N^\gamma = V_1^\alpha T_1^\beta N^\gamma \quad .$$

If the equation for the internal energy is not known, we do not know the final pressure. However, the pressure must decrease because the isothermal compressibility is always positive.

Note that the free expansion of a gas is a spontaneous process and irreversible process.

65.10.16
Stima Della Temperatura Di Equilibrio Di Un Pianeta

©|B.W.Carroll & D.A.Ostlie, *An Introduction To Modern Astrophysics*, ..., ..., 1stEd., ...|???||

In prima approssimazione si può supporre che la temperatura di un pianeta dipenda dal bilancio energetico dell'energia che riceve per irraggiamento dal Sole e di quella che emette verso lo spazio, supponendo di considerare sia il pianeta che il Sole due corpi neri. Si consideri il caso della Terra. L'energia irradiata dal Sole vale $P_S = 3.846{\cdot}10^{26}$ W, il raggio equatoriale medio del Sole vale $R = 6.96{\cdot}10^8$ m e la distanza media tra la Terra e il Sole vale $d = 1.496{\cdot}10^{11}$ m. Si supponga che una frazione $a = 30\%$ (l'albedo della Terra) dell'energia che arriva sulla Terra venga riflessa e la restante assorbita.

1. Supponendo che la temperatura del pianeta sia uniforme, determinare l'espressione del valore della temperatura di equilibrio T nel caso il pianeta ruoti così velocemente che si possa considerare uniforme la temperatura su tutta la sua superficie.
2. Determinare l'espressione del valore della temperatura di equilibrio T nel caso il pianeta ruoti in modo da rivolgere sempre la stessa faccia verso il Sole.
3. Si commentino i risultati nel caso della Terra.

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65.10.17
Scudo Termico

Si supponga di avere due piani paralleli a piccola distanza, che si comportano come un corpo nero, a temperature $T_{\mathtt{H}}$ e $T_{\mathtt{C}}$, con $T_{\mathtt{H}} > T_{\mathtt{C}}$. Il flusso di energia tra i due piani è $J = \sigma\left(T_{\mathtt{H}}^4 - T_{\mathtt{C}}^4\right)$.

1. Se si inserisce un terzo piano parallelamente ai due (schermo), anche esso schematizzabile con un corpo nero, la temperatura di questo raggiungerà un valore intermedio, T_x. Determinare T_x in funzione di $T_{\mathtt{H}}$ e $T_{\mathtt{C}}$ e dimostrare che il flusso di energia viene dimezzato dall'inserimento del terzo piano.
2. Si generalizzi il risultato al caso in cui si inseriscono N piani indipendenti.
3. Si supponga che lo schermo sia opaco ($\tau = 0$) e abbia riflettività ϱ. Determinare T_x in funzione di $T_{\mathtt{H}}$ e $T_{\mathtt{C}}$ calcolare il flusso di energia dopo l'inserimento del terzo piano.

Quello descritto è il principio dello scudo termico usato per ridurre la trasmissione di calore per irraggiamento.

WORKED SOLUTION

Il piano intermedio riceve un flusso $J = \sigma\left(T_{\mathtt{H}}^4\right)$ da parte del piano più caldo e un flusso $J = \sigma\left(T_{\mathtt{C}}^4\right)$ da parte del piano più freddo. E cederà due flussi $J = \sigma\left(T_x^4\right)$ verso entrambi i piani. La condizione di equilibrio termico del piano intermedio fornisce dunque

$$T_x^4 = \frac{T_{\mathtt{H}}^4 + T_{\mathtt{C}}^4}{2}$$

Da questo si deduce che il flusso netto di calore risulta dimezzato.

$$J = \sigma\left(T_{\mathtt{H}}^4 - T_x^4\right) = \sigma\left(T_x^4 - T_{\mathtt{C}}^4\right) = \sigma\frac{T_{\mathtt{H}}^4 - T_{\mathtt{C}}^4}{2}$$

Nel caso di inserimento di N scudi termici (corpi neri) il flusso risulta ridotto di un fattore $N + 1$

$$J' = J/\left(N + 1\right)$$

Se lo scudo ha reflectivity ϱ, la temperatura di equilibrio non cambia, mentre il flusso di calore viene moltplicato per un fattore $1 - \varrho$.

65.10.18
Cooling of the Earth

A material made of radioactive nuclides may increase its temperature due to conversion of intrinsic particle energy (where particles are nuclei) into internal kinetic energy.

It is though that the cooling of the Earth is slowed down by this process of conversion of intrinsic unstable nuclei energy into internal kinetic energy which tends to increase the temperature.

65.10.19
Greenhouse Effect

The greenhouse effect is presented as an example of the consequences of the wavelength dependence of radiation properties.

DRAFT IN PROGRESS

65.10.20
Flusso Di Calore Tra Due Sfere Concentriche

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Si consideri un oggetto sferico a simmetria sferica. Assumendo il coefficiente di conduzione termica k costante si dimostri che il flusso di calore in direzione radiale tra due sfere concentriche di raggio rispettivamente r_1, r_2 con $r_1 < r_2$ e temperatura T_1, T_2 è dato da

$$J = \frac{4\pi k r_1 r_2 (T_1 - T_2)}{r_2 - r_1} \tag{65.10.02}$$

65.10.21
Espansioni Quasi-Statica E Non Di Un Gas Ideale

Un cilindro con pistone senza attrito contiene $V_0 = 1\ \text{ft}^3$ di un gas perfetto alla pressione di $200 psia$ ($1\ \text{psia} = 1 lbf/\ \text{in}^2 = 6.895{\cdot}10^3 a$). Il pistone è tenuto fermo da una forza esterna. Il fondo del cilindro ha la parete diatermica ed è a contatto con un serbatoio di temperatura.

- Si calcoli il lavoro fatto nel caso la forza esterna sia diminuita gradualmente (trasformaizone quasi-statica) fino a che il volume aumenta di un fattore $\alpha > 1$.
- Si calcoli il lavoro fatto nel caso il modulo della forza esterna sia bruscamente diminuito dello stesso fattore α (trasformaizone non quasi-statica).

WORKED SOLUTION

Si definisca il gas come sistema.

- Nel caso la forza esterna venga diminuita gradualmente (trasformazione quasi-statica) il lavoro fatto sul sistema dall'ambiente vale
$$\mathcal{W}' = -\mathcal{W}'_g = -\int_{V_0}^{V} p\, \mathrm{d}V = -p_0 V_0 \log\left[\frac{V}{V_0}\right] = -p_0 V_0 \log[\alpha] < 0$$
Il lavoro fatto dall'ambiente sul gas è negativo cioè il gas fa lavoro espandendosi. Si noti che, essendo la trasformazione quasi-statica, il lavoro può essere calcolato indifferentemente usando le variabili di stato del sistema o le variabili relative all'ambiente. Si ha poi $V = \alpha V_0$ e quindi, essendo la temperatura costante, $\alpha p = p_0$.
- Se il modulo della forza esterna viene diminuito bruscamente di un fattore α il processo non è quasi-statico e il lavoro può solo essere calcolato usando le variabili relative all'ambiente. Lo stato finale, raggiunto dopo una fase transitoria più o meno lunga che non sappiamo descrivere, è lo stesso che se la trasformazione è quasi-statica in quanto la temperatura finale è la stessa (per il contatto termico con il serbatoio di temperatura) e la pressione finale anche. Dopo una fase transitoria il sistema si porterà nello stesso stato finale del caso precedente. La forza esterna applicata al pistone durante l'espansione è costante e vale Ap_0/α dove A è l'area del pistone. Ne segue
$$\mathcal{W}'' = -p_0 V_0 \left(\frac{\alpha - 1}{\alpha}\right) < 0$$

Si noti che è sempre

$$|\mathcal{W}''| < |\mathcal{W}'|$$

cioè il lavoro fatto dall'ambiente sul sistema nel caso non quasi-statico, in valore asoluto, è minore di quello nel caso quasi-statico; ovvero, il lavoro fatto dal sistema nel caso non quasi-statico, in valore asoluto,è minore che nel caso quasi-statico.

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65.10.22
Flusso Di Calore Attraverso Un Muro

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|25.46||

Si stimi la quantità di calore che viene ceduta all'ambiente esterno in un freddo giorno d'inverno attraverso un muro di mattoni di lunghezza $6.2\ \mathrm{m}$ e altezza $3.8\ \mathrm{m}$ e spessore $32\ \mathrm{cm}$ se la temperatura interna vale $26°\mathrm{C}$ e quella esterna vale $-18°\mathrm{C}$. Si assuma per la conduttività termica del muro il valore $0.74\ \mathrm{W/\ m\ K}$.

65.10.23
Flusso Di Calore Attraverso Una Parete Composta

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|§ 25.7E6 25.52||

Si consideri una parete composta formata da due lastre di materiale differente poste a contatto. Le lastre hanno sezione A, spessore L_1 ed L_2 e conduttività termica k_1 e k_2. Le temperature delle due pareti esterne sono T_1 e T_2 con $T_2 > T_1$.

1. Determinare il flusso di calore H che si instaura in regime stazionario attraverso la parete composta.
2. Determinare la temperatura T_x che si determina alla superficie di contatto delle due lastre.
3. Generalizzare il risultato la caso di n lastre di spessore L_j e **conductivity** termica k_j con $j = 1, \ldots n$.

WORKED SOLUTION

1. $H = A\,(T_2 - T_1)\,/\,(L_1/k_1 + L_2/k_2)$;
2. $T_x = (R_1 T_1 + R_2 T_2)\,/\,(T_1 + T_2)$ con $R_j \equiv L_j/k_j$;
3. $H = A\,(T_2 - T_1)\,/\left(\sum_j R_j\right)$.

65.10.24
Finestre a Doppio Vetro

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|25.55||

Il vetro di una finestra ha una superficie di $1.4\ \mathrm{m}^2$ e spessore $3.0\ \mathrm{mm}$. La temperatura esterna è $-20\ °\mathrm{F}$ e quella interna $70\ °\mathrm{F}$.

1. Calcolare il flusso di calore verso l'esterno. Si consideri una finestra a doppio vetro avente lo stesso spessore complessivo di vetro ma con uno strato di aria di $7.5\ \mathrm{cm}$ tra i due vetri.
2. Quale è il flusso di calore verso l'esterno supponendo che solo la conduzione sia significativa?

WORKED SOLUTION

1. $2.38 \cdot 10^4\ \mathrm{W}$;
2. $25\ \mathrm{W}$.

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65.10.25
Ice Formation

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|25.58||

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 4.20||

Un contenitore di acqua è stato lasciato in un ambiente freddo la cui temperatura è $-10°\mathrm{C}$ finché non si è formato uno strato di ghiaccio in superficie di spessore $5\ \mathrm{cm}$. Calcolare il tasso di formazione del ghiaccio (in $\mathrm{cm}/\ \mathrm{ora}$) sulla superficie inferiore della lastra di ghiaccio assumendo per la **conductivity** termica e densità del ghiaccio i valori $k = 1.7\ \mathrm{W}/\ \mathrm{m\ K}$ e $\rho = 0.92\ \mathrm{g}/\mathrm{cm}^3$. Si supponga che non ci sia flusso di calore attraverso le pareti del contenitore.

65.10.26
Elastic Wire

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|22.17||

©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|3.19 3.29||

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 2.7, 3.8, 3.9||

Si consideri una rotaia d'acciaio di sezione uniforme $A = 20\ \mathrm{cm}^2$ che a riposo e alla temperatura ambiente $T_0 = 20\ °\mathrm{C}$ ha lunghezza $L_0 = 12\ \mathrm{m}$.

1. Di quanto varia la lunghezza della rotaia per variazioni stagionali di temperatura $\Delta T \simeq 50°\mathrm{C}$?
2. Quale è la forza che esercita la rotaia se la temperatura varia di $\Delta T = 25°\mathrm{C}$ e la rotaia è mantenuta a lunghezza costante L_0. Si consideri la rotaia come una sbarra elastica ideale e si assuma per l'acciaio $\alpha = 11{\cdot}10^{-6}°\mathrm{C}^{-1}$ ed $Y = 2{\cdot}10^{11}\ \mathrm{N}/\ \mathrm{m}^2$.
3. The tension in a wire is increased quasi-statically and isothermally. If the length, cross-sectional area, and isothermal Young modulus of the wire remain practically constant, calculate the work.

65.10.27
Enthalpy of Formation of Water

©|D.V.Schroeder, *Thermal Physics*, 1999, Addison-Wesley, ...Ed., ...|Introduction to thermal physics - § 1|*Adapted from*|

Consider the chemical reaction in which hydrogen and oxygen gas combine to form liquid water. For each mole of water produced, $\Delta\mathcal{H} = -286\ \mathrm{kJ}$, referred to as the enthalpy of formation of water, because water is being formed out of elemental constituents in their most stable states. The numerical values provided by data tables usually assume that both the reactants and the product are at room temperature and atmospheric pressure. If one mole of hydrogen burns, then $-\Delta\mathcal{H}$ is the amount of heat produced. Nearly all of this energy comes from the thermal and chemical energy of the molecules themselves, but a small amount comes from work done by the atmosphere as it collapses to fill the space left behind by the consumed gases.

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65.11 Exercises Problems and Physical Applications

65-001 Heat

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 4.4|*Adapted from*|

The amount of water in a lake may be increased by action of underground springs, by inflow from a river, and by rain. It may be decreased by various outflows and by evaporation.

1. How much rain is there in the lake?
2. How much water in the lake is due to rain?
3. What concept is analogous to **rain in the lake**?

65-002 Determination of an Equation of State From Thermoelastic Coefficients

Consider a real gas whose coefficients of isothermal compressibility and isobaric expansion are given respectively by:

$$\kappa_{\mathrm{T}} = \frac{RT}{p^2 V} \qquad \beta = \frac{1}{V}\left(\frac{a}{T^2} + \frac{R}{p}\right) \quad .$$

Determine the equation of state.

65-003 Work for a Van Der Waals Gas

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 3.4||

Calculate the work done upon expansion of n moles of a Van der Waals gas quasi-statically and isothermally from volume V_{I} to a volume V_{F}.

65-004 Nuclear Fusion in Stars

This § is referenced at pages:
[2476]

Read § 60-011.

Energy Production in Stars comes from Thermonuclear Reactions. The main process is 4 hydrogen nuclei fusion into one helium nucleus with $A = 1$.

Determine the energy produced.

65-005 Heat-Capacity Depending on the Temperature

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 4.16 - 4.17||

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65-006 Heat-Capacity at Constant Volume and Internal Energy

Is it always true that $\mathrm{d}\mathcal{U} = C_{\mathrm{V}}\ \mathrm{d}T$?

65-007 Calore Specifico Gas Perfetto

Determine the specific heat for a ideal gas undergoing a quasi-static transformation with $pV^{\alpha} =$ constant, for different values of α.

65-008 A Photon Gas

©|Lim Yung-Kuo, *Problems And Solutions On Thermodynamics And Statistical Physics*, 1990, World Scientific, ...Ed., ...

65-009 Thermal Capacities for a Condensed Phase

Show that, for any condensed phase,

$$C_{\mathrm{p}} \approx C_{\mathrm{V}} \quad .$$

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65.12 External Documents

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CHAPTER 66

Entropy and the Second Principle

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©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|||

©|The Presumed Extensivity of Entropy|§ 5|*Entropy and the Time Evolution of Macroscopic Systems - Walter T. Grandy, Jr.*|

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66.01 Introduction

The main goal of this section is to relate the path functions đQ and đW to state functions, in order to exploit the path-independence of state functions; this is accomplished by introducing entropy.

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J.P.Ansermet *et al.*
We realise now that temperature (and pressure) cannot be the only thermodynamic variable that characterises thermal processes. The melting process of an ice cube suggests the need to introduce a thermodynamic quantity that is different from temperature (and pressure). It is indeed clear that an ice cube needs heat for it to melt. However, the melting occurs at a fixed temperature (and pressure).
Thus, a variable different than temperature needs to be used to characterise this process.

The first principle alone does not explain why some phenomena conserve the energy and can happen, while others conserve the energy but do not happen, such as:

- when two bodies at different temperatures are put in contact, the temperatures are equalized and never happens that the temperature difference increase (thermal irreversibility);
- free expansion of a gas in vacuum is never seen to go backward, even if the laws of mechanics are time-reversible and no mechanical law forbids it (mechanical irreversibility);
- a piece of paper can burn, but paper is never seen to be recreated from its ashes (chemical irreversibility), that is no Phoenix.

We need something more than the first principle to understand why such processes only appear to be unidirectional in time, while all fundamental laws of macroscopic physics (basically ElectroMagnetism and gravitation) are invariant under time-reversal.

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Vladislav Ĉápek and Daniel P. Sheehan
The second law has more common formulations than any other physical law.
Despite, or perhaps because of, its fundamental importance, no single formulation has risen to dominance. This is a reflection of its many facets and applications, its protean nature, its colorful and confused history, but also its many unresolved foundational issues.
Formulations can be categorized roughly into five categories, depending on whether they involve: device and process impossibilities; engines; equilibrium; entropy; or mathematical sets and spaces.
We will now consider twenty-one standard (and non-standard) formulations of the second law.
This survey is by no means exhaustive.
Carnot theorem is occasionally but not widely cited as the second law. Usually it is deduced from the Kelvin-Planck or Clausius statements.
All natural processes are irreversible. Irreversibility is an essential feature of natural processes and it is the essential thermodynamic characteristic defining the direction of time; e.g., omelettes do not spontaneously unscramble; redwood trees do not un-grow; broken Ming vases do not reassemble; the dead do not come back to life. An irreversible process is, by definition, not quasi-static (reversible); it cannot be undone without additional irreversible changes to the universe.
Irreversibility is so undeniably observed as an essential behavior of the physical world that it is put forward by numerous authors in second law statements.
In many thermodynamic texts, natural and irreversible are equated, in which case this formulation is tautological; however, as a reminder of the essential content of the law, it is unsurpassed. In fact, it is so deeply understood by most scientists as to be superfluous.

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The macroscopic properties of an isolated non-static system eventually assume static values.
Gyftopolous and Beretta - Among all the states of a system with given values of energy, the amounts of constituents and the parameters, there is one and only one stable equilibrium state. Moreover, starting from any state of a system it is always possible to reach a stable equilibrium state with arbitrary specified values of amounts of constituents and parameters by means of a reversible weight process.
One can distinguish between stable and unstable static (or equilibrium) states, depending on whether they persist over time intervals significant for some particular purpose in hand. For instance, to say "Diamonds are forever" is to assume much. Diamond is a metastable state of carbon under everyday conditions; at elevated temperatures it reverts to graphite. In a large enough vacuum, graphite will evaporate into a vapor of carbon atoms and they, in turn, will thermally ionize into a plasma of electrons and ions. After $\approx 10^{33}$ years, the protons might decay, leaving a tenuous soup of electrons, positrons, photons, and neutrinos. Which of these is a stable equilibrium? None or each, depending on the time scale and environment of interest. By definition, a stable static state is one that can change only if its surroundings change, but still, time is a consideration. To a large degree, equilibrium is a matter of taste, time, and convenience.
Gyftopoulos and Beretta emphasise one and only one stable equilibrium state. This is echoed by others, notably by Mackey who reserves this caveat for his strong form of the second law.
Gibbs - For the equilibrium of an isolated system, it is necessary and sufficient that in all possible variations of the state of the system which do not alter its energy, the variation of its entropy shall either vanish or be negative. In other words, thermodynamic equilibrium for an isolated system is the state of maximum entropy. Although Gibbs does not refer to this as a statement of the second law, per se, this maximum entropy principle conveys its essential content.
Entropy Properties - Every thermodynamic system has two properties (and perhaps others): an intensive one, absolute temperature T, that may vary spatially and temporally in the system; and an extensive one, entropy S. Together they satisfy the following three conditions:

1. The entropy change $đ\mathbb{S}$ during time interval $đt$ is the sum of: entropy flow through the boundary of the system $đ\mathbb{S}_e$ and entropy production within the system, $đ\mathbb{S}_i$; that is, $đ\mathbb{S} = đ\mathbb{S}_e + đ\mathbb{S}_i$.
2. Heat flux (not matter flux) through a boundary at uniform temperature T results in entropy change $đ\mathbb{S}_e = \delta\mathbb{Q}/T$.
3. For reversible processes within the system, $đ\mathbb{S}_i = 0$, while for irreversible processes, $đ\mathbb{S}_i > 0$.

This version is a starting point for some approaches to irreversible thermodynamics.
In addition to academic formulations there are also many folksy aphorisms that capture aspects of the law. Many are catchphrases for more formal statements. Although loathe to admit it, most of these are used as primary rules of thumb by working scientists. Most are anonymous; when possible, we try to identify them with academic forms. Among these are:

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- Disorder tends to increase, (Clausius, Planck).
- Heat goes from hot to cold, (Clausius).
- There are no perfect heat engines, (Carnot).
- There are no perfect refrigerators, (Clausius).
- Murphy Law (and corollary), (Murphy 1947).
  - If anything can go wrong it will.
  - Situations tend to progress from bad to worse.
- A mess expands to fill the space available.
- The only way to deal with a can of worms is to find a bigger can.
- Laws of Poker in Hell:
  - Poker exists in Hell. (Zeroth Law)
  - You can't win. (First Law)
  - You can't break even. (Second Law)
  - You can't leave the game. (Third Law)
- Messes don't go away by themselves. (Mom)
- Perpetual motion machines are impossible. (Nearly everyone)

The discovery of thermodynamic entropy as a state function is one of the triumphs of nineteenth-century theoretical physics. Inasmuch as the second law is one of the central laws of nature, its handmaiden - entropy - is one of the most central physical concepts. It can pertain to almost any system with more than a few particles, thereby subsuming nearly everything in the universe from nuclei to super-clusters of galaxies.
Roughly, entropy is a quantitative macroscopic measure of microscopic disorder. It is the only major physical quantity predicated and reliant upon wholesale ignorance of the system it describes. This approach is simultaneously its greatest strength and its Achilles heel.
A working-man summary of standard properties can be extracted from Gyftopoulous and Beretta. Classical entropy must:

- be well defined for every system and state;
- be invariant for any reversible adiabatic process and increase for any irreversible adiabatic process;
- be additive and sub-additive for all systems, subsystems and states;
- be non-negative, and vanish for all states described by classical mechanics;
- have one and only one state corresponding to the largest value of entropy;
- be such that graphs of entropy versus energy for stable equilibria are smooth and concave; and
- reduce to relations that have been established experimentally.

There is no completely satisfactory definition of entropy. To some degree, every definition is predicated on physical ignorance of the system it describes and, therefore, must rely on powerful ad-hoc assumptions to close the explanatory gap. These limit their scopes of validity. Let us review a few examples. The Boltzmann-Gibbs entropy assumes equal a priori probability either of phase space or ensemble space. While this is a reasonable assumption for simple equilibrium systems like the ideal gas and Lorentz gas, it is known to fail for large classes of systems, especially at disequilibrium; the molecular chaos ansatz is similarly suspect. It is not known what the necessary conditions are for ergodicity. The thermodynamic limit, which is presumed or necessary for most quantum and classical thermodynamic formalisms, on its face cannot be completely realistic, particularly since it ignores boundary conditions that are known to be pivotal for many thermodynamic behaviors. Extensivity, also presumed for most entropies, is ostensibly violated by systems that exhibit long-range order and fields. These include systems from nuclei up to the largest scale structures of the universe. Information entropies are hobbled by lack of general definitions of order, disorder and complexity. Finally, as it is deduced from thermodynamics,

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the notion of entropy is critically dependent of the presumed validity of the second law. Among the many foundational issues thwarting a general definition of physical entropy, none is more urgent than extending entropy into the non-equilibrium regime. After all, changes in the world are primarily irreversible nonequilibrium processes, but even the most basic nonequilibrium properties, like transport coefficients, cannot be reliably predicted in general.

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66.02 A Bird's Eye View

©|J.M.Powers, *Lecture Notes On Thermodynamics*, ..., ..., ...Ed., WEB - URL.|||

The first principle, among other things, has provided unification of the concepts of work and heat, showing they are both a mean to exchange energy. However, there was an important difference between work and heat: work is an exchange of energy related to the macroscopic Force|Torque exerted on the system, while heat is an energy exchange that can exist in the absence of any macroscopic Force|Torque. The second principle emphasizes and clarifies precisely this difference.

So, let us ask whether there is a way to directly relate the path functions, work and heat, to some other state functions, in addition to combining the two through the first law. In fact, we have two path functions, but so far, only one relation to a state function: a second one is needed.

For the quasi-static work caused by a volume change, we know that, although the work depends on path, division of the work by the pressure gives a change in a state function, the volume. In other words, the reciprocal pressure is an integrating factor that converts work into an exact differential.

We should look for a similar integrating factor for heat.

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The second law of thermodynamics. First is the definition of the entropy, which relates a path function to a new state function. For closed systems:

$$đ\mathcal{S} \equiv \frac{\delta \mathcal{Q}_R}{T} \qquad \text{with } \mathcal{Q}_R \text{ the heat receive on any reversible path} \quad .$$

The second part prescribes the observed limits on the directions of adiabatic processes in closed systems (i.e., it identifies those states that are accessible and inaccessible by adiabatic processes):

- $đ\mathcal{S} > 0$, for irreversible adiabatic changes of state;
- $đ\mathcal{S} = 0$, for reversible adiabatic changes of state;
- $đ\mathcal{S} < 0$, for changes of state that cannot be realized adiabatically.

The second part of the second law divides a state diagram into three parts (a line and two areas), as illustrated on the TS diagram in Figure 2.9. We emphasize that the second law (2.3.7) does not preclude the system entropy from decreasing; in fact, the entropy must decrease whenever heat is removed from a system during a work-free process. Let us compare the second law (2.3.7) with the first law (2.2.4): the first law asserts that energy is a conserved quantity, but in contrast, the second law asserts that entropy may not be conserved.
Entropy is created during irreversible processes.

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Conservation of mass and energy, by themselves, admit as possibilities phenomena which are not observed in nature: we need another axiom.
In fact, a variety of phenomena which may satisfy mass and energy conservation, but are not observed in nature. Some include:

- water running uphill without an external assist;
- CO_2 and H_2O reacting spontaneously to form CH_4 and O_2;
- air separating into its constituents spontaneously.

Let us summarize some more reasons for studying the second law:

- it predicts the direction in time of processes;
- it aids in determining equilibrium conditions;
- it allows one to determine peak performance of practical devices;
- it enables one to frame analysis of the factors which inhibit the realization of peak performance;
- it allows a rational definition of the absolute temperature scale;
- it has implications beyond engineering in physics, philosophy, economics, computer science, etc..

Second law of thermodynamics: The entropy of an isolated system can never decrease with time. This definition begs the question, what is entropy? Let us loosely define it here as a measure of the so-called randomness (or disorder) of a system, with high randomness corresponding to high entropy. Low randomness or low disorder often corresponds to low entropy.
Interpreted in another way, structure or order requires energy input to be realized, while over time, without continued maintenance, structure and order decay. The formulation of the second law we adopt will be robust enough to prevent us from predicting water to run uphill, methane to spontaneously form from carbon dioxide and water, or air to separate into its constituents. It will also be seen to be an important principle for predicting the optimal behavior of a wide variety of engineering devices. All that said, it should be noted that the equivalence of entropy with disorder, while useful and common, is likely not universal. Certainly Wright characterizes it as "a highly contentious opinion" and discusses counter-examples, especially as related to molecular level phenomena. Wright arguments are reinforced by Styer, who concludes that the notion of "entropy as disorder" be used only in conjunction with the notion of "entropy as freedom." Freedom here is to be interpreted as the ability to acquire a variety of states: if only one state is available, freedom is severely restricted, and entropy is low; if many states are available, freedom is widespread, and entropy is high. Both terms, "disorder" and "freedom" are shown by Styer to have alternate interpretations which render both imperfect metaphors for entropy.
Second law of thermodynamics: "Heat cannot, of itself, pass from a colder to a hotter body."
Second law of thermodynamics: It is impossible for any system to operate in a thermodynamic cycle and deliver a net amount of work to its surroundings while receiving an energy transfer by heat from a single thermal reservoir.
In informal language, the Kelvin-Planck statement says: you can turn all the work into heat, but you cannot turn all the heat into work.
We shall find it useful to have in hand definitions for so-called reversible and irreversible processes. Let us take

- Reversible process: A process in which it is possible to return both the system and surroundings to their original states.
- Irreversible process: a process in which it is impossible to return both the system and surroundings to their original states.

Now, it may be possible to restore the system to its original state but not the surroundings (or the surroundings to its original state but not the system). Such a process is irreversible. We shall often study reversible processes as they represent an ideal of the most we can ever hope to achieve. Some common engineering idealizations of reversible processes include frictionless motion and ideal inviscid flow. Now, everything in the real world deviates from the ideal.
The informal statement of the Kelvin-Planck version of the second law is that you cannot turn all the heat into work. Now, an engineer often wants to harvest as much as possible of the thermal energy of combustion and convert it into mechanical energy. Kelvin-Planck simply says we cannot

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have it all. But it lets us have some! In fact if we only harvest a portion of the thermal energy for work and reject the rest in the form of thermal energy, we can satisfy the second law.
Now, recall we previewed the idea of thermal efficiency, η:

$$\boxed{\eta \equiv \frac{\text{what you want}}{\text{what you pay for}}} \quad .$$

It is possible to prove the following corollary to the Kelvin-Planck statement:

$$\boxed{\eta_{\texttt{irreversible}} \leq \eta_{\texttt{reversible}}} \quad ,$$

for cycles operating between the same thermal reservoirs.
And there is a second corollary, applicable for two different cycles, both reversible, and both operating between the same thermal reservoirs:

$$\boxed{\eta_{\texttt{reversible;1}} = \eta_{\texttt{reversible;2}}} \quad .$$

The second corollary to the Kelvin-Planck statement holds that all reversible engines operating between the same thermal reservoirs have the same efficiency. This is independent of any details of the cycle or the materials involved. This implies that the thermal efficiency, should depend only on the character of the reservoirs involved *But, a thermal reservoir is uniquely defined by its temperature. -* Editor's note. Specifically, we will define η in terms of what we will call the temperature of the reservoir. This is the classical macroscopic interpretation of temperature. Later statistical theories give it the additional interpretation as a measure of the average translational kinetic energy of molecules of the system. But that is not our approach here! Now, we might suppose that this new thermodynamic property, temperature, should somehow be a measure of how much heat is transferred from one reservoir to another. Moreover, each reservoir will have its own temperature. The hot reservoir will have temperature $T_{\texttt{H}}$; the cold reservoir will have temperature $T_{\texttt{C}}$. So we are then saying that

$$\eta \equiv f[T_{\texttt{H}}, T_{\texttt{C}}]$$

As of yet, this functional form is unspecified. Substituting this form into our earlier equation we get

$$f[T_{\texttt{H}}, T_{\texttt{C}}] = 1 - \frac{|\mathcal{Q}_{\texttt{L}}|}{|\mathcal{Q}_{\texttt{H}}|} \quad .$$

This can only be true if $|\mathcal{Q}_{\texttt{L}}|$ and $|\mathcal{Q}_{\texttt{H}}|$ have some relation to $T_{\texttt{C}}$ and $T_{\texttt{H}}$. So let us propose a useful definition. We insist that our temperatures take the form of that for a Carnot cycle

$$\frac{T_{\texttt{H}}}{T_{\texttt{C}}} \equiv \frac{|\mathcal{Q}_{\texttt{L}}|}{|\mathcal{Q}_{\texttt{H}}|} \quad .$$

This is just a definition that cannot be argued. Its utility will be seen as its justification, but nothing more. It is valid only in the context of a Carnot cycle, and not for other cycles. Our logic train is that we observe heat engines, such as steam engines seen by Carnot in the early 1800s, doing work as a result of heat transfers. That effect, work, must have a cause. And we are going to assert that the cause is affiliated with a temperature difference.

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66.03 Conversion of Work Into Heat and Vice-Versa

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When two stones are rubbed together under water, the work done against the force of friction is transformed into internal energy tending to produce a rise of temperature of the stones. As soon as the temperature of the stones rises above that of the surrounding water, however, there is heating of the water. If the mass of water is large enough, then there will be no appreciable rise of temperature, and the water can be regarded as a heat reservoir, as discussed in Sec. 4.10.
Since the state of the stones is the same at the end of the process as at the beginning, the net result of the process is merely the conversion of mechanical work into heat. Similarly, when an electric current is maintained in a resistor immersed either in running water or in a very large mass of water, there is also a conversion of electric work into heat, without any change in the thermodynamic coordinates of the wire.
In general, work of any kind may be done on a system in contact with a reservoir, causing heat to leave the system without altering the state of the system. The system acts merely as an intermediary. It is apparent from the first law that the work is equal to the heat; in other words, the transformation of work into heat is accomplished with 100 percent efficiency. Moreover, this transformation can be continued indefinitely.
To study the opposite process, namely, the conversion of heat into work, we must also have at hand a process, or series of processes, by means of which such a conversion may continue indefinitely without involving any resulting changes in the state of the system. At first thought, it might appear that the isothermal expansion of an ideal gas might be a suitable process to consider in discussing the conversion of heat into work. In this case, there is no change of internal energy, since the temperature remains constant, and, therefore, heat has been converted completely into work. This process, however, involves a change of state of the gas. The volume increases and the pressure decreases until atmospheric pressure is reached, at which point the process stops. Therefore, the process of isothermal expansion cannot be used indefinitely.
Beware, the word heat is used in a sloppy way, as it would be more precise to use internal energy. - Editor's note.

66.04 Entropy and the Second Law of Thermodynamics

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Historically, several statements of the second principle of thermodynamics exist. Many of them are related to the functioning of thermal machines, as they date back to the beginning of the industrial era.

Here we use a more recent and more general approach, which immediately introduces an extensive and additive, but not conservative, state quantity, entropy.

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The usefulness of the concept of entropy and the Second Law depends on our ability to define entropy for a physical system in a calculable way. If the entropy of a reference or standard state is defined, then the entropy of an arbitrary state can be obtained through any reversible process that transforms the reference/standard state to the desired state. In practice δQ is measured with the knowledge of the **Heat-Capacity** .
In a real system, the transformation from the reference/standard state to the desired state occurs in a finite time and involves irreversible processes along the path. In this process, the entropy of the system, and hence of the universe, increases. **In classical thermodynamics it is assumed that every irreversible transformation that a system undergoes can also be achieved through a reversible transformation for which the equation defining the change of entropy is valid. In other words, it is assumed that every irreversible transformation that results in a certain change in the entropy of the system can be exactly reproduced through a reversible process in which the entropy change is solely due to the exchange of heat.** Since the change in entropy of the system depends only on the initial and final states, the change in entropy calculated using a reversible path will be equal to the entropy change produced by the irreversible processes.
However, it must be noted that a reversible transformation from an initial state to the final state may give the right value for the change in entropy of the system, but it leaves the entropy of the universe unchanged. In a reversible process, the change in entropy of the system is compensated by the opposite change in the entropy of the exterior, leaving the entropy of the universe unchanged. On the other hand, the naturally occurring irreversible transformation increases the entropy of the universe. Some authors restrict the above assumption to transformations between equilibrium states: this restriction excludes chemical reactions, in which the transformations are often from a non-equilibrium state to an equilibrium state.
...
In the classical formulation of entropy it is often stated that, along the irreversible path, the entropy may not be a function of the total energy and the total volume and hence it is not defined. However, for a large class of systems, the notion of local equilibrium makes entropy a well-defined quantity, even if it is not a function of the total energy and volume.
...
The basis of the modern approach is the notion of local equilibrium. For a very large class of systems that are not in thermodynamic equilibrium, thermodynamic quantities such as temperature, concentration, pressure and internal energy, remain well-defined concepts locally, that is, one could meaningfully formulate a thermodynamic description of a system in which intensive variables such as temperature and pressure are well defined in each elemental volume, and extensive variables such as entropy and internal energy are replaced by their corresponding densities. Thermodynamic variables can thus be functions of position and time. This is the assumption of local equilibrium. There are, of course, systems in which this assumption is not a good approx-

imation, but such systems are exceptional. In most hydrodynamic and chemical systems, local equilibrium is an excellent approximation. Modern computer simulations of molecular dynamics have shown that if initially the system is in such a state that temperature is not well defined, then in a very short time (few molecular collisions) the system relaxes to a state in which temperature is a well-defined quantity.

The modern approach bridges classical thermodynamics to the modern thermodynamics of irreversible processes.

The entropy of a system can vary:

- by external operations (transfers across the border of either heat or matter);
- by internal phenomena (creation of entropy).

For any closed system, an extensive, additive and not conserved function of the thermodynamic state exists, called entropy, $\mathcal{S} = \mathcal{S}[X, Y, T]$, such that, for every transformation between two states, (1) and (2), corresponding to times t_1 and t_2, with $t_2 > t_1$:

$$\boxed{\Delta\mathcal{S} = \mathcal{S}_i + \mathcal{S}_e \qquad \mathcal{S}_e \equiv \int_{t_1}^{t_2} \frac{\delta\,\mathcal{Q}}{T} \qquad \mathcal{S}_i \geq 0 \qquad \text{for each subsystem:} \qquad \Delta\mathcal{S}_i \geq 0} \quad ,$$

with the following meaning.

- The received entropy, $\mathcal{S}_e$, is the variation of entropy due to exchange of heat with the environment, which may be either positive or negative, and calculated by dividing each exchanged heat, $\delta\,\mathcal{Q}$, by the external temperature of the external heat reservoirs at the point of the closed surface delimiting the system where the heat enters/exits the system.
- The intensive state function T depending on the system temperature, measured as any empirical temperature, is called absolute thermodynamic temperature and it is always positive. It is a universal function of the empirical temperature of the system as recorded by measuring some arbitrary property. It constitutes a particular temperature scale, still to be defined. We can assume the system small enough that the temperature on the border can be considered uniform; otherwise, integration relates to both duration and the set of points forming the border. The thermodynamic temperature is given here without any other specification. It will be seen that it identifies with the temperature defined from the ideal gas equation [a].
- The entropy created inside the system, $\mathcal{S}_i$, is due to irreversible processes, with $\mathcal{S}_i \geq 0$ and $\mathcal{S}_i = 0$ if and only if all processes internal to the systems are reversible. The term has the same sign as the time interval and therefore it physically determines the arrow of time, makes it possible to qualify real phenomena as irreversible and provides a macroscopic interpretation of the concept of entropy directly related to time.
- It is also postulated that the internal entropy production in non-negative for every subsystem, $\Delta\mathcal{S}_i \geq 0$, so that it is not allowed that a negative entropy change is compensated by a larger positive change somewhere else.
- Whenever an infinitesimal piece of system is considered, it must be remembered that it has to be small, on a macroscopic scale, but large on the microscopic scale, in such a way that the properties of infinitesimal piece can be obtained by sensible averaging over its microscopic properties.

[a] Logical path. 1) Empirical arbitrary temperature. 2) Constant volume ideal gas thermometer, as a reference thermometer. 3) Ideal gas thermometer as an abstraction. 4) Discovery that absolute temperature is the same as the ideal gas temperature.

The equality $\mathcal{S}_i = 0$ corresponds to so-called reversible limit transformations, for which the meaning of the passage of time has no influence. In these unreal cases, for which time does not exist, the balance is reduced to:

$$\boxed{\Delta\mathcal{S} = \mathcal{S}_e \equiv \int_{t_1}^{t_2} \frac{\delta\,\mathcal{Q}_R}{T}} \quad .$$

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However, the interest of reversible evolutions is considerable, because they make it possible to carry out the calculations of entropy changes for any real transformation in a path-independent way. In fact, experimental observations have indicated that by dividing $\delta \mathcal{Q}_R$ by a uniform and continuous function $T[\theta]$ of an empirical temperature θ, called the absolute thermodynamic temperature, one obtains an integral which is independent of the path and may therefore be identified with a state function, entropy.

Note that, as for state functions, only differences of entropy are defined.

It is also worth emphasizing, that $\mathrm{d}\mathcal{S}$, is the exact differential of the entropy, while $\delta \mathcal{S}_i$ and $\delta \mathcal{S}_e$ are not exact differential; only for reversible transformations $\delta \mathcal{S}_e$ is an exact differential.

Some special cases.

- Isolated System (thermally Isolated System): $\Delta \mathcal{S} = \mathcal{S}_i \geq 0$. The principle of increase of entropy for Isolated Systems. Note that every system can be extended to a larger Isolated System by including in the definition of system all the surroundings interacting with the system.
- steady system: $0 = \Delta \mathcal{S} = \mathcal{S}_i + \mathcal{S}_e$. The entropy created is given off to the surroundings.

It is important to point-out that the inequality $\mathcal{S}_i \geq 0$ does not prevent that the entropy of open or closed systems may decrease. This happens for processes for which the change of $\mathcal{S}_e$ decreases with a larger rate, in absolute value than the rate $\mathcal{S}_i$ increases.

If the system is open the additional contribution due to matter exchange should be included.

$$\frac{\mathrm{d}\Delta \mathcal{S}}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{S}_e}{\mathrm{d}t} + \frac{\mathrm{d}\mathcal{S}_i}{\mathrm{d}t} = -\oiint_{\partial\Omega} \mathbf{J_s} \cdot \mathrm{d}\mathbf{S} + \iiint_{\Omega} \sigma_{\mathbf{s}} \cdot \mathrm{d}V \quad ,$$

with

- $\mathbf{J_s}$: entropy flux, that is entropy per unit time per unit area perpendicular to the direction of $\mathbf{J_s}$;
- $\sigma_{\mathbf{s}} \geq 0$: entropy produced per unit time per unit volume.

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Thermodynamics as formulated in the twentieth century, what we chose to call Modern Thermodynamics is a theory of irreversible processes in contrast to nineteenth century thermodynamics, which is a theory of states (thermostatics). In the latter, thermodynamics is confined to initial and final states, the processes that transform a state are not central to the theory; they are treated as another subject, kinetics . In Modern Thermodynamics, however, entropy-increasing irreversible processes are central to the theory. In it, we find additional tools, such as the rate of entropy production, to characterize and analyse non-equilibrium systems, and new results, such as the Onsager reciprocal relations, that are applicable to the thermoelectric effect, thermal diffusion and other cross-effects. The context in which thermodynamics is being taught is rapidly changing.

The First Principle considers all transformations as equally possible, provided energy is conserved. It does not take into account the irreversibility of a process and does not not say anything on the stability of equilibriums states.

The Second Principle defines the privileged sense in which the transformations can take place and specify the equilibrium conditions of the thermodynamic state of the system. It is an empirical law established by experimental observations. It is promoted to a postulate which can be approached in different ways and give place to various statements.

PREPRINT

66.04.01
Measuring the Entropy

Entropy cannot be measured directly. However, it can be measured indirectly from the **Heat-Capacity** . The zero temperature entropy is provided by the third law of thermodynamics.

66.04.02
Thermodynamic Equilibrium From Maximum Entropy

Read § 66.13.17, § 66.13.18.

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66.05 Reversibility and Irreversibility

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There is irreversibility if and only if there is production of entropy.

There are several causes of irreversibility, that is production of entropy.

The first of these causes is constituted by all of the viscous or solid friction forces, whose work is systematically transformed into internal energy or heat. The second is, in the absence of external field strength, the non-uniformity of the quantities intensive in the system, for example particle density, temperature or pressure.

A transformation between two states of thermodynamic equilibrium is said to be quasi-static if it occurs in such a way that the system can always be considered in thermodynamic equilibrium with the external environment during the whole transformation. In this case the macroscopic parameters of the system are always well-defined during the transformation, and temperature, pressure, chemical potential coincide with those of the external environment; moreover, they are uniform across the system, as all subsystems must be in equilibrium among each other as well.

A quasi-static transformation is reversible if it can be traced back in the reverse direction so that, at the end, the system status and the status of the whole universe are unchanged. A transformation is only reversible if it is quasi-static. A quasi-static transformation is reversible if dissipative effects are missing: any form of friction/viscosity, effects of hysteresis, diffusion, deformations, etc.

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66.06 Irreversibility of Natural Processes

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66.07 Local Equilibrium for Extended Systems

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The definitions given for internal (and total) energy and enthalpy, as well as for entropy and for any other thermodynamic state function, can be extended to states of non-equilibrium. In fact, as long as temperature changes, density changes, pressure changes and changes in any other intensive thermodynamic variables, are small enough in space and slow enough over time, these can be defined and measured locally, in space and in the time.

In this case all the intensive variables (such as temperature, pressure, density ....) become functions of position and time, while extensive variables (such as internal energy, parental and entropy) are replaced by density functions (either per volume unit or mass unit) which depend on position and time.

These states are called local thermodynamic equilibrium states.

For these states, entropy is defined as the volume integral of the entropies of individual infinitesimal parts, a priori dependent on time:

$$\mathcal{U}[t] \equiv \iiint \mathfrak{u}[\mathbf{x},t]\,\mathrm{d}V \qquad \mathcal{H}[t] \equiv \iiint \mathfrak{h}[\mathbf{x},t]\,\mathrm{d}V \qquad \mathcal{S}[t] \equiv \iiint \mathfrak{s}[\mathbf{x},t]\,\mathrm{d}V \quad .$$

The equation of state still applies, but locally, in space and time.

At the local thermodynamic equilibrium, all the intensive thermodynamic variables are homogeneous, and each arbitrary subset is in thermodynamic equilibrium with the adjacent parts.

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66.08 Entropy Outside Equilibrium

66.09 Fundamental Thermodynamic Relation - Gibbs Equation

Dal primo principio della termodinamica (65.05.06) e dal secondo principio della termodinamica (66.11.08) si ricava, per una trasformazione infinitesima reversibile di un sistema pVT:

$$\boxed{\mathrm{d}\mathcal{U} = T\,\mathrm{d}\mathcal{S} - p\,\mathrm{d}V} \quad . \tag{66.09.01}$$

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Essendo una relazione tra variabili di stato questa vale per una qualunque trasformazione tra due stati assegnati. Solo nel caso di trasformazione reversibile però si può interpretare $T\,\mathrm{d}\mathcal{S}$ come il calore assorbito dal sistema e $-p\,\mathrm{d}V$ come il lavoro fatto sul sistema. In generale infatti il calore e il lavoro dipendono dalla trasformazione e solo la somma del calore assorbito e del lavoro fatto ne è indipendente.

The identification of $-p\,\mathrm{d}V$ as the mechanical work and of $T\,\mathrm{d}\mathcal{S}$ as the heat transfer is valid only for reversible processes.

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Note that this equation was derived by assuming a reversible process. This equation can therefore be integrated for any reversible process, for during such a process the state of the substance can be identified at any point during the process. We also note that Eq. (66.09.01) deals only with state properties. Suppose we have an irreversible process taking place between the given initial and final states. The properties of a substance depend only on the state, and therefore the changes in the properties during a given change of state are the same for an irreversible process as for a reversible process. Therefore, Eq. (66.09.01) is often applied to an irreversible process between two given states, but the integration of Eq. (66.09.01) is performed along a reversible path between the same two states.

For isochoric/isobaric reversible processes:

$\mathrm{d}\mathcal{U} = \delta\mathcal{Q}_{\mathrm{R}} = C_{\mathrm{V}}\,\mathrm{d}T$ isochoric processes , $\mathrm{d}\mathcal{H} = \delta\mathcal{Q}_{\mathrm{R}} = C_{\mathrm{p}}\,\mathrm{d}T$ isobaric processes .

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66.10 Thermal Engines - Refrigerators - Heat-Pumps

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A thermal engine is a device which uses the heat taken from a high-temperature reservoir to let a thermodynamic system generate work while rejecting some heat into a cooler reservoir. The thermodynamic system repeatedly transforms in a thermodynamic cycle, that is any closed path with the final state coincident with the initial state.

For any system such that work is given by the generalized macroscopic coordinates (Y, X) the work is represented, in the generalized work plane, $Y - X$, by the area of the region surrounded by the path of the cycle. It is important to take into account the sign of the work, to define whether positive/negative work on the system is given by a path in the clockwise/counter-clockwise direction. For a pVT system, work on the system during a quasi-static transformation is $\delta\mathcal{W} = -p\,\mathrm{d}V$, that is, positive/negative for compression/expansion. Therefore, for a simple counter-clockwise path, the work on the system is positive. For a thermal engine to do work, therefore, the path must be clockwise.

A thermal engine working with only one thermal reservoir, that is, an engine producing work while extracting heat from only one thermal reservoir, cannot exist due to the second law, Kelvin-Planck statement.

On the contrary, it is possible to fully convert work into heat.

The next simplest thermal engine uses two thermal reservoirs at different temperatures. If the engine is a reversible one, it is a Carnot engine. By definition, a Carnot engine is a reversible engine operating between only two thermal reservoirs.

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Not Everything is a Heat Engine

It must be emphasized that the discussion applies to heat engines and not otherwise. In particular, consider an electrochemical fuel cell. If it's done right, the efficiency of such a thing vastly exceeds $1 - T_{\mathrm{C}}/T_{\mathrm{H}}$. That's OK, because it's not a heat engine. Even though it takes in fuel and puts out work, it is not a heat engine. As an even more familiar example, consider a plain old electrochemical battery driving an electric motor. If it's done right, the efficiency of such a thing vastly exceeds $1 - T_{\mathrm{C}}/T_{\mathrm{H}}$.

It is not worth the trouble to try to understand such things in terms of heat, or in terms of classical thermodynamics generally. I doubt it is even possible. It is However, straightforward to understand such things in terms of modern (post-1898) thermodynamics, i.e. statistical mechanics. Rather than getting bogged down trying to define heat, formalize everything in terms of energy and entropy instead. Entropy is defined in terms of probability, as a sum over states. In a battery, the state that stores the energy doesn't even have a temperature, and it isn't in equilibrium with your thermometer. Let us be clear: For a heat reservoir, its entropy is proportional to its energy, and the constant of proportionality is the inverse temperature. For a battery, that's just not true. You can stick a lot of energy into a battery without changing its temperature.

66.10.01
Efficiency and Coefficients of Performance

In general:

$$\text{efficiency} \equiv \frac{\text{(what you want)}}{\text{(what you pay for)}} \quad .$$

When working with two thermal reservoirs, there are three possibilities:

- one wants to produce work: thermal engine;
- one wants to cool something, by taking heat away: refrigerator;
- one wants to heat something, by injecting heat: heater (a heat-pump).

66.10.02
Efficiency of a Thermal Engine

Il rendimento di un ciclo di una macchina termica che opera tra due sorgenti di calore a temperatura $T_{\mathtt{H}}$ e $T_{\mathtt{C}}$ ($T_{\mathtt{C}} < T_{\mathtt{H}}$), è definito dalla relazione

$$\boxed{\eta \equiv \frac{|\mathcal{W}|}{|\mathcal{Q}_{\mathtt{H}}|} = 1 - \frac{|\mathcal{Q}_{\mathtt{C}}|}{|\mathcal{Q}_{\mathtt{H}}|} \leq 1} \quad , \tag{66.10.01}$$

dove $|\mathcal{Q}_{\mathtt{H}}| > 0$ è il calore assorbito dalla sorgente calda, $|\mathcal{Q}_{\mathtt{C}}| > 0$ è il calore ceduto alla sorgente fredda, mentre $|\mathcal{W}| > 0$ è il lavoro fatto dalla macchina termica.

Tale definizione è motivata dal fatto che in una macchina termica si cerca di massimizzare il lavoro ottenuto minimizzando il calore estratto alla sorgente calda.

Note that only two of the three $\mathcal{Q}_{\mathtt{C}}$, $\mathcal{Q}_{\mathtt{H}}$ and $\mathcal{W}$ are independent quantities.

Il massimo rendimento di una macchina termica che opera tra due sorgenti a temperature $T_{\mathtt{H}}$ e $T_{\mathtt{C}}$ ($T_{\mathtt{C}} < T_{\mathtt{H}}$) è quello di una macchina di Carnot ed è dato dal teorema di Carnot

$$\eta_{\mathtt{max}} = 1 - \frac{T_{\mathtt{C}}}{T_{\mathtt{H}}} \quad .$$

66.10.03
Coefficient of Performance of a Refrigerator

Il rendimento di un ciclo di una macchina frigorifera è definito dalla relazione

$$\boxed{\omega_{\mathtt{C}} \equiv \frac{|\mathcal{Q}_{\mathtt{C}}|}{|\mathcal{W}|} = \frac{|\mathcal{Q}_{\mathtt{H}}| - |\mathcal{W}|}{|\mathcal{W}|}} \quad , \tag{66.10.02}$$

dove $|\mathcal{Q}_{\mathtt{C}}| > 0$ è il calore assorbito dalla sorgente fredda, $|\mathcal{Q}_{\mathtt{H}}| > 0$ è il calore ceduto alla sorgente calda e $|\mathcal{W}| > 0$ il lavoro che occorre fare sul frigorifero. Tale definizione è basata sul fatto che in un frigorifero si cerca di minimizzare il lavoro che è necessario compiere per sottrarre una certa quantità di calore alla sorgente fredda (raffreddamento).

Note that only two of the three $\mathcal{Q}_{\mathtt{C}}$, $\mathcal{Q}_{\mathtt{H}}$ and $\mathcal{W}$ are independent.

Il massimo rendimento di una macchina frigorifera che opera tra due sorgenti di calore a temperature $T_{\mathtt{H}}$ e $T_{\mathtt{C}}$ ($T_{\mathtt{C}} < T_{\mathtt{H}}$) è quello di una macchina di Carnot ed è dato dal teorema di Carnot

$$\max\{\omega_{\mathtt{C}}\} = \frac{T_{\mathtt{C}}}{T_{\mathtt{H}} - T_{\mathtt{C}}} \quad .$$

Il rendimento di una macchina frigorifera può essere un numero maggiore o minore di uno.

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66.10.04
Coefficient of Performance of a Heat-Pump

Il rendimento di un ciclo di una pompa di calore è definito dalla relazione

$$\omega_{\mathtt{H}} \equiv \frac{|\mathcal{Q}_{\mathtt{H}}|}{|\mathcal{W}|} = \frac{|\mathcal{Q}_{\mathtt{C}}| + |\mathcal{W}|}{|\mathcal{W}|} \quad , \tag{66.10.03}$$

dove $|\mathcal{Q}_{\mathtt{C}}| > 0$ è il calore assorbito dalla sorgente fredda, $|\mathcal{Q}_{\mathtt{H}}| > 0$ è il calore ceduto alla sorgente calda e $|\mathcal{W}| > 0$ il lavoro che occorre fare sulla pompa di calore. Tale definizione è basata sul fatto che in una pomap di calore si cerca di minimizzare il lavoro che è necessario compiere per fornire una certa quantità di calore alla sorgente calda (riscaldamento).

Note that only two of the three $\mathcal{Q}_{\mathtt{C}}$, $\mathcal{Q}_{\mathtt{H}}$ and $\mathcal{W}$ are independent.

Il massimo rendimento di una pompa di calore che opera tra due sorgenti di calore a temperature $T_{\mathtt{H}}$ e $T_{\mathtt{C}}$ ($T_{\mathtt{C}} < T_{\mathtt{H}}$) è quello di una macchina di Carnot ed è dato dal teorema di Carnot

$$\max\{\omega_{\mathtt{H}}\} = \frac{T_{\mathtt{H}}}{T_{\mathtt{H}} - T_{\mathtt{C}}} \quad .$$

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66.11 Historical Statements of the Second Law - Kelvin-Planck and Clausius

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66.11.01 Carnot Cycle and Theorems

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What is the maximum thermal efficiency that can be achieved by an engine operating between reservoirs, inasmuch as efficiency 1 is not allowed by the second law? What are the characteristics of such an engine? Of what effect is the nature of the working substance?

66.11.01.01 Single-Reservoir Thermal Engine

A thermal engine working with a single reservoir does not exists thanks to Kelvin-Planck formulation of the first law. This can be formalized as:

$$\mathcal{W} \geq 0 \quad \Leftrightarrow \quad \mathcal{Q} \leq 0 \quad \Leftrightarrow \quad \frac{\mathcal{Q}}{T} \leq 0 \quad .$$

In preparation for the forthcoming generalization, for one heat reservoirs and arbitrary cycle, one has:

$$\frac{\mathcal{Q}_1}{T_1} \leq 0 \qquad \text{Kelvin-Planck statement} \quad .$$

66.11.01.02 Thermal Engine Operating Between Two Reservoirs

A reversible thermal engine that works between two heat reservoirs only is said a Carnot machine. The cycle of a Carnot machine necessarily consists of two isothermal and two adiabatic transformations.

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A Carnot engine operates between two reservoirs in a particularly simple way. All the absorbed heat enters the system at a constant high temperature, namely, that of the hotter reservoir. Also, all the rejected heat leaves the system at a constant low temperature, that of the cooler reservoir. Notice that the exchange of heat in the Carnot cycle is unlike the situation for idealized real engines, which need a series of reservoirs to exchange heat during the constant-volume or constant-pressure processes in order to achieve reversibility.
In the Carnot cycle, a single reservoir exchanges heat during a constant-temperature process. Since all four processes are reversible, the Carnot engine is a reversible engine.

A Carnot machine can be based on any kind of thermodynamic system, for instance a gas, a liquid, liquid-vapor in equilibrium. Carnot discovery that the reversible engine produces maximum work amounts to the statement that its efficiency is maximum. This efficiency is independent of the properties of the engine and is a function only of the temperatures of the hot and the cold reservoirs.

Examples of Carnot cycles include:

- a gas, not necessarily and ideal gas;
- any pure substance;
- a two-phase system, such as steam and water in equilibrium;
- an photon gas.

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A Carnot cycle consists of reversible processes only, and therefore it may be performed in either direction. When it is performed in a direction opposite to the one needed to produce work, the cycle is a refrigeration cycle.

Carnot Theorems

The Carnot engine has the maximum possible efficiency but it is totally useless; in fact the isothermal processes must take place quasi-statically, that is infinitely slowly in order for the working material to remain in thermal equilibrium with the reservoirs. Therefore, the work done per cycle is finite but the cycle duration is infinite so that the engine power is zero.

Carnot theorem states that: a Carnot engine operating between two reservoirs has an efficiency, η_{R}, larger or equal than any other engine operating between the same two reservoirs, η_{I},

$$\boxed{\eta_{\mathrm{I}} \leq \eta_{\mathrm{R}}} \quad .$$

Imagine a Carnot engine, R, and any other engine, I, which might be irreversible, working between the same two reservoirs and adjusted so that they both extract the same heat from the hot reservoir. Assume, by absurd, that $\eta_{\mathrm{I}} > \eta_{\mathrm{R}}$: the work produced by the engine I is therefore larger than the work produced by engine R. Now, let the engine I drive the Carnot engine R backward as a Carnot refrigerator. The engine and the refrigerator coupled together in this way constitute a self-contained machine, since all the work needed to operate the refrigerator is supplied by the engine. Actually, not all the work produced by the machine I is needed to run the Carnot engine backward, and the remaining work is free for use. Moreover, the hot reservoir is actually irrelevant, as it gives the same heat it receives. Therefore the coupled machine produces a net work by only extracting heat from the single cold reservoir, this violating Kelvin-Planck statement and the hypothesis is false. Therefore:

$$\eta_{\mathrm{I}} \leq \eta_{\mathrm{R}} \quad .$$

Carnot corollary to Carnot theorem states that: all Carnot engines operating between two reservoirs have the same efficiency,

$$\boxed{\eta_{\mathrm{R1}} = \eta_{\mathrm{R2}}} \quad .$$

In fact, it is sufficient to do twice the reasoning used to demonstrate the theorem, by reverting the two reversible engines in turn, to obtain the result.

Carnot theorems translate into:

$$\boxed{\frac{\mathcal{Q}_{\mathrm{H}}}{T_{\mathrm{H}}} + \frac{\mathcal{Q}_{\mathrm{C}}}{T_{\mathrm{C}}} \leq 0} \quad ,$$

with the equality that applies to reversible Carnot engines only.

In preparation for the forthcoming generalization, for two heat reservoirs and a Carnot cycle, one has:

$$\boxed{\frac{\mathcal{Q}_1}{T_1} + \frac{\mathcal{Q}_2}{T_2} \leq 0} \quad .$$

66.11.01.03 Thermal Engine Operating With Many Reservoirs (Discrete Set)

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It can be shown that:

$$\boxed{\sum_i \frac{\mathcal{Q}_i}{T_i} \leq 0} \quad .$$

WORK-IN-PROGRESS

66.11.01.04 Thermal Engine Operating With Many Reservoirs (Continuous Set)

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©|ABejan|arg2|*arg3*|

The final step in this line of generalization is to consider the possibility of a continuous variation of system boundary temperature as the cycle is executed in contact with an infinite sequence of temperature reservoirs, each contributing a heat interaction of size $\delta\,\mathcal{Q}$.

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If we are given an arbitrary reversible process, between two states, it is always possible to find a reversible zig-zag path process between the same two states, consisting of an adiabatic process followed by an isothermal process followed by an adiabatic process, such that the heat transferred during the isothermal segment is the same as that transferred during the original process and the work done is the same as that of the original process.

It can be shown that:

$$\oint \frac{\delta\,\mathcal{Q}}{T} \leq 0 \quad .$$

66.11.02 Reversible Part of the Second Law and Entropy

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|§ 8.1, 8.2, 8.5, 8.6||

For every reversible cyclic process during which a system reversibly absorbs the amount of heat $đ\mathcal{Q}_R$ from the environment via temperature sources with temperature T:

$$\oint \frac{\delta\,\mathcal{Q}_R}{T} = 0 \quad . \tag{66.11.01}$$

→ 2641

It follows that there exists a function of the thermodynamic coordinates of a system whose value at the final state minus its value at the initial state equals

$$\int_1^2 \frac{\delta\,\mathcal{Q}_R}{T} \quad .$$

Si può allora introdurre la funzione di stato entropia come segue. La variazione di entropia tra due stati (1) e (2) di un sistema è definita da

$$\Delta\mathcal{S}_{1\to 2} \equiv \int_1^2 \frac{\delta\,\mathcal{Q}_R}{T} \quad , \tag{66.11.02}$$

dove l'integrale va valutato lungo una qualunque trasformazione reversibile che connette i due stati.

It must be emphasized that: any process is isoentropic if and only if it is both adiabatic and reversible.

Entropy is an additive quantity.

WORK-IN-PROGRESS

66.11.02.01 TS Diagram

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66.11.03
Irreversible Part of the Second Law

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Per ogni processo ciclico reversibile durante il quale un sistema assorbe le quantità di calore δQ_R dall'ambiente tramite sorgenti a temperatura T si ha

$$\oint \frac{\delta Q_R}{T} = 0 \quad .$$

Per ogni processo ciclico irreversibile durante il quale un sistema assorbe le quantità di calore δQ dall'ambiente tramite sorgenti a temperatura T si ha

$$_I\oint \frac{\delta Q}{T} < 0 \quad . \tag{66.11.03}$$

→ 2641

This notation of equation (66.11.03) is a symbolic one, since an irreversible path is not a succession of states of equilibrium and cannot be indicated on, say, a pV diagram.

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Per ogni processo ciclico durante il quale un sistema assorbe le quantità di calore δQ dall'ambiente tramite sorgenti a temperatura T si ha

$$\oint \frac{\delta Q}{T} \leq 0 \quad , \tag{66.11.04}$$

→ 2641 2642

dove il segno di uguaglianza vale se e solo se la trasformazione è reversibile. In equation (66.11.04):

- for a reversible transformation all the quantities refer to the system (and are equal to those of the surrounding) and the equality sign applies;
- for an irreversible cycle all the quantities refer to the auxiliary reservoirs and the inequality sign applies.

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In order to avoid misunderstandings, as to the meaning of equation (66.11.01), the equality for reversible cycles, and equation (66.11.03), the inequality for irreversible cycles, we must point out that T represents the temperature of the reservoir or auxiliary reversible engine that provides the small quantity of heat δQ, and is not necessarily equal to the temperature T' of the system (or part of the system) that receives the heat δQ.
Indeed, if the cycle is irreversible, then $T' < T$ when δQ is positive, because heat cannot flow spontaneously from a cooler to a hotter body. Similarly, $T' > T$ when δQ is negative.
On the other hand, if the cycle is reversible, then we must always have $T = T'$, because an exchange of heat between two bodies at different temperatures is not reversible. In equation (66.11.01), we may take T to be the temperature of the reservoir or auxiliary surroundings and also the temperature of the part of the system that absorbs the heat δQ. In equation (66.11.01), all quantities refer to the system itself, and the integral is evaluated for a closed path taken by the reversible

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system. In equation (66.11.04), everything refers to the auxiliary reservoirs, and the integral is evaluated for a closed range of reservoir temperatures through which, incidentally, the irreversible system is brought back to its initial state.

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Note that reversible isoentropic implies adiabatic, and isoentropic adiabatic implies reversible. But, irreversible isoentropic is not adiabatic $\delta Q < 0$ and implies that heat flows out of the system. Of course, if the process is irreversible adiabatic, then it is not isoentropic (T not constant) and implies that the temperature increases. Finally, if the process is isoentropic adiabatic, then it is not irreversible $\delta \mathcal{Q}_R = 0$ and implies that the process is reversible.

La diseguaglianza di Clausius si può riscrivere, in termini della variazione di entropia, come

$$\Delta \mathcal{S}_{1\to 2} \geq \int_1^2 \frac{\delta \mathcal{Q}}{T} \quad . \tag{66.11.05}$$

→ 2642

The Second Law states that the change of the entropy of a system is always greater or equal to the entropy supplied to a system in heat form.

Questa fornisce il valore minimo della variazione di entropia di un sistema per una trasformazione qualunque tra due stati di equilibrio (1) e (2), $\Delta \mathcal{S}_{1\to 2}$. La temperatura che appare nella diseguaglianza è la temperatura delle sorgenti con cui il sistema scambia le quantità di calore $\delta \mathcal{Q}$, atrraveros una parte della supercie chiusa che racchiude il sistema.

Equation (66.11.05) shows that, in an irreversible process, the entropy absorbed from the surroundings is less than the increase of entropy of the system: some entropy is created by the irreversible process. Therefore, the entropy of an isolated system is always increased by the occurrence of an irreversible process. Or, the total entropy of the universe is always increased when an irreversible process takes place.

Dalla diseguaglianza di Clausius (66.11.05) si può riscrivere la variazione di entropia di un sistema in una trasformazione generica tra due stati di equilibrio (1) e (2), come

$$\Delta \mathcal{S}_{1\to 2} = \int_1^2 \frac{\delta \mathcal{Q}}{T} + \Delta \mathcal{S}_0 \quad , \tag{66.11.06}$$

che esprime la variazione di entropia come una parte dovuta allo scambio di calore con l'ambiente esterno ed una parte, $\Delta \mathcal{S}_0$, che ne è indipendente con

$$\Delta \mathcal{S}_0 \geq 0 \quad . \tag{66.11.07}$$

Si osservi che, benché il calore non sia una funzione di stato, il calore diviso la temperatura termodinamica assoluta a cui viene scambiato in una trasformazione reversibile è una funzione di stato. Di conseguenza la capacità termica divisa per la temperatura termodinamica assoluta a cui viene scambiato il calore è una funzione di stato e può perciò essere espressa in termini di funzioni di stato nella relazione (**??**).

66.11.03.01 Principle of Increase of Entropy

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|§ 8.11||

Per una trasformazione reversibile:

$$\boxed{\delta \mathcal{Q}_R = T \, \mathrm{d}\mathcal{S}} \quad . \tag{66.11.08}$$

→ 2634

Ne segue che l'entropia di un sistema termicamente isolato non può diminuire. Quindi l'equilibrio termodinamico di un sistema isolato è caratterizzato da entropia massima.

Il caso in cui l'entropia di un sistema termicamente isolato abbia un massimo relativo ma non assoluto in una certa configurazione è detto stato metastabile. Lo stato è stabile per piccole variazioni della configurazione ma la stabilità non è assoluta.

LET'S INTERROGATE

Un sistema termicamente isolato può esistere in uno stato di quasi-equilibrio quando il sistema non è in equilibrio termodinamico ma la trasformazione di evoluzione verso lo stato di equilibrio termodinamico è così lenta che il sistema appare in equilibrio nella scala dei tempi considerata.

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The entropy change of the universe was found to be positive for each of the irreversible processes treated so far. We are led to believe, therefore, that whenever an irreversible process occurs, the entropy of the universe increases. To establish this proposition, known as the entropy principle, in a general manner, it is sufficient to confine our attention to adiabatic processes only, since we have already seen that the entropy principle is true for all processes involving the irreversible transfer of heat.

66.11.04
Thermodynamic Temperature and Absolute Zero

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|§ 7.5, 7.6, 7.7||

Read § 66.13.29 for the calculation of the efficiency of A Carnot engine by using a ideal gas.

The efficiency of a Carnot engine only depends od the temperature of the two reservoirs and therefore, in general:

$$\frac{\mathcal{Q}_H}{\mathcal{Q}_C} = \psi[T_H, T_C] \implies \psi[T_H, T_C]\psi[T_C, T_H] = 1 \quad .$$

Given three thermal reservoirs at temperatures, T_1, T_2, T_3, with $T_1 < T_2 < T_3$ and three Carnot engine operating between the three pairs of reservoirs, one has:

$$\frac{\mathcal{Q}_1}{\mathcal{Q}_2} = \psi[T_1, T_2] \quad ,$$

$$\frac{\mathcal{Q}_2}{\mathcal{Q}_3} = \psi[T_2, T_3] \quad ,$$

$$\frac{\mathcal{Q}_1}{\mathcal{Q}_3} = \psi[T_1, T_3] \quad ,$$

$$\frac{\mathcal{Q}_1}{\mathcal{Q}_3} = \frac{\mathcal{Q}_1\mathcal{Q}_2}{\mathcal{Q}_3\mathcal{Q}_2} = \psi[T_1, T_3] = \psi[T_1, T_2]\psi[T_2, T_3] \quad ,$$

$$\psi[T_1, T_2] = \frac{\psi[T_1, T_3]}{\psi[T_2, T_3]} \quad ,$$

$$\psi[T_1, T_2] = \frac{\phi[T_1]}{\phi[T_2]} \quad .$$

WORK IN PROGRESS

66.11.04.01 Absolute Zero and Carnot Efficiency

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|§ 7.8||

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If any system undergoes a reversible isothermal process without transfer of heat, the temperature at which this process takes place is called absolute zero. In other words, at absolute zero, an isotherm and an adiabatic are identical. It should be noticed that the definition of absolute zero holds for all substances and is, therefore, independent of the specific properties of any arbitrarily chosen substance. Furthermore, the definition is in terms of purely macroscopic concepts. No reference is made to atoms or molecules. Whether absolute zero may be achieved is a question that is left to experiment.

...

For a Carnot engine to have an efficiency of 1, it is clear that the temperature of the cold reservoirs must be zero. Only when the lower reservoir is at absolute zero will all the heat be converted into work. Since nature does not provide us with a reservoir at absolute zero, a heat engine with efficiency 1 is a practical impossibility.

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66.12 Frome Mechanics to Thermodynamics

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66.13 Examples and Physical Applications

66.13.01 Identification of the Absolute Thermodynamic Temperature With the Ideal Gas One

Ideal gas:

$$pV = nRT^\star \qquad \mathcal{U} = \mathcal{U}[T^\star] \quad ,$$

where $T^\star$ is by definition the ideal gas temperature. The ideal gas temperature has been used so far, but called simply T, anticipating what will be shown here, that it is coincident with the thermodynamic absolute temperature.

Reversible transformation:

$$\begin{aligned}
T &= T[T^\star] \quad , \\
\mathrm{d}\mathcal{U} = T\,\mathrm{d}\mathcal{S} - \frac{nRT^\star}{V}\,\mathrm{d}V &= C_{\mathrm{V}}[T^\star]\,\mathrm{d}T^\star \quad , \\
\mathrm{d}\mathcal{S} &= \frac{nRT^\star}{VT}\,\mathrm{d}V + \frac{C_{\mathrm{V}}[T^\star]}{T}\,\mathrm{d}T^\star \quad , \\
\Longrightarrow\ 0 = \left(\frac{\partial}{\partial V}\right)_{T^\star} \frac{C_{\mathrm{V}}[T^\star]}{T} &= \left(\frac{\partial}{\partial T^\star}\right)_{V} \frac{nRT^\star}{VT} \quad , \\
\Longrightarrow\ T \propto T^\star \qquad \text{and we can choose:} \qquad & \boxed{T = T^\star} \quad .
\end{aligned}$$

66.13.02 Internal Energy and Entropy of Ideal Gases

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Equazione di stato di un gas ideale:

$$\boxed{pV = Nk_{\mathrm{B}}T = nRT = \frac{m}{\mathcal{M}_0}RT} \quad .$$

Energia di un gas ideale:

$$\boxed{\left(\frac{\partial \mathcal{U}}{\partial V}\right)_T = \left(\frac{\partial \mathcal{U}}{\partial p}\right)_T = 0 \Longrightarrow \mathcal{U} = \mathcal{U}[T] \Longrightarrow \left\{c_{\mathrm{V}} = c_{\mathrm{V}}[T] \qquad c_{\mathrm{p}} = c_{\mathrm{p}}[T]\right\}} \quad .$$

La variazione di energia interna di un gas ideale è data dalla relazione

$$\boxed{\Delta\mathcal{U} = n\int_{T_0}^{T} c_{\mathrm{V}}[T]\,\mathrm{d}T = nc_{\mathrm{V}}[\widetilde{T}]\,(T - T_0) \qquad \text{for a suitable } \widetilde{T} \text{ such that } T_0 \le \widetilde{T} \le T} \quad . \tag{66.13.01}$$

If c_{V} does not depend on temperature:

$$\boxed{\Delta\mathcal{U} = nc_{\mathrm{V}}\,(T - T_0)} \quad . \tag{66.13.02}$$

La variazione di entropia di un gas ideale vale: vale:

$$\boxed{\Delta\mathcal{S} = nc_{\mathrm{V}}[\widetilde{T}]\log\left[\frac{T}{T_0}\right] + nR\log\left[\frac{V}{V_0}\right] = nc_{\mathrm{p}}[\widetilde{T}]\log\left[\frac{T}{T_0}\right] - nR\log\left[\frac{p}{p_0}\right] \qquad \text{for a } \widetilde{T} \text{ such that } T_0 \le \widetilde{T} \le T} \quad . \tag{66.13.03}$$

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66.13.03
Internal Energy and Entropy of Incompressible Solid or Liquid

©|J.M.Powers, *Lecture Notes On Thermodynamics*, ..., ..., ...Ed., WEB - URL.|par 8.5||

Note that under most conditions encountered by in the laboratory, the internal energy can be regarded as a function of T alone for all substances.

$$V[p,T] = V_0 = \text{constant} \qquad \mathrm{d}V = 0 \quad ,$$
$$\mathcal{U} = \mathcal{U}[T] \quad ,$$
$$\mathrm{d}\mathcal{U} = C_{\mathsf{V}}[T]\,\mathrm{d}T \quad ,$$
$$\mathrm{d}\mathcal{S} = \frac{C_{\mathsf{V}}[T]}{T}\,\mathrm{d}T \quad .$$

Note the expression is the same as for a ideal gas at costt volume, because in both cases the internla enegy does not depedn on volume.

66.13.04
Compressione Reversibile E Non Di Un Gas Ideale

©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|2.12||

Un kg-mole ($= 1{\cdot}10^3$ mol) di gas ideale viene compressa da una pressione $p_0 = 1$ atm alla pressione $p = 10$ atm in un sistema cilindro-pistone in modo che la temperatura iniziale e finale del gas sia la stessa, pari a $T = 127°\mathrm{C}$. Calcolare la variazione di entropia del gas, dell'ambiente esterno e la variazione totale di entropia dell'universo se il processo è meccanicamente reversibile e l'ambiente esterno è costituito da una sorgente di calore a temperatura T; se il processo è meccanicamente reversibile e l'ambiente esterno è costituito da una sorgente di calore a temperatura $T' = 27°\mathrm{C}$; se il processo è meccanicamente irreversibile e richiede $\alpha = 20\%$ di lavoro in più rispetto alla compressione reversibile e l'esterno è costituito da una sorgente di calore a temperatura T'.

WORKED SOLUTION

La variazione di entropia di un gas ideale le cui capacità termiche sono indipendenti dalla temperatura (nell'intervallo di temperatura di interesse) vale

$$\Delta\mathcal{S} = nc_{\mathsf{V}} \log\left[\frac{T_2}{T_1}\right] + nR\log\left[\frac{V_2}{V_1}\right]$$

Ne segue per il gas

$$\Delta\mathcal{S}_g = nR\log\left[\frac{V_2}{V_1}\right] = nR\log\left[\frac{p_1}{p_2}\right] = -19.1{\cdot}10^3\ \mathrm{J/kg}$$

Il gas cede calore alla sorgente esterna alla stessa temperatura del gas. La quantità di calore ricevuta dal gas vale

$$\mathcal{Q} = \Delta\mathcal{U} + \mathcal{W}$$

ma essendo la temperatura finale e iniziale del gas ideale uguali fra loro si ha $\Delta\mathcal{U} = 0$ e quindi

$$\mathcal{Q} = \mathcal{W} = nRT\log\left[\frac{V_2}{V_1}\right]$$

La variazione di entropia della sorgente vale dunque

$$\Delta\mathcal{S}_s = \frac{-Q}{T} = -nR\log\left[\frac{V_2}{V_1}\right] = 19.1{\cdot}10^3\ \mathrm{J/kg}$$

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$$\Delta \mathcal{S}_g + \Delta \mathcal{S}_s = 0$$

come deve essendo il processo reversibile sia dal punto di vista meccanico che termico.
Nel caso in cui la sorgente esterna sia a temperatura diversa ma la compressione ancora meccanicamente reversibile la variazione di entropia del gas non cambia essendo gli stati iniziale e finale del gas gli stessi che nel caso precedente. La stessa quantità di calore però è ora ceduta ad una sorgente a temperatura T' la cui variazione di entropia vale dunque

$$\Delta \mathcal{S}'_s = \frac{-Q}{T'} = -nR\frac{T}{T'}\log\left[\frac{V_2}{V_1}\right] = 25.5{\cdot}10^3 \text{ J/kg}$$

Ne segue

$$\Delta \mathcal{S}'_g + \Delta \mathcal{S}'_s = \Delta \mathcal{S}_g + \Delta \mathcal{S}'_s = 6.38{\cdot}10^3 \text{ J/kg} > 0 \qquad (66.13.04)$$

→ 2648

In tale caso la variazione totale di entropia è positiva essendo lo scambio di calore con il serbatoio esterno irreversibile.
La diseguaglianza di Clausius può essere applicata al sistema gas ottenendo

$$\Delta \mathcal{S}_g = nR\log\left[\frac{V_2}{V_1}\right] \geq nR\frac{T}{T'}\log\left[\frac{V_2}{V_1}\right] = \frac{\mathcal{Q}}{T'} = -\Delta \mathcal{S}'_s$$

che è soddisfatta essendo di fatto equivalente alla (66.13.04). Si noti che in questo caso la variazione di entropia del gas è sempre data da $\mathcal{Q}/T$.
Nel caso la compressione meccanica sia irreversibile e richieda $\alpha = 20\%$ di lavoro in più che il caso reversibile la quantità di calore che il gas cede alla sorgente di calore vale

$$\mathcal{Q} = \mathcal{W}(1+\alpha)$$

La variazione di entropia del gas è la stessa calcolata nei due casi precedenti in quanto lo stato iniziale e finale del gas sono gli stessi. La variazione di entropia della sorgente vale però ora

$$\Delta \mathcal{S}'_s = \frac{-\mathcal{Q}}{T'} = -nR\frac{T}{T'}(1+\alpha)\log\left[\frac{V_2}{V_1}\right] = 30.6{\cdot}10^3 \text{ J/kg}$$

e la variazione totale di entropia dell'universo vale

$$\Delta \mathcal{S}''_g + \Delta \mathcal{S}''_s = \Delta \mathcal{S}_g + \Delta \mathcal{S}''_s = 11.5{\cdot}10^3 \text{ J/kg} > 0 \qquad (66.13.05)$$

→ 2648

Si noti in questo caso che la variazione di entropia del gas non è data da $\mathcal{Q}/T$ perché il processo è meccanicamente irreversibile e ciò significa che è internamente irreversibile. La sorgente termica non subisce invece alcuna irreversibilità interna. L'irreversibilità connessa con lo scambio termico è infatti esterna sia al gas che al serbatoio di temperatura. Calcolando $\mathcal{Q}/T$ per il gas si ottiene

$$\frac{\mathcal{Q}}{T} = -23.0{\cdot}10^3 \text{ J/kg}$$

che rappresenta la variazione di entropia del sistema causata dallo scambio termico. Tuttavia questa non è la variazione totale di entropia del sistema gas in quanto questo subisce anche un aumento di entropia dovuto alla irreversibilità meccanica interna, che non sappiamo calcolare direttamente e che parzialmente compensa la diminuzione di entropia dovuta alla cessione di calore ed è tale che la variazione totale di entropia è quella calcolata sopra utilizzando il fatto che l'entropia è una funzione di stato. Anche in questo caso si può applicare la diseguaglianza di Clausius al sistema gas ottenendo

$$\Delta \mathcal{S}_g = nR\log\left[\frac{V_2}{V_1}\right] \geq nR\frac{T}{T'}(1+\alpha)\log\left[\frac{V_2}{V_1}\right] = \frac{\mathcal{Q}}{T'} = -\Delta \mathcal{S}''_s$$

che è soddisfatta essendo di fatto equivalente alla (66.13.05).
Il grado di irreversibilità dei processi è misurato dal valore di $\Delta \mathcal{S}_{\mathtt{tot}}$ ed è maggiore nel terzo caso rispetto la secondo in quanto nel terzo caso esiste, oltre alla fonte di irreversibilità del secondo processo, una fonte di irreversibilità addizionale.

66.13.05
Unit of Entropy - Entropy Generator

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It is instructive to calculate a unit of entropy, one joule per kelvin, in order to gain a feeling for this new variable. Consider the Joule paddle wheel apparatus. The system is one kilogram of water at room temperature, T. The surroundings are the adiabatic cylindrical wall and top, the diathermic bottom in contact with a heat reservoir, also at temperature T, and the paddle wheels. A slowly falling mass $m = 29.9\ \mathrm{kg}$ causes the paddle wheels to turn, so the portion of the boundary formed by the paddle wheels moves. The falling mass does work on the system, which tends to experience an increase in the temperature of the water. However, the diathermic bottom prevents the temperature from rising by removing energy from the system in the form of heat. What is the change of entropy of the water and reservoir?

If the system is water: entropy is produced by converting mechanical energy into heat given to the heat reservoir. The system water does not change its status.

If the system includes the Joule paddle wheel apparatus and the heat reservoir and anything else involved, then the system is adiabatically isolated and it is an entropy generator.

66.13.06
Thermalisation of Two Blocks

66.13.07
Mono-Thermal Expansion of a Ideal Gas

©|J.P.Pérez *et al.*, *Thermodynamique*, ..., DUNOD, ...Ed., WEB - URL.|IV.2.a, IV.2.b||

A ideal gas in thermal contact with a thermal reservoir at temperature T_0 changes its volume from V_1 to V_2, moving from one equilibrium state to another one. Except for the contact with the thermal reservoir, the gas is surrounded by adiabatic walls.

Whatever the process, the entropy change of the gas is determined by the initial and final states.

$$\Delta\mathcal{U} = 0 = \mathcal{W} + \mathcal{Q} \equiv \mathcal{W} + (\mathcal{Q}_{\mathtt{cnd}} + \mathcal{Q}_{\mathtt{cnv}} + \mathcal{Q}_{\mathtt{rad}}) \quad ,$$

$$0 \leq \Delta\mathcal{S}_{\mathtt{U}} = \Delta\mathcal{S} + \Delta\mathcal{S}_{\mathtt{T}} = nR\log\left[\frac{V_2}{V_1}\right] + \frac{-\mathcal{Q}}{T_0} \quad ,$$

$$\Longrightarrow \mathcal{Q} \leq nRT_0\log\left[\frac{V_2}{V_1}\right] \Longrightarrow (-\mathcal{W}) \leq nRT_0\log\left[\frac{V_2}{V_1}\right] \quad .$$

Maximum work (and heat) happens for a reversible transformation. In this case, and only in this case, work can be calculated from the properties of the gas:

$$-\mathcal{Q} = \mathcal{W} = -nRT_0\log\left[\frac{V_2}{V_1}\right] \Longrightarrow \Delta\mathcal{S}_{\mathtt{U}} = 0 \quad .$$

IN-PROGRESS

66.13.08
Entropy of Mixing Between Two Ideal Gases at the Same Temperature

©|J.P.Pérez *et al.*, *Thermodynamique*, ..., DUNOD, ...Ed., WEB - URL.|V.2.b||

66.13.09
Entropy Production by a Ohmic Conductor With Steady Current

©|J.P.Pérez *et al.*, *Thermodynamique*, ..., DUNOD, ...Ed., WEB - URL.|V.2.g||

Consider a resistor R where a steady current I flows and heat transfers energy to the atmosphere, considered as a thermal reservoir at temperature T_0.

$$\Delta\mathcal{S} = 0 \qquad \text{steady state} \Longrightarrow \text{the thermodynamic state of the resistor does not change} \quad ,$$

$$\Delta\mathcal{S}_{\mathrm{T}} = \frac{RI^2\Delta t}{T_0} \quad ,$$

$$\Delta\mathcal{S}_{\mathrm{U}} = 0 + \frac{RI^2\Delta t}{T_0} \geq 0 \tag{66.13.06}$$

The system resistor is a entropy producer: entropy is produced by converting electric energy into heat given to the heat reservoir atmosphere. The system resistor does not change its status.

Note that, the ambient temperature is different (smaller) than the temperature of the resistor.

Show that, in case of two resistors in parallel, instead of one, the current is divided between the resistors in such a way that the entropy production is minimum, for a fixed I from the battery.

66.13.10
Variazione Di Entropia Dell'acqua Che Congela

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Una quantità di acqua di massa $m = 1$ kg congela mentre è alla temperatura di $T = 0$°C.

1. Determinare la variazione di entropia dell'acqua durante questo processo.

WORKED SOLUTION

1. $\Delta\mathcal{S} = -2.9 \cdot 10^2 cal/$ K.

66.13.11
Il Cubetto Di Ghiaccio Nel Lago

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Un cubetto di ghiaccio di massa $m = 12.6$ g e temperatura -10°C viene messo entro un lago la cui temperatura è $+15$°C. Calcolare la variazione di entropia dell'universo dopo che si è ristabilito l'equilibrio termico.

WORKED SOLUTION

1. $+0.95$ Joule/ K.

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66.13.12
Variazione Di Entropia per La Pioggia Che Cade in Un Lago

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Una goccia di pioggia di massa $m = 1{\cdot}10^{-3}$ kg cade entro un lago da un'altezza $h = 100$ m mischiandosi con l'acqua del lago e raggiungendo l'equilibrio termodinamico. Calcolare la variazione di entropia associata a tale processo se il lago l'atmosfera e la goccia di pioggia si trovano inizialmente tutti alla stessa temperatura $T = 20$ °C.

66.13.13
Produzione Di Entropia Di Un Paracadutista

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Un paracadutista di massa $m = 80$ kg scende ad una velocità costante di $v = 5$ ms.

1. Supponendo che il paracadutista e l'aria circostante si trovino alla temperatura costante di $T = 20$°C stimare l'aumento di entropia per unità di tempo.

WORKED SOLUTION

1. $\Delta\mathcal{S}/\Delta t = mgv/T = 13$ J/ K.

66.13.14
Niagara Falls

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©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|1.14||

At Niagara Falls, 5700 m^3/s of water falls through a vertical distance of $50m$, dissipating all of its gravitational energy. Calculate the rate of increase of entropy contributed by this falling water if the temperature of the environment is $T = 20$ °C.

66.13.15
Produzione Di Entropia per Il Riscaldamento Domestico

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In una giornata invernale calore viene ceduto all'ambiente esterno da un appartamento tramite le pareti ad un ritmo di $2.5{\cdot}10^4 calor/$ ora. La temperatura interna alla casa è $T_{\text{I}} = 21$ °C e la temperatura esterna è $T_e = -5$ °C.

1. Determinare la produzione di entropia per unità di tempo causata da questo processo.

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66.13.16
Maximum Entropy for Isolated Systems

66.13.17
Entropy Maxima

This § is referenced at pages:
[2630]

A rigid adiabatic cylindrical container has a moving diathermic wall dividing its volume into two parts of volume V_1 and V_2 with $V_0 = V_1 + V_2 =$ constant, with the two parts containing n_1 and n_2 moles at initial temperatures T_1 and T_2. Assume the molar heats at constant volume of the two gases, c_1 and c_2, are independent of the temperature.

1. The wall is left free: determine the equilibrium condition and entropy.
2. The wall is removed: determine the final state and its entropy.
3. Compare the entropies in the two cases.

Constraints:

$$V_0 = V_1 + V_2 = \text{constant} \quad ,$$
$$\mathcal{U}_0 = \mathcal{U}_1 + \mathcal{U}_2 = \text{constant} \quad .$$

1. Equilibrium state:

$$T_1' = T_2' \equiv T^\star = \frac{c_1 n_1 T_1 + c_2 n_2 T_2}{c_1 n_1 + c_2 n_2} \quad ,$$
$$p_1 = p_2 \quad ,$$
$$\frac{V_1'}{n_1} = \frac{V_2'}{n_2} \qquad V_1' = \frac{V_0 n_1}{n_1 + n_2} \qquad V_2' = \frac{V_0 n_2}{n_1 + n_2} \quad ,$$
$$\Delta\mathcal{S} = n_1 c_1 \log\left[\frac{T^\star}{T_1}\right] + n_2 c_2 \log\left[\frac{T^\star}{T_2}\right] + n_1 R \log\left[\frac{V_1'}{V_1}\right] + n_2 R \log\left[\frac{V_2'}{V_2}\right] \quad .$$

2. Equilibrium state:

$$T_1' = T_2' \equiv T^\star = \frac{c_1 n_1 T_1 + c_2 n_2 T_2}{c_1 n_1 + c_2 n_2} \quad ,$$
$$\frac{p_1}{n_1} = \frac{p_2}{n_2} \quad ,$$
$$V_1' = V_0 \qquad V_2' = V_0 \quad ,$$
$$\Delta\mathcal{S} = n_1 c_1 \log\left[\frac{T^\star}{T_1}\right] + n_2 c_2 \log\left[\frac{T^\star}{T_2}\right] + n_1 R \log\left[\frac{V_0}{V_1}\right] + n_2 R \log\left[\frac{V_0}{V_2}\right] \quad .$$

3. Clearly the entropy is larger in the second case as it is easy to see by comparing the two expressions. In fact, one might imagine the two operations in sequence: by removing the constraint wall, the system proceeds towards a more stable state with larger entropy.

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66.13.18
Thermal Mechanical Chemical Balance of an Isolated System

This § is referenced at pages:
[2630]

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©|H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|§ 2.5||

Un cilindro chiuso a pareti rigide e adiabatiche contiene un pistone mobile, conduttore di calore e permeabile, che divide il volume in due parti. Le due parti sono occupate rispettivamente da n_1 ed n_2 moli di un fluido comprimibile.

Si noti che il fluido non deve necessariamente essere un gas perfetto in quanto nessuna proprietà specifica dell'equazione di stato dei due sotto-sistemi deve essere utilizzata.

1. Determinare la relazione che esiste all'equilibrio tra le temperature T_1, T_2 nei due vani del cilindro nel caso in cui il pistone sia fisso e impermeabile.
2. Determinare la relazione che esiste all'equilibrio tra le pressioni p_1, p_2 e le temperature T_1, T_2 nei due vani del cilindro nel caso in cui il pistone è libero di muoversi senza attrito e impermeabile.
3. Determinare la relazione che esiste all'equilibrio tra le pressioni i potenziali chimici, μ_1, μ_2, le pressioni p_1, p_2 e le temperature T_1, T_2 nei due vani del cilindro nel caso in cui il pistone è libero di muoversi senza attrito ed è permeabile alle molecole dei due gas.

WORKED SOLUTION

Il sistema complessivo è isolato termicamente (pareti adiabatiche) e meccanicamente (non scambia lavoro con l'esterno) ed è chiuso.
All'equilibrio l'entropia di un sistema termicamente isolato è massima.
Si ricava allora

$$\mathrm{d}\mathcal{S} = \frac{\mathrm{d}\mathcal{U}}{T} + \frac{p}{T}\,\mathrm{d}V$$

L'entropia del sistema complessivo è data dalla somma delle entropie dei due sotto-sistemi essendo l'additività dell'entropia un assunto fondamentale della termodinamica. Il differenziale dell'entropia del sistema totale vale allora

$$\mathrm{d}\mathcal{S} = \mathrm{d}\mathcal{S}_1 + \mathrm{d}\mathcal{S}_2 = \frac{\mathrm{d}\mathcal{U}_1}{T_1} + \frac{p_1}{T_1}\,\mathrm{d}V_1 + \frac{\mathrm{d}\mathcal{U}_2}{T_2} + \frac{p_2}{T_2}\,\mathrm{d}V_2 \qquad (66.13.07)$$

→ 2653 2654

e all'equilibrio vale zero (entropia massima).

1. Essendo i volumi dei due vani fissati i due sotto-sistemi non compiono lavoro l'uno sull'altro (se la trasformazine è reversibile si può dire: $\mathrm{d}V_1 = \mathrm{d}V_2 = 0$; in generale, volume fissato implica zero lavoro contro le forze di pressione). Poiché il sistema è isolato l'energia totale è costante.

$$\mathcal{U}_1 + \mathcal{U}_2 = \text{constant} \implies \mathrm{d}\mathcal{U}_1 + \mathrm{d}\mathcal{U}_2 = 0$$

Sostituendo nell'espressione (66.13.07) si trova

$$\mathrm{d}\mathcal{S} = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)\mathrm{d}\mathcal{U}_1$$

L'uguaglianza a zero di $\mathrm{d}\mathcal{S}$ deve valere per variazioni arbitrarie di $\mathrm{d}\mathcal{U}_1$. Ne segue che all'equilibrio termodinamico si ha

$$T_1 = T_2$$

E si è ricavata la condizione di equilibrio termico dei due sotto-sistemi.
Supponendo che i due vani siano soggetti ad una differenza di temperatura piccola ma finita il sistema non è in equilibrio. Per l'evoluzione del sistema si può scrivere allora l'espressione

della variazione di entropia della trasformazione che segue

$$\Delta \mathcal{S} \simeq \left(\frac{1}{T_1} - \frac{1}{T_2}\right) \Delta \mathcal{U}_1$$

Allora ne segue, affinche una trasfomrazione spontanea possa avvenire:

$$\Delta \mathcal{S} \geq 0 \implies \left(\frac{1}{T_1} - \frac{1}{T_2}\right) \Delta \mathcal{U}_1 \geq 0 \implies \qquad \Delta \mathcal{U}_1 < 0 \text{ e } \left(\frac{1}{T_1} - \frac{1}{T_2}\right) \text{ devono avere lo stesso segno} \quad .$$

cioè l'energia interna del sistema a temperatura maggiore diminuisce e fluisce sotto forma di calore verso il sistema a temperatura minore la cui temperatura quindi cresce. Il sistema evolve dunque verso lo stato di equilibrio termodinamico.

2. Nel caso in cui il pistone sia mobile i volumi dei due vani non sono più fissi ma lo è la somma dei due volumi V_1 e V_2. Ne segue che ora i vincoli al sistema sono ora due. Assumendo che la trasformazione verso l'equilibrio sia reversibile:

$$\begin{aligned} \mathcal{U}_1 + \mathcal{U}_2 = \text{constant} &\implies \mathrm{d}\mathcal{U}_1 + \mathrm{d}\mathcal{U}_2 = 0 \quad , \\ V_1 + V_2 = \text{constant} &\implies \mathrm{d}V_1 + \mathrm{d}V_2 = 0 \quad . \end{aligned}$$

Sostituendo nell'espressione (66.13.07) si trova ora

$$\mathrm{d}\mathcal{S} = \left(\frac{1}{T_1} - \frac{1}{T_2}\right) \mathrm{d}\mathcal{U}_1 + \left(\frac{p_1}{T_1} - \frac{p_2}{T_2}\right) \mathrm{d}V_1 = 0$$

L'uguaglianza a zero di $\mathrm{d}\mathcal{S}$ deve valere per variazioni arbitrarie di $\mathrm{d}V_1$ e $\mathrm{d}\mathcal{U}_1$. Ne segue che all'equilibrio termodinamico si ha

$$T_1 = T_2 \qquad p_1 = p_2$$

In questo secondo caso oltre alla condizione di equilibrio termico si è ricavata pure la condizione di equilibrio meccanico tra i due sotto-sistemi.
La risposta era evidente dall'inizio essendo le due condizioni trovate le condizioni di equilibrio termico e meccanico del sistema. Tuttavia il problema mostra come tali condizioni possano essere ricavate in modo del tutto generale a partire dal Secondo Principio della Termodinamica.

3. Nel caso generale in cui i due sotto-sistemi possano scambiare pure materia

$$\begin{aligned} \mathcal{U}_1 + \mathcal{U}_2 = \text{constant} &\implies \mathrm{d}\mathcal{U}_1 + \mathrm{d}\mathcal{U}_2 = 0 \quad , \\ V_1 + V_2 = \text{constant} &\implies \mathrm{d}V_1 + \mathrm{d}V_2 = 0 \quad , \\ N_1 + N_2 = \text{constant} &\implies \mathrm{d}N_1 + \mathrm{d}N_2 = 0 \quad . \end{aligned}$$

tra loro si ottiene all'equilibrio termodinamico la condizioni aggiuntiva dell'equilibrio chimico di uguaglianza dei potenziali chimici di tutte le spece (k) presenti

$$\boxed{T_1 = T_2 \qquad p_1 = p_2 \qquad \mu_1^{(k)} = \mu_2^{(k)}} \quad .$$

Occorre sottolineare che la condizione $\mathrm{d}\mathcal{S} = 0$ è solo necessaria e non sufficiente per il massimo dell'entropia e che quello che si è determinato è solo la condizione di stazionarietà dell'entropia. Occorrerebbe verificare anche che la soluzione trovata corrisponde effettivamente ad un massimo. In generale per un sistema complesso potranno esistere anche configurazioni di massimo relativo corrispondenti a stati metastabili. In tali stati il sistema è in equilibrio stabile purché non intervengano perturbazioni troppo grosse.

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66.13.19
Lavoro Ottenibile Da Una Macchina Non Ciclica

©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|2.16||

Si abbia un dispositivo che produce lavoro.

1. Determinare il lavoro ottenibile $\mathcal{W}$ se il dispositivo (non una macchina termica ciclica) produce lavoro scambiando calore solo con due serbatoi a temperature T_{H} e T_{C}, con $T_{\text{H}} > T_{\text{C}}$, ma le sue proprietà hanno un netto cambiamento durante il processo.
2. Determinare il lavoro ottenibile $\mathcal{W}$ se il dispositivo è una macchina termica ciclica che lavora tra gli stessi due serbatoi.

66.13.20
Sistema Termodinamico a Tre Corpi

Si considerino tre corpi di capacità termica C indipendente dalla temperatura, che si trovano all'inizio alle temperature T_1, T_2 e T_3. Calcolare la massima temperatura a cui è possibile portare uno dei tre corpi senza fare lavoro sul sistema dall'esterno.

66.13.21
Espansione Di Un Gas Ideale in Presenza Di Attrito

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One mole of an ideal gas expands isothermally from $p_1 = 2$ bar to $p_2 = 1$ bar in the piston-and-cylinder device shown in Fig. 2-10. The device is surrounded by the atmosphere, which exerts a constant pressure of $p_A = 1$ bar on the outer face of the piston. Moreover, the device is always in thermal equilibrium with the atmosphere, which constitutes a heat reservoir at $T = 300$ K. During the expansion process, a frictional force is exerted on the piston and this force varies in such a way that it always almost balances the net pressure forces on the piston. Thus the piston moves very slowly and experiences negligible acceleration. The piston and cylinder are good heat conductors. Determine the entropy change of the gas, the entropy change of the atmosphere, and the total entropy change brought about by the process.

66.13.22
Ciclo Reversibile: Frigorifero Pompa Di Calore E Macchina Termica

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1. Mostrare come l'efficienza η di una macchina termica reversibile ideale che opera tra due sorgenti è legata al coefficiente di prestazione ω_{C} del frigorifero ideale ottenuto eseguendo il ciclo nel verso opposto.
2. Mostrare come l'efficienza η di una macchina termica reversibile ideale che opera tra due sorgenti è legata al coefficiente di prestazione ω_{H} della pomap di calore ideale ottenuta eseguendo il ciclo nel verso opposto.

WORKED SOLUTION

1. $\eta = 1/(1+\omega_{\text{C}})$.
2. $\eta = ...$

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66.13.23
Macchina Di Carnot Composta

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In una macchina di Carnot a due stadi la quantità di calore $|\mathcal{Q}_1|$ è assorbita a temperatura T_1, viene fatto un lavoro $\mathcal{W}_1$ e si cede la quantità di calore $|\mathcal{Q}_2|$ ad un serbatoio a temperatura T_2. Il secondo stadio assorbe il calore ceduto dal primo stadio, fa un lavoro $\mathcal{W}_2$ e cede la quantità di calore $|\mathcal{Q}_3|$ ad un serbatoio a temperatura più bassa T_3.

1. Dimostrare che l'efficienza della macchina composta è $\eta = (T_1 - T_3)/T_1$.

66.13.24
Pompa Di Calore per Riscaldamento

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Una pompa di calore è una macchina termica che, fungendo da frigorifero, sottrae calore da un serbatoio a bassa temperatura e fornisce calore ad un serbatoio a temperatura più elevata. Una pompa di calore può essere usata per riscaldare l'interno di un edificio sottraendo calore all'ambiente esterno. Si supponga che la temperatura esterna sia $T_e = -10°\mathrm{C}$, che la temperatura sia da mantenere a $T_\mathrm{I} = 22°\mathrm{C}$ all'interno dell'edificio e che sia necessario fornire calore all'ambiente interno ad una potenza di $16\ \mathrm{kW}$ per compensare le normali dispersioni di calore. Qual'è la potenza minima che bisogna fornire alla pompa di calore per ottenere il risultato desiderato ?

WORKED SOLUTION

Il rendimento di un frigorifero è definito dalla relazione

$$\omega_\mathrm{C} \equiv \frac{\mathcal{Q}_\mathrm{C}}{\mathcal{W}} = \frac{\mathcal{Q}_\mathrm{H} - \mathcal{W}}{\mathcal{W}}$$

dove $\mathcal{Q}_\mathrm{C} > 0$ è il calore assorbito dalla sorgente fredda, $\mathcal{Q}_\mathrm{H} > 0$ è il calore ceduto alla sorgente calda ed $\mathcal{W} > 0$ il lavoro che occorre fare sul frigorifero. Tale definizione è basata sul fatto che in un frigorifero si cerca di minimizzare il lavoro che è necessario compiere per sottrarre una certa quantità di calore al serbatoio freddo. Il massimo rendimento di una macchina frigorifera che opera tra due serbatoi a temperature T_H e T_C ($T_\mathrm{C} < T_\mathrm{H}$) è dato dal teorema di Carnot

$$\max \omega_\mathrm{C} = \frac{T_\mathrm{C}}{T_\mathrm{H} - T_\mathrm{C}} \quad .$$

Si noti che il rendimento di un frigorifero ideale tende all'infinito se le temperature dei due serbatoi di calore tendono a diventare uguali per cui è facile trasferire calore da un sistema freddo ad uno più caldo se la differenza di temperatura è piccola. Viceversa se, a parità di T_H, la temperatura del sistema freddo a cui si deve sottrarre calore tende allo zero assoluto il rendimento tende a zero per cui è sempre più difficile raffreddare un corpo freddo verso lo zero assoluto.
Ne segue il lavoro minimo che occorre fornire

$$\omega = \frac{\mathcal{Q}_\mathrm{H} - \mathcal{W}}{\mathcal{W}} \leq \omega_\mathrm{max} \implies \mathcal{W} \geq \frac{\mathcal{Q}_\mathrm{H}}{\omega_\mathrm{max} + 1} \implies \frac{\mathcal{W}}{\Delta t} \geq \frac{\mathcal{Q}_\mathrm{H}}{\Delta t}\left(\frac{1}{\omega_\mathrm{max} + 1}\right) = 1.7\ \mathrm{kW}$$

è dunque possibile fornire calore all'interno di un edificio usando un lavoro $(\omega_\mathrm{max} + 1)$ volte inferiore. Il minimo teorico calcolato sarà certamente inferiore alla richiesta di un frigorifero reale per il quale si ha tipicamente $\omega \approx 5$. In ogni caso la pompa di calore permette di riscaldare la casa con una potenza molto inferiore a quella necessaria a fornire lo stesso calore alla casa trasformando in calore tutto il lavoro fatto su una macchina termica, ad esempio un riscaldatore elettrico. In questo caso infatti il calore fornito all'ambiente va pagato tutto in termini di lavoro speso. Un

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frigorifero è dunque un'ottimo riscaldatore di ambiente, in quanto fornendo un certo lavoro si cede all'ambiente una quantità di calore superiore. La stessa pompa di calore può poi essere usata come condizionatore quando la temperatura esterna è superiore a quella interna a differenza di un riscaldatore elettrico.

66.13.25
Otto Cycle

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|||

©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|2.6||

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Il ciclo di un motore a quattro tempi a benzina può essere idealizzato con il ciclo Otto standard. Questo è un ciclo reversibile consistente in due trasformazioni isocore e due adiabatiche di aria, in quanto l'aria è il componente predominante della miscela combustibile e combusta. L'aria è assunta come gas perfetto e tutti gli scambi di calore avvengono con serbatoi a temperatura fissata.

1. Determinare il rendimento del ciclo.

WORKED SOLUTION

Il ciclo Otto consiste di sei passi.

- Ingresso della miscela ($0 \to 1$): la miscela aria-benzina entra nel cilindro a pressione atmosferica e occupa tutto il volume V_1.
- Compressione ($1 \to 2$): la miscela aria-benzina viene compressa adiabaticamente (perché la compressione è rapida) fino al volume della camera di combustione V_2 e relativa pressione p_2 con relativo aumento della temperatura.
- Scoppio ($2 \to 3$): la miscela è soggetta a combustione che causa un repentino aumento di pressione e temperatura a volume essenzialmente costante, $V_3 = V_2$, data la rapidità della combustione. In questa schematizzazione di ciclo reversibile l'aumento di temperatura è causato da cessione di calore $\mathcal{Q}_1$ da parte di opportuni serbatoi di calore esterni.
- Espansione ($3 \to 4$): la miscela combusta calda si espande adiabaticamente finché si raggiunge il volume $V_4 = V_1$.
- Scarico ($4 \to 1$): La valvola di scarico si apre e la miscela combusta si espande alla pressione atmosferica in un processo rapido che è supposto isocoro e con la caduta di temperatura schematizzata da cessione di calore ad una serie di serbatoi opportuni.
- Eliminazione della miscela ($1 \to 0$): il pistone ora spinge la maggior parte del gas combusto restante fuori dal cilindro in un processo isobaro a pressione atmosferica.

Il primo e l'ultimo passo del ciclo sono supposti coincidenti e percorsi in senso opposto cosicché si cancellano lasciando solo il ciclo $1-2-3-4-5$. Durante la fase $2-3$ la miscela riceve calore da una serie di serbatoi opportuni a temperatura superiore a quelle della miscela mentre durante la fase $4-1$ cede calore ad una serie di serbatoi opportuni a temperatura inferiore a quella della miscela. In entrambi i casi non compie lavoro. Viceversa nei due tratti adiabatici $1-2$ e $3-4$ viene scambiato lavoro con l'ambiente. Definendo il rapporto di compressione del ciclo

$$r \equiv \frac{V_1}{V_2} = \frac{V_4}{V_3}$$

l'adiabaticità e reversibilità delle trasformazioni $1-2$ e $3-4$ fornisce

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1} \Longrightarrow T_2 = T_1 r^{\gamma-1} \quad ,$$
$$T_3 V_3^{\gamma-1} = T_4 V_4^{\gamma-1} \Longrightarrow T_3 = T_4 r^{\gamma-1}$$

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Si ha allora

$$\begin{aligned}\mathcal{Q}_{2-3} &= \mathcal{U}_3 - \mathcal{U}_2 = C_{\mathtt{V}}\,(T_3 - T_2) = C_{\mathtt{V}}\,(T_4 - T_1)\,r^{\gamma-1} \quad ,\\ \mathcal{Q}_{4-1} &= \mathcal{U}_1 - \mathcal{U}_4 = C_{\mathtt{V}}\,(T_1 - T_4)\end{aligned}$$

Si ha inoltre

$$\mathcal{W}_{1-2} = \int_1^2 p\,\mathrm{d}V = p_1 V_1^{\gamma}\int_1^2 \frac{\mathrm{d}V}{V^{\gamma}} = \frac{p_1 V_1^{\gamma}}{1-\gamma}\left(V_2^{1-\gamma} - V_1^{1-\gamma}\right) = \frac{p_1 V_1}{1-\gamma}\left(r^{\gamma-1} - 1\right) \quad ,$$

$$\mathcal{W}_{3-4} = \int_3^4 p\,\mathrm{d}V = p_4 V_4^{\gamma}\int_3^4 \frac{\mathrm{d}V}{V^{\gamma}} = \frac{p_4 V_4^{\gamma}}{1-\gamma}\left(V_4^{1-\gamma} - V_3^{1-\gamma}\right) = \frac{p_4 V_4}{1-\gamma}\left(1 - r^{\gamma-1}\right)$$

Dalla definizione di rendimento di un ciclo di una macchina termica

$$\eta \equiv \frac{\mathcal{W}}{\mathcal{Q}_{\mathtt{H}}}$$

dove $\mathcal{Q}_{\mathtt{H}} > 0$ è il calore assorbito dalla sorgente calda mentre $\mathcal{W} > 0$ è il lavoro fatto dalla macchina termica. si ha allora

$$\eta = \frac{\mathcal{W}_{1-2} + L_{3-4}}{\mathcal{Q}_{2-3}} = \frac{p_1 V_1\left(r^{\gamma-1} - 1\right) + p_4 V_4\left(1 - r^{\gamma-1}\right)}{C_{\mathtt{V}}\,(1-\gamma)\,(T_4 - T_1)\,r^{\gamma-1}} = 1 - \frac{1}{r^{\gamma-1}}$$

Il rendimento del ciclo cresce al crescere del rapporto di compressione. In pratica però il rapporto di compressione non può essere aumentato oltre un certo limite per evitare pre-combustione della miscela, cioè combustione prima del punto 2, fenomeno che può danneggiare il motore e diminuisce la potenza sviluppata. Usando un valore tipico di $r = 5$ e per l'aria $\gamma = 1.4$ si ha $\eta = 0.48$. Questo è solo un valore limite superiore teorico che non è ottenibile da alcun motore reale. Per $r = 5$ l'efficienza reale di un motore può essere dell'ordine di 0.25. Nel caso reale infatti entreranno in gioco tutta un a serie di fenomeni che rendono il ciclo irreversibile quali attriti, turbolenze, cadute di pressione, effetti dinamici relativi ad accelerazioni e decelerazioni flussi di calore attraverso differenze finite di temperatura, e altro.

IN PROGRESS

66.13.26
Diesel Cycle

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©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|2.6||

66.13.27
Steam Engine

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66.13.28
Stirling Cycle

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66.13.29
Efficiency of a Carnot Engine Working With a Ideal Gas

This § is referenced at pages:
[2643]

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|§ 7.7||

66.13.30
Efficiency of a Carnot Engine Working With a Photon Gas

Calculate the efficiency of a Carnot Engine explicitly by using a photon gas as a medium.

66.13.31
Isoentropic Curves

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 8.17|*arg3*|

1. Prove that two isentropic curves do not intersect for systems of two independent variables.
2. Show that isentropic curves do generally intersect for systems with more than two independent variables.

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66.14 Exercises Problems and Physical Applications

66-001 Coefficients of Performance of Refrigerator and Heat-Pump

Show that:

$$\omega_{\mathrm{H}} = \omega_{\mathrm{C}} + 1 \quad .$$

66-002 Carnot Cycle in the TS Plane

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 8.1||

Using the TS Representation, derive the expression for the efficiency of a Carnot engine directly from a TS diagram.

66-003 Entropy for Heating of a Solid/liquid

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 8.8||

66-004 Debye Law

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 8.10||

66-005 Non Quasi-Static Expansion of a Gas

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 8.11||

66-006 Expansion of a Gas

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|problem 8.12||

66-007 Entropy Changes for Different Processes of a Block of Copper

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CHAPTER 67

Thermodynamics in Action

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Physical quantities such as internal energy, enthalpy and entropy are not directly measurable. It is therefore important that numerical values of properties that are not directly measurable can often be calculated from the numerical values of other properties that are easily measurable, at least in principle, such as pressure, volume and temperature.

Many of such relations are presented in this section. Unless stated otherwise, **energy**, in short, will always mean: internal energy.

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67.01 Thermodynamics Potentials

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One can build a number of of other functions of state, starting from internal energy and enthalpy, simply by adding various other combinations of the functions of state in such a way as to give the resulting quantity the dimensions of energy. Technically, this is a Legendre transformation, read § 67.13. potential. The result is a plethora of thermodynamic functions. All these thermodynamic potentials carry the same information but the information is encoded in different ways in the different potentials.

Basic state functions (all energies):

$\mathcal{U} \equiv \mathcal{H} - pV \implies \boxed{\mathrm{d}\mathcal{U} = +T\,\mathrm{d}\mathcal{S} - p\,\mathrm{d}V}$ Energy (strictly speaking: internal energy) (67.01.01) → 2664

$\mathcal{H} \equiv \mathcal{U} + pV \implies \boxed{\mathrm{d}\mathcal{H} = +T\,\mathrm{d}\mathcal{S} + V\,\mathrm{d}p}$ Entalphy (67.01.02) → 2664

$\mathcal{F} \equiv \mathcal{U} - T\mathcal{S} \implies \boxed{\mathrm{d}\mathcal{F} = -\mathcal{S}\,\mathrm{d}T - p\,\mathrm{d}V}$ Helmholtz free energy (Helmholtz function) (67.01.03) → 2664

$\mathcal{G} \equiv \mathcal{H} - T\mathcal{S} \implies \boxed{\mathrm{d}\mathcal{G} = -\mathcal{S}\,\mathrm{d}T + V\,\mathrm{d}p}$ Gibbs free entalphy (Gibbs function) . (67.01.04) → 2664

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Poirer 14.2 Helmholtz and Gibbs Free Energies
The two new free energy/enthalpy quantities, serve as practical tools. They enable the Second Law to be applied directly to the system, rather than the total system-thus making it possible to forget about the surroundings entirely.
The free energies also describe the extent to which the internal energy can be transformed into useful work.
Note also that the free energies are obtained by subtracting the conjugate variable product $T\mathcal{S}$ from $\mathcal{U}$ and $\mathcal{H}$ (respectively), rather than adding it. This is a critical point: since all four functions are positive, free energy/enthalpy values are always less than the corresponding energy/enthalpy. To some extent, this reflects the Second Law fact that only some of the energy can be converted into useful work.
Since the independent variables T and p are generally preferred to T and V, $\mathcal{G}$ is more important than $\mathcal{F}$, in practice. This is especially true for thermodynamic processes that occur at constant T and p such as phase transitions and chemical reactions. Therefore we will often single out $\mathcal{G}$ for special treatment even though similar analyses might hold for $\mathcal{F}$, or even $\mathcal{U}$ and $\mathcal{H}$.

67.01.01 Fundamental Relations and Natural Variables

This § is referenced at pages:
[Never referenced.]

If the thermodynamic potentials are known as a function of their natural variables, all the thermodynamic functions can be deduced by making suitable derivatives. On the other hand, if the thermodynamic potentials are known as a function of other variables, it is necessary to do also some integral and therefore arbitrary functions appear from determine otherwise.

The natural variables of thermodynamic potentials are the variables in terms of which the fundamental relationships are expressed, as in equations (67.01.01), (67.01.02), (67.01.03), (67.01.04):

$$\boxed{\mathcal{U} = \mathcal{U}[\mathcal{S}, V]} \quad ,$$

$$\boxed{\mathcal{H} = \mathcal{H}[\mathcal{S}, p]} \quad ,$$

$$\boxed{\mathcal{F} = \mathcal{F}[T, V]} \quad ,$$

$$\boxed{\mathcal{G} = \mathcal{G}[T, p]} \quad .$$

In addition:

$$\boxed{\mathrm{d}\mathcal{U} = +T\,\mathrm{d}\mathcal{S} - p\,\mathrm{d}V \implies \mathcal{U} = \mathcal{U}[\mathcal{S}, V] \implies T = \left(\frac{\partial \mathcal{U}}{\partial \mathcal{S}}\right)_V} \quad ,$$

$$\boxed{\mathrm{d}\mathcal{S} = \frac{\mathrm{d}\mathcal{U}}{T} + \frac{p\,\mathrm{d}V}{T} \implies \mathcal{S} = \mathcal{S}[\mathcal{U}, V]} \quad .$$

→ 2774

67.01.02 (Internal) Energy

67.01.03 Enthalpy

67.01.04 Helmholtz Free Energy

67.01.05 Gibbs Free Enthalpy

67.01.06 Maxwell Relations

From equations (67.01.01), (67.01.02), (67.01.03), (67.01.04), one has:

$$\boxed{T = +\left(\frac{\partial \mathcal{U}}{\partial \mathcal{S}}\right)_V} \qquad \boxed{p = -\left(\frac{\partial \mathcal{U}}{\partial V}\right)_{\mathcal{S}}} \tag{67.01.05}$$

→ 2705

$$\boxed{T = +\left(\frac{\partial \mathcal{H}}{\partial \mathcal{S}}\right)_p} \qquad \boxed{V = +\left(\frac{\partial \mathcal{H}}{\partial p}\right)_{\mathcal{S}}} \tag{67.01.06}$$

$$\boxed{\mathcal{S} = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_V} \qquad \boxed{p = -\left(\frac{\partial \mathcal{F}}{\partial V}\right)_T} \tag{67.01.07}$$

$$\boxed{\mathcal{S} = -\left(\frac{\partial \mathcal{G}}{\partial T}\right)_p} \qquad \boxed{V = +\left(\frac{\partial \mathcal{G}}{\partial p}\right)_T} \tag{67.01.08}$$

Da queste si derivano le relazioni di Maxwell (come conseguenza dell'uguaglianza delle derivate miste di una funzione di due variabili differenziabile due volte) la cui importanza sta nel fatto che legano

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grandezza misurabili (p, V, T) con l'entropia che non è direttamente misurabile.

$$+\left(\frac{\partial T}{\partial V}\right)_{\mathcal{S}} = -\left(\frac{\partial p}{\partial \mathcal{S}}\right)_{V} \quad ,$$

$$+\left(\frac{\partial T}{\partial p}\right)_{\mathcal{S}} = +\left(\frac{\partial V}{\partial \mathcal{S}}\right)_{p} \tag{67.01.09}$$

$$+\left(\frac{\partial \mathcal{S}}{\partial V}\right)_{T} = +\left(\frac{\partial p}{\partial T}\right)_{V} \tag{67.01.10}$$

→ 2668

$$-\left(\frac{\partial \mathcal{S}}{\partial p}\right)_{T} = +\left(\frac{\partial V}{\partial T}\right)_{p} \tag{67.01.11}$$

→ 2668 2679

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67.02 TdS Equations

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Write the fundamental relation both with energy and enthalpy and then consider iso-core and iso-bare transformations:

$$\mathrm{d}\mathcal{U} = T\ \mathrm{d}\mathcal{S} \qquad \text{fundamental relation with } \mathcal{U} \text{ at constant volume} \quad ,$$
$$\mathrm{d}\mathcal{H} = T\ \mathrm{d}\mathcal{S} \qquad \text{fundamental relation with } \mathcal{H} \text{ at constant pressure} \quad .$$

It follows:

$$\boxed{C_{\mathrm{V}} = \left(\frac{\partial \mathcal{U}}{\partial T}\right)_V = T\left(\frac{\partial \mathcal{S}}{\partial T}\right)_V} \quad , \qquad \rightarrow 2666$$

$$\boxed{C_{\mathrm{p}} = \left(\frac{\partial \mathcal{H}}{\partial T}\right)_p = T\left(\frac{\partial \mathcal{S}}{\partial T}\right)_p} \quad .$$

Write the differential of entropy first as a function of T and V, after as a function of T and p, multiply by T, use equations (67.02), (67.02), and finally use **Maxwell Equations** to obtain the first and second TdS equations:

$$\boxed{T\ \mathrm{d}\mathcal{S} = C_{\mathrm{V}}\ \mathrm{d}T + T\left(\frac{\partial p}{\partial T}\right)_V\ \mathrm{d}V} \quad , \quad , \qquad \rightarrow 2667\ 2669$$

$$\boxed{T\ \mathrm{d}\mathcal{S} = C_{\mathrm{p}}\ \mathrm{d}T - T\left(\frac{\partial V}{\partial T}\right)_p\ \mathrm{d}p} \quad , \quad , \qquad \rightarrow 2667\ 2669$$

$$\boxed{T\ \mathrm{d}\mathcal{S} = C_{\mathrm{V}}\left(\frac{\partial T}{\partial p}\right)_V\ \mathrm{d}p + C_{\mathrm{p}}\left(\frac{\partial T}{\partial V}\right)_p\ \mathrm{d}V} \quad ,$$

dette equazioni per l'entropia o equazioni $T\ \mathrm{d}\mathcal{S}$.

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67.03 Equations for Thermal Capacities

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67.03.01
Constant Volume Thermal Capacity and Stability of Thermal Equilibrium

It can be shown that

$$\boxed{C_{\mathtt{V}} \geq 0} \quad . \tag{67.03.01}$$

→ 2721

Actually, it can be shown that $C_{\mathtt{V}} \geq 0$ is the condition for thermal stability of a pVT system. Read § 67.06.04.

67.03.02
Constant Volume and Constant Pressure Thermal Capacities

Read § 65.08.

67.03.03
Difference Between Constant Pressure and Constant Volume Thermal Capacities

La relazione generale tra le capacità termiche a pressione e volume costante di un sistema pVT a composizione costante si ricava dalle relazioni (67.02), (67.02) ed è data dalla relazione

$$\boxed{C_{\mathtt{p}} - C_{\mathtt{V}} = T \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_p = -T \left(\frac{\partial p}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p^2 = \frac{\beta^2 V T}{\kappa_{\mathtt{T}}}} \quad . \tag{67.03.02}$$

→ 2583 2667

L'importanza della relazione sta nel fatto che permette di esprimere $C_{\mathtt{V}}$, difficilmente misurabile, in termini di grandezze più facilmente misurabili.

In general, the functions on the right-side, β and $\kappa_{\mathtt{T}}$, depend on pVT.

It is worth noting that the difference $C_{\mathtt{p}} - C_{\mathtt{V}}$ requires the knowledge of the equation of state only, even if thermal capacities are partial derivatives of the internal energy and enthalpy.

As, in all generality,

$$\boxed{\kappa_{\mathtt{T}} \geq 0} \quad ,$$

it follows that

$$\boxed{C_{\mathtt{p}} \geq C_{\mathtt{V}}} \quad .$$

Equation (67.03.02) implies that:

$$\{T \to 0\} \implies \{C_{\mathtt{p}} \to C_{\mathtt{V}}\} \quad .$$

Equation (67.03.02) also implies that $C_{\mathtt{p}} = C_{\mathtt{V}}$ whenever $\kappa_{\mathtt{T}} = 0$.

67.03.04
Ratio Between Constant Pressure and Constant Volume Thermal Capacities

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It can be shown that:

$$\boxed{\gamma \equiv \frac{C_{\mathrm{p}}}{C_{\mathrm{V}}} = \frac{\kappa_{\mathrm{T}}}{\kappa_{\mathrm{S}}} \geq 0} \quad . \tag{67.03.03}$$

Therefore:

$$\boxed{0 \leq C_{\mathrm{V}} \leq C_{\mathrm{p}}} \quad ,$$

as $C_{\mathrm{V}} > 0$.

67.03.05
Derivatives of Constant Pressure and Constant Volume Thermal Capacities

Dalle relazioni (67.01.10), (67.01.11), (**??**) si deducono le relazioni che danno la variazione delle capacità termiche

$$\boxed{\left(\frac{\partial C_{\mathrm{V}}}{\partial V}\right)_T = +T\left(\frac{\partial^2 p}{\partial T^2}\right)_V} \quad , \quad , \qquad \rightarrow 2583$$

$$\boxed{\left(\frac{\partial C_{\mathrm{p}}}{\partial p}\right)_T = -T\left(\frac{\partial^2 V}{\partial T^2}\right)_p} \quad . \qquad \rightarrow 2583$$

67.03.06
Thermal Capacities From G and F

$$\boxed{C_{\mathrm{V}} = -T\left(\frac{\partial^2 \mathcal{F}}{\partial T^2}\right)_V} \quad , \quad ,$$

$$\boxed{C_{\mathrm{p}} = -T\left(\frac{\partial^2 \mathcal{G}}{\partial T^2}\right)_p} \quad .$$

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67.04 Equations for Internal Energy and Entalphy

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Si deducono scrivendo il differenziale dell'energia interna o dell'entalpia, usando l'espressione differenziale del primo principio e le equazioni dell'entropia (67.02), (67.02). La loro importanza sta nel fatto che legano grandezze misurabili (p, V, T) con l'energia interna e l'entalpia che non sono direttamente misurabili.

The derivation of the following expressions requires the second law, as the first law alone is not enough.

67.04.01 Internal Energy

This § is referenced at pages:
[2583]

$$\left(\frac{\partial \mathcal{U}}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p \quad ,$$

$$\left(\frac{\partial \mathcal{U}}{\partial p}\right)_T = -T\left(\frac{\partial V}{\partial T}\right)_p - p\left(\frac{\partial V}{\partial p}\right)_T \quad , \qquad \rightarrow 2583$$

$$\left(\frac{\partial \mathcal{U}}{\partial T}\right)_p = +C_{\mathrm{p}} - p\left(\frac{\partial V}{\partial T}\right)_p \quad .$$

67.04.02 Entalphy

This § is referenced at pages:
[Never referenced.]

$$\left(\frac{\partial \mathcal{H}}{\partial p}\right)_T = V - T\left(\frac{\partial V}{\partial T}\right)_p \quad ,$$

$$\left(\frac{\partial \mathcal{H}}{\partial V}\right)_T = +T\left(\frac{\partial p}{\partial T}\right)_V + V\left(\frac{\partial p}{\partial V}\right)_T \quad ,$$

$$\left(\frac{\partial \mathcal{H}}{\partial T}\right)_V = +C_{\mathrm{v}} + V\left(\frac{\partial p}{\partial T}\right)_V \quad .$$

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67.05 Equations for Compressibilities

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67.05.01
Isothermal Compressibility and Mechanical Equilibrium

It can be shown that:

$$\boxed{\kappa_{\mathrm{T}} \geq 0} \quad . \tag{67.05.01}$$

Actually, it can be shown that $\kappa_{\mathrm{T}} \geq 0$ is the condition for mechanical stability of a pVT system. Read § 67.06.04 ■

67.05.02
Isothermal and Adiabatic Compressibility

This § is referenced at pages:
[1492]

I moduli di compressibilità isoterma e adiabatica sono definiti come

$$\boxed{\kappa_{\mathrm{T}} \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \geq 0} \quad , \quad , \qquad \rightarrow 2721$$

$$\boxed{\kappa_{\mathrm{S}} \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S} \quad .$$

The first term can be obtained from measured pVT data. The second term can be measured as it is related to the sound speed in the material.

67.05.03
Difference Between Isothermal and Adiabatic Compressibility

It can be shown that:

$$\boxed{\kappa_{\mathrm{T}} - \kappa_{\mathrm{S}} = \frac{TV\beta^2}{C_{\mathrm{p}}}} \quad . \tag{67.05.02}$$

As, in all generality,

$$\boxed{C_{\mathrm{p}} \geq 0} \quad ,$$

it follows that

$$\boxed{\kappa_{\mathrm{T}} \geq \kappa_{\mathrm{S}}} \quad .$$

67.05.04
Ratio Between Isothermal and Adiabatic Compressibility

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It can be shown that:

$$\boxed{\gamma \equiv \frac{\kappa_{\mathrm{T}}}{\kappa_{\mathrm{S}}} = \frac{C_{\mathrm{p}}}{C_{\mathrm{V}}} \geq 1} \quad . \tag{67.05.03}$$

Therefore:

$$\boxed{0 \leq \kappa_{\mathrm{S}} \leq \kappa_{\mathrm{T}}} \quad ,$$

as $\kappa_{\mathrm{T}} > 0$.

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67.06 Thermodynamic Equilibrium

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©|D.V.Schroeder, *Thermal Physics*, 1999, Addison-Wesley, ...Ed., ...|5.2||

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67.06.01 Maximum Work

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The propensity of physical systems to increase their entropy can be channelled to deliver useful work. All such applications are governed by the maximum work theorem. Consider a system that is to be taken from a specified initial state to a specified final state. Also available are two auxiliary systems, into **one of which work can be transferred**, and into **the other of which heat can be transferred**. Then the maximum work theorem states that **for all processes leading from the specified initial state to the specified final state of the primary system, the delivery of work is maximum (and the delivery of heat is minimum) for a reversible process**. Furthermore the delivery of work (and of heat) is identical for every reversible process.

The repository system into which work is delivered is called a **reversible work source**, $\mathcal{W}$. Reversible work sources are defined as systems enclosed by adiabatic impermeable walls and characterized by relaxation times sufficiently short that all processes within them are essentially quasi-static. From the thermodynamic point of view the conservative (non-frictional) systems considered in the theory of mechanics are reversible work sources.

The repository system into which heat is delivered is called a **reversible heat reservoir**, H. Reversible heat reservoirs are defined as systems enclosed by rigid impermeable walls and characterized by relaxation times sufficiently short that all processes of interest within them are essentially quasi-static. Note that infinite thermal capacity (implying fixed temperature) is not included in the definition of reversible heat reservoirs. If the temperature of the reversible heat reservoir is T_{H} the transfer of heat $\delta\mathcal{Q}$ to the reversible heat reservoir increases its entropy according to the quasi-static relationship $\delta\mathcal{Q} = T_{\mathrm{H}}\,\mathrm{d}\mathcal{S}$. The external interactions of a reversible heat reservoir accordingly are fully described by its **Heat-Capacity** $C(T_{\mathrm{H}})$ (the definition of the reversible heat reservoir implies that this **Heat-Capacity** is at constant volume, but we shall not so indicate by an explicit subscript). The energy change of the reversible heat reservoir is $\mathrm{d}\mathcal{U} = \delta\mathcal{Q} = C(T_{\mathrm{H}})\,\mathrm{d}T_{\mathrm{H}}$ and the entropy change is $\mathrm{d}\mathcal{S} = C(T_{\mathrm{H}})\,\mathrm{d}T_{\mathrm{H}}/T_{\mathrm{H}}$.

The proof of the maximum work theorem is almost immediate. Consider two processes. Each leads to the same energy change, $\Delta\mathcal{U}$, and the same entropy change, $\Delta\mathcal{S}$, within the primary subsystem (S), for these are determined by the specified initial and final states. The two processes differ only in the apportionment of the energy difference between the reversible work source and the reversible heat reservoir. But the process that delivers the maximum possible work to the reversible work source correspondingly delivers the least possible heat to the reversible heat reservoir, and therefore leads to the least possible entropy increase of the reversible heat reservoir

(and thence of the entire system). The absolute minimum of $\Delta\mathcal{S}_{\mathtt{total}}$ for all possible processes, is attained by any reversible process (for all of which $\Delta\mathcal{S}_{\mathtt{total}} = 0$).

$$\mathrm{d}\mathcal{U} \quad \text{fixed} \quad \text{all quantities referred to the primary sub-system } S \quad , \quad ,$$

$$\mathrm{d}\mathcal{S} \quad \text{fixed} \quad \text{all quantities referred to the primary sub-system } S \quad , \quad ,$$

$$\mathrm{d}\mathcal{U} = \delta\mathcal{Q} + \delta\mathcal{W} \quad \text{all quantities referred to the primary sub-system } S \quad , \quad ,$$

$$\delta\mathcal{W}_{S\to A} \equiv -\delta\mathcal{W} = \delta\mathcal{Q} - \mathrm{d}\mathcal{U} \quad \text{all quantities referred to the primary sub-system } S \quad , \quad ,$$

$$\text{assume:} \quad \mathrm{d}\mathcal{S}_A = 0 \quad \text{reversible work source} \quad , \quad ,$$

$$\text{assume:} \quad \mathrm{d}\mathcal{S}_H = -\frac{\delta\mathcal{Q}}{T_{\mathtt{H}}} \quad \text{reversible heat reservoir, } H\text{, at temperature } T_{\mathtt{H}} \quad ,$$

$$\mathrm{d}\mathcal{S}_U = \mathrm{d}\mathcal{S}_H + \mathrm{d}\mathcal{S}_A + \mathrm{d}\mathcal{S} \geq 0 \quad ,$$

$$\mathrm{d}\mathcal{S}_U = \mathrm{d}\mathcal{S}_H + \mathrm{d}\mathcal{S}_A + \mathrm{d}\mathcal{S} = 0 \quad \text{ifoif all reversible transformations} \quad ,$$

$$-\frac{\delta\mathcal{Q}}{T_{\mathtt{H}}} + \mathrm{d}\mathcal{S} \geq 0 \quad ,$$

$$-\frac{\mathrm{d}\mathcal{U} - \delta\mathcal{W}}{T_{\mathtt{H}}} + \mathrm{d}\mathcal{S} \geq 0 \quad ,$$

$$\frac{\mathrm{d}\mathcal{U} - \delta\mathcal{W}}{T_{\mathtt{H}}} \leq \mathrm{d}\mathcal{S} \quad ,$$

$$\boxed{\delta\mathcal{W}_{S\to A} \equiv -\delta\mathcal{W} \leq T_{\mathtt{H}}\,\mathrm{d}\mathcal{S} - \mathrm{d}\mathcal{U}}$$

Note: $\mathrm{d}\mathcal{S} > 0$ does not imply that something irreversible is happening in the system, $\mathrm{d}\mathcal{S}_i > 0$; it might be just heat entering the system, and $\mathrm{d}\mathcal{S} > 0$ is compensated by an entry decrease of the heat reservoir.

67.06.02
Reservoirs - Heat Volume Particle

- A heat reservoir is any system in thermal equilibrium, so large that it is possible to extract/dump heat from/to the heat reservoir without changing its temperature: changing entropy without affecting its temperature. It is sometimes called **temperature reservoir**. It is sometimes better called **entropy reservoir**.
- A volume reservoir is any system in mechanical equilibrium, so large that it is possible to extract/dump volume from/to the volume reservoir without changing its pressure: changing volume without affecting its pressure. It is sometimes called **pressure reservoir**.
- A particle reservoir is any system in chemical equilibrium, so large that it is possible to extract/dump particles from/to the particle reservoir without changing its chemical potential: changing number of particles without affecting its chemical potential.

67.06.03
Equilibrium for Thermodynamic Systems in Different Conditions

©|N.W.Tschoegl, *Fundamentals Of Equilibrium And Steady-State Thermodynamics*, 2000, Elsevier, ...Ed., ...|extensive||

In general, the reversible work done on a system will be the sum of different contributions:

$$\delta\mathcal{W} \equiv -p\,\mathrm{d}V + \sum_i Y_i\,\mathrm{d}X_i \qquad \mathrm{d}\mathcal{U} = \delta\mathcal{Q} - p\,\mathrm{d}V + \sum_i Y_i\,\mathrm{d}X_i \equiv \delta\mathcal{Q} - p\,\mathrm{d}V + \mathcal{W}' \quad .$$

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67.06.03.01 Contact With a Heat Reservoir

©|N.W.Tschoegl, *Fundamentals Of Equilibrium And Steady-State Thermodynamics*, 2000, Elsevier, ...Ed., ...|extensive||

Il lavoro totale, $\mathcal{W}_{S\to A} \equiv -\mathcal{W}$, che può compiere un sistema che sia a contatto termico con l'ambiente a temperatura T_0 all'inizio alla fine di un insieme di trasformazioni arbitrarie tra due stati di equilibrio fissati e tali che le variazioni di energia interna ed entropia del sistema siano $\Delta\mathcal{U}$ e $\Delta\mathcal{S}$, è limitato superiormente dalla relazione:

$$\Delta\mathcal{S}_{\mathcal{U}} \geq 0 \implies \boxed{\mathcal{W}_{S\to A} \equiv -\mathcal{W} \leq -\left(\Delta\mathcal{U} - T_0\Delta\mathcal{S}\right) \equiv -\Delta\mathcal{F}} \quad . \tag{67.06.01}$$

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Il lavoro totale massimo, tra tutte le possibili trasformazioni tra due stati di equilibrio arbitrari ma fissati, si ha per una trasformazione reversibile.

Il sistema esterno che assorbe o produce lavoro si assume essere totalmente reversibile.

Se un sistema non compie/riceve lavoro, se ne deduce quindi che l'energia libera di Helmoltz può solo diminuire; ovvero tra tutti i possibili stati finali, fissato lo stato iniziale, ce ne saranno alcuni con energia libera invariata (se e solo se le trasformaioni sono tutte reversibili) ed altri con energia libera minore (se e solo se almeno una trasforamzione è irreversibile).

At the thermodynamic equilibrium, the value of any unconstrained parameter of a system which is in thermal contact with a heat reservoir and does no work, is such that the free energy is minimized:

$$\boxed{\mathrm{d}\mathcal{F}\Big|_T = 0 \qquad d^2\mathcal{F}\Big|_T \geq 0}$$

It is important to emphasize that equation (67.06.01) clarifies that, for reversible transformations, the work done by a system in contact with a heat reservoir is:

$$\Delta\mathcal{S}_{\mathcal{U}} = 0 \implies \boxed{-\mathcal{W} = -\Delta\mathcal{F}} \quad . \tag{67.06.02}$$

Since $\mathcal{F}$ characterizes the work that can be extracted from a system in contact with a thermal reservoir, the relation $\mathcal{F} < \mathcal{U}$, from the definition, tells us that not all internal energy can be converted into work.

This equation is basic in the study of system such as Polarizable|Magnetizable matter in external fields, given that ElectroMagnetism provides the work for mono-thermal processes.

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S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...
Note that if the system is mechanically isolated from its surroundings, so that no work can be applied or extracted, then $\mathrm{d}\mathcal{F} \leq 0$ Thus any change in $\mathcal{F}$ will be negative. As the system settles down towards equilibrium, all processes will tend to force $\mathcal{F}$ downwards. Once the system has reached equilibrium, $\mathcal{F}$ will be constant at this minimum level. Hence equilibrium can only be achieved by minimizing $\mathcal{F}$.

67.06.03.02 Contact With a Heat and Volume Reservoir

©|N.W.Tschoegl, *Fundamentals Of Equilibrium And Steady-State Thermodynamics*, 2000, Elsevier, ...Ed., ...|extensive||

Il lavoro totale utile, $\mathcal{W}'_{S\to A} \equiv -\mathcal{W}'$, che può compiere un sistema che sia a contatto termico con l'ambiente a temperatura T_0 e a contatto con l'ambiente a pressione esterna p_0, all'inizio alla fine di un insieme di trasformazioni arbitrarie tra due stati di equilibrio fissati e tali che le variazioni di energia interna ed entropia del sistema siano $\Delta\mathcal{U}$ e $\Delta\mathcal{S}$, è limitato superiormente dalla relazione:

$$\Delta\mathcal{S}_{\mathcal{U}} \geq 0 \implies \boxed{\mathcal{W}'_{S\to A} \equiv -\mathcal{W}' \leq -\left(\Delta\mathcal{U} - T_0\Delta\mathcal{S} + p_0\Delta V\right) = -\Delta\mathcal{G}} \quad . \tag{67.06.03}$$

→ 2675

Il lavoro totale utile massimo, tra tutte le possibili trasformazioni tra due stati di equilibrio arbitrari ma fissati, si ha per una trasformazione reversibile. Tale lavoro è quello utile, in quanto si sottrae l'inevitabile lavoro contro la pressione esterna.

Il sistema esterno che assorbe o produce lavoro si assume essere totalmente reversibile.

UNDER PEER REVIEW

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Se un sistema non compie/riceve lavoro, se ne deduce quindi che l'entalpia libera di Gibbs può solo diminuire; ovvero tra tutti i possibili stati finali, fissato lo stato iniziale, ce ne saranno alcuni con entalpia libera invariata (se e solo se le trasformaioni sono tutte reversibili) ed altri con energia libera minore (se e solo se almeno una trasforamzione è irreversibile).

At the thermodynamic equilibrium, the value of any unconstrained parameter of a system coupled with a heat reservoir through a diathermal wall and with a volume reservoir through a movable wall and does no work, is such that the free entalpy is minimized:

$$\boxed{\mathrm{d}\mathcal{G}\Big|_{p,T} = 0 \qquad d^2\mathcal{G}\Big|_{p,T} \geq 0}$$

It is important to emphasize that equation (67.06.03) clarifies that, for reversible transformations, the useful work done by a system in contact with a heat reservoir and a volume reservoir is:

$$\Delta\mathcal{S}_{\mathcal{U}} = 0 \implies \boxed{-\mathcal{W}' = -\Delta\mathcal{G}} \quad . \tag{67.06.04}$$

Since $\mathcal{G}$ characterizes the useful work that can be extracted from a system in contact with a thermal reservoir and a volume reservoir, the relation $\mathcal{G} < \mathcal{H}$, from the definition, tells us that not all entalphy can be converted into useful work.

This equation is basic in the study of system such as phase transitions and chemical reactions which are typically mono-thermal and mon-bare processes.

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This means that the maximum amount of work is achieved by a reversible (that is quasi-static) process connecting the initial and final states. Keep in mind that the system is assumed to exchange heat with the reservoir at T and that this assumption puts a constraint on the quasi-static process that results in the maximum amount of work. In fact, such a process is uniquely determined for a particular choice for the initial and the final states. More specifically, we must first follow the adiabatic going through the initial state until the temperature of the system reaches the temperature T of the reservoir and then go through the quasi-static isothermal process at the temperature T to reach the adiabatic that goes through the final state and finally follow this adiabatic to arrive at the final state. This is the same construction as for the Carnot cycle.

© - QUOTE

For a pure substance in a single phase the two variables T and p or T and V define the thermodynamic state, which therefore cannot change if both variables are held fixed.
It is important to realize that equations (67.06.01), (67.06.03) applies only to more complicated situations such as phase transitions and chemical reactions.

67.06.04
Summary of the Main Stability Conditions at Thermodynamic Equilibrium

This § is referenced at pages:
[2667, 2670]

- For given $\mathcal{U}$ and V: $\mathcal{S}$ is maximum.
- For given $\mathcal{H}$ and p: $\mathcal{S}$ is maximum.
- For given T and V: $\mathcal{F}$ is minimum.
- For given T and p: $\mathcal{G}$ is minimum.

Since temperature is a more convenient variable than entropy, the last two conditions are more useful than the first two, but all of them are as fundamental as any other.

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Stability conditions:

$$\boxed{\kappa_{\mathrm{T}} > 0 \qquad C_{\mathrm{V}} > 0} \quad ,$$

and

$$\boxed{\kappa_{\mathrm{T}} \geq \kappa_{\mathrm{S}} \geq 0 \qquad C_{\mathrm{p}} \geq C_{\mathrm{V}} \geq 0} \quad .$$

© - QUOTE

H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...
Thus both heat capacities and both compressibilities must be positive in a stable system. Addition of heat, either at constant pressure or at constant volume, necessarily increases the temperature of a stable system, the more so at constant volume than at constant pressure. And decreasing the volume, either isothermally or iso-entropically, necessarily increases the pressure of a stable system, the more so isothermally than iso-entropically.

The condition for stability of diffusional equilibrium turns out to be:

$$\boxed{\frac{\partial \mu_k}{\partial N_k} \geq 0} \quad .$$

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67.07 Le Chatelier Principle

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67.08 Equilibrium of Phases of a Single Pure Substance

This § is referenced at pages:
[2546]

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|||

A phase is a portion of a system that possesses well defined physical and chemical properties, of homogeneous composition and with well-defined border, so that composition, pressure, volume and temperature can be defined overall.

A pure substance is a system made up of a single chemical species. It can exist in different physical states: gaseous, liquid, solid; we also say under different phases, that is to say under different homogeneous parts characterized by the same physical and chemical properties. When a pure substance evolves from one state of equilibrium to another, we sometimes witness a sudden modification of some of its physical, mechanical, optical, electric properties, etc. It is said to undergo a change of state or a phase transition. This phenomenon is frequently observed in nature; we know well the solidification of water as well as its vaporization, but the passage of iron from state α to state γ, by heating, is also a phase transition.

A phase transition is a condition such that some discontinuity in some physical property happens.

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A phase transition is an abrupt change in a system that occurs over a small range in a control variable. For thermodynamic phase transitions, typical control variables are the "intensive variables" of temperature, pressure, or magnetic field. Thermodynamic phase transitions in materials and condensed matter, the subject of this book, occur when there is a singularity in the free energy function of the material, or in one of the derivatives of the free energy function. Accompanying a phase transition are changes in some physical properties and structure of the material, and changes in properties or structure are the usual way that a phase transition is discovered. There is a very broad range of systems that can exhibit phase transitions, extending from atomic nuclei to traffic flow or politics. For many systems it is a challenge to find reliable models of the free energy, however, so thermodynamic analyses are not available.

A non trivial example is iron recalescence, the transition of pure iron from state, where the metallic network is cubic centered, in another state, where the network is cubic with centered faces. If we pass a current through an iron wire so that the wire deforms and becomes red, $T > 1200$ K, and then remove the current, the wire cools and shorten. We observe that the contraction of the wire marks a stopping time when the temperature is $T = 1179$ K. Heat is ejected due to the phase transition.

In the gaseous state, a pure substance or a mixture of pure substances form only one phase. In the liquid state, a pure substance or a mixture of pure substances form, as a rule, only one phase, except in some cases, such as helium, which has two different liquid phases. In the solid state, a pure substance often presents itself under different crystal structures: each of these allotropic varieties then constitute a solid phase (water, iron, for instance).

Experience of water refreezing. A wire stretched by masses hanging at its ends is placed on a bar of ice. Under the action of the strong pressure exerted by the wire, the temperature of fusion decreases and the ice melts on contact with the wire. As the pressure of the wire stops the liquid thus formed refreezes above the wire. Finally the wire completely crosses the block of ice.

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The importance of $\mathcal{G}$ derives from the fact that its natural variables are temperature, pressure, and composition. Classical solution thermodynamics is an experimental science, and T, p, and x (the set of mole fractions) are the variables most susceptible to measurement and control in the laboratory. Moreover, these are the independent variables favoured by engineers for use in the design of chemical processes.

It is an experimental observation that the phase transitions take place at constant temperature and pressure, whereby during a phase transition the Gibbs function remains constant, that is:

$$\boxed{\mathrm{d}\mathcal{G} = 0} \qquad \text{at phase equilibrium} \quad .$$

Le funzioni di Gibbs molari delle due fasi sono quindi uguali nelle condizioni in cui c'è equilibrio di fase. In fact considering an infinitesimal change of number of moles of the two phases , n^α and n^β, but at constant number of moles, n, for a closed system, indicando con α e β due diverse fasi, si ha:

$$\mathrm{d}\mathcal{G} = g^\alpha[p,T]\ \mathrm{d}n^\alpha + g^\beta[p,T]\ \mathrm{d}n^\beta = 0 \qquad \text{at } \mathrm{d}p = \mathrm{d}T = 0, \text{ with } n^\alpha + n^\beta = n \quad .$$

Date certe condizioni di pressione e temperatura la fase stabile è quella a cui corrisponde il valore minimo dell'energia di Gibbs molare.

Dalla definizione di $\mathcal{G}$, equazione (67.01.11), si ha che la fase stabile corrisponde al migliore compromesso tra minimizzare l'energia interna, minimizzare il prodotto pV (cioè V, se p è data) e massimizzare $T\mathcal{S}$ (cioè $\mathcal{S}$, se T è data).

$$g^\alpha = g^\beta \quad ,$$

dove $h^{\beta\alpha}_{\cdot\cdot}$ è l'entalpia di transizione dalla fase β alla fase α, e T la temperatura di transizione. Si ha

$$h^{\beta\alpha}_{\cdot\cdot} + h^{\alpha\beta}_{\cdot\cdot} = 0 \quad .$$

Inoltre

$$s^\alpha = -\left(\frac{\partial g^\alpha}{\partial T}\right)_p$$

per cui $\mathcal{G}$ ha una cuspide al punto di transizione (discontinuità nella derivata prima).

This equality of the free mass enthalpies of the two phases results in the relation $p = p(T)$ between the pressure and temperature. As a result, if a pure substance is balanced in two phases, any isothermal evolution of this equilibrium is necessarily isobaric and reciprocal.

If we want to find pressures and temperatures such that phase equilibrium is possible, we should look for infinitesimal changes, $\mathrm{d}p$ and $\mathrm{d}T$, such that equality of the molar Gibbs function is maintained:

$$\mathrm{d}g^\alpha = \mathrm{d}g^\beta \implies -s^\alpha\ \mathrm{d}T + v^\alpha\ \mathrm{d}p = -s^\beta\ \mathrm{d}T + v^\beta\ \mathrm{d}p \quad .$$

Per un sistema a due fasi si ha quindi l'equazione di Clapeyron

$$\boxed{\frac{\mathrm{d}p}{\mathrm{d}T} = \frac{s^\alpha - s^\beta}{v^\alpha - v^\beta} = \frac{h^{\beta\alpha}_{\cdot\cdot}}{T\,(v^\alpha - v^\beta)}} \quad .$$

The molar/specific latent heat of a pure substance, or molar/specific enthalpy of transition from phase 1 to phase 2, is the heat necessary to achieve, reversibly, at a temperature and a constant pressures, the phase transition of the unit of mole/mass of this substance, from phase 1 to phase 2.

As the molar/specific volumes, as well as the vapor pressure, only depend on temperature, phase transition enthalpies are functions of temperature. In other words, general thermodynamic functions depend on two macroscopic variables, but transition enthalpies are defined on the vapor pressure curve, so that they only depend on one variable.

Per un sistema a più fasi:

$$g^\alpha = g^\beta = \ldots = g^\tau \quad ,$$

which, in general, allows only discrte solutions for **triple points** at a specified value of (p,T).

Let us consider the unit of mole/mass of pure substance, entirely in the solid state, in the vicinity of the triple point. Let us consider an isothermal and isobaric reversible evolution which first brings the

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substance to the state liquid, then in the gaseous state and finally brings it back to the initial solid state. For this cyclical development, the variation in mole/mass enthalpy of the system is zero. Therefore:

$$\Delta h_{lg} + \Delta h_{gs} + \Delta h_{sl} = 0 \quad .$$

67.08.01
Approximate Solution for Equilibrium With Vapor

L'equazione di Clapeyron per l'equilibrio liquido-vapore e solido-vapore, trattando il vapore come un gas ideale, trascurando il volume molare della fase liquida/solida rispetto a quello della fase gassosa e considerando il calore latente di vaporizzazione/sublimazione indipendente dalla temperatura si scrive

$$p[T] = p_0 \exp\left[-\frac{\lambda^{(\mathtt{mol})}_{\mathtt{vap}}}{R}\left(\frac{1}{T}-\frac{1}{T_0}\right)\right] = p_0 \exp\left[-\frac{M\lambda^{(\mathtt{sp})}_{\mathtt{vap}}}{R}\left(\frac{1}{T}-\frac{1}{T_0}\right)\right], \tag{67.08.01}$$

$$p[T] = p_0 \exp\left[-\frac{\lambda^{(\mathtt{mol})}_{\mathtt{sub}}}{R}\left(\frac{1}{T}-\frac{1}{T_0}\right)\right] = p_0 \exp\left[-\frac{M\lambda^{(\mathtt{sp})}_{\mathtt{sub}}}{R}\left(\frac{1}{T}-\frac{1}{T_0}\right)\right]. \tag{67.08.02}$$

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67.09 Equilibrium of Phases of Many Substances

The criterion for phase equilibrium in pVT systems of uniform T and p, with m substances in different phases, $\alpha\beta\ldots\pi$, is concisely stated in terms of the chemical potential:

$$\mu_k^\alpha = \mu_k^\beta = \ldots = \mu_k^\pi \qquad k = 1\ldots m \quad .$$

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67.10 Chemical Potential for Open Systems

©|N.W.Tschoegl, *Fundamentals Of Equilibrium And Steady-State Thermodynamics*, 2000, Elsevier, ...Ed., ...|extensive||

Nel caso di un sistema aperto (cioè un sistema che può acquistare o cedere materia) il primo principio della termodinamica va generalizzato per includere il cambiamento di energia indotto dal flusso di materia. Per un sistema composto di vari componenti, ciascuno con numero di moli n_k e/o numero di particelle N_k, si ha

$$\mathrm{d}\mathcal{U} = T\,\mathrm{d}\mathcal{S} - p\,\mathrm{d}V + \sum_k \mu_k\,\mathrm{d}N_k \qquad \mu_k \equiv \left(\frac{\partial \mathcal{U}}{\partial N_k}\right)_{\mathcal{S},V,N_j} \tag{67.10.01}$$

→ 2682

dove nella derivazione parziale si tiene fisso il numero di particelle, N_j, di tutti i componenti esclusa la specie di cui si valuta il potenziale chimico, k.

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The terms in equation (67.10.01) are interpreted as types of energy fluxes that change the internal energy of the system. In addition to the reversible work and the reversible heat, the third term is an energy flux known as the reversible chemical work.

Analogamente si introducono i potenziali chimici partendo dalle altre relazioni fondamentali:

$$\mathrm{d}\mathcal{U} = +T\,\mathrm{d}\mathcal{S} - p\,\mathrm{d}V + \sum_k \mu_k\,\mathrm{d}N_k \qquad \mu_k \equiv \left(\frac{\partial \mathcal{U}}{\partial N_k}\right)_{\mathcal{S},V,N_j} \quad , \quad ,$$

$$\mathrm{d}\mathcal{H} = +T\,\mathrm{d}\mathcal{S} + V\,\mathrm{d}p + \sum_k \mu_k\,\mathrm{d}N_k \qquad \mu_k \equiv \left(\frac{\partial \mathcal{H}}{\partial N_k}\right)_{p,\mathcal{S},N_j} \quad , \quad ,$$

$$\mathrm{d}\mathcal{F} = -\mathcal{S}\,\mathrm{d}T - p\,\mathrm{d}V + \sum_k \mu_k\,\mathrm{d}N_k \qquad \mu_k \equiv \left(\frac{\partial \mathcal{F}}{\partial N_k}\right)_{V,T,N_j} \quad , \quad ,$$

$$\mathrm{d}\mathcal{G} = -\mathcal{S}\,\mathrm{d}T + V\,\mathrm{d}p + \sum_k \mu_k\,\mathrm{d}N_k \qquad \mu_k \equiv \left(\frac{\partial \mathcal{G}}{\partial N_k}\right)_{T,p,N_j} \quad .$$

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Phase transitions and chemical reactions, which have the same initial and final pressure and temperature, dictate the common practice of defining the chemical potential in terms of the Gibbs function. although it can be defined from the other characteristic functions as well. The chemical potential is a state function because it is derived from another state function.

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Using the idea of extensive and intensive quantities, we can now derive another useful relation involving the Gibbs free enthalpy. First recall the definition of chemical potential This equation says that if you add one particle to a system, holding the temperature and pressure fixed, the Gibbs free enthalpy of the system increases by μ. If you keep adding more particles, each one again adds μ to the Gibbs free energy. Now you might think that during this procedure the value

of μ could gradually change, so that by the time you've doubled the number of particles, μ has a very different value from when you started. But in fact, if T and p are held fixed, this can't happen: each additional particle must add exactly the same amount to $\mathcal{G}$, because $\mathcal{G}$ is an extensive quantity that must simply grow in proportion to the number of particles. The constant of proportionality, according to equation is simply μ:

$$\mathcal{G} = N\mu$$

This amazingly simple equation gives us a new interpretation of the chemical potential, at least for a pure system with only one type of particle: μ is just the Gibbs free energy per particle.

The preceding argument is subtle, so please think it through carefully. Perhaps the best way to understand it is to think about why the same logic can't be applied to the Helmholtz free energy. The problem here is that to increase $\mathcal{F}$ by an amount μ, you have to add a particle while holding the temperature and volume fixed. Now, as you add more and more particles, μ does gradually change, because the system is becoming more dense. It's true that $\mathcal{F}$ is an extensive quantity, but this does not imply that $\mathcal{F}$ doubles when you double the density of the system, holding its volume fixed. In the previous section it was crucial that the two variables being held fixed T and p, were both intensive, so that all extensive quantities could grow in proportion to N.

For a system containing more than one type of particle, the equation generalizes in a natural way:

$$\boxed{\mathcal{G} = \sum_k \mu_k N_k} \quad .$$

The proof is the same as before, except that we imagine building up the system in infinitesimal increments keeping the proportions of the various species fixed throughout the process. This result does not imply, however, that $\mathcal{G}$ for a mixture is simply equal to the sum of the $\mathcal{G}$ for the pure components. In fact the μ_k in are generally different from their values for the corresponding pure substances.

IN PROGRESS

67.11 Other Thermodynamic Systems

©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|||

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67.11.01 Chemical Reactions

©|S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...|||

In chemistry, chemical reactions in the laboratories are usually at constant atmospheric pressure. In this case, as discussed in section § 65.08.02,

$$\Delta\mathcal{H} = \mathcal{Q} \quad ,$$

and hence $\Delta\mathcal{H}$ is the reversible heat added to the system: the heat absorbed by the reaction.

- If $\Delta\mathcal{H} < 0$, the reaction is called exothermic and heat will be emitted.
- If $\Delta\mathcal{H} > 0$, the reaction is called endothermic and heat will be absorbed.

However, this does not explain whether or not a chemical reaction will actually proceed.

Most often, reactions occurs at constant temperature and pressure. In fact, even if the temperature may change during a reaction, the final products usually thermalize with the environment to the original temperature, and therefore, we can use functions of state. Therefore a chemical system will minimize the Gibbs function, so that, if the Gibbs function decreases, the reaction may spontaneously occur.

© - QUOTE

However, one may also need to consider the kinetics of the reaction. Often a reaction has to pass via a metastable intermediate state, which may have a higher Gibbs function, so the system cannot spontaneously lower its Gibbs function without having it slightly raised first. This gives a reaction an activation energy that must be added before the reaction can proceed, even though the completion of the reaction gives you all that energy back and more.

67.11.02 ElectroMagnetic Radiation

©|H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|3.6||

Read § 65.09.01.06.

67.11.03
Rubber Band

This § is referenced at pages:
[2553]

©|H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|3.7||

For a rubber band a simple good model is:

$$\mathcal{U} = cL_0T \quad , \tag{67.11.01}$$

→ 2685

and

$$\tau = bT\left(\frac{L-L_0}{L_1-L_0}\right) \qquad \text{for } L_0 \le L \le L_1 \quad . \tag{67.11.02}$$

→ 2685

The corresponding fundamental equation for the rubber band is:

$$\mathcal{S} = \mathcal{S}_0 + cL_0 \log\left[\frac{\mathcal{U}}{\mathcal{U}_0}\right] - \frac{b\,(L-L_0)^2}{2\,(L_1-L_0)} \quad ,$$

where b and c are constants.

1. For the rubber band model, calculate the fractional change in $(L-L_0)$ that results from an increase ΔT temperature, at constant tension. Express the result in terms of the length and the temperature.
2. A rubber band is stretched by an amount $\mathrm{d}L$, at constant T. Calculate the heat transfer $\mathrm{d}\mathcal{Q}$ to the rubber band. Also calculate the work done. How are these related and why?
3. If the energy of the un-stretched rubber band, equation (67.11.01), were found to increase quadratically with T, instead of linearly, would equation (67.11.02) require alteration? Again find the fundamental equation of the rubber band.

67.11.04
Elastic Wire

This § is referenced at pages:
[2565]

Si consideri una sbarra elastica, cioè un sistema solido one-dimensionale descrivibile come un sistema elastico ideale. Si definisce il modulo di Young isotermo, E,

$$\frac{1}{E} \equiv \frac{1}{X}\left(\frac{\partial X}{\partial \tau}\right)_T \tag{67.11.03}$$

che ha le dimensioni di una pressione ed è necessariamente non negativo. Infatti se E fosse negativo significherebbe che comprimendo la sbarra ($\Delta X < 0$) aumenta la tensione ($\Delta\tau > 0$) e quindi la sbarra eserciterebbe una forza di trazione ai propri estremi tendente ad accorciarla ulteriormente rendendo instabile il sistema per fluttuazioni di lunghezza.

Read § 67-002

DRAFT - IN PROGRESS

67.11.05
Magnetic Systems

©|H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|§ 3.8|*Excellent*|

Read § 67.12.

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An idiosyncrasy of magnetic systems becomes evident if we attempt to consider problems about the condition of equilibrium of two subsystems following the removal of a constraint.
We soon discover that we do not have the capability of constraining the magnetic moment; in practice the magnetic moment is always unconstrained! We can specify and control the magnetic field applied to a sample Uust as we can control the pressure), and we thereby can bring about a desired value of the magnetic moment. We can even hold that value of the magnetic moment constant by monitoring its value and by continually adjusting the magnetic field-again, just as we might keep the volume of a system constant by a feedback mechanism that continually adjusts the external pressure. But that is very different from simply enclosing the= system in a restrictive wall. There exist no walls restrictive with respect to magnetic moment. Despite the fact that the magnetic moment is an unconstrainable variable, the over-all structure of thermodynamic theory still applies. The fundamental equation, the equations of state, the Gibbs-Duhem, and the Euler relations maintain their mutual relationships. The nonavailability of walls restrictive to magnetic moment can be viewed as a mere experimental quirk, that does not significantly influence the applicability of thermodynamic theory.

67.11.06
Ideal Solutions

This § is referenced at pages:
[2545]

© - QUOTE

M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...
The correlation and prediction of the properties of real materials is greatly facilitated when one has a standard or model of ideal behavior to which real behavior can be compared. Such models should be simple, and they should conform to the behavior of real materials at least in some limiting condition. Thus, as we have seen, the ideal gas serves as a useful model of the behavior of gases, and it represents real-gas behavior in the limit as p approaches zero.
Another important model is the ideal solution, which serves as a standard of behavior for the composition dependence of the properties of mixtures.
Thus we define an ideal solution as one for which

$$\mu_k = g_k + RT \log [x_k]$$

for all constituent species at all temperatures, pressures, and compositions as a function of the Gibbs energy of pure k at the T and p of the mixture and in the same physical state (real gas, liquid, or solid) as the mixture.
Ideal-gas mixtures are ideal solutions. However, ideal solutions are not necessarily ideal-gas mixtures. Nevertheless, many features of the ideal solution are superficially similar to those of the ideal-gas mixture.

Una soluzione ideale è una soluzione tale che il potenziale chimico del componente k -esimo in soluzione è dato dalla relazione

$$\boxed{\mu_k = g_k + RT \log [x_k]} \tag{67.11.04}$$

→ 2720

dove x_k è la frazione molare del componente k -esimo e g_k è la funzione di Gibbs molare del componente k -esimo.

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67.12 ThermoDynamics of Systems in Electric|Magnetic Fields

This § is referenced at pages:
[1946, 1947, 2686]

©|R.Balian, *From Microphysics To Macrophysics*, 2007, Springer, ...Ed., ...|§ 6.6.5||

©|F.Reif, *Fundamentals Of Statistical And Thermal Physics*, 2009, Waveland Press, ...Ed., ...|§ 11.1, 11.2||

©|V.Heine|**The thermodynamics of bodies in static ElectroMagnetic fields** |*Math. Proc. Camb. Philos. Soc, Volume ...*

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|||

©|S.Bobbio, *Electrodynamics of Materials*, 2000, Academic Press, 1stEd., ...||*Definitive*|

©|A.Zangwill, *Modern electrodynamics*, 2012, CUP, 1stEd., ...||*Excellent!*|

©|O.Narayan and A. P. Young|**Free energies in the presence of Electric|Magnetic fields** - Am.J.Phys, , ..., ..., ...Ed., ...

Read § 41.02.01.01 for additional topics, § **??**.

The problem with long-range interactions can be readily exposed by observign thar additiviy of the internla elergy is lost:

$$\mathcal{U}_{A+B} = \mathcal{U}_A + \mathcal{U}_{AB} + \mathcal{U}_B \neq \mathcal{U}_A + \mathcal{U}_B \quad ,$$

due to the appearance of an interaction term, $\mathcal{U}_{AB}$ whicih is negligible in the case of short-range interactions.

By keeping this in mind, as avery step of classucal thermodynamics must be reviewed under this new light, it is owever possible to bypass the problem by rearranging the three terms:

$$\mathcal{U}_{A+B} = \mathcal{U}_A + (\mathcal{U}_{AB} + \mathcal{U}_B) = (\mathcal{U}_A + \mathcal{U}_{AB}) + \mathcal{U}_B \quad ,$$

so that additivity is recovered but soem arbotrairt, and possibily ambiguitym, is introdcued. In other words, instead of considering the two sysmtes, A and B, we consider either the two systems A and B plus the interaction energy between A and B, or the two systems A plus the interaction energy between A and B and B.

It must be emphasized that the following discussion is limited to the case of Polarizable and|or Magnetizable materials, and not to fixed Polarization/Magnetization bodies, which need a different dedicated treatment. No hysteresis is allowed as well, which needs a different dedicated treatment as well.

Moreover, an **homogeneous and isotropic system** is assumed.

In case the material is non homogeneous, the usual local treatment should be carried on to extend the treatment.

In case the material is non isotropic, the relations between ElectroMagnetic fields becomes tensor relations and the usual tensor treatment should be carried on to extend the treatment.

Pressure and volume are thermodynamics variables for every material system. Even if they will be neglected in the present treatment, the formalism of thermodynamics can be used to include pressure and volume changes.

Finally, in order to simplify the mathematical technicalities, very small samples are assumed.

As usual, thermodynamics provides general relations, while statistical physics allows to determine the particular form of the thermodynamic functions according to the specific model.

© - QUOTE

... Guggenheim has applied thermodynamics to a static ElectroMagnetic system, which includes the whole infinite field and all bodies in it, the latter in thermal equilibrium with one another at an absolute temperature T. ...

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... Expressions for the ElectroMagnetic contributions to the Helmholtz free energy and the entropy follow from ..., and any other thermodynamic function may be derived from these two as required. The following conditions apply to Guggenheim treatment and also to the present work: steady-state conditions and slow reversible thermodynamic processes are assumed; all bodies are constrained to constant volume; Electric|Magnetic susceptibilities maybe anisotropic, vary as functions of position, and may depend on the temperature and field strengths; hysteresis effects are absent. ...
... there is one point that needs emphasizing. The work ... involves integrations over all space where the fields do not effectively vanish. This work is stored by the system as a whole, and not necessarily with an energy density $\mathbf{E}\cdot\delta\mathbf{D}+\mathbf{H}\cdot\delta\mathbf{B}$. Thus previous results apply to the complete region of the field, without giving any indication of how the work, free energy and entropy may be distributed between the different bodies. It has not always been recognized in the past that there is no physical significance in assuming the energy to be distributed through space in any definite way. The different parts of the field are all closely and automatically coupled, because of the **action at a distance** nature of ElectroMagnetic forces. The only independent thermodynamic quantities of ElectroMagnetic origin that are directly physically observable are the heat flowing in or out of the individual bodies, and the total net work done on the whole region of the field during any rearrangement of charges, currents and magnets. Thus the individual free energy of one body has no physical significance itself. However, associated with each body there is a definite entropy, from which its individual properties can be deduced; for instance, the cooling on adiabatic deMagnetization. ...
... the thermodynamic behavior of any one of the bodies in the ElectroMagnetic field is determined by introducing the principle that its behavior depends only on the field over the region of that body itself. Hence another system is considered instead, consisting of the particular body, with all other material media being replaced by free space; a certain redistribution of charges and currents ensures that the field strengths inside the body are exactly the same as before. The thermodynamic behavior of this auxiliary system is completely determined by Guggenheim results, because there is only the one body to which heat and temperature changes can refer. ...
... The formulae of the last section will now be re-expressed in terms of the externally applied field. Consider a single body, at a uniform temperature, in a static ElectroMagnetic field produced by some set of charges and currents. ...
... The infinitesimal work, $\delta\mathcal{W}$ done in introducing or rearranging charges or currents may be transformed to:

$$\delta\mathcal{W}=\iiint\limits_{\infty}(\mathbf{E}_0\cdot\delta\mathbf{D}_0)\,\mathrm{d}V+\iiint\limits_{\infty}(\mathbf{H}_0\cdot\delta\mathbf{B}_0)\,\mathrm{d}V+\iiint\limits_{V}(-\mathbf{P}\cdot\delta\mathbf{E}_0)\,\mathrm{d}V+\iiint\limits_{V}(+\mathbf{B}_0\cdot\delta\mathbf{M})\,\mathrm{d}V\quad.\tag{67.12.01}$$

The difficulty in this case is that the long-range ElectroMagnetic forces, makes impossible to unambiguously disentangle the field/material/interaction energies.

One starts from the calculation of the work, that is the change of energy, following ElectroMagnetism, and, following thermodynamics, equates this to the adiabatic work, according to the first law of thermodynamics; in fact, thermodynamics tells us that the adiabatic work is a function of state (and only adiabatic work) is a work not depending on the path.

© - QUOTE

We may illustrate the point by a simple example due to Purcell. Consider a volume of gas confined in one part of a cylinder and bounded on one side by a piston attached to a compressed spring. The spring exerts a force $|F|=pA$ on the piston; here p is the pressure and A the area of the piston. For small displacements of the piston from the position shown we may take F as constant.
If, as is usual, we take the gas to be the thermodynamic system, we have on heating the gas: $\mathrm{d}\mathcal{U}_G=\delta\mathcal{Q}+\delta\mathcal{W}=\delta\mathcal{Q}-|F|\Delta x$ as work has to be done to compress more the spring

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($\delta \mathcal{W} = -|F|\Delta x < 0$).
We may, however, take the thermodynamic system as composed of both the gas and the spring. Then no external work is done, but $đ\mathcal{U}_G + đ\mathcal{U}_S = \delta \mathcal{Q}$
These equations are consistent each other. We see that the two ways of dividing up the internal energy of the system have identical thermodynamic consequences; the difference is merely a question of bookkeeping.

Read § 41.01 and § 41.02 for simple examples explicitly calculated. Read § 41.04.06 for explicit discussion of Polarization/Magnetization by orientation for permanent **Dipoles** .

67.12.01
Materials in External ElectroMagnetic Fields

The ElectroMagnetic properties of any substance are not independent of their thermal properties, like other properties, such as physical properties (density for instance), mechanical properties (the Young module, for instance) and optical properties (the **index of refraction** , for instance).

When dealing with electric or magnetic substances in ElectroMagnetic fields one must introduce a fundamental function, the entropy and|or the internal energy, dependent of ElectroMagnetic variables in order to characterize the ElectroMagnetic and thermal properties of the material.

The problem of the choice of the ThermoDynamics variables is a very subtle one because ElectroMagnetic interactions (that is charges and currents) **interact at distance** by means of the ElectroMagnetic fields. As a consequence **the separation between different parts of the system cannot be carried on in a unique way**. The very definition of work is thus affected.

Note that in all the following considerations the self-energy of a point charges or **Dipoles** (either electric or magnetic) or any other fixed-structure object (a fixed structure) does not enter explicitly into the energy considerations as long as the structure does not change. As long as the structure does not change the self-energy stays constant.

There exist different methods and procedures for the definition of the system and of the pair of conjugate variables, differing in the way the ElectroMagnetic fields are taken into account.

It is useful to remind that, when dealing with ElectroMagnetism in matter, the ElectroMagnetic fields and propersts are the **macroscopic ElectroMagnetic fields and propertis**, resluting from the spatial average of the microscopic ElectroMagnetic fields, averaging to be done on a scale small at the macroscopi level but large at the microscopic level. The spatial average thus condicted allow to get rid of the variation of the ElectroMagnetic fields and propertis at the microscopic scale.

This approach implies that the Polarization and Magnetization vectors, $\mathbf{P}$ and $\mathbf{M}$, are slowly varying at the macroscopic scale and therefore the total average charge and current density can be split into the contribution from free charges and those from **Electric Dipole** | **Magnetic Dipole** :

$$\rho = \rho^{\mathrm{F}} + \rho^{\mathrm{P}} = \rho^{\mathrm{F}} - \mathbf{div}\mathbf{P} \qquad \mathbf{j} = \mathbf{j}^{\mathrm{F}} + \mathbf{j}^{\mathrm{P}} + \mathbf{j}^{\mathrm{M}} = \mathbf{j}^{\mathrm{F}} + \partial_t \mathbf{P} + \mathbf{rot}\mathbf{M} \quad .$$

Tha charge conservation law and **Maxwell Equations** apply as well.

Note that in some cases it might be necessary to consider higher orders multipolar contributions in addition to the **Electric Dipole** | **Magnetic Dipole** .

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67.12.02
Matter in Electric Fields

This § is referenced at pages:
[2549, 2572]

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The total work received during an adiabatic transformation by the total system made of the external conductors and charges (the external sources), the field and the Polarizable material is calculated:

$$\boxed{\delta \mathcal{W}_{\text{TOT}} = \int_{\infty} (\mathbf{E} \cdot \delta \mathbf{D}) \, dV} \quad . \tag{67.12.02}$$

→ 2691 2692 2693 2694

This work to build up the fields and Polarization is produced by the external generators, if any, and|or by the external agents keeping the system in almost-equilibrium at any time in order to implement a quasi-static transformation.

The electric field E appears as the intensive variable.

The product of the electric field D with the infinitesimal volume element appears as the extensive variable for the whole system.

Up to now the procedure is totally unambiguous.

One can then try to split the work in equation (67.12.02) into contributions from the different parts of the system.

The work in equation (67.12.02) contains both the work done to Polarize the material and the energy used to produce the field.

In the case of a linear Polarizable material:

$$\boxed{\delta \mathcal{W}_{\text{TOT}} = \frac{1}{2} \int_{\infty} \delta (\mathbf{D} \cdot \mathbf{E}) \, dV} \quad . \tag{67.12.03}$$

67.12.02.01 Constant Charges

This § is referenced at pages:
[2693]

Consider the case when the electric field is produced by a set of fixed charges. Let the electric field produced in absence of the Polarizable material be $\mathbf{E}_0$. This is different from $\mathbf{E}_1$ of § 67.12.02.02. The work to produce the electric field in absence of the Polarizable material is:

$$\delta \mathcal{W}^{0}_{\text{EMPTY}} = \varepsilon_0 \int_{\infty} (\mathbf{E}_0 \cdot \delta \mathbf{E}_0) \, dV \quad . \tag{67.12.04}$$

System Made of the Polarizable Material Plus Its Interaction With the Applied Field

It seems to be natural to split the total work into two parts. The first part consists of the part related to the sources and the field when no Polarizable material is present. The second part of the work is related to the Polarizable material and to its interaction with the electric field. The change of this second part during an adiabatic transformation is thus:

$$\delta \mathcal{W}_0 \equiv \delta \mathcal{W}_{\text{TOT}} - \delta \mathcal{W}^{0}_{\text{EMPTY}} \quad ,$$

which can be written as

$$\boxed{\delta \mathcal{W}_0 = - \int_{\text{MAT}} (\mathbf{P} \cdot \delta \mathbf{E}_0) \, dV} \quad . \tag{67.12.05}$$

→ 2691 2692

The two conjugate variables appearing in the definition (67.12.05) of the work received by the Polarizable material are the applied field, $\mathbf{E}_0$, and the Polarization of the material. The work received by the material

is expressed as an integral over the volume of the Polarizable material only, in direct contrast with the total work in equation (67.12.02), expressed as an integral over the whole space.

If one knows the equation of state relating $\mathbf{P}$ to $\mathbf{E}$ (or, equivalently, $\mathbf{E}_0$) one can find a thermodynamic potential by integrating (67.12.05). The Polarization is usually expressed as a function of the applied field $\mathbf{E}_0$ and the temperature. This introduces naturally a free energy, $\mathcal{F}^0[T, \mathbf{E}_0]$. This is the suitable thermodynamic potential to describe the Polarizable material in the field produced by given sources, those which would produce the field $\mathbf{E}_0$ in absence of the Polarizable material. The free energy can thus be expressed as:

$$\mathcal{F}^0[T, \mathbf{E}_0] - \mathcal{F}^0[T, \mathbf{0}] = -\int_0^1 \mathrm{d}\lambda \int_{\mathtt{MAT}} (\,\mathbf{P}[T, \lambda\mathbf{E}_0]\cdot\mathbf{E}_0\,)\ \mathrm{d}V \quad . \tag{67.12.06}$$

→ 2692

The difference in equation (67.12.06) is the total work supplied to bring the Polarizable material from infinity to the region where the field is $\mathbf{E}_0$ when the Polarizable material is not there, at constant temperature and for fixed values of the external sources, at fixed charges.

The Legendre transform gives the internal energy of the system consisting of the material and of its interaction with the applied field $\mathbf{E}_0$ which would be created by the external sources alone:

$$\mathcal{U}^0[\mathcal{S}, \mathbf{E}_0] = \mathcal{F}^0[T, \mathbf{E}_0] + T\mathcal{S} \quad . \tag{67.12.07}$$

→ 2692

The analogous study at the microscopic level can be best carreid on by resticting to a partial Hamiltonian describing the Polarizable material in the applied field $\mathbf{E}_0$ only, instead of writing a global Hamiltonian of the Polarizable material and the outside sources. This Hamiltonian depnds on $\mathbf{E}_0$ only through the term:

$$-\int \left(\hat{\mathbf{P}}\cdot\mathbf{E}_0\right)\ \mathrm{d}V \quad , \tag{67.12.08}$$

→ 2692 2695

where $\int \hat{\mathbf{P}}\ \mathrm{d}V$ is the sum of the microscopic **Electric Dipoles** . It contains other terms accountig for interaxctions between the latter. This approach is the microscopic conterpart of waht describred so far. Infact, if $\mathbf{E}_0$ is varyed slowly and there is no heat exchange with the sourrounding world, the change in the Hamiltonian can be identified with (67.12.08) where $\mathbf{P}$ is interpreted as the statistical average of $\hat{\mathbf{P}}$. By regarding this energy variation as as work in an adiabatic transformation we can identify $\langle\hat{H}\rangle$ with the macroscopic internal energy (67.12.07). The replacement of the applied field $\mathbf{E}_0$ in the macroscopic **Maxwell Equations** by the total field $\mathbf{E}$ is the result of the interaction between **Dipoles** .

System Made of the Polarizable Material Only

The contribution of the interaction of the Polarizable material with the applied field can be subtracted as follows. We no longer include the interaction term:

$$-\int \left(\mathbf{E}_0\cdot\hat{P}\right)\ \mathrm{d}V \quad .$$

The work received by the Polarizable material itself is:

$$\boxed{\delta\mathcal{W} = \int (\,\mathbf{E}_0\cdot\,\delta\mathbf{P}\,)\ \mathrm{d}V} \quad . \tag{67.12.09}$$

Forces and Torques on a Polarizable Material in the Electric Field at Constant Charges

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At constant charge:

$$\boxed{F_\xi = -\left(\frac{\partial W}{\partial \xi}\right)\bigg|_{\mathcal{Q}} = \int_{\mathtt{Volume}} (\,\mathbf{P}\cdot\mathbf{grad}\,)\,\mathbf{E}_0\ \mathrm{d}V} \quad . \tag{67.12.10}$$

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67.12.02.02 Constant Voltages

This § is referenced at pages:
[2691]

Consider the case when the electric field is produced by a set of fixed potentials. Let the electric field produced in absence of the Polarizable material be $\mathbf{E}_1$. This is different from $\mathbf{E}_0$ of § 67.12.02.01. The work to produce the electric field in absence of the Polarizable material is:

$$\delta \mathcal{W}^{\,1}_{\texttt{EMPTY}} = \varepsilon_0 \int\limits_{\texttt{AllSpace}} (\,\mathbf{E}_1 \cdot \delta \mathbf{E}_1\,)\; \mathrm{d}V \quad . \tag{67.12.11}$$

System Made of the Polarizable Material Plus Its Interaction With the Applied Field

It seems to be natural to split the total work into two parts. The first part consists of the part related to the sources and the field when no Polarizable material is present. The second part of the work is related to the Polarizable material and to its interaction with the electric field. The change of this second part during an adiabatic transformation is thus:

$$\delta W_{\,1} \equiv \delta W_{\,\texttt{TOT}} - \delta \mathcal{W}^{\,1}_{\texttt{EMPTY}} \quad ,$$

which can be written as

$$\boxed{\delta W_{\,1} = + \int\limits_{\texttt{MAT}} (\,\mathbf{E}_1 \cdot \delta \mathbf{P}\,)\; \mathrm{d}V} \quad . \tag{67.12.12}$$

→ 2693

The two conjugate variables appearing in the definition (67.12.12) of the work received by the Polarizable material are the applied field, $\mathbf{E}_1$, and the Polarization of the material. The work received by the material is expressed as an integral over the volume of the Polarizable material only, in direct contrast with the total work in equation (67.12.02), expressed as an integral over the whole space.

If one knows the equation of state relating $\mathbf{P}$ to $\mathbf{E}$ (or, equivalently, $\mathbf{E}_1$) one can find a thermodynamic potential by integrating (67.12.12). The Polarization is usually expressed as a function of the applied field $\mathbf{E}_1$ and the temperature. This introduces naturally a free energy, $\mathcal{F}^1[T, \mathbf{E}_1]$. This is the suitable thermodynamic potential to describe the Polarizable material in the field produced by given sources, those which would produce the field $\mathbf{E}_1$ in absence of the Polarizable material. The free energy can thus be expressed as:

$$\mathcal{F}^1[T, \mathbf{E}_1] - \mathcal{F}^1[T, \mathbf{0}] = + \int\limits_0^1 \mathrm{d}\lambda \int\limits_{\texttt{MAT}} (\,\mathbf{E}_1[T, \lambda \mathbf{P}] \cdot \mathbf{P}\,)\; \mathrm{d}V \quad . \tag{67.12.13}$$

→ 2693

The difference in equation (67.12.13) is the total work supplied to bring the Polarizable material from infinity to the region where the field is $\mathbf{E}_1$ when the Polarizable material is not there, at constant temperature and for fixed values of the external sources, at fixed potentials.

The Legendre transform gives the internal energy of the system consisting of the material and of its interaction with the applied field $\mathbf{E}_1$ which would be created by the external sources alone:

$$\mathcal{U}^1[\mathcal{S}, \mathbf{E}_1] = \mathcal{F}^1[T, \mathbf{E}_1] + T\mathcal{S} \quad . \tag{67.12.14}$$

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System Made of the Polarizable Material Only

$$\delta W_{\,1} = -\int_{\text{MAT}} (\,\mathbf{P}\cdot\,\delta\mathbf{E}_1\,)\;\mathrm{d}V \quad . \tag{67.12.15}$$

Forces and Torque on a Polarizable Material in the Electric Field at Constant Voltages

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$$F_\xi = +\left(\frac{\partial W}{\partial \xi}\right)\bigg|_V = \int_{\text{Volume}} (\,\mathbf{P}\cdot\mathbf{grad}\,)\,\mathbf{E}_0\;\mathrm{d}V \quad . \tag{67.12.16}$$

DRAFT IN PROGRESS

67.12.02.03 Interaction of Polarization With Electric Fields

The interaction of an **Electric Dipole** $\mathbf{p}$ with the electric field is given by:

$$\mathcal{U}' = -\,\mathbf{p}\cdot\mathbf{E}_0 \quad .$$

The **Dipole** $\mathbf{p}$ interacts with the external field $\mathbf{E}_0$. The fact that the real electric field is not $\mathbf{E}_0$ but $\mathbf{E} \neq \mathbf{E}_0$ is due to the presence of the Polarizable material itself and therefore it does not enter in the interaction field energy.

67.12.02.04 First Law of ThermoDynamics

Assume isothermal conditions.

Assume fixed field sources, that is fixed charge distribution. The external generators do no work to keep the fixed charge distribution.

The first law of thermodynamics, for a small volume of matter ΔV, reads:

M := matter

M+F := matter plus field interaction

$$\mathcal{U}_{\text{F}} = \frac{\varepsilon_0}{2}(\,\mathbf{E}_0\cdot\mathbf{E}_0)\,\Delta V$$

$$\delta\mathcal{U}_{\text{M}} = \delta\Omega - p\,\mathrm{d}V + (\,\mathbf{E}_0\cdot\,\delta\mathbf{P}\,)\,\Delta V \qquad \delta\mathcal{F}_{\text{M}} = -\mathcal{S}\,\mathrm{d}T - p\,\mathrm{d}V + (\,\mathbf{E}_0\cdot\,\delta\mathbf{P}\,)\,\Delta V \quad \text{does not include field interaction}$$

$$\delta\mathcal{U}_{\text{M+F}} = \delta\Omega - p\,\mathrm{d}V - (\,\mathbf{P}\cdot\,\delta\mathbf{E}_0\,)\,\Delta V \qquad \delta\mathcal{F}_{\text{M+F}} = -(\,\mathbf{P}\cdot\,\delta\mathbf{E}_0\,)\,\Delta V \quad \text{does include field interact}$$

Both relations are correct; the difference is that the internal energy is defined in (M+F) and (M) for different thermodynamic systems:

$$\delta\mathcal{U}_{\text{M+F}} - \delta\mathcal{U}_{\text{M}} = -\Delta V\,\delta(\,\mathbf{P}\cdot\mathbf{E}_0) \quad .$$

Clearly, the transformation between $\mathcal{U}_{\text{M+F}}$ and $\mathcal{U}_{\text{M}}$ is a Legendre transformation, similar to the transformation between energy and entalphy.

The different definitions of the **energy of the Polarizable material** share the energy in a different way between the different parts of the complete system formed by the external conductors, the field and the Polarizable material.

Only the global expression (67.12.02), which includes the field sources, has an intrinsic significance. All other expressions are convenient computational tools. The external sources are eliminated in all cases, but this elimination leads to different results depending on the nature of the sources (either fixed charges or potentials). Moreover, in both cases one may include or not the interaction of the matter with the field.

In all cases the conjugate variables are the Polarization, $\mathbf{P}$, and the so-called **applied field**, $\mathbf{E}_0$ or $\mathbf{E}_1$, that is the field which would be produced by the fixed charges or **ElectroMagnetic potentials** if there were no Polarizable material. Note also that the role of the **displacement** is played in any case by the variable under the δ, which is either $\mathbf{P}$ or the applied field. The role of the **force** is played by the other variable. It is important to distinguish these cases in order to identify the natural variables of the thermodynamic potentials, which are always the **displacement** variables.

The point of view of fixed charges is better suited to the microscopic point of view. In fact the applied field $\mathbf{E}_0$ does not include the Polarization of the Polarizable material. It occurs in the Hamiltonian as in equation (67.12.08).

67.12.03 Matter in Magnetic Fields

This § is referenced at pages:
[2549, 2573]

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The total work received during an adiabatic transformation by the total system made of the external currents (the external sources), the field and the magnetic substance is calculated:

$$\delta W_{\text{TOT}} = \int_{\infty} (\mathbf{H}\cdot\delta\mathbf{B})\ \mathrm{d}V \quad . \tag{67.12.17}$$

→ 2695 2696

This work to build up the fields and Magnetization is produced by the external generators, if any, and|or by the external agents keeping the system in almost-equilibrium at any time in order to implement a quasi-static transformation.

The magnetic field H appears as the intensive variable.

The product of the magnetic field B with the infinitesimal volume element appears as the extensive variable for the whole system.

Up to now the procedure is totally unambiguous.

One can then try to split the work in equation (67.12.17) into contributions from the different parts of the system.

The work in equation (67.12.17) contains both the work done to Magnetize the material and the energy used to produce the field.

In the case of a linear Magnetizable material:

$$\delta W_{\text{TOT}} = \frac{1}{2}\int_{\infty} \delta(\mathbf{H}\cdot\mathbf{B})\ \mathrm{d}V \quad . \tag{67.12.18}$$

67.12.03.01 Constant Magnetic Flux

Let the currents free to change. If the magnetic substance is removed from the fields the equation of current generating circuit is:

$$\varepsilon = RI + L\dot{I} \quad .$$

Assume R to be negligible, $R \simeq 0$. In any case one always neglects the energy dissipated by Joule effect, because it has nothing to do with energy of the matter in the field. Until the material moves: $\delta\phi_B \neq 0$ and $\dot{I} \neq 0$. Therefore I is changing. As soon as the material stops moving $\dot{I} = 0$ and the current remains constant. The current at the end of the moving process is different from the one at the beginning. This is the analogous of the constant voltage case for electric fields.

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System Made of the Magnetic Substance Plus Its Interaction With the Applied Field

Let $\mathbf{B}_0$ be the magnetic field which would occur if the Magnetizable substance were removed without any action on the external currents. Furthermore suppose that the external currents are produced by superconducting wires in order to neglect the Joule dissipation and avoid the need for external generators.

$$\boxed{\delta W_{\,0} = -\int\limits_{\text{MAT}} (\,\mathbf{M}\cdot\,\delta\mathbf{B}_0\,)\;\mathrm{d}V} \quad . \tag{67.12.19}$$

The substance is brought into the field without any action on the external currents. Currents are therefore induced and hence the current changes. This work includes this part of energy.

System Made of the Magnetic Substance Only

$$\boxed{\delta W_{\,0} = -\int\limits_{\text{MAT}} (\,\mathbf{B}_0\cdot\,\delta\mathbf{M}\,)\;\mathrm{d}V} \quad .$$

67.12.03.02 Constant External Currents

Let $\mathbf{B}_1$ be the magnetic field which would occur if the Magnetizable substance were removed while keeping constant external currents.

System Made of the Magnetic Substance Plus Its Interaction With the Applied Field

The work received by the substance, including its interaction with the field, for an arbtrary change of parameters is obtained by subtracting from the general expression (67.12.17) the work when no material is present:

$$\int\limits_{\infty} (\,\mathbf{H}\cdot\,\delta\mathbf{B}\,)\;\mathrm{d}V - \frac{1}{\mu_0}\int (\,\mathbf{B}_1\cdot\,\delta\mathbf{B}_1\,)\;\mathrm{d}V \quad .$$

$$\boxed{\delta W = \int (\,\mathbf{B}_1\cdot\,\delta\mathbf{M}\,)\;\mathrm{d}V} \quad . \tag{67.12.20}$$

Using this expression we get by integration the total work released by the generators to maintain the external currents constant and by an observer to bring the substance from infinity into the field.

System Made of the Magnetic Substance Only

If one is interested only in the work done on the substance itself with the external currents being kept constant and not in the lectric work provided by the generators the latter must be substracted. The latter is given by:

$$\begin{aligned} &\delta\int \mathbf{j}\cdot\mathbf{A}\;\mathrm{d}V && \text{if the magntic substance is present} \quad , \\ &\delta\int \mathbf{j}_1\cdot\mathbf{A}_1\;\mathrm{d}V && \text{if the magntic substance is not present} \end{aligned}$$

$$\boxed{\delta W = -\int (\,\mathbf{M}\cdot\,\delta\mathbf{B}_1\,)\;\mathrm{d}V} \quad . \tag{67.12.21}$$

This point of view is the best suited for the microscopic study, with the external field $\mathbf{B}_1$ produced by external constant currents.

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Forces and Torques on a Magnetic Substance in the Magnetic Field at Constant Currents

©|J.D.Jackson, *Classical Electrodynamics*, 1999, J.Wiley & Sons, 3rdEd., ...|6.2||

©|J.P.Pérez *et al.*, *Thermodynamique*, ..., DUNOD, ...Ed., WEB - URL.|23.3||

$$F_\xi = + \left(\frac{\partial W}{\partial \xi} \right) \Big|_J = \int_{\texttt{Volume}} (\mathbf{M}\cdot\mathbf{grad})\, \mathbf{B}_1 \, \mathrm{d}V \quad . \tag{67.12.22}$$

In this case the energy change is the total energy change due to the introduction of the material in the magnetic field. it includes the energy delivered by the generators to keep constant currents, like in the case of constant electric potentials.

67.12.03.03 Interaction of Magnetization With Magnetic Fields

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 2 - § 15.2||

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The - Let us name it - **mechanical** interaction interaction energy of a **Magnetic Dipole** $\mathbf{m}$ with the magnetic field is given by:

$$\mathcal{U}' = -\mathbf{m}\cdot\mathbf{B}_1 \quad .$$

The **Magnetic Dipole** $\mathbf{m}$ interacts with the external field $\mathbf{B}_1$. This energy does not include the work done by any generator to keep the currents constant.

The total energy of the system **Magnetic Dipole** plus current sources, on the other hand, is:

$$\mathcal{U}'' = +\mathbf{m}\cdot\mathbf{B}_1 \quad .$$

This energy does include the work done by generators to keep all the currents constant (the current of the **Magnetic Dipole** and the constant currents generating the external field).

The fact that the real magnetic field, $\mathbf{B}$, is not $\mathbf{B}_1$ but $\mathbf{B} \neq \mathbf{B}_1$ is due to the field produced by the **Magnetic Dipole** itself and therefore it does not enter in the interaction field energy.

67.12.03.04 First Law of Thermodynamics

Assume isothermal conditions.

Assume fixed field sources, that is fixed current distribution. The external generators do work to keep the fixed current distribution.

The first law of thermodynamics, for a small volume of matter ΔV, reads:

M := matter

M+F := matter plus field interaction

$$\mathcal{U}_{\texttt{F}} = \frac{1}{2\mu_0} (\mathbf{B}_1\cdot\mathbf{B}_1)\, \Delta V$$

$$\delta\mathcal{U}_{\texttt{M}} = \delta\mathcal{Q} - p\,\mathrm{d}V - (\mathbf{M}\cdot\delta\mathbf{B}_1)\,\Delta V \qquad \delta\mathcal{F}_{\texttt{M}} = -\mathcal{S}\,\mathrm{d}T - p\,\mathrm{d}V - (\mathbf{M}\cdot\delta\mathbf{B}_1)\,\Delta V \quad \text{does not include field interaction}$$

$$\delta\mathcal{U}_{\texttt{M+F}} = \delta\mathcal{Q} - p\,\mathrm{d}V + (\mathbf{B}_1\cdot\delta\mathbf{M})\,\Delta V \qquad \delta\mathcal{F}_{\texttt{M+F}} = -\mathcal{S}\,\mathrm{d}T - p\,\mathrm{d}V + (\mathbf{B}_1\cdot\delta\mathbf{M})\,\Delta V \quad \text{does include field interac}$$

Both relations are correct; the difference is that the internal energy is defined in (M+F) and (M) for different thermodynamic systems:

$$\delta\mathcal{U}_{\texttt{M+F}} - \delta\mathcal{U}_{\texttt{M}} = +\Delta V\, \delta(\mathbf{M}\cdot\mathbf{B}_1) \quad .$$

Clearly, the transformation between $\mathcal{U}_{\texttt{M+F}}$ and $\mathcal{U}_{\texttt{M}}$ is a Legendre transformation, similar to the transformation between energy and enthalpy.

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Actually, as the ElectroMagnetic relations apply at constant temperature only, all the previous relations do not represent the internal energy, but the free Helmholtz energy.

67.12.04
Summary of Thermo-Mechanics of Matter in ElectroMagnetic Fields

In the following tables, the last two columns show the work per unit volume of matter.

67.12.04.01 Summary of ThermoMechanics of Matter in Electric Fields

	Work by Generators	Matter plus Interaction with Fields	Matter Only
Fixed charges	no	$-(\mathbf{P}\cdot\delta\mathbf{E}_0)$	$+(\mathbf{E}_0\cdot\delta\mathbf{P})$
Fixed voltages	yes	$+(\mathbf{E}_1\cdot\delta\mathbf{P})$	$-(\mathbf{P}\cdot\delta\mathbf{E}_1)$

Table 67.11: Work per unit volume on matter in electric fields for infinitesimal changes of the field configuration.

67.12.04.02 Summary of ThermoMechanics of Matter in Magnetic Fields

	Work by Generators	Matter plus Interaction with Fields	Matter Only
Fixed fluxes	no	$-(\mathbf{M}\cdot\delta\mathbf{B}_0)$	$+(\mathbf{B}_0\cdot\delta\mathbf{M})$
Fixed currents	yes	$+(\mathbf{B}_1\cdot\delta\mathbf{M})$	$-(\mathbf{M}\cdot\delta\mathbf{B}_1)$

Table 67.12: Work per unit volume on matter in magnetic fields for infinitesimal changes of the field configuration.

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67.13 Legendre Transform

This § is referenced at pages:
[2663]

©|R.K. P. Zia, Edward F. Redish, Susan R. McKay|Making sense of the Legendre transform|*Am.J.Phys*, , ..., ..., ...*Ed.*, ..., ...

The Legendre transform, in its basic form, shows how to define a function that contains the same information as the smooth strictly convex/concave function $F[x]$, but as a function of $F'[x]$:

$$G[s] \equiv sx[s] - F[x[s]] \Longrightarrow \quad \{ \quad G'[s] = x[s] \quad \text{and} \quad F'[x] = s[x] \quad \} \quad .$$

Moreover:

$$G[s] + F[x] = sx \qquad G''[s] = x'[s] \qquad F''[x] = s'[x] \quad .$$

The Legendre transform appears frequently in thermodynamics when different variables are traded for their conjugates.

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67.14 Examples and Physical Applications

67.14.01 Ice Fusion by Adiabatic Reversible Compression

67.14.02 Polytropic Transformation for a Pure Substance

Per esprimere la relazione tra pressione e volume in una trasformazione quasi statica generica di una sostanza pura si usa spesso l'equazione empirica:

$$pV^{\delta} = \text{constant} = p_0 V_0^{\delta} \quad , \tag{67.14.01}$$

(detta trasformazione politropica), dove δ è una costante adimensionale opportuna che definisce la trasformazione.

- Il caso $\delta = 0$ corrisponde ad una trasformazione isobara.
- Il caso $\delta = 1$ corrisponde ad una trasformazione isoterma, nel caso di un gas ideale.
- Il caso $\delta \longrightarrow +\infty$ corrisponde ad una trasformazione isocora.
- Il caso $\delta = \gamma =$ indipendente dalla temperatura corrisponde ad una trasformazione adiabatica

It is implicit that the transformation is reversible, or at least quasi-static, as we have an equation assuming that p and V are properties of the system and are defined at each stage.

1. Show that the work is
$$\mathcal{W} = \frac{p_0 V_0}{\delta - 1}\left(\left(\frac{p}{p_0}\right)^{(\delta-1)/\delta} - 1\right) = \frac{p_0 V_0}{\delta - 1}\left(\left(\frac{V}{V_0}\right)^{1-\delta} - 1\right) \qquad \text{per } \delta \neq 1 \quad .$$
Verify it is coincident with the known values for $\delta = 0, 1, \gamma, +\infty$.
2. Calculate the specific heat along a polytropic transformation and verify it is coincident with the known values for $\delta = 0, 1, \gamma, +\infty$.

67.14.03 Properties of a Ideal Gas

An experiment by Joule in the mid-19th century established that the energy of a low density gas is independent of its volume.

1. Show that for a ideal gas internal energy and|or enthalpy only depends on the temperature.
2. Show that for a ideal gas: $c_{\mathsf{p}}[T] - c_{\mathsf{v}}[T] = R$.
3. Determine the fundamental equation for the energy.

67.14.04 Properties of Clausius Fluid

L'equazione di stato di Clausius per un gas è

$$\boxed{p\,(v - b) = RT \qquad \Leftrightarrow \qquad p\,(V - nb) = nRT} \quad . \tag{67.14.02}$$

dove b è una costante caratteristica del gas.

Demonstrate the following statements.

1. L'energia interna specifica per un gas di Clausius è data dalla relazione
$$\boxed{\left(\frac{\partial u}{\partial v}\right)_T = 0 \qquad u = c_{\mathsf{v}} T + \text{constant}} \quad , \tag{67.14.03}$$
→ 2701
la stessa del gas ideale: perchè?
2. The work on a isothermal transformation.

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3. Per un gas di Clausius si ha la relazione
$$\boxed{c_{\mathrm{p}} - c_{\mathrm{v}} = R} \quad .$$
4. It can be shown that:
$$\boxed{\Delta \mathcal{S} = nR \log \left[\frac{V - nb}{V_0 - nb} \right] + \int_{T_0}^{T} \frac{C_{\mathrm{v}}}{T} \mathrm{d}T} \quad .$$
5. L'equazione per un processo adiabatico quasi-statico è
$$\boxed{p(V - nb)^{\gamma} = \text{constant}} \quad .$$
6. Il modulo di compressibilità isoterma di un gas di Clausius è dato da
$$\kappa_{\mathrm{T}} = \frac{(v - b)}{pv} \quad .$$
7. L'efficienza di una macchina termica di Carnot che usa un gas di Clausius come sostanza di lavoro è la stessa che del gas ideale, come si deduce dalla (67.14.03).

67.14.05
Properties of Van Der Waals Fluid

©|J.M.Powers, *Lecture Notes On Thermodynamics*, ..., ..., ...Ed., WEB - URL.|11.5, 11.7||

©|H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|3.5 excellent!!!!||

L'equazione di stato di un fluido di Van der Waals è
$$\boxed{\left(p + \frac{a}{v^2}\right)(v - b) = RT \qquad \Leftrightarrow \qquad \left(p + \frac{n^2 a}{V^2}\right)(V - nb) = nRT} \quad , \tag{67.14.04}$$
→ 2713
dove a e b sono costanti caratteristiche del gas.

The rationale is:

- The volume really available to the molecules is not the full volume of the container, V, but it is reduced by an amount proportional to the number of molecules;
- long-range weakly attractive **Van Der Waals forces** reduce the momenta of the molecules hitting the walls of the volume and thus the pressure and every molecule is slowed by all the others so that the correction term is proportional to the square on the number of molecules.

Clearly, the pressure correction term make the Van der Waals model non extensive.

Demonstrate the following statements.

1. Find $c_{\mathrm{p}} - c_{\mathrm{v}}$ for a **Van der Waals fluid** . Per un fluido di Van der Waals si ha la relazione
$$\boxed{c_{\mathrm{p}} - c_{\mathrm{v}} = R \left(\frac{RTv^3}{RTv^3 - 2a(v - b)^2} \right)} \quad .$$
2. Find that the most general expression of $\mathcal{U}[T, V]$ for a **Van der Waals fluid** is:
$$\boxed{u = c_{\mathrm{v}} T - \frac{a}{v} + \text{constant}} \tag{67.14.05}$$

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WORKED SOLUTION

In fact:

$$\left(\frac{\partial \mathcal{U}}{\partial V}\right)_T = a\frac{n^2}{V^2} \quad ,$$

$$\mathrm{d}\mathcal{U} = C_{\mathrm{V}}\,\mathrm{d}T + a\frac{n^2}{V^2}\,\mathrm{d}V \quad ,$$

→ 2702

$$\left(\frac{\partial C_{\mathrm{V}}}{\partial V}\right)_T = 0 \qquad \text{from } \mathrm{d}\mathcal{U} \text{ being an exact differential} \quad ,$$

$$\boxed{\mathcal{U} - \mathcal{U}_0 = \int_{T_0}^{T} C_{\mathrm{V}}[T]\,\mathrm{d}T - an^2\left(\frac{1}{V} - \frac{1}{V_0}\right)} \quad .$$

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It should, perhaps, be noted that equation (2) is integrable term by term, despite our injunction that such an approach generally is not possible. The segregation of the independent variables T and V in separate terms in equation (2) is a fortunate but unusual simplification which permits term by term integration in this special case.

It make sense that for a **Van der Waals fluid** the energy relations are different from a ideal gas, because the pressure-correcting term is linked to internal interaction energy of the molecules and it does affect the energetic of the molecules.

3. It can be shown that:

$$\boxed{\Delta\mathcal{S} = nR\log\left[\frac{V - nb}{V_0 - nb}\right] + \int_{T_0}^{T}\frac{C_{\mathrm{V}}}{T}\,\mathrm{d}T} \quad .$$

4. L'equazione per un processo adiabatico quasi-statico è

$$\boxed{T(V - nb)^{R/c_{\mathrm{V}}} = \text{constant}} \quad ,$$

whenever c_{V} is fixed and where it is important to note that the exponent is not $\gamma - 1$, as for the ideal gas; in fact for a **Van der Waals fluid** $c_{\mathrm{p}} - c_{\mathrm{V}} \neq R$.

5. L'equazione per un processo adiabatico quasi-statico può esere anche scritta:

$$\boxed{\left(p + \frac{an^2}{V^2}\right)(V - nb)^{1+R/c_{\mathrm{V}}} = \text{constant}} \quad ,$$

whenever c_{V} is fixed.

6. Calcolare il coefficiente di dilatazione termica a pressione costante:

$$\boxed{\beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p} \quad .$$

7. Calcolare il modulo di compressibilità isoterma:

$$\boxed{\kappa_{\mathrm{T}} \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T} \quad .$$

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Here a and b are two empirical constants characteristic of the particular gas. In strictly quantitative terms the success of the equation has been modest, and for detailed practical applications it has been supplanted by more complicated empirical equations with five or more empirical constants. Nevertheless the Van der Waals equation is remarkably successful in representing the qualitative features of real fluids, including the gas-liquid phase transition.
We shall see later that the ideal **Van der Waals fluid** is unstable in certain regions of temperature and pressure, and that it spontaneously separates into two phases ("liquid" and "gas").

67.14.06
Properties of Dieterici Fluid

Find the main thermodynamical properties for a fluid substance satisfying the Dieterici equation of state (64.07.04).

67.14.07
Capacità Termica Di Liquido/Vapore Saturato

©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|4.11||

$$C_{\mathtt{sat}} = C_{\mathtt{V}} + \frac{\beta T}{\kappa_{\mathtt{T}}}\left(\frac{\partial V}{\partial T}\right)_{sat} \quad ,$$
$$C_{\mathtt{sat}} = C_{\mathtt{p}} - \beta T V\left(\frac{\partial p}{\partial T}\right)_{sat}$$

67.14.08
Supercooled Water

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Very pure liquid water can be supercooled at atmospheric pressure to temperatures well below $T = 0°\mathrm{C}$. Assume that a mass of water has been cooled as a liquid to $T = -5\,°\mathrm{C}$. A small crystal of ice (whose mass is negligible) is added to seed the supercooled liquid. If the subsequent change of state occurs adiabatically and at constant (atmospheric) pressure, what fraction of the system solidifies and what is the entropy change (per unit mass) of the system and of the ambient?

- Latent heat of fusion of water at $T = 0°\mathrm{C}$: $\lambda = 333.4\ \mathrm{kJ/kg}$;
- **Heat-Capacity** of water in the range $-5°\mathrm{C} \le T \le 0°\mathrm{C}$: $C = 4.2\ \mathrm{kJ/kg/K}$.

WORKED SOLUTION

The final state of the system is presumed to be the equilibrium state of a mixture of ice and water at atmospheric pressure. This state exists at $T = 0°\mathrm{C}$. The change of state occurs in a closed system at constant pressure and this process is adiabatic, and therefore $\Delta\mathcal{H} = 0$. Note that the system is not isolated, as it is subject to a fixed external pressure.
Since $\Delta\mathcal{H}$ is a property change its value is independent of path, and, therefore, for purposes of calculation we may consider an arbitrary path made up of two reversible steps: heating of the full mass up to $T = 0°\mathrm{C}$ followed by solidification of a fraction $f \equiv \Delta m/m$ of the mass.
The entalphy required to heat the liquid water is taken by the entalphy released on freezing of the

water:

$$0 = \Delta\mathcal{H} = \Delta\mathcal{H}_1 + \Delta\mathcal{H}_2 = m\int_{T_1}^{T_2} c_{\mathrm{p}}\,\mathrm{d}T - fm\lambda = 0 \quad ,$$

$$c_{\mathrm{p}}\,(T_2 - T_1) - f\lambda = 0 \qquad \text{if } c_{\mathrm{p}} \text{ is fixed} \quad .$$

For the entropy:

$$\Delta\mathcal{S}_{\mathrm{U}} = \Delta\mathcal{S} + \Delta\mathcal{S}_{\mathrm{A}} = m\int_{T_1}^{T_2} \frac{c_{\mathrm{p}}}{T}\,\mathrm{d}T - \frac{fm\lambda}{T} \quad ,$$

$$\frac{\Delta\mathcal{S}_{\mathrm{U}}}{m} = c_{\mathrm{p}}\log\left[\frac{T_2}{T_1}\right] - \frac{f\lambda}{T_2} \quad .$$

It is found:

- $f = 0.063$.
- $\Delta\mathcal{S} = 0.7$ kJ/kg/K; $\Delta\mathcal{S} = 0$ ambient (adiabatic walls).

Should one find $f > 1$ the conclusion would be that the final state has only liquid water at a temperature larger than the fusion temperature.
Note this does not describe the real process, but just a process with the same initial and final states.

67.14.09
Superheated Steam

©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|4.19||

67.14.10
Capacità Termica Di Acqua Più Il Contenitore

©|M.dePodesta, *Understanding The Properties Of Matter*, 2002, CRC Press, 2ndEd., ...|11.30||

Una robusta scatola di metallo contiene acqua allo stato solido ed è riscaldata lentamente da $T_1 = -20°\mathrm{C}$ a $T_2 = +20°\mathrm{C}$ mentre la capacità termica dell'oggetto composto è misurata continuamente. Se non si ha a disposizione nessun metodo per tenere conto separatamente della quantità di calore usata come calore latente di fusione e di quella usata per innalzare la temperatura si concluderebbe che c'è un'anomalia nell'andamento della capacità termica. Discutere qualitativamente l'andamento della capacità termica.

WORKED SOLUTION

A temperatura inferiore a $T = 0°\mathrm{C}$ la capacità termica è circa costante ed uguale alla somma delle capacità termiche della scatola e del ghiaccio. Allo zero diventa infinita. A temperatura superiore a $T = 0°\mathrm{C}$ la capacità termica è circa costante ed uguale alla somma delle capacità termiche della scatola e dell'acqua.

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67.14.11
Fundamental Equation and Natural Variables for Internal Energy

The distinction between equations which are fundamental and those which are not fundamental, can be easily explained by comparing the two equations

$$\mathcal{U} = \mathcal{U}[V, \mathcal{S}, N_1, \dots N_n] \quad , \qquad \rightarrow 2705$$
$$\mathcal{U} = \mathcal{U}[V, T, N_1, \dots N_n] \quad . \qquad \rightarrow 2705$$

Given the relation (67.01.05)

$$T = \left(\frac{\partial \mathcal{U}}{\partial \mathcal{S}}\right)_{V,N_k} \quad ,$$

it is clear that temperature can be readily derived from (67.14.11). On the other hand, deriving the entropy from equation (67.14.11) requires to solve the differential equation

$$\mathcal{U} = \mathcal{U}[V, \left(\frac{\partial \mathcal{U}}{\partial \mathcal{S}}\right)_{V,N_k}, N_1, \dots N_n] \quad ,$$

which requires in general appropriate boundary conditions.

67.14.12
Fundamental Equation of the Ideal Gas in the Entropy Representation

©|N.W.Tschoegl, *Fundamentals Of Equilibrium And Steady-State Thermodynamics*, 2000, Elsevier, ...Ed., ...|detailed||

The molar fundamental equation of the single-component ideal gas in the entropy representation is:

$$s = s[\mathfrak{u}, v] = s_0 + (f[\mathfrak{u}] - f[\mathfrak{u}_0]) + R \log [v/v_0] \quad ,$$

where the subscript denotes a suitably chosen reference state. The function $f[\mathfrak{u}]$ is a function of $\mathfrak{u}$ only and its explicit form differs from gas to gas but it is not known in general:

$$f[\mathfrak{u}] = f[\mathfrak{u}_0] + \int_{T_0}^{T} \frac{c_\mathfrak{v}}{T} \, \mathrm{d}T \quad .$$

67.14.13
Fundamental Equation for a Van Der Waals Fluid

The fundamental equation of a **Van der Waals fluid** is:

$$\boxed{s = R \log \left[(v-b)\,(u + a/v)^{(c_\mathfrak{v}/R)}\right] + s_0} \quad ,$$

where s_0 is a constant; it can be applied, as a particular case, also to a ideal or Clausius fluid.

67.14.14
Properties of an Ideal Photon Gas

©|H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|3.6||

©|S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...|Exercise 23.6||

First note that a photon gas is a really ideal gas of massless (so relativistic) particles.

Un gas di fotoni, gas ideale di particelle ultra-relativistiche a massa nulla, è caratterizzato dall'equazione di stato:

$$p = \frac{1}{3}\alpha T^4 \quad , \qquad (67.14.06)$$

dalla ulteriore equzione di stato che fornisce il numero di fotoni, variabile non indipendente:

$$N = \beta V T^3 \quad ,$$

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e dall'equazione per l'energia interna:

$$\mathcal{U} = \alpha V T^4 \quad , \tag{67.14.07}$$

→ 2707

con $\alpha = 7.6{\cdot}10^{-16}\ \mathrm{J/(m^3\cdot K^4)}$ e $\beta = 2.0{\cdot}10^{7}\ \mathrm{1/(m^3\cdot K^3)}$, entrambe costanti universali.

1. Rappresentare le isoterme nel piano pV e calcolare la capacità termica a volume costante per $V = 1{\cdot}10^{10}\ \mathrm{km^3}$ e $T = 2{\cdot}10^{7}\ \mathrm{K}$ (temperature e volume tipici di una stella).
2. Calcolare la variazione di entropia per una generica trasformazione isoterma e per una generica trasformazione isocora.
3. Show that the entropy can be written as:
$$\mathcal{S}[\mathcal{U}, V] = \frac{4}{3}\left(\alpha V \mathcal{U}^3\right)^{1/4} \quad ,$$
in terms of its natural variables.
4. Show that:
$$pV = \left(\frac{\zeta\,[4]}{\zeta\,[3]}\right) N[V,T] k_{\mathrm{B}} T \simeq (0.9004) N[V,T] k_{\mathrm{B}} T \quad .$$
Beware that, even if the equation resembles the one for the molecular ideal gas, for the photon gas $N = N[V,T]$ is not an independent parameter which can be fixed at will, but it is determined by volume and temperature.
5. Show that $\mathcal{G} = 0$; calculate the chemical potential via the internal energy.
6. Calculate $\mathcal{H}$ and $\mathcal{F}$ and show that: $\mathcal{U} = -3\mathcal{F}$, $pV = \mathcal{U}/3$, $\mathcal{S} = 4\mathcal{U}/(3T)$; verify the Gibbs-Duhem relation: $\mathcal{U} = T\mathcal{S} - pV$.
7. Evaluate c_{v} and c_{p}.
8. The universe is considered by cosmologists to be an expanding ElectroMagnetic cavity containing radiation that now is at a temperature of $T = 2.7$ K. What will be the temperature of the radiation when the volume of the universe is twice its present value? Assume the expansion to be iso-entropic (this being a non-obvious prediction of cosmological model calculations).
9. Assuming the ElectroMagnetic radiation filling the universe to be in equilibrium at $T = 2.7$ K, what is the pressure associated with this radiation?
10. Estimate the energy per cubic meter inside a typical room at ambient conditions due to thermal radiation from the environment.

WORKED SOLUTION

$N = N[V,T]$ is not a free parameter, as it is given by V and T. Therefore the system depends on two variables, and not on two plus N. Moreover, as pressure and temperature are related independently from volume, the permissible pairs of independent variables are only: (V,T) and (V,p), but not (p,T).

1. A horizontal line.
2. $$\mathrm{d}\mathcal{S} = \frac{\mathrm{d}\mathcal{U}}{T} + \frac{p}{T}\,\mathrm{d}V \quad ,$$
$$V = \text{constant} \implies \mathrm{d}\mathcal{S} = 4\alpha T^2 V\,\mathrm{d}T = \ldots \quad ,$$
$$T = \text{constant} \implies \mathrm{d}\mathcal{S} = \alpha\,\mathrm{d}V T^3 + \frac{\alpha}{3}T^3\,\mathrm{d}V = \ldots \tag{67.14.08}$$
3. Using equations (65.09.01.06), (65.09.01.06):
$$\mathrm{d}\mathcal{S} = \frac{\mathrm{d}\mathcal{U}}{T} + \frac{p}{T}\,\mathrm{d}V = \left(\frac{\alpha V}{\mathcal{U}}\right)^{1/4}\mathrm{d}\mathcal{U} + \frac{\alpha}{3}\left(\frac{\mathcal{U}}{\alpha V}\right)^{3/4}\mathrm{d}V = \frac{4}{3}\,\mathrm{d}\left(\alpha V \mathcal{U}^3\right)^{1/4} \quad ,$$
$$\implies \mathcal{S} - \mathcal{S}_0 = \frac{4}{3}\left(\alpha V \mathcal{U}^3\right)^{1/4} = \frac{4}{3}\alpha V T^3 = \frac{4\mathcal{U}}{3T} \quad .$$
4. ...
5. ...

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6. As $\mathcal{G} = 0$ it is clear that the chemical potential is zero. This can be cross-checked with the calculation from the internal energy.

$$\mathcal{S}[\mathcal{U}, V] = \frac{4}{3}\left(\alpha V \mathcal{U}^3\right)^{1/4} \quad ,$$

$$\mu \equiv \left(\frac{\partial \mathcal{U}}{\partial N}\right)_{\mathcal{S},V} = 0 \quad .$$

7. Use equation (67.14.07):

$$C_{\mathrm{v}} = 4\alpha V T^3 = 4\frac{\alpha}{\beta} N[V,T] \propto T^3 \implies c'_{\mathrm{v}} = 4\frac{\alpha}{\beta} \qquad \text{for a single photon} \quad .$$

For a photon gas, C_{p} does not exist, because one cannot vary the temperature at constant pressure for a photon gas.

8. ...

67.14.15
Fundamental Equation for Gas Ideale a Capacità Termiche Costanti

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Verificre che per un gas ideale le cui capacità termiche siano costanti valgono le seguenti relazioni (relazioni fondamentali):

$$\mathcal{U}[\mathcal{S}, V] = c_{1\mathrm{u}} V^{(-R/C_{\mathrm{v}})} \exp\left(\mathcal{S}/C_{\mathrm{v}}\right) + c_{2\mathrm{u}} \quad ,$$

$$\mathcal{H}[\mathcal{S}, p] = c_{1\mathrm{h}} p^{(+R/C_{\mathrm{p}})} \exp\left(\mathcal{S}/C_{\mathrm{p}}\right) + c_{2\mathrm{h}} \quad ,$$

dove le c_{k} sono costanti.

67.14.16
Fundamental Equation of a Model System (1)

©|DArovas|2.7.6 Example system|*Excellent example fully developed.*|

The fundamental equation of a model system is:

$$\mathcal{U}[\mathcal{S}, V, N] = a\frac{\mathcal{S}^3}{NV} \quad ,$$

where a is a known constant.

Determine the equation of state, the expression for the chemical potential, the thermal capacities and compressibilities.

67.14.17
Fundamental Equation of a Model System (2)

©|H.B.Callen, *Thermodynamics*, 1985, J.Wiley & Sons, 2ndEd., ...|3.2-1 Example system||

The fundamental equation of a model system is:

$$\mathcal{U}[\mathcal{S}, V, N] = b\frac{\mathcal{S}^4}{NV^2} \quad ,$$

where b is a known constant.

Determine the equation of state, the expression for the chemical potential, the thermal capacities and compressibilities.

LATEX-PROGRESS

WORKED SOLUTION

$b = \left(v_0^2 \theta\right) / R^3$

67.14.18
Adiabatic Free Expansion of a Gas

67.14.19
Gibbs Free Enthalpy and Chemical Potential

1. Show that for a system made by one single chemical species, the chemical potential per molecule is equal to the Gibbs free enthalpy per molecule: $\mu = \mathcal{G}/N$.
2. Show that for a system made by more than one chemical species, the chemical potential per molecule is not equal to the Gibbs free enthalpy per molecule: $\mu_j \neq \mathcal{G}/N_j$.
3. Show that the extensivity of volume, entropy, number of particles, energies and Euler theorem imply:
$$\mathcal{U} = T\mathcal{S} - pV + \sum_k \mu_k N_k \implies \mathcal{G} = \sum_k \mu_k N_k$$
4. Show that the previous result provides the Gibbs-Duhem relation:
$$\mathcal{S}\, \mathrm{d}T - V\, \mathrm{d}p + \sum_k \mu_k\, \mathrm{d}N_k = 0 \quad .$$

67.14.20
Gibbs-Helmholtz Equations

Show that:
$$\mathcal{U} = -T^2 \left(\frac{\partial(\mathcal{F}/T)}{\partial T} \right)_V ,$$
$$\mathcal{H} = -T^2 \left(\frac{\partial(\mathcal{G}/T)}{\partial T} \right)_p .$$

67.14.21
Variazione Della Pressione Atmosferica Con L'altezza

This § is referenced at pages:
[1540, 1540]

©|D.V.Schroeder, *Thermal Physics*, 1999, Addison-Wesley, ...Ed., ...|problem 1.16, 1.40, 5.45||

Si consideri l'aria un gas ideale di massa molecolare media $\mathcal{M}_0 = 29\ \mathrm{g/mol}$, si assuma una atmosfera isoterma a $T = 288\ \mathrm{K}$ e la pressione a livello del mare pari a $p_0 = 1\ \mathrm{atm}$. Assume a fixed acceleration of gravity.

1. Calculate the pressure as a function of the height above the sea-level assuming isothermal atmosphere.
2. Assume the composition of the isothermal atmosphere at sea-level is $f_0 = 0.8\ N_2$ and $f_1 = 0.2\ O_2$. Determine the change of composition as a function of the height.
3. Calculate mean gravitational potential energy per molecule and the specific heat of a column of isothermal atmosphere.
4. Calculate the pressure as a function of the height above the sea-level assuming a Polytropic $p - V$ relation (including adiabatic atmosphere with fixed thermal capacities).

DRAFT-LW-PROGRESS

Read also § 28-012.

WORKED SOLUTION

1. Hydrostatic equilibrium:

$$\frac{\mathrm{d}p[z]}{\mathrm{d}z} = -g\rho[z] \qquad \text{mechanics} \quad ,$$

$$\frac{\mathrm{d}p[\rho[z], T[z]]}{\mathrm{d}z} = -g\rho[z] \qquad \text{thermodynamics}$$

Formula barometrica per atmosfera isoterma composta di un gas ideale di massa molecolare $\mathcal{M}_0$:

$$\boxed{p[z] = p_0 \exp\left[-\frac{\mathcal{M}_0 g z}{RT}\right] = p_0 \exp\left[-\frac{mgz}{k_{\mathrm{B}} T}\right] \qquad \text{the barometric equation}} \quad , \qquad (67.14.09)$$

che scritta nella forma tipica di un decadimento esponenziale

$$\boxed{p[z] = p_0 \exp\left[-\frac{z}{\ell}\right] \qquad \ell \equiv \frac{RT}{\mathcal{M}_0 g} = \frac{k_{\mathrm{B}} T}{mg}} \qquad (67.14.10)$$

da una lunghezza di decadimento della pressione pari a $\ell \simeq 8.4$ km.

2. For the composition af the isothermal atmosphere made of a mixture of ideal gases, one finds:

$$\eta_{\mathtt{N}} \equiv \eta_{\mathtt{N}} \qquad \ell \equiv \frac{RT}{\mathcal{M}_0 g} = \frac{k_{\mathrm{B}} T}{mg} \quad ,$$

$$\eta_{\mathtt{N}}[z] = \eta_{\mathtt{N}0} \exp\left[-z/\ell\right] = \frac{p_0}{k_{\mathrm{B}} T} \exp\left[-z/\ell\right] \qquad \ell \eta_{\mathtt{N}0} = \int_0^\infty \eta_{\mathtt{N}}[z]\, \mathrm{d}z \qquad \ell^2 \eta_{\mathtt{N}0} = \int_0^\infty z \eta_{\mathtt{N}}[z]\, \mathrm{d}z \qquad N = A \int_0^\infty \eta_{\mathtt{N}}[z]\, \mathrm{d}z = \eta_{\mathtt{N}0} A \ell \quad ,$$

$$\langle z \rangle = \ell = \frac{RT}{\mathcal{M}_0 g} = \frac{k_{\mathrm{B}} T}{mg} \quad .$$

The latter equation shows that heavier molecules have a lower mean altitude. For instance, CO_2, is heavier.

3. The total energy of a column of section A is:

$$E = N c_{\mathtt{V}} T + \int_0^\infty mNgz\eta_{\mathtt{N}}[z]\, \mathrm{d}z = N c_{\mathtt{V}} T + mNg\langle z \rangle = N c_{\mathtt{V}} T + N k_{\mathrm{B}} T = N c_{\mathtt{V}} T + \eta_{\mathtt{N}0} A \ell k_{\mathrm{B}} T \quad ,$$

$$\langle \phi \rangle = k_{\mathrm{B}} T \qquad c = c_{\mathtt{V}} + k_{\mathrm{B}} = c_{\mathtt{p}} \qquad \text{specific heat per molecule} \quad .$$

4. polytropic relation for a ideal gas of index γ:

$$p\mathcal{M}_0 = \rho RT \qquad pV^\gamma = \text{constant} \qquad p[T] = KT^{(\gamma/(\gamma-1))} \quad .$$

Therefore, derivation of the $p[T]$ relation gives:

$$\frac{\mathrm{d}p[z]}{\mathrm{d}z} = \frac{K\gamma}{\gamma - 1} T^{(1/(\gamma-1))} \frac{\mathrm{d}T[z]}{\mathrm{d}z} \quad ,$$

$$\frac{\mathrm{d}p[z]}{\mathrm{d}z} = -g\rho[z] = -\frac{g\mathcal{M}_0}{R} \frac{p[z]}{T[z]} = -\frac{Kg\mathcal{M}_0}{R} T^{(1/(\gamma-1))} \quad ,$$

$$\Longrightarrow \frac{\gamma}{\gamma - 1} \frac{\mathrm{d}T[z]}{\mathrm{d}z} = -\frac{g\mathcal{M}_0}{R} \qquad \text{for } \gamma \neq 1 \quad ,$$

$$\Longrightarrow \frac{\mathrm{d}T[z]}{\mathrm{d}z} = \frac{g\mathcal{M}_0}{R} \frac{1-\gamma}{\gamma} \quad ,$$

$$T[z] = \frac{g\mathcal{M}_0}{R} \frac{1-\gamma}{\gamma} (z - z_0) + T[z_0] \quad .$$

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67.14.22
A Saturated Liquid-Vapor Water Mixture

©|YACengel|Example 4-4 and start from example 4-5.||

67.14.23
Equilibrio Di Fase Tra La Grafite E Il Diamante

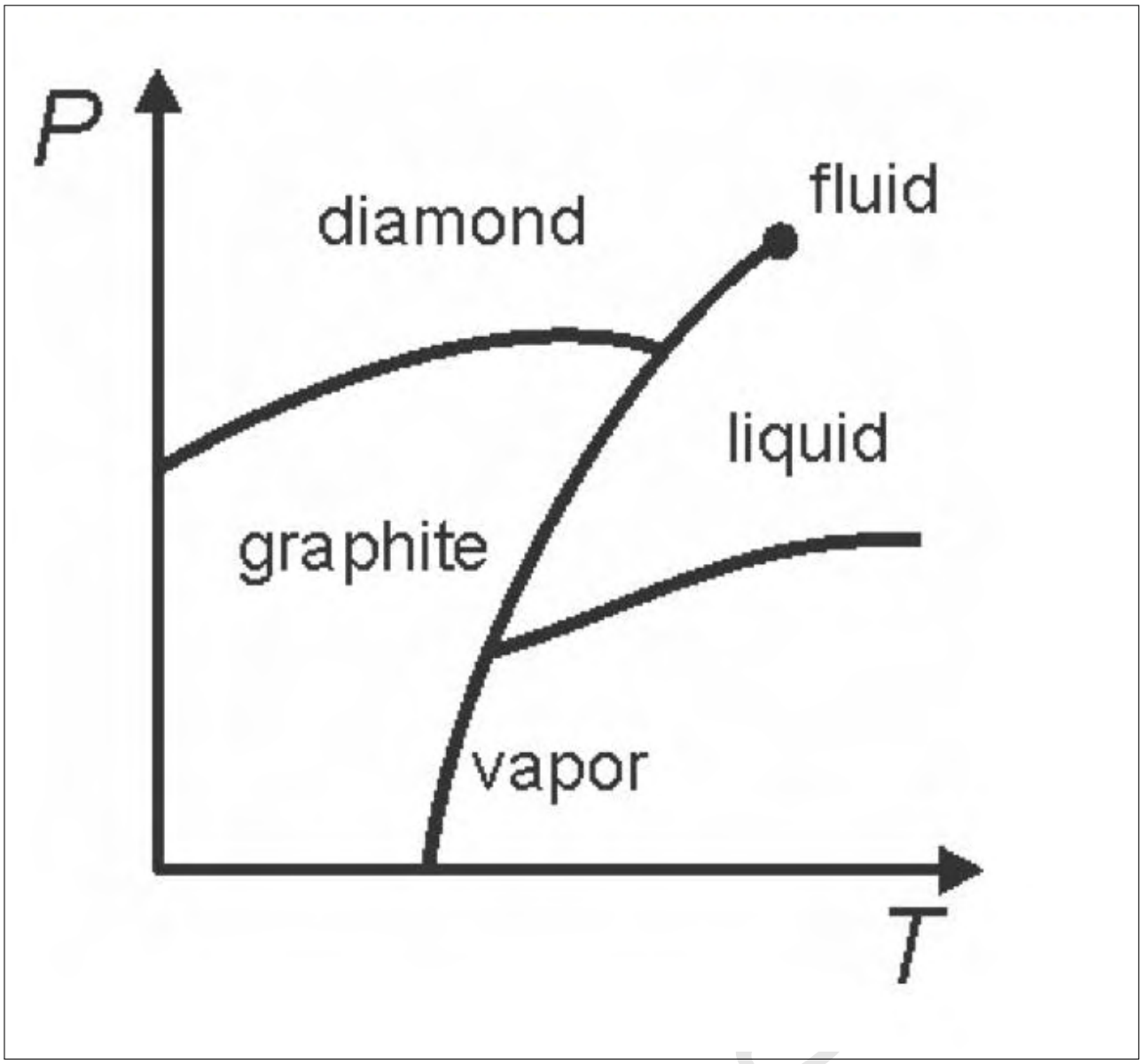
©|M.W.Zemansky & R.H.Dittman, *Heat And Thermodynamics*, 1997, McGraw-Hill, 7thEd., ...|Diagramma di stato in § x.y||

©|D.Halliday & R.Resnick & K.S.Krane, *Fisica*, 2003/2004, C.E.A., 5thEd., ...|17.11||

Using the qualitative phase-diagram in figure answer the questions.

- How many triple points are shown in the phase-diagram and what phases can coexist at each triple point?
- Is it possible to say which solid phase, graphite or carbon, has a higher density?
- Synthetic diamond can be made from graphite: how is it possible?
- Can graphite and diamond be sublimated?
- The standard Gibbs energy of the formation of diamond, at $T_0 = 298$ K and $p_0 = 1$ atm, is $g_d = 2.9$ kJ/mol greater than that of graphite. Which of these two allotropes is more stable at T_0 and p_0?
- Is graphite or liquid carbon denser?
- Why are there two triple points in this diagram?

DRAFT-IN-PROGRESS



Sapendo che a $T_0 = 25°\mathrm{C}$ e $p_0 = 1\ \mathrm{atm}$ i volumi specifici di diamante e grafite sono rispettivamente:

$$v_d = 2.85{\cdot}10^{-1}\ \mathrm{cm}^3/\mathrm{g} \qquad v_g = 4.44{\cdot}10^{-1}\ \mathrm{cm}^3/\mathrm{g}\ ,$$

mentre i rispettivi coefficienti di compressibilità isoterma valgono:

$$\kappa_d = 0.16{\cdot}10^{-6}\ \mathrm{atm}^{-1} \qquad \kappa_g = 3.0{\cdot}10^{-6}\ \mathrm{atm}^{-1}\ ,$$

mentre

$$g^d = 2.390{\cdot}10^5\ \mathrm{J/kg} \qquad g^g = 0\ \mathrm{J/kg} \qquad \text{chosen as zero, as it is the stable phase}\ .$$

Determinare quale pressione è necessaria per trasformare in diamante la grafite, a $T_0 = 25°\mathrm{C}$.

WORKED SOLUTION

Poiché si opera a temperatura costante

$$đg_T = v\ đp_T$$

e la funzione di Gibbs specifica cresce al crescere della pressione. A pressioni e temperature ambiente l'energia di Gibbs specifica del diamante è maggiore di quella della grafite per cui lo stato stabile è la grafite. Tuttavia al crescere della pressione l'energia di Gibbs specifica del diamante cresce meno rapidamente di quella della grafite, a causa dei diversi valori dei volumi specifici, per cui ad una certa pressione il diamante e la grafite potrebbero esistere in equilibrio di fase. Si noti che, in generale, v non è indipendente dalla pressione per cui la dipendenza dalla pressione della

funzione di Gibbs specifica, $g(p)$, non è una funzione lineare, eccetto che si possa considerare v fissato. In generale, quindi, non sono quindi sufficienti queste sole considerazioni a dedurre che esiste una pressione alla quale le due funzioni di Gibbs assumono lo stesso valore, ma occorre determinare il volume specifico in funzione della pressione.
Si ha

$$v = v[T,p] \qquad \left.\frac{\partial v}{\partial p}\right|_T = -\kappa_{\mathrm{T}} v$$

Essendo la temperatura fissata si può sostituire la derivata parziale con quella totale e integrare.
Per variazioni di pressione non troppo alte, $|\Delta p\kappa| \ll 1$, da verificare una volta ottenuto il risultato, si può approssimare l'equazione precedente con

$$v[p,T_0] \simeq v_0\left(1-\kappa_{\mathrm{T}}\left(p-p_0\right)\right)$$

Si noti che il volume specifico non può essere considerato costante, nell'intervallo di pressione considerato, ma il valore di κ_{T}, cioè la variazione relativa del volume specifico la variare della pressione può essere considerato costante.
Ne segue

$$g - g^0 = v_0\left(p-p_0\right) - \frac{v_0\kappa}{2}\left(p-p_0\right)^2$$

che, definendo $\Delta p \equiv p - p_0$, diventa

$$g - g^0 = v_0\left(\Delta p\right) - \frac{v_0\kappa}{2}\left(\Delta p\right)^2$$

Allora si ottiene per la grafite e il diamante

$$g^g[p,T_0] = v_0^g\left(\Delta p\right) - \frac{v_0^g\kappa_g\left(\Delta p\right)^2}{2} + g_0^g \quad ,$$

$$g^d[p,T_0] = v_0^d\left(\Delta p\right) - \frac{v_0^d\kappa_d\left(\Delta p\right)^2}{2} + g_0^d \quad .$$

Uguagliando le due espressioni e risolvendo rispetto alla pressione incognita si determina la minore delle soluzioni per p, $p^\star = 1.53\cdot 10^3$ atm.
Il valore trovato è consistente con l'approssimazione, $|\Delta p\kappa| \ll 1$, fatta nel corso del calcolo.
A questo livello di approssimazione, l'andamento $g[p]$ è una parabola con il vertice in alto e derivata prima positiva a p_0.
Per avere un idea dell'ordine di grandezza di tale pressioni si può stimare a quale profondità sotto il livello del suolo si ottiene tale pressione assumendo per la Terra una densità costante pari alla densità media $\rho = 5.5$ g/cm^3. Si trova una profondità di $\simeq 2.7$ km.
Si noti che tutte queste considerazioni si riferiscono a temperatura costante.
La spiegazione dell'esistenza del diamante, fase solida instabile del carbonio a temperatura e pressione ambiente sta nel fatto che il diamante è uno stato metastabile e/o che la transizione verso la grafite è estremamente lenta.
In entrambi i casi si deve pensare che un diamante non è per sempre....

67.14.24
Calori Specifici Di Un Fluido Di Van Der Waals

L'energia interna specifica per un fluido di Van der Waals è data dalla relazione

$$u = c_{\mathtt{V}} T - \frac{a}{v} + \text{constant}$$

1. Dimostrare, usando l'espressione dell'energia interna specifica, che per un fluido di Van der Waals si ha la relazione
$$c_{\mathtt{p}} - c_{\mathtt{V}} = R \left(\frac{RTv^3}{RTv^3 - 2a(v-b)^2} \right) \quad .$$
2. Dimostrare la stessa relazione facendo uso solo uso dell'equazione di stato (67.14.04) e della relazione generale tra le capacità termiche a pressione e volume costante di un sistema pVT a composizione costante:
$$\boxed{C_{\mathtt{p}} - C_{\mathtt{V}} = T \left(\frac{\partial p}{\partial T} \right)_V \left(\frac{\partial V}{\partial T} \right)_p} \quad .$$

67.14.25
Equilibrio Liquido-Vapore Entro Un Recipiente Rigido Riscaldato

©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|E4.2 4.7 4.8||

Si discuta la seguente situazione, tenendo presente l'andamento tipico del diagramma pV di una sostanza pura. Un recipiente rigido e sigillato è riempito con una miscela liquido-vapore di una sostanza pura. Il recipiente viene scaldato.

1. Discutere qualitativamente il comportamento della miscela liquido-vapore nel caso il recipiente contenga inizialmente solo una (arbitrariamente) piccola quantità di liquido.
2. Discutere qualitativamente il comportamento della miscela liquido-vapore nel caso il recipiente contenga inizialmente solo una (arbitrariamente) piccola quantità di vapore.
3. Discutere qualitativamente il comportamento della miscela liquido-vapore nel caso il recipiente contenga inizialmente una miscela liquido-vapore di volume pari al volume critico.
4. Il volume totale del recipiente vale $V = 1\ \mathrm{m}^3$, e contiene $V_l = 0.05\ \mathrm{m}^3$ di liquido saturo in equilibrio con $V_l = 0.95\ \mathrm{m}^3$ di vapore saturo alla pressione di $1{\cdot}10^5 a$. Quale è la minima quantità di calore $\mathcal{Q}_g$ che deve essere fornita all'acqua affinché il liquido si vaporizzi completamente?
5. Il volume totale del recipiente vale $V = 10\ \mathrm{ft}^3$ e contiene liquido e vapore d'acqua saturi in equilibrio a $200\ \mathrm{psia}$. Se il 99% della massa è liquido quale è la minima quantità di calore $\mathcal{Q}_l$ che deve essere fornita all'acqua affinché il recipiente diventi pieno di liquido

67.14.26
Problems From H. B. Callen

GO TO AFTER included File

PROBLEMS

1.10-1. The following ten equations are purported to be fundamental equations of various thermodynamic systems. However, five are inconsistent with one or more of postulates II, III, and IV and consequently are not physically acceptable. In each case qualitatively sketch the fundamental relationship between S and U (with N and V constant). Find the five equations that are not physically permissible and indicate the postulates violated by each.

The quantities v_0, θ, and R are positive constants, and in all cases in which fractional exponents appear only the real positive root is to be taken.

a) $S = \left(\frac{R^2}{v_0\theta}\right)^{1/3} (NVU)^{1/3}$

b) $S = \left(\frac{R}{\theta^2}\right)^{1/3} \left(\frac{NU}{V}\right)^{2/3}$

c) $S = \left(\frac{R}{\theta}\right)^{1/2} \left(NU + \frac{R\theta V^2}{v_0^2}\right)^{1/2}$

d) $S = \left(\frac{R^2\theta}{v_0^3}\right) V^3/NU$

e) $S = \left(\frac{R^3}{v_0\theta^2}\right)^{1/5} [N^2VU^2]^{1/5}$

f) $S = NR \ln(UV/N^2R\theta v_0)$

g) $S = \left(\frac{R}{\theta}\right)^{1/2} [NU]^{1/2} \exp(-V^2/2N^2v_0^2)$

h) $S = \left(\frac{R}{\theta}\right)^{1/2} (NU)^{1/2} \exp\left(-\frac{UV}{NR\theta v_0}\right)$

i) $U = \left(\frac{v_0\theta}{R}\right) \frac{S^2}{V} \exp(S/NR)$

j) $U = \left(\frac{R\theta}{v_0}\right) NV \left(1 + \frac{S}{NR}\right) \exp(-S/NR)$

1.10-2. For each of the five physically acceptable fundamental equations in problem 1.10-1 find U as a function of S, V, and N.

GO TO BEFORE included File

67.14.27
Temperatura Di Ebollizione Dell'acqua in Montagna

Si consideri l'aria un gas ideale di massa molecolare media $M = 29\ \mathrm{g/mol}$, si assuma una atmosfera isoterma a $T = 288\ \mathrm{K}$ e la pressione a livello del mare pari a $p_0 = 1\ \mathrm{atm}$. La massa molecolare dell'acqua vale $M_{\mathtt{acq}} = 18\ \mathrm{g/mol}$ e il calore latente di vaporizzazione dell'acqua vale $\lambda_{\mathtt{vap}}^{(\mathtt{spe})} = 540\ \mathrm{cal/gr} = 2.259{\cdot}10^6\ \mathrm{J/kg}$ a $T = 100\ ^\circ\mathrm{C}$.

1. Valutare la temperatura T_1 di ebollizione dell'acqua in una località ad un'altezza $h = 1500\ \mathrm{m}$ rispetto al livello del mare.
2. Valutare la temperatura di ebollizione dell'acqua in una pentola a pressione la cui valvola sia tarata alla pressione differenziale di $\Delta p = 1\ \mathrm{atm}$.

WORKED SOLUTION

Per valutare la temperatura di ebollizione dell'acqua in una località ad un'altezza h rispetto al livello del mare occorre dapprima determinare la pressione atmosferica in quella località.
Con i dati del testo si trova $p(h) = 0.837\ \mathrm{atm}$.
Si può ora determinare a quale temperatura la tensione di vapore dell'acqua, calcolata tramite l'equazione di Clapeyron, uguaglia la pressione atmosferica richiesta. L'equazione di Clapeyron per l'equilibrio liquido-vapore, trattando il vapore come un gas ideale, trascurando il volume specifico della fase liquida rispetto a quello della fase gassosa e considerando il calore latente di vaporizzazione indipendente dalla temperatura si scrive

$$p(T) = p_0 \exp\left[-\frac{\lambda_{\mathtt{vap}}^{(\mathtt{mol})}}{R}\left(\frac{1}{T}-\frac{1}{T_0}\right)\right] = p_0 \exp\left[-\frac{M\lambda_{\mathtt{vap}}^{(\mathtt{spe})}}{R}\left(\frac{1}{T}-\frac{1}{T_0}\right)\right] \tag{67.14.11}$$

→ 2716

La relazione tra il generico calore molare e specifico per una massa m e numero di moli n di una sostanza di massa molecolare M è data da

$$m\lambda^{(\mathtt{spe})} = n\lambda^{(\mathtt{mol})} \Longrightarrow \lambda^{(\mathtt{mol})} = \frac{m}{n}\lambda^{(\mathtt{spe})} = M\lambda^{(\mathtt{spe})} \tag{67.14.12}$$

Ne segue, risolvendo per T, $T_1 = 95.1^\circ\mathrm{C}$.
Analogamente si può rispondere alla seconda domanda la sola differenza è che ora occorre trovare la temperatura alla quale la tensione di vapore vale $2\ \mathrm{atm}$ che risulta essere $T_2 = 121^\circ\mathrm{C}$.

WORKED SOLUTION

1. $T_1 = 95.1^\circ\mathrm{C}$
2. $T_2 = 121^\circ\mathrm{C}$.

DRAFT

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67.14.28
Variazione Del Punto Di Fusione Del Ghiaccio

La densità del ghiaccio vale $\rho_{\mathtt{gh}} = 0.92\ \mathrm{g/cm^3}$, la densità dell'acqua vale $\rho_{\mathtt{ac}} = 1\ \mathrm{g/cm^3}$ e il calore latente di fusione vale $\lambda_{\mathtt{fus}} = 80\ \mathrm{cal/g}$ a $0\ ^\circ\mathrm{C}$.

1. Calcolare la variazione di pressione Δp necessaria per far diminuire di $\Delta T = 1^\circ\mathrm{C}$ la temperatura di fusione del ghiaccio a pressione atmosferica.
2. Calcolare quale deve essere lo spessore h di una lastra di ghiaccio a $0^\circ\mathrm{C}$ affinché la temperatura di fusione del ghiaccio alla base si abbassi di $\Delta T = 1^\circ\mathrm{C}$.

Dati

Water triple point: $T = 273.16\ \mathrm{K} = 0.01\,^\circ\mathrm{C}$; $p = 612\ \mathrm{Pa} = 0.006\ \mathrm{atm}$.
Water freezing point at $1\ \mathrm{atm}$: $T = 273.15\ \mathrm{K} = 0.00\,^\circ\mathrm{C}$; $p = 101.325\ \mathrm{kPa} = 1\ \mathrm{atm}$.

CRA

WORKED SOLUTION

Si noti che la variazione di temperatura richiesta è piccola rispetto alla temperatura stessa. è quindi possibile risolvere l'equazione di Clapeyron in modo approssimato supponendo $\mathrm{d}p/\,\mathrm{d}T$ costante. Si può utilizzare la forma approssimativa finita dell'equazione di Clapeyron (67.14.11)

$$\boxed{\frac{\Delta p}{\Delta T} \simeq \frac{\lambda_{\mathtt{fus}}}{T\Delta v} < 0} \qquad (67.14.13)$$

→ 2716

dato che la variazione della temperatura richiesta (variabile indipendente) è piccola. Si possono allora considerare costanti, nell'intervallo di temperatura considerato, sia il calore latente che i volumi specifici mentre la variazione relativa delle temperatura assoluta è piccola. Di conseguenza il valore della derivata della tensione di vapore saturo rispetto alla temperatura è approssimativamente costante e si può usare la relazione approssimata (67.14.13). Con i dati del testo si trova $\Delta p = 1.41{\cdot}10^7 a = 139\ \mathrm{atm}$. Si noti che la variazione di pressione è grande nonostante il piccolo intervallo di temperatura considerato. Questo è dovuto al grande valore della variazione della pressione in funzione della temperatura per la curva di coesistenza dell'acqua

$$\boxed{\frac{\Delta p}{\Delta T} \simeq \frac{\lambda_{\mathtt{fus}}}{T\Delta v}} \simeq -1.4{\cdot}10^7\ \mathrm{Pa/K} < 0$$

Una conseguenza di ciò è la notevole differenza di pressione tra il punto del ghiaccio (punto a cui si ha equilibrio ghiaccio-acqua a pressione atmosferica) e il punto triplo dell'acqua, nonostante la piccola differenza di temperatura, $\Delta T = 0.01\ \mathrm{K}$.
Un calcolo più esatto si può fare facilmente a partire dall'equazione di Clapeyron supponendo di poter considerare costanti, nell'intervallo di temperatura considerato, il calore latente e i volumi specifici e integrando l'equazione differenziale risultante:

$$\frac{\mathrm{d}p}{\mathrm{d}T} = \frac{\lambda_{\mathtt{fus}}}{T\Delta v} \Longrightarrow \boxed{\Delta p = \frac{\lambda_{\mathtt{fus}}}{\Delta v} \log\left[1 + \frac{\Delta T}{T_0}\right]} \ .$$

La lastra di ghiaccio deve esercitare sulla sua base questa pressione addizionale affinché la temperatura di fusione alla base si abbassi della quantità richiesta. L'altezza h della lastra di ghiaccio si ricava dunque dalla

$$\boxed{\Delta p = \rho_{\mathtt{gh}} g h}$$

Ne segue $h = 1.5\ \mathrm{km}$.

WORKED SOLUTION

1. $\Delta p = 1.41{\cdot}10^7 a = 139\ \mathrm{atm}$
2. $h = 1.5\ \mathrm{km}$.

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67.14.29
Temperature Di Flash Di Un Liquido Infiammabile

©|F.H.Crawford, *Heat Thermodynamics And Statistical Physics*, 1963, Harcourt Brace and World, ...Ed., ...|§ 11.9||

This § is referenced at pages:
[2720]

Il punto di flash di un liquido infiammabile volatile è definito come quella temperatura alla quale la tensione di vapore del liquido è tale da fornire alla superficie del liquido esposto all'atmosfera una concentrazione di vapore uguale al limite di infiammabilità inferiore $l_{\mathtt{inf}}$, cioè alla concentrazione in volume critica al di sopra della quale il vapore costituisce una miscela infiammabile con l'aria.

La sostanza n-pentano ha $l_{\mathtt{inf}} = 1.4\%$ (in volume) e tensione di vapore $p_1 = 10$ mmHg a $T_1 = -50.1°\mathrm{C}$ e $p_2 = 40$ mmHg a $T_2 = -29.2$ °C.

1. Determinare il punto di flash di n-pentano.
2. Determinare il punto di ebollizione di n-pentano.
3. Determinare il valore del calore latente specifico di evaporazione medio di n-pentano nell'intervallo di temperatura considerato.
4. Discuss the effect of atomospheric air on the vapor pressure.

WORKED SOLUTION

1. Occorre determinare a quale temperatura la concentrazione dei vapori di n-pentano in prossimità della superficie del liquido corrisponde a $l_{\mathtt{inf}}$. Supponendo che sia il vapore che l'aria circostante siano trattabili come gas perfetti e assumendo temperatura uniforme, la pressione parziale è proporzionale alla concentrazione in volume

$$p = \left(\frac{n}{V}\right) RT$$

e la pressione totale è quella atmosferica $p_{\mathtt{atm}}$. La pressione parziale è dunque proporzionale alla concentrazione in volume. La pressione parziale corrispondente alla composizione cercata è dunque

$$p_{\mathtt{inf}} = p_{\mathtt{atm}} l_{\mathtt{inf}} = 10.6 \text{ mmHg}$$

Quindi quando il liquido è in equilibrio con il suo vapore ad una pressione $p_{\mathtt{inf}}$ la concentrazione del vapore in atmosfera corrisponde a $l_{\mathtt{inf}}$ e si ha il punto di flash. Occorre ora determinare a quale temperatura la tensione di vapore assume il valore $p_{\mathtt{inf}}$. L'interpolazione lineare tra due punti di una funzione $y(x)$ è data dalla formula

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) \tag{67.14.14}$$

da cui, usando $x \equiv T$ e $y \equiv p$, si trova

$$T - T_1 = \left(\frac{T_2 - T_1}{p_2 - p_1}\right)(p - p_1)$$

da cui si ricava $T = -49.7°\mathrm{C}$. Tuttavia sappiamo che l'andamento della tensione di vapore in funzione della temperatura non è lineare ma è approssimativamente lineare l'andamento del logaritmo della tensione di vapore in funzione di $1/T$ dove la temperatura è la temperatura assoluta

$$\log[p] = A - \frac{B}{T} \tag{67.14.15}$$

Effettuiamo allora l'interpolazione in termini di $\pi = \log[p]$ e $\theta = 1/T$. Si ha allora $\theta_1 \equiv 1/T_1 = 4.48{\cdot}10^{-3}$ K^{-1}, $\theta_2 \equiv 1/T_2 = 4.10{\cdot}10^{-3}$ K^{-1} e, misurando p in mmHg,

$\pi \equiv \log[p] = 2.36$, $\pi_1 \equiv \log[p_1] = 2.30$, $\pi_2 \equiv \log[p_2] = 3.69$ e la formula da applicare è ora

$$\theta - \theta_1 = \left(\frac{\theta_2 - \theta_1}{\pi_2 - \pi_1}\right)(\pi - \pi_1) \tag{67.14.16}$$

→ 2718

da cui si deduce $T = -49.3\,°\mathrm{C}$. Tale secondo valore, pur approssimato, è più vicino del precedente al valore che si otterrebbe usando una curva più completa della tensione di vapore e che darebbe $T_{\mathtt{flash}} = -49.44$. Si noti che il valore approssimativo ottenuto con l'interpolazione lineare in termini di p e T è relativamente preciso perché cade molto vicino ad uno dei due punti dati.

2. Il punto di ebollizione si ricava dalla stessa formula (67.14.16) usata ora come formula di estrapolazione per determinare la temperatura alla quale la tensione di vapore vale $1\ \mathrm{atm}$. Si trova $T = 31.4°\mathrm{C}$ che differisce alquanto dalla temperatura di ebollizione reale $T_{\mathtt{eb}} = 36.1°\mathrm{C}$. Ciò è dovuto al fatto che la linearità di $\log[p]$ in funzione di $1/T$ è solo approssimata e l'estrapolazione fatta è molto al di fuori dell'intervallo di cui sono dati i valori.
3. Il calore latente di vaporizzazione medio nell'intervallo considerato può essere ricavato dai dati del problema e dall'equazione di Clapeyron

$$\log\left[\frac{p_2}{p_1}\right] = \frac{\lambda M}{R}\left(\frac{1}{T_1} - \frac{1}{T_2}\right) \tag{67.14.17}$$

Si trova $\lambda = 4.16{\cdot}10^5\ \mathrm{J/kg}$.
Una soluzione più precisa del problema si potrebbe ottenere usando valori della tensione di vapore per più valori di temperatura anziché solo due. Questo permetterebbe di tenere conto della non esatta linearità dell'andamento della curva del logaritmo della tensione di vapore in funzione del reciproco della temperatura assoluta.

4. Si osservi che in questo problema si è ignorato l'effetto dell'aria sulla tensione di vapore. L'unico effetto dell'introduzione dell'aria a temperatura costante è quello di modificare la pressione alla superficie del liquido che non è più uguale alla tensione di vapore ma diventa uguale alla pressione atmosferica. Viceversa, in approssimazione di gas perfetto, l'introduzione dell'aria non influenza in alcun modo il vapore in modo diretto. Tuttavia l'effetto sul potenziale chimico del liquido della variazione di pressione è piccolo è può essere, in prima approssimazione, trascurato. Immaginiamo infatti di partire dall'equilibrio liquido-vapore della sostanza pura e introdurre a temperatura costante l'aria fino ad una pressione totale $p_{\mathtt{TOT}}$ in un recipiente. La variazione della funzione di Gibbs molare del liquido, causata dalla variazione di pressione, deve essere compensata da una uguale variazione della funzione di Gibbs molare del vapore affinché sussista ancora equilibrio di fase

$$\Delta g_{\mathtt{L}} = v_{\mathtt{L}}\Delta p_{\mathtt{L}} = v_{\mathtt{V}}\Delta p_{\mathtt{V}} = \Delta g_{\mathtt{V}}$$

dove $\Delta p_{\mathtt{L}} = p_{\mathtt{TOT}} - p^0_{\mathtt{V}}$ e si considerano $v_{\mathtt{L}}$ e $v_{\mathtt{V}}$ costanti. Si può allora ricavare $\Delta p_{\mathtt{V}}$ che risulta molto minore di $\Delta p_{\mathtt{L}}$ a causa del rapporto dei volumi specifici che da per l'acqua un fattore $\approx 5{\cdot}10^{-5}$ nel rapporto tra l'aumento della pressione totale del liquido e la variazione della pressione del vapore.
Un secondo effetto che si è ignorato è il fatto che aggiungendo aria una parte di questa passa in soluzione nell'acqua per cui a rigore anche la fase liquida del sistema è composta da due componenti.

WORKED SOLUTION

1. $T = -49.3°\mathrm{C}$;
2. $T = 31.4°\mathrm{C}$
3. $\lambda = 4.16{\cdot}10^5\ \mathrm{J/kg}$.

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67.14.30
Stima Della Temperatura Di Equilibrio Di Un Pianeta

©|B.W.Carroll & D.A.Ostlie, *An Introduction To Modern Astrophysics*, ..., ..., 1stEd., ...|???||

In prima approssimazione si può supporre che la temperatura di un pianeta dipenda dal bilancio energetico dell'energia che riceve per irraggiamento dal Sole e di quella che emette verso lo spazio, supponendo di considerare sia il pianeta che il Sole due corpi neri. Si consideri il caso della Terra. L'energia irradiata dal Sole vale $P_S = 3.846 \cdot 10^{26}$ W, il raggio equatoriale medio del Sole vale $R = 6.96 \cdot 10^{8}$ m e la distanza media tra la Terra e il Sole vale $d = 1.496 \cdot 10^{11}$ m. Si supponga che una frazione $a = 30\%$ (l'albedo della Terra) dell'energia che arriva sulla Terra venga riflessa e la restante assorbita.

1. Supponendo che la temperatura del pianeta sia uniforme, determinare l'espressione del valore della temperatura di equilibrio T nel caso il pianeta ruoti così velocemente che si possa considerare uniforme la temperatura su tutta la sua superficie.
2. Determinare l'espressione del valore della temperatura di equilibrio T nel caso il pianeta ruoti in modo da rivolgere sempre la stessa faccia verso il Sole.
3. Si commentino i risultati nel caso della Terra.

67.14.31
Equazione Di Stato Di Un Solido O Liquido

Per descrivere il comportamento di un solido o di un liquido si può spesso usare l'equazione di stato

$$V = V_0 + 3\alpha V_0 (T - T_0) - \kappa_T V_0 (p - p_0) \tag{67.14.18}$$

dove V_0 è il volume alla pressione p_0 e temperatura T_0 e α e κ_T sono costanti.

1. Giustificare l'espressione e interpretare i coefficienti che vi compaiono.
2. Verificare che C_V dipende solo dalla temperatura.
3. Determinare l'espressione dell'energia interna.
4. Determinare l'espressione dell'entropia.

67.14.32
Innalzamento Della Temperatura Di Ebollizione Di Una Soluzione

©|M.W.Zemansky & M.M.Abbot & H.C.VanNess, *Fondamenti Di Termodinamica per Ingegneri*, ..., ..., ...Ed., ...|11.34||

This § is referenced at pages:
[2720]

Si scioglie una piccola quantità di zucchero z in acqua (liquida). La soluzione è in equilibrio di fase con il vapor d'acqua puro. Si assuma che la soluzione si comporti come una soluzione ideale, cioè che il potenziale chimico del componente k -esimo in soluzione sia dato dalla relazione

$$\mu_k = g_k + RT \log [x_k]$$

dove x_k è la frazione molare del componente k -esimo e g_k è la funzione di Gibbs molare del componente k -esimo.

1. Motivare il fatto che si può considerare trascurabile la frazione di zucchero allo stato di vapore.
2. Mostrare che l'equazione dell'equilibrio di fase è $g_a^v = g_a^l + RT \log [1 - x_z]$, dove g_a^v è la funzione di Gibbs molare del vapor d'acqua puro, g_a^l è la funzione di Gibbs molare del liquido, e x_z la frazione molare dello zucchero nella soluzione.
3. Mostrare che per un variazione infinitesima di x_z a pressione costante si ha
$$-s_a^v \, \mathrm{d}T = -s_a^l \, \mathrm{d}T + R \log [1 - x_z] \, \mathrm{d}T + RT \, \mathrm{d}\log [1 - x_z] \quad .$$
4. Combinando i risultati dei punti precedenti mostrare che il risultato della domanda precedente si riduce a $0 = \left(h_a^v - h_a^l\right) \, \mathrm{d}T + RT^2 \, \mathrm{d}\log [1 - x_z]$

DRAFT - PRELIMINARY VERSION

5. Osservando che $x_z \ll 1$ e che $h_a^v - h_a^l \equiv \Delta h_a^{lv}$, il calore latente di vaporizzazione dell'acqua pura, mostrare che il punto di ebollizione della soluzione si innalza approssimativamente della quantità $\Delta T = x_z RT^2/\Delta h_a^{lv}$.

67.14.33
Abbassamento Della Temperatura Di Congelamento Di Una Soluzione

©|M.W.Zemansky & M.M.Abbot & H.C.VanNess, *Fondamenti Di Termodinamica per Ingegneri*, ..., ..., ...Ed., ...|11.35||

This § is referenced at pages:
[Never referenced.]

Si scioglie una piccola quantità di zucchero z in acqua (liquida). La soluzione è in equilibrio di fase con il ghiaccio puro. Si assuma che la soluzione si comporti come una soluzione ideale (equazione (67.11.04)). Seguendo la falsariga del problema § 67.14.32 dimostrare che l'abbassamento del punto di congelamento della soluzione è dato approssimativamente da $\Delta T = -x_z RT^2/\Delta h_a^{sl}$, dove Δh_a^{sl} è il calore latente di fusione dell'acqua.

67.14.34
Effetto Dell'aria Sull'equilibrio Liquido Vapore Di Una Sostanza

©|F.H.Crawford, *Heat Thermodynamics And Statistical Physics*, 1963, Harcourt Brace and World, ...Ed., ...|§ 11.9||

©|D.V.Sivuchin, , ..., ..., ...Ed., ...|||

Vedere il problema § 67.14.29.

67.14.35
Calore Specifico Lungo Una Trasformazione Qualunque

©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|4.10||

67.14.36
Tensione Di Vapore Di Una Bolla Di Liquido

©|M.Abbott & H.VanNess, *Schaum's Outline Of Thermodynamics*, 1989, McGraw-Hill, 2ndEd., ...|4.21||

67.14.37
Liquid-Solid Phase Boundary for Water

©|D.Arovas|2.12.3||

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67.14.38
Phase Transitions of an Ideal Gas

The stability criteria, equations (67.05.02), (67.03.01), are intrinsically satisfied for an ideal gas. Therefore ideal gases does not exhibit phenomena of phase transitions nor critical points.

67.14.39
Cooling

67.14.39.01 Gas Cooling

This § is referenced at pages:
[Never referenced.]

A working substance (gas) in a piston-cylinder. Let us cool it, starting from a thermostat at the initial temperature. Practical experimental problems are not considered.

1. Isothermal reversible compression in contact with the thermostat: work is done on the substance and heat is given off to the thermostat; try to make the final volume as small as possible.
2. Adiabatic reversible expansion after thermal insulation from the surroundings: the substance does work and its internal energy (and thus temperature) decreases; try to make the final pressure as small as possible.
3. If you can use (part of) the cooled substance as a new thermostat at a starting temperature you may repeat the cycle.

67.14.39.02 Elastic Cooling

This § is referenced at pages:
[Never referenced.]

67.14.39.03 Magnetic Cooling

This § is referenced at pages:
[Never referenced.]

From table 67.12 we have, for matter only and fixed currents:

$$\delta \mathcal{W} = -(\mathbf{M}\cdot \delta \mathbf{B}_1)$$

the work done on the sample per unit volume when the field $\mathbf{B}_1$ changes.

Note that, to keep a parallelism with a pVT system, we have to identify, formally and only formally, $\mathbf{B}_1$ with volume and $\mathbf{M}$ with pressure.

Consider a paramagnetic salt.

For an infinitesimal transformation, the balance of energy per unit volume gives, for the heat received, as $\mathcal{U} = -MB$ (see for instance equation (41.04.06)):

$$\delta \mathcal{Q} = \mathrm{d}\mathcal{U} - \delta \mathcal{W} = \mathrm{d}(-MB) + M\,\mathrm{d}B = -B\,\mathrm{d}M = T\,\mathrm{d}\mathcal{S} \quad .$$

- Isothermal Magnetization (by increasing the external magnetic field); work is done on the system.
- Adiabatic (isoentropic) deMagnetization; no work.

Plane $\mathcal{S}T$, or better $T\mathcal{S}$.

Recall that **Langevin equation** is the equation of state.

It can be shown, working as for a pVT system, that a Maxwell relation applies:

$$\left(\frac{\partial \mathcal{S}}{\partial B}\right)_T = \left(\frac{\partial M}{\partial T}\right)_B \leq 0 \quad ,$$

negative from **Langevin equation** and easily interpreted as negative in the probabilistic order/disorder entropy interpretation.

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67.15 Exercises Problems and Physical Applications

67-001 Internal Energy as a Function of Non-Natural Variables

©|S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...|Exercise 16.4||

Suppose that one knows the function $\mathcal{U}[T,V]$, that is the internal energy is known in terms of variables that are not all its natural variables Show that this leads to the following expression for the pressure:

$$\frac{p}{T} = f[V] + \int \left(\frac{\partial \mathcal{U}}{\partial V}\right)_T \frac{\mathrm{d}T}{T^2} \quad ,$$

where $f[V]$ is some unknown function of the volume.

67-002 Heat Capacities for an Elastic Wire in Tension

This § is referenced at pages:
[2685]

For an elastic wire under tension, derive the equations for the **Heat-Capacity** at constant extensive variable, length, and the **Heat-Capacity** at constant intensive variable, tension per unit area.

67-003 First Law and Jacobian

©|J.M.Powers, *Lecture Notes On Thermodynamics*, ..., ..., ...Ed., WEB - URL.|§ 9.5||

Use the first law to show that any closed path in the pV plane maps into a closed path with the same enclosed area.

67-004 Maxwell Relations and Jacobian

©|S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...|§ 16||

Consider a reversible cycle, represented in both the pV and TS planes. The fundamental equation (66.09.01), for the closed cycle, implies:

$$0 = \oint \mathrm{d}\mathcal{U} = \oint T\,\mathrm{d}\mathcal{S} - \oint p\,\mathrm{d}V \implies \oint T\,\mathrm{d}\mathcal{S} = \oint p\,\mathrm{d}V \implies \iint \mathrm{d}T\,\mathrm{d}\mathcal{S} = \iint \mathrm{d}p\,\mathrm{d}V \quad .$$

Show that the above relation is able to generate all **Maxwell Equations** via the Jacobian.

67-005 A New Kind of Matter

©|Lim Yung-Kuo, *Problems And Solutions On Thermodynamics And Statistical Physics*, 1990, World Scientific, ...Ed., ...

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67-006 Rubber Bands

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CHAPTER 68

Elements of Kinetic Theory and Transport Phenomena

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©|Berkeley Physics Course, Vol. 5, *Statistical Physics*, 1965, McGraw-Hill, ...Ed., ...|§ 8||
©|F.Reif, *Fundamentals Of Statistical And Thermal Physics*, 2009, Waveland Press, ...Ed., ...|||

©|J.P.Pérez *et al.*, *Thermodynamique*, ..., DUNOD, ...Ed., WEB - URL.|2-3-4-5||

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68.01 Introduction

© - QUOTE

R.Balian, *From Microphysics To Macrophysics*, 2007, Springer, ...Ed., ...
Kinetic theory is based upon the analysis of the **elementary mechanical processes**, **free motion of the molecules between two successive collisions**, **collisions** between those molecules, and, furthermore, upon calculating **averages over the system of gas molecules** and striking the balance.

Important preliminary concepts:

- Cross-section: read § 30.06.
- Mean free path: read § 30.06.01.
- Beer-Lambert law: see equation (46.04.01).
- Virial Theorem: read § 30.07.
- Collisions, Kinetic Temperature and thermalization: read § 64.04.04.02.

© - QUOTE

©|L.D.Landau *et al.*, *Statistical Physics*, ..., ..., ...Ed., WEB - URL - ISBN: 0-08-023039-3.|§ 29|*excellent*|

Let us consider the probability distribution for the momenta, and once again emphasise the very important fact that in classical statistics this distribution does not depend on the nature of the interaction of particles within the system or on the nature of the external field, and so can be expressed in a form applicable to all bodies.

WORK IN PROGRESS

68.02 Mean Free Path Collision Time and Scattering Cross Section

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©|F.Reif, *Fundamentals Of Statistical And Thermal Physics*, 2009, Waveland Press, ...Ed., ...|||

Let us consider N all-identical molecules of a dilute gas in a volume V and neglect all interactions among molecules. We will treat the molecules as rigid spheres of finite size to allow collisions to occur and start considering collision process between two hard spheres.

If the diameter of the molecules is d, the effective cross-sectional area for scattering is πd^2. In fact, focusing on a single molecule as it makes its way through the gas, while traveling a distance ℓ and sweeping out a volume $\Delta V = \pi d^2 \ell$ it will collide with any other molecule whose center lies within this volume.

The mean free path is defined as the average distance traveled by the molecule between each collision; therefore for one collision, using the relative velocity to account for relative motion of the molecules, and $\ell = \lambda$:

$$\eta_{\mathbb{N}} \pi d^2 \lambda = \eta_{\mathbb{N}} \pi d^2 \langle u \rangle \tau = 1 = \eta_{\mathbb{N}} \pi d^2 \sqrt{2} \langle v \rangle \tau \implies \boxed{\lambda = \langle v \rangle \tau = \frac{1}{\eta_{\mathbb{N}} \pi d^2 \sqrt{2}}} \quad .$$

Note that:

- $\eta_{\mathbb{N}} \pi d^2 \langle u \rangle \tau = 1$ requires $\langle u \rangle$, that is the relative molecule velocity; it is a molecule moving inside a set of fixed targets with speed given by the relative speed.
- $\lambda = \langle v \rangle \tau$ needs $\langle v \rangle$, the real molecule velocity.

A more refined analysis requires the use of the concept of cross-section, giving:

$$\boxed{\lambda = \langle v \rangle \tau = \frac{1}{\eta_{\mathbb{N}} \sigma \sqrt{2}}} \quad . \tag{68.02.01}$$

Read § 30.06.01.

Dilute gas means: $\lambda \gg d$.

Summarizing the relations between cross-section and interaction rate, for a totally random set of identical molecules in a dilute gas, is:

$$\boxed{\lambda = \langle v \rangle \tau = \frac{\langle v \rangle}{\langle u \rangle} \frac{1}{\eta_{\mathbb{N}} \sigma} = \frac{1}{\sqrt{2} \eta_{\mathbb{N}} \sigma} \qquad \mathbf{u} \equiv \mathbf{v}_2 - \mathbf{v}_1 \qquad \langle u \rangle \approx \sqrt{2} \langle v \rangle} \quad .$$

Examples

Mean free path in atmosphere at NTP: $\lambda \simeq 0.07\ \mu\text{m} \gg d$. Therefore the approximations based on relatively infrequent encounters among molecules (dilute gas) are justified

68.02.01 Kinetic Diameter

©|WEB - URL |Molecular Physics|§ 2|

The kinetic diameter is a measure of molecules that expresses the likelihood that a molecule in a dilute gas will collide with another molecule. It is an indication of the size of the molecule as a projectile/target. The kinetic diameter is not the same as atomic diameter defined in terms of the size of the atom's electron shell, which is generally a lot smaller, depending on the exact definition used. Rather, it is the size of the sphere of influence that can lead to a scattering event. In other words, it is the size which would give the cross-section in case the interaction would be a black-disk one.

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Kinetic diameter it is obviously related to the mean free path of molecules in the gas and is defined by the relations of this section.

Examples

Mean diameters as derived from the co-volume in **Van der Waals** equation:

- water: 0.29 nm;
- O_2: 0.29 nm;
- H_2: 0.28 nm;
- CO_2: 0.32 nm;
- C_6H_6: 0.45 nm.

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68.03 Classical Maxwell-Boltzmann Velocity Distribution

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 1|*§ 40*|

©|Berkeley Physics Course, Vol. 5, *Statistical Physics*, 1965, McGraw-Hill, ...Ed., ...|||

©|F.Reif, *Fundamentals Of Statistical And Thermal Physics*, 2009, Waveland Press, ...Ed., ...|7.9, 7.13||

©|S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...|§ 5||

Full isotropy and homogeneity is assumed. Zero total linear momentum, $\mathbf{P} = 0$, and zero total angular momentum, $\mathbf{J} = 0$, are assumed.

© - QUOTE

©|R.Balian, *From Microphysics To Macrophysics*, 2007, Springer, ...Ed., ...|§ 9.2.1||

It is remarkable that the Maxwell distribution is thus valid in the, either gas or liquid, fluid phases, even at higher densities. The spatial correlations between molecules become very strong in the liquid, but their velocities remain statistically decoupled, at least as long as the classical approximation is valid.

The Maxwell-Boltzmann velocity distribution follows immediately from the Boltzmann-Gibbs factor,

$$\exp\left[-\frac{H[q,p]}{k_{\mathrm{B}}T}\right] \quad ,$$

whenever **classical statistical mechanics** applies, by integrating out the potential energy terms of the Hamiltonian as long as potential energy only depends only on coordinates and not on momenta.

The Center-Of-Mass coordinates of molecules are usually well described by classical statistical physics, while rotational and vibrational usually are not.

WORK IN PROGRESS

68.03.01
Heuristic Derivation

The following hypotheses are assumed.

- The diameter of the molecules are much smaller than the distance between them (dilute gas).
- The collisions between molecules are elastic.
- The positions and velocities of the molecules are initially at random (homogeneous and isotropic in position and velocity).
- The three components of the velocity of a particle are independent: $f[v_x, v_y, v_z] = h[v_x]h[v_y]h[v_z]$.

It follows:

$$\mathrm{d}N \equiv f[v_x, v_y, v_z]\ \mathrm{d}v_x\ \mathrm{d}v_y\ \mathrm{d}v_z \equiv h[v_x]h[v_y]h[v_z]\ \mathrm{d}v_x\ \mathrm{d}v_y\ \mathrm{d}v_z \equiv 4\pi v^2\phi[v]\ \mathrm{d}V \equiv \Phi[v]\ \mathrm{d}V \qquad \text{thanks to isotropy} \quad ,$$

$$h[v_x]h[v_y]h[v_z] = f[v_x, v_y, v_z] \equiv \Phi[v] \qquad \text{thanks to isotropy} \quad ,$$

$$\implies h'[v_x]h[v_y]h[v_z] \propto \Phi'[v]\frac{\mathrm{d}v}{\mathrm{d}v_x} = \Phi'[v]\frac{v_x}{v} \quad ,$$

$$\implies \frac{1}{v_x}\frac{h'[v_x]}{h[v_x]} \propto \frac{1}{v}\frac{\Phi'[v]}{\Phi[v]} \quad ,$$

$$\implies \frac{1}{v_x}\frac{h'[v_x]}{h[v_x]} \propto \frac{1}{v}\frac{\Phi'[v]}{\Phi[v]} \equiv \lambda \qquad \lambda \text{ is independent from } v \text{ and } v_x\text{, as well as from } v_y \text{ and } v_z \quad , \quad ,$$

$$\implies \Phi'[v] = \lambda v \Phi[v] \quad ,$$

$$\implies \Phi[v] = \exp\left(\frac{\lambda}{2}v^2\right) \quad ,$$

$$\text{and similarly:} \qquad h[v_k] \propto \exp\left(\frac{\lambda}{2}v_k^2\right) \quad .$$

Note that the derivation does not require to assume that the molecules move between collisions without interacting at a constant speed in a straight line (no interactions among molecules). In fact, the Maxwell-Boltzmann distribution is valid for any system of classical particles at equilibrium.

Note that some of the results in kinetic theory do not depend on the explicit expressions of the distribution functions.

68.03.02
Single-Particle Velocity Distribution

Common choices for the normalization of the distribution functions include:

- normalization to N, the total number of particles;
- normalization to one (the distribution function is a probability function);
- normalization to the number density of particles, the concentration, $\eta_{\mathbb{N}}$.

The expressions in orthonormal Cartesian Coordinates Coordinate System of the single-particle dis-

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tribution functions, are as follows:

$$\mathcal{D}[v_x, v_y, v_z] \equiv f[v_x, v_y, v_z]\, \mathrm{d}v_x\, \mathrm{d}v_y\, \mathrm{d}v_z = h[v_x]h[v_y]h[v_z]\, \mathrm{d}v_x\, \mathrm{d}v_y\, \mathrm{d}v_z \qquad \text{homogeneity, independent from } x, y, z \quad , \quad ,$$

$$1 = \iiint \mathcal{D}[v_x, v_y, v_z] = \iiint_{-\infty}^{+\infty} f[v_x, v_y, v_z]\, \mathrm{d}v_x\, \mathrm{d}v_y\, \mathrm{d}v_z \qquad \text{normalization} \quad , \quad ,$$

$$\langle v_x \rangle = \langle v_y \rangle = \langle v_z \rangle = 0 \quad , \quad ,$$

$$\mathcal{D}[v_k] = h[v_k]\, \mathrm{d}v_k \qquad \text{homogeneity, independent from } x \quad , \quad ,$$

$$1 = \int_{-\infty}^{+\infty} h[v_k]\, \mathrm{d}v_k \quad , \quad ,$$

$$\langle v_k \rangle = 0 \quad , \quad ,$$

$$\boxed{h[v_k] = \sqrt{\frac{m}{2\pi k_\mathrm{B} T}} \exp\left[-\frac{m v_k^2}{2 k_\mathrm{B} T}\right]} \quad , \quad ,$$

$$\langle v_k^2 \rangle = \frac{k_\mathrm{B} T}{m} \quad .$$

The expressions in spherical coordinates of the single-particle distribution functions, are as follows:

$$\mathcal{D}[v, \theta, \phi] \equiv \frac{g[v]}{4\pi}\, \mathrm{d}V\, \mathrm{d}\Omega \qquad \text{isotropy, independent from } \theta \text{ and } \phi \quad , \quad ,$$

$$1 = \iiint \mathcal{D}[v, \theta, \phi] = \iiint \frac{g[v]}{4\pi}\, \mathrm{d}V\, \mathrm{d}\Omega = \int_0^{+\infty} g[v]\, \mathrm{d}V \qquad \text{normalization} \quad , \quad ,$$

$$\boxed{\mathcal{D}[v] = g[v]\, \mathrm{d}V = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2 k_\mathrm{B} T}\right)^{3/2} v^2 \exp\left[-\frac{m v^2}{2 k_\mathrm{B} T}\right] \mathrm{d}V = 4\pi \left(\frac{m}{2\pi k_\mathrm{B} T}\right)^{3/2} v^2 \exp\left[-\frac{m v^2}{2 k_\mathrm{B} T}\right] \mathrm{d}V} \quad , \quad ,$$

$$\langle v^2 \rangle = \frac{3 k_\mathrm{B} T}{m} \quad , \quad ,$$

$$\langle v \rangle = \sqrt{\frac{8 k_\mathrm{B} T}{\pi m}} \propto \sqrt{\frac{T}{m}} \quad , \quad ,$$

$$v_\mathrm{RMS} = \sqrt{\frac{3 k_\mathrm{B} T}{m}} \propto \sqrt{\frac{T}{m}} \quad , \quad ,$$

$$v_\mathrm{MAX} = \sqrt{\frac{2 k_\mathrm{B} T}{m}} \propto \sqrt{\frac{T}{m}} \quad , \quad ,$$

$$v_\mathrm{MAX} < \langle v \rangle < v_\mathrm{RMS} \quad .$$

The relation with the kinetic temperature, assuming the fluid has zero overall (Center-Of-Mass) velocity, is:

$$\boxed{\frac{1}{2} m \langle v^2 \rangle \equiv \frac{3}{2} k_\mathrm{B} T \qquad \text{kinetic temperature} \qquad \text{if} \qquad \langle v \rangle = 0} \quad ,$$

read § 64.04.04.01.

Note that for a real gas in a real container the distribution of velocities would be generally different near the walls, where isotropy and homogeneity are lost. In fact, for instance, near the walls particles directing towards the wall might be slowed down by the bulk particles, and|or possibly being accelerated by the interaction with the particles of the walls.

So, the Single-Particle Velocity Distribution is a Gaussian.

The Single-Particle Velocity squared Distribution is a chi-square distribution.

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68.03.03
Mathematical Relations for the Maxwell-Boltzmann Distribution

The following functions are normalized to one in $(-\infty, +\infty)$ ($\Re\mathfrak{e}\,[a > 0]$ and $\Re\mathfrak{e}\,[b > 0]$):

$$\frac{\sqrt{a}\exp -aw^2}{\sqrt{\pi}} \quad , \quad ,$$

$$\frac{2a^{3/2}w^2\exp -aw^2}{\sqrt{\pi}} \quad , \quad ,$$

$$\frac{\sqrt{a}\exp -\dfrac{b^2}{4a} - aw^2 - bw}{\sqrt{\pi}} \quad .$$

The following functions are normalized to one in $(0, +\infty)$ ($\Re\mathfrak{e}\,[a > 0]$ and $\Re\mathfrak{e}\,[b > 0]$):

$$\frac{2\sqrt{a}\exp -aw^2}{\sqrt{\pi}} \quad , \quad ,$$

$$2aw\exp -aw^2 \quad , \quad ,$$

$$\frac{4a^{3/2}w^2\exp -aw^2}{\sqrt{\pi}} \quad , \quad ,$$

$$2a^2w^3\exp -aw^2 \quad , \quad ,$$

$$\frac{8a^{5/2}w^4\exp -aw^2}{3\sqrt{\pi}} \quad , \quad ,$$

$$\frac{2\sqrt{a}\exp -\dfrac{b^2}{4a} - aw^2 - bw}{\sqrt{\pi}\,\mathrm{erfc}\left(\dfrac{b}{2\sqrt{a}}\right)} \quad .$$

The correct normalization factors for the most common integrals can be calculated as follows:

$$\int_{-\infty}^{+\infty} \exp -az^2\,\mathbb{d}z = \sqrt{\frac{\pi}{a}} \qquad \Re\mathfrak{e}\,[a > 0] \quad ,$$

$$\int_{0}^{+\infty} z\exp -az^2\,\mathbb{d}z = \frac{1}{2a} \qquad \Re\mathfrak{e}\,[a > 0] \quad ,$$

$$\int_{-\infty}^{+\infty} z^2\exp -az^2\,\mathbb{d}z = \frac{1}{2}\sqrt{\frac{\pi}{a^{3/2}}} \qquad \Re\mathfrak{e}\,[a > 0] \quad .$$

68.03.04
Towards Interacting Particles

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©|R.Balian, *From Microphysics To Macrophysics*, 2007, Springer, ...Ed., ...|§ 9||

In § 7 and 8 we restricted ourselves to the study of low density gases. When a gas is compressed, its molecules get closer to one another and the approximation which consists in neglecting their interactions is no longer valid. As the density increases there thus appear corrections to the ideal gas laws and we shall now give a theory for those. However, the occurrence of interaction terms in the Hamiltonian gives rise to a new mathematical difficulty: the partition function can no longer be factorized into the contributions of the single molecules. Calculating it involves more or less

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well controlled approximations, even when one introduces simplified models for the interactions (§ 9.1 and § 9.2).
The interactions between the molecules play a more interesting role, namely, they produce a phenomenon which is remarkable, although well known from experience: liquefaction. We know that if the temperature is not too high - more precisely, if it does not lie above the critical temperature - compressing a gas makes it abruptly go over into the liquid state, when one reaches the saturated vapor pressure. If we try to forget that this is a trivial fact and try to examine it critically, we find that it has surprising aspects: the **thermodynamic quantities show a discontinuity or a singularity, and the fluid separates into two phases, the liquid and the vapor** one. How can we explain that these two phases coexist and how can we explain their qualitatively different properties, if they consist of the same molecules which interact with the same forces? How can we derive from a single microscopic theory a discontinuous macroscopic behavior? The same questions crop up for the numerous other instances of phase transitions which one observes on a macroscopic scale: solidification, the possibility to obtain various liquid phases for mixtures or various crystalline phases for solids, ferromagnetism, superconductivity, and so on. Their **discontinuous nature is difficult to understand when we start from the microscopic structure** of matter, and this has given rise to a **long controversy: is there a sudden change in the forces between the elementary constituents? or is it possible to explain the existence of phases which are macroscopically so different starting from a single microscopic model? The second solution has won the day** thanks to many researches which constitute one of the most spectacular contributions of statistical physics. **We cite amongst the first important stages the elaboration of approximate microscopic theories of magnetism (P.Weiss, 1924; W.L. Bragg and E.J.Williams, 1934) and of liquefaction (J.Yvon, J.E. Mayer, 1937), which showed the appearance of a phase transition as a cooperative phenomenon where, for instance, the magnetic moments orient themselves all in the same direction below the Curie temperature due to their mutual interactions. A decisive conceptual step was the first rigorous solution of a microscopic model which shows a phase transition, the two-dimensional Ising model (L.Onsager, 1944). More recently, the singular behavior of the thermodynamic functions near a critical point, which had been known and studied experimentally for a long time, found its explanation thanks to the work by K.Wilson (1971).**
...
In § 7.1 we saw that the ideal gas model, which is sufficient for describing rarefied mono-atomic gases, is based upon three approximations: the internal degrees of freedom of the molecules are frozen in, **their center of mass motion has a classical nature and there are no interactions between the molecules**. The first condition was abandoned in § 8 when we wanted to study rarefied poly-atomic gases. On the other hand, we showed in § 7.1.3 that **the second approximation - the use of classical statistical mechanics for the translational degrees of freedom - is valid not only for gases, but also for almost all liquids**. However, we must take into account the interactions between the molecules, if we want to study the properties of a compressed gas or of a liquid.
It is difficult to make a simple model of the interactions between poly-atomic molecules as their collisions can produce transitions from one state to another in each of the molecules. The interactions thus affect the internal structure of the molecules. The situation is simpler for mono-atomic fluids and we shall restrict ourselves to those in the present chapter. In all cases the interactions between the molecules are due to the Coulomb forces between their constituents, the electrons and the nuclei. In the case of a mono-atomic gas, the electrons remain frozen in into their ground state at the temperatures of interest; when two atoms approach one another, the deformation of the electron cloud produces an effective potential between them.
We sketched the theory of this interaction: attractive for long distances apart and repulsive for short distances apart.
...
Of course, even though the elementary forces between the constituents, the nuclei and electrons, which are Coulomb forces, are two-body forces, the effective interactions between the atoms

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also contain three-, four-, ... body forces. However, at the usual gas densities and even at liquid densities, the probability for three or more atoms to be sufficiently close to one another so that the effects of these three-, four-, ... body forces can come into play is small. We shall therefore restrict ourselves mainly to a study of Hamiltonians which, apart from the kinetic energy, contain solely the binary potential which we have just described and which is a function of the relative positions of the molecular nuclei, taken in pairs.
Thus, we can write for the effective Hamiltonian

$$H = \sum_i \frac{p_i^2}{2m_i} + \frac{1}{2}\sum_{a\neq b} \Phi\left(|\mathbf{r}_a - \mathbf{r}_b|\right) \qquad (68.03.01)$$

→ 2734

where the potential energy Φ of the interaction between the molecules depends only on their distance apart since they are mono-atomic, and has the shape shown in Fig. 9.1. **This potential has a quantum mechanical origin as its calculation involves crucially the freezing- in of the electronic degrees of freedom into their ground state.** Nevertheless, the effective Hamiltonian, which results from eliminating the motion of the electrons and which depends on the positions and momenta of the nuclei of the atoms in the gas, can be treated classically for all mono-atomic fluids - bar helium at low temperatures - as we saw in § 7.1.3. The fluid is therefore finally satisfactorily described by a model of classical point particles with the Hamiltonian (68.03.01).
...
The presence of the interaction not only changes (through T) the thermodynamic or (through μ) the chemical quantities, but also the equation of state, which now differs from that for a ideal gas. Unfortunately, the canonical partition function which occurs in the grand canonical potential ... for each particle is a multiple integral over a very large number of variables, and its evaluation is impossible, even for the simplest models for the potential Φ. We must therefore in what follows appeal to various approximations without always being able to give a rigorous justification for them.
Let us bear in mind that ... in the grand canonical potential ... is based solely on two approximations: the interactions between the molecules are taken into account by a binary isotropic potential, which is realistic for a mono-atomic gas, and we use the classical limit. This expression is therefore justified not only for the gas phase, but also for the liquid phase of the substance studied.
As we indicated at the start of this section, statistical mechanics thus provides a single expression, which contains a potential Φ, fixed once and for all, and which should describe the liquid-vapor transition. Nevertheless, **nothing enables us to discern from the grand canonical potential ... the presence of a line of singularities which would separate the two phases, liquid and vapor. The theory of the transition and of the critical point must start from the grand canonical potential ... but we expect that it will be difficult mathematically. To give an idea of how complicated it is we note that even the proof itself of the extensivity is by no means easy.**
...
One property remains very simple, notwithstanding the presence of inter-molecular interactions, namely, the velocity distribution.
...
exactly as in the case of a system without interactions. **It is remarkable that the Maxwell distribution is thus valid in the, gas or liquid, fluid phases, even at higher densities. The spatial correlations between molecules become very strong in the liquid, but their velocities remain statistically decoupled, at least as long as the classical approximation is valid.**

The Center-Of-Mass motion is always a classical motion in non-extreme conditions. The only approximation needed to derive the Maxwell-Boltzmann distribution is the classical approximation for fluids. Obviously, for crystalline solids, the classical approximation is not good, as the crystal is a quantum system.

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68.04 Rate of Collisions Against a Surface

This § is referenced at pages:
[2736, 2737]

©|Berkeley Physics Course, Vol. 5, *Statistical Physics*, 1965, McGraw-Hill, ...Ed., ...|§ 8||
©|F.Reif, *Fundamentals Of Statistical And Thermal Physics*, 2009, Waveland Press, ...Ed., ...|||

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68.04.01
Jet-Like (Collinear) Velocity Distribution

Consider a fixed plane portion of a surface, either real or purely geometric, $\mathbb{d}\mathbf{S} \equiv \widehat{\mathbf{n}}\, \mathbb{d}\mathbf{S}$, parallel to the xy plane, $\widehat{\mathbf{n}} = \{0, 0, 1\}$, and a steady flow for a velocity distribution along the positive z and uniform in the xy plane, with uniform and constant velocity, v:

$$\boxed{\frac{\mathbb{d}N}{\mathbb{d}\mathbf{S}\ \mathbb{d}t} = \eta_{\mathbb{N}} v} \quad . \tag{68.04.01}$$

→ 2735

Compare with the more general equation (16.06.16): a different notation is used in equation (68.04.01) as this is a simpler situation.

It is useful to note that equation (68.04.01), as it should be, is the same equation, with the proper reinterpretation of physics quantities, as equation (65.09.01.06), when the speed of light is replaced to $\langle v \rangle$ and the energy density is replaced to the number density.

68.04.02
Isotropic Velocity Distribution

Consider a fixed plane portion of a surface, either real or purely geometric, $\mathbb{d}\mathbf{S} \equiv \widehat{\mathbf{n}}\, \mathbb{d}\mathbf{S}$, and a steady flow for a isotropic velocity distribution.

A crude calculation, assuming that all particles are moving at the mean speed and 1/6 of them (that is 1/6 for each orthogonal direction) are hitting the wall, gives the rough approximation:

$$\boxed{\frac{\mathbb{d}N}{\mathbb{d}\mathbf{S}\ \mathbb{d}t} \approx \frac{1}{6}\eta_{\mathbb{N}}\langle v \rangle} \quad .$$

The full calculations gives:

$$\mathbb{d}N[v, \theta, \phi] \equiv N \left(g[v]\ \mathbb{d}V \frac{\mathbb{d}\Omega}{4\pi} \right) \left(\frac{\mathbb{d}\mathbf{S} \cos[\theta]\ v\ \mathbb{d}t}{V} \right) = \quad ,$$

integrating in azimuth and polar angle: $\quad 2\pi \frac{1}{2}\, \mathbb{d}\mathbf{S}\ \mathbb{d}t \frac{N}{V} \frac{1}{4\pi} \int v g[v]\ \mathbb{d}V = \quad ,$

$$\mathbb{d}N = \frac{1}{4}\eta_{\mathbb{N}}\langle v \rangle\ \mathbb{d}\mathbf{S}\ \mathbb{d}t \qquad \Leftrightarrow \qquad \boxed{\frac{\mathbb{d}N}{\mathbb{d}\mathbf{S}\ \mathbb{d}t} = \frac{1}{4}\eta_{\mathbb{N}}\langle v \rangle} \quad .$$

Strictly speaking, the above derivation is only valid for **ideal gases**, as the module of the velocity is assumed to be constant, while for a real gas the particles approaching the walls are typically attracted and slowed-down by the bulk particles.

68.04.03
Effusion From a Small Hole

The molecules inside the container, in thermodynamic equilibrium, have a isotropic velocity distribution. By effusion from a small hole, one means the case when the hole is so small not to perturb the local equilibrium near the hole itself.

These relationships are applicable in the case the mean free path in the gas within the container is much larger than the linear dimension of the hole, to prevent the presence of the hole to alter the equilibrium distribution within the container, in case a particle suffers many interactions near the hole.

Furthermore, the mean free path in the gas must be much smaller than the linear dimensions of the container to prevent the walls from having a predominant role in the mechanics of the system, as in the case the bumps occur between the particles and the walls only.

68.04.03.01 Effusion Into Vacuum From a Small Hole

L'effusione di un gas attraverso un piccolo buco di sezione $\mathrm{d}\mathbf{S}$ in un contenitore è data dalla relazione

$$\frac{\mathrm{d}N}{\mathrm{d}\mathbf{S}\,\mathrm{d}t} = \frac{1}{4}\eta_{\mathbb{N}}\langle v\rangle \equiv \frac{1}{4}\eta_{\mathbb{N}}\langle v\rangle \quad , \tag{68.04.02}$$

→ 2736

dove $\eta_{\mathbb{N}}$ è la concentrazione e $\langle v\rangle$ la velocità media delle molecole del gas entro il recipiente.

It is useful to note that equation (68.04.02), as it should be, is the same equation, with the proper reinterpretation of physics quantities, as equation (65.09.01.06), when the speed of light is replaced to $\langle v\rangle$ and the energy density is replaced to the number density.

For a ideal gas it can be written as:

$$\frac{1}{4}\eta_{\mathbb{N}}\langle v\rangle = \frac{p}{\sqrt{2\pi m k_{\mathrm{B}} T}} \quad . \tag{68.04.03}$$

68.04.03.02 Cross-Effusion From a Small Hole

If there is gas on two containers on both sides of the hole, one must balance the passages in both directions. In the effusion regime, $\lambda \gg D$, the system will achieve equilibrium when the two molecular fluxes balance. If the temperature is uniform, a rate of effusion which is proportional to the difference in the densities, that is, to the difference in pressure at the two sides is found.

68.04.03.03 Cross-Effusion From a Large Hole

In the hydrodynamics regime, $\lambda \ll D$, at equilibrium the pressures balance.

68.04.04
Kinetic Pressure - Molecular Perfect Gas Equation of State

This § is referenced at pages:
[2749, 2749]

Consider a fixed plane portion of a surface, either real or purely geometric, $\mathrm{d}\mathbf{S} \equiv \hat{\mathbf{n}}\,\mathrm{d}\mathbf{S}$.

Pressure is the rate of linear momentum given off per unit area, by a molecule whose Center-Of-Mass has velocity $\mathbf{v}$.

Perfect gas: $\mathcal{K} = \mathcal{U}$

The same calculations as in section § 68.04, adding the additional factor $\Delta q_{\perp} = 2mv_{\perp} = 2mv\cos[\theta]$, provides:

$$\mathrm{d}(\Delta q_{\perp})[v,\theta,\phi,\ \mathrm{d}\mathbf{S},\ \mathrm{d}t] \equiv N\left(g[v]\,\mathrm{d}V\frac{\mathrm{d}\Omega}{4\pi}\right)(2mv\cos[\theta])\left(\frac{\mathrm{d}\mathbf{S}\cos[\theta]\,v\,\mathrm{d}t}{V}\right) = (2\pi)\left(\frac{1}{3}\right)(\mathrm{d}\mathbf{S}\,\mathrm{d}t)\frac{2mN}{V}\frac{1}{4\pi}\int v^2 g[v]\,\mathrm{d}V \quad ,$$

$$p = \frac{2}{3}\eta_{\mathbb{N}}\frac{m}{2}\langle v^2\rangle = \frac{2}{3}\frac{\mathcal{K}}{V} = \frac{2}{3}\frac{\mathcal{U}}{V} \quad .$$

→ 2754

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If the gas is not ideal, the particles about to collide the walls will be slowed-down by the bulk of the fluid; it is the same argument as for the corrective term of Van der Waals equation of state. Therefore the calculation of pressure is not strictly valid because, even if the Maxwell distribution is still valid for a real fluid, it is valid in the bulk fluid, not at the edges where surface effects might appear.

68.04.05
Kinetic Pressure - Photon Gas Equation of State

This § is referenced at pages:
[Never referenced.]

©|Lim Yung-Kuo, *Problems And Solutions On Thermodynamics And Statistical Physics*, 1990, World Scientific, ...Ed., ...

Consider a fixed plane portion of a surface, either real or purely geometric, $\mathbb{d}\mathbf{S} \equiv \hat{\mathbf{n}}\, \mathbb{d}\mathbf{S}$.
Pressure is the rate of linear momentum given off per unit area, by a photon with linear momentum $\mathbf{q}$.
The same calculations as in section § 68.04, adding the additional factor $\Delta q_\perp = 2q_\perp = 2q\cos[\theta]$:

$$\mathbb{d}(\Delta q_\perp)[\epsilon,\theta,\phi,\ \mathbb{d}\mathbf{S},\ \mathbb{d}t] \equiv N\left(g[\epsilon]\,\mathbb{d}\epsilon\frac{\mathbb{d}\Omega}{4\pi}\right)(2q\cos[\theta])\left(\frac{\mathbb{d}\mathbf{S}\cos[\theta]\,c\,\mathbb{d}t}{V}\right) = (2\pi)\left(\frac{1}{3}\right)(c\,\mathbb{d}\mathbf{S}\,\mathbb{d}t)\,\frac{2N}{cV}\frac{1}{4\pi}\int \epsilon g[\epsilon]\,\mathbb{d}\epsilon \quad ,$$

$$\boxed{p = \frac{1}{3}\eta_{\mathbb{N}}\langle\epsilon\rangle = \frac{1}{3}\frac{\mathcal{U}}{V}} \quad .$$

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68.05 Molecular Transport Processes in the Mean Free Path Approximation

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 1 - § 41, 42 and 43.||

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R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...
We have considered so far only the molecular motions ... in thermal equilibrium. We want now to discuss what happens when things are near, but not exactly in, equilibrium. In a situation far from equilibrium, things are extremely complicated, but in a situation very close to equilibrium we can easily work out what happens. To see what happens, we must, however, return to the kinetic theory. Statistical mechanics and thermodynamics deal with the equilibrium situation, but away from equilibrium we can only analyse what occurs atom by atom, so to speak.

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We will assume that the interaction is very short ranged compared to the mean free path. For a low density gas of N particles contained in a volume V, the interaction energy between particles is negligible compared to the kinetic energy of the particles.
The mean free path, λ, of a particle is the average distance it travels between collisions. We shall assume that collisions occur at random in the gas, so a particle has the same chance of collision in any interval of length $(\ell, \ell + \mathbb{d}\mathbf{L})$. The average number of collisions per unit length is $1/\lambda$. The probability that a collision occurs in an interval $\mathbb{d}\mathbf{L}$ is therefore $\mathbb{d}\mathbf{L}/\lambda$.

There are two distinct type of processes:

- mobility, the drift of particles due to outside forces;
- diffusion, the spreading determined only by the internal forces, the random collisions.

In general, for many particles, the motion will be the superposition between a drift, that is motion of the Center-Of-Mass of any infinitesimal part, and a diffusive motion, that is motion around the Center-Of-Mass.

68.05.01
Equation of Continuity

Let us suppose $\mathcal{J}$, a scalar, is the volume density of a conserved quantity, $\varrho \equiv \mathcal{J}$. L'equazione di continuità (equazione (21.03.03) or equation (28.03.01)), si esprime in termine della densità di volume della grandezza conservata, ϱ, e del flusso della grandezza conservata, $\mathbf{j}$:

$$\boxed{\varrho[\mathbf{x},t] \qquad \mathbf{j}[\mathbf{x},t] \qquad \mathbf{div}\mathbf{j} + \partial_t \varrho = 0} \quad . \tag{68.05.01}$$

→ 2220 2740

Let us suppose instead that $\mathcal{J}$, a scalar, is the volume density of a non-conserved quantity, $\varrho \equiv \mathcal{J}$.

L'equazione di continuità (equazione (21.03.03) or equation (28.03.01)), in tal caso, oltre alla densità di volume della grandezza conservata, ϱ, e all flusso della grandezza conservata, $\mathbf{j}$, must include the amount of $\mathcal{C}$ generate per unit volume per unit time, $\sigma[\mathbf{x},t]$, so that it becomes:

$$\boxed{\varrho[\mathbf{x},t] \qquad \mathbf{j}[\mathbf{x},t] \qquad \sigma[\mathbf{x},t] \qquad \mathbf{div}\mathbf{j} + \partial_t \varrho = \sigma} \quad . \tag{68.05.02}$$

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68.05.02
Convective Motion - the Drift of Particles

The motion is caused by an external field of forces, such as in the case of the motion of electric charges in the presence of an electric field or massive particles in the gravity field. It is assumed that the motion is slowed by a viscous resistance, so that, in steady conditions, the velocity is constant: the drift velocity.

In general, the drift velocity is a function of the external force, $\mathbf{F}$: $\mathbf{v} = \mathbf{v}[\mathbf{F}]$:

$$\varrho \equiv \frac{\mathrm{d}\mathcal{C}}{\mathrm{d}V} \qquad \text{the amount of transported quantity per unit volume} \quad , \quad ,$$

$$\mathbf{j} \equiv \varrho\mathbf{v} = \varrho\mu\mathbf{F} \qquad \text{the current density} \quad , \quad ,$$

$$\boxed{\mathbf{v} \equiv \mu\mathbf{F} \qquad \mu = \frac{\tau}{m}} \quad ,$$

e la velocità è data dall'effetto combinato del campo di forza che provoca il moto e dalle interazione con le altre particelle che rallentano, in termini della mobilità, μ.

68.05.03
Diffusive Motion - the Diffusion of Particles

Consider a small volume of matter and, applying the **Koënig theorem** for kinetic energy, separate the Center-Of-Mass motion, the convective motion, from the diffusive random motion in the [glo:CMRF]Center-of-Momenta Re
The description of the system in the [glo:CMRF]Center-of-Momenta Reference Frame, the frame comoving with the fluid, leaves the motion due to the molecular random fluctuations around the Center-Of-Mass velocity.

In generale ad ogni variabile di stato intensiva, $\mathcal{J}$, è associata una variabile coniugata, $\mathcal{C}$, tale che per ottenere uniformità nella variabile $\mathcal{J}$ occorre un flusso della corrispondente variabile coniugata $\mathcal{C}$. In generale ogni cambiamento di $\mathcal{J}$ avviene tramite un flusso di $\mathcal{C}$.

In generale, se $\mathbf{j}$ è il flusso di $\mathcal{C}$, la relazione più semplice, quando $\mathcal{J}$ è una grandezza scalare e le proprietà di diffusione sono omogenee e isotrope, è data da:

$$\boxed{|\mathbf{j}| = \frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\mathbf{S}_\perp\,\mathrm{d}t} \qquad \mathbf{j} = -K\mathbf{grad}\mathcal{J} \qquad K > 0 \qquad [\mathrm{K}] = \mathrm{m}^2/\mathrm{s}} \quad ,$$

dove K è un'opportuna costante.

In generale si definiscono la variabile, $\mathcal{J}$, e il flusso di $\mathcal{C}$ in modo che un flusso entrante una superficie chiusa causi un aumento della variabile intensiva all'intenro del volume (tenendo costanti le eventuali altre grandezze fisiche del problema). Inoltre il flusso deve andare in direzione opposta al gradiente, se la variabile intensiva deve tendere a diventare uniforme.

Si hanno dunque in generale le seguenti proprietà per un moto diffusivo:

- la legge dell'evoluzione verso l'equilibrio, caratterizzato da un valore uniforme di $\mathcal{J}$;
- la relazione di positività $\partial\mathcal{J}/\partial\mathcal{C} > 0$, (con le eventuali altre variabili costanti);
- la relazione flusso e gradiente: $\mathbf{j}\cdot\mathbf{grad}\mathcal{J} \leq 0$.

Si dimostra che due qualunque delle proprietà precedenti implicano la terza.

In the case of diffusive motion, even in the absence of external forces, the current density, $\mathbf{j}$, depends on $\mathcal{J}$, via the relation:

$$\boxed{j_p[\mathbf{x},t] = -K^{\cdot\cdot}_{pq}[\mathbf{x},t]\frac{\partial\mathcal{J}}{\partial x^q}[\mathbf{x},t] \qquad \text{with } \mathbb{K} \text{ positive definite}} \quad , \qquad (68.05.03)$$

→ 2740

which is **a constitutive equation**, defined by a suitable second-order tensor $\mathbb{K}$, which is positive definite due to $\mathbf{j}\cdot\mathbf{grad}\mathcal{J} \leq 0$.

The usual simplification might apply in any give problem, whenever the system is either homogeneous or isotropic, and|or does not explicitly depend on time.

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68.05.04
Diffusion Equation

This § is referenced at pages:
[1994]

Suppose $\mathcal{C}$ is a conserved quantity. Whenever $\mathbb{K}$ is a fixed tensor proportional to the identity, that is the system is homogeneous and isotropic and independent of time, the diffusive current density in equation (68.05.03) can be inserted into the continuity equation (68.05.01) finding:

$$\boxed{K\nabla^2\mathcal{J} = \partial_t\mathcal{J}} \quad , \tag{68.05.04}$$

that is the diffusion equation.

The Green function of the diffusion equation in n-dimension is:

$$\boxed{G[\mathbf{x}, t, \mathbf{x}_0, t_0] = \frac{\exp -\dfrac{|\mathbf{x}-\mathbf{x}_0|^2}{4K(t-t_0)}}{(4\pi K(t-t_0))^{n/2}}\Theta(t-t_0) \qquad t > t_0} \quad ,$$

representing a point source at $\mathbf{x} = \mathbf{x}_0$ and $t = t_0$.

Figure 68.11 shows the Green function of the 1D diffusion equation at different times.

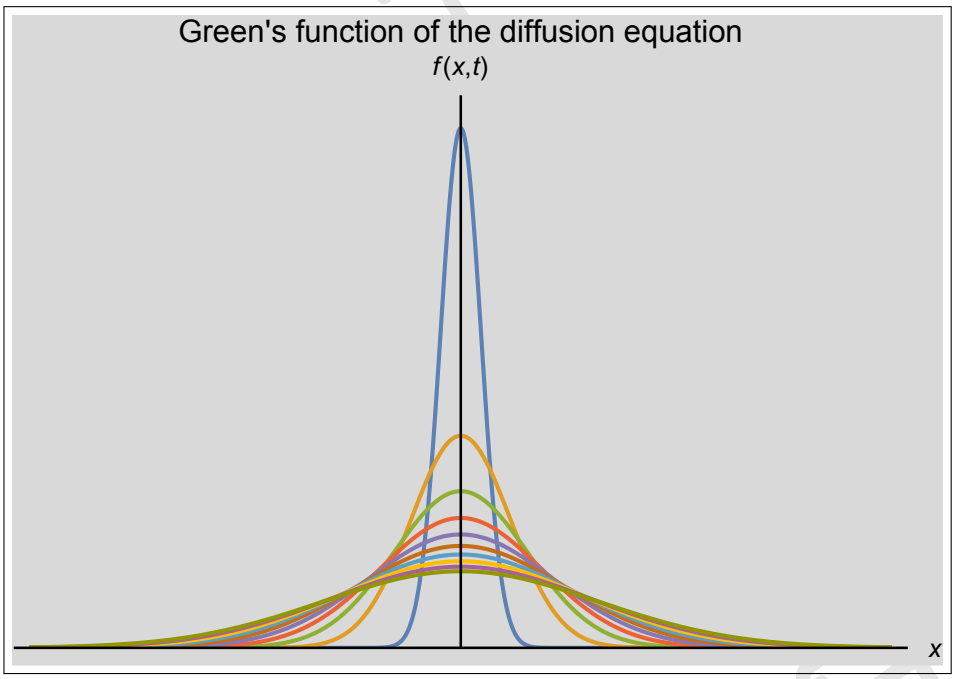


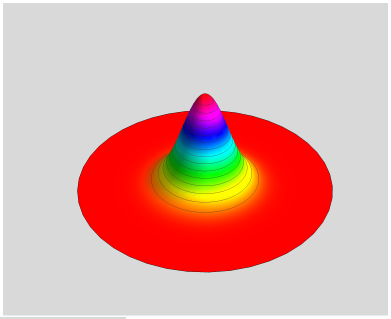
Figure 68.11: Green function of the 1D diffusion equation

The Green function of the 1D diffusion equation at different times.

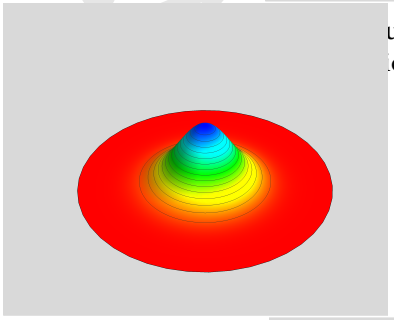
Figure 68.12 shows the Green function of the 2D diffusion equation at different times. Some properties of the Green function of the diffusion equation on the line, plane and space are:

1. 1D: $\langle r\rangle = 2\sqrt{Kt/\pi}$; $\langle r^2\rangle = 2Kt$;
2. 2D: $\langle r\rangle = \sqrt{\pi Kt}$; $\langle r^2\rangle = 4Kt$;
3. 3D: $\langle r\rangle = 4\sqrt{Kt/\pi}$; $\langle r^2\rangle = 6Kt$.

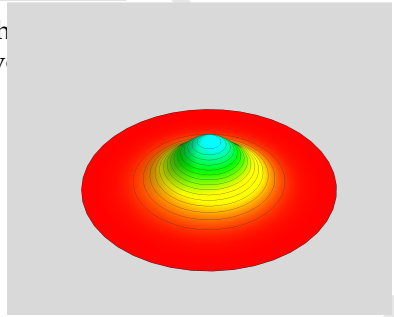
Note that the diffusion equation is not compatible with relativistic causality, as the Green function is non zero at arbitrarily large distances for any time $t > 0$.



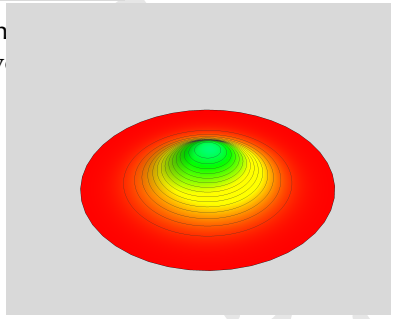
unction of the 2D diffu-
on - time evolution (1)



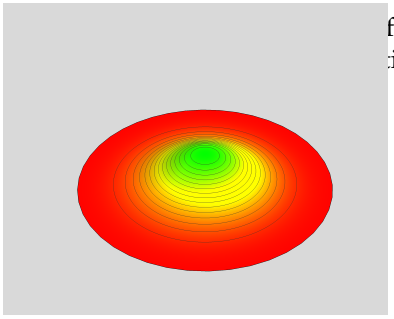
(b) Green function of th
sion equation - time ev



(c) Green function of th
sion equation - time ev



function of the 2D diffu-
ion - time evolution (4)



(e) Green function of the 2D diffu-
sion equation - time evolution (5)

Figure 68.12: Green function of the 2D diffusion equation at different times.

68.05.05
Diffusion - Crude Calculation in the Mean Free Path Approximation

$$j_z = -\frac{1}{3}\eta_{\mathbb{N}}\langle v\rangle\lambda\frac{\partial Q}{\partial z} \quad .$$

68.05.06
Diffusion of Energy - Constitutive Equation for the Energy Flux (Heat)

This § is referenced at pages:
[2602]

Read also § 65.09.02.

Nel caso in cui esista un gradiente di temperatura (variabile intensiva ϱ) si genera un flusso di energia (variabile coniugata $\mathcal{C}$), il calore, che tende a rendere uniforme la temperatura. La relazione, per un materiale isotropo, è data dalla legge di Fourier, che regola il calore (flusso di energia):

$$\mathbf{j}[\mathbf{x},t] = -F[\mathbf{x},t]\mathbf{grad}(T[\mathbf{x},t]) \qquad F \geq 0, \text{ thermal } \mathbf{conductivity} \quad , \tag{68.05.05}$$

dove F è il coefficiente di **conductivity** termica, $T[\mathbf{x},t](\equiv \varrho[\mathbf{x},t])$ la temperatura e $\mathbf{j}$ il flusso di energia, il calore, $Q(\equiv \mathcal{C})$.

Nel caso più generale di materiale non isotropo la condicibilità diventa un tensore di secondo ordine.

A crude calculation, assuming that all particles are moving at the mean speed and 1/6 of them (that is 1/6 for each orthogonal direction) are hitting the wall, gives the rough approximation:

$$F \approx \frac{1}{3}\eta_{\mathbb{N}}\langle v\rangle c\lambda \quad .$$

The thermal **conductivity** for gases does not depend on density. In fact, the heat transport is proportional to the density and to the mean free path, but the mean free path is inversely proportional on the density.

68.05.07
Diffusion of Linear Momentum - Constitutive Equation for the Linear Momentum Flux

©|S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...|§ 9.1||

As linear momentum is a vector, the flux of linear momentum is a second order tensor.

Il fenomeno della viscosità è data dal trasporto di quantità di moto, grandezza vettoriale. See equation (28.05.03). In generale, la relazione tra gradiente della quantità di moto e il flusso di quantità di moto (sforzo di taglio) è una relazione tensoriale: read § 32.08.03.

In the simple case of a dilute gas, the coefficient of viscosity can be calculated readily on the basis of kinetic theory.

A crude calculation, assuming that all particles are moving at the mean speed and 1/6 of them (that is 1/6 for each orthogonal direction) are hitting the area of the plane from above. But molecules which cross the plane from below have, on the average, experienced their last collision at a distance given by the mean free path below the plane, where the component of the velocity parallel to the plane is larger/smaller. The simple calculation gives the rough approximation:

$$\eta \approx \frac{1}{3}\eta_{\mathbb{N}}\langle v\rangle\lambda m \quad . \tag{68.05.06}$$

→ 2742

The exact calculation gives the same result as equation (68.05.06).

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68.05.08
Diffusion of Mass - Constitutive Equation for the Mass Flux

One speaks of self-diffusion whenever the diffusing molecules are, except for being labelled, identical to the remaining molecules of the substance. The more general and complicated situation would be that of mutual diffusion, where the diffusing and background molecules are unlike.

Nel caso della diffusione la più semplice relazione tra flusso (densità di corrente) e concentrazione è lineare ed è data dalla legge di Fick

$$\mathbf{j}[\mathbf{x},t] = -D[\mathbf{x},t]\mathbf{grad}(\eta_{\mathbb{N}}[\mathbf{x},t]) \qquad D>0 \quad , \tag{68.05.07}$$

dove D è il coefficiente di diffusione, $\eta_{\mathbb{N}}[\mathbf{x},t](\equiv \varrho[\mathbf{x},t])$ la concentrazione e $\mathbf{j}$ il flusso di numero di particelle $N(\equiv \mathcal{C})$.

In non isotropic materials, D, is a second-rank tensors, the diffusivity.

68.05.08.01 Self-Diffusion

Self-diffusion: labelled molecules diffuse amongst unlabelled, but otherwise identical, molecules.

Consider a gas in equilibrium and look at the motion of a given molecule, tagged but otherwise identical to the others, such as considering molecules with atomic nuclei made of different isotopes. These molecules follows straight trajectories interrupted by collisions that change its direction of motion, and, after a few collisions, the molecules will be mixed with the others.

The tagged molecule is said to follow a random walk; this is the self-diffusion process.

Example: diffusion of a radio-tracer isotope that, while chemically identical to the host element, is easily tracked; however, the mass of the different isotopes are slightly different.

The coefficient of self-diffusion is:

$$\boxed{D \approx \frac{1}{3}\langle v\rangle\lambda} \quad . \tag{68.05.08}$$

→ 2743

68.05.08.02 Mutual-Diffusion

Diffusion processes, are processes such that particles of a certain kind diffuse in a medium made of particles of a different kind.

Consider the diffusion of molecules of one type (type 1, mass m_1, diameter d_1) amongst molecules of another type (type 2, mass m_2, diameter d_2). In this case the diffusion constant D_{12} is the same as equation (68.05.08) with the diameter of the molecule replaced by The average diameter and the mass replaced by the reduced mass.

Example: a perfume diffusing in a room.

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68.05.09
Diffusion of Particle - Constitutive Equation for the Particle Flux

It is totally analogous to the diffusion of mass, just replacing the conserved mass with the conserved number of particles.

68.05.10
Diffusion in External Fields

Total current:

$$\boxed{\mathbf{j} = \varrho\mu\mathbf{F} - K\mathbf{grad}(\eta_{\mathbb{N}}[\mathbf{x}, t])} \quad .$$

In addition, the continuity equation applies.

In steady conditions, $\mathbf{j} = 0$.

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68.06 Brownian Motion

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 1|*§ 41*|

Read § 68.08.15, § 68.08.16.

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68.07 Detailed Balance

©|J.Oxenius, Kinetic Theory Of Particles And Photons|§ 1|*Excellent full clear discussion.*|

The reciprocity relation,

$$\boxed{\wp[i \to f] = \wp[f \to i]} \quad ,$$

is not generally valid, not even when the Hamiltonian of the system is invariant under space reflection and time-reversal. However, it holds in the framework of perturbation theory, and thus applies to radiative processes and to collisions in the Born approximation. Moreover, the reciprocity relation holds for all collision processes involving non wave-Polarized particles, provided that the Hamiltonian is invariant under space reflection and time-reversal.

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68.08 Examples and Physical Applications

68.08.01 Pompa Di Gas

©|I.E.Irodov, *Problems In General Physics*, 1988, MIR publishers Moscow, ...Ed., ...|2.8||

Un contenitore rigido di volume V contiene n moli di un gas a temperatura T e deve essere evacuato tramite una pompa che evacua un volume per unità di tempo $c = \Delta V/\Delta t$ (misurato alla pressione del gas a quell'istante) costante e indipendente dalla pressione. Determinare l'andamento temporale della pressione entro il contenitore.

WORKED SOLUTION

Si supponga che all'inizio di ogni ciclo il gas occupi uniformemente tutto il volume a disposizione incluso il volume ΔV della camera della pompa che verrà evacuato durante il ciclo. Si suppone cioè che al termine di ogni ciclo il gas occupi l'intero volume in un tempo molto più breve del periodo della pompa e che all'inizio del ciclo successivo il gas abbia raggiunto uno stato di equilibrio termodinamico.
Il numero di moli evacuate ad ogni ciclo vale allora

$$-\Delta n = \frac{n(t)\Delta V}{V}$$

da cui si trova, dividendo per l'intervallo di tempo di durata di un ciclo Δt,

$$-\frac{\Delta n}{\Delta t} = \frac{n(t)}{V}\frac{\Delta V}{\Delta t} = c\frac{n(t)}{V}$$

Passando al limite per Δt che tende a zero, nell'ipotesi di poter considerare il processo continuo, cioè che $\Delta V \ll V$ (ovvero $\Delta n \ll n$), si ha

$$-\frac{\mathrm{d}n}{\mathrm{d}t} = c\frac{n(t)}{V}$$

che risolta per c costante da

$$\boxed{n(t) = n_0 \exp\left[-\frac{ct}{V}\right]} \qquad (68.08.01)$$

→ 2747

e analogo andamento per la pressione.
Se l'approssimazione di processo continuo non è valida occorre trattare in tutta generalità il processo discreto. Sia n_k il numero di moli nel recipiente al termine del ciclo k-esimo. Si ha allora la corrispondenza

$$t = k\Delta t \qquad t = 0 \text{ per } k = 0$$

tra il tempo e il numero di cicli di pompaggio. Si ha allora

$$n_{k+1} = n_k\,(1-\alpha) \qquad k \geq 0 \qquad \alpha \equiv \frac{\Delta V}{V} < 1$$

che ha per soluzione

$$n_k = n_0\,(1-\alpha)^k = n_0\,(1-\alpha)^{t/\Delta t} = n_0\,(1-\alpha)^{ct/\alpha V} = n_0\,(1-\alpha)^{\frac{c\Delta t k}{\alpha V}}$$

data la condizione iniziale n_0 all'inizio del pompaggio. Al limite per $\alpha \to 0$, mantenedo k, costante si ottiene l'andamento continuo in funzione del tempo (68.08.01). Si osservi che $\alpha V = c\Delta t$.
Si noti che la sola ipotesi fatta è che all'inizio di ogni ciclo il gas occupi uniformemente l'intero volume. Non si richiede che il gas sia perfetto. Non si richiede che la temperatura del gas non

cambi. Se il gas non è perfetto le sue espansioni ad occupare l'intero contenitore tipicamente ne diminuiranno l'energia interna e quindi la temperatura. Se si vuole conoscere l'andamento della pressione entro il contenitore, anziché quello del numero di moli occorre sapere come varia la temperatura del gas durante il processo.

68.08.02
Gas Effusion With Two Containers

©|Lim Yung-Kuo, *Problems And Solutions On Thermodynamics And Statistical Physics*, 1990, World Scientific, ...Ed., ...

A rigid and thermal isolated box of volume $2V$ is divided into two equal parts by a thin, heat-conducting partition. One side contains an ideal gas of molecules at atmospheric pressure and $T = 293$ K. A small hole of surface A is opened in the partition, small enough so that thermal equilibrium between the two sides is maintained via heat conduction through the partition.

1. Calculate the number of particles, pressure and temperature as functions of time in both halves of the box.
2. Suppose the partition is a non-conductor of heat. Discuss briefly and qualitatively any deviations from the time-dependence of number of particles, temperature and pressure found in the previous question.

68.08.03
Graham Law

©|S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...|§ 7||

Demonstrate the empirical relation known as Graham law of effusion which states that the rate of effusion is inversely proportional to the square root of the mass of the effusing molecule.

68.08.04
Effusion - Gas Compared

©|S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...|example 7.2||

How much faster does helium gas effuse out of a small hole than N_2? Assume two identical containers with both gases starting at the same temperature and pressure.

68.08.05
Isotope Separation by Effusion

©|S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...|Example 7.1||

Consider a mixture of ideal gases effusing from the same hole in a container with diathemal and rigid walls, in sucha away that the temperature is constant. Determine the ratio of the component gases outside the container for gas UF_6, considering the two isotopes with $A = 235$ and $A = 238$ of uranium, as a function of the initial ratio inside the container and, from that, the enrichment factor.

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68.08.06
Flux of Gas at NTP

©|S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...|example 7.3||

Calculate the particle flux for N_2 gas at NTP.

68.08.07
Isotope Separation by Ultracentrifuging

68.08.08
Molecular Beams

©|S.J.Blundell & K.M.Blundell, *Concepts In Thermal Physics*, 2010, Oxford, 2ndEd., ...|example 7.5||

©|WEB - URL |||

What is the mean kinetic energy and mean speed of gas molecules effusing out of a small hole?

WORKED SOLUTION

$$\langle \mathrm{k} \rangle = 2k_{\mathrm{B}}T; \langle v \rangle = \sqrt{\frac{9\pi k_{\mathrm{B}}T}{8m}}.$$

68.08.09
Distribution of Relative Velocities

©|Lim Yung-Kuo, *Problems And Solutions On Thermodynamics And Statistical Physics*, 1990, World Scientific, ...Ed., ...

Show that the RMS relative speed of molecules in a gas made of all identical particles is $\langle u \rangle = \sqrt{2\langle v^2 \rangle}$.

68.08.10
Equation of State of a Ideal Molecular Gas

This § is referenced at pages:
[Never referenced.]

It can be derived from the pressure: read § 68.04.04, for a molecular gas, and read § 68.04.04, for a photon gas.

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68.08.11
Doppler Broadening of Spectral Lines

This § is referenced at pages:
[2751]

©|F.Reif, *Fundamentals Of Statistical And Thermal Physics*, 2009, Waveland Press, ...Ed., ...|problem 7.22||
©|Lim Yung-Kuo, *Problems And Solutions On Thermodynamics And Statistical Physics*, 1990, World Scientific, ...Ed., ...

Atoms|Molecules in thermal motion act as sources in motion, when emitting ElectroMagnetic radiation, so that the measured frequency shows a spectrum related to the velocity distribution of the Atoms|Molecules. When looking at the spectral lines of hot gas Atoms|Molecules, the limit on resolution is often given by this effect.

In fact, the light emitted, with frequency ω', by Atoms|Molecules traveling with radial velocity with respect to the detector, will be detected with frequency, ω, differing from those of Atoms|Molecules at rest, due to the **Doppler shift** .

Consider a set of Atoms|Molecules with no macroscopic velocity with respect to the observer, that is with zero velocity of the Center-Of-Mass in the Rest Frame of rest of the observer. In this Rest Frame, the velocity distribution of the Atoms|Molecules, as seen from the observer, is the Maxwell-Boltzmann distribution of velocities. Show that a Gaussian shape of the spectral line $I[\omega]$ is produced starting from emission at fixed frequency ω'.

PRELIM-PROGRESS

68.08.12
Doppler Broadening and Red/blue Shift of Spectral Lines

Extend the results of § 68.08.11 to the situation where the gas has a macroscopic movement, that is, a Center-Of-Mass motion, with radial component of the velocity $V_r \lessgtr 0$ with respect to the observer.

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68.08.13
Escape of Molecules From the Atmosphere

©|Lim Yung-Kuo, *Problems And Solutions On Thermodynamics And Statistical Physics*, 1990, World Scientific, ...Ed., ...

© - QUOTE

R.Balian, *From Microphysics To Macrophysics*, 2007, Springer, ...Ed., ...
The composition of planetary atmospheres is directly affected by the molecular velocity distribution. In fact, the escape velocity from a star, $\sqrt{2\mathcal{G}M/R}$, where $\mathcal{G}$ is the gravitational constant, depends on the mass, M, and the radius, R, of the star. Molecules with larger velocities can escape from the gravitational field of the star. This escape is easier for light molecules and that it is helped by high temperatures. This enables us to understand why there is no hydrogen in the Earth atmosphere, even though that element is so very abundant in the Universe. If the temperature at the top of the atmosphere is sufficiently high and if the molecules are sufficiently light, we have in equilibrium an appreciable fraction of molecules with velocities above the escape velocity, which for the Earth is $11\ \mathrm{km/s}$. Those molecules can escape and at equilibrium will be replaced by others which, in their turn, escape. This has happened for hydrogen. For oxygen and nitrogen the molecular masses are sufficiently large that the losses are insignificant, even at the rather high temperatures prevailing when the Earth was formed. On the Moon the escape velocity is so low that the whole atmosphere has had time to escape.

© - QUOTE

MüllerAndWWeiss EntropyAndEnergy
The atmosphere of a planet is a case in point for the competition of energy and entropy. Energy tends to assemble all atmospheric molecules on the surface of the planet while entropy tends to spread the molecules evenly throughout space. In the end the entropy wins and planets will be bare of atmosphere. This will happen earlier for a hot planet than for a cold one. Earth, which is a temperate planet in the solar system, has already lost all light gases but it holds on - for the time being - to the heavier ones like nitrogen, oxygen and argon.
We consider an atmosphere of total mass of a planet as an ideal gas of constant and uniform temperature. Therefore the internal energy is constant. The density distribution in the atmosphere will arrange itself so as to minimize the available free energy which in this case has only two essential parts: The potential energy and the entropy. The potential energy will tend to assemble the atoms of the atmosphere on the surface of the planet, because that is where the potential energy has its minimum. On the other hand, the entropy will tend to become maximal by scattering the atoms through infinite space. Thus the planetary atmosphere is subject to opposing tendencies and it provides a paradigm for the competition of energy and entropy. In order to investigate this competition suggestively we convert the available free energy from a functional of to a function of a single variable by considering a model atmosphere which is constrained between the planetary surface and an imagined roof of fixed height H. We consider this model atmosphere as being in partial equilibrium in the sense that the available free energy under the roof is minimal. In this manner .... become functions of the single variable H and we investigate their extrema in order to determine the density distribution in full unconstrained equilibrium.

68.08.14
Thermal Excitation of Hydrogen Atoms

©|D.V.Schroeder, *Thermal Physics*, 1999, Addison-Wesley, ...Ed., ...|§ 6.1||

© - QUOTE

©|D.V.Schroeder, *Thermal Physics*, 1999, Addison-Wesley, ...Ed., ...|§ 6.1||

As a simple application of Boltzmann factor, let us consider a Hydrogen atom in the atmosphere of the Sun, where the temperature is about 5800 K.
Let us compare the probability of finding the hydrogen atom in one of its first excited states (s_2) to the probability of finding it in the ground state (s_1). The ratio of probabilities is the ratio of Boltzmann factors, and the difference in energy is 10.2 eV while $k_B T$ is 0.50 eV. So, the ratio of probabilities to find the atom in one specific state s_2 with respect state s_1, is approximately $1.4 \cdot 10^{-9}$. However, there are four such excited states, s_2, all with the same energy but different angular momentum, so, the ratio of probabilities to find the atom in any one state s_2 with respect state s_1, is approximately $5.6 \cdot 10^{-9}$, about 5.6 atoms in the first excited state for every billion in the ground state.

68.08.15
Uni-Dimensional Random Walk

This § is referenced at pages:
[2745, 2754]

Read also section § 15-024.

WORKED SOLUTION

Binomial distribution. movement along the rectilinear abscissa s.

$$p + q = 1 \quad ,$$
$$\Delta s = +\lambda \qquad \text{with probability } p \quad ,$$
$$\Delta s = -\lambda \qquad \text{with probability } q \quad ,$$
$$\langle s \rangle = (p - q)\,\lambda \qquad \text{single step} \quad ,$$
$$\langle s^2 \rangle = (p + q)\,\lambda^2 = \lambda^2 = (p + q)^2\,\lambda^2 \qquad \text{single step, first passage from direct calculation} \quad ,$$
$$\sigma^2 = \langle s^2 \rangle - \langle s \rangle^2 = 4pq\lambda^2 \qquad \text{single step} \quad ,$$
$$\sigma^2 = \langle s^2 \rangle - \langle s \rangle^2 = \lambda^2 \qquad \text{for the symmetrical case: } p = q = 1/2 \quad ,$$
$$S_n = \sum_{k=1}^{n} s_k \qquad \text{position after } n \text{ steps} \quad ,$$
$$\langle S_n \rangle = n\,(p - q)\,\lambda \quad , \quad ,$$
$$\langle S_n \rangle = 0 \qquad \text{for the symmetrical case: } p = q = 1/2 \quad ,$$
$$\sigma^2[S_n] = n\sigma^2[s] = 4npq\lambda^2 \quad ,$$
$$\sigma^2[S_n] = n\lambda^2 \qquad \text{for the symmetrical case: } p = q = 1/2 \quad .$$

→ 2754

68.08.16
Three-Dimensional Random Walk

This § is referenced at pages:
[2745]

Read also section § 68.08.15.

WORKED SOLUTION

Use (68.08.15). Model the path as if the particle, at each step, makes a 1D jump along all the three x, y, z directions. Assume $p = q = 1/2$. All the steps and in all directions, are made of independent variables.

$$\mathbf{S}_n = \sum_{k=1}^{n} \mathbf{s}_k \qquad \text{position after } n \text{ steps} \quad , \quad ,$$

$$\langle \mathbf{S}_n \rangle = 0 \quad , \quad ,$$

$$\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle = \lambda^2 \quad , \quad ,$$

$$\sigma^2[\mathbf{S}_n] = \langle |\mathbf{S}_n|^2 \rangle - \langle \mathbf{S}_n \rangle^2 = n \langle |\mathbf{s}|^2 \rangle = 3n\sigma^2[\mathbf{s}] = 3n\lambda^2 \quad .$$

68.08.17
Equation of State for a Mono-Atomic Ideal Gas

When equation (68.04.04) is applied to a mono-atomic ideal gas, one has:

$$\boxed{N k_{\mathrm{B}} T = pV = \frac{2}{3}\mathcal{K} = \frac{2}{3}\mathcal{U}} \quad .$$

The above relation is equation (65.09.01.06), to be compared with the photon gas equation (65.09.01.06).

The same equation of state can be deduced by the Virial theorem, read § 30.07.04.02.

68.08.18
Convective Motion - Electric Conduction

The transport of electric charge provides Ohm law, whenever the **conductivity** does not depend on the electric field.

$$\boxed{\mathbf{j}[\mathbf{x}, t] = -\sigma[\mathbf{x}, t]\mathbf{grad}(\Phi[\mathbf{x}, t])} \qquad (68.08.02)$$

DRAFT IN PROGRESS

68.09 Exercises Problems and Physical Applications

68-001 Kinetic Energy Distribution

Show that the distribution of kinetic energy is:

$$\mathrm{d}N[\mathtt{k}, \mathtt{k} + \mathrm{d}\mathtt{k}] = \frac{N\sqrt{\pi}}{2} (k_{\mathrm{B}}T)^{3/2} \sqrt{\mathtt{k}} \exp\left[-\frac{\mathtt{k}}{k_{\mathrm{B}}T}\right] \mathrm{d}\mathtt{k} \quad .$$

Check the normalization constant.

68-002 Inverse Velocity Distribution

Show that the distribution of the inverse velocity is:

$$\langle \frac{1}{v} \rangle = \frac{4}{\pi \langle v \rangle} \quad .$$

68-003 Velocity Distribution of Parallel and Perpendicular Velocity Components

Show that the distribution of the parallel and perpendicular components velocity is:

$$\mathrm{d}N[v_{\parallel}, v_{\parallel} + \mathrm{d}v_{\parallel}; v_{\perp}, v_{\perp} + \mathrm{d}v_{\perp}] \propto \exp\left[-\frac{\mathtt{k}}{k_{\mathrm{B}}T}\right] v_{\perp} \, \mathrm{d}v_{\parallel} \, \mathrm{d}v_{\perp} \quad . \tag{68.09.01}$$

Note that, while $v_{\parallel}$ ranges from $-\infty$ to $+\infty$, $v_{\perp}$ ranges from 0 to $+\infty$.

Calculate the normalization constant.

68-004 Kinetic Energy of Effusing Particles

Calculate the mean value of the kinetic energy of particles effusing from a small hole, assuming that the variation of the number of particles inside the container and the change of temperature is negligible.

WORKED SOLUTION

$\langle \mathtt{k} \rangle = 2k_{\mathrm{B}}T$

68-005 Atomic and Molecular Jets

Calculate the RMS speed of a nitrogen molecules at room NPT.

WORKED SOLUTION

$v_{\mathrm{RMS}} \approx 500 \text{ m/s}.$

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68-006 Atomic and Molecular Jets

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68-007 Effusion

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68-008 Gas Diffusion in a Room

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You have just very gently exhaled a helium atom in this room. Calculate how long it will take to diffuse with a reasonable probability to some point on a spherical surface of radius $R = 1$ m surrounding your head. Assume STP. Assume $\sigma = 10^{-20}$ m^2.

68-009 Effusion

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A box of volume $2V$ is divided into halves by a thin partition. The left side contains a ideal gas at pressure p_0 and the right side is initially vacuum. A small hole of area A is punched in the partition. What is the pressure in the left hand side as a function of time? Assume the temperature is constant on both sides. Express your answer in terms of the average velocity.

68-010 Molecular Beams

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A beam of energetic neutral hydrogen atoms is coming through a hole in the wall of plasma confinement device. Describe the apparatus that you would use to measure the energy distribution of these atoms.

68-011 Emptying in Vacuum

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At time $t = 0$, a thin walled vessel of volume V, kept at constant temperature, contains N_0 ideal gas molecules which begin to leak out through a small hole of area A. Assuming negligible pressure outside the vessel, calculate the number of molecules leaving through the hole per unit time and the number remaining at time t. .

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CHAPTER 69

Elements of Statistical Physics

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©|F.Reif, *Fundamentals Of Statistical And Thermal Physics*, 2009, Waveland Press, ...Ed., ...|||
©|R.Balian, *From Microphysics To Macrophysics*, 2007, Springer, ...Ed., ...|§ 3, 4||
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The statistical interpretation of entropy links the thermodynamics concept of entropy to the statistical laws which underlie physical systems consisting of a large number of particles. It was only with great difficulty accepted by the international scientific community of the time.

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R.Balian, *From Microphysics To Macrophysics*, 2007, Springer, ...Ed., ...
In § 3 we defined the statistical entropy $S(D)$ which measures the random nature, or disorder, of a density operator. In those two chapters we assumed that the latter was given. However, in order actually to be able to calculate the properties of a system which has been prepared in some given way we must know how to assign to it a density operator representing the physical situation that we want to describe. This problem of the choice of D will be solved in the present chapter for thermodynamic equilibrium states. In order to find the general form, the so-called Boltzmann-Gibbs distribution, of the density operators, or the densities in phase, describing these states, we shall use a postulate of a statistical nature which is similar to the criteria used in statistics to find the unbiased probability law for a set of random events. We introduce in this way a general prediction method (§ 4.1.3). This method leads us to represent a system in thermodynamic equilibrium by the most disordered macro-state compatible with the macroscopic data (§ 4.1).

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69.01 Introduction

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69.01.01 Micro-State and Macro-State

The macroscopic state (**macro-state**) of a system can be characterized, globally, using few quantities directly accessible to our senses.

The microscopic state (**micro-state**) of a system is one state compatible with its macroscopic state that can be described using all the many physical quantities required and defined at the microscopic scale and therefore not accessible to our senses.

A system is generally subject to constraints, that is, to conditions that its macroscopic variables must satisfy, for example: if the system is isolated, its energy is constant; if the system is closed, it is the number of material particles that remains constant; if the walls of the system are rigid, its volume is constant.

Most often, a very large number of microscopic states correspond to the same macroscopic state. This number differs according to whether the particles are distinguishable, or not; they are so if they have different physical characteristics. If they are identical, they are classically always distinguishable, thanks to classical determinism so that, once a particle has been labelled it can followed and identified forever; in quantum physics they are always indistinguishable as the concept of trajectory is lost. When the particles are indistinguishable, the number of microscopic states corresponding to one macroscopic state, is smaller than for the corresponding distinguishable case.

The number of accessible microscopic states corresponding to a macroscopic state is a fundamental quantity in statistical physics.

Macroscopic variables are averages of suitable sets of microscopic variables. This average value should be understood as an average over a large number of systems identical to our system; this generally coincides with the time average of the magnitude over one single system (**ergodic hypothesis**).

Properties of the macro-state are obtained by integrating (averaging) over a large number of variables which seem to be less relevant to what we see in the macroscopic world. That is, the macroscopic theory is an effective theory of an underlining more complete theory. The same happens in fundamental physics.

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The theory of processes relating to the attainment of an equilibrium state is called kinetics. It is not part of statistical physics proper, which deals only with systems in statistical equilibrium.
Statistical physics applies to macroscopic bodies having a very large number of degrees of freedom (order of $\mathcal{N}_A$).
Even if we could solve the equations of motion in a general form, it would be impossible to substitute in the general solution the initial conditions for the velocities and coordinates of all the particles.
When the number of particles is very large, new types of regularity appear.
Let a given macroscopic mechanical system have s degrees of freedom: the position of points of the system in space be described by s generalized coordinates, q_i, $i = 1, 2, ..., s$.
The state of the system at a given instant will be defined by the values at that instant of the s generalized coordinates q_i and the s corresponding velocities, $\dot{q}_i$, $i = 1, 2, ..., s$. However, in statistics it is customary to describe a system by its generalized coordinates and momenta, p_i,

$i = 1, 2, ..., s$, not velocities, since this affords a number of advantages.
Let us now consider a macroscopic body or system of bodies, and assume that the system is isolated [a], i.e. does not interact with any other bodies. A part of the system, which is very small compared with the whole system but still macroscopic, may be imagined to be separated from the rest; clearly, when the number of particles in the whole system is sufficiently large, the number in a small part of it may still be very large. Such relatively small but still macroscopic parts will be called subsystems. A subsystem is again a mechanical system, but not an isolated one; on the contrary, it interacts in various ways with the other parts of the system. Because of the very large number of degrees of freedom of the other parts, these interactions will be very complex and intricate. Thus the state of the subsystem considered will vary with time in a very complex and intricate manner.
An exact solution for the behavior of the subsystem can be obtained only by solving the mechanical problem for the entire isolated system, i.e. by setting up and solving all the differential equations of motion with given initial conditions, which, as already mentioned, is an impracticable task. Fortunately, it is just this very complicated manner of variation of the state of subsystems which, though rendering the methods of mechanics inapplicable, allows a different approach to the solution of the problem.
A fundamental feature of this approach is the fact that, because of the extreme complexity of the external interactions with the other parts of the system, during a sufficiently long time the subsystem considered will be many times in every possible state. This may be more precisely formulated as follows. Let $\Delta q \Delta p$ denote some small volume of the phase space of the subsystem, corresponding to coordinates q_i and momenta p_i lying in short intervals $\Delta q_i \Delta p_i$. We can say that, in a sufficiently long time T, the extremely intricate phase trajectory passes many times through each such volume of phase space. Let Δt be the part of the total time T during which the subsystem was in the given volume of phase space $\Delta q \Delta p$. When the total time T increases indefinitely, some limit is assumed to exist:

$$\equiv \lim_{T\to\infty} \frac{\Delta t}{T} \quad .$$

This quantity may clearly be regarded as the probability that, if the subsystem is observed at an arbitrary instant, it will be found in the given volume of phase space $\Delta q \Delta p$.
...
This probability, $đw$, may be written:

$$đw = \rho[q_1, ..., q_s; p_1, ..., p_s]\ đq_1 ... đq_s\ đp_1 ... đp_s$$

where $\rho[q_1, ..., q_s; p_1, ..., p_s]$ is a function of all the coordinates and momenta. The function $\rho[q_1, ..., q_s; p_1, ..., p_s]$ which represents the density of the probability distribution in phase space, is called the statistical distribution function.
This function must be obviously normalised.
The following circumstance is extremely important in statistical physics. The statistical distribution of a given subsystem does not depend on the initial state of any other small part of the same system, since over a sufficiently long time the effect of this initial state will be entirely outweighed by the effect of the much larger remaining parts of the system. It is also independent of the initial state of the particular small part considered, since in time this part passes through all possible states, any of which can be taken as the initial state. Without having to solve the mechanical problem for a system (taking account of initial conditions), we can therefore find the statistical distribution for small parts of the system.
...
The determination of the statistical distribution for any subsystem is in fact the fundamental problem of statistical physics. In speaking of "small parts" of a isolated system, we must bear in mind that the macroscopic bodies with which we have to deal are usually themselves such "small parts" of a large isolated system consisting of these bodies together with the external medium which surrounds them.

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...
The averaging with respect to the distribution function (called statistical averaging) frees us from the necessity of following the variation with time of the actual value of the physical quantity $f[p,q]$ in order to determine its mean value.
...
In practice, however, when statistical physics is applied to macroscopic bodies, its probabilistic nature is not usually apparent. The reason is that, if any macroscopic body (in external conditions independent of time) is observed over a sufficiently long period of time, it is found that all physical quantities describing the body are practically constant (and equal to their mean values) and undergo appreciable changes relatively very rarely; we mean, of course, macroscopic quantities describing the body as a whole or macroscopic parts of it, but not individual particles. This result, which is fundamental to statistical physics, follows from very general considerations and becomes more and more nearly valid as the body considered becomes more complex and larger. In terms of the statistical distribution, we can say that, if by means of the distribution function we construct the probability distribution function for various functions $f[p,q]$, this function will have an extremely sharp maximum around the mean value, and will be appreciably different from zero only in the immediate vicinity of this point.
...
If a closed macroscopic system is in a state such that in any macroscopic subsystem the macroscopic physical quantities are to a high degree of accuracy equal to their mean values, the system is said to be in a state of statistical equilibrium (or thermodynamic or thermal equilibrium). It is seen from the foregoing that, if a closed macroscopic system is observed for a sufficiently long period of time, it will be in a state of statistical equilibrium for much the greater part of this period. If, at any initial instant, a closed macroscopic system was not in a state of statistical equilibrium (if, for example, it was artificially disturbed from such a state by means of an external interaction and then left to itself, becoming again a closed system), it will necessarily enter an equilibrium state. The time within which it will reach statistical equilibrium is called the relaxation time. In using the term "sufficiently long" intervals of time, we have meant essentially times long compared with the relaxation time.
...
The subsystems previously discussed are not themselves isolated systems; on the contrary, they are subject to the continuous interaction of the remaining parts of the system. But since these parts, which are small in comparison with the whole of the large system, are themselves macroscopic bodies also, we can still suppose that over not too long intervals of time they behave approximately as isolated systems. For the particles which mainly take part in the interaction of a subsystem with the surrounding parts are those near the surface of the subsystem; the relative number of such particles, compared with the total number of particles in the subsystem, decreases rapidly when the size of the subsystem increases, and when the latter is sufficiently large the energy of its interaction with the surrounding parts will be small in comparison with its internal energy. Thus we may say that the subsystems are quasi-isolated. It should be emphasised once more that this property holds only over not too long intervals of time. Over a sufficiently long interval of time, the effect of interaction of subsystems, However, weak, will ultimately appear. Moreover, it is just this relatively weak interaction which leads finally to the establishment of statistical equilibrium. The fact that different subsystems may be regarded as weakly interacting has the result that they may also be regarded as statistically independent.
By statistical independence we mean that the state of one subsystem does not affect the probabilities of various states of the other subsystems.
Let us consider any two subsystems, A and B:

$$\boxed{\rho_{A+B} = \rho_A \rho_B} \quad .$$

A similar relation is valid for a group of several subsystems.
The converse statement is clearly also true: if the probability distribution for a compound system

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is a product of factors, each of which depends only on quantities describing one part of the system, then the parts concerned are statistically independent, and each factor is proportional to the probability of the state of the corresponding part.
If f_A and f_B are two physical quantities relating to two different subsystems, then it follows immediately that:

$$\langle f_A + f_B \rangle = \langle f_A \rangle + \langle f_B \rangle \quad ,$$
$$\langle f_A f_B \rangle = \langle f_A \rangle \langle f_B \rangle \quad ,$$
$$\langle \Delta f_A \rangle = \langle \Delta f_B \rangle = 0 \quad ,$$
$$\langle \Delta f_A \Delta f_B \rangle = 0 \quad ,$$
$$\langle (\Delta f)^2 \rangle \equiv \langle (\Delta f_A + \Delta f_B)^2 \rangle = (\Delta f_A)^2 + (\Delta f_B)^2 \quad .$$

It follows that, as N increases, the mean square also increases in proportion to N. The relative fluctuation is therefore inversely proportional to $1/\sqrt{N}$:

$$\frac{\langle (\Delta f)^2 \rangle^{1/2}}{\langle f \rangle} \sim \frac{1}{\sqrt{N}} \quad .$$

[a] ©||L.D.Landau *et al.*, *Statistical Physics*, ..., ..., ...Ed., WEB - URL - ISBN: 0-08-023039-3.||

: uses the term **closed system** where nowadays the term **isolated system** is normally used.

69.01.02
Phase-Space

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Let a given macroscopic system have s degrees of freedom, with $s \approx \mathcal{N}_A$.
Let the positions of points of the system in space be described by s generalized coordinates, q_i, $i = 1, 2, ..., s$
The state of the system at any given instant will be defined by the values, at that instant, of the s generalized coordinates $q_i[t]$ and the s corresponding velocities, $\dot{q}_i[t]$, $i = 1, 2, ..., s$.
However, in statistics it is customary to describe the system by its generalized coordinates, $q_i[t]$, and momenta, $p_i[t]$, $i = 1, 2, ..., s$, not velocities, since this affords a number of advantages.
This is the so-called micro-state, as from the microscopic approach of thermodynamics.
The classical volume in phase space is defined as:

$$\mathrm{d}\Phi \equiv \frac{\mathrm{d}q_1 \ldots \mathrm{d}q_s \, \mathrm{d}p_1 \ldots \mathrm{d}p_s}{h^{2s}} \quad ,$$

by dividing by h^{2s}. so to make it dimensionless.

69.01.03
Distribution Function

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The distribution function in phase-space gives the probability, in general time-dependent, to find the system in a certain infinitesimal volume of the phase space, around $(q_1, \ldots, q_s; p_1, \ldots, p_s)$:

$$\mathrm{d}\mathcal{P}[q_1, \ldots, q_s; p_1, \ldots, p_s; t] = \mathcal{F}[q_1, \ldots, q_s; p_1, \ldots, p_s; t] \, \mathrm{d}q_1 \ldots \mathrm{d}q_s \, \mathrm{d}p_1 \ldots \mathrm{d}p_s \qquad 1 = \int \mathrm{d}\mathcal{P} \quad .$$

By integrating out any set of variables one finds the marginal (or reduced) distribution function in the remaining variables.
This process is the same as the general **effective theory** approach: you have a large number of variables, some of which are not accessible, others are not interesting, so one integrates them out, leaving a theory which is applicable whenever the ignored degrees of freedom are not relevant. This is the same as:

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- approximating the Earth shape as a regular geometric shape, instead of all the details of mountains, seas, ...;
- ignoring the effects of asteroids on the orbit of an Earth satellite, possbily adding an effective perturbation....

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69.02 Boltzmann-Gibbs Distribution

This § is referenced at pages:
[Never referenced.]

©|R.P.Feynman *et al.*, *The Feynman's Lectures On Physics*, 1964, Addison-Wesley, ...Ed., ...|Vol. 1|*§ 40*|
©|Berkeley Physics Course, Vol. 5, *Statistical Physics*, 1965, McGraw-Hill, ...Ed., ...|||

69.02.01 Micro-Canonical Ensemble

Classical Systems

In classical statistical physics, consider a isolated system described by the generalized coordinates $q \equiv \{q_i\}, i = 1, 2, ..., s$ and momenta $p \equiv \{p_i\}, i = 1, 2, ..., s$.

The probability to find the system in a micro-state defined by the hyper-volume $\mathrm{d}q\ \mathrm{d}p$ of its phase space is given by:

$$\boxed{\mathrm{d}\wp \propto \mathrm{d}q\ \mathrm{d}p} \quad . \tag{69.02.01}$$

If the system is made of N identical (indistinguishable) particles, the above probability must be divided by $N!$.

That is, all equal volumes in the phase space are equally probable.

Quantum Systems

In quantum statistical physics, consider a system which is isolated, has a continuous energy spectrum; the same recipe as for classical systems applies:

$$\boxed{\mathrm{d}\wp \propto \mathrm{d}q\ \mathrm{d}p} \quad .$$

For a quantum system which is isolated, has discrete states labeled by s, energy levels ϵ_s, and multiplicity of each level g_s, the probability to find the system in state s is given by:

$$\boxed{\wp_s \propto g_s} \quad . \tag{69.02.02}$$

That is, all discrete states are equally probable.

The extension to systems with both discrete and continuous spectrum follows trivially.

69.02.02 Canonical Ensemble

Classical Systems

In classical statistical physics, consider a system described by the generalized coordinates $q \equiv \{q_i\}, i = 1, 2, ..., s$ and momenta $p \equiv \{p_i\}, i = 1, 2, ..., s$, which is in contact with a heat reservoir at temperature T and has Hamiltonian $H[q, p]$.

The probability to find the system in a micro-state defined by the hyper-volume $\mathrm{d}q\ \mathrm{d}p$ of its phase space is given by:

$$\boxed{\mathrm{d}\wp \propto \exp\left[-\frac{H[q,p]}{k_{\mathrm{B}}T}\right] \mathrm{d}q\ \mathrm{d}p} \quad . \tag{69.02.03}$$

If the system is made of N identical (indistinguishable) particles, the above probability must be divided by $N!$.

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Quantum Systems

In quantum statistical physics, consider a system which is in contact with a heat reservoir at temperature T and having a continuous energy spectrum; the same recipe as for classical systems applies:

$$\mathrm{d}\wp \propto g_s \exp\left[-\frac{H[q,p]}{k_{\mathrm{B}}T}\right] \mathrm{d}q\, \mathrm{d}p \quad .$$

For a quantum system which is in contact with a heat reservoir at temperature T, has discrete states labelled by s, energy levels ϵ_s and multiplicity of each level g_s, the probability to find the system in state s is given by:

$$\wp_s \propto \exp\left[-\frac{\epsilon_s}{k_{\mathrm{B}}T}\right] \quad . \tag{69.02.04}$$

The extension to systems with both discrete and continuous spectrum follows trivially.

69.02.02.01 Equipartition Theorem

This § is referenced at pages:
[2539]

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One of the problems of statistical mechanics consists in determining how the energy of a system is distributed over its various degrees of freedom. The general answer to this question is obtained by writing down that the temperatures associated with the independent degrees of freedom become equal. The result takes a particularly simple form for all problems in classical statistical mechanics where the Hamiltonian is quadratic in each of the phase space variables which occur in it.

In thermal equilibrium the internal energy, $\mathcal{U} = \langle H \rangle$ is the sum of the average energies, $\langle h_i \rangle$, associated with the s degrees of freedom. **The energy equipartition theorem states that in the canonical system framework the internal energy per independent quadratic degree of freedom, $\langle h_i \rangle$, is equal to $k_{\mathrm{B}}T/2$, whatever the value of the constants coefficients of the second degree Hamiltonian polynomial. The total energy is thus equally distributed over all degrees of freedom and it is proportional to the temperature**:

$$\langle aq^2 \rangle = \langle bp^2 \rangle = \frac{k_{\mathrm{B}}T}{2} \quad . \tag{69.02.05}$$

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69.03 Toy Examples

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Without any loss of generality, consider an isolated system.

Macroscopic state: state of a system which can be globally characterized by a few macroscopic variables.

Microscopic state: state compatible with the macroscopic state and described by a large number of microscopic variables, not directly accessible; every microscopic state is known to within a certain extent and with some quantic uncertainly: the smallest volume in phase-space for any pair of conjugate variables is dictated by quantum physics:

$$\boxed{\Delta q \Delta p \geq \frac{\hbar}{2}} \quad .$$

The system will have in general states, labelled by s, with energy E_s and multiplicity, g_s, typically infinite in number, with either discrete or continuous spectrum.

The possible states must be compatible with the conserved quantities of the system, the mechanical invariant quantities, such as energy, linear momentum and angular momentum.

69.03.01 Three Particles With Four States

Consider a system of three particles, A, B, C, whose total energy is $E = 3\varepsilon$ and any particle can have energies $e = 0$, $e = \varepsilon$, $e = 2\varepsilon$, $e = 3\varepsilon$. The macroscopic state is characterized by its total energy E.

The micro-states are characterized by stating for every particle which state is in.

The number of microscopic states corresponding to the same macroscopic state are calculated as follows.

For distinguishable particles:

- all three particles have energy $e = \varepsilon$ (1 state);
- one particle has energy $e = 3\varepsilon$ and the others zero (3 states);
- one particle has energy $e = 2\varepsilon$, another one has energy $e = \varepsilon$, the third one has zero energy (6 states).

In total, 10 different microscopic states are accessible for the given macroscopic state.

For indistinguishable particles:

- all three particles have energy $e = \varepsilon$ (1 state);
- one particle has energy $e = 3\varepsilon$ and the others zero (1 state);
- one particle has energy $e = 2\varepsilon$, another one has energy $e = \varepsilon$, the third one has zero energy (1 state).

In total, 3 different microscopic states are accessible for the given macroscopic state.

69.03.02 Many Distinguishable Particles With Two States

The system of N particles with two states, for each of the considered particles, is very important in physics. It is indeed a simple and effective model to study the behavior of many significant physical systems.

Consider two different such systems:

- N particles located inside a rigid and adiabatic container ideally divided in two identical sub-volumes, with n_+ particle inside the right container and n_- particle inside the left container. Consider, for instance, the center of mass position $z_{\text{CM}} \simeq \ell\,(n_+ - n_-)$
- N identical spin 1/2 particles, with n_+ with spin up and n_- with spin down, so that $s_z = (n_+ - n_-)\,\hbar/2$; for instance paramagnetic particles in an external uniform and constant magnetic field, with energies $\pm\mu B_z$.

The macroscopic state is defined by: $(n_+ - n_-)$. Nothing else is relevant on the macroscopic scale.

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How many corresponding microscopic states correspond to any given macroscopic state, that is for any given values of n_+ particles on the right and n_- on the left. The number of micro-states of the system is the number of ways to get n_+ particles on the right side, among the N total particles, and n_- on the left side, that is to say the number of combinations:

$$n_+ + n_- = N \qquad g = \binom{N}{n_+} = \binom{N}{n_-} = \frac{N!}{n_+! n_-!}$$

For $p = q = 1/2$ and $N = 10$ with $2^{10} = 1024$, $\sigma = 1.6$, possible configurations:

n_+	g	,
0	1	,
1	10	,
2	45	,
3	120	,
4	210	,
5	252	,
6	210	,
7	120	,
8	45	,
9	10	,
10	1	,

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For $p = q = 1/2$ and $N = 1000$, with $2^{1000} = 1.1 \cdot 10^{301}$, $\sigma = 1.58$, possible configurations:

n_+	g	,
484.	1.62051×10^{299}	,
485.	1.72409×10^{299}	,
486.	1.82697×10^{299}	,
487.	1.92826×10^{299}	,
488.	2.02704×10^{299}	,
489.	2.12239×10^{299}	,
490.	2.21335×10^{299}	,
491.	2.29899×10^{299}	,
492.	2.37843×10^{299}	,
493.	2.45080×10^{299}	,
494.	2.51529×10^{299}	,
495.	2.57119×10^{299}	,
496.	2.61784×10^{299}	,
497.	2.65471×10^{299}	,
498.	2.68137×10^{299}	,
499.	2.69749×10^{299}	,
500.	2.70288×10^{299}	,
501.	2.69749×10^{299}	,
502.	2.68137×10^{299}	,
503.	2.65471×10^{299}	,
504.	2.61784×10^{299}	,
505.	2.57119×10^{299}	,
506.	2.51529×10^{299}	,
507.	2.45080×10^{299}	,
508.	2.37843×10^{299}	,
509.	2.29899×10^{299}	,
510.	2.21335×10^{299}	,
511.	2.12239×10^{299}	,
512.	2.02704×10^{299}	,
513.	1.92826×10^{299}	,
514.	1.82697×10^{299}	,
515.	1.72409×10^{299}	,
516.	1.62051×10^{299}	,

For air at NTP the average molecular speed is $v \approx 500$ m/s. Considering 1/3 of mulecules with velociy mainly along the long direction of the box, each molecule moves to the other half every $\approx (1/3)(L/v)$

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69.04 Statistical Entropy and Missing Information

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This § is referenced at pages:
[2772]

Statistical entropy is also called **(missing) information entropy**.

In order to assign a number to the amount of information which is missing because the expected event is random define:

$$p_i : i = 1, \ldots, M \qquad p_i \geq 0 \qquad \sum_{i=1}^{M} p_i = 1 \qquad \mathcal{S}[p_i] \equiv -k_B \sum_{i=1}^{M} p_i \ln [p_i] \equiv -k_B \langle \ln [p] \rangle \geq 0 \quad .$$

Some properties follows.

1. $$f[p] \equiv -p \log p \qquad \text{for } 0 \leq p \leq 1 \quad ,$$
$$0 \leq f[p] \leq e^{-1} \quad .$$
2. Minimum entropy.
The quantity $\mathcal{S}$ reaches a minimum, equal to zero, whenever anyone of the events is a certainty. One gains no information, when one knows in advance which message one is about to receive.
3. Maximum entropy.
For fixed M, $\mathcal{S}$ is maximum when $p_i = 1/M$. In fact, $\mathcal{S} \leq k_B \ln [M]$: find the stationary points of $\mathcal{S}[p_i] - \lambda \left(\sum_{i=1}^{M} p_i \right)$; by equating to zero the gradient, one finds that the stationary point correspond to all p_i are equal which implies $p_i = 1/M$ and $\mathcal{S}[p_i = 1/M] = k_B \ln [M] \geq 0$; as the second derivative is negative, the point is maximum. Indeed, we know least about the system in this case. More generally, $\mathcal{S}$ increases whenever any two probabilities get closer to one another.
4. Increase of entropy.
In the case of M equiprobable events the statistical entropy increases with M.
5. Impossibility of Events.
If some events have zero probability, they can just as well be left out of the reckoning when we evaluate the uncertainty.
6. Symmetry. As the M events are treated on the same footing, is a symmetric function of its M arguments.
7. Additivity. If we consider the occurrence of any one among the set of events $1, \ldots, m$ as a single event A, with probability $q_A = p_1 + \ldots + p_m$, and similarly one of the events $m+1, \ldots, M$ as a single event B, it follows that
$$\mathcal{S}[p_1, \ldots, p_m, p_{m+1}, \ldots, p_M] = \mathcal{S}[q_A, q_B] + q_A \mathcal{S}[\frac{p_1}{q_A}, \ldots, \frac{p_m}{q_A}] + q_B \mathcal{S}[\frac{p_{m+1}}{q_B}, \ldots, \frac{p_M}{q_B}] \quad .$$
This equation expresses that the average information acquired in two stages must be added: the first term corresponds to the choice between A and B, and the next two terms to the choices inside the groups A and B.
8. Sub-additivity. Consider a composite event corresponding to the combination of two events belonging one to a set $a = \{\ldots, m, \ldots, M\}$ and the other to a $b = \{\ldots, n, \ldots, N\}$. If the composite event $\{m, n\}$ has a probability, p_{mn}, the probability of the event m is $p^a_m = \sum_n p_{mn}$ and that of event n is $p^b_n = \sum_m p_{mn}$ One can show that the statistical entropies associated with these probability laws satisfy the so-called sub-additivity inequality:
$$\mathcal{S}[p^a_1, \ldots, p^a_m, \ldots, p^a_M] + \mathcal{S}[p^b_1, \ldots, p^b_n, \ldots, p^b_N] \geq \mathcal{S}[p_{11}, \ldots, p_{mn}, \ldots, p_{MN}] \quad .$$

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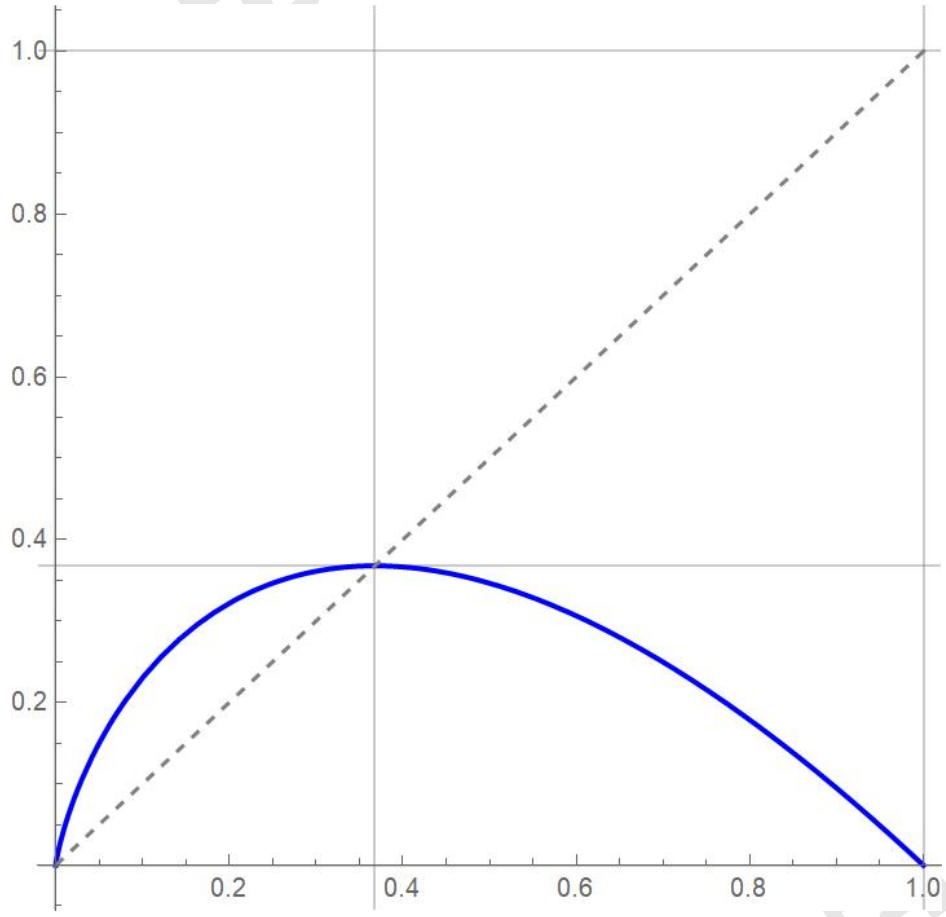
The equal sign holds when the sets a and b are statistically independent, that is, when there is no correlation between the events m and n, in which case the factorization of the probabilities, $p_{mn} = p^a_m p^b_n$ entails the additivity of the statistical entropies. The inequality thus expresses that we lack less information about a composite system when we know the correlations between the events m and n than when we only know the separate probabilities of these events.

9. Concavity [1].
10. Shannon theorems [2].
11. Consider a isolated system made of two independent sub-systems: the number of microscopic states accessible to the system, Ω, is given by

$$\boxed{\Omega = \Omega_1 \Omega_2 \implies \mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2} \quad , \tag{69.04.01}$$

so that entropies are additive. The hypothesis means that the surface interaction energy is negligible with respect to the bulk volume energy, that is short range forces.

The less the information in the probability law, the larger the entropy.



[1] See R.Balian, *From Microphysics To Macrophysics*, 2007, Springer, ...Ed., ...
[2] See R.Balian, *From Microphysics To Macrophysics*, 2007, Springer, ...Ed., ...

69.05 Maximum Entropy Principle

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The probability distribution for micro-states of any system at thermodynamic equilibrium corresponds to maximizing the entropy subject to all the constraints.

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69.06 Micro-Canonical Ensemble

As nothing distinguishes the different microscopic states corresponding to the same macroscopic state of an isolated system, Boltzmann enunciated the following simple hypothesis, known as micro-canonical hypothesis: for an isolated system the energy is fixed and all microscopic states corresponding to the same macroscopic state have the same probability.

According the maximum entropy principle approach, and considering that all and only the microstates corresponding to the given total energy are possible, all the others having zero probability and being thus irrelevant according to the impossibility of events in section § 69.04, the maximum entropy this corresponds to all equal probabilities for all the compatible microstates. In fact, the only constraint is, in this case, the normalization to one of the probabilities. Therefore the entropy becomes:

$$\mathcal{S} = k_{\mathrm{B}} \sum_{s=1}^{M} \frac{1}{M} \log\left[M\right] = k_{\mathrm{B}} \log\left[M\right] \equiv k_{\mathrm{B}} \log\left[\Omega\right] \quad .$$

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69.07 Canonical Ensemble

So far we have only considered isolated systems. The assumptions made in this context, however, make possible to deduce the Boltzmann distribution, that is to say, the probability that a system, in a state s of energy E_s, interacting with the outside environment at the temperature T is in thermodynamic equilibrium with the latter.

By maximizing the entropy with the two constraints of the sum of the probabilities and the fixed mean value for the energy, one finds:

$$\mathcal{S} = -k_{\mathrm{B}} \sum_{s=1}^{M} (p_s \ln [p_s]) + k_{\mathrm{B}}\mu \left(1 - \sum_{s=1}^{M} p_s\right) + k_{\mathrm{B}}\lambda \left(\langle E\rangle - \sum_{s=1}^{M} p_s E_s\right) \quad ,$$

$$\frac{\partial \mathcal{S}}{\partial p_s} = -k_{\mathrm{B}} \left(\ln [p_s] + 1 - \mu - \lambda E_s\right) = 0 \implies p_s = \exp \lambda E_s \exp \mu - 1 \quad ,$$

$$\boxed{\mathcal{Z} \equiv \sum_{s=1}^{M} \exp \lambda E_s} \quad ,$$

$$\sum_{s=1}^{M} p_s = 1 \implies \exp \mu - 1 \sum_{s=1}^{M} \exp \lambda E_s = 1 \implies \text{to determine } \mu \qquad \boxed{p_s = \frac{\exp \lambda E_s}{\mathcal{Z}}} \quad ,$$

$$\boxed{p_s[E_s] \propto \exp [-\beta E_s] \qquad \beta \equiv \frac{1}{k_{\mathrm{B}} T}} \qquad \text{to determine } \lambda \quad .$$

$$\mathcal{S}[p_s] = -k_{\mathrm{B}} \sum_{s=1}^{M} p_s \ln [p_s] \quad ,$$

$$\boxed{p_s[E_s] \propto \exp -\beta E_s \qquad \beta \equiv \frac{1}{k_{\mathrm{B}} T} \qquad \mathcal{Z} = \sum_{s=1}^{M} \exp -\beta E_s \qquad p_s[E_s] = \frac{\exp -\beta E_s}{\mathcal{Z}}} \quad ,$$

$$\mathrm{d}\mathcal{S}[p_s] = -k_{\mathrm{B}} \sum_{s=1}^{M} (\,\mathrm{d}p_s \ln [p_s] + \,\mathrm{d}p_s) = -k_{\mathrm{B}} \sum_{s=1}^{M} \mathrm{d}p_s \ln [p_s] \quad ,$$

$$\langle E\rangle = \sum_{s=1}^{M} p_s E_s \qquad \langle E\rangle = -\frac{\partial}{\partial \beta} \ln [\mathcal{Z}] \qquad \text{to determine } \beta \text{ for any given } \langle E\rangle$$

$$\mathrm{d}E = \sum_{s=1}^{M} \mathrm{d}p_s E_s + \sum_{s=1}^{M} p_s \,\mathrm{d}E_s$$

$$\mathrm{d}\mathcal{S}[p_s] = -k_{\mathrm{B}} \sum_{s=1}^{M} \mathrm{d}p_s \ln [p_s] = -k_{\mathrm{B}} \sum_{s=1}^{M} \mathrm{d}p_s \ln \left[\frac{\exp -\beta E_s}{\mathcal{Z}}\right] = -k_{\mathrm{B}} \sum_{s=1}^{M} \mathrm{d}p_s \ln [\exp -\beta E_s] + k_{\mathrm{B}} \sum_{s=1}^{M} \mathrm{d}p_s \ln [\mathcal{Z}] =$$

$$= -k_{\mathrm{B}} \sum_{s=1}^{M} \mathrm{d}p_s \ln [\exp -\beta E_s] = +\beta k_{\mathrm{B}} \sum_{s=1}^{M} \mathrm{d}p_s E_s = \frac{1}{T} \sum_{s=1}^{M} E_s \,\mathrm{d}p_s \equiv \frac{\delta \mathcal{Q}}{T}$$

$$\boxed{\delta \mathcal{Q} = \sum_{s=1}^{M} E_s \,\mathrm{d}p_s \qquad \text{reversible}} \quad ,$$

$$\boxed{\delta \mathcal{W} = \sum_{s=1}^{M} p_s \,\mathrm{d}E_s \qquad \text{reversible}} \quad . \tag{69.07.01}$$

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$$\mathrm{d}E = \sum_s \mathrm{d}p_s E_s + \sum_s p_s \, \mathrm{d}E_s \quad ,$$

$$\mathrm{d}\mathcal{S} = -\sum_s \mathrm{d}p_s \ln [p_s] \quad ,$$

$$\mathrm{d}\mathcal{S} = \frac{1}{T} \sum_s E_s \, \mathrm{d}p_s \equiv \frac{\delta \mathcal{Q}}{T} \quad ,$$

$$\delta \mathcal{Q} = \sum_s E_s \, \mathrm{d}p_s \qquad \text{reversible} \quad ,$$

$$\delta W = \sum_s p_s \, \mathrm{d}E_s \qquad \text{reversible} \quad .$$

The precise meaning of the different nature of the transfer of energy by heat and by work is now clear:

- in the heat transfer, only the probabilities (that is the occupation numbers) of the various states are changed, not the values of the energies;
- in the work transfer, the energy levels are changed while the probabilities (that is the occupation numbers) of the levels remain the same.

In general, the energy levels depend on the external parameters, such as volume. Therefore the maximized entropy will depend on the external parameters, for instance volume, and on the average energy, to be identified with the internal energy here: $\langle E \rangle \equiv \mathcal{U}$. As from equation (67.01.01), considering a system with volume as the unique external parameter, one has

$$\mathcal{S} = \mathcal{S}[\mathcal{U}, V] \quad .$$

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69.08 Grand-Canonical Ensemble

By maximizing the entropy with the three constraints of the sum of the probabilities, the fixed mean value for the energy and the fixed mean values of all the components, one finds:

$$p\left[E, n_a, N_b, \ldots\right] = \frac{\exp\left[-\beta E + \mu_a N_a + \mu_b N_b + \ldots\right]}{Z} \quad .$$

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69.09 General Boltzmann-Gibbs Distribution

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The expression, which is called the Boltzmann-Gibbs distribution, represents the general form of the density operators in thermodynamic equilibrium when the averages of the constants of the motion A_i are given:

$$p_j = \frac{\exp\left[-\sum_i \lambda_i A_i[j]\right]}{\mathcal{Z}} \qquad \mathcal{Z} = \sum_{j=1}^{M} \exp\left[-\sum_i \lambda_i A_i[j]\right] \qquad \langle A_i \rangle = -\frac{\partial}{\partial \lambda_i} \ln\left[\mathcal{Z}\left[\lambda_j\right]\right] \quad .$$

69.09.01 Non-Zero Total Linear Momentum

$$p\left[E, \mathbf{P}\right] = \frac{\exp\left[-\beta E + \boldsymbol{\lambda}\cdot\mathbf{P}\right]}{\mathcal{Z}} \qquad \mathbf{v} = \boldsymbol{\lambda}/\beta \quad .$$

The entropy of a system at equilibrium in the absence of external fields remains unchanged if the system is subjected to a uniform translation (**Galilei-Invarian***).

69.09.02 Non-Zero Total Angular Momentum

$$p\left[E, \mathbf{J}\right] = \frac{\exp\left[-\beta E + \boldsymbol{\lambda}\cdot\mathbf{J}\right]}{\mathcal{Z}} \qquad \boldsymbol{\omega} = \boldsymbol{\lambda}/\beta \quad .$$

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69.10 Mathematical Tools

$$\boxed{\ln[n!] \approx n \ln[n] - n} \quad .$$

$$\boxed{n! \simeq \sqrt{2\pi n} n^n \exp -n} \quad .$$

$$\boxed{\log[x] \leq x - 1} \quad .$$

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CHAPTER 70

Non-Equilibrium Thermodynamics

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CHAPTER 71

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This section consists of 30 longer form questions intended to synthesize material from the book and occasionally extend the range of topics covered: WEB - URL

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CHAPTER I

THE FUNDAMENTAL PRINCIPLES OF STATISTICAL PHYSICS

§ 1. Statistical distributions

Statistical physics, often called for brevity simply *statistics*, consists in the study of the special laws which govern the behaviour and properties of macroscopic bodies (that is, bodies formed of a very large number of individual particles, such as atoms and molecules). To a considerable extent the general character of these laws does not depend on the mechanics (classical or quantum) which describes the motion of the individual particles in a body, but their substantiation demands a different argument in the two cases. For convenience of exposition we shall begin by assuming that classical mechanics is everywhere valid.

In principle, we can obtain complete information concerning the motion of a mechanical system by constructing and integrating the equations of motion of the system, which are equal in number to its degrees of freedom. But if we are concerned with a system which, though it obeys the laws of classical mechanics, has a very large number of degrees of freedom, the actual application of the methods of mechanics involves the necessity of setting up and solving the same number of differential equations, which in general is impracticable. It should be emphasised that, even if we could integrate these equations in a general form, it would be completely impossible to substitute in the general solution the initial conditions for the velocities and coordinates of all the particles.

At first sight we might conclude from this that, as the number of particles increases, so also must the complexity and intricacy of the properties of the mechanical system, and that no trace of regularity can be found in the behaviour of a macroscopic body. This is not so, however, and we shall see below that, when the number of particles is very large, new types of regularity appear.

These *statistical laws* resulting from the very presence of a large number of particles forming the body cannot in any way be reduced to purely mechanical laws. One of their distinctive features is that they cease to have meaning when applied to mechanical systems with a small number of degrees of

freedom. Thus, although the motion of systems with a very large number of degrees of freedom obeys the same laws of mechanics as that of systems consisting of a small number of particles, the existence of many degrees of freedom results in laws of a different kind.

The importance of statistical physics in many other branches of theoretical physics is due to the fact that in Nature we continually encounter macroscopic bodies whose behaviour can not be fully described by the methods of mechanics alone, for the reasons mentioned above, and which obey statistical laws.

In proceeding to formulate the fundamental problem of classical statistics, we must first of all define the concept of *phase space*, which will be constantly used hereafter.

Let a given macroscopic mechanical system have s degrees of freedom: that is, let the position of points of the system in space be described by s coordinates, which we denote by q_i, the suffix i taking the values 1, 2, ..., s. Then the state of the system at a given instant will be defined by the values at that instant of the s coordinates q_i and the s corresponding velocities $\dot{q}_i$. In statistics it is customary to describe a system by its coordinates and momenta p_i, not velocities, since this affords a number of very important advantages. The various states of the system can be represented mathematically by points in *phase space* (which is, of course, a purely mathematical concept); the coordinates in phase space are the coordinates and momenta of the system considered. Every system has its own phase space, with a number of dimensions equal to twice the number of degrees of freedom. Any point in phase space, corresponding to particular values of the coordinates q_i and momenta p_i of the system, represents a particular state of the system. The state of the system changes with time, and consequently the point in phase space representing this state (which we shall call simply the *phase point* of the system) moves along a curve called the *phase trajectory*.

Let us now consider a macroscopic body or system of bodies, and assume that the system is closed, i.e. does not interact with any other bodies. A part of the system, which is very small compared with the whole system but still macroscopic, may be imagined to be separated from the rest; clearly, when the number of particles in the whole system is sufficiently large, the number in a small part of it may still be very large. Such relatively small but still macroscopic parts will be called *subsystems*. A subsystem is again a mechanical system, but not a closed one; on the contrary, it interacts in various ways with the other parts of the system. Because of the very large number of degrees of freedom of the other parts, these interactions will be very complex and intricate. Thus the state of the subsystem considered will vary with time in a very complex and intricate manner.

An exact solution for the behaviour of the subsystem can be obtained only

by solving the mechanical problem for the entire closed system, i.e. by setting up and solving all the differential equations of motion with given initial conditions, which, as already mentioned, is an impracticable task. Fortunately, it is just this very complicated manner of variation of the state of subsystems which, though rendering the methods of mechanics inapplicable, allows a different approach to the solution of the problem.

A fundamental feature of this approach is the fact that, because of the extreme complexity of the external interactions with the other parts of the system, during a sufficiently long time the subsystem considered will be many times in every possible state. This may be more precisely formulated as follows. Let $\Delta p\,\Delta q$ denote some small "volume" of the phase space of the subsystem, corresponding to coordinates q_i and momenta p_i lying in short intervals Δq_i and Δp_i. We can say that, in a sufficiently long time T, the extremely intricate phase trajectory passes many times through each such volume of phase space. Let Δt be the part of the total time T during which the subsystem was in the given volume of phase space $\Delta p\,\Delta q$.† When the total time T increases indefinitely, the ratio $\Delta t/T$ tends to some limit

$$w = \lim_{T\to\infty} \Delta t/T. \tag{1.1}$$

This quantity may clearly be regarded as the probability that, if the subsystem is observed at an arbitrary instant, it will be found in the given volume of phase space $\Delta p\,\Delta q$.

On taking the limit of an infinitesimal phase volume‡

$$\mathrm{d}q\,\mathrm{d}p = \mathrm{d}q_1\,\mathrm{d}q_2 \ldots \mathrm{d}q_s\,\mathrm{d}p_1\,\mathrm{d}p_2 \ldots \mathrm{d}p_s, \tag{1.2}$$

we can define the probability $\mathrm{d}w$ of states represented by points in this volume element, i.e. the probability that the coordinates q_i and momenta p_i have values in given infinitesimal intervals between q_i, p_i and $q_i+\mathrm{d}q_i$, $p_i+\mathrm{d}p_i$. This probability $\mathrm{d}w$ may be written

$$\mathrm{d}w = \varrho(p_1, \ldots, p_s, q_1, \ldots, q_s)\,\mathrm{d}p\,\mathrm{d}q, \tag{1.3}$$

where $\varrho(p_1, \ldots, p_s, q_1, \ldots, q_s)$ is a function of all the coordinates and momenta; we shall usually write for brevity $\varrho(p, q)$ or even ϱ simply. The function ϱ, which represents the "density" of the probability distribution in phase space, is called the *statistical distribution function*, or simply the

† For brevity, we shall usually say, as is customary, that the system "is in the volume $\Delta p\,\Delta q$ of phase space", meaning that the system is in states represented by phase points in that volume.

‡ In what follows we shall always use the conventional notation $\mathrm{d}p$ and $\mathrm{d}q$ to denote the products of the differentials of all the momenta and all the coordinates of the system respectively.

distribution function, for the body concerned. This function must obviously satisfy the *normalisation condition*

$$\int \varrho\,\mathrm{d}p\,\mathrm{d}q = 1 \tag{1.4}$$

(the integral being taken over all phase space), which simply expresses the fact that the sum of the probabilities of all possible states must be unity.

The following circumstance is extremely important in statistical physics. The statistical distribution of a given subsystem does not depend on the initial state of any other small part of the same system, since over a sufficiently long time the effect of this initial state will be entirely outweighed by the effect of the much larger remaining parts of the system. It is also independent of the initial state of the particular small part considered, since in time this part passes through all possible states, any of which can be taken as the initial state. Without having to solve the mechanical problem for a system (taking account of initial conditions), we can therefore find the statistical distribution for small parts of the system.

The determination of the statistical distribution for any subsystem is in fact the fundamental problem of statistical physics. In speaking of "small parts" of a closed system, we must bear in mind that the macroscopic bodies with which we have to deal are usually themselves such "small parts" of a large closed system consisting of these bodies together with the external medium which surrounds them.

If this problem is solved and the statistical distribution for a given subsystem is known, we can calculate the probabilities of various values of any physical quantities which depend on the states of the subsystem (i.e. on the values of its coordinates q and momenta p). We can also calculate the mean value of any such quantity $f(p, q)$, which is obtained by multiplying each of its possible values by the corresponding probability and integrating over all states. Denoting the averaging by a bar, we can write

$$\bar{f} = \int f(p, q)\,\varrho(p, q)\,\mathrm{d}p\,\mathrm{d}q, \tag{1.5}$$

from which the mean values of various quantities can be calculated by using the statistical distribution function.†

The averaging with respect to the distribution function (called *statistical averaging*) frees us from the necessity of following the variation with time of the actual value of the physical quantity $f(p, q)$ in order to determine its mean value. It is also obvious that, by the definition (1.1) of the probability, the statistical averaging is exactly equivalent to a time averaging. The latter

† In this book we shall denote averaging by a bar over the symbol or by angle brackets: $\bar{f}$ or $\langle f\rangle$, being influenced in this solely by convenience in writing the formulae. The second way is preferable for writing the mean values of lengthy expressions.

would involve following the variation of the quantity with time, establishing the function $f = f(t)$, and determining the required mean value as

$$\bar{f} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\, dt.$$

The foregoing discussion shows that the deductions and predictions concerning the behaviour of macroscopic bodies which are made possible by statistical physics are probabilistic. In this respect statistical physics differs from (classical) mechanics, the deductions of which are entirely deterministic. It should be emphasised, however, that the probabilistic nature of the results of classical statistics is not an inherent property of the objects considered, but simply arises from the fact that these results are derived from much less information than would be necessary for a complete mechanical description (the initial values of the coordinates and momenta are not needed).

In practice, however, when statistical physics is applied to macroscopic bodies, its probabilistic nature is not usually apparent. The reason is that, if any macroscopic body (in external conditions independent of time) is observed over a sufficiently long period of time, it is found that all physical quantities describing the body are practically constant (and equal to their mean values) and undergo appreciable changes relatively very rarely; we mean, of course, macroscopic quantities describing the body as a whole or macroscopic parts of it, but not individual particles.† This result, which is fundamental to statistical physics, follows from very general considerations (to be discussed in § 2) and becomes more and more nearly valid as the body considered becomes more complex and larger. In terms of the statistical distribution, we can say that, if by means of the function $\varrho(p, q)$ we construct the probability distribution function for various values of the quantity $f(p, q)$, this function will have an extremely sharp maximum for $f = \bar{f}$, and will be appreciably different from zero only in the immediate vicinity of this point.

Thus, by enabling us to calculate the mean values of quantities describing macroscopic bodies, statistical physics enables us to make predictions which are valid to very high accuracy for by far the greater part of any time interval which is long enough for the effect of the initial state of the body to be entirely eliminated. In this sense the predictions of statistical physics become

† We may give an example to illustrate the very high degree of accuracy with which this is true. If we consider a region in a gas which contains, say, only 1/100 gram-molecule, we find that the mean relative variation of the energy of this quantity of matter from its mean value is only $\sim 10^{-11}$. The probability of finding (in a single observation) a relative deviation of the order of 10^{-6}, say, is given by a fantastically small number, $\sim 10^{-3\times 10^{15}}$.

practically determinate and not probabilistic. (For this reason, we shall henceforward almost always omit the bar when using mean values of macroscopic quantities.)

If a closed macroscopic system is in a state such that in any macroscopic subsystem the macroscopic physical quantities are to a high degree of accuracy equal to their mean values, the system is said to be in a state of *statistical equilibrium* (or *thermodynamic* or *thermal equilibrium*). It is seen from the foregoing that, if a closed macroscopic system is observed for a sufficiently long period of time, it will be in a state of statistical equilibrium for much the greater part of this period. If, at any initial instant, a closed macroscopic system was not in a state of statistical equilibrium (if, for example, it was artificially disturbed from such a state by means of an external interaction and then left to itself, becoming again a closed system), it will necessarily enter an equilibrium state. The time within which it will reach statistical equilibrium is called the *relaxation time*. In using the term "sufficiently long" intervals of time, we have meant essentially times long compared with the relaxation time.

The theory of processes relating to the attainment of an equilibrium state is called *kinetics*. It is not part of statistical physics proper, which deals only with systems in statistical equilibrium.

§ 2. Statistical independence

The subsystems discussed in § 1 are not themselves closed systems; on the contrary, they are subject to the continuous interaction of the remaining parts of the system. But since these parts, which are small in comparison with the whole of the large system, are themselves macroscopic bodies also, we can still suppose that over not too long intervals of time they behave approximately as closed systems. For the particles which mainly take part in the interaction of a subsystem with the surrounding parts are those near the surface of the subsystem; the relative number of such particles, compared with the total number of particles in the subsystem, decreases rapidly when the size of the subsystem increases, and when the latter is sufficiently large the energy of its interaction with the surrounding parts will be small in comparison with its internal energy. Thus we may say that the subsystems are *quasi-closed*. It should be emphasised once more that this property holds only over not too long intervals of time. Over a sufficiently long interval of time, the effect of interaction of subsystems, however weak, will ultimately appear. Moreover, it is just this relatively weak interaction which leads finally to the establishment of statistical equilibrium.

The fact that different subsystems may be regarded as weakly interacting has the result that they may also be regarded as statistically independent.

By *statistical independence* we mean that the state of one subsystem does not affect the probabilities of various states of the other subsystems.

Let us consider any two subsystems, and let $dp^{(1)}\,dq^{(1)}$ and $dp^{(2)}\,dq^{(2)}$ be volume elements in their phase spaces. If we regard the two subsystems together as one composite subsystem, then the statistical independence of the subsystems signifies mathematically that the probability of the composite subsystem's being in its phase volume element $dp^{(12)}\,dq^{(12)} = dp^{(1)}\,dq^{(1)}.\,dp^{(2)}\,dq^{(2)}$ can be written as the product of the probabilities for the two subsystems to be respectively in $dp^{(1)}\,dq^{(1)}$ and $dp^{(2)}\,dq^{(2)}$, each of these probabilities depending only on the coordinates and momenta of the subsystem concerned. Thus we can write

$$\varrho_{12}\,dp^{(12)}\,dq^{(12)} = \varrho_1\,dp^{(1)}\,dq^{(1)}\cdot\varrho_2\,dp^{(2)}\,dq^{(2)},$$

or

$$\varrho_{12} = \varrho_1\varrho_2, \tag{2.1}$$

where ϱ_{12} is the statistical distribution of the composite subsystem, and ϱ_1, ϱ_2 the distribution functions of the separate subsystems. A similar relation is valid for a group of several subsystems.†

The converse statement is clearly also true: if the probability distribution for a compound system is a product of factors, each of which depends only on quantities describing one part of the system, then the parts concerned are statistically independent, and each factor is proportional to the probability of the state of the corresponding part.

If f_1 and f_2 are two physical quantities relating to two different subsystems, then from (2.1) and the definition (1.5) of mean values it follows immediately that the mean value of the product f_1f_2 is equal to the product of the mean values of the quantities f_1 and f_2 separately:

$$\overline{f_1f_2} = \bar{f}_1\cdot\bar{f}_2. \tag{2.2}$$

Let us consider a quantity f relating to a macroscopic body or to a part of it. In the course of time this quantity varies, fluctuating about its mean value. We may define a quantity which represents the average range of this fluctuation. The mean value of the difference $\Delta f = f-\bar{f}$ is not suitable for this purpose, since the quantity f varies from its mean value in both directions, and the difference $f-\bar{f}$, which is alternately positive and negative, has mean value zero regardless of how often f undergoes considerable deviations from its mean value. The required characteristic may conveniently be defined as the mean square of this difference. Since $(\Delta f)^2$ is always positive, its mean value tends to zero only if $(\Delta f)^2$ itself tends to zero; that is, the mean value

† Provided, of course, that these subsystems together still form only a small part of the whole closed system.

is small only when the probability of considerable deviations of f from $\bar{f}$ is small. The quantity $\langle(\Delta f)^2\rangle^{1/2}$ is called the *root-mean-square* (*r.m.s.*) *fluctuation* of the quantity f. Multiplying out the square $(f-\bar{f})^2$ shows that

$$\langle(\Delta f)^2\rangle = \overline{f^2}-(\bar{f})^2, \tag{2.3}$$

i.e. the r.m.s. fluctuation is determined by the difference between the mean square of the quantity and the square of its mean.

The ratio $\langle(\Delta f)^2\rangle^{1/2}/\bar{f}$ is called ther *relative fluctuation* of the quantity f. The smaller this ratio is, the more negligible is the proportion of time during which the body is in states where the deviation of f from its mean value is a considerable fraction of the mean value.

We shall show that the relative fluctuations of physical quantities decrease rapidly when the size of the bodies (that is, the number of particles) to which they relate increases. To prove this, we first note that the majority of quantities of physical interest are additive. This property is a consequence of the fact that the various parts of a body are quasi-closed systems, and signifies that the value of such a quantity for the whole body is the sum of its values for the various (macroscopic) parts of the body. For example, since the internal energies of these parts are, as shown above, large compared with their interaction energies, it is sufficiently accurate to assume that the energy of the whole body is equal to the sum of the energies of its parts.

Let f be such an additive quantity. We imagine the body concerned to be divided into a large number N of approximately equal small parts. Then

$$\bar{f} = \sum_{i=1}^{N}\bar{f}_i,$$

where the quantities f_i relate to the individual parts of the body.

It is clear that, as the size of the body increases, $\bar{f}$ increases approximately in proportion to N. Let us also determine the r.m.s. fluctuation of f. We have

$$\langle(\Delta f)^2\rangle = \left\langle\left(\sum_i \Delta f_i\right)^2\right\rangle.$$

Because of the statistical independence of the different parts of the body, the mean values of the products $\Delta f_i\,\Delta f_k$ are

$$\overline{\Delta f_i\,\Delta f_k} = \overline{\Delta f_i}\cdot\overline{\Delta f_k} = 0 \qquad (i \neq k),$$

since each $\overline{\Delta f_i} = 0$. Hence

$$\langle(\Delta f)^2\rangle = \sum_{i=1}^{N}\langle(\Delta f_i)^2\rangle. \tag{2.4}$$

It follows that, as N increases, the mean square $\langle(\Delta f)^2\rangle$ also increases in proportion to N. The relative fluctuation is therefore inversely proportional

to $\sqrt{N}$:

$$\frac{\langle(\Delta f)^2\rangle^{1/2}}{\bar{f}} \propto \frac{1}{\sqrt{N}}. \tag{2.5}$$

On the other hand, if we consider a homogeneous body to be divided into parts of a given small size, it is clear that the number of parts will be proportional to the total number of particles (molecules) in the body. Hence the result can also be stated by saying that the relative fluctuation of any additive quantity f decreases inversely as the square root of the number of particles in a macroscopic body, and so, when the number of these is sufficiently large, the quantity f itself may be regarded as practically constant in time and equal to its mean value. This conclusion has already been used in § 1.

§ 3. Liouville's theorem

Let us now return to a further study of the properties of the statistical distribution function, and suppose that a subsystem is observed over a very long interval of time, which we divide into a very large (in the limit, infinite) number of equal short intervals between instants t_1, t_2, ... At each of these instants the subsystem considered is represented in its phase space by a point A_1, A_2, ... The set of points thus obtained is distributed in phase space with a density which in the limit is everywhere proportional to the distribution function $\varrho(p, q)$. This follows from the significance of the latter function as determining the probabilities of various states of the subsystem.

Instead of considering points representing states of one subsystem at different instants t_1, t_2, ..., we may consider simultaneously, in a formal manner, a very large (in the limit, infinite) number of exactly identical subsystems,† which at some instant, say $t = 0$, are in states represented by the points A_1, A_2, ...

We now follow the subsequent movement of the phase points which represent the states of these subsystems over a not too long interval of time, such that a quasi-closed subsystem may with sufficient accuracy be regarded as closed. The movement of the phase points will then obey the equations of motion, which involve the coordinates and momenta only of the particles in the subsystem.

It is clear that at any instant t these points will be distributed in phase space according to the same distribution function $\varrho(p, q)$, in just the same way as at $t = 0$. In other words, as the phase points move about in the course of time, they remain distributed with a density which is constant at any given point and is proportional to the corresponding value of ϱ.

† Such an imaginary set of identical systems is usually called a *statistical ensemble*.

This movement of phase points may be formally regarded as a steady flow of a "gas" in phase space of $2s$ dimensions, and the familiar equation of continuity may be applied, which expresses the constancy of the total number of "particles" (in this case, phase points) in the gas. The ordinary equation of continuity is

$$\partial\varrho/\partial t + \mathrm{div}\,(\varrho\mathbf{v}) = 0,$$

where ϱ is the density and $\mathbf{v}$ the velocity of the gas. For steady flow, we have

$$\mathrm{div}\,(\varrho\mathbf{v}) = 0.$$

For a space of $2s$ dimensions, this will become

$$\sum_{i=1}^{2s} \frac{\partial}{\partial x_i}(\varrho v_i) = 0.$$

In the present case the "coordinates" x_i are the coordinates q and momenta p, and the "velocities" $v_i = \dot{x}_i$ are the time derivatives $\dot{q}$ and $\dot{p}$ given by the equations of motion. Thus we have

$$\sum_{i=1}^{s}\left[\frac{\partial}{\partial q_i}(\varrho\dot{q}_i) + \frac{\partial}{\partial p_i}(\varrho\dot{p}_i)\right] = 0.$$

Expanding the derivatives gives

$$\sum_{i=1}^{s}\left[\dot{q}_i\frac{\partial\varrho}{\partial q_i} + \dot{p}_i\frac{\partial\varrho}{\partial p_i}\right] + \varrho\sum_{i=1}^{s}\left[\frac{\partial\dot{q}_i}{\partial q_i} + \frac{\partial\dot{p}_i}{\partial p_i}\right] = 0. \tag{3.1}$$

With the equations of motion in Hamilton's form:

$$\dot{q}_i = \partial H/\partial p_i, \quad \dot{p}_i = -\partial H/\partial q_i,$$

where $H = H(p, q)$ is the Hamiltonian for the subsystem considered, we see that

$$\partial\dot{q}_i/\partial q_i = \partial^2 H/\partial q_i\,\partial p_i = -\partial\dot{p}_i/\partial p_i.$$

The second term in (3.1) is therefore identically zero. The first term is just the total time derivative of the distribution function. Thus

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = \sum_{i=1}^{s}\left(\frac{\partial\varrho}{\partial q_i}\dot{q}_i + \frac{\partial\varrho}{\partial p_i}\dot{p}_i\right) = 0. \tag{3.2}$$

We therefore reach the important conclusion that the distribution function is constant along the phase trajectories of the subsystem. This is *Liouville's theorem*. Since quasi-closed subsystems are under dicussion, the result is valid only for not too long intervals of time, during which the subsystem behaves as if closed, to a sufficient approximation.

§ 4. The significance of energy

It follows at once from Liouville's theorem that the distribution function must be expressible entirely in terms of combinations of the variables p and q which remain constant when the subsystem moves as a closed subsystem. These combinations are the *mechanical invariants* or *integrals of the motion*, which are the first integrals of the equations of motion. We may therefore say that the distribution function, being a function of the mechanical invariants, is itself an integral of the motion.

It proves possible to restrict very considerably the number of integrals of the motion on which the distribution function can depend. To do this, we must take into account the fact that the distribution ϱ_{12} for a combination of two subsystems is equal to the product of the distribution functions ϱ_1 and ϱ_2 of the two subsystems separately: $\varrho_{12} = \varrho_1\varrho_2$. Hence

$$\log \varrho_{12} = \log \varrho_1 + \log \varrho_2, \tag{4.1}$$

so that the logarithm of the distribution function is an additive quantity. We therefore reach the conclusion that the logarithm of the distribution function must be not merely an integral of the motion, but an additive integral of the motion.

As we know from mechanics, there exist only seven independent additive integrals of the motion: the energy, the three components of the momentum vector and the three components of the angular momentum vector. We shall denote these quantities for the ath subsystem (as functions of the coordinates and momenta of the particles in it) by $E_a(p, q)$, $\mathbf{P}_a(p, q)$, $\mathbf{M}_a(p, q)$ respectively. The only additive combination of these quantities is a linear combination of the form

$$\log \varrho_a = \alpha_a + \beta E_a(p, q) + \boldsymbol{\gamma}\cdot\mathbf{P}_a(p, q) + \boldsymbol{\delta}\cdot\mathbf{M}_a(p, q) \tag{4.2}$$

with constant coefficients α_a, β, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, of which β, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ must be the same for all subsystems in a given closed system.

We shall return in Chapter III to a detailed study of the distribution (4.2); here we need note only the following points. The coefficient α_a is just the normalisation constant, given by the condition $\int \varrho_a \, \mathrm{d}p^{(a)} \, \mathrm{d}q^{(a)} = 1$. The constants β, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, involving seven independent quantities altogether, may be determined from the seven constant values of the additive integrals of the motion for the whole closed system. Thus we reach a conclusion very important in statistical physics. The values of the additive integrals of the motion (energy, momentum and angular momentum) completely define the statistical properties of a closed system, i.e. the statistical distribution of any of its subsystems, and therefore the mean values of any physical quantities relating to them. These seven additive integrals of the motion replace the

unimaginable multiplicity of data (initial conditions) which would be required in the approach from mechanics.

The above arguments enable us at once to set up a simple distribution function suitable for describing the statistical properties of a closed system. Since, as we have seen, the values of non-additive integrals of the motion do not affect these properties, the latter can be described by any function ϱ which depends only on the values of the additive integrals of the motion for the system and which satisfies Liouville's theorem. The simplest such function is ϱ = constant for all points in phase space which correspond to given constant values of the energy (E_0), momentum ($\mathbf{P}_0$) and angular momentum ($\mathbf{M}_0$) of the system (regardless of the values of the non-additive integrals) and $\varrho = 0$ at all other points. It is clear that a function so defined will certainly remain constant along a phase trajectory of the system, i.e. will satisfy Liouville's theorem.

This formulation, however, is not quite exact. The reason is that the points defined by the equations

$$E(p, q) = E_0, \quad \mathbf{P}(p, q) = \mathbf{P}_0, \quad \mathbf{M}(p, q) = \mathbf{M}_0 \tag{4.3}$$

form a manifold of only $2s-7$ dimensions, not $2s$ like the phase volume. Consequently, if the integral $\int \varrho \, \mathrm{d}p \, \mathrm{d}q$ is to be different from zero, the function $\varrho(p, q)$ must become infinite at these points. The correct way of writing the distribution function for a closed system is

$$\varrho = \text{constant} \times \delta(E-E_0)\,\delta(\mathbf{P}-\mathbf{P}_0)\,\delta(\mathbf{M}-\mathbf{M}_0). \tag{4.4}$$

The presence of the delta functions† ensures that ϱ is zero at all points in phase space where one or more of the quantities E, $\mathbf{P}$, $\mathbf{M}$ is not equal to the given value E_0, $\mathbf{P}_0$ or $\mathbf{M}_0$. The integral of ϱ over the whole of a phase volume which includes all or part of the above-mentioned manifold of points is finite. The distribution (4.4) is called *microcanonical*.‡

The momentum and angular momentum of a closed system depend on its motion as a whole (uniform translation and uniform rotation). We can therefore say that the statistical state of a system executing a given motion depends only on its energy. In consequence, energy is of exceptional importance in statistical physics.

† The definition and properties of the delta function are given, for example, in *Quantum Mechanics*, § 5.

‡ It should be emphasised once more that this distribution is not the true statistical distribution for a closed system. Regarding it as the true distribution is equivalent to asserting that, in the course of a sufficiently long time, the phase trajectory of a closed system passes arbitrarily close to every point of the manifold defined by equations (4.3). But this assertion (called the *ergodic hypothesis*) is certainly not true in general.

In order to exclude the momentum and angular momentum from the subsequent discussion we may use the following device. We imagine the system to be enclosed in a rigid "box" and take coordinates such that the "box" is at rest. Under these conditions the momentum and angular momentum are not integrals of the motion, and the only remaining additive integral of the motion is the energy. The presence of the "box", on the other hand, clearly does not affect the statistical properties of small parts of the system (subsystems). Thus for the logarithms of the distribution functions of the subsystems, instead of (4.2), we have the still simpler expressions

$$\log \varrho_a = \alpha_a + \beta E_a(p, q). \tag{4.5}$$

The microcanonical distribution for the whole system is

$$\varrho = \text{constant} \times \delta(E - E_0). \tag{4.6}$$

So far we have assumed that the closed system as a whole is in statistical equilibrium; that is, we have considered it over times long compared with its relaxation time. In practice, however, it is usually necessary to discuss a system over times comparable with or even short relative to the relaxation time. For large systems this can be done, owing to the existence of what are called *partial* (or *incomplete*) *equilibria* as well as the complete statistical equilibrium of the entire closed system. Such equilibria occur because the relaxation time increases with the size of the system, and so the separate small parts of the system attain the equilibrium state considerably more quickly than equilibrium is established between these small parts. This means that each small part of the system is described by a distribution function of the form (4.2), with the parameters β, γ, δ of the distribution having different values for different parts. In such a case the system is said to be in *partial equilibrium*. In the course of time, the partial equilibrium gradually becomes complete, and the parameters β, γ, δ for each small part slowly vary and finally become equal throughout the closed system.

Another kind of partial equilibrium is also of frequent occurrence, namely that resulting from a difference in the rates of the various processes occurring in the system, not from a considerable difference in relaxation time between the system and its small parts. One obvious example is the partial equilibrium in a mixture of several substances involved in a chemical reaction. Owing to the comparative slowness of chemical reactions, equilibrium as regards the motion of the molecules will be reached, in general, considerably more rapidly than equilibrium as regards reactions of molecules, i.e. as regards the composition of the mixture. This enables us to regard the partial equilibria of the mixture as equilibria at a given (actually non-equilibrium) chemical composition.

The existence of partial equilibria leads to the concept of *macroscopic states* of a system. Whereas a mechanical microscopic description of the system specifies the coordinates and momenta of every particle in the system, a macroscopic description is one which specifies the mean values of the physical quantities determining a particular partial equilibrium, for instance the mean values of quantities describing separate sufficiently small but macroscopic parts of the system, each of which may be regarded as being in a separate equilibrium.

§ 5. The statistical matrix

Turning now to the distinctive features of quantum statistics, we may note first of all that the purely mechanical approach to the problem of determining the behaviour of a macroscopic body in quantum mechanics is of course just as hopeless as in classical mechanics. Such an approach would require the solution of Schrödinger's equation for a system consisting of all the particles in the body, a problem still more hopeless, one might even say, than the integration of the classical equations of motion. But even if it were possible in some particular case to find a general solution of Schrödinger's equation, it would be utterly impossible to select and write down the particular solution satisfying the precise conditions of the problem and specified by particular values of an enormous number of different quantum numbers. Moreover, we shall see below that for a macroscopic body the concept of stationary states itself becomes to some extent arbitrary, a fact of fundamental significance.

Let us first elucidate some purely quantum-mechanical features of macroscopic bodies as compared with systems consisting of a relatively small number of particles.

These features amount to an extremely high density of levels in the energy eigenvalue spectrum of a macroscopic body. The reason for this is easily seen if we note that, because of the very large number of particles in the body, a given quantity of energy can, roughly speaking, be "distributed" in innumerable ways among the various particles. The relation between this fact and the high density of levels becomes particularly clear if we take as an example a macroscopic body consisting of a "gas" of N particles which do not interact at all, enclosed in some volume. The energy levels of such a system are just the sums of the energies of the individual particles, and the energy of each particle can range over an infinite series of discrete values.† It is clear that, on choosing in all possible ways the values of the N terms in this sum, we shall

† The separations between successive energy levels of a single particle are inversely proportional to the square of the linear dimensions L of the volume enclosing it ($\sim \hbar^2/mL^2$, where m is the mass of the particle and $\hbar$ the quantum constant).

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k; its value is†

$$k = 1.38\times10^{-16}\ \text{erg/deg}.$$

In all subsequent formulae the temperature will be assumed measured in energy units. To convert to the temperature measured in degrees, in numerical calculations, we need only replace T by kT. The continual use of the factor k, whose only purpose is to indicate the conventional units of temperature measurement, would merely complicate the formulae.

If the temperature is in degrees, the factor k is usually included in the definition of entropy:

$$S = k\log\Delta\Gamma, \tag{9.2}$$

instead of (7.7), in order to avoid the appearance of k in the general relations of thermodynamics. Then formula (9.1) defining the temperature, and therefore all the general thermodynamic relations derived subsequently in this chapter, are unaffected by the change to degrees.

Thus the rule for conversion to degrees is to substitute in all formulae

$$T \to kT, \quad S \to S/k. \tag{9.3}$$

§ 10. Macroscopic motion

As distinct from the microscopic motion of molecules, the *macroscopic motion* is that in which the various macroscopic parts of a body participate as a whole. Let us consider the possibility of macroscopic motion in a state of thermodynamic equilibrium.

Let the body be divided into a large number of small (but macroscopic) parts, and let M_a, E_a and $\mathbf{P}_a$ denote the mass, energy and momentum of the ath part. The entropy S_a of each part is a function of its internal energy, i.e. the difference between its total energy E_a and the kinetic energy $P_a^2/2M_a$ of its macroscopic motion.‡ The total entropy of the body can therefore be written

$$S = \sum_a S_a(E_a - P_a^2/2M_a). \tag{10.1}$$

† For reference, we may also give the conversion factor between degrees and electron-volts:

$$1\ \text{eV} = 11{,}606\ \text{deg}.$$

‡ The fact that the entropy of a body is a function only of its internal energy follows at once from Galileo's relativity principle; the number of quantum states, and therefore the statistical weight (whose logarithm is the entropy), must be the same in all inertial frames of reference, and in particular that in which the body is at rest.

We shall assume that the body is a closed system. Then its total momentum and angular momentum are conserved, as well as its energy:

$$\sum_a \mathbf{P}_a = \text{constant}, \quad \sum_a \mathbf{r}_a\times\mathbf{P}_a = \text{constant}, \tag{10.2}$$

where $\mathbf{r}_a$ is the radius vector of the ath part. In a state of equilibrium, the total entropy S of the body as a function of the momenta $\mathbf{P}_a$ has a maximum subject to the conditions (10.2). Using the familiar Lagrange's method of undetermined multipliers, we find the necessary conditions for a maximum by equating to zero the derivatives with respect to $\mathbf{P}_a$ of the sum

$$\sum_a \{S_a + \mathbf{a}\cdot\mathbf{P}_a + \mathbf{b}\cdot\mathbf{r}_a\times\mathbf{P}_a\}, \tag{10.3}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. Differentiation of S_a with respect to $\mathbf{P}_a$ gives,† by the definition of the temperature,

$$\frac{\partial}{\partial\mathbf{P}_a} S_a\left(E_a - \frac{P_a^2}{2M_a}\right) = -\frac{\mathbf{P}_a}{M_a T} = -\frac{\mathbf{v}_a}{T},$$

where $\mathbf{v}_a = \mathbf{P}_a/M_a$ is the velocity of the ath part of the body. Differentiation of (10.3) therefore gives

$$-\mathbf{v}_a/T + \mathbf{a} + \mathbf{b}\times\mathbf{r}_a = 0,$$

or

$$\mathbf{v}_a = \mathbf{u} + \boldsymbol{\Omega}\times\mathbf{r}_a, \tag{10.4}$$

where $\mathbf{u} = T\mathbf{a}$ and $\boldsymbol{\Omega} = T\mathbf{b}$ are constant vectors.

This result has a simple physical significance. If the velocities of all the parts of a body are given by formula (10.4) with the same $\mathbf{u}$ and $\boldsymbol{\Omega}$, this means that we have a translational motion of the body as a whole with constant velocity $\mathbf{u}$ and a rotation of the body as a whole with constant angular velocity $\boldsymbol{\Omega}$. Thus we arrive at the important result that in thermodynamic equilibrium a closed system can execute only a uniform translational and rotational motion as a whole. No internal macroscopic motion is possible in a state of equilibrium.‡

† The derivative with respect to a vector is to be understood as another vector whose components are equal to the derivatives with respect to the components of the first vector.

‡ To avoid misunderstanding, an important exception to this rule should be noted: superfluid liquid helium cannot rotate as a whole. This effect will be discussed elsewhere in the *Course of Theoretical Physics*. Here we shall only mention that the proof given above is invalid in that case, because the velocity distribution is subject to a further condition (the superfluid flow is a potential flow) and the entropy maximum is to be sought under this condition.

In what follows we shall usually consider bodies at rest, and the energy E will accordingly be the internal energy of the body.

So far we have made use only of the necessary condition for a maximum of entropy as a function of the momenta, but not of the sufficient condition to be imposed on the second derivatives. It is easy to see that the latter condition leads to the very important result that the temperature must be positive: $T > 0$.† To deduce this, it is not in fact necessary to calculate the second derivatives; instead, we can argue as follows.

Let us consider a body forming a closed system, at rest as a whole. If the temperature were negative, the entropy would increase with decreasing argument. Since the entropy tends to increase, the body would spontaneously seek to break up into dispersing parts (with total momentum $\Sigma \mathbf{P}_a = 0$), so that the argument of each $\mathbf{S}_a$ in the sum (10.1) should take its least possible value. In other words, bodies in equilibrium could not exist with $T < 0$.

The following point should be noted, however. Although the temperature of a body or any part of it can never be negative, there may exist partial equilibria in which the temperature corresponding to a particular group of degrees of freedom of the body is negative. This is further discussed in § 73.

§ 11. Adiabatic processes

Among the various kinds of external interactions to which a body is subject, those which consist in a change in the external conditions form a special group. By "external conditions" we mean in a wide sense various external fields. In practice the external conditions are most often determined by the fact that the body must have a prescribed volume. In one sense this case may also be regarded as a particular type of external field, since the walls which limit the volume are equivalent in effect to a potential barrier which prevents the molecules in the body from escaping.

If the body is subject to no interactions other than changes in external conditions, it is said to be *thermally isolated*. It must be emphasised that, although a thermally isolated body does not interact directly with any other bodies, it is not in general a closed system, and its energy may vary with time.

In a purely mechanical way, a thermally isolated body differs from a closed system only in that its Hamiltonian (the energy) depends explicitly on the time: $E = E(p, q, t)$, because of the variable external field. If the body also interacted directly with other bodies, it would have no Hamiltonian of its own, since the interaction would depend not only on the coordinates of the molecules of the body in question but also on those of the molecules in the other bodies.

† The temperature $T = 0$ (absolute zero) corresponds to $-273.15°C$.

This leads to the result that the law of increase of entropy is valid not only for closed systems but also for a thermally isolated body, since here we regard the external field as a completely specified function of coordinates and time, and in particular neglect the reaction of the body on the field. That is, the field is a purely mechanical and not a statistical object, whose entropy can in this sense be taken as zero. This proves the foregoing statement.

Let us suppose that a body is thermally isolated, and is subject to external conditions which vary sufficiently slowly. Such a process is said to be *adiabatic*. We shall show that, in an adiabatic process, the entropy of the body remains unchanged, i.e. the process is reversible.

We shall describe the external conditions by certain parameters which are given functions of time. For example, suppose that there is only one such parameter, which we denote by λ. The time derivative $\mathrm{d}S/\mathrm{d}t$ of the entropy will depend in some manner on the rate of variation $\mathrm{d}\lambda/\mathrm{d}t$ of the parameter λ. Since $\mathrm{d}\lambda/\mathrm{d}t$ is small, we can expand $\mathrm{d}S/\mathrm{d}t$ in powers of $\mathrm{d}\lambda/\mathrm{d}t$. The zero-order term in this expansion, which does not involve $\mathrm{d}\lambda/\mathrm{d}t$, is zero, since if $\mathrm{d}\lambda/\mathrm{d}t = 0$ then $\mathrm{d}S/\mathrm{d}t = 0$ also, because the entropy of a closed system in thermodynamic equilibrium must remain constant under constant external conditions. The first-order term, which is proportional to $\mathrm{d}\lambda/\mathrm{d}t$, must also be zero, since this term changes sign with $\mathrm{d}\lambda/\mathrm{d}t$, whereas $\mathrm{d}S/\mathrm{d}t$ is always positive, according to the law of increase of entropy. Hence it follows that the expansion of $\mathrm{d}S/\mathrm{d}t$ begins with the second-order term, i.e. for small $\mathrm{d}\lambda/\mathrm{d}t$ we have

$$\mathrm{d}S/\mathrm{d}t = A(\mathrm{d}\lambda/\mathrm{d}t)^2,$$

or

$$\mathrm{d}S/\mathrm{d}\lambda = A\,\mathrm{d}\lambda/\mathrm{d}t.$$

Thus, when $\mathrm{d}\lambda/\mathrm{d}t$ tends to zero, so does $\mathrm{d}S/\mathrm{d}\lambda$, which proves that the adiabatic process is reversible.

It must be emphasised that, although an adiabatic process is reversible, not every reversible process is adiabatic. The condition for a process to be reversible requires only that the total entropy of the whole of a closed system be constant, while the entropies of its individual parts may either increase or decrease. In an adiabatic process, a stronger condition holds: the entropy of a body which is only a part of a closed system also remains constant.

We have defined an adiabatic process as one which is sufficiently slow. More precisely, we can say that the external conditions must change so

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Thus we see that the correction to the classical distribution function for the momenta is equivalent to adding to the kinetic energy in the exponent an expression quadratic in the momenta, with coefficients depending on the law of interaction between the particles in the body.

If it is desired to find the probability distribution for any one momentum p_i, then (33.17) must be integrated with respect to all the other momenta. All the terms involving the squares p_k^2 $(k \neq i)$ will then give constants negligible compared with unity, while the terms containing products of different momenta give zero. The result is, again in exponential form,

$$dw_{p_i} = \text{constant} \times \exp\left\{-\frac{p_i^2}{2m_iT}\left[1-\frac{\hbar^2}{12T^3m_i}\overline{\left(\frac{\partial U}{\partial q_i}\right)^2}\right]\right\} dp_i. \quad (33.19)$$

Thus the distribution obtained differs from the Maxwellian only in that the true temperature T is replaced by a somewhat higher "effective temperature":

$$T_{\text{eff}} = T + \frac{\hbar^2}{12T^2m_i}\overline{\left(\frac{\partial U}{\partial q_i}\right)^2}.$$

Similarly we can calculate the corrected coordinate distribution function by integrating I with respect to the momenta:

$$dw_q = \text{constant} \times dq \int I\, dp.$$

The same calculations as led to (33.13) give

$$dw_q = \text{constant} \times \exp\left\{-\frac{1}{T}\left[U-\frac{\hbar^2}{24T^2}\sum_i\frac{1}{m_i}\left(\frac{\partial U}{\partial q_i}\right)^2+\frac{\hbar^2}{12T}\sum_i\frac{1}{m_i}\frac{\partial^2 U}{\partial q_i^2}\right]\right\} dq. \quad (33.20)$$

§ 34. The Gibbs distribution for rotating bodies

The problem of the thermodynamic relations for rotating bodies has already been considered in § 26. Let us now see how the Gibbs distribution is to be formulated for rotating bodies. This will complete the investigation of their statistical properties. As regards the uniform translational motion, Galileo's relativity principle shows that, as already mentioned in § 26, this motion has only a trivial effect on the statistical properties and so needs no special consideration.

In a system of coordinates rotating with the body, the usual Gibbs distribution is valid; in classical statistics,

$$\varrho = (2\pi\hbar)^{-s} e^{[F'-E'(p,\,q)]/T}, \quad (34.1)$$

where $E'(p, q)$ is the energy of the body in this system, as a function of the coordinates and momenta of its particles, and F' the free energy in the same system (which, of course, is not the same as the free energy of the body when at rest). The energy $E'(p, q)$ is related to the energy $E(p, q)$ in a fixed system by

$$E'(p, q) = E(p, q) - \boldsymbol{\Omega}\cdot\mathbf{M}(p, q), \quad (34.2)$$

where $\boldsymbol{\Omega}$ is the angular velocity of rotation and $\mathbf{M}(p, q)$ the angular momentum of the body (see § 26). Substituting (34.2) in (34.1), we find the Gibbs distribution for a rotating body in the form†

$$\varrho = (2\pi\hbar)^{-s} e^{[F'-E(p,\,q)+\boldsymbol{\Omega}\cdot\mathbf{M}(p,\,q)]/T}. \quad (34.3)$$

In classical statistics the Gibbs distribution for a rotating body can also be represented in another form. To obtain this, we use the following expression for the energy of the body in the rotating coordinate system:

$$E' = \Sigma\,\tfrac{1}{2}mv'^2 - \tfrac{1}{2}\Sigma m(\boldsymbol{\Omega}\times\mathbf{r})^2 + U, \quad (34.4)$$

where the $\mathbf{v}'$ are the velocities of the particles relative to the rotating system, and the $\mathbf{r}$ their radius vectors (see *Mechanics*, § 39). Denoting by

$$E_0(\mathbf{v}', \mathbf{r}) = \Sigma\,\tfrac{1}{2}mv'^2 + U \quad (34.5)$$

the part of the energy which is independent of $\boldsymbol{\Omega}$, we obtain the Gibbs distribution in the form

$$\varrho = (2\pi\hbar)^{-s}\exp\left\{\frac{1}{T}\left[F'-E_0(\mathbf{v}', \mathbf{r})+\frac{1}{2}\Sigma m(\boldsymbol{\Omega}\times\mathbf{r})^2\right]\right\}.$$

The function ϱ determines the probability corresponding to the element of phase space $dx_1\, dy_1\, dz_1 \ldots dp'_{1x}\, dp'_{1y}\, dp'_{1z} \ldots$, where $\mathbf{p}' = m\mathbf{v}' + m\boldsymbol{\Omega}\times\mathbf{r}$ (see *Mechanics*, § 39). Since, in obtaining the differentials of the momenta, we must regard the coordinates as constant, $d\mathbf{p}' = m\, d\mathbf{v}'$, and the probability distribution expressed in terms of the coordinates and velocities of the particle is

$$dw = C\exp\left\{\frac{F'}{T}-\frac{1}{T}\left[E_0(\mathbf{v}', \mathbf{r})-\Sigma\frac{1}{2}m(\boldsymbol{\Omega}\times\mathbf{r})^2\right]\right\} \times dx_1\, dy_1\, dz_1 \ldots dv'_{1x}\, dv'_{1y}\, dv'_{1z} \ldots, \quad (34.6)$$

where C denotes for brevity the factor $(2\pi\hbar)^{-s}$ together with the product of the particle masses which appears when we go from the momentum differentials to the velocity differentials.

† The distribution (34.3), like the ordinary Gibbs distribution, is fully in agreement with the result (4.2) derived in § 4 from Liouville's theorem: the logarithm of the distribution function is a linear function of the energy and angular momentum of the body.

For a body at rest we have

$$dw = Ce^{[F-E_0(\mathbf{v},\mathbf{r})]/T}\, dx_1\, dy_1\, dz_1 \ldots dv_{1x}\, dv_{1y}\, dv_{1z} \ldots, \qquad (34.7)$$

with the same expression (34.5) for $E_0(\mathbf{v}, \mathbf{r})$, now a function of the velocities in the fixed coordinate system. Thus we see that the Gibbs distribution for the coordinates and velocities for a rotating body differs from that for a body at rest only by the additional potential energy $-\frac{1}{2}\Sigma m(\Omega\times\mathbf{r})^2$. In other words, as regards the statistical properties of the body, the rotation is equivalent to the existence of an external field corresponding to the centrifugal force. The statistical properties are not affected by the Coriolis force.

It should be emphasised, however, that this last result applies only to classical statistics. In the quantum case the expression

$$\hat{w} = \exp[(F'-\hat{H}+\Omega\cdot\hat{\mathbf{M}})/T] \qquad (34.8)$$

gives the statistical operator corresponding to (34.3) for a rotating body. Formally we can reduce this operator to a form analogous to (34.6), the velocities $\mathbf{v}'$ being replaced by the operators $\hat{\mathbf{v}}' = \hat{\mathbf{p}}'/m - \Omega\times\mathbf{r}$, but the components of this vector operator do not commute, unlike those of the operator $\hat{\mathbf{v}}$ in the fixed system. The statistical operators corresponding to the expressions (34.6) and (34.7) will therefore in general be markedly different from each other, quite apart from the fact that one of them contains the centrifugal energy.

§ 35. The Gibbs distribution for a variable number of particles

So far we have always tacitly assumed that the number of particles in a body is some given constant, and have deliberately passed over the fact that in reality particles may be exchanged between different subsystems. In other words, the number N of particles in a subsystem will necessarily fluctuate about its mean value. In order to formulate precisely what we mean by the number of particles, we shall use the term *subsystem* to refer to a part of the system which is enclosed in a fixed volume. Then N will denote the number of particles within that volume.†

Thus the problem arises of generalising the Gibbs distribution to bodies with a variable number of particles. Here we shall write the formulae for bodies consisting of identical particles; the further generalisation to systems containing different particles is obvious (§ 85).

† In deriving the Gibbs distribution in § 28 we have in essence already understood subsystems in this sense; in going from (28.2) to (28.3) we differentiated the entropy whilst regarding the volume of the body (and therefore of the medium) as constant.

The distribution function now depends not only on the energy of the quantum state but also on the number N of particles in the body, and the energy levels E_{nN} are of course themselves different for different N (as indicated by the suffix N). The probability that the body contains N particles and is in the nth state will be denoted by w_{nN}.

The form of this function can be determined in exactly the same way as the function w_n in § 28. The only difference is that the entropy of the medium is now a function not only of its energy E' but also of the number N' of particles in it: $S' = S'(E', N')$. Writing $E' = E^{(0)} - E_{nN}$ and $N' = N^{(0)} - N$ (where N is the number of particles in the body, and $N^{(0)}$ the given total number of particles in the entire closed system, which is large compared with N), we have in accordance with (28.2)

$$w_{nN} = \text{constant}\times\exp\{S'(E^{(0)}-E_{nN}, N^{(0)}-N)\};$$

the quantity $\Delta E'$ is regarded as constant, as in § 28.

Next, we expand S' in powers of E_{nN} and N, again taking only the linear terms. Equation (24.5), in the form

$$dS = \frac{dE}{T} + \frac{P}{T}\, dV - \frac{\mu}{T}\, dN,$$

shows that $(\partial S/\partial E)_{V,N} = 1/T$, $(\partial S/\partial N)_{E,V} = -\mu/T$. Hence

$$S'(E^{(0)}-E_{nN}, N^{(0)}-N) \cong S'(E^{(0)}, N^{(0)}) - \frac{E_{nN}}{T} + \frac{\mu N}{T},$$

the chemical potential μ (and the temperature) being the same for the body and the medium, from the conditions of equilibrium.

Thus we obtain for the distribution function the expression

$$w_{nN} = Ae^{(\mu N - E_{nN})/T}. \qquad (35.1)$$

The normalisation constant A can be expressed in terms of the thermodynamic quantities in the same way as in § 31. The entropy of the body is

$$S = -\langle \log w_{nN}\rangle = -\log A - \frac{\mu\bar{N}}{T} + \frac{\bar{E}}{T},$$

and so

$$T\log A = \bar{E} - TS - \mu\bar{N}.$$

But $\bar{E} - TS = F$, and the difference $F - \mu\bar{N}$ is the thermodynamic potential Ω. Thus $T\log A = \Omega$, and (35.1) may be rewritten as

$$w_{nN} = e^{(\Omega+\mu N - E_{nN})/T}. \qquad (35.2)$$

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71.04 From A. Ghosh and P. Mukhopadhyay

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[1527]

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©|L.D.Landau *et al.*, *Statistical Physics*, ..., ..., ...Ed., WEB - URL - ISBN: 0-08-023039-3.|||

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71.05 From W. J. Beek and K. M. K. Muttzall and J. W. Van Heuven

71.06 From J.P.Perez

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Part H

Additional Topics

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Part I

Appendix

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APPENDICES A

Mathematics

A Mathematics 2803

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VECTOR IDENTITIES[4]

Notation: f, g, are scalars; $\mathbf{A}$, $\mathbf{B}$, etc., are vectors; T is a tensor; I is the unit dyad.

(1) $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{B}\times\mathbf{C}\cdot\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} = \mathbf{C}\times\mathbf{A}\cdot\mathbf{B}$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{C}\times\mathbf{B})\times\mathbf{A} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

(3) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 0$

(4) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$

(5) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D}$

(6) $\nabla(fg) = \nabla(gf) = f\nabla g + g\nabla f$

(7) $\nabla\cdot(f\mathbf{A}) = f\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla f$

(8) $\nabla\times(f\mathbf{A}) = f\nabla\times\mathbf{A} + \nabla f\times\mathbf{A}$

(9) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$

(10) $\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

(11) $\mathbf{A}\times(\nabla\times\mathbf{B}) = (\nabla\mathbf{B})\cdot\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

(12) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(13) $\nabla^2 f = \nabla\cdot\nabla f$

(14) $\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$

(15) $\nabla\times\nabla f = 0$

(16) $\nabla\cdot\nabla\times\mathbf{A} = 0$

If $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ are orthonormal unit vectors, a second order tensor T can be written in the dyadic form

(17) $\mathsf{T} = \sum_{i,j} T_{ij}\mathbf{e}_i\mathbf{e}_j$

In cartesian coordinates the divergence of a tensor is a vector with components

(18) $(\nabla\cdot\mathsf{T})_i = \sum_j (\partial T_{ji}/\partial x_j)$

[This definition is required for consistency with Eq. (29)]. In general

(19) $\nabla\cdot(\mathbf{AB}) = (\nabla\cdot\mathbf{A})\mathbf{B} + (\mathbf{A}\cdot\nabla)\mathbf{B}$

(20) $\nabla\cdot(f\mathsf{T}) = \nabla f\cdot\mathsf{T} + f\nabla\cdot\mathsf{T}$

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APPENDICES B

International System of Units

B International System of Units 2805

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[1003, 1030, 1039, 1223, 1523, 1523, 1687, 1691, 1748, 1987, 2062, 2388, 2396, 2522]

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SI

A concise summary of the International System of Units, SI

Metrology is the science of measurement and its application. Metrology includes all theoretical and practical aspects of measurement, whatever the measurement uncertainty and field of application.

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The International Bureau of Weights and Measures (BIPM) was established by Article 1 of the Metre Convention, which was signed on 20 May 1875. It is charged with providing the basis for a single, coherent system of measurements to be used throughout the world and it operates under the authority of the International Committee of Weights and Measures (CIPM). The decimal metric system, dating from the time of the French Revolution, was based in 1799 on the metre and the kilogram. Under the terms of the Metre Convention, new international prototypes of the metre and kilogram were manufactured and formally adopted by the first General Conference on Weights and Measures (CGPM) in 1889. In 1960 the 11th CGPM formally defined and established the International System of Units (SI). Since then the SI has been periodically updated to take account of advances in science and the need for measurements in new domains. The last major revision was adopted by the 26th CGPM (2018), which decided that the SI would be based on the fixed numerical values of a set of seven **defining constants** from which the definitions of the seven base units of the SI would be deduced**.** This document is a summary of the **SI Brochure**, a publication produced by the BIPM which gives a detailed explanation of the current status of the SI.

The SI is the system of units in which:

- the unperturbed ground state hyperfine transition frequency of the caesium 133 atom $\Delta\nu_{\mathrm{Cs}}$ is 9 192 631 770 Hz,
- the speed of light in vacuum c is 299 792 458 m/s,
- the Planck constant h is $6.626\,070\,15 \times 10^{-34}$ J s,
- the elementary charge e is $1.602\,176\,634 \times 10^{-19}$ C,
- the Boltzmann constant k is $1.380\,649 \times 10^{-23}$ J/K,
- the Avogadro constant N_{A} is $6.022\,140\,76 \times 10^{23}$ mol^{-1},
- the luminous efficacy of monochromatic radiation of frequency 540×10^{12} Hz, K_{cd}, is 683 lm/W,

where the hertz, joule, coulomb, lumen, and watt, with unit symbols Hz, J, C, lm, and W, respectively, are related to the units second, metre, kilogram, ampere, kelvin, mole, and candela, with unit symbols s, m, kg, A, K, mol, and cd, respectively, according to $\mathrm{Hz} = \mathrm{s}^{-1}$, $\mathrm{J} = \mathrm{kg\ m^2\ s^{-2}}$, $\mathrm{C} = \mathrm{A\ s}$, $\mathrm{lm} = \mathrm{cd\ m^2\ m^{-2}} = \mathrm{cd\ sr}$, and $\mathrm{W} = \mathrm{kg\ m^2\ s^{-3}}$.

These definitions specify the exact numerical value of each constant when its value is expressed in the corresponding SI unit. By fixing the exact numerical value the unit becomes defined, since the product of the ***numerical value*** and the ***unit*** has to equal the ***value*** of the constant, which is invariant.

The defining constants have been chosen such that, when taken together, their units cover all of the units of the SI. In general, there is no one-to-one correspondence between the defining constants and the SI base units, except for the caesium frequency $\Delta\nu_{\mathrm{Cs}}$ and the Avogadro constant N_{A} . Any SI unit is a product of powers of these seven constants and a dimensionless factor.

For example: using $\mathrm{Hz} = \mathrm{s}^{-1}$, one metre can be derived from the speed of light c and caesium frequency $\Delta\nu_{\mathrm{Cs}}$

$$1\,\mathrm{m} = \left(\frac{c}{299\,792\,458}\right)\mathrm{s} = \frac{9\,192\,631\,770}{299\,792\,458}\frac{c}{\Delta\nu_{\mathrm{Cs}}} \approx 30.663\,319\frac{c}{\Delta\nu_{\mathrm{Cs}}}\cdot$$

The concept of base units and derived units was used to define the SI until 2018. These categories, although no longer essential in the SI, are maintained in view of their convenience and widespread use. The definitions of the base units, which follow from the definition of the SI in terms of the seven defining constants, are given in Table 1.

Table 1 *The seven base units of the SI*

Quantity	SI unit
time	The **second**, symbol s, is the SI unit of time. It is defined by taking the fixed numerical value of the caesium frequency $\Delta\nu_{Cs}$, the unperturbed ground-state hyperfine transition frequency of the caesium 133 atom, to be 9 192 631 770 when expressed in the unit Hz, which is equal to s^{-1}.
length	The **metre**, symbol m, is the SI unit of length. It is defined by taking the fixed numerical value of the speed of light in vacuum c to be 299 792 458 when expressed in the unit m s^{-1}, where the second is defined in terms of $\Delta\nu_{Cs}$.
mass	The **kilogram**, symbol kg, is the SI unit of mass. It is defined by taking the fixed numerical value of the Planck constant h to be $6.626\,070\,15 \times 10^{-34}$ when expressed in the unit J s, which is equal to kg m^2 s^{-1}, where the metre and the second are defined in terms of c and $\Delta\nu_{Cs}$.
electric current	The **ampere**, symbol A, is the SI unit of electric current. It is defined by taking the fixed numerical value of the elementary charge e to be $1.602\,176\,634 \times 10^{-19}$ when expressed in the unit C, which is equal to A s, where the second is defined in terms of $\Delta\nu_{Cs}$.
thermodynamic temperature	The **kelvin**, symbol K, is the SI unit of thermodynamic temperature. It is defined by taking the fixed numerical value of the Boltzmann constant k to be $1.380\,649 \times 10^{-23}$ when expressed in the unit J K^{-1}, which is equal to kg m^2 s^{-2} K^{-1}, where the kilogram, metre and second are defined in terms of h, c and $\Delta\nu_{Cs}$.
amount of substance	The **mole**, symbol mol, is the SI unit of amount of substance. One mole contains exactly $6.022\,140\,76 \times 10^{23}$ elementary entities. This number is the fixed numerical value of the Avogadro constant, N_A, when expressed in the unit mol^{-1} and is called the Avogadro number. The amount of substance, symbol n, of a system is a measure of the number of specified elementary entities. An elementary entity may be an atom, a molecule, an ion, an electron, any other particle or specified group of particles.
luminous intensity	The **candela**, symbol cd, is the SI unit of luminous intensity in a given direction. It is defined by taking the fixed numerical value of the luminous efficacy of monochromatic radiation of frequency 540×10^{12} Hz, K_{cd}, to be 683 when expressed in the unit lm W^{-1}, which is equal to cd sr W^{-1}, or cd sr kg m^{-2} s^3, where the kilogram, metre and second are defined in terms of h, c and $\Delta\nu_{Cs}$.

All other quantities may be called "derived quantities" and are measured using derived units, which can be written as products of powers of base units. Twenty-two derived units are given a special name as listed in Table 2.

Table 2 *Derived units with special names in the SI*

Derived quantity	Name of derived unit	Symbol for unit	Expression in terms of other units
plane angle	radian	rad	m/m
solid angle	steradian	sr	m^2/m^2
frequency	hertz	Hz	s^{-1}
force	newton	N	kg m s^{-2}
pressure, stress	pascal	Pa	N/m^2 = kg m^{-1} s^{-2}
energy, work, amount of heat	joule	J	N m = kg m^2 s^{-2}
power, radiant flux	watt	W	J/s = kg m^2 s^{-3}
electric charge	coulomb	C	A s
electric potential difference	volt	V	W/A = kg m^2 s^{-3} A^{-1}
capacitance	farad	F	C/V = kg^{-1} m^{-2} s^4 A^2
electric resistance	ohm	Ω	V/A = kg m^2 s^{-3} A^{-2}
electric conductance	siemens	S	A/V = kg^{-1} m^{-2} s^3 A
magnetic flux	weber	Wb	V s = kg m^2 s^{-2} A^{-1}
magnetic flux density	tesla	T	Wb/m^2 = kg s^{-2} A^{-1}
inductance	henry	H	Wb/A = kg m^2 s^{-2} A^{-2}
Celsius temperature	degree Celsius	°C	K
luminous flux	lumen	lm = cd sr	cd sr
illuminance	lux	lx =cd sr m^{-2}	lm/m^2
activity referred to a radionuclide	becquerel	Bq	s^{-1}
absorbed dose, kerma	gray	Gy	J/kg = m^2 s^{-2}
dose equivalent	sievert	Sv	J/kg = m^2 s^{-2}
catalytic activity	katal	kat	mol s^{-1}

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Although the hertz and the becquerel are both equal to the reciprocal second, hertz is used only for periodic phenomena, and becquerel is used only for stochastic processes in radioactive decay.

The unit of Celsius temperature is the degree Celsius, °C, which is equal in magnitude to the kelvin, K, the unit of thermodynamic temperature. The quantity Celsius temperature t is related to thermodynamic temperature T by the equation $t/^{\circ}\mathrm{C} = T/\mathrm{K} - 273.15$.

The sievert is also used for the quantities 'directional dose equivalent' and 'personal dose equivalent'.

There are many more quantities than units. For each quantity, there is only one SI unit (although this may often be expressed in different ways by using the special names), while the same SI unit may be used to express the values of several different quantities (for example the SI unit J/K may be used to express the value of both heat capacity and entropy). It is therefore important not to use the unit alone to specify the quantity. This applies both to scientific texts and also to measuring instruments (i.e. an instrument read-out should indicate both the quantity concerned and the unit).

There are quantities with the unit one, 1, i.e. ratios of two quantities of the same kind. For example, refractive index is the ratio of two speeds, and relative permittivity is the ratio of the permittivity of a dielectric medium to that of free space. There are also quantities with the character of a count, for example, the number of cellular or biomolecular entities. These quantities also have the unit one. The unit one is by nature an element of any system of units. Quantities with the unit one can therefore be considered as traceable to the SI. However, when expressing the values of dimensionless quantities, the unit 1 is not written.

Decimal multiples and sub-multiples of SI units

A set of prefixes have been adopted for use with the SI units in order to express the values of quantities that are either much larger than, or much smaller than, the SI unit when used without any prefix. They can be used with any SI unit. The SI prefixes are listed in Table 3.

Table 3 ***The SI prefixes***

Factor	Name	Symbol	Factor	Name	Symbol
10^{1}	deca	da	10^{-1}	deci	d
10^{2}	hecto	h	10^{-2}	centi	c
10^{3}	kilo	k	10^{-3}	milli	m
10^{6}	mega	M	10^{-6}	micro	μ
10^{9}	giga	G	10^{-9}	nano	n
10^{12}	tera	T	10^{-12}	pico	p
10^{15}	peta	P	10^{-15}	femto	f
10^{18}	exa	E	10^{-18}	atto	a
10^{21}	zetta	Z	10^{-21}	zepto	z
10^{24}	yotta	Y	10^{-24}	yocto	y

When the prefixes are used, the prefix name and the unit name are combined to form a single word. Similarly, the prefix symbol and the unit symbol are written without any space to form a single symbol, which may itself be raised to any power. For example, we may write: kilometre, km; microvolt, μV; or femtosecond, fs.

When the SI units are used without any prefixes, the resulting set of units is described as having **coherent** in the following sense: when only coherent units are used, equations between the numerical values of quantities take exactly the same form as the equations between the quantities themselves. The use of a coherent set of units has technical advantages, for example in algebraic calculus (see the SI Brochure).

The kilogram, kg, is problematic because the name already includes a prefix, for historical reasons. Multiples and sub-multiples of the kilogram are written by combining prefixes with the gram: thus we write milligram, mg, not microkilogram, μkg.

Units outside the SI

The SI is the only system of units that is universally recognized, so it has a distinct advantage for establishing international dialogue. The use of the SI, as a standard system of units, simplifies the teaching of science. For these reasons, the use of SI units is recommended in all fields of science and technology. Other units, i.e. non-SI units, are generally defined in terms of SI units using conversion factors.

Nonetheless, some non-SI units are still widely used. A few, such as the minute, hour and day as units of time, will always be used because they are part of our culture. Others are used for historical reasons, to meet the needs of special interest groups, or because there is no convenient SI alternative. It will always remain the prerogative of a scientist to use the units that are considered to be best suited to the purpose. However, when non-SI units are used, the correspondence to the SI should always be quoted. A selection of non-SI units is listed in Table 4 with their conversion factors to the SI. For a more comprehensive list, see the SI Brochure.

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Table 4 *A selection of non-SI units*

Quantity	Unit	Symbol	Relation to SI
time	minute	min	1 min = 60 s
time	hour	h	1 h = 3600 s
time	day	d	1 d = 86 400 s
volume	litre	L or l	1 L = 1 dm^3
mass	tonne	t	1 t = 1000 kg
energy	electronvolt (*e*/C) J	eV	1 eV = 1.602 176 634 × 10^{-19} J

When units are named after an individual their symbol should begin with a capital letter (for example: ampere, A; kelvin, K; hertz, Hz; or coulomb, C). In all other cases, except the litre, they begin with a lower case letter (for example: metre, m; second, s; or mole, mol). The symbol for the litre is an exception; either a lower case letter 'l' or a capital 'L' may be used, the capital is allowed in this case to avoid confusion between the lower case letter l and the number one, 1.

The language of science: using the SI to express the values of quantities

The value of a quantity is written as the product of a number and a unit. The number multiplying the unit is the numerical value of the quantity in that unit. A single space is always left between the number and the unit. The numerical value depends on the choice of unit, so that the same value of a quantity may have different numerical values when expressed in different units, as in the examples provided below.

The speed of a bicycle is approximately
v = 5.0 m/s = 18 km/h.
The wavelength of one of the yellow lines of the sodium doublet is
$\lambda = 5.896 \times 10^{-7}$ m = 589.6 nm.

Quantity symbols are written using italic (slanting) type, and they are generally single letters of the Latin or Greek alphabet. Either capital or lower case letters may be used, and additional information on the quantity may be added as a subscript or as information in brackets.

Authorities such as the International Organization for Standardization (ISO) and international scientific unions such as the International Union of Pure and Applied Physics (IUPAP) and the International Union of Pure and Applied Chemistry (IUPAC) have specified recommended symbols for many quantities. Examples include:

T	for thermodynamic temperature
C_p	for heat capacity at constant pressure
x_i	for the mole fraction (amount fraction) of species *i*
μ_r	for relative permeability.

Unit symbols are written using roman (upright) type, regardless of the type used in the surrounding text. They are mathematical entities and not abbreviations; unit symbols are never followed by a full stop (except at the end of a sentence) nor by an 's' for the plural. The use of the correct form for unit symbols is mandatory, and is illustrated by examples in the SI Brochure. Unit symbols may be more than a single letter. They are written in lower case letters, the exception being that the first letter is a capital when the unit is named after a person. However, when the name of a unit is spelled in full, it should begin with a lower case letter (except at the beginning of a sentence), to distinguish the unit from the person (for example a temperature of 293 kelvin).

When writing the value of a quantity as the product of a numerical value and a unit, both the number and the unit may be treated by the ordinary rules of algebra. For example, the equation T = 293 K may equally be written as T/K = 293. This procedure is described as the use of quantity calculus, or the algebra of quantities. It is often useful to use the ratio of a quantity to its unit as the heading for columns in a table, or for labelling the axes of a graph, so that the entries in the table or the labels for the tick marks on the axes are all simply denoted by numbers. The example (Table 5) shows a table of the velocity squared versus pressure, with the columns labelled in this way.

Table 5 *Example of column heading in a table of the velocity squared versus pressure*

p/kPa	$v^2/(m/s)^2$
48.73	94766
72.87	94771
135.42	94784

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In forming products or quotients of unit symbols the normal rules of algebra apply. In forming products of unit symbols, a space should be left between units (or alternatively a half-high centred dot can be used as a multiplication symbol). The importance of the space should be noted: the product of a metre and a second is denoted by m s (with a space), but ms (without a space) is used to denote a millisecond. In addition, when forming complicated products of units, brackets or negative exponents should be used to avoid ambiguities. For example, the molar gas constant R is given by:

$$pV_m/T = R = 8.314 \text{ Pa m}^3 \text{ mol}^{-1} \text{ K}^{-1}$$
$$= 8.314 \text{ Pa m}^3/(\text{mol K}).$$

When formatting numbers, the decimal marker may be either a point (i.e. a full stop) or a comma, depending on the circumstances. For documents in the English language a point is usual, but for many languages and in many countries a comma is usual.

When a number has many digits, it is customary to group the digits into threes about the decimal point to aid readability. This is not essential, but it is often done and is generally helpful. When this format is used, the groups of three digits should be separated only by a space; neither a point nor a comma should be used. The uncertainty in the numerical value of a quantity may often be conveniently shown by giving the uncertainty in the least significant digits in brackets after the number.

For example: The value of the electron mass is given in the 2014 CODATA listing of fundamental constants as

$$m_e = 9.109\,383\,56(11) \times 10^{-31} \text{ kg},$$

where 11 is the standard uncertainty in the final digits quoted for the numerical value.

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Bureau International des Poids et Mesures

For further information, see the BIPM website, or the **SI Brochure** 9th edition, which is available at

www.bipm.org

This summary has been prepared by the Comité Consultatif des Unités (CCU) of the Comité International des Poids et Mesures (CIPM), and is published by the BIPM.

APPENDICES C

Physics Constants

C Physics Constants 2812

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CODATA RECOMMENDED VALUES OF THE FUNDAMENTAL PHYSICAL CONSTANTS: 2018

NIST SP 961 (May 2019)

An extensive list of constants is available on the NIST Physics Laboratory Web site physics.nist.gov/constants. For numerical values a number in parentheses, if present, is the one-standard-deviation uncertainty in the last two digits. For units with square brackets the full descriptions of m^{-1} and m are cycles per meter and meter per cycle, respectively. For the first radiation constant the full description of m^2 is m^{-2} $(m/cycle)^4$.

Quantity	Symbol	Numerical value	Unit
speed of light in vacuum	c	299 792 458 (exact)	m s^{-1}
Newtonian constant of gravitation	G	$6.674\,30(15) \times 10^{-11}$	m^3 kg^{-1} s^{-2}
Planck constant	h	$6.626\,070\,15 \times 10^{-34}$ (exact)	J Hz^{-1}
in eV s		$4.135\,667\,696\ldots \times 10^{-15}$	eV Hz^{-1}
	$\hbar$	$1.054\,571\,817\ldots \times 10^{-34}$	J s
in eV s		$6.582\,119\,569\ldots \times 10^{-16}$	eV s
elementary charge	e	$1.602\,176\,634 \times 10^{-19}$ (exact)	C
vacuum magnetic permeability $4\pi\alpha\hbar/e^2c$	μ_0	$1.256\,637\,062\,12(19) \times 10^{-6}$	N A^{-2}
$\mu_0/(4\pi \times 10^{-7})$		$1.000\,000\,000\,55(15)$	N A^{-2}
vacuum electric permittivity $1/\mu_0c^2$	ϵ_0	$8.854\,187\,8128(13) \times 10^{-12}$	F m^{-1}
Josephson constant $2e/h$	K_J	$483\,597.848\,4\ldots \times 10^9$	Hz V^{-1}
von Klitzing constant $\mu_0c/2\alpha = 2\pi\hbar/e^2$	R_K	25 812.807 45 . . .	Ω
magnetic flux quantum $2\pi\hbar/(2e)$	Φ_0	$2.067\,833\,848\ldots \times 10^{-15}$	Wb
Bohr magneton $e\hbar/2m_e$	μ_B	$9.274\,010\,0783(28) \times 10^{-24}$	J T^{-1}
in eV T^{-1}		$5.788\,381\,8060(17) \times 10^{-5}$	eV T^{-1}
nuclear magneton $e\hbar/2m_p$	μ_N	$5.050\,783\,7461(15) \times 10^{-27}$	J T^{-1}
in eV T^{-1}		$3.152\,451\,258\,44(96) \times 10^{-8}$	eV T^{-1}
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,5693(11) \times 10^{-3}$	
inverse fine-structure constant	α^{-1}	137.035 999 084(21)	
Rydberg frequency $\alpha^2m_ec^2/2h = E_h/2h$	cR_∞	$3.289\,841\,960\,2508(64) \times 10^{15}$	Hz
energy equivalent in eV		13.605 693 122 994(26)	eV
Rydberg constant	R_∞	10 973 731.568 160(21)	$[m^{-1}]$
Bohr radius $\hbar/\alpha m_ec = 4\pi\epsilon_0\hbar^2/m_ee^2$	a_0	$5.291\,772\,109\,03(80) \times 10^{-11}$	m
Hartree energy $\alpha^2m_ec^2 = e^2/4\pi\epsilon_0a_0 = 2(cR_\infty)h$	E_h	$4.359\,744\,722\,2071(85) \times 10^{-18}$	J
in eV		27.211 386 245 988(53)	eV
electron mass	m_e	$9.109\,383\,7015(28) \times 10^{-31}$	kg
in u		$5.485\,799\,090\,65(16) \times 10^{-4}$	u
energy equivalent in MeV	m_ec^2	0.510 998 950 00(15)	MeV
electron-muon mass ratio	m_e/m_μ	$4.836\,331\,69(11) \times 10^{-3}$	
electron-proton mass ratio	m_e/m_p	$5.446\,170\,214\,87(33) \times 10^{-4}$	
electron charge to mass quotient	$-e/m_e$	$-1.758\,820\,010\,0(53) \times 10^{11}$	$C\,kg^{-1}$
reduced Compton wavelength $\hbar/m_ec = \alpha a_0$	$\lambda\!\!\!^{-}_C$	$3.861\,592\,6790(12) \times 10^{-13}$	m
Compton wavelength	λ_C	$2.426\,310\,238\,67(73) \times 10^{-12}$	[m]
classical electron radius α^2a_0	r_e	$2.817\,940\,325(13) \times 10^{-15}$	m
Thomson cross section $(8\pi/3)r_e^2$	σ_e	$0.665\,245\,8732(60) \times 10^{-29}$	m^2
electron magnetic moment	μ_e	$-9.284\,764\,704(28) \times 10^{-24}$	J T^{-1}
to Bohr magneton ratio	μ_e/μ_B	$-1.001\,159\,652\,181\,28(18)$	
to nuclear magneton ratio	μ_e/μ_N	$-1838.281\,971\,88(11)$	
electron magnetic moment anomaly $\lvert\mu_e\rvert/\mu_B - 1$	a_e	$1.159\,652\,181\,28(18) \times 10^{-3}$	
electron g-factor $-2(1 + a_e)$	g_e	$-2.002\,319\,304\,362\,56(35)$	
electron-proton magnetic moment ratio	μ_e/μ_p	−658.210 687 89(20)	
muon mass in u	m_μ	0.113 428 9259(25)	u
energy equivalent in MeV	$m_\mu c^2$	105.658 3755(23)	MeV
muon-electron mass ratio	m_μ/m_e	206.768 2830(46)	
muon magnetic moment	μ_μ	$-4.490\,448\,30(10) \times 10^{-26}$	J T^{-1}
to Bohr magneton ratio	μ_μ/μ_B	$-4.841\,970\,47(11) \times 10^{-3}$	
to nuclear magneton ratio	μ_μ/μ_N	−8.890 597 03(20)	
muon magnetic moment anomaly $\lvert\mu_\mu\rvert/(e\hbar/2m_\mu) - 1$	a_μ	$1.165\,920\,89(63) \times 10^{-3}$	

Quantity	Symbol	Numerical value	Unit
muon g-factor $-2(1 + a_\mu)$	g_μ	−2.002 331 8418(13)	
muon-proton magnetic moment ratio	μ_μ/μ_p	−3.183 345 142(71)	
proton mass	m_p	$1.672\,621\,923\,69(51) \times 10^{-27}$	kg
in u		1.007 276 466 621(53)	u
energy equivalent in MeV	m_pc^2	938.272 088 16(29)	MeV
proton-electron mass ratio	m_p/m_e	1836.152 673 43(11)	
proton magnetic moment	μ_p	$1.410\,606\,797\,36(60) \times 10^{-26}$	J T^{-1}
to nuclear magneton ratio	μ_p/μ_N	2.792 847 344 63(82)	
proton magnetic shielding correction $1-\mu'_p/\mu_p$ (H_2O, sphere, 25 °C)	σ'_p	$25.689(11) \times 10^{-5}$	
proton gyromagnetic ratio $2\mu_p/\hbar$	γ_p	$2.675\,221\,8744(11) \times 10^8$	s^{-1} T^{-1}
		42.577 478 518(18)	MHz T^{-1}
shielded proton gyromagnetic ratio $2\mu'_p/\hbar$ (H_2O, sphere, 25 °C)	γ'_p	$2.675\,153\,151(29) \times 10^8$	s^{-1} T^{-1}
		42.576 384 74(46)	MHz T^{-1}
neutron mass in u	m_n	1.008 664 915 95(49)	u
energy equivalent in MeV	m_nc^2	939.565 420 52(54)	MeV
neutron-proton mass ratio	m_n/m_p	1.001 378 419 31(49)	
neutron magnetic moment	μ_n	$-9.662\,3651(23) \times 10^{-27}$	J T^{-1}
to nuclear magneton ratio	μ_n/μ_N	−1.913 042 73(45)	
deuteron mass in u	m_d	2.013 553 212 745(40)	u
deuteron energy equivalent in MeV	m_dc^2	1875.612 942 57(57)	MeV
deuteron-proton mass ratio	m_d/m_p	1.999 007 501 39(11)	
deuteron magnetic moment	μ_d	$4.330\,735\,094(11) \times 10^{-27}$	J T^{-1}
to nuclear magneton ratio	μ_d/μ_N	0.857 438 2338(22)	
helion (^{3}He nucleus) mass in u	m_h	3.014 932 247 175(97)	u
energy equivalent in MeV	m_hc^2	2808.391 607 43(85)	MeV
shielded helion magnetic moment (gas, sphere, 25 °C)	μ'_h	$-1.074\,553\,090(13) \times 10^{-26}$	J T^{-1}
to Bohr magneton ratio	μ'_h/μ_B	$-1.158\,671\,471(14) \times 10^{-3}$	
to nuclear magneton ratio	μ'_h/μ_N	−2.127 497 719(25)	
alpha particle mass in u	m_α	4.001 506 179 127(63)	u
energy equivalent in MeV	$m_\alpha c^2$	3727.379 4066(11)	MeV
Boltzmann constant	k	$1.380\,649 \times 10^{-23}$ (exact)	J K^{-1}
Avogadro constant	N_A	$6.022\,140\,76 \times 10^{23}$ (exact)	mol^{-1}
atomic mass constant $\frac{1}{12}m(^{12}C) = 1$ u	m_u	$1.660\,539\,066\,60(50) \times 10^{-27}$	kg
energy equivalent in MeV	m_uc^2	931.494 102 42(28)	MeV
Faraday constant N_Ae	F	96 485.332 12 . . .	C mol^{-1}
molar gas constant N_Ak	R	8.314 462 618 . . .	J mol^{-1} K^{-1}
in eV K^{-1}		$8.617\,333\,262\ldots \times 10^{-5}$	eV K^{-1}
molar volume of ideal gas RT/p ($T = 273.15$ K, $p = 101.325$ kPa)	V_m	$22.413\,969\,54\ldots \times 10^{-3}$	m^3 mol^{-1}
Stefan-Boltzmann constant $\pi^2k^4/60\hbar^3c^2$	σ	$5.670\,374\,419\ldots \times 10^{-8}$	W m^{-2} K^{-4}
first radiation constant $2\pi hc^2$	c_1	$3.741\,771\,852\ldots \times 10^{-16}$	[W m^2]
second radiation constant hc/k	c_2	$1.438\,776\,877\ldots \times 10^{-2}$	[m K]
Wien displacement law constant $b = \lambda_{max}T = c_2/4.965\,114\,231\ldots$	b	$2.897\,771\,955\ldots \times 10^{-3}$	[m K]
Cu x unit: $\lambda(\mathrm{Cu\,K}\alpha_1)/1\,537.400$	xu(Cu Kα$_1$)	$1.002\,076\,97(28) \times 10^{-13}$	m
Mo x unit: $\lambda(\mathrm{Mo\,K}\alpha_1)/707.831$	xu(Mo Kα$_1$)	$1.002\,099\,52(53) \times 10^{-13}$	m

Energy equivalents

$[1\ m^{-1}]c = 299\,792\,458$ Hz	(1 Hz)$h/k = 4.799\,243\,073\ldots \times 10^{-11}$ K	(1 J) $= 6.241\,509\,074\ldots \times 10^{18}$ eV	(1 eV)$/c^2 = 1.073\,544\,102\,33(32) \times 10^{-9}$ u
$[1\ m^{-1}]hc/k = 1.438\,776\,877\ldots \times 10^{-2}$ K	(1 Hz)$h = 4.135\,667\,696\ldots \times 10^{-15}$ eV	(1 eV) $= 1.602\,176\,634 \times 10^{-19}$ J	(1 kg) $= 6.022\,140\,7621(18) \times 10^{26}$ u
$[1\ m^{-1}]hc = 1.239\,841\,984\ldots \times 10^{-6}$ eV	(1 K)$k/hc = 69.503\,480\,04\ldots$ $[m^{-1}]$	(1 eV)$/hc = 8.065\,543\,937\ldots \times 10^5$ $[m^{-1}]$	(1 u) $= 1.660\,539\,066\,60(50) \times 10^{-27}$ kg
$[1\ m^{-1}]h/c = 1.331\,025\,050\,10(40) \times 10^{-15}$ u	(1 K)$k/h = 2.083\,661\,912\ldots \times 10^{10}$ Hz	(1 eV)$/h = 2.417\,989\,242\ldots \times 10^{14}$ Hz	(1 u)$c/h = 7.513\,006\,6104(23) \times 10^{14}$ $[m^{-1}]$
(1 Hz)$/c = 3.335\,640\,951\ldots \times 10^{-9}$ $[m^{-1}]$	(1 K)$k = 8.617\,333\,262\ldots \times 10^{-5}$ eV	(1 eV)$/k = 1.160\,451\,812\ldots \times 10^4$ K	(1 u)$c^2 = 9.314\,941\,0242(28) \times 10^8$ eV

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NIST National Institute of Standards and Technology • U.S. Department of Commerce

Fundamental Physical Constants — Frequently used constants

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
speed of light in vacuum	c	299 792 458	m s^{-1}	exact
Newtonian constant of gravitation	G	6.674 30(15) $\times 10^{-11}$	m^3 kg^{-1} s^{-2}	2.2×10^{-5}
Planck constant	h	6.626 070 15 $\times 10^{-34}$	J Hz^{-1}	exact
	$\hbar$	1.054 571 817 ... $\times 10^{-34}$	J s	exact
elementary charge	e	1.602 176 634 $\times 10^{-19}$	C	exact
vacuum magnetic permeability $4\pi\alpha\hbar/e^2c$	μ_0	1.256 637 062 12(19) $\times 10^{-6}$	N A^{-2}	1.5×10^{-10}
vacuum electric permittivity $1/\mu_0c^2$	ϵ_0	8.854 187 8128(13) $\times 10^{-12}$	F m^{-1}	1.5×10^{-10}
Josephson constant $2e/h$	K_J	483 597.848 4 ... $\times 10^9$	Hz V^{-1}	exact
von Klitzing constant $\mu_0c/2\alpha = 2\pi\hbar/e^2$	R_K	25 812.807 45 ...	Ω	exact
magnetic flux quantum $2\pi\hbar/(2e)$	Φ_0	2.067 833 848 ... $\times 10^{-15}$	Wb	exact
conductance quantum $2e^2/2\pi\hbar$	G_0	7.748 091 729 ... $\times 10^{-5}$	S	exact
electron mass	m_e	9.109 383 7015(28) $\times 10^{-31}$	kg	3.0×10^{-10}
proton mass	m_p	1.672 621 923 69(51) $\times 10^{-27}$	kg	3.1×10^{-10}
proton-electron mass ratio	m_p/m_e	1836.152 673 43(11)		6.0×10^{-11}
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	7.297 352 5693(11) $\times 10^{-3}$		1.5×10^{-10}
inverse fine-structure constant	α^{-1}	137.035 999 084(21)		1.5×10^{-10}
Rydberg frequency $\alpha^2m_ec^2/2h$	cR_∞	3.289 841 960 2508(64) $\times 10^{15}$	Hz	1.9×10^{-12}
Boltzmann constant	k	1.380 649 $\times 10^{-23}$	J K^{-1}	exact
Avogadro constant	N_A	6.022 140 76 $\times 10^{23}$	mol^{-1}	exact
molar gas constant N_Ak	R	8.314 462 618 ...	J mol^{-1} K^{-1}	exact
Faraday constant N_Ae	F	96 485.332 12 ...	C mol^{-1}	exact
Stefan-Boltzmann constant $(\pi^2/60)k^4/\hbar^3c^2$	σ	5.670 374 419 ... $\times 10^{-8}$	W m^{-2} K^{-4}	exact
Non-SI units accepted for use with the SI				
electron volt (e/C) J	eV	1.602 176 634 $\times 10^{-19}$	J	exact
(unified) atomic mass unit $\frac{1}{12}m(^{12}C)$	u	1.660 539 066 60(50) $\times 10^{-27}$	kg	3.0×10^{-10}

* The energy of a photon with frequency ν expressed in unit Hz is $E = h\nu$ in J. Unitary time evolution of the state of this photon is given by $\exp(-iEt/\hbar)|\varphi\rangle$, where $|\varphi\rangle$ is the photon state at time $t = 0$ and time is expressed in unit s. The ratio $Et/\hbar$ is a phase.

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Fundamental Physical Constants — Extensive Listing

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Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
		UNIVERSAL		
speed of light in vacuum	c	299 792 458	m s^{-1}	exact
vacuum magnetic permeability $4\pi\alpha\hbar/e^2c$	μ_0	$1.256\,637\,062\,12(19) \times 10^{-6}$	N A^{-2}	1.5×10^{-10}
$\mu_0/(4\pi \times 10^{-7})$		1.000 000 000 55(15)	N A^{-2}	1.5×10^{-10}
vacuum electric permittivity $1/\mu_0c^2$	ϵ_0	$8.854\,187\,8128(13) \times 10^{-12}$	F m^{-1}	1.5×10^{-10}
characteristic impedance of vacuum $\mu_0 c$	Z_0	376.730 313 668(57)	Ω	1.5×10^{-10}
Newtonian constant of gravitation	G	$6.674\,30(15) \times 10^{-11}$	m^3 kg^{-1} s^{-2}	2.2×10^{-5}
	$G/\hbar c$	$6.708\,83(15) \times 10^{-39}$	$(\mathrm{GeV}/c^2)^{-2}$	2.2×10^{-5}
Planck constant*	h	$6.626\,070\,15 \times 10^{-34}$	J Hz^{-1}	exact
		$4.135\,667\,696\ldots \times 10^{-15}$	eV Hz^{-1}	exact
	$\hbar$	$1.054\,571\,817\ldots \times 10^{-34}$	J s	exact
		$6.582\,119\,569\ldots \times 10^{-16}$	eV s	exact
	$\hbar c$	197.326 980 4 ...	MeV fm	exact
Planck mass $(\hbar c/G)^{1/2}$	m_P	$2.176\,434(24) \times 10^{-8}$	kg	1.1×10^{-5}
energy equivalent	$m_\mathrm{P}c^2$	$1.220\,890(14) \times 10^{19}$	GeV	1.1×10^{-5}
Planck temperature $(\hbar c^5/G)^{1/2}/k$	T_P	$1.416\,784(16) \times 10^{32}$	K	1.1×10^{-5}
Planck length $\hbar/m_\mathrm{P}c = (\hbar G/c^3)^{1/2}$	ℓ_P	$1.616\,255(18) \times 10^{-35}$	m	1.1×10^{-5}
Planck time $l_\mathrm{P}/c = (\hbar G/c^5)^{1/2}$	t_P	$5.391\,247(60) \times 10^{-44}$	s	1.1×10^{-5}
		ELECTROMAGNETIC		
elementary charge	e	$1.602\,176\,634 \times 10^{-19}$	C	exact
	$e/\hbar$	$1.519\,267\,447\ldots \times 10^{15}$	A J^{-1}	exact
magnetic flux quantum $2\pi\hbar/(2e)$	Φ_0	$2.067\,833\,848\ldots \times 10^{-15}$	Wb	exact
conductance quantum $2e^2/2\pi\hbar$	G_0	$7.748\,091\,729\ldots \times 10^{-5}$	S	exact
inverse of conductance quantum	G_0^{-1}	$12\,906.403\,72\ldots$	Ω	exact
Josephson constant $2e/h$	K_J	$483\,597.848\,4\ldots \times 10^{9}$	Hz V^{-1}	exact
von Klitzing constant $\mu_0 c/2\alpha = 2\pi\hbar/e^2$	R_K	$25\,812.807\,\mathrm{R}\ldots$	Ω	exact
Bohr magneton $e\hbar/2m_\mathrm{e}$	μ_B	$9.274\,010\,078\,3(28) \times 10^{-24}$	J T^{-1}	3.0×10^{-10}
		$5.788\,381\,8060(17) \times 10^{-5}$	eV T^{-1}	3.0×10^{-10}
	μ_B/h	$1.399\,624\,493\,61(42) \times 10^{10}$	Hz T^{-1}	3.0×10^{-10}
	μ_B/hc	46.686 447 783 14	[m^{-1} T^{-1}]†	3.0×10^{-10}
	μ_B/k	0.671 713 815 63(20)	K T^{-1}	3.0×10^{-10}
nuclear magneton $e\hbar/2m_\mathrm{p}$	μ_N	$5.050\,783\,7461(15) \times 10^{-27}$	J T^{-1}	3.1×10^{-10}
		$3.152\,451\,258\,44(96) \times 10^{-8}$	eV T^{-1}	3.1×10^{-10}
	μ_N/h	7.622 593 2291(23)	MHz T^{-1}	3.1×10^{-10}
	μ_N/hc	$2.542\,623\,413\,53(78) \times 10^{-2}$	[m^{-1} T^{-1}]†	3.1×10^{-10}
	μ_N/k	$3.658\,267\,7756(11) \times 10^{-4}$	K T^{-1}	3.1×10^{-10}
		ATOMIC AND NUCLEAR		
		General		
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,5693(11) \times 10^{-3}$		1.5×10^{-10}
inverse fine-structure constant	α^{-1}	137.035 999 084(21)		1.5×10^{-10}
Rydberg frequency $\alpha^2 m_\mathrm{e}c^2/2h = E_\mathrm{h}/2h$	cR_∞	$3.289\,841\,960\,2508(64) \times 10^{15}$	Hz	1.9×10^{-12}
energy equivalent	hcR_∞	$2.179\,872\,361\,1035(42) \times 10^{-18}$	J	1.9×10^{-12}
		13.605 693 122 994(26)	eV	1.9×10^{-12}
Rydberg constant	R_∞	10 973 731.568 160(21)	[m^{-1}]†	1.9×10^{-12}
Bohr radius $\hbar/\alpha m_\mathrm{e}c = 4\pi\epsilon_0\hbar^2/m_\mathrm{e}e^2$	a_0	$5.291\,772\,109\,03(80) \times 10^{-11}$	m	1.5×10^{-10}
Hartree energy $\alpha^2 m_\mathrm{e}c^2 = e^2/4\pi\epsilon_0 a_0 = 2hcR_\infty$	E_h	$4.359\,744\,722\,2071(85) \times 10^{-18}$	J	1.9×10^{-12}
		27.211 386 245 988(53)	eV	1.9×10^{-12}
quantum of circulation	$\pi\hbar/m_\mathrm{e}$	$3.636\,947\,5516(11) \times 10^{-4}$	m^2 s^{-1}	3.0×10^{-10}

Fundamental Physical Constants — Extensive Listing

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
	$2\pi\hbar/m_e$	$7.273\,895\,1032(22) \times 10^{-4}$	$m^2\ s^{-1}$	3.0×10^{-10}
		Electroweak		
Fermi coupling constant‡	$G_F/(\hbar c)^3$	$1.166\,3787(6) \times 10^{-5}$	GeV^{-2}	5.1×10^{-7}
weak mixing angle§ θ_W (on-shell scheme)				
$\sin^2\theta_W = s_W^2 \equiv 1 - (m_W/m_Z)^2$	$\sin^2\theta_W$	0.222 90(30)		1.3×10^{-3}
		Electron, e^-		
electron mass	m_e	$9.109\,383\,7015(28) \times 10^{-31}$	kg	3.0×10^{-10}
		$5.485\,799\,090\,65(16) \times 10^{-4}$	u	2.9×10^{-11}
energy equivalent	$m_e c^2$	$8.187\,105\,7769(25) \times 10^{-14}$	J	3.0×10^{-10}
		0.510 998 950 00(15)	MeV	3.0×10^{-10}
electron-muon mass ratio	m_e/m_μ	$4.836\,331\,69(11) \times 10^{-3}$		2.2×10^{-8}
electron-tau mass ratio	m_e/m_τ	$2.875\,85(19) \times 10^{-4}$		6.8×10^{-5}
electron-proton mass ratio	m_e/m_p	$5.446\,170\,214\,87(33) \times 10^{-4}$		6.0×10^{-11}
electron-neutron mass ratio	m_e/m_n	$5.438\,673\,4424(26) \times 10^{-4}$		4.8×10^{-10}
electron-deuteron mass ratio	m_e/m_d	$2.724\,437\,107\,462(96) \times 10^{-4}$		3.5×10^{-11}
electron-triton mass ratio	m_e/m_t	$1.819\,200\,062\,251(90) \times 10^{-4}$		5.0×10^{-11}
electron-helion mass ratio	m_e/m_h	$1.819\,543\,074\,573(79) \times 10^{-4}$		4.3×10^{-11}
electron to alpha particle mass ratio	m_e/m_α	$1.370\,933\,554\,787(45) \times 10^{-4}$		3.3×10^{-11}
electron charge to mass quotient	$-e/m_e$	$-1.758\,820\,010\,76(53) \times 10^{11}$	$C\ kg^{-1}$	3.0×10^{-10}
electron molar mass $N_A m_e$	$M(e), M_e$	$5.485\,799\,0888(17) \times 10^{-7}$	$kg\ mol^{-1}$	3.0×10^{-10}
reduced Compton wavelength $\hbar/m_e c = \alpha a_0$	$\lambda\!\!\!^{-}_C$	$3.861\,592\,6796(12) \times 10^{-13}$	m	3.0×10^{-10}
Compton wavelength	λ_C	$2.426\,31\,238\,67(73) \times 10^{-12}$	[m]†	3.0×10^{-10}
classical electron radius $\alpha^2 a_0$	r_e	$2.817\,940\,262(13) \times 10^{-15}$	m	4.5×10^{-10}
Thomson cross section $(8\pi/3)r_e^2$	σ_e	$0.665\,245\,8321(60) \times 10^{-29}$	m^2	9.1×10^{-10}
electron magnetic moment	μ_e	$-9.284\,764\,7043(28) \times 10^{-24}$	$J\ T^{-1}$	3.0×10^{-10}
to Bohr magneton ratio	μ_e/μ_B	$-1.001\,159\,652\,181\,28(18)$		1.7×10^{-13}
to nuclear magneton ratio	μ_e/μ_N	$-1838.281\,971\,88(11)$		6.0×10^{-11}
electron magnetic moment anomaly $\lvert\mu_e\rvert/\mu_B - 1$	a_e	$1.159\,652\,181\,28(18) \times 10^{-3}$		1.5×10^{-10}
electron g-factor $-2(1 + a_e)$	g_e	$-2.002\,319\,304\,361\,56(35)$		1.7×10^{-13}
electron-muon magnetic moment ratio	μ_e/μ_μ	206.766 9883(46)		2.2×10^{-8}
electron-proton magnetic moment ratio	μ_e/μ_p	−658.210 687 89(20)		3.0×10^{-10}
electron to shielded proton magnetic moment ratio (H_2O, sphere, 25 °C)	μ_e/μ'_p	−658.227 5971(72)		1.1×10^{-8}
electron-neutron magnetic moment ratio	μ_e/μ_n	960.920 50(23)		2.4×10^{-7}
electron-deuteron magnetic moment ratio	μ_e/μ_d	−2143.923 4915(56)		2.6×10^{-9}
electron to shielded helion magnetic moment ratio (gas, sphere, 25 °C)	μ_e/μ'_h	864.058 257(10)		1.2×10^{-8}
electron gyromagnetic ratio $2\lvert\mu_e\rvert/\hbar$	γ_e	$1.760\,859\,630\,23(53) \times 10^{11}$	$s^{-1}\ T^{-1}$	3.0×10^{-10}
		28 024.951 4242(85)	$MHz\ T^{-1}$	3.0×10^{-10}
		Muon, μ^-		
muon mass	m_μ	$1.883\,531\,627(42) \times 10^{-28}$	kg	2.2×10^{-8}
		0.113 428 9259(25)	u	2.2×10^{-8}
energy equivalent	$m_\mu c^2$	$1.692\,833\,804(38) \times 10^{-11}$	J	2.2×10^{-8}
		105.658 3755(23)	MeV	2.2×10^{-8}
muon-electron mass ratio	m_μ/m_e	206.768 2830(46)		2.2×10^{-8}
muon-tau mass ratio	m_μ/m_τ	$5.946\,35(40) \times 10^{-2}$		6.8×10^{-5}
muon-proton mass ratio	m_μ/m_p	0.112 609 5264(25)		2.2×10^{-8}

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Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
muon-neutron mass ratio	m_μ/m_n	0.112 454 5170(25)		2.2×10^{-8}
muon molar mass $N_A m_\mu$	$M(\mu), M_\mu$	$1.134\,289\,259(25) \times 10^{-4}$	kg mol^{-1}	2.2×10^{-8}
reduced muon Compton wavelength $\hbar/m_\mu c$	$\lambda\!\!\!^{-}_{C,\mu}$	$1.867\,594\,306(42) \times 10^{-15}$	m	2.2×10^{-8}
muon Compton wavelength	$\lambda_{C,\mu}$	$1.173\,444\,110(26) \times 10^{-14}$	[m]†	2.2×10^{-8}
muon magnetic moment	μ_μ	$-4.490\,448\,30(10) \times 10^{-26}$	J T^{-1}	2.2×10^{-8}
to Bohr magneton ratio	μ_μ/μ_B	$-4.841\,970\,47(11) \times 10^{-3}$		2.2×10^{-8}
to nuclear magneton ratio	μ_μ/μ_N	$-8.890\,597\,03(20)$		2.2×10^{-8}
muon magnetic moment anomaly $\lvert\mu_\mu\rvert/(e\hbar/2m_\mu) - 1$	a_μ	$1.165\,920\,89(63) \times 10^{-3}$		5.4×10^{-7}
muon g-factor $-2(1 + a_\mu)$	g_μ	$-2.002\,331\,8418(13)$		6.3×10^{-10}
muon-proton magnetic moment ratio	μ_μ/μ_p	$-3.183\,345\,142(71)$		2.2×10^{-8}
Tau, τ^-				
tau mass¶	m_τ	$3.167\,54(21) \times 10^{-27}$	kg	6.8×10^{-5}
		1.907 54(13)	u	6.8×10^{-5}
energy equivalent	$m_\tau c^2$	$2.846\,84(19) \times 10^{-10}$	J	6.8×10^{-5}
		1776.86(12)	MeV	6.8×10^{-5}
tau-electron mass ratio	m_τ/m_e	3477.23(23)		6.8×10^{-5}
tau-muon mass ratio	m_τ/m_μ	16.8170(11)		6.8×10^{-5}
tau-proton mass ratio	m_τ/m_p	1.893 76(13)		6.8×10^{-5}
tau-neutron mass ratio	m_τ/m_n	1.891 15(13)		6.8×10^{-5}
tau molar mass $N_A m_\tau$	$M(\tau), M_\tau$	$1.907\,54(13) \times 10^{-3}$	kg mol^{-1}	6.8×10^{-5}
reduced tau Compton wavelength $\hbar/m_\tau c$	$\lambda\!\!\!^{-}_{C,\tau}$	$1.110\,538(75) \times 10^{-16}$	m	6.8×10^{-5}
tau Compton wavelength	$\lambda_{C,\tau}$	$6.977\,1(47) \times 10^{-16}$	[m]†	6.8×10^{-5}
Proton, p				
proton mass	m_p	$1.672\,621\,92[illegible](51) \times 10^{-27}$	kg	3.1×10^{-10}
		1.007 27[illegible]66 6[illegible]1(53)	u	5.3×10^{-11}
energy equivalent	$m_p c^2$	$1.503\,[illegible]77\,6[illegible]98(46) \times 10^{-10}$	J	3.1×10^{-10}
		938.272[illegible]88 16[illegible]	MeV	3.1×10^{-10}
proton-electron mass ratio	m_p/m_e	1836.152 67[illegible]43(11)		6.0×10^{-11}
proton-muon mass ratio	m_p/m_μ	8.880 243 3[illegible](20)		2.2×10^{-8}
proton-tau mass ratio	m_p/m_τ	0.528 051(36)		6.8×10^{-5}
proton-neutron mass ratio	m_p/m_n	0.998 623 478 12[illegible]9)		4.9×10^{-10}
proton charge to mass quotient	e/m_p	9.578 833 1560(29)[illegible]$\times 10^7$	C kg^{-1}	3.1×10^{-10}
proton molar mass $N_A m_p$	$M(\mathrm{p}), M_p$	1.007 276 466 27(31) $\times$ [illegible]$^{-3}$	kg mol^{-1}	3.1×10^{-10}
reduced proton Compton wavelength $\hbar/m_p c$	$\lambda\!\!\!^{-}_{C,p}$	2.103 089 103 36(64) $\times$ 10^{-}[illegible]	m	3.1×10^{-10}
proton Compton wavelength	$\lambda_{C,p}$	1.321 409 855 39(40) $\times$ 1[illegible]$^{-15}$	[illegible]	3.1×10^{-10}
proton rms charge radius	r_p	$8.414(19) \times 10^{-16}$	[illegible]	2.2×10^{-3}
proton magnetic moment	μ_p	1.410 606 797 36(60) $\times$ 10^{-}[illegible]	J T^{-1}	4.2×10^{-10}
to Bohr magneton ratio	μ_p/μ_B	$1.521\,032\,202\,30(46) \times 10^{-3}$		3.0×10^{-10}
to nuclear magneton ratio	μ_p/μ_N	2.792 847 344 63(82)		2.9×10^{-10}
proton g-factor $2\mu_p/\mu_N$	g_p	5.585 694 6893(16)		[illegible] $\times 10^{-10}$
proton-neutron magnetic moment ratio	μ_p/μ_n	$-1.459\,898\,05(34)$		2.4[illegible] $\times 10^{-7}$
shielded proton magnetic moment (H_2O, sphere, 25 °C)	μ'_p	$1.410\,570\,560(15) \times 10^{-26}$	J T^{-1}	1.1 $\times 10^{-8}$
to Bohr magneton ratio	μ'_p/μ_B	$1.520\,993\,128(17) \times 10^{-3}$		1.1×10^{-8}
to nuclear magneton ratio	μ'_p/μ_N	2.792 775 599(30)		1.1×10^{-8}
proton magnetic shielding correction $1 - \mu'_p/\mu_p$ (H_2O, sphere, 25 °C)	σ'_p	$2.5689(11) \times 10^{-5}$		4.2×10^{-4}

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Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
proton gyromagnetic ratio $2\mu_p/\hbar$	γ_p	$2.675\,221\,8744(11) \times 10^8$	$s^{-1}\ T^{-1}$	4.2×10^{-10}
		$42.577\,478\,518(18)$	MHz T^{-1}	4.2×10^{-10}
shielded proton gyromagnetic ratio $2\mu'_p/\hbar$ (H_2O, sphere, 25 °C)	γ'_p	$2.675\,153\,151(29) \times 10^8$	$s^{-1}\ T^{-1}$	1.1×10^{-8}
		$42.576\,384\,74(46)$	MHz T^{-1}	1.1×10^{-8}
Neutron, n				
neutron mass	m_n	$1.674\,927\,498\,04(95) \times 10^{-27}$	kg	5.7×10^{-10}
		$1.008\,664\,915\,95(49)$	u	4.8×10^{-10}
energy equivalent	$m_n c^2$	$1.505\,349\,762\,87(86) \times 10^{-10}$	J	5.7×10^{-10}
		$939.565\,420\,52(54)$	MeV	5.7×10^{-10}
neutron-electron mass ratio	m_n/m_e	$1838.683\,661\,73(89)$		4.8×10^{-10}
neutron-muon mass ratio	m_n/m_μ	$8.892\,484\,06(20)$		2.2×10^{-8}
neutron-tau mass ratio	m_n/m_τ	$0.528\,779(36)$		6.8×10^{-5}
neutron-proton mass ratio	m_n/m_p	$1.001\,378\,419\,31(49)$		4.9×10^{-10}
neutron-proton mass difference	$m_n - m_p$	$2.305\,574\,35(82) \times 10^{-30}$	kg	3.5×10^{-7}
		$1.388\,449\,33(49) \times 10^{-3}$	u	3.5×10^{-7}
energy equivalent	$(m_n - m_p)c^2$	$2.072\,146\,89(74) \times 10^{-13}$	J	3.5×10^{-7}
		$1.293\,332\,36(46)$	MeV	3.5×10^{-7}
neutron molar mass $N_A m_n$	$M(n), M_n$	$1.008\,664\,915\,60(57) \times 10^{-3}$	kg mol^{-1}	5.7×10^{-10}
reduced neutron Compton wavelength $\hbar/m_n c$	$\lambda\!\!\!^{-}_{C,n}$	$2.100\,194\,1552(12) \times 10^{-16}$	m	5.7×10^{-10}
neutron Compton wavelength	$\lambda_{C,n}$	$1.319\,590\,905\,81(75) \times 10^{-15}$	[m]†	5.7×10^{-10}
neutron magnetic moment	μ_n	$-9.662\,3651(23) \times 10^{-27}$	J T^{-1}	2.4×10^{-7}
to Bohr magneton ratio	μ_n/μ_B	$-1.041\,875\,63(25) \times 10^{-3}$		2.4×10^{-7}
to nuclear magneton ratio	μ_n/μ_N	$-0.913\,042\,73(45)$		2.4×10^{-7}
neutron g-factor $2\mu_n/\mu_N$	g_n	$-3.826\,085\,45(90)$		2.4×10^{-7}
neutron-electron magnetic moment ratio	μ_n/μ_e	$1.040\,668\,82(25) \times 10^{-3}$		2.4×10^{-7}
neutron-proton magnetic moment ratio	μ_n/μ_p	$-0.684\,979\,34(16)$		2.4×10^{-7}
neutron to shielded proton magnetic moment ratio (H_2O, sphere, 25 °C)	μ_n/μ'_p	$-0.684\,996\,94(16)$		2.4×10^{-7}
neutron gyromagnetic ratio $2\lvert\mu_n\rvert/\hbar$	γ_n	$1.832\,471\,71(43) \times 10^8$	$s^{-1}\ T^{-1}$	2.4×10^{-7}
		$29.164\,693\,(69)$	MHz T^{-1}	2.4×10^{-7}
Deuteron, d				
deuteron mass	m_d	$3.343\,583\,7724(\ldots) \times 10^{-27}$	kg	3.0×10^{-10}
		$2.013\,553\,212\,745(40)$	u	2.0×10^{-11}
energy equivalent	$m_d c^2$	$3.005\,063\,231\,02(91) \times 10^{-10}$	J	3.0×10^{-10}
		$1875.612\,942\,57(57)$	MeV	3.0×10^{-10}
deuteron-electron mass ratio	m_d/m_e	$3670.482\,967\,88(13)$		3.5×10^{-11}
deuteron-proton mass ratio	m_d/m_p	$1.999\,007\,501\,39(11)$		5.6×10^{-11}
deuteron molar mass $N_A m_d$	$M(d), M_d$	$2.013\,553\,212\,05(61) \times 10^{-3}$	kg mol^{-1}	3.0×10^{-10}
deuteron rms charge radius	r_d	$2.127\,99(74) \times 10^{-15}$	m	3.5×10^{-4}
deuteron magnetic moment	μ_d	$4.330\,735\,094(11) \times 10^{-27}$	J T^{-1}	2.6×10^{-9}
to Bohr magneton ratio	μ_d/μ_B	$4.669\,754\,570(12) \times 10^{-4}$		2.6×10^{-9}
to nuclear magneton ratio	μ_d/μ_N	$0.857\,438\,2338(22)$		2.6×10^{-9}
deuteron g-factor μ_d/μ_N	g_d	$0.857\,438\,2338(22)$		2.6×10^{-9}
deuteron-electron magnetic moment ratio	μ_d/μ_e	$-4.664\,345\,551(12) \times 10^{-4}$		2.6×10^{-9}
deuteron-proton magnetic moment ratio	μ_d/μ_p	$0.307\,012\,209\,39(79)$		2.6×10^{-9}
deuteron-neutron magnetic moment ratio	μ_d/μ_n	$-0.448\,206\,53(11)$		2.4×10^{-7}

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Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
		Triton, t		
triton mass	m_t	$5.007\,356\,7446(15) \times 10^{-27}$	kg	3.0×10^{-10}
		$3.015\,500\,716\,21(12)$	u	4.0×10^{-11}
energy equivalent	$m_t c^2$	$4.500\,387\,8060(14) \times 10^{-10}$	J	3.0×10^{-10}
		$2808.921\,132\,98(85)$	MeV	3.0×10^{-10}
triton-electron mass ratio	m_t/m_e	$5496.921\,535\,73(27)$		5.0×10^{-11}
triton-proton mass ratio	m_t/m_p	$2.993\,717\,034\,14(15)$		5.0×10^{-11}
triton molar mass $N_A m_t$	$M(t), M_t$	$3.015\,500\,715\,17(92) \times 10^{-3}$	kg mol^{-1}	3.0×10^{-10}
triton magnetic moment	μ_t	$1.504\,609\,5202(30) \times 10^{-26}$	J T^{-1}	2.0×10^{-9}
to Bohr magneton ratio	μ_t/μ_B	$1.622\,393\,6651(32) \times 10^{-3}$		2.0×10^{-9}
to nuclear magneton ratio	μ_t/μ_N	$2.978\,962\,4656(59)$		2.0×10^{-9}
triton g-factor $2\mu_t/\mu_N$	g_t	$5.957\,924\,931(12)$		2.0×10^{-9}
		Helion, h		
helion mass	m_h	$5.006\,412\,7796(15) \times 10^{-27}$	kg	3.0×10^{-10}
		$3.014\,932\,247\,175(97)$	u	3.2×10^{-11}
energy equivalent	$m_h c^2$	$4.499\,539\,4125(14) \times 10^{-10}$	J	3.0×10^{-10}
		$2808.391\,607\,43(85)$	MeV	3.0×10^{-10}
helion-electron mass ratio	m_h/m_e	$5495.885\,280\,07(24)$		4.3×10^{-11}
helion-proton mass ratio	m_h/m_p	$2.993\,152\,671\,67(13)$		4.4×10^{-11}
helion molar mass $N_A m_h$	$M(h), M_h$	$3.014\,932\,246\,13(91) \times 10^{-3}$	kg mol^{-1}	3.0×10^{-10}
helion magnetic moment	μ_h	$-1.074\,617\,532(13) \times 10^{-26}$	J T^{-1}	1.2×10^{-8}
to Bohr magneton ratio	μ_h/μ_B	$-1.158\,740\,958(14) \times 10^{-3}$		1.2×10^{-8}
to nuclear magneton ratio	μ_h/μ_N	$-2.127\,625\,307(25)$		1.2×10^{-8}
helion g-factor $2\mu_h/\mu_N$	g_h	$-4.255\,250\,615(50)$		1.2×10^{-8}
shielded helion magnetic moment (gas, sphere, 25 °C)	μ'_h	$-1.074\,553\,0\,0(13) \times 10^{-26}$	J T^{-1}	1.2×10^{-8}
to Bohr magneton ratio	μ'_h/μ_B	$-1.15\,\,671\,\,1(14) \times 10^{-3}$		1.2×10^{-8}
to nuclear magneton ratio	μ'_h/μ_N	$-2.127\,\,7\,719(25)$		1.2×10^{-8}
shielded helion to proton magnetic moment ratio (gas, sphere, 25 °C)	μ'_h/μ_p	$-0.761\,766\,\,618(\,\,)$		1.2×10^{-8}
shielded helion to shielded proton magnetic moment ratio (gas/H_2O, spheres, 25 °C)	μ'_h/μ'_p	$-0.761\,786\,1313(\,\,3)$		4.3×10^{-9}
shielded helion gyromagnetic ratio $2\lvert\mu'_h\rvert/\hbar$ (gas, sphere, 25 °C)	γ'_h	$2.037\,894\,569(24) \times 10^{8}$	$^{-1}$ T^{-1}	1.2×10^{-8}
		$32.434\,099\,42(38)$	MHz T^{-1}	1.2×10^{-8}
		Alpha particle, α		
alpha particle mass	m_α	$6.644\,657\,3357(20) \times 10^{-2}$	k	3.0×10^{-10}
		$4.001\,506\,179\,127(63)$	u	1.6×10^{-11}
energy equivalent	$m_\alpha c^2$	$5.971\,920\,1914(18) \times 10^{-10}$	J	3.0×10^{-10}
		$3727.379\,4066(11)$	MeV	3.0×10^{-10}
alpha particle to electron mass ratio	m_α/m_e	$7294.299\,541\,42(24)$		3.3×10^{-11}
alpha particle to proton mass ratio	m_α/m_p	$3.972\,599\,690\,09(22)$		5.5×10^{-11}
alpha particle molar mass $N_A m_\alpha$	$M(\alpha), M_\alpha$	$4.001\,506\,1777(12) \times 10^{-3}$	kg mol^{-1}	3.0×10^{-10}
		PHYSICOCHEMICAL		
Avogadro constant	N_A	$6.022\,140\,76 \times 10^{23}$	mol^{-1}	exact
Boltzmann constant	k	$1.380\,649 \times 10^{-23}$	J K^{-1}	exact
		$8.617\,333\,262\ldots \times 10^{-5}$	eV K^{-1}	exact
	k/h	$2.083\,661\,912\ldots \times 10^{10}$	Hz K^{-1}	exact

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Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
	k/hc	69.503 480 04 ...	$[m^{-1}\ K^{-1}]^{\dagger}$	exact
atomic mass constant$^{\|}$				
$m_u = \frac{1}{12}m(^{12}C) = 2h/cR_\infty\alpha^2 A_r(e)$	m_u	$1.660\,539\,066\,60(50) \times 10^{-27}$	kg	3.0×10^{-10}
energy equivalent	$m_u c^2$	$1.492\,418\,085\,60(45) \times 10^{-10}$	J	3.0×10^{-10}
		931.494 102 42(28)	MeV	3.0×10^{-10}
molar mass constant$^{\|}$	M_u	$0.999\,999\,999\,65(30) \times 10^{-3}$	kg mol^{-1}	3.0×10^{-10}
molar mass$^{\|}$ of carbon-12 $A_r(^{12}C)M_u$	$M(^{12}C)$	$11.999\,999\,9958(36) \times 10^{-3}$	kg mol^{-1}	3.0×10^{-10}
molar Planck constant	$N_A h$	$3.990\,312\,712 \ldots \times 10^{-10}$	J Hz^{-1} mol^{-1}	exact
molar gas constant $N_A k$	R	8.314 462 618 ...	J mol^{-1} K^{-1}	exact
Faraday constant $N_A e$	F	96 485.332 12 ...	C mol^{-1}	exact
standard-state pressure		100 000	Pa	exact
standard atmosphere		101 325	Pa	exact
molar volume of ideal gas RT/p				
$T = 273.15$ K, $p = 100$ kPa	V_m	$22.710\,954\,64 \ldots \times 10^{-3}$	m^3 mol^{-1}	exact
or standard-state pressure				
Loschmidt constant N_A/V_m	n_0	$2.651\,645\,804 \ldots \times 10^{25}$	m^{-3}	exact
molar volume of ideal gas RT/p				
$T = 273.15$ K, $p = 101.325$ kPa	V_m	$22.413\,969\,54 \ldots \times 10^{-3}$	m^3 mol^{-1}	exact
or standard atmosphere				
Loschmidt constant N_A/V_m	n_0	$2.686\,780\,111 \ldots \times 10^{25}$	m^{-3}	exact
Sackur-Tetrode (absolute entropy) constant**				
$\frac{5}{2} + \ln[(m_u k T_1/2\pi\hbar^2)^{3/2} k T_1/p_0]$				
$T_1 = 1$ K, $p_0 = 100$ kPa	S_0/R	$-1.151\,70\ldots\,37\,06(45)$		3.9×10^{-10}
or standard-state pressure				
$T_1 = 1$ K, $p_0 = 101.325$ kPa		$-1.\ldots\,8(45)$		3.9×10^{-10}
or standard atmosphere				
Stefan-Boltzmann constant				
$(\pi^2/60)k^4/\hbar^3 c^2$	σ	$5.670\,374\,41\ldots \times 10^{-8}$	W m^{-2} K^{-4}	exact
first radiation constant for spectral				
radiance $2hc^2\,\mathrm{sr}^{-1}$	c_{1L}	$1.191\,042\,972 \ldots \times 10^{-16}$	[W m^2 sr^{-1}]††	exact
first radiation constant $2\pi hc^2 = \pi\,\mathrm{sr}\,c_{1L}$	c_1	$3.741\,771\,852 \ldots \times 10^{-16}$	[W m^2]††	exact
second radiation constant hc/k	c_2	$1.438\,776\,877 \ldots \times 10^{-2}$	[m K]†	exact
Wien displacement law constants				
$b = \lambda_{max} T = c_2/4.965\,114\,231...$	b	$2.897\,771\,955 \ldots \times 10^{-3}$	[m K]†	exact
$b' = \nu_{max}/T = 2.821\,439\,372...\,c/c_2$	b'	$5.878\,925\,757 \ldots \times 10^{10}$	Hz K^{-1}	exact

* The energy of a photon with frequency ν expressed in unit Hz is $E = h\nu$ in J. Unitary time evolution of the state of this photon is given by $\exp(-iEt/\hbar)|\varphi\rangle$, where $|\varphi\rangle$ is the photon state at time $t = 0$ and time is expressed in unit s. The ratio $Et/\hbar$ is a phase.

† The full description of m^{-1} is cycles or periods per meter and that of m is meter per cycle (m/cycle). The scientific community is aware of the implied use of these units. It traces back to the conventions for phase and angle and the use of unit Hz versus cycles/s. No solution has been agreed upon.

‡ Value recommended by the Particle Data Group (Tanabashi, *et al.*, 2018).

§ Based on the ratio of the masses of the W and Z bosons m_W/m_Z recommended by the Particle Data Group (Tanabashi, *et al.*, 2018). The value for $\sin^2\theta_W$ they recommend, which is based on a variant of the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme, is $\sin^2\hat{\theta}_W(M_Z) = 0.231\,22(4)$.

¶ This and other constants involving m_τ are based on $m_\tau c^2$ in MeV recommended by the Particle Data Group (Tanabashi, *et al.*, 2018).

$^{\|}$ The relative atomic mass $A_r(X)$ of particle X with mass $m(X)$ is defined by $A_r(X) = m(X)/m_u$, where $m_u = m(^{12}C)/12 = 1$ u is the atomic mass constant and u is the unified atomic mass unit. Moreover, the mass of particle X is $m(X) = A_r(X)$ u and the molar mass of X is $M(X) = A_r(X)M_u$, where $M_u = N_A$ u is the molar mass constant and N_A is the Avogadro constant.

** The entropy of an ideal monoatomic gas of relative atomic mass A_r is given by $S = S_0 + \frac{3}{2}R\ln A_r - R\ln(p/p_0) + \frac{5}{2}R\ln(T/\mathrm{K})$.

†† The full description of m^2 is $\mathrm{m}^{-2} \times (\mathrm{m/cycle})^4$. See also footnote for m^{-1}.

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APPENDICES D

Periodic table of the Elements

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APPENDICES E

Atomic electronic structure of the elements

E Atomic electronic structure of the elements 2824

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5. Electronic Structure of the Elements

Table 5.1. Reviewed 2011 by J.E. Sansonetti (NIST). The electronic configurations and the ionization energies are from the NIST database, "Ground Levels and Ionization Energies for the Neutral Atoms," W.C. Martin, A. Musgrove, S. Kotochigova, and J.E. Sansonetti, `http://www.nist.gov/pml/data/ion_energy.cfm`. The electron configuration for, say, iron indicates an argon electronic core (see argon) plus six $3d$ electrons and two $4s$ electrons.

	Element		Electron configuration ($3d^5$ = five $3d$ electrons, *etc.*)		Ground state $^{2S+1}L_J$	Ionization energy (eV)
1	H	Hydrogen	$1s$		$^2S_{1/2}$	13.5984
2	He	Helium	$1s^2$		$^1S_0$	24.5874
3	Li	Lithium	(He) $2s$		$^2S_{1/2}$	5.3917
4	Be	Beryllium	(He) $2s^2$		$^1S_0$	9.3227
5	B	Boron	(He) $2s^2$ $2p$		$^2P_{1/2}$	8.2980
6	C	Carbon	(He) $2s^2$ $2p^2$		$^3P_0$	11.2603
7	N	Nitrogen	(He) $2s^2$ $2p^3$		$^4S_{3/2}$	14.5341
8	O	Oxygen	(He) $2s^2$ $2p^4$		$^3P_2$	13.6181
9	F	Fluorine	(He) $2s^2$ $2p^5$		$^2P_{3/2}$	17.4228
10	Ne	Neon	(He) $2s^2$ $2p^6$		$^1S_0$	21.5645
11	Na	Sodium	(Ne) $3s$		$^2S_{1/2}$	5.1391
12	Mg	Magnesium	(Ne) $3s^2$		$^1S_0$	7.6462
13	Al	Aluminum	(Ne) $3s^2$ $3p$		$^2P_{1/2}$	5.9858
14	Si	Silicon	(Ne) $3s^2$ $3p^2$		$^3P_0$	8.1517
15	P	Phosphorus	(Ne) $3s^2$ $3p^3$		$^4S_{3/2}$	10.4867
16	S	Sulfur	(Ne) $3s^2$ $3p^4$		$^3P_2$	10.3600
17	Cl	Chlorine	(Ne) $3s^2$ $3p^5$		$^2P_{3/2}$	12.9676
18	Ar	Argon	(Ne) $3s^2$ $3p^6$		$^1S_0$	15.7596
19	K	Potassium	(Ar) $4s$		$^2S_{1/2}$	4.3407
20	Ca	Calcium	(Ar) $4s^2$		$^1S_0$	6.1132
21	Sc	Scandium	(Ar) $3d$ $4s^2$	Transition elements	$^2D_{3/2}$	6.5615
22	Ti	Titanium	(Ar) $3d^2$ $4s^2$		$^3F_2$	6.8281
23	V	Vanadium	(Ar) $3d^3$ $4s^2$		$^4F_{3/2}$	6.7462
24	Cr	Chromium	(Ar) $3d^5$ $4s$		$^7S_3$	6.7665
25	Mn	Manganese	(Ar) $3d^5$ $4s^2$		$^6S_{5/2}$	7.4340
26	Fe	Iron	(Ar) $3d^6$ $4s^2$		$^5D_4$	7.9024
27	Co	Cobalt	(Ar) $3d^7$ $4s^2$		$^4F_{9/2}$	7.8810
28	Ni	Nickel	(Ar) $3d^8$ $4s^2$		$^3F_4$	7.6399
29	Cu	Copper	(Ar) $3d^{10}$ $4s$		$^2S_{1/2}$	7.7264
30	Zn	Zinc	(Ar) $3d^{10}$ $4s^2$		$^1S_0$	9.3942
31	Ga	Gallium	(Ar) $3d^{10}$ $4s^2$ $4p$		$^2P_{1/2}$	5.9993
32	Ge	Germanium	(Ar) $3d^{10}$ $4s^2$ $4p^2$		$^3P_0$	7.8994
33	As	Arsenic	(Ar) $3d^{10}$ $4s^2$ $4p^3$		$^4S_{3/2}$	9.7886
34	Se	Selenium	(Ar) $3d^{10}$ $4s^2$ $4p^4$		$^3P_2$	9.7524
35	Br	Bromine	(Ar) $3d^{10}$ $4s^2$ $4p^5$		$^2P_{3/2}$	11.8138
36	Kr	Krypton	(Ar) $3d^{10}$ $4s^2$ $4p^6$		$^1S_0$	13.9996
37	Rb	Rubidium	(Kr) $5s$		$^2S_{1/2}$	4.1771
38	Sr	Strontium	(Kr) $5s^2$		$^1S_0$	5.6949
39	Y	Yttrium	(Kr) $4d$ $5s^2$		$^2D_{3/2}$	6.2173
40	Zr	Zirconium	(Kr) $4d^2$ $5s^2$		$^3F_2$	6.8339
41	Nb	Niobium	(Kr) $4d^4$ $5s$		$^6D_{1/2}$	6.7589
42	Mo	Molybdenum	(Kr) $4d^5$ $5s$		$^7S_3$	7.0924
43	Tc	Technetium	(Kr) $4d^5$ $5s^2$		$^6S_{5/2}$	7.28
44	Ru	Ruthenium	(Kr) $4d^7$ $5s$		$^5F_5$	7.3605
45	Rh	Rhodium	(Kr) $4d^8$ $5s$		$^4F_{9/2}$	7.4589
46	Pd	Palladium	(Kr) $4d^{10}$		$^1S_0$	8.3369
47	Ag	Silver	(Kr) $4d^{10}$ $5s$		$^2S_{1/2}$	7.5762
48	Cd	Cadmium	(Kr) $4d^{10}$ $5s^2$		$^1S_0$	8.9938

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49	In	Indium	(Kr) $4d^{10}5s^2$	$5p$				$^2P_{1/2}$	5.7864
50	Sn	Tin	(Kr) $4d^{10}5s^2$	$5p^2$				$^3P_0$	7.3439
51	Sb	Antimony	(Kr) $4d^{10}5s^2$	$5p^3$				$^4S_{3/2}$	8.6084
52	Te	Tellurium	(Kr) $4d^{10}5s^2$	$5p^4$				$^3P_2$	9.0096
53	I	Iodine	(Kr) $4d^{10}5s^2$	$5p^5$				$^2P_{3/2}$	10.4513
54	Xe	Xenon	(Kr) $4d^{10}5s^2$	$5p^6$				$^1S_0$	12.1298
55	Cs	Cesium	(Xe)	$6s$				$^2S_{1/2}$	3.8939
56	Ba	Barium	(Xe)	$6s^2$				$^1S_0$	5.2117
57	La	Lanthanum	(Xe) $5d$	$6s^2$				$^2D_{3/2}$	5.5769
58	Ce	Cerium	(Xe) $4f$ $5d$	$6s^2$				$^1G_4$	5.5387
59	Pr	Praseodymium	(Xe) $4f^3$	$6s^2$				$^4I_{9/2}$	5.473
60	Nd	Neodymium	(Xe) $4f^4$	$6s^2$				$^5I_4$	5.5250
61	Pm	Promethium	(Xe) $4f^5$	$6s^2$				$^6H_{5/2}$	5.582
62	Sm	Samarium	(Xe) $4f^6$	$6s^2$				$^7F_0$	5.6437
63	Eu	Europium	(Xe) $4f^7$	$6s^2$				$^8S_{7/2}$	5.6704
64	Gd	Gadolinium	(Xe) $4f^7$ $5d$	$6s^2$				$^9D_2$	6.1498
65	Tb	Terbium	(Xe) $4f^9$	$6s^2$				$^6H_{15/2}$	5.8638
66	Dy	Dysprosium	(Xe) $4f^{10}$	$6s^2$				$^5I_8$	5.9389
67	Ho	Holmium	(Xe) $4f^{11}$	$6s^2$				$^4I_{15/2}$	6.0215
68	Er	Erbium	(Xe) $4f^{12}$	$6s^2$				$^3H_6$	6.1077
69	Tm	Thulium	(Xe) $4f^{13}$	$6s^2$				$^2F_{7/2}$	6.1843
70	Yb	Ytterbium	(Xe) $4f^{14}$	$6s^2$				$^1S_0$	6.2542
71	Lu	Lutetium	(Xe) $4f^{14}$ $5d$	$6s^2$				$^2D_{3/2}$	5.4259
72	Hf	Hafnium	(Xe) $4f^{14}5d^2$	$6s^2$				$^3F_2$	6.8251
73	Ta	Tantalum	(Xe) $4f^{14}5d^3$	$6s^2$				$^4F_{3/2}$	7.5496
74	W	Tungsten	(Xe) $4f^{14}5d^4$	$6s^2$				$^5D_0$	7.8640
75	Re	Rhenium	(Xe) $4f^{14}5d^5$	$6s^2$				$^6S_{5/2}$	7.8335
76	Os	Osmium	(Xe) $4f^{14}5d^6$	$6s^2$				$^5D_4$	8.4382
77	Ir	Iridium	(Xe) $4f^{14}5d^7$	$6s^2$				$^4F_{9/2}$	8.9670
78	Pt	Platinum	(Xe) $4f^{14}5d^9$	$6s$				$^3D_3$	8.9588
79	Au	Gold	(Xe) $4f^{14}5d^{10}$	$6s$				$^2S_{1/2}$	9.2255
80	Hg	Mercury	(Xe) $4f^{14}5d^{10}$	$6s^2$				$^1S_0$	10.4375
81	Tl	Thallium	(Xe) $4f^{14}5d^{10}6s^2$	$6p$				$^2P_{1/2}$	6.1082
82	Pb	Lead	(Xe) $4f^{14}5d^{10}6s^2$	$6p^2$				$^3P_0$	7.4167
83	Bi	Bismuth	(Xe) $4f^{14}5d^{10}6s^2$	$6p^3$				$^4S_{3/2}$	7.2855
84	Po	Polonium	(Xe) $4f^{14}5d^{10}6s^2$	$6p^4$				$^3P_2$	8.414
85	At	Astatine	(Xe) $4f^{14}5d^{10}6s^2$	$6p^5$				$^2P_{3/2}$	
86	Rn	Radon	(Xe) $4f^{14}5d^{10}6s^2$	$6p^6$				$^1S_0$	10.7485
87	Fr	Francium	(Rn)	$7s$				$^2S_{1/2}$	4.0727
88	Ra	Radium	(Rn)	$7s^2$				$^1S_0$	5.2784
89	Ac	Actinium	(Rn) $6d$	$7s^2$				$^2D_{3/2}$	5.3807
90	Th	Thorium	(Rn) $6d^2$	$7s^2$				$^3F_2$	6.3067
91	Pa	Protactinium	(Rn)$5f^2$ $6d$	$7s^2$				$^4K_{11/2}$	5.89
92	U	Uranium	(Rn)$5f^3$ $6d$	$7s^2$				$^5L_6$	6.1938
93	Np	Neptunium	(Rn)$5f^4$ $6d$	$7s^2$				$^6L_{11/2}$	6.2657
94	Pu	Plutonium	(Rn)$5f^6$	$7s^2$				$^7F_0$	6.0260
95	Am	Americium	(Rn)$5f^7$	$7s^2$				$^8S_{7/2}$	5.0738
96	Cm	Curium	(Rn)$5f^7$ $6d$	$7s^2$				$^9D_2$	6.0914
97	Bk	Berkelium	(Rn)$5f^9$	$7s^2$				$^6H_{15/2}$	6.1979
98	Cf	Californium	(Rn)$5f^{10}$	$7s^2$				$^5I_8$	6.2817
99	Es	Einsteinium	(Rn)$5f^{11}$	$7s^2$				$^4I_{15/2}$	6.3676
100	Fm	Fermium	(Rn)$5f^{12}$	$7s^2$				$^3H_6$	6.50
101	Md	Mendelevium	(Rn)$5f^{13}$	$7s^2$				$^2F_{7/2}$	6.58
102	No	Nobelium	(Rn)$5f^{14}$	$7s^2$				$^1S_0$	6.65
103	Lr	Lawrencium	(Rn)$5f^{14}$	$7s^2$	$7p$?			$^2P_{1/2}$?	4.9?
104	Rf	Rutherfordium	(Rn)$5f^{14}6d^2$	$7s^2$?				$^3F_2$?	6.0?

Lanthanides: La, Ce, Pr, Nd, Pm, Sm, Eu, Gd, Tb, Dy, Ho, Er, Tm, Yb, Lu

Transition elements: Hf, Ta, W, Re, Os, Ir, Pt, Au, Hg

Actinides: Ac, Th, Pa, U, Np, Pu, Am, Cm, Bk, Cf, Es, Fm, Md, No, Lr

* The usual *LS* coupling scheme does not apply for these three elements. See the introductory note to the NIST table from which this table is taken.

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APPENDICES F

Atomic and Nuclear properties of the materials

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6. Atomic and Nuclear Properties of Materials

Table 6.1 Abridged from `pdg.lbl.gov/AtomicNuclearProperties` by D.E. Groom (2017). See web pages for more detail about entries in this table and for several hundred others. Parentheses in the dE/dx and density columns indicate gases at 20° C and 1 atm. Boiling points are at 1 atm. Refractive indices n are evaluated at the sodium D line blend (589.2 nm); values $\gg$1 in brackets indicate $(n-1)\times10^6$ for gases at 0° C and 1 atm.

Material	Z	A	$\langle Z/A\rangle$	Nucl.coll. length λ_T {g cm^{-2}}	Nucl.inter. length λ_I {g cm^{-2}}	Rad.len. X_0 {g cm^{-2}}	$dE/dx\|_{\rm min}$ { MeV g^{-1}cm^2}	Density {g cm^{-3}} ({g ℓ^{-1}})	Melting point (K)	Boiling point (K)	Refract. index @ Na D
H_2	1	1.008[illegible]	0.99212	42.8	52.0	63.05	(4.103)	0.071(0.084)	13.81	20.28	1.11[132.]
D_2	1	2.[illegible]4101[illegible]64(8)	0.49650	51.3	71.8	125.97	(2.053)	0.169(0.168)	18.7	23.65	1.11[138.]
He	2	4.00[illegible](2)	0.49967	51.8	71.0	94.32	(1.937)	0.125(0.166)		4.220	1.02[35.0]
Li	3	6.9[illegible](2)	0.43221	52.2	71.3	82.78	1.639	0.534	453.6	1615.	
Be	4	9.[illegible]21831(5)	0.44384	55.3	77.8	65.19	1.595	1.848	1560.	2744.	
C diamond	6	12.01[illegible]	0.49955	59.2	85.8	42.70	1.725	3.520			2.419
C graphite	6	12.01[illegible]7(8)	0.49955	59.2	85.8	42.70	1.742	2.210	Sublimes at 4098. K		
N_2	7	14.007([illegible]	0.[illegible]9976	61.1	89.7	37.99	(1.825)	0.807(1.165)	63.15	77.29	1.20[298.]
O_2	8	15.999(3)	0.5[illegible]002	61.3	90.2	34.24	(1.801)	1.141(1.332)	54.36	90.20	1.22[271.]
F_2	9	18.9984031[illegible](6)	0.47372	65.0	97.4	32.93	(1.676)	1.507(1.580)	53.53	85.03	[195.]
Ne	10	20.1797(6)	0.4955[illegible]	65.7	99.0	28.93	(1.724)	1.204(0.839)	24.56	27.07	1.09[67.1]
Al	13	26.9815385(7)	0.481[illegible]	69.7	107.2	24.01	1.615	2.699	933.5	2792.	
Si	14	28.0855(3)	0.[illegible]848	70.2	108.4	21.82	1.664	2.329	1687.	3538.	3.95
Cl_2	17	35.453(2)	0.47951	73.8	115.7	19.28	(1.630)	1.574(2.980)	171.6	239.1	[773.]
Ar	18	39.948(1)	0.45059	75.7	119.7	19.55	(1.519)	1.396(1.662)	83.81	87.26	1.23[281.]
Ti	22	47.867(1)	0.45961	78.8	126.2	16.16	1.477	4.540	1941.	3560.	
Fe	26	55.845(2)	0.46557	81.7	132.1	13.84	1.451	7.874	1811.	3134.	
Cu	29	63.546(3)	0.45636	84.2	137.3	12.86	1.403	8.960	1358.	2835.	
Ge	32	72.630(1)	0.44053	86.[illegible]	143.0	12.25	1.370	5.323	1211.	3106.	
Sn	50	118.710(7)	0.42119	[illegible]8.2	166.7	8.82	1.263	7.310	505.1	2875.	
Xe	54	131.293(6)	0.41129	90.8	[illegible]2.1	8.48	(1.255)	2.953(5.483)	161.4	165.1	1.39[701.]
W	74	183.84(1)	0.40252	110.4	191.[illegible]	6.76	1.145	19.300	3695.	5828.	
Pt	78	195.084(9)	0.39983	112.[illegible]	19[illegible].7	6.54	1.128	21.450	2042.	4098.	
Au	79	196.966569(5)	0.40108	112.5	196.3	6.46	1.134	19.320	1337.	3129.	
Pb	82	207.2(1)	0.39575	114.1	199.[illegible]	6.37	1.122	11.350	600.6	2022.	
U	92	[238.02891(3)]	0.38651	118.6	209.0	6.00	1.081	18.950	1408.	4404.	
Air (dry, 1 atm)			0.49919	61.3	90.1	[illegible].2	(1.815)	(1.205)		78.80	[289]
Shielding concrete			0.50274	65.1	97.5	26.5	1.711	2.300			
Borosilicate glass (Pyrex)			0.49707	64.6	96.[illegible]	28.1[illegible]	1.696	2.230			
Lead glass			0.42101	95.9	158.0	[illegible].87	1.255	6.220			
Standard rock			0.50000	66.8	101.3	26.54	1.688	2.650			
Methane (CH_4)			0.62334	54.0	73.8	[illegible].47	(2.417)	(0.667)	90.68	111.7	[444.]
Ethane (C_2H_6)			0.59861	55.0	75.9	45.66	(2.304)	(1.263)	90.36	184.5	
Propane (C_3H_8)			0.58962	55.3	76.7	45.[illegible]7	[illegible].262)	0.493(1.868)	85.52	231.0	
Butane (C_4H_{10})			0.59497	55.5	77.1	45.23	(2.278)	(2.489)	134.9	272.6	
Octane (C_8H_{18})			0.57778	55.8	77.8	45.00	2.1[illegible]	0.703	214.4	398.8	
Paraffin ($CH_3(CH_2)_{n\approx23}CH_3$)			0.57275	56.0	78.3	44.85	[illegible].088	[illegible].930			
Nylon (type 6, 6/6)			0.54790	57.5	81.6	41.92	1.9[illegible]	1.[illegible]8			
Polycarbonate (Lexan)			0.52697	58.3	83.6	41.50	1.[illegible]86	1.20			
Polyethylene ($[CH_2CH_2]_n$)			0.57034	56.1	78.5	44.77	2.0[illegible]	0.8[illegible]			
Polyethylene terephthalate (Mylar)			0.52037	58.9	84.9	39.95	1.848	[illegible]40			
Polyimide film (Kapton)			0.51264	59.2	85.5	40.58	1.820	1.[illegible]			
Polymethylmethacrylate (acrylic)			0.53937	58.1	82.8	40.55	1.929	1.[illegible]9			1.49
Polypropylene			0.55998	56.1	78.5	44.77	2.041	[illegible].90			
Polystyrene ($[C_6H_5CHCH_2]_n$)			0.53768	57.5	81.7	43.79	1.936	1.[illegible]6			1.59
Polytetrafluoroethylene (Teflon)			0.47992	63.5	94.4	34.84	1.671	2.2[illegible]			
Polyvinyltoluene			0.54141	57.3	81.3	43.90	1.956	1.03			1.58
Aluminum oxide (sapphire)			0.49038	65.5	98.4	27.94	1.647	3.970	2327	[illegible]273.	1.77
Barium flouride (BaF_2)			0.42207	90.8	149.0	9.91	1.303	4.893	1641.	2533.	1.47
Bismuth germanate (BGO)			0.42065	96.2	159.1	7.97	1.251	7.130	[illegible]		2.15
Carbon dioxide gas (CO_2)			0.49989	60.7	88.9	36.20	1.819	(1.842)			[449.]
Solid carbon dioxide (dry ice)			0.49989	60.7	88.9	36.20	1.787	1.563	Sublimes at 194.7 K		
Cesium iodide (CsI)			0.41569	100.6	171.5	8.39	1.243	4.510	894.2	[illegible]53.	1.79
Lithium fluoride (LiF)			0.46262	61.0	88.7	39.26	1.614	2.635	1121.	1946.	1.39
Lithium hydride (LiH)			0.50321	50.8	68.1	79.62	1.897	0.820	965.		
Lead tungstate ($PbWO_4$)			0.41315	100.6	168.3	7.39	1.229	8.300	1403.		2.20
Silicon dioxide (SiO_2, fused quartz)			0.49930	65.2	97.8	27.05	1.699	2.200	1986.	3223.	1.46
Sodium chloride (NaCl)			0.47910	71.2	110.1	21.91	1.847	2.170	1075.	1738.	1.54
Sodium iodide (NaI)			0.42697	93.1	154.6	9.49	1.305	3.667	933.2	1577.	1.77
Water (H_2O)			0.55509	58.5	83.3	36.08	1.992	1.000	273.1	373.1	1.33
Silica aerogel			0.50093	65.0	97.3	27.25	1.740	0.200	(0.03 H_2O, 0.97 SiO_2)		

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Material	Dielectric constant ($\kappa = \epsilon/\epsilon_0$) () is ($\kappa$–1)×$10^6$ for gas	Young's modulus [10^6 psi]	Coeff. of thermal expansion [10^{-6}cm/cm-°C]	Specific heat [cal/g-°C]	Electrical resistivity [$\mu\Omega$cm(@°C)]	Thermal conductivity [cal/cm-°C-sec]
H_2	(253.9)	—	—	—	—	—
He	(64)	—	—	—	—	—
Li	—	—	56	0.86	8.55(0°)	0.17
Be	—	37	12.4	0.436	5.885(0°)	0.38
C	[illegible]	0.7	0.6–4.3	0.165	1375(0°)	0.057
N_2	(5[illegible].5)	—	—	—	—	—
O_2	(495)	—	—	—	—	—
Ne	(127)	—	—	—	—	—
Al	[illegible]	10	23.9	0.215	2.65(20°)	0.53
Si	1[illegible]	16	2.8–7.3	0.162	—	0.20
Ar	[illegible].7)	—	—	—	—	—
Ti	—	16.8	8.5	0.126	50(0°)	—
Fe	—	2[illegible].5	11.7	0.11	9.71(20°)	0.18
Cu	—	16	16.5	0.092	1.67(20°)	0.94
Ge	16.0	[illegible]	5.75	0.073	—	0.14
Sn	—	6	20	0.052	11.5(20°)	0.16
Xe	—	—	—	—	—	—
W	—	50	4.4	0.032	5.5(20°)	0.48
Pt	—	21	8.9	0.032	9.83(0°)	0.17
Pb	—	2.6	29.3	0.038	20.65(20°)	0.083
U	—	—	[illegible].1	0.028	29(20°)	0.064

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APPENDICES G

Table of isotopes

G Table of isotopes 2831

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Atomic Weights and Isotopic Compositions for All Elements

Column Descriptions

Isotope			Relative Atomic Mass	Isotopic Composition	Standard Atomic Weight	Notes
1	H	1	1.00782503223(9)	0.999885(70)	[1.00784,1.00811]	m
	D	2	2.01410177812(12)	0.000115(70)		
	T	3	3.0160492779(24)			
2	He	3	3.0160293201(25)	0.00000134(3)	4.002602(2)	g,r
		4	4.00260 5413(6)	0.99999866(3)		
3	Li	6	6. 51228 74(16)	0.0759(4)	[6.938,6.997]	m
		7	7.016003 66(45)	0.9241(4)		
4	Be	9	9.01218304 (82	1	9.0121831(5)	
5	B	10	10.012 369 (41)	0.199(7)	[10.806,10.821]	m
		11	11.00930 6(45)	0.801(7)		
6	C	12	12.0000000	0.9893(8)	[12.0096,12.0116]	
		13	13.00335483507 5)	0. 107(8)		
		14	14.003241988 .0)			
7	N	14	14.00307400443( )	0.99 8(20)	[14.00643,14.00728]	
		15	15.00010889888(64)	0.0036 20)		
8	O	16	15.99491461957(17)	0.9975 16	[15.99903,15.99977]	
		17	16.99913175650(69)	0. 038(1)		
		18	17.99915961286(76)	0. 0205(14)		
9	F	19	18.99840316273(92)	1	18.998403163(6)	
10	Ne	20	19.9924401762(17)	0.9048(3)	20.17 7(6)	g,m
		21	20.993846685(41)	0.0027(1)		
		22	21.991385114(18)	0.0925(3)		
11	Na	23	22.9897692820(19)	1	2.9897 8(2)	
12	Mg	24	23.985041697(14)	0.7899(4)	[2 30 ,24 307]	
		25	24.985836976(50)	0.1000(1)		
		26	25.982592968(31)	0.1101(3)		
13	Al	27	26.98153853(11)	1	26.9815385(7)	
14	Si	28	27.97692653465(44)	0.92223(19)	[28.084,28.086]	
		29	28.97649466490(52)	0.04685(8)		
		30	29.973770136(23)	0.03092(11)		
15	P	31	30.97376199842(70)	1	30.973761998(5)	
16	S	32	31.9720711744(14)	0.9499(26)	[32.059,32.076]	
		33	32.9714589098(15)	0.0075(2)		
		34	33.967867004(47)	0.0425(24)		
		36	35.96708071(20)	0.0001(1)		
17	Cl	35	34.968852682(37)	0.7576(10)	[35.446,35.457]	m
		37	36.965902602(55)	0.2424(10)		
18	Ar	36	35.967545105(28)	0.003336(21)	39.948(1)	g,r
		38	37.96273211(21)	0.000629(7)		
		40	39.9623831237(24)	0.996035(25)		
19	K	39	38.9637064864(49)	0.932581(44)	39.0983(1)	
		40	39.963998166(60)	0.000117(1)		
		41	40.9618252579(41)	0.067302(44)		
20	Ca	40	39.962590863(22)	0.96941(156)	40.078(4)	g
		42	41.95861783(16)	0.00647(23)		
		43	42.95876644(24)	0.00135(10)		
		44	43.95548156(35)	0.02086(110)		
		46	45.9536890(24)	0.00004(3)		
		48	47.95252276(13)	0.00187(21)		
21	Sc	45	44.95590828(77)	1	44.955908(5)	
22	Ti	46	45.95262772(35)	0.0825(3)	47.867(1)	
		47	46.95175879(38)	0.0744(2)		
		48	47.94794198(38)	0.7372(3)		
		49	48.94786568(39)	0.0541(2)		
		50	49.94478689(39)	0.0518(2)		
23	V	50	49.94715601(95)	0.00250(4)	50.9415(1)	
		51	50.94395704(94)	0.99750(4)		
24	Cr	50	49.94604183(94)	0.04345(13)	51.9961(6)	
		52	51.94050623(63)	0.83789(18)		
		53	52.94064815(62)	0.09501(17)		
		54	53.93887916(61)	0.02365(7)		

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25	Mn	55	54.93804391(48)	1	54.938044(3)	
26	Fe	54	53.93960899(53)	0.05845(35)	55.845(2)	
		56	55.93493633(49)	0.91754(36)		
		57	56.93539284(49)	0.02119(10)		
		58	57.93327443(53)	0.00282(4)		
27	Co	59	58.93319429(56)	1	58.933194(4)	
28	Ni	58	57.93534241(52)	0.68077(19)	58.6934(4)	r
		60	59.93078588(52)	0.26223(15)		
		61	60.93105557(52)	0.011399(13)		
		62	61.92834537(55)	0.036346(40)		
		64	63.9279[illegible]82(58)	0.009255(19)		
29	Cu	63	[illegible]29597[illegible]2(56)	0.6915(15)	63.546(3)	r
		65	[illegible].92778[illegible]0(71)	0.3085(15)		
30	Zn	64	63.929142[illegible](71)	0.4917(75)	65.38(2)	r
		66	65.92[illegible]38[illegible]94)	0.2773(98)		
		67	66.927[illegible]277[illegible]96)	0.0404(16)		
		68	67.92484[illegible]5(98)	0.1845(63)		
		70	69.9253192(21)	0.0061(10)		
31	Ga	69	68.9255735(13)	0.6[illegible]108(9)	69.723(1)	
		71	70.92470258([illegible]	0.[illegible]92(9)		
32	Ge	70	69.92424875(90)	0.[illegible]57[illegible]27)	72.630(8)	
		72	71.922075826(81)	0.2745([illegible]		
		73	72.923458956(61)	0.0775(12)		
		74	73.921177761(13)	0.3650[illegible]0)		
		76	75.921402726(19)	0.[illegible]5(12)		
33	As	75	74.92159457(95)	1	74.921595(6)	
34	Se	74	73.922475934(15)	0.0089(4)	78.971(8)	r
		76	75.919213704(17)	0.0937(29)		
		77	76.919914154(67)	0.0763(16)		
		78	77.91730928(20)	0.2377(28)		
		80	79.9165218(13)	0.4961(41)		
		82	81.9166995(15)	0.0873(22)		
35	Br	79	78.9183376(14)	0.5069(7)	[7[illegible]90[illegible],7[illegible].907]	
		81	80.9162897(14)	0.4931(7)		
36	Kr	78	77.92036494(76)	0.00355(3)	83.798(2)	g,m
		80	79.91637808(75)	0.02286(10)		
		82	81.91348273(94)	0.11593(31)		
		83	82.91412716(32)	0.11500(19)		
		84	83.9114977282(44)	0.56987(15)		
		86	85.9106106269(41)	0.17279(41)		
37	Rb	85	84.9117897379(54)	0.7217(2)	85.4678(3)	g
		87	86.9091805310(60)	0.2783(2)		
38	Sr	84	83.9134191(13)	0.0056(1)	87.62(1)	g,r
		86	85.9092606(12)	0.0986(1)		
		87	86.9088775(12)	0.0700(1)		
		88	87.9056125(12)	0.8258(1)		
39	Y	89	88.9058403(24)	1	88.90584(2)	
40	Zr	90	89.9046977(20)	0.5145(40)	91.224(2)	g
		91	90.9056396(20)	0.1122(5)		
		92	91.9050347(20)	0.1715(8)		
		94	93.9063108(20)	0.1738(28)		
		96	95.9082714(21)	0.0280(9)		
41	Nb	93	92.9063730(20)	1	92.90637(2)	
42	Mo	92	91.90680796(84)	0.1453(30)	95.95(1)	g
		94	93.90508490(48)	0.0915(9)		
		95	94.90583877(47)	0.1584(11)		
		96	95.90467612(47)	0.1667(15)		
		97	96.90601812(49)	0.0960(14)		
		98	97.90540482(49)	0.2439(37)		
		100	99.9074718(11)	0.0982(31)		
43	Tc	97	96.9063667(40)		[98]	
		98	97.9072124(36)			
		99	98.9062508(10)			
44	Ru	96	95.90759025(49)	0.0554(14)	101.07(2)	g
		98	97.9052868(69)	0.0187(3)		
		99	98.9059341(11)	0.1276(14)		
		100	99.9042143(11)	0.1260(7)		
		101	100.9055769(12)	0.1706(2)		
		102	101.9043441(12)	0.3155(14)		
		104	103.9054275(28)	0.1862(27)		
45	Rh	103	102.9054980(26)	1	102.90550(2)	

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46	Pd	102	101.9056022(28)	0.0102(1)	106.42(1)	g
		104	103.9040305(14)	0.1114(8)		
		105	104.9050796(12)	0.2233(8)		
		106	105.9034804(12)	0.2733(3)		
		108	107.9038916(12)	0.2646(9)		
		110	109.90517220(75)	0.1172(9)		
47	Ag	107	106.9050916(26)	0.51839(8)	107.8682(2)	g
		109	108.9047553(14)	0.48161(8)		
48	Cd	106	105.9064599(12)	0.0125(6)	112.414(4)	g
		108	107.9041834(12)	0.0089(3)		
		110	109.90300661(61)	0.1249(18)		
		111	110.90418287(61)	0.1280(12)		
		112	111.902[illegible]287(60)	0.2413(21)		
		113	112.9[illegible]440[illegible]13(45)	0.1222(12)		
		114	113.90336[illegible]09(43)	0.2873(42)		
		116	115.9047[illegible]515(17)	0.0749(18)		
49	In	113	112.90406[illegible]34(9[illegible])	0.0429(5)	114.818(1)	
		115	114.9[illegible]878764(2)	0.9571(5)		
50	Sn	112	111.90482[illegible]87(61)	0.0097(1)	118.710(7)	g
		114	113.90278[illegible]7(10)	0.0066(1)		
		115	114.903344[illegible]6)	0.0034(1)		
		116	115.9017428[illegible](1[illegible]	0.1[illegible]54(9)		
		117	116.90295398([illegible]2)	0.0[illegible]7(7)		
		118	117.90160657(54)	[illegible].2422[illegible]9)		
		119	118.90331117(78)	0.[illegible]51[illegible]4)		
		120	119.90220163(97)	0.3258([illegible]		
		122	121.9034438(26)	0.0463(3)		
		124	123.9052766(11)	0.0579[illegible])		
51	Sb	121	120.9038120(30)	0.5[illegible]21(5)	121.760(1)	g
		123	122.9042132(23)	0.4279(5)		
52	Te	120	119.9040593(33)	0.0009(1)	127.60(3)	g
		122	121.9030435(16)	0.0255(12)		
		123	122.9042698(16)	0.0089(3)		
		124	123.9028171(16)	0.0474(14)		
		125	124.9044299(16)	0.0707(15)		
		126	125.9033109(16)	0.1884(25)		
		128	127.90446128(93)	0.3174(8)		
		130	129.906222748(12)	0.3408(62)		
53	I	127	126.9044719(39)	1	126.90447([illegible]	
54	Xe	124	123.9058920(19)	0.000952(3)	131.293(6)	g,[illegible]
		126	125.9042983(38)	0.000890(2)		
		128	127.9035310(11)	0.019102(8)		
		129	128.9047808611(60)	0.264006(82)		
		130	129.903509349(10)	0.040710(13)		
		131	130.90508406(24)	0.212324(30)		
		132	131.9041550856(56)	0.269086(33)		
		134	133.90539466(90)	0.104357(21)		
		136	135.907214484(11)	0.088573(44)		
55	Cs	133	132.9054519610(80)	1	132.90545196(6)	
56	Ba	130	129.9063207(28)	0.00106(1)	137.327(7)	
		132	131.9050611(11)	0.00101(1)		
		134	133.90450818(30)	0.02417(18)		
		135	134.90568838(29)	0.06592(12)		
		136	135.90457573(29)	0.07854(24)		
		137	136.90582714(30)	0.11232(24)		
		138	137.90524700(31)	0.71698(42)		
57	La	138	137.9071149(37)	0.0008881(71)	138.90547(7)	g
		139	138.9063563(24)	0.9991119(71)		
58	Ce	136	135.90712921(41)	0.00185(2)	140.116(1)	g
		138	137.905991(11)	0.00251(2)		
		140	139.9054431(23)	0.88450(51)		
		142	141.9092504(29)	0.11114(51)		
59	Pr	141	140.9076576(23)	1	140.90766(2)	
60	Nd	142	141.9077290(20)	0.27152(40)	144.242(3)	g
		143	142.9098200(20)	0.12174(26)		
		144	143.9100930(20)	0.23798(19)		
		145	144.9125793(20)	0.08293(12)		
		146	145.9131226(20)	0.17189(32)		
		148	147.9168993(26)	0.05756(21)		
		150	149.9209022(18)	0.05638(28)		
61	Pm	145	144.9127559(33)		[145]	
		147	146.9151450(19)			
62	Sm	144	143.9120065(21)	0.0307(7)	150.36(2)	g
		147	146.9149044(19)	0.1499(18)		
		148	147.9148292(19)	0.1124(10)		
		149	148.9171921(18)	0.1382(7)		
		150	149.9172829(18)	0.0738(1)		

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		152	151.9197397(18)	0.2675(16)		
		154	153.9222169(20)	0.2275(29)		
63	Eu	151	150.9198578(18)	0.4781(6)	151.964(1)	g
		153	152.9212380(18)	0.5219(6)		
64	Gd	152	151.9197995(18)	0.0020(1)	157.25(3)	g
		154	153.9208741(17)	0.0218(3)		
		155	154.9226305(17)	0.1480(12)		
		156	155.9221312(17)	0.2047(9)		
		157	156.9239686(17)	0.1565(2)		
		158	157.9241123(17)	0.2484(7)		
		160	159.9270624(18)	0.2186(19)		
65	Tb	159	158.925[illegible]47(19)	1	158.92535(2)	
66	Dy	156	15[illegible].92428[illegible]7(17)	0.00056(3)	162.500(1)	g
		158	1[illegible]7.9244[illegible]09(21)	0.00095(3)		
		160	15[illegible].9[illegible]2046[illegible]	0.02329(18)		
		161	160.92694[illegible](20)	0.18889(42)		
		162	161.9[illegible]05[illegible](20)	0.25475(36)		
		163	162.92[illegible]738[illegible](20)	0.24896(42)		
		164	163.9291[illegible]9(20)	0.28260(54)		
67	Ho	165	164.930328[illegible]	1	164.93033(2)	
68	Er	162	161.9287884(2[illegible]	0.[illegible]139(5)	167.259(3)	g
		164	163.9292088(20)	[illegible].0160[illegible](3)		
		166	165.9302995(22)	0.[illegible]3350[illegible](36)		
		167	166.9320546(22)	0.22869[illegible]		
		168	167.9323767(22)	0.26978(1[illegible]		
		170	169.9354702(26)	0.1491[illegible]36)		
69	Tm	169	168.9342179(22)	1	168.93422(2)	
70	Yb	168	167.9338896(22)	0.00123(3)	173.054(5)	g
		170	169.9347664(22)	0.02982(39)		
		171	170.9363302(22)	0.1409(14)		
		172	171.9363859(22)	0.2168(13)		
		173	172.9382151(22)	0.16103(63)		
		174	173.9388664(22)	0.32026(80)		
		176	175.9425764(24)	0.12996(83)		
71	Lu	175	174.9407752(20)	0.97401(13)	174.967[illegible](1)	g
		176	175.9426897(20)	0.02599(13)		
72	Hf	174	173.9400461(28)	0.0016(1)	178.49(2)	
		176	175.9414076(22)	0.0526(7)		
		177	176.9432277(20)	0.1860(9)		
		178	177.9437058(20)	0.2728(7)		
		179	178.9458232(20)	0.1362(2)		
		180	179.9465570(20)	0.3508(16)		
73	Ta	180	179.9474648(24)	0.0001201(32)	180.94788(2)	
		181	180.9479958(20)	0.9998799(32)		
74	W	180	179.9467108(20)	0.0012(1)	183.84(1)	
		182	181.94820394(91)	0.2650(16)		
		183	182.95022275(90)	0.1431(4)		
		184	183.95093092(94)	0.3064(2)		
		186	185.9543628(17)	0.2843(19)		
75	Re	185	184.9529545(13)	0.3740(2)	186.207(1)	
		187	186.9557501(16)	0.6260(2)		
76	Os	184	183.9524885(14)	0.0002(1)	190.23(3)	g
		186	185.9538350(16)	0.0159(3)		
		187	186.9557474(16)	0.0196(2)		
		188	187.9558352(16)	0.1324(8)		
		189	188.9581442(17)	0.1615(5)		
		190	189.9584437(17)	0.2626(2)		
		192	191.9614770(29)	0.4078(19)		
77	Ir	191	190.9605893(21)	0.373(2)	192.217(3)	
		193	192.9629216(21)	0.627(2)		
78	Pt	190	189.9599297(63)	0.00012(2)	195.084(9)	
		192	191.9610387(32)	0.00782(24)		
		194	193.9626809(10)	0.3286(40)		
		195	194.9647917(10)	0.3378(24)		
		196	195.96495209(99)	0.2521(34)		
		198	197.9678949(23)	0.07356(130)		
79	Au	197	196.96656879(71)	1	196.966569(5)	
80	Hg	196	195.9658326(32)	0.0015(1)	200.592(3)	
		198	197.96676860(52)	0.0997(20)		
		199	198.96828064(46)	0.1687(22)		
		200	199.96832659(47)	0.2310(19)		
		201	200.97030284(69)	0.1318(9)		
		202	201.97064340(69)	0.2986(26)		
		204	203.97349398(53)	0.0687(15)		

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81	Tl	203	202.9723446(14)	0.2952(1)	[204.382,204.385]	
		205	204.9744278(14)	0.7048(1)		
82	Pb	204	203.9730440(13)	0.014(1)	207.2(1)	g,r
		206	205.9744657(13)	0.241(1)		
		207	206.9758973(13)	0.221(1)		
		208	207.9766525(13)	0.524(1)		
83	Bi	209	208.9803991(16)	1	208.98040(1)	
84	Po	209	208.9824308(20)		[209]	
		210	209.9828741(13)			
85	At	210	209.9871479(83)		[210]	
		211	210.987[illegible]66(30)			
86	Rn	211	21[illegible].99060[illegible]1(73)		[222]	
		220	[illegible]0.0113[illegible]1(23)			
		222	22[illegible]782[illegible]			
87	Fr	223	223.0[illegible]36[illegible]25		[223]	
88	Ra	223	223.0185[illegible]3(27)		[226]	
		224	224.0202[illegible]0(23)			
		226	226.025410[illegible]			
		228	228.0310707(26)			
89	Ac	227	227.0277523(25)		[227]	
90	Th	230	230.0331341(19)		232.0377(4)	g
		232	232.0380558(21)	1		
91	Pa	231	231.0358842(24)	1	231.03588(2)	
92	U	233	233.0396355(29)		238.02891(3)	g,m
		234	234.0409523(19)	0.000054(5)		
		235	235.0439301(19)	0.007204(6)		
		236	236.0455682(19)			
		238	238.0507884(20)	0.992742(10)		
93	Np	236	236.046570(54)		[237]	
		237	237.0481736(19)			
94	Pu	238	238.0495601(19)		[2[illegible]]	
		239	239.0521636(19)			
		240	240.0538138(19)			
		241	241.0568517(19)			
		242	242.0587428(20)			
		244	244.0642053(56)			
95	Am	241	241.0568293(19)			
		243	243.0613813(24)			
96	Cm	243	243.0613893(22)			
		244	244.0627528(19)			
		245	245.0654915(22)			
		246	246.0672238(22)			
		247	247.0703541(47)			
		248	248.0723499(56)			
97	Bk	247	247.0703073(59)			
		249	249.0749877(27)			
98	Cf	249	249.0748539(23)			
		250	250.0764062(22)			
		251	251.0795886(48)			
		252	252.0816272(56)			
99	Es	252	252.082980(54)			
100	Fm	257	257.0951061(69)			
101	Md	258	258.0984315(50)			
		260	260.10365(34#)			
102	No	259	259.10103(11#)			
103	Lr	262	262.10961(22#)			
104	Rf	267	267.12179(62#)			
105	Db	268	268.12567(57#)			
106	Sg	271	271.13393(63#)			
107	Bh	272	272.13826(58#)			
108	Hs	270	270.13429(27#)			
109	Mt	276	276.15159(59#)			
110	Ds	281	281.16451(59#)			

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111	Rg	280	280.16514(61#)
112	Cn	285	285.17712(60#)
113	Nh	284	284.17873(62#)
114	Fl	289	289.19042(60#)
115	Mc	288	288.19274(62#)
116	Lv	293	293.20449(60#)
117	Ts	292	292.20746(75#)
118	Og	294	294.213 (71#)

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APPENDICES H

Properties of Matter: data tables

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CHAPTER 2

Tables

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by Michael de Podesta

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Table 2.1 The properties of particles that are treated as fundamental in this book. The most important properties of the particles for understanding the properties of matter are the first two rows of the table: mass and electric charge. The internal angular momentum (spin) and magnetic moment of the particles are discussed in the text below.

Property	Units	Electron	Neutron	Proton
Mass	Atomic mass units u = 1.661×10^{-27} kg	5.485×10^{-4} ≈1/1836	1.0085	1.0071
Electric charge	Proton charge = 1.602×10^{-19} C	−1	0	+1
Magnetic moment	Bohr magneton $\mu_B = 9.274 \times 10^{-24}$ J T^{-1}	1.001	1.0419×10^{-3}	1.521×10^{-3}
Magnetic moment	Nuclear magneton $\mu_N = 5.051 \times 10^{-27}$ J T^{-1}	1837.8	1.913	2.793
Intrinsic (spin) angular momentum	Planck constant divided by 2π $\hbar = 1.054 \times 10^{-34}$ J s	½	½	½
Electric dipole moment	Cm	0	0	0
Lifetime		Stable	Stable within nuclei half life ≈ 15 minutes in free space.	Stable

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Table 2.2. The elements with atomic numbers up to 105 together with their date of discovery. The term 'Old' as a date of discovery indicates that the element was known in antiquity. The names of the elements tell many fascinating stories about their discovery.

Z	Element, symbol, date of discovery	Origin of name
1	Hydrogen, H, (1766)	Greek: *Hydros Genes*: meaning Water Forming
2	Helium, He, (1895)	Greek: *Helios* meaning Sun
3	Lithium, Li, (1817)	Greek: *Lithos* meaning Stone
4	Beryllium, Be, (1797)	Greek: *Beryllos* meaning Beryl
5	Boron, B, (1808)	Arabic: *Buraq*
6	Carbon, C, (Old)	Latin: *Carbo* meaning Charcoal
7	Nitrogen, N, (1772)	Greek: *Nitron Genes* meaning Nitre Forming
8	Oxygen, O, (1774)	Greek: *Oxy Genes* meaning Acid Forming
9	Fluorine, F, (1886)	Latin: *Fluere* meaning To Flow
10	Neon, Ne, (1898)	Greek: *Neos* meaning New
11	Sodium, Na, (1807)	English: Soda: The symbol comes from the Latin *Natrium*
12	Magnesium, Mg, (1755)	Greek: Magnesia, a district in Thessaly
13	Aluminium, Al, (1825)	Latin: *alumen* meaning alum
14	Silicon, Si, , (1824)	Latin: *Silicis* meaning flint
15	Phosphorus, P, (1669)	Greek: Phosphorus meaning Bringer of Light
16	Sulphur, S, (Old)	Sanskrit: *Sulvere* meaning Sulphur
17	Chlorine, Cl, (1774)	Greek: *Chloros* meaning Pale Green
18	Argon, Ar, (1894)	Greek: *Argos* meaning Inactive
19	Potassium, K, (1807)	English: Potash: The symbol comes from the Latin *Kalium*
20	Calcium, Ca, (1808)	Latin: *Calix* meaning Lime
21	Scandium, Sc, (1879)	Latin: *Scandia* meaning Scandinavia
22	Titanium, Ti, (1791)	*Titans*, Sons of the Earth Goddess.
23	Vanadium, V, (1801)	*Vanadis*, Scandinavian goddess
24	Chromium, Cr, (1780)	Greek: *Chroma* meaning Colour
25	Manganese, Mn, (1774)	Latin: *Magnes* meaning Magnet
26	Iron, Fe, (Old)	Saxon: Iron: The symbol comes from the Latin *Ferrum*
27	Cobalt, Co, (1735)	German: *kobald* meaning Goblin
28	Nickel, Ni, (1751)	German: *Kupfernickel* meaning either Devil's Copper or St Nicholas' Copper
29	Copper, Cu, (Old)	Latin: *Cuprum* meaning Cyprus
30	Zinc, Zn, (1400)	German: *Zink*
31	Gallium, Ga, (1875)	Latin: *Gallia* meaning France
32	Germanium, Ge, (1886)	Latin: *Germania* meaning German
33	Arsenic, As, (1280)	Greek: *Arsenikon* meaning Yellow Orpiment
34	Selenium, Se, (1817)	Greek: *Selene* meaning Moon
35	Bromine, Br, (1826)	Greek: *Bromos* meaning Stench
36	Krypton, Kr, (1898)	Greek: *Kryptos* meaning Hidden
37	Rubidium, Rb, (1861)	Latin: *Rubidius* meaning Deepest Red
38	Strontium, Sr, (1790)	English: *Strontian* in Scotland
39	Yttrium, Y, (1794)	The town of Ytterby in Sweden
40	Zirconium, Zr, (1789)	Arabic: *Zargun* meaning Gold Colour
41	Niobium, Nb, (1801)	Greek: *Niobe*, a daughter of Tantalus: Also called Columbium in USA
42	Molybdenum, Mo, (1781)	Greek: *Molybdos* meaning Lead
43	Technetium, Tc, (1937)	Greek: *Technikos* meaning Artificial
44	Ruthenium, Ru, (1808)	Latin: *Ruthenia* meaning Russia
45	Rhodium, Rh, (1803)	Greek: *Rhodon* meaning Rose
46	Palladium, Pd, (1803)	The asteroid *Pallas*
47	Silver, Ag, (Old)	Saxon: *Siolfur* meaning Silver: The symbol comes from the Latin *Argentum*
48	Cadmium, Cd, (1817)	Latin: *Cadmia* meaning *Calomine*
49	Indium, In, (1863)	Indigo
50	Tin, Sn, (Old)	Saxon: Tin: The symbol comes from the Latin *Stannum*
51	Antimony, Sb, (Old)	Greek: *Anti+Monos* meaning not alone. The symbol is from Latin *Stibium*
52	Tellurium, Te, (1782)	Latin: *Tellus* meaning Earth
53	Iodine, I, (1811)	Greek: *Iodes* meaning Violet
54	Xenon, Xe, (1898)	Greek: *Xenos* meaning Stranger
55	Caesium, Cs, (1860)	Latin: *Caesius* meaning Sky Blue
56	Barium, Ba, (1808)	Greek: *Barys* meaning Heavy
57	Lanthanum, La, (1839)	Greek: *Lanthanein* meaning To Lie Hidden
58	Cerium, Ce, (1803)	*Ceres*, an asteroid discovered in 1801
59	Praseodymium, Pr, (1885)	Greek: *Prasios Didymos* meaning Green Twin
60	Neodymium, Nd, (1885)	Greek: *Neos Didymos* meaning New Twin
61	Promethium, Pm, (1945)	Greek: *Prometheus*
62	Samarium, Sm, (1879)	The mineral samarskite
63	Europium, Eu, (1901)	Europe
64	Gadolinium, Gd, (1880)	J. Gadolin, a Finnish chemist
65	Terbium, Tb, (1843)	The town of Ytterby in Sweden
66	Dysprosium, Dy, (1886)	Greek: *Dysprositos* meaning Hard To Obtain

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Z	Element, symbol, date of discovery	Origin of name
67	Holmium, Ho, (1878)	Latin: *Holmia* meaning Stockholm
68	Erbium, Er, (1842)	The town of Ytterby in Sweden
69	Thulium, Tm, (1879)	*Thule*, meaning Ancient Scandinavia: The Uttermost North
70	Ytterbium, Yb, (1878)	The town of Ytterby in Sweden
71	Lutetium, Lu, (1907)	Latin: *Lutetia* meaning Paris
72	Hafnium, Hf, (1923)	Latin: *Hafnia* meaning Copenhagen
73	Tantalum, Ta, (1802)	Greek: *Tantalos*, the father of Niobe
74	Tungsten, W, (1783)	Swedish: *Tung Sten* meaning Heavy Stone: The symbol comes from the alternative name Wolfram
75	Rhenium, Re, (1925)	Latin: *Rhenus* meaning Rhine
76	Osmium, Os, (1803)	Greek: *Osme* meaning Smell
77	Iridium, Ir, (1803)	Latin: Iris meaning Rainbow
78	Platinum, Pt, (Old)	Spanish: *Platina* meaning Silver
79	Gold, Au, (Old)	Saxon: Gold
80	Mercury, Hg, (Old)	Latin: The planet Mercury: The symbol comes from the Latin *Hydragyrum* meaning Liquid Silver
81	Thallium, Tl, (1861)	Greek: *Thallos* meaning Green Twig
82	Lead, Pb, (Old)	Saxon: Lead: The symbol comes from the Latin *Plumbum*
83	Bismuth, Bi, (1450)	German: *Bisemutem*
84	Polonium, Po, (1898)	Poland
85	Astatine, At, (1940)	Greek: *Astatos* meaning unstable
86	Radon, Rn, (1900)	Radium
87	Francium, Fr, (1939)	France
88	Radium, Ra, (1898)	Latin: Radius meaning Ray
89	Actinium, Ac, (1899)	Greek: *aktinos* meaning Ray
90	Thorium, Th, (1815)	*Thor* The Scandinavian god of war
91	Protractinium, Pa, (1917)	Greek: *Protos* meaning First
92	Uranium, U, (1789)	The planet Uranus
93	Neptunium, Np, (1940)	The planet Neptune
94	Plutonium, Pu, (1940)	The planet Pluto
95	Americium, Am, (1944)	English: America
96	Curium, Cm, (1944)	Pierre and Marie Curie
97	Berkelium, Bk, (1949)	English: Berkeley
98	Californium, Cf, (1950)	English: California
99	Einsteinium, Es, (1952)	Albert Einstein
100	Fermium, Fm, (1952)	Enrico Fermi
101	Mendelevium, Md, (1955)	Dmitri Mendeleyev
102	Nobelium, No, (1958)	Alfred Nobel
103	Lawrencium, Lr, (1961)	Ernest O. Lawrence
104	Rutherfordium, Rf, (1964)	Ernest Rutherford
105	Dubnium, Db, (1967)	The town of Dubna, home to a centre for nuclear research

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Table 2.3 Photon energies, frequencies, and wavelengths.

Frequency (Hz)	**Wavelength** (m)	**Energy** (eV)	**Comment**
10^{6}	3×10^{2}	4.14×10^{-9}	Radio broadcasts
10^{7}	3×10^{1}	4.14×10^{-8}	
10^{8}	[illegible]	4.14×10^{-7}	Television broadcasts
10^{9}	3×10^{-1}	4.14×10^{-6}	A gigahertz: microwave ovens and mobile phones
10^{10}	3×10^{-2}	4.14×10^{-5}	Infra-red
10^{11}	3×10^{-3}	[illegible]	Infra-red
10^{12}	3×10^{-4}	[illegible] 10^{-3}	A terahertz: Infra-red: typical frequency of atomic vibration
6.6×10^{12}	4.55×10^{-4}	2.5×10^{-2}	Infra-red: corresponds to processes occurring at around room temperature (290K)
10^{13}	3×10^{-5}	4.14×10^{-2}	Infra-red
4×10^{14}	7.5×10^{-7}	1.654	Red light. Corresponds to processes involving electrons in the outer (valence) shells of atoms
10^{15}	3×10^{-7}	4.14	Blue light: corresponds to processes involving electrons in the outer (valence) shells of atoms
10^{16}	3×10^{-8}	4.14×10^{1}	Ultra-violet light
10^{17}	3×10^{-9}	4.14×10^{2}	Ultra-violet light
10^{18}	3×10^{-10}	4.14×10^{3}	Ultra-violet light
10^{19}	3×10^{-11}	4.14×10^{4}	X-rays
10^{20}	3×10^{-12}	4.14×10^{5}	X-rays: corresponds to processes involving electrons in the inner shells of atoms
10^{21}	3×10^{-13}	4.14×10^{6}	X-rays
10^{22}	3×10^{-14}	4.14×10^{7}	X-rays
10^{23}	3×10^{-15}	4.14×10^{8}	Gamma rays: Corresponds to processes that occur within nuclei

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Table 3.1 The SI base units. Notice that, with the exception of the kilogram, the definitions are in terms of physical phenomena and not defining artefacts. Although the definitions seem obscure, the language is carefully chosen in order to make accurate realisations of the standards feasible.

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Quantity: Unit (abbreviation)	Definition
Time: second (s)	The second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between two hyperfine levels of the ground state of the caesium-133 atom.
Length: metre (m)	The metre is the length of the path travelled by light in vacuum during a time interval 1/299,792,458 of a second. Note: This defines 299,792,458 ms^{-1} as the *exact* speed of light in a vacuum.
Mass: kilogram (kg)	The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram
Electric Current: ampere (A)	The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton, per metre of length.
Thermodynamic temperature: kelvin (K)	The kelvin, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of pure water.
Amount of substance: mole (mol)	The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12.
Luminous Intensity: candela (cd)	The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation 540×10^{12} hertz and that has a radiant intensity of 1/683 watt per steradian

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Quantity	Name	Symbol	Expression in terms of SI base units
plane angle	radian	rad	$m\ m^{-1} = 1$
solid angle	steradian	sr	$m^2\ m^{-2} = 1$

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Table 3.3 SI derived units with special names. The name of the units are all written with lower case letters (with the exception of degree Celsius), but that the symbols for the units have upper case letters: be careful to distinguish between seimens (S) and seconds (s). The symbol for the ohm, Ω, is the greek letter 'W', called omega.

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Quantity	Name	Symbol	Expression in terms of other units	Expression in terms of SI base units
frequency	ertz	Hz		s^{-1}
force	newton	N		m kg s^{-2}
pressure		Pa	N m^{-2}	m^{-1} kg s^{-2}
stress				
energy	oule	J	N m	m^2 kg s^{-2}
work				
quantity of heat				
power	watt	W	J s^{-1}	m^2 kg s^{-3}
radiant flux				
electric charge	coulomb	C		s A
quantity of electricity				
electrical potential	volt	V	W A^{-1}	m^2 kg s^{-3} A^{-1}
potential difference				
electromotive force				
capacitance	farad	F	C V^{-1}	m^2 kg $^{-1}$ s^4 A^{-1}
electric resistance	ohm	Ω	V A^{-1}	m^2 kg s^{-3} A^2
electric conductance	siemens	S	A V^{-1}	m^2 kg $^{-1}$ s^3 A^{-1}
magnetic flux	weber	Wb	V	m^2 kg s^{-2} A^{-1}
magnetic flux density	tesla	T	Wb m^{-2}	kg s^{-2} A^{-1}
inductance	henry	H	Wb	m^2 kg s^{-2} A^{-2}
Celsius temperature	degree celsius	°C		K
luminous flux	lumen	lm		cd sr
illuminance	lux	lx	lm m^{-2}	m^{-2} cd sr

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Table 5.1 **The density of various gases at STP in units of kg m^{-3}. The lines in the table separate gases of monatomic, diatomic and polyatomic molecules.**

Gas	A (u)	Density (kg m^{-3})
Helium, He	4.0030	0.1786
Neon, Ne	20.180	0.9003
Argon, Ar	39.948	1.782
Krypton, Kr	83.800	3.739
Xenon, Xe	131.29	5.858
Hydrogen, H_2	2.0160	0.08995
Nitrogen, N_2	28.014	1.250
Oxygen, O_2	31.998	1.428
Chlorine, Cl_2	70.906	3.164
Methane, CH_4	16.043	0.7158
Ethane, C_2H_6	30.070	1.342
Propane, C_3H_8	44.097	1.968

Table 5.2 **The major components of *dry* atmospheric air. Typically water vapour is also present at a level of roughly 0.5%.**

Gas	Molecular mass	% by volume
Nitrogen, N_2	28.01	78.09
Oxygen, O_2	32.00	20.95
Argon, Ar	39.95	0.93
Carbon dioxide, CO_2	44.01	0.03

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Table 5.3 **The molar volume of various gases at STP in units of 10^{-3} m^3. The lines in the table separate gases of monatomic, diatomic and polyatomic molecules.**

Gas	Molar density (m^{-3})	Mass of 1 mol ($\times 10^{-3}$ kg)	Molar volume ($\times 10^{-3}$ m^3)
Helium, He	44.6158	4.0030	22.4136
Neon, Ne	44.6152	20.180	22.4139
Argon, Ar	44.6162	39.948	22.4134
Krypton, Kr	44.6168	83.800	22.4131
Xenon, Xe	44.6174	131.29	22.4128
Hydrogen, H_2	44.6160	2.0160	22.4135
Nitrogen, N_2	44.6168	28.014	22.4131
Oxygen, O_2	44.6162	31.998	22.4134
Chlorine, Cl_2	44.6172	70.906	22.4129
Methane, CH_4	44.6170	16.043	22.4130
Ethane, C_2H_6	44.6178	30.070	22.4126
Propane, C_3H_8	44.6182	44.097	22.4124

Table 5.4 **Values of the expansivity coefficients β_V and β_P for gases whose initial pressure is 0.1333 MPa at 0 °C, valid in the temperature range 0 °C to 100 °C. The pressure 0.1333 MPa is a little greater than normal atmospheric pressure.**

Gas	β_V (°C^{-1})	β_P (°C^{-1})
Helium, He	3.6580×10^{-3}	3.6605×10^{-3}
Hydrogen, H_2	3.6588×10^{-3}	3.6613×10^{-3}
Nitrogen, N_2	3.6735×10^{-3}	3.6760×10^{-3}
Air	3.6728×10^{-3}	3.6753×10^{-3}
Neon, Ne	3.6600×10^{-3}	3.6625×10^{-3}

Table 5.5 **Comparison of experimental and theoretical expansivities of gases. See also Table 5.4.**

Gas	β_V (°C^{-1})	% difference between theory and experiment
Helium, He	3.6580×10^{-3}	– 0.082
Hydrogen, H_2	3.6588×10^{-3}	– 0.060
Nitrogen, N_2	3.6735×10^{-3}	+ 0.342
Air	3.6728×10^{-3}	+ 0.323
Neon, Ne	3.6600×10^{-3}	– 0.027

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Table 5.6 The molar heat capacities at constant pressure C_P(J K^{-1} mol^{-1}) for the monatomic noble gases. These data are graphed in Figure 5.3.

- The shaded figures correspond to data taken in the liquid or solid phase. For each gas the boiling temperature and melting temperature are separated by less than 5 K.
- The data between the two double lines is from a separate source from the rest of the table. Notice that the extra measurement resolution still shows agreement between the heat capacities of the different gases.

T(K)	He	Ne	Ar	Kr	Xe
50	—	—	24.8	25.1	25.1
100	—	—	20.8	31.6	28.2
150	—	—	20.8	20.8	33.6
200	—	—	20.8	20.8	20.8
298.15	20.786	20.786	20.786	20.786	20.786
400	20.8	20.8	20.8	20.8	20.8
600	20.8	20.8	20.8	20.8	20.8
800	20.8	20.8	20.8	20.8	20.8
1000	20.8	20.8	20.8	20.8	20.8
1500	20.8	20.8	20.8	20.8	20.8
2000	20.8	20.8	20.8	20.8	20.8
2500	20.8	20.8	20.8	20.8	20.8

Table 5.7 The molar heat capacities at constant pressure C_P(J K–1 mol–1) for some diatomic gases. These data are graphed in Figure 5.4.

- The shaded figures correspond to data taken in the liquid or solid phase.

T(K)	H_2	0_2	N_2	F_2	Cl_2	Br_2	I_2
50	—	46.1	41.5	—	29.2	33.6	35.2
100	—	29.1	29.1	—	42.3	43.6	45.6
150	—	29.1	29.1	—	51.0	49.2	4 6
200	—	29.1	29.1	—	54.2	53.8	51.5
400	29.2	30.1	29.2	33.0	35.3	36.7	8 3
600	29.3	32.1	30.1	35.2	36.6	37.3	37.6
800	29.6	33.7	31.4	36.3	37.2	37.5	37.8
1000	30.2	34.9	32.7	37.0	37.5	37.7	37.9
1500	32.3	36.6	34.9	37.9	38.0	38.0	38.2
2000	34.3	37.8	36.0	38.4	38.3	38.2	38.5
2500	36.0	38.9	36.0	38.8	38.6	38.5	38.8

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Table 5.8 The ratio of the principal heat capacities ($\gamma = C_P/C_V$) of some gases. The shaded results correspond to a pressure of 200 atmospheres (20 MPa). The notes below each section of the table summarise the results for that class of gases. There appears to be a trend towards a reduction in γ as the temperature is increased. Where no temperature shown, the temperature of the measurement is not known but is probably either 0 °C or close to 20 °C.

Gas	T(°C)	T(K)	γ
Monatomic gases			
He	0.0	273.20	1.630
Ar	0.0	273.20	1.667
Ne	19.0	292.20	1.642
Kr	19.0	292.20	1.689
Xe	19.0	292.20	1.666
Hg	310.0	583.20	1.666
All the above results are close to 1.66			
Diatomic gases			
H_2	10.0	283.20	1.407
N_2	20.0	293.20	1.401
O_2	10.0	283.20	1.400
CO	1800.0	2073.2	1.297
NO	—	—	1.394
Most of the above results are close to 1.4			
C: Air			
Air	-79.3	193.90	1.405
Air	10.0	283.20	1.401
Air	500.0	773.20	1.357
Air	900.0	1173.2	1.320
Air	0.0	273.20	1.828
Air	-79.3	193.90	2.333
Most of the above results are close to 1.4 except for those shaded.			

Gas	T(°C)	T(K)	γ
Triatomic gases			
O_3	—	—	1.290
H_20	100.0	373.20	1.334
CO_2	10.0	283.20	1.300
CO_2	300.0	573.20	1.220
CO_2	500.0	773.20	1.200
NH_3	—	—	1.336
N_20	—	—	1.324
H_2S	—	—	1.340
CS_2	—	—	1.239
SO_2	20.0	293.20	1.260
SO_2	500.0	773.20	1.200
All the above results are close to 1.3			
Polyatomic gases			
CH_4	—	—	1.313
C_2H_6	—	—	1.220
C_3H_8	—	—	1.130
C_2H_2	—	—	1.260
C_2H_4	—	—	1.264
C_6H_6	20.0	293.20	1.400
C_6H_6	99.7	372.90	1.105
CHCl	30.0	303.20	1.110
$CHCl_3$	99.8	373.00	1.150
CCl_4	—	—	1.130
The above results are between 1.1 and 1.4			

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Table 5.9 The number of degrees of freedom p for the molecules of a variety of gases predicted from the measured value of γ according to Equation 5.48. Values are plotted only for those gases included in Table 5.8.

Gas	γ	T(K)	p	Gas	γ	T(K)	p
A: Some monatomic gases				**D: Some triatomic gases**			
He	1.630	273.20	3.17	O_3	1.290	—	6.90
Ar	[illegible]	273.20	3.00	H_20	1.334	373.20	5.99
Ne	1.64[illegible]	292.20	3.12	CO_2	1.300	283.20	6.67
Kr	1.[illegible]9	[illegible]2.20	2.90	CO_2	1.220	573.20	9.09
Xe	1.66[illegible]	292.20	3.00	CO_2	1.200	773.20	10.00
Hg	1.666	583[illegible]0	3.00	NH_3	1.336	—	5.95
				N_20	1.324	—	6.17
B: Some diatomic gases				H_2S	1.340	—	5.88
H_2	1.407	283.20	4.91	CS_2	1.239	—	8.37
N_2	1.401	293.20	4.99	SO_2	1.260	293.20	7.69
O_2	1.400	283[illegible]0	5.00	SO_2	1.200	773.20	10.00
CO	1.297	2073.2	6.73				
NO	1.394	—	5.08	**E: Some polyatomic gases**			
				CH_4	1.313	—	6.39
C: Air				C_2H_6	1.220	—	9.09
Air	1.405	193.90	4.94	C_3H_8	1.130	—	15.4
Air	1.401	283.20	4.99	C_2H_2	1.260	—	7.69
Air	1.357	773.20	[illegible].60	C_2H_4	1.264	—	7.58
Air	1.320	1173.2	6.[illegible]	C_6H_6	1.400	293.20	5.00
Air	1.828	273.20	2.42	C_6H_6	1.105	372.90	19.0
Air	2.333	193.90	1.50	$CHCl_3$	1.110	303.20	18.2
				$CHCl_3$	1.150	373.00	13.3
				[illegible]	1.130	—	15.4

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Table 5.11 Measured values of the thermal conductivities of some gases. The units are 10^{-2} W m^{-1} K^{-1}. For example, the thermal conductivity of argon at 273.2 K is 1.63×10^{-2} W m^{-1} K^{-1}.

	Temperature (K)				
Gas	**73.2**	**173.2**	**273.2**	**373.2**	**1273**
Monatomic gases					
Helium, He	5.95	10.45	14.22	17.77	41.90
Neon, Ne	1.74	3.37	4.65	5.66	12.80
Argon, Ar	—	1.09	1.63	2.12	5.00
Krypton, Kr	—	0.57	0.87	1.15	2.90
Xenon, Xe	—	0.34	0.52	0.70	1.90
Radon, Ra	—	—	0.33	0.45	—
Diatomic gases					
Hydrogen, H_2	5.09	11.24	16.82	21.18	—
Fluorine, F_2	—	1.56	2.54	3.47	—
Chlorine, Cl_2	—	—	0.79	1.15	—
Bromine, Br_2	—	—	0.40	0.60	—
Nitrogen, N_2	—	1.59	2.40	3.09	7.40
Oxygen, O_2	—	1.59	2.45	3.23	8.60
Carbon monoxide, CO	—	1.51	2.32	3.04	—
Air, N_2/O_2	—	1.58	2.41	3.17	7.60
Polyatomic gases					
Ammonia, NH_4	-	—	2.18	3.38	—
Carbon dioxide, CO_2	-	—	1.45	2.23	7.90
Ethane, C_2H_6	—	1.80	—	—	—
Ethene, C_2H_4	—	1.40	—	—	—
Methane, CH_4	—	1.88	3.02	—	—
Sulpur dioxide, SO_2	—	—	0.77	—	—
Water/Steam, H_20	—	—	1.59	2.35	—

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Table 5.12 Calculated and experimental values for κ for argon at various temperatures. Also shown is the inferred value(a) for the molecular diameter.

T (K)	Data (W m^{-1} K^{-1})	Prediction (W m^{-1} K^{-1})	Ratio	a (nm)
173.2	1.09×10^{-2}	5.23×10^{-3}	2.08	0.21
273.2	1.63×10^{-2}	6.57×10^{-3}	2.48	0.19
373.2	2.12×10^{-2}	7.68×10^{-3}	2.76	0.18
273	5.00×10^{-2}	14.19×10^{-3}	3.52	0.16

Table 5.13 Results from an analysis of the thermal conductivity data assuming the data has the form $\kappa = AT^{x}$. The significance of a is discussed in the text.

Gas	A	x	a (nm)
Helium, He	30.91×10^{-4}	0.685	0.108
Neon, Ne	9.14×10^{-4}	0.695	0.198
Argon, Ar	2.34×10^{-4}	0.754	0.391
Krypton, Kr	0.93×10^{-4}	0.806	0.620
Xenon, Xe	0.42×10^{-4}	0.857	0.923
Radon, Ra	0.12×10^{-4}	0.[illegible]	1.73

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Table 5.14 The speed of sound in a selection of gases listed in order of increasing molecular mass M in atomic mass units u. The shaded entries in the table are gases that have a 'partner' gas in the table with the same molecular mass. See the text for more details.

Gas	M	T(K)	c_{sound} (ms^{-1})
Hydrogen, H_2	2.0	273.2	1286
Helium, He	4.0	273.2	971.9
Deuterium, D_2	4.0	273.2	890
Methane, CH_4	16.0	273.2	430
Ammonia, NH_3	18.0	273.2	415
Water (steam), H_2O	18.0	373.2	473
Water (steam), H_2O	18.0	407.2	494
Fluorine, F_2	19.0	373.2	332
Heavy Water (steam), D_2O	20.0	373.2	451
Neon, Ne	20.2	273.2	434
Acetylene, C_2H_2	26.0	273.2	329
Nitrogen, N_2	28.0	273.2	337
Carbon monoxide, CO	28.0	273.2	337
Ethylene, C_2H_4	28.0	273.2	318
Ethane, C_2H_6	30.0	283.2	308
Ethane, C_2H_6	30.0	304.2	316
Nitric oxide, NO	30.0	283.2	324
Nitric oxide, NO	30.0	289.2	334
Oxygen, O_2	32.0	303.2	332
Methanol, CH_3OH	32.0	370.2	335
Hydrogen sulphide, H_2S	34.1	273.2	310
Hydrogen chloride, HCl	36.5	273.2	296
Argon, Ar	40.0	273.2	307.8
Nitrous oxide, N_2O	44.0	308.2	268
Propane, C_3H_8	44.0	273.2	238
Carbon dioxide, CO_2	44.0	273.2	259
Ethanol, C_2H_5OH	46.0	320.2	238
Sulphur dioxide, SO_2	64.0	273.2	211
Chlorine, Cl_2	70.9	293.2	209
Carbon disulphide, CS_2	76.0	273.2	192
Benzene, C_6H_6	78.0	273.2	177
Bromine, Br_2	79.9	331.2	149
Hydrogen bromide, HBr	80.9	273.2	200
Krypton, Kr	83.8	273.2	213
Cyclohexane, C_6H_{12}	84.0	303.2	181
Hydrogen iodide, HI	127.9	273.2	157
Xenon, Xe	131.3	273.2	170
Sulphur hexafluoride, SF_6	146.0	284.2	133
Carbon tetrachloride, CCl_4	153.8	370.2	145
Iodine, I_2	263.8	453.2	138

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Table 5.15 Details of gases whose molecules have relative molecular mass of 4 and 28. The table enables a detailed comparison of theoretical expectations and experimental results for the dependence of the speed of sound upon molecular complexity.

Gas	M	Number of atoms per molecule	Expected γ	T(K)	c_{sound} (theoretical) $\sqrt{(\gamma RT/M)}$	c_{sound} (experimental) Table 5.14
He	4.0	1	1.667 (p=3)	273.2	972.8	971.9
D_2	4.0	2	1.400 (p=5)	273.2	891.5	890.0
N_2	28	2	1.400 (p=5)	273.2	336.9	337.0
CO	28	2	1.400 (p=5)	273.2	336.9	337.0
CH_2CH_2	28	6	1.2 (p=10?)	273.2	≈312	318.0

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Table 5.16 The relative dielectric permittivity ε of various gases at atmospheric pressure (1.013×10^5 Pa). For pressures below atmospheric pressure ε varies linearly with pressure. The relative permittivity of vacuum is exactly 1, and all the gases in the table have values of ε within 1% of unity. The table shows the value of $10^4\,(\varepsilon - 1)$, which clearly shows the variation between gases. The table also shows the relative molecular mass of the molecules of the gas.

Different experimenters find different values of ε and the data for $10^4\,(\varepsilon - 1)$ should all be treated as accurate to only about 10%. The entry for ethanol has two alternative values to indicate two particularly divergent values for $10^4\,(\varepsilon - 1)$. For other entries I have taken averages of tabulated results, or ignored entries in tables that were clearly in error.

The data refer to values obtained with electric fields oscillating at radio frequencies, $\approx 10^6$ Hz. The shaded entries in the table, i.e. helium, hydrogen, argon, oxygen, nitrogen, and air, are typical results for ε valid from DC up to optical frequencies $\approx 10^{15}$ Hz. The variation over that range is within ±2 of the least significant figure in the table.

Gas	*M*	*T* (°C)	$10^4(\varepsilon - 1)$
Monatomic gases			
Helium, He	4.0	20	0.65
Neon, Ne	20.2	0	1.3
Argon, Ar	40.0	20	5.16
Mercury, Hg	200.6	180	7.4
Mercury, Hg	200.6	180	7.4
Diatomic gases			
Hydrogen, H_2	2.0	0	2.72
Hydrogen, H_2	2.0	20	2.54
Nitrogen, N_2	28.0	20	5.47
Oxygen, O_2	32.0	20	4.94
Air (dry, no CO_2)	28.8	20	5.36
Carbon monoxide, CO	28.0	25	6.92
Triatomic gases			
Carbon dioxide, CO_2	44.0	0	9.88
Carbon dioxide, CO_2	44.0	25	9.22
Carbon dioxide, CO_2	44.0	100	7.23
Nitrous oxide, N_2O	44.0	25	11
Water (steam) H_2O	18.0	100	60
Polyatomic gases			
Ethane, C_2H_6	30.0	0	15
Benzene, C_6H_6	65.0	100	32.7
Methanol, CH_3OH	32.0	100	33
Ethanol, C_2H_5OH	44.0	100	61 or 46
Ammonia, NH_3	18.0	0	8.34
Ammonia, NH_3	18.0	100	4.87

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Table 5.18 The refractive index of various gases as $10^6(n_{light}-1)$ together with the molecular weight of the molecules of the gas. The data refer to gases at STP (P = 0.1013 MPa: T = 0 °C). The refractive index is that appropriate to the bright yellow 'D' lines in the emission spectrum of sodium vapour and varies slightly with frequency.

Gas	*M*	$(n_{light}-1) \times 10^6$
Hydrogen, H_2	2	132
Helium, He	4	36
Methane, CH_4	18	444
Water vapour, H_2O	18	254
Ammonia, NH_4	18	376
Neon, Ne	20	67
Nitrogen, N_2	28	297
Carbon monoxide, CO	28	338
Ar	29	293
Nitric oxide, NO	30	297
Oxygen, O_2	32	271
Methanol, CH_3OH	32	586
Hydrogen sulphide, H_2S	34	633
Hydrogen chloride, HCl	36	447
Fluorine, F_2	38	195
Argon, Ar	40	281
Nitrous oxide, N_2O	44	516
Carbon dioxide, CO_2	44	451
Ethanol, C_2H_5OH	46	878
Sulphur dioxide, SO_2	64	686
Chlorine, Cl_2	71	773
Carbon disulphide, CS_2	76	481
Benzene, C_6H_6	78	1762
Hydrogen bromide, HBr	81	570
Krypton, Kr	84	427
Hydrogen iodide, HI	128	906
Xenon, Xe	131	702
Bromine, Br_2	160	1132

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Table 5.19 Comparison of the experimental values of the refractive index of gases with the prediction of their refractive index based on Equation 2.17. Before comparing the data, the dielectric constant data have been corrected to STP using factors discussed in §5.6.2. The first three entries in the table are for non-polar gases and the last two are for polar gases. Notice the good agreement between theory and experiment for the non-polar gases, and the massive disagreement for water vapour.

Gas	$10^4(\varepsilon - 1)$	T	Correction factor	$10^4(\varepsilon - 1)$ (STP)	Prediction $10^6(\sqrt{\varepsilon} - 1)$	Experiment $10^6(n_{\text{light}} - 1)$
Non-polar gases						
He	0.65	20	293/273	0.70	35	36
Ne	1.3	0	1	1.3	65	67
Ar	5.5	20	293/273	5.54	277	281
Polar gases						
NH_3	8.34	0	1	8.34	416	376
H_2O	60	100	$(293/273)^2$	69.1	3449	254

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Table 7.1 The approximate values of the density of some solids.

Solid	ρ(kg m^{-3})
Metals	
Aluminium/Dural	2700–2800
Phosphor–bronze	8900
Brass	8400–8500
Gold (22 carat)	17500
Gold (9 carat)	11300
Mild steel	7900
Stainless steel	7700–7800
Wrought iron	7800
Invar	8000
Platinum/Iridium	21500
Wood	
Balsa	200
Pine	500
Oak	700
Beech	750
Teak	850
Ebony	1200
Natural materials	
Amber	1100
Beeswax	950
Granite	2700
Ice	920
Coal	1.4–1.6
Mica	2800

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Table 7.2 The density of the elements (kg m^{-3}). Also shown is the atomic number Z and the atomic weight A in units of the atomic mass unit u = 1.66×10^{-27} kg. For example, the density of *magnesium*, whose atoms each contain 12 protons, is 1.738×10^{3} kg m^{-3}. The mass of an atom of magnesium is $24.31 \times 1.66 \times 10^{-27}$ kg. The densities of elements that are normally gaseous at room temperature are evaluated at a temperature just below their freezing point at atmospheric pressure. For helium, which does not solidify at atmospheric pressure at any temperature, the density is evaluated at 4.2 K and 25 atmospheres (25×10^{5} Pa) pressure which is sufficient to cause solidification.

Z	Element and symbol	A	Density	Z	Element and symbol	A	Density
1	Hydrogen, H	1.008	89				
2	Helium, He	4.003	120	51	Antimony, Sb	121.7	6692
3	Lithium , Li	6.941	533	52	Tellurium, Te	127.6	6247
4	Beryllium, Be	9.012	1846	53	Iodine, I	126.9	4953
5	Boron, B	10.81	2466	54	Xenon, Xe	131.3	3560
6	Carbon (graphite), C	12.01	2266	55	Caesium, Cs	132.9	1900
6	Carbon (diamond), C	12.01	3513	56	Barium, Ba	137.3	3594
7	Nitrogen, N	14.01	1035	57	Lanthanum, La	138.9	6174
8	Oxygen, O	16.00	1460	58	Cerium, Ce	140.1	6711
9	Fluorine, F	19.00	1140	59	Praseodymium, Pr	140.9	6779
10	Neon, Ne	20.18	1442	60	Neodymium, Ne	144.2	7000
11	Sodium, Na	22.99	966	61	Promethium, Pm	145.0	7220
12	Magnesium, Mg	24.31	1738	62	Samarium, Sm	150.4	7536
13	Aluminium, Al	26.98	2698	63	Europium, Eu	152.0	5248
14	Silicon, Si	28.09	2330	64	Gadolinium, Gd	157.2	7870
15	Phosphorus, P	30.97	1820	65	Terbium, Tb	158.9	8267
16	Sulphur, S	32.06	2086	66	Dysprosium, Dy	162.5	8531
17	Chlorine, Cl	35.45	2030	67	Holmium, Ho	164.9	8797
18	Argon, Ar	39.95	1656	68	Erbium, Er	167.3	9044
19	Potassium, K	39.10	862	69	Thulium, Th	168.9	9325
20	Calcium, Ca	40.08	1530	70	Ytterbium, Yb	173.0	6966
21	Scandium, Sc	44.96	2992	71	Lutetium, Lu	175.0	9842
22	Titanium, Ti	47.90	4508	72	Hafnium, Hf	178.5	13276
23	Vanadium, V	50.94	6090	73	Tantalum, Ta	180.9	16670
24	Chromium, Cr	52.00	7194	74	Tungsten, W	183.9	19254
25	Manganese, Mn	54.94	7473	75	Rhenium, Re	186.2	21023
26	Iron, Fe	55.85	7873	76	Osmium, Os	190.2	22580
27	Cobalt, Co	58.93	8800	77	Iridium, Ir	192.2	22550
28	Nickel, Ni	58.70	8907	78	Platinum, Pt	195.1	21450
29	Copper, Cu	63.55	8933	79	Gold, Au	197.0	19281
30	Zinc, Zn	65.38	7135	80	Mercury, Hg	200.6	13546
31	Gallium, Ga	69.72	5905	81	Thallium, Tl	204.4	11871
32	Germanium, Ge	72.59	5323	82	Lead, Pb	207.2	11343
33	Arsenic, As	74.92	5776	83	Bismuth, Bi	209.0	9803
34	Selenium, Se	78.96	4808	84	Polonium, Po	209.0	9400
35	Bromine, Br	79.90	3120	85	Astatine, At	210.0	—
36	Krypton, Kr	83.80	3000	86	Radon, Rn	222.0	4400
37	Rubidium, Rb	85.47	1533	87	Francium, Fr	223.0	—
38	Strontium, Sr	87.62	2583	88	Radium, Ra	226.0	5000
39	Yttrium, Y	88.91	4475	89	Actinium, Ac	227.0	10060
40	Zirconium, Zr	91.22	6507	90	Thorium, Th	232.0	11725
41	Niobium, Nb	92.91	8578	91	Protractinium, Pa	231.0	15370
42	Molybdenum, Mo	95.94	10222	92	Uranium, U	238.0	19050
43	Technetium, Tc	97.00	11496	93	Neptunium, Np	237.0	20250
44	Ruthenium, Ru	101.1	12360	94	Plutonium, Pu	244.0	19840
45	Rhodium, Rh	102.9	12420	95	Americium, Am	243.0	13670
46	Palladium, Pd	106.4	11995	96	Curium, Cm	247.0	13300
47	Silver, Ag	107.9	10500	97	Berkelium, Bk	247.0	14790
48	Cadmium, Cd	112.4	8647	98	Californium, Cf	251.0	15100
49	Indium, In	114.8	7290	99	Einsteinium, Es	254.0	—
50	Tin, Sn	118.7	7285	100	Fermium, Fm	257.0	—

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Table 7.3 The atomic number Z, atomic mass A and the density ρ of the lanthanide elements extracted from Table 7.2. The row marked $\%A$ is the % density increase (compared with La) expected if the separation between atoms is unchanged and only the atomic mass changes. The row marked $\%\rho$ is the % density increase (compared with La) actually found. It shows that the 59% density increase is much greater than can be explained by the 26% increase in atomic mass alone.

	La	Ce	Pr	Nd	Pm	Sm	Eu	Gd	Tb	Dy	Ho	Er	Tm	Yb	Lu
Z	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71
A	138.9	140.1	140.9	144.2	145	150.4	152.0	157.3	158.9	162.5	164.9	167.3	168.9	173.0	175.0
%*A*	0	0.87	1.44	3.84	4.39	8.28	9.40	13.21	14.41	16.99	18.74	20.41	21.62	24.57	25.96
ρ	6174	6711	6773	7000	7220	7536	5248	7870	8267	8531	8797	9044	9325	6966	9842
%ρ	0	8.7	9.7	13.4	16.9	22.1	-15.0	27.4	33.9	38.2	42.5	46.5	51.4	12.8	59.4

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Table 7.4 The bulk modulus of the elements in their solid state. The temperature of the measurements varies considerably and there are discrepancies of the up to 50% in figures from different sources.

Z	Element	*B* (GPa)	Z	Element	*B* (GPa)	Z	Element	*B* (GPa)
1	Hydroge	0.2	29	Copper	137.8	59	Praseodymium	30.6
2	Heli	0.1	30	Zinc	72.0	60	Neodymium	32.7
3	Lithiu	11.1	31	Gallium	56.9	61	Promethium	35.0
4	Beryllium	100.3	32	Germanium	7.7	62	Samarium	39.4
5	Boron	178.0	33	Arsenic	22.0	63	Europium	14.7
6	Carbon (diam nd)	542.0	34	Selenium	8.3	64	Gadolinium	38.3
6	Carbon (graphite)	33.0	35	Bromine	1.9	65	Terbium	39.9
7	Nitrogen	1.2	36	Krypton	3.5	66	Dysprosium	38.4
8	Oxygen		37	Rubidium	1.9	67	Holmium	39.7
9	Fluorine		38	Strontium	1.2	68	Erbium	41.1
10	Neon	1.1	39	Yttrium	36.6	69	Thulium	39.7
11	Sodium	6.4	40	Zirconium	83.3	70	Ytterbium	13.3
12	Magnesium	44.7	41	Niobium	170.2	71	Lutetium	41.1
13	Aluminum	75.	42	Molybdenum	231.0	72	Hafnium	109.0
14	Silicon	98.8	43	Technetium	297.0	73	Tantalum	200.0
15	Phosphorous (Red)	10.9	44	Ruthenium	320.8	74	Tungsten	323.2
15	Phosphorous(White)	4.9	45	Rh dium	270.4	75	Rhenium	372.0
16	Sulphur	17.8	46	Palla ium	182.0	76	Osmium	418.0
17	Chlorine		47	Silver	100.7	77	Iridium	355.0
18	Argon	2.7	48	Cadm m	41.6	78	Platinum	228.0
19	Potassium	3.1	49	Indiu	41.1	79	Gold	217.0
20	Calcium	17.2	50	Tin	58.2	80	Mercury	25.0
21	Scandium	43.5	51	Antimony	42.0	81	Thallium	35.9
22	Titanium	105.1	52	Tellurium	23.0	82	Lead	45.8
23	Vanadium	161.9	53	Iodine	7.7	83	Bismuth	31.3
24	Chromium	160.1	54	Xenon	3.6	84	Polonium	26.0
25	Manganese	118.0	55	Caesium	1.6	29	Copper	137.8
26	Iron	169.8	56	Barium	10	30	Zinc	72.0
27	Cobalt	191.4	57	Lanthanum	2 3			
28	Nickel	186.0	58	Cerium	2 9			

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Table 7.5 Values of the bulk modulus of the noble gas solids calculated according to Equation 7.6, compared with experimental data from Table 7.4.

	Substance			
	Ne	**Ar**	**Kr**	**Xe**
σ ($\times 10^{-10}$ m)	2.74	3.44	3.65	3.98
ε ($\times 10^{-3}$ eV)	3.1	10.3	14.0	20.0
$75.13\varepsilon/\sigma^3$($\times 10^9$ Pa)	1.81	3.18	3.46	3.81
Data	1.1	2.7	3.5	3.6
Ratio (theory/expt)	1.65	1.18	0.99	1.06

Table 7.6 Values of the bulk modulus of the alkali metals calculated according to Equation 7.11, compared with experimental data from Table 7.4.

	Substance				
	Li	**Na**	**K**	**Rb**	**Cs**
n ($\times$ [illegible] m^{-3})	4.63	2.53	1.33	1.08	0.86
ε_F ($\times$ eV)	4.7	3.14	2.05	1.78	1.53
$2n\varepsilon_F/3$($\times 10^9$ Pa)	23.2	8.5	2.9	2.06	1.41
Data	11.1	6.4	3.1	1.9	1.6
Ratio (theory/expt)	[illegible]	1.33	0.94	1.08	0.88

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Table 7.7 The coefficient of linear expansivity α for various solids at temperatures around room temperature. (*Kaye & Laby*). The volume expansivity of the elements is given by $\beta = 3\alpha$ as shown in Example 7.4.

Elemental metals	**α ($°C^{-1}$)**	**Miscellaneous**	**α ($°C^{-1}$)**	**Alloys**	**α ($°C^{-1}$)**
Aluminium (Al)	23	Brick	3–10	Brass (68% Cu/32% Zn)	18–19
Antimony (Sb)	≈ 11	Cement and concrete	10–14	Bronze (80% Cu/20% Sn)	17–18
Bismuth (Bi)	[illegible]	Marble	3–15	Constantan (60% Cu/40% Ni)	15–17
Cadmium (Cd)	[illegible]	Lead glass (46% pbo)	≈ 8	Duralumin (95% Al/4% Cu)	23
Chromium (Cr)	≈ 7	Typical glass	≈ 8–10	Magnalium (90% Al/10% Mg)	≈ 23
Cobalt (Co)	≈ 12	Porcelain	2–6	Nickel steel(10% Ni/90%Fe)	13
Copper (Cu)	16	Silica	0.4	Nickel steel(36% Ni/64%Fe)	0–1.5
Gold (Au)	[illegible]	Typical wood (along grain)	3–5	Nickel steel(43% Ni/57%Fe)	7.9
Iridium (Ir)	6.5	Typical wood (across grain)	35–60	Nickel steel(58% Ni/42%Fe)	11.4
Iron (Fe)	11.7			Carbon steel	≈11
Lead (Pb)	29	**Plastics**		Stainless steel (74%Fe/18%Cr/8%Ni)	29
Magnesium (Mg)	25	Epoxy resins	45–65	Phosphor-bronze	17
Nickel (Ni)	12.8	Epoxy resins	45–65	Platinum–Iridium (90% Pt/10% Ir)	8.7
Palladium (Pd)	≈ 11	Polycarbonates	66		
Platinum (Pt)	8.9	Low-density polyethylene	40–150	**Carbon**	
Rhodium (Rh)	8.4	Medium-density polyethylene	80–220	Diamond	1.0
Silver (Ag)	19	High density polyethylene	200–360	Graphite (polycrystalline)	7.1
Tantalum (Ta)	6.5	Natural rubber	220		
Thallium (Tl)	≈ 28	Hard rubber	60		
Tin (Sn)	≈ 21	Perspex	50–90		
Titanium (Ti)	≈ 9	Nylon	80–280		
Tungsten (W)	4.5	Polystyrene	[illegible]		
Vanadium (V)	≈ 8	Polyvinyl chloride (pvc)	70–[illegible]		
Zinc (Zn)	≈ 30				

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Table 7.8 Expected and experimentally determined values the coefficient of linear expansivity thermal expansivity α for some alloys and their component metals.

Alloy composition	Expected (see text)	Experimental α ($^{\circ}C^{-1}$)
Aluminium alloys		
Duralumin (95% Al/4% Cu)	22.5×10^{-6}	23×10^{-6}
Magnalium (90% Al/10% Mg)	23.2×10^{-6}	$\approx 23 \times 10^{-6}$
Aluminium	—	23×10^{-6}
Copper	—	16.7×10^{-6}
Magnesium	—	$\approx 25 \times 10^{-6}$
Copper alloys		
Brass (68% Cu/32% Zn)	21×10^{-6}	$18\text{-}19 \times 10^{-6}$
Bronze (80% Cu/20% Sn)	17.6×10^{-6}	$17\text{-}18 \times 10^{-6}$
Constantan (60% Cu/40% Ni)	15.1×10^{-6}	$15\text{-}17 \times 10^{-6}$
Copper	—	16.7×10^{-6}
Zinc	—	$\approx 30 \times 10^{-6}$
Tin	—	$\approx 21 \times 10^{-6}$
Ni	—	12.8×10^{-6}
Platinum alloy		
Platinum-Iridium (90% Pt/10% Ir)	8.66×10^{-6}	8.7×10^{-6}
Platinum	—	8.9×10^{-6}
Iridium	—	6.5×10^{-6}
Iron alloys		
Nickel steel (10% Ni/90% Fe)	11.8×10^{-6}	13×10^{-6}
Nickel steel(36% Ni/64%Fe)	2.1×10^{-6}	$0\text{–}1.5 \times 10^{-6}$
Nickel steel(43% Ni/57%Fe)	12.2×10^{-6}	7.9×10^{-6}
Nickel steel(58% Ni/42%Fe)	12×10^{-6}	11.4×10^{-6}
Stainless steel (74%Fe/18% Cr/8%Ni)	10.9×10^{-6}	29×10^{-6}
Iron	—	11.7×10^{-6}
Nickel	—	12.8×10^{-6}
Chromium	—	7×10^{-6}

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Table 7.9 The speed of sound in solids at 20 °C showing c_L, the speed of longitudinal waves, and c_T, the speed of transverse (shear waves).

	Speed of sound	
Elemental metals	c_L(ms^{-1})	c_T(ms^{-1})
Aluminium, Al	6374	3111
Beryllium, Be	12890	8880
Cadmium, Cd	2780	–
Chromium, Cr	6608	4005
Copper, Cu	4759	2325
Gold, Au	3240	1200
Iron, Fe	5957	3224
Lead, Pb	2160	700
Magnesium, Mg	5823	3163
Manganese, Mn	4600	—
Molybdenum, Mo	6475	3505
Nickel, Ni	5700	3000
Niobium, Nb	5068	2092
Platinum, Pt	3260	1730
Silver, Ag	3704	1698
Tantalum, Ta	4159	2036
Tin, Sn	3380	1594
Titanium, Ti	6130	3182
Tungsten, W	5221	2887
Uranium, U	3370	1940
Vanadium, V	6023	2774
Zinc, Zn	4187	2421
Zirconium, Zr	4650	2250
Insulators	c_L(ms^{-1})	c_T(ms^{-1})
Carbon (diamond)	18350	9200
Glass (crown)	5661	3420
Glass (heavy flint)	5260	2960
Glass (pyrex)	5640	3280
Quartz crystal X-cut	5720	—
Quartz fused	5970	3755
Concrete	4250–5250	—
Ice (-20°C)	≈3840	
Plastics	c_L(ms^{-1})	c_T(ms^{-1})
Polyethylene	2000	3111
Polystyrene	2350	1120
PVC	2300	–
Rubber	1600	4005

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Table 7.10 The molar heat capacity at constant pressure C_P of the elements at room temperature 25 °C (298.15K). The shaded data are elements that are either liquids or gases at this temperature.

Z	Element	A	ρ (kg m^{-3})	C_P (J K mol^{-1})	Z	Element	A	ρ (kg m^{-3})	C_P (J K mol^{-1})
1	Hydrogen, H	1.008	89	28.824	49	Indium, In	114.8	7290	26.74
2	Helium, He	4.003	120	20.786	50	Tin, Sn	118.7	7285	26.99
3	Lithium, Li	6.941	533	24.770	51	Antimony, Sb	121.7	6692	25.23
4	Beryllium, Be	9.012	1846	16.44	52	Tellurium, Te	127.6	6247	25.73
5	Boron, B	10.81	2466	11.09	53	Iodine, I	126.9	4953	54.438
6	Carbon (graphite), C	12.01	2266	8.53	54	Xenon, Xe	131.3	3560	20.786
6	Carbon (diamond), C	12.01	3513	6.11	55	Caesium, Cs	132.9	1900	32.17
7	Nitrogen, N	14.01	1035	29.125	56	Barium, Ba	137.3	3594	28.07
8	Oxygen, O	16.00	1460	29.355	57	Lanthanum, La	138.9	6174	27.11
9	Fluorine, F	19.00	1140	31.300	58	Cerium, Ce	140.1	6711	26.94
10	Neon, Ne	20.18	1442	20.786	59	Praseodymium, Pr	140.9	6779	27.20
11	Sodium, Na	22.99	966	28.24	60	Neodymium, Ne	144.2	7000	27.45
12	Magnesium, Mg	24.31	1738	24.89	61	Promethium, Pm	145.0	7220	26.81
13	Aluminium, Al	26.98	2698	24.35	62	Samarium, Sm	150.4	7536	29.54
14	Silicon, Si	28.09	2329	20.0	63	Europium, Eu	152.0	5248	27.66
15	Phosphorus, P	30.97	1820	23.84	64	Gadolinium, Gd	157.2	7870	37.03
16	Sulphur, S	32.06	2086	22.64	65	Terbium, Tb	158.9	8267	28.91
17	Chlorine, Cl	35.45	2030	33.90	66	Dysprosium, Dy	162.5	8531	28.16
18	Argon, Ar	39.95	1656	20.786	67	Holmium, Ho	164.9	8797	27.15
19	Potassium, K	39.10	862	29.58	68	Erbium, Er	167.3	9044	28.12
20	Calcium, Ca	40.08	1530	25.31	69	Thulium, Th	168.9	9325	27.03
21	Scandium, Sc	44.96	2992	25.52	70	Ytterbium, Yb	173.0	6966	26.74
22	Titanium, Ti	47.90	4508	25.02	71	Lutetium, Lu	175.0	9842	26.86
23	Vanadium, V	50.94	6090	24.89	72	Hafnium, Hf	178.5	13276	25.73
24	Chromium, Cr	52.00	7194	23.35	73	Tantalum, Ta	180.9	16670	25.36
25	Manganese, Mn	54.94	7473	26.32	74	Tungsten, W	183.9	19254	24.27
26	Iron, Fe	55.85	7873	25.10	75	Rhenium, Re	186.2	21023	25.48
27	Cobalt, Co	58.93	8800	24.81	76	Osmium, Os	190.2	22580	24.70
28	Nickel, Ni	58.70	8907	26.07	77	Iridium, Ir	192.2	22550	25.10
29	Copper, Cu	63.55	8933	24.44	78	Platinum, Pt	195.1	21450	25.86
30	Zinc, Zn	65.38	7135	25.40	79	Gold, Au	197.0	19281	25.42
31	Gallium, Ga	69.72	5905	25.86	80	Mercury, Hg	200.6	13546	27.98
32	Germanium, Ge	72.59	5323	23.35	81	Thallium, Tl	204.4	11871	26.32
33	Arsenic, As	74.92	5776	24.64	82	Lead, Pb	207.2	11343	26.44
34	Selenium, Se	78.96	4808	25.36	83	Bismuth, Bi	208.9	9803	25.52
35	Bromine, Br	79.90	3120	75.69	84	Polonium, Po	209	9400	25.75
36	Krypton, Kr	83.80	3000	20.79	85	Astatine, At	210		
37	Rubidium, Rb	85.47	1533	31.06	86	Radon, Rn	222	4400	20.786
38	Strontium, Sr	87.62	2583	26.40	87	Francium, Fr	223	2400	31.70
39	Yttrium, Y	88.91	4475	26.53	88	Radium, Ra	226	5000	25.76
40	Zirconium, Zr	91.22	6507	25.36	89	Actinium, Ac	227	10070	27.20
41	Niobium, Nb	92.91	8578	24.60	90	Thorium, Th	232	11724	27.32
42	Molybdenum, Mo	95.94	10222	24.06	91	Protractinium, Pa	231	15370	27.20
43	Technetium, Tc	97	11496	25.88	92	Uranium, U	238	19050	27.66
44	Ruthenium, Ru	101.1	12360	24.06	93	Neptunium, Np	237	20250	29.62
45	Rhodium, Rh	102.9	12420	24.98	94	Plutonium, Pu	244	19840	32.80
46	Palladium, Pd	106.4	11995	25.98	95	Americium, Am	243	13670	25.86
47	Silver, Ag	107.9	10500	25.35	96	Curium, Cm	247	1330	27.70
48	Cadmium, Cd	112.4	8647	25.98					

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Table 7.11 The predicted value of the heat capacity of monatomic solids according to the Debye theory. Also tabulated is the fraction of the high temperature limiting value ($3R$) expected at the temperature indicated.

T/Θ_D	C(T) J K^{-1} mol^{-1}	C(T) /R
0	0	0
0.01	1.944×10^{-3}	7.7927×10^{-5}
0.02	1.555×10^{-2}	6.2342×10^{-4}
0.03	5.248×10^{-2}	2.1040×10^{-3}
0.04	0.1244	4.9873×10^{-3}
0.05	0.2430	9.7408×10^{-3}
0.06	0.4198	1.6829×10^{-2}
0.07	0.6658	2.6693×10^{-2}
0.0	0.9903	3.9702×10^{-2}
0.09	1.399	5.6074×10^{-2}
0.1	1.891	7.5821×10^{-2}
0.2	9.195	0.36863
0.3	15.158	0.60770
0.4	18.604	0.74585
0.5	20.588	0.82541
0.6	21.795	0.87380
0.7	22.572	0.90495
0.8	23.098	0.92603
0.9	23.466	0.94089
1.0	23.729	0.95173
1.1	23.842	0.95987
1.2	24.098	0.96612
1.3	24.221	0.97103
1.4	24.318	0.97495
1.5	24.398	0.97813
1.6	24.463	0.98074
1.7	24.517	0.98291
1.8	24.562	0.98474
1.9	24.601	0.98629
2.0	24.634	0.98761

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Table 7.12 The Debye temperatures θ_D of several elements as determined by analysis of the T^3 behaviour of their low-temperature heat capacity (Equation 7.61).

Element	Z	Θ_D (K)	Element	Z	Θ_D (K)
Beryllium	4	1440	Zirconium	40	291
C(Diamond)	6	2230	Molybdenum	42	450
Magnesium	12	400	Silver	47	225
Aluminium	13	428	Cadmium	48	209
Titanium	22	420	Tin	50	200
Vanadium	23	380	Tantalum	73	240
Chromium	24	630	Tungsten	74	400
Manganese	25	410	Platinum	78	240
Iron	26	470	Gold	79	165
Nickel	28	450	Lead	82	105
Copper	29	343	Uranium	92	207

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Table 7.13 The electrical resistivity of the elements which are solid at around room temperature. Take care with the exponents of values in this table which vary from entry to entry and column to column by 46 orders of magnitude.

Z	Element	ρ(Ω m)	σ(S m^{-1})	Z	Element	ρ(Ω m)	σ(S m^{-1})
1	Hydrogen, H	—	—	49	Indium, In	8.37×10^{-8}	1.19×10^{7}
2	Helium, He	—	—	50	Tin, Sn	1.1×10^{-7}	9.1×10^{6}
3	Lithium, Li	8.55×10^{-8}	1.17×10^{7}	51	Antimony, Sb	3.9×10^{-7}	2.56×10^{6}
4	Beryllium, Be	4×10^{-8}	2.5×10^{7}	52	Tellurium, Te	0.00436	229
5	Boron, B	18000	5.56×10^{5}	53	Iodine, I	1.37×10^{-7}	7.30×10^{8}
6	Carbon (diamond), C	10^{11}	10^{-11}	54	Xenon, Xe	—	—
7	Nitrogen, N	—	—	55	Caesium, Cs	2×10^{-7}	5×10^{6}
8	Oxygen, O	—	—	56	Barium, Ba	5×10^{-7}	2×10^{6}
9	Fluorine, F	—	—	57	Lanthanum, La	5.7×10^{-7}	1.75×10^{6}
10	Neon, Ne	—	—	58	Cerium, Ce	7.3×10^{-7}	1.37×10^{6}
11	Sodium, Na	4.2×10^{-8}	2.38×10^{7}	59	Praseodymium, Pr	6.8×10^{-7}	1.47×10^{6}
12	Magnesium, Mg	4.38×10^{-8}	2.28×10^{7}	60	Neodymium, Ne	6.4×10^{-7}	1.56×10^{6}
13	Aluminium, Al	2.65×10^{-8}	3.77×10^{7}	61	Promethium, Pm	5×10^{-7}	2×10^{6}
14	Silicon, Si	0.001	1000	62	Samarium, Sm	9.4×10^{-7}	1.06×10^{6}
15	Phosphorus, P	1×10^{-9}	1×10^{9}	63	Europium, Eu	9×10^{-7}	1.11×10^{6}
16	Sulphur, S	2×10^{15}	5×10^{-16}	64	Gadolinium, Gd	1.34×10^{-6}	7.46×10^{5}
17	Chlorine, Cl	—	—	65	Terbium, Tb	1.14×10^{-6}	8.77×10^{5}
18	Argon, Ar	—	—	66	Dysprosium, Dy	5.7×10^{-7}	1.75×10^{6}
19	Potassium, K	6.15×10^{-8}	1.63×10^{7}	67	Holmium, Ho	8.7×10^{-7}	1.15×10^{6}
20	Calcium, Ca	3.43×10^{-8}	2.92×10^{7}	68	Erbium, Er	8.7×10^{-7}	1.15×10^{6}
21	Scandium, Sc	6.1×10^{-7}	1.64×10^{6}	69	Thulium, Th	7.9×10^{-7}	1.27×10^{6}
22	Titanium, Ti	4.2×10^{-7}	2.38×10^{6}	70	Ytterbium, Yb	2.9×10^{-7}	3.45×10^{6}
23	Vanadium, V	2.48×10^{-7}	4.03×10^{6}	71	Lutetium, Lu	7.9×10^{-7}	1.27×10^{6}
24	Chromium, Cr	1.27×10^{-7}	7.87×10^{6}	72	Hafnium, Hf	3.51×10^{-7}	2.85×10^{6}
25	Manganese, Mn	1.85×10^{-6}	5.41×10^{5}	73	Tantalum, Ta	1.25×10^{-7}	8.03×10^{6}
26	Iron, Fe	9.71×10^{-8}	1.03×10^{7}	74	Tungsten, W	5.65×10^{-8}	1.77×10^{7}
27	Cobalt, Co	6.24×10^{-8}	1.60×10^{7}	75	Rhenium, Re	1.93×10^{-7}	5.18×10^{6}
28	Nickel, Ni	6.84×10^{-8}	1.46×10^{7}	76	Osmium, Os	8.12×10^{-8}	1.23×10^{7}
29	Copper, Cu	1.67×10^{-8}	5.98×10^{7}	77	Iridium, Ir	5.3×10^{-8}	1.89×10^{7}
30	Zinc, Zn	5.92×10^{-8}	1.69×10^{7}	78	Platinum, Pt	1.06×10^{-7}	9.43×10^{6}
31	Gallium, Ga	2.7×10^{-7}	3.70×10^{6}	79	Gold, Au	2.35×10^{-8}	4.26×10^{7}
32	Germanium, Ge	0.46	2.1739	80	Mercury, Hg	9.41×10^{-7}	1.06×10^{6}
33	Arsenic, As	2.6×10^{-7}	3.85×10^{6}	81	Thallium, Tl	1.8×10^{-7}	5.56×10^{6}
34	Selenium, Se	0.01	100	82	Lead, Pb	2.07×10^{-7}	4.84×10^{6}
35	Bromine, Br	—	—	83	Bismuth, Bi	1.07×10^{-6}	9.36×10^{5}
36	Krypton, Kr	—	—	84	Polonium, Po	1.4×10^{-6}	7.14×10^{5}
37	Rubidium, Rb	1.25×10^{-7}	8×10^{6}	85	Astatine, At	—	—
38	Strontium, Sr	2.3×10^{-7}	4.35×10^{6}	86	Radon, Rn	—	—
39	Yttrium, Y	5.7×10^{-7}	1.75×10^{6}	87	Francium, Fr	—	—
40	Zirconium, Zr	4.21×10^{-7}	2.37×10^{6}	88	Radium, Ra	1×10^{-6}	1×10^{6}
41	Niobium, Nb	1.25×10^{-7}	8×10^{6}	89	Actinium, Ac	—	—
42	Molybdenum, Mo	5.2×10^{-8}	1.92×10^{7}	90	Thorium, Th	1.3×10^{-7}	7.69×10^{6}
43	Technetium, Tc	2.26×10^{-7}	4.42×10^{6}	91	Protractinium, Pa	1.77×10^{-7}	5.65×10^{6}
44	Ruthenium, Ru	7.6×10^{-8}	1.32×10^{7}	92	Uranium, U	3.08×10^{-7}	3.25×10^{6}
45	Rhodium, Rh	4.51×10^{-8}	2.22×10^{7}	93	Neptunium, Np	1.22×10^{-6}	8.20×10^{5}
46	Palladium, Pd	1.08×10^{-7}	9.26×10^{6}	94	Plutonium, Pu	1.46×10^{-6}	6.85×10^{5}
47	Silver, Ag	1.59×10^{-8}	6.29×10^{7}	95	Americium, Am	6.8×10^{-7}	1.4706×10^{6}
48	Cadmium, Cd	6.83×10^{-8}	1.46×10^{7}				

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Table 7.14 The resistivities (Ω m) of three alloys at around room temperature is shown in centre of the three tables below. On either side of the data for the alloy, are the resistivities of the component elements.

Component 1	Alloy	Component 2
Cu	**Cu(Zn)**	**Zn**
1.55×10^{-8}	6.3×10^{-8}	5.5×10^{-8}
Pt	**Pt(10% Ir)**	**Ir**
9.81×10^{-8}	24.8×10^{-8}	4.7×10^{-8}
Pt	**Pt(10% Rh)**	**Rh**
1.31×10^{-8}	18.7×10^{-8}	4.3×10^{-8}

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Table 7.15 Examples of substances which display superconducting behaviour below the temperature shown.

Substance	Alloy
Low-temperature superconductors: elements	
Aluminium	1.75
Lead	7.2
Niobium	9.25
Tin	3.72
Vanadium	5.4
Low-temperature superconductors: alloys	
V_3Si	17.1
Nb_3Sn	18.3
[illegible]	39
High-temperature superconductors	
$YBa_2Cu_3O_{7-\delta}$	93
$Hg_1Ba_2Ca_2Cu_3O_{10}$	133

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Table 7.16 The relative dielectric permittivity ε of various insulators (and semiconductors) The relative permittivity of vacuum is exactly 1. All measurements refer to 20°C, but are insensitive to small changes $\approx \pm 10$°C around this temperature.

Substance		ε
Elements		
Silicon	Si	11.9
Germanium	Ge	16.0
Ceramics		
Alumina	Al_2O_3	8.5
Strontium titanate	$SrTiO_3$	200
Strontium zirconate	$SrZrO_3$	38
Glass		
Quartz	SiO_2	4.5
Borosilicate glass	SiO_2 with BO	4 – 5
Lead glass	SiO_2 with PbO	7
Plastics		
Polyethylene		2.3
Polystyrene		2.6
Polytetrafluoroethylene	PTFE	2.1
Polyamide	Nylon	3 – 4

Table 7.17 Typical orders of magnitude of the resistivity of some insulating substances at around room temperature. The data correspond to values of ρ determined one minute after the electric field is applied.

Insulator	ρ (Ω m)	Insulator	ρ (Ω m)
Alumina Al_2O_3	$10^9 – 10^{12}$	Paper	$\approx 10^{10}$
Quartz SiO_2	$\approx 10^{16}$	PTFE	$10^{15} – 10^{19}$
Diamond C	$10^{10} – 10^{11}$	Polystyrene	10^{15} - 10^{19}
Boron B	$10^{10} – 10^{11}$	Varnish	10^7
Iodine I_2	10^{13}	Soil	$10^2 – 10^4$
Glass	$10^9 – 10^{12}$	Distilled water	$10^2 – 10^5$

Table 7.18 Typical values (and ranges of values) of the dielectric strength of some insulating substances.

Insulator	Vm^{-1}
Alumina, Al_2O_3	$10 – 35 \times 10^6$
Sapphire, Al_2O_3	17×10^6
Quartz, SiO_2	$25 – 40 \times 10^6$
Beryllia	$10 – 14 \times 10^6$

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Table 7.19 Thermal conductivity κ of solid elements (W K^{-1} m^{-1}) as a function of absolute temperature. The shaded entries refer to data above the melting temperature of the element. The labels M, I and SC stand for metal, insulator and semiconductor respectively. The two results for phosphorus at 173.2 K correspond to different crystal structures known as 'black' and 'yellow' phosphorus respectively.

Element and type of material		Temperature (K)				
		73.2 K	173.2 K	273.2 K	373.2 K	1273 K
Lithium, Li	M	94	86	82	47	59
Beryllium, Be	M	367	218	168	129	93
Boron, B		72	32	19	11	10
Carbon (Graphite), C	I	70–2[illegible]	80–230	75–195	50–130	35–70
Carbon (Diamond), C	I	1700–490[illegible]	1000–2600	700–1700	—	—
Sodium, Na	M	14[illegible]	142	88	78	60
Magnesium, Mg	M	160	157	154	150	—
Aluminium, Al	M	241	2[illegible]6	240	233	92
Silicon, Si	SC	33[illegible]	1[illegible]8	108	65	32
Phosphorous, P	I	20	13/0.25	0.18	0.16	—
Sulphur, S	I	0.39	0.29	0.15	0.17	—
Potassium, K	M	105	10[illegible]	53	45	32
Scandium, Sc	M	15	16			
Titanium, Ti	M	26	22	21	19	21
Vanadium, V	M	32	31	31	33	38
Chromium, Cr	M	120	96.5	91	82	66
Manganese, Mn	M	7	8	—	—	—
Iron, Fe	M	99	83.5	72	56	34
Cobalt, Co	M	130	105	89	69	53
Nickel, Ni	M	113	94	83	67	71
Copper, Cu	M	420	403	395	3[illegible]	354
Zinc, Zn	M	117	117	112	104	66
Gallium, Ga	M	43	41	33	[illegible]	—
Germanium, Ge	SC	113	67	46.5	29	17.5
Selenium (*c*-axis), Se	I	6.8	4.8	4.8	[illegible]	—
Rubidium, Rb	M	59	58	32	29	[illegible]
Yttrium, Y	M	16.5	17	—	—	—
Zirconium, Zr	M	26	23	22	21	2[illegible]
Niobium, Nb	M	53	53	55	58	[illegible]4
Molybdenum, Mo	M	145	139	135	127	1[illegible]3
Technetium, Tc	M	—	51	50	50	—
Ruthenium, Ru	M	123	117	115	108	[illegible]
Rhodium, Rh	M	156	151	147	137	—
Palladium, Pd	M	72	72	73	79	93
Silver, Ag	M	432	428	422	407	377
Cadmium, Cd	M	100	97	95	89	445
Indium, In	M	92	84	76	42	—
Tin, Sn	M	76	68	63	32	40
Antimony, Sb	M	33	25.5	22	19	27
Tellurium(*c*-axis), Te	I	5.1	3.6	2.9	2.4	6.3

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Table 7.20 Thermal conductivity of a variety materials ($WK^{-1}\,m^{-1}$). The tables refer to metallic alloys, refractory materials, i.e. those suitable for use in high temperatures without degradation, and a selection of everyday materials.

	173.2K	273.2K	373.2K	573.2K	873.2K	973.2K	1473.2K
Brass ($Cu_{70}Zn_{30}$)	89	106	128	146	—	—	—
Bronze ($Cu_{90}Sn_{10}$)	—	53	60	80	—	—	—
Carbon steel	48	50	48.5	54.5	—	30.5	—
Silicon steel	—	25	28.5	31	—	28	—
Stainless steel	—	24.5	25	25.5	—	24.8	—
Alumina (Al_2O_3)	—	40	28	—	9.2	—	5.7
Beryllia (BeO)	—	300	213	—	61	—	22
Fire brick	–	—	—	—	1.1	—	1.3
Silica (SiO_2) fused quartz	–	1.33	1.48	—	2.4	—	—
Zirconia (ZrO_2)	—	—	1.8	—	2.0	—	2.2

Substance	κ ($WK^{-1}\,m^{-1}$)	Substance	κ ($WK^{-1}\,m^{-1}$)	Substance	κ ($WK^{-1}\,m^{-1}$)
Brick wall	≈1	Porcelain	1.5	Glass wool	0.037
Plaster	≈0.13	Rubber	≈0.2	Cotton wool	0.03
Timber	≈ 0.15	Polystyrene	≈0.1	Sheep's wool	0.05
Balsa wood	≈0.06	Glass (crown)	1.1	Nylon	0.25
Paper	0.06	Glass (flint)	0.85	Epoxy resins	≈0.2
Cardboard	0.21	Glass (pyrex)	1	Cellular polystyrene	≈0.04

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Table 9.1 The density of some elements at their melting temperatures in the liquid state. Also given is the ratio of the liquid density to the density of the solid at 25 °C (Table 7.2). The four elements which contract on melting: (silicon, gallium, germanium and bismuth) are shaded.

Z	Element	A	Liquid density (kg m^{-3})	Ratio of liquid/solid density
3	Lithium	6.941	516	0.968
5	Boron	10.81	2080	0.843
11	Sodium	22.99	930	0.962
12	Magnesium	24.31	1580	0.909
13	Aluminium	26.98	2400	0.889
[illegible]	Silicon	28.09	2525	1.080
16	Sulphur	32.06	1819	0.872
19	[illegible]otassium	39.10	824	0.955
[illegible]	Ca[illegible]cium	40.08	1365	0.892
22	Ti[illegible]ium	47.90	4130	0.916
23	Vanad[illegible]m	50.94	5550	0.878
25	M[illegible]gan[illegible]se	54.94	6430	0.860
26	[illegible]ron	55.85	7100	0.901
28	Nickel	58.70	7800	0.875
29	Copper	63.55	8000	0.895
30	Zinc	[illegible]5.38	6600	0.925
31	Gallium	[illegible]69.72	6113.6	1.035
32	Germaniu[illegible]	72.5[illegible]	5530	1.038
34	Selenium	7[illegible]	4000	0.832
37	Rubidium	85.47	[illegible].470	0.959
40	Zirconium	[illegible].22	5800	0.891
41	Niobium	92.91	7830	0.913
42	Molybdenum	95.94	[illegible]350	0.915
44	Ruthenium	101.1	1090[illegible]	0.889
45	Rhodium	102.9	1[illegible]0	0.874
46	Palladium	106.4	[illegible]0700	[illegible].892
47	Silver	107.9	9300	0[illegible]86
48	Cadmium	112.4	8020	[illegible].927
50	Tin	118.7	6980	0.9[illegible]
51	Antimony	121.7	6490	[illegible].970
52	Tellurium	127.6	5770	0.92[illegible]
55	Caesium	132.9	1845	[illegible].971
56	Barium	137.3	3323	0.92[illegible]
72	Hafnium	178.5	12000	0.90[illegible]
73	Tantalum	180.9	15000	0.900
74	Tungsten	183.9	17600	0.914
75	Rhenium	186.2	18800	0.894
76	Osmium	190.2	20100	0.890
77	Iridium	192.2	20000	0.887
78	Platinum	195.1	19700	0.918
79	Gold	197.0	17320	0.898
81	Thallium	204.4	11290	0.951
82	Lead	207.2	10690	0.942
83	Bismuth	209.0	10050	1.025
92	Uranium	238.0	17907	0.940

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Table 9.2 The density of substances that are liquids at room temperature. The table gives the name of the substance, the chemical formula for its molecules, the relative molecular mass of each molecule, the density and the temperature of the density measurement. Only the last three entries in the table are *inorganic*.

Liquid and chemical formula		MW	Density (kg m^{-3})	
Organic liquids				
Methanol	CH_3OH	32	791	@20°C
Ethanol	C_2H_5OH	46	789	@20°C
Propan-1-ol	C_3H_7OH	60	804	@20°C
Propan-2-ol	C_3H_7OH	60	786	@20°C
2 Methyl-propan-1-ol	C_4H_9OH	74	817	@20°C
2 Methyl-propan-2-ol	C_4H_9OH	74	789	@20°C
Butan-1-ol	C_4H_9OH	74	810	@20°C
Butan-2-ol	C_4H_9OH	74	808	@20°C
2 Methyl-butan-1-ol	$C_5H_{11}OH$	88	816	@20°C
2 Methyl-butan-2-ol	$C_5H_{11}OH$	88	809	@20°C
Pentanol	$C_5H_{11}OH$	88	813	@20°C
Octanol	$C_8H_{17}OH$	130	827	@20°C
Aniline	C_6H_7N	86	1026	@15°C
Acetone	C_3H_6O	58	787	@25°C
Benzene	C_6H_6	78	879	@20°C
Inorganic liquids				
Carbon disulphide	CS_2	76	1293	@0 °C
Carbon tetrachloride	CCl_4	154	1632	@0 °C
Water (see Table 9.3)	H_2O	18	1000	@0 °C

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Table 9.3 The density of water (H_2O) and heavy water (D_2O) as a function of temperature at atmospheric pressure.

T (°C)	H_2O	D_2O	T (°C)	H_2O	D_2O
0	999.84	—	40	992.22	1100.0
2	999.94	—	45	—	1097.9
4	999.97	—	50	988.04	1095.7
5	—	1105.6	55	—	1093.3
6	999.94	—	60	983.20	1090.6
8	999.85	—	65	—	1087.8
10	999.70	1106.0	70	977.77	1084.8
15	—	1105.9	75	—	1081.6
20	998.20	1105.3	80	971.79	1078.2
25	—	1104.4	85	—	1074.7
30	995.65	1103.2	90	965.31	1071.1
35	—	1101.7	95	—	1067.4
			100	958.36	1063.5

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Table 9.4 The bulk modulus of some liquids at the pressure and temperature shown. The pressure is shown in units of atmospheres, where one atmosphere is approximately 0.1 MPa.

Liquid and formula	***P*** **(Atm)**	***B*** **(GPa)**	***T*** **(°C)**
Organic liquids			
Methanol, CH_3OH	37	0.97	14.7
Ethanol, C_2H_5OH	1	1.32	0
Propan-1-ol, C_3H_7OH	8	1.04	17.7
Propan-2-ol, C_3H_7OH	8	0.983	17.8
Butan-1-ol, C_4H_9OH	8	1.13	17.4
Butan-2-ol, C_4H_9OH	8	1.03	17.9
Ethe[illegible]	1	0.689	0
Ether	1000	1.56	0
Benzene, C_6H_6	8	1.10	17.9
Inorganic liquids			
Carbon disulphide, CS_2	8	1.16	15.6
Carbon tetrachloride, CCl_4	1	1.12	20
Water, H_2O	1	2.05	15
Water, H_2O	1000	2.75	15
Water, H_2O	2500	3.88.	14.2

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Table 9.5 The coefficient of volume expansivity β for various liquids at temperatures around room temperature. The shaded column shows the value of the volume expansivity of the corresponding solid substance. N/A indicates that data is not available.

Substance		MW	*T* (°C)	β (°C^{-1}) Liquid	β (°C^{-1}) Solid
Organic liquids					
Acetic acid	CH_3COOH	60	20	107×10^{-5}	N/A
Acetone	CH_3COCH_3	58	20	143×10^{-5}	N/A
Methanol	CH_3OH	46	20	119×10^{-5}	N/A
Ethanol	C_2H_5OH	32	20	108×10^{-5}	N/A
Aniline	C_6H_5N	86	20	85×10^{-5}	N/A
Benzene	C_6H_6	78	20	121×10^{-5}	N/A
Toluene	$C_6H_5CH_3$	92	20	107×10^{-5}	N/A
Inorganic liquids					
Carbon Disulphide	CS_2	76	20	119×10^{-5}	N/A
Carbon Tetrachloride	CCl_4	154	20	122×10^{-5}	N/A
Water	H_2O	18	20	21×10^{-5}	N/A
Metals					
Lithium	Li	7	400-1125	19×10^{-5}	16.8×10^{-5} @ 20 °C
Sodium	Na	39	96.5	25×10^{-5}	21.2×10^{-5} @ 20 °C
Potassium	K	39.1	64 - 1400	29×10^{-5}	24.9×10^{-5} @ 20 °C
Rubidium	Rb	133	39	30×10^{-5}	27.0×10^{-5} @ 20 °C
Copper	Cu	63.6	1084	10×10^{-5}	4.95×10^{-5} @ 20 °C
Copper	Cu	63.6	1084	10×10^{-5}	6.09×10^{-5} @ 527 °C
Mercury	Hg	200.6	-39 - 100	18.1×10^{-5}	N/A

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Table 9.6 The speed of sound in liquids showing c_L, the speed of longitudinal waves. For the elements, where possible, the data for the solid state (taken from Table 7.12) is also included, in the shaded column, for comparison. Data for ice is also included.

Substance	T (°C)	c_L (ms^{-1})	Substance	T (°C)	c_L (ms^{-1})	Substance	T (°C)	c_L (ms^{-1})	c_L (ms^{-1})
Organic Liquids			***Elements***			***Elements***			
Acetic acid	20	1173	Hydrogen, H_2	–258	1242	Cadmium, Cd	360	2150	2780
Acetone	20	1190	Helium, He	–269	211	Copper,Cu	1350	3350	4759
Methanol	20	1121	Nitrogen, N_2	–189	745	Gallium, Ga	50	2740	
Ethanol	20	1162	Oxygen, O_2	–186	950	Mercury,Hg	20	1454	
Propanol	20	1204	Sodium, Na	110	2520	Silver, Ag	1150	2630	3704
Butanol	20	1258	Potassium, K	80	1869	Tin, Sn	240	2470	3380
iso-Pentanol	20	1255	Rubidium, Rb	50	1427	Zinc, Zn	450	2700	4187
Hexanol	20	1331	Caesium, Cs	40	980				
Hexanol	20	1331							
Heptanol	20	1343							
Water	0	1402							
Ice	–20	3840							

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Table 9.7 The viscosity η of various substance in their liquid state in units of mPa s as a function of the temperature in °C. To obtain the viscosity in units of Pa s, multiply the entries in this table by 10^{-3}. For example, the viscosity of mercury at 25 °C is 1.528×10^{-3} Pa s.

	Temperature (°C)												
Substance	**–10[illegible]**	**–50**	**0**	**25**	**30**	**50**	**75**	**100**	**400**	**600**	**700**	**800**	**1100**
Acetic acid	—	—	—	1.116	1.037	0.792	0.591	0.457	—	—	—	—	—
Acetone	—	—	0.402	0.310	0.295	0.247	0.200	0.165	—	—	—	—	—
Benzene	—	—	—	0.603	0.562	0.436	0.332	0.263	—	—	—	—	—
Carbon disulphide	2.13[illegible]	0.[illegible]96	0.445	0.357	0.343	—	—	—	—	—	—	—	—
Methanol	—	[illegible].258	0.797	0.543	0.507	0.392	0.294	0.227	—	—	—	—	—
Ethanol	98.9[illegible]	8.31[illegible]	1.[illegible]73	1.084	0.983	0.684	0.459	0.323	—	—	—	—	—
Sodium	—	[illegible]	—	—	—	—	—	0.680	0.286	0.215	0.192	0.174	—
Potassium	—	—	—	—	—	—	—	0.458	0.224	0.172	0.155	0.141	—
Mercury	—	—	1.61[illegible]	[illegible].528	1.497	1.401	1.322	1.255	—	—	—	—	—
Tin	—	—	—	—	—	—	—	—	1.33	1.04	0.950	0.890	0.780

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Table 9.9 The surface energy or surface tension of various substances in their liquid state (10^{-3} N m^{-1}) at a given temperature in °C. For example, the surface tension of benzene is 28.88×10^{-3} N m^{-1}.

Substance	Temperature (°C)	γ (mN m^{-1})
Acetic acid	20	27.59
Acetone	20	23.46
Benzene	20	28.88
Carbon disulphide	20	32.32
Methanol	20	22.50
Ethanol	20	22.39
Water	20	72.75
Sodium	100	209.9
Potassium	65	110.9
Mercury	25	485.5
Lead	350	444.5
Aluminium	700	900
Gold	1100	1120

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Table 9.13 The heat capacities at constant pressure C_P for a selection of substances that are liquids at around room temperature. The table records the substance name and chemical formula, the relative molecular mass of its constituent molecules, the number of atoms per molecule, and the temperature at which the measurement is made. The molar heat capacity is then recorded as in J K^{-1} and as a multiple of the gas constant R.

Substance		*MW*	*N*	*T* (°C)	C_P (J K^{-1} mol^{-1})	C_P (*R*)
Organic liquids						
Methanol	CH_3OH	32	6	12	80.64	9.7
Ethanol	C_2H_5OH	46	9	0	105.3	12.7
Ethanol	C_2H_5OH	46	9	20	113.4	13.6
Ethanol	C_2H_5OH	46	9	40	124.7	15.0
Propanol	C_3H_7OH	60	12	18	138.0	16.6
Acetic acid	$C_2H_4O_2$	60	8	20	124.3	15.0
Acetone	C_3H_6O	58	10	20	124.7	15.0
Aniline	C_6H_7N	93	14	15	199.9	24.0
Benzene	C_6H_6	78	12	10	110.8	13.3
Benzene	C_6H_6	78	12	40	138.1	16.6
Bromoethane	C_2H_5Br	109	8	20	100.8	12.1
Chloroform	$CHCl_3$	120	5	20	113.8	13.7
Cyclohexane	C_6H_{12}	84	16	20	156.5	18.8
1,2 Dichloroethane	$C_2H_4Cl_2$	98	8	20	129.3	15.6
Dichloromethane	$C_2H_2Cl_2$	85	6	20	100.0	12.0
Ethanadiol	$C_2H_6O_2$	62	10	20	149.8	18.0
Ethyl acetate	$C_4H_8O_2$	88	8	20	170.1	20.5
Ethyl nitrate	$C_2H_5O_3N$	91	1	20	170.3	20.5
Formamide	CH_3ON	45	6	20	107.6	12.9
Formic acid	CH_2O_2	46	5	20	99.0	11.9
Nitromethane	CH_3O_2N	61	7	20	106.0	12.7
Nitroethane	$C_2H_5O_2N$	75	1	0	134.2	16.1
Toluene	C_7H_8	92	15	18	153.6	18.5
Inorganic liquids						
Arsenic trifluoride	AsF_3	132	4	20	126.6	15.2
Boron trichloride	BCl_3	118	4	20	106.7	12.8
Bromine	Br_2	160	2	20	75.7	9.11
Carbon disulphide	CS_2	76	3	20	75.7	9.11
Hydrogen cyanide	HCN	27	3	20	70.6	8.49
Water	H_2O	18	3	0	75.9	9.13
Heavy water	D_2O	20	3	0	84.3	10.1
Mercury	Hg	201	1	20	28.0	3.37
Hydrazine	N_2H_4	32	6	20	98.0	11.9
Silicon tetrachloride	$SiCl_4$	170	5	20	145.3	17.5
Tin tetrachloride	$SnCl_4$	261	5	20	165.3	19.9
Titanium tetrachloride	$TiCl_4$	190	5	20	145.2	17.5

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Table 9.14 Thermal conductivity of miscellaneous non-metallic liquids in units of $WK^{-1}\,m^{-1}$. The data is given at two temperatures T_1 and T_2, and varies roughly linearly between these two temperatures. (Figure 9.32 (a)).

Liquid	T_1	T_2	K_1	K_2
Acetone	193	333	0.198	0.146
Aniline	293		0.172	
Benzene	293	323	0.147	0.137
Methanol	233	333	0.223	0.186
Ethanol	233	353	0.189	0.150
N-butanol	213	353	0.167	0.106
N-propanol	233	353	0.168	0.148
Toluene	193	353	0.159	0.119
Carbon tetrachloride	253	333	0.115	0.102
Water	273	353	0.561	0.673
Xenon	173	223	0.07	0.05

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Table 9.15 Thermal conductivity (W K^{-1} m^{-1}) of elemental metals in their liquid state. Shaded entries refer to the solid state. The data are graphed in Figure 9.33.

Liquid		173 K	273 K	373 K	573 K	973 K	K_L/K_S(%)
Lithium	Li	98	86	82	47	59	57
Sodium	Na	141	142	88	78	60	62
Potassium	K	105	104	53	45	32	51
Rubidium	Rb	59	58	32	29	22	55
Caesium	Cs	37	36	20	20.6	17.7	56
Mercury	Hg	29.5	7.8	9.4	11.7	—	26
Aluminium	Al	241	236	240	233	92	39
Bismuth	Bi	11	8.2	7.2	13	17	181
Gallium	Ga	43	41	33	45	—	80
Tin	Sn	76	68	63	32	40	51

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Table 9.16 Thermal conductivity ($WK^{-1}\,m^{-1}$) and electrical resistivity (Ω m) of elemental metals in their liquid state. Also evaluated is the quantity $\rho\kappa/T$ known as the *Lorentz number* and has theoretical value of 2.45×10^{-8} ($W\,\Omega\,K^{-2}$).

	373 K			573 K			973 K		
Liquid	ρ	κ	$\rho\kappa/T$	ρ	κ	$\rho\kappa/T$	ρ	κ	$\rho\kappa/T$
Sodium	9.7×10^{-8}	88	2.3×10^{-8}	16.8×10^{-8}	78	2.3×10^{-8}	39.2×10^{-8}	60	2.4×10^{-8}
Potassium	17.5×10^{-8}	43	2.5×10^{-8}	28.2×10^{-8}	45	2.2×10^{-8}	66.4×10^{-8}	32	2.2×10^{-8}
Rubidium	27.5×10^{-8}	32	2.4×10^{-8}	48×10^{-8}	29	2.4×10^{-8}	99×10^{-8}	22	2.2×10^{-8}
Caesium	43.5×10^{-8}	23	2.3×10^{-8}	67×10^{-8}	20.6	2.4×10^{-8}	134×10^{-8}	17.7	2.4×10^{-8}
Mercury	103.5×10^{-8}	14	2.3×10^{-8}	128×10^{-8}	11.7	2.6×10^{-8}	214×10^{-8}	—	—

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Table 9.17 The resistivity ($\times 10^{-8}\ \Omega$ m) of elemental metals with low melting points. The shaded data above the line in the table refers to the metals in the solid state and data below line refer to data in the liquid state. The last row of the table shows the ratio of the resistivities in the solid and liquid states. The figure is derived from the ratio of the last datum in the solid region to the first datum in the liquid region.

T(K)	Na	K	Rb	Cs	Hg
0	0	0	0	0	0
78.2	0.76	1.30	2.59	4.1	5.8
273.2	4.33	6.49	11.5	18.8	94.1
373.2	9.51	15.8	27.3	44.5	103.5
[illegible]73.2	17.4	27.7	45.1	67.3	128
97[illegible]2	38.9	64.7	93	128	214
1473[illegible]	88	165	250	338	630
ρ_S/ρ_L (%)	46	41	42	42	6

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Table 9.19 The results of calculations of the molecular polarisability of non-polar molecules based on dielectric constant data for both liquid and gaseous states. The value of on Equation 9.51 $\alpha/\varepsilon_o = (\varepsilon - 1)/n$ with n estimated by either Equation 9.49 or 9.50 as appropriate. The data for the densities of liquid hydrogen, nitrogen and oxygen are estimates based on a 10% decrease of the density of the solid. See Table 5.16 for gas data and Table 9.18 for liquid data. The gas data refer to atmospheric pressure (1.013×10^5 Pa). Notice that the inferred value of α is quite similar in liquid and gaseous states.

		Liquid				Gas			
Substance		ρ (kg m^{-3})	ε–1	n (×10^{28} m^3)	α/ε_o (×10^{-30})	ρ (kg m^{-3})	ε–1	n (×10^{28} m^3)	α/ε_o (×10^{-30})
Argon	40	≈1410	0.53	2.12	25	293	5.16	2.50	21
Helium	4	≈120	0.048	1.81	2.65	293	0.65	2.50	2.6
Hydrogen	2	≈80	0.22	2.41	9.5	293	2.54	2.50	10.2
Nitrogen	28	≈930	0.45	2.00	22.5	293	5.47	2.50	21.9
Oxygen	32	≈1300	0.507	2.45	20.5	293	4.94	2.50	19.8

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Table 9.20 The results of the calculations of the permanent molecular dipole moment (in C m) of polar molecules according to Equation 9.53. The gas data refer to atmospheric pressure (1.013×10^5 Pa).

		Liquid					Gas			
Substance	***M***	***T*** **(K)**	***ρ*** **(kg m^{-3})**	***ε*–1**	***n*** **(×10^{28} m^3)**	**p (×10^{-30})**	***T*** **(K)**	***ε*–1**	***n*** **(×10^{28} m^3)**	**p (×10^{-30})**
Methanol	32	298	791	31.6	1.49	15.2	373	57	1.97	6.29
Ethanol	46	298	789	23.3	1.03	15.7	373	61 or 78	1.97	6.5 or 7.4
Water	18	293	1000	79.4	3.35	16.0	373	60	1.97	6.45

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Table 9.18 The relative dielectric permittivity ε of various insulating liquids. The relative permittivity of vacuum is exactly 1.

Substance	MW	*T*	ε – 1	ε
Argon, Ar	40	82 K	0.53	1.53
Helium, He	4	4.19 K	0.048	1.048
Hydrogen, H_2	2	20.4 K	0.228	1.228
Nitrogen, N_2	28	70 K	0.45	1.45
Oxygen, O_2	32	80 K	0.507	1.507
Methanol, CH_3OH	32	25 °C	31.6	32.6
Ethanol, C_2H_5OH	46	25 °C	23.3	24.3
Propanol, C_3H_7OH	60	25 °C	19.1	20.1
Butanol, C_4H_9OH	74	20 °C	16.8	17.8
Pentanol, $C_5H_{11}OH$	88	25 °C	12.9	13.9
Hexanol, $C_6H_{13}OH$	102	25 °C	12.3	13.3
Aniline, C_6H_7N	86	20 °C	5.90	6.90
Acetone, C_3H_6O	58	25 °C	19.7	20.7
Carbon disulphide, CS_2	76	20 °C	1.64	2.64
Water, H_2O	18	20 °C	79.4	80.4

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Table 9.19 The results of calculations of the molecular polarisability of non-polar molecules based on dielectric constant data for both liquid and gaseous states. The value of on Equation 9.51 $\alpha/\varepsilon_o = (\varepsilon - 1)/n$ with n estimated by either Equation 9.49 or 9.50 as appropriate. The data for the densities of liquid hydrogen, nitrogen and oxygen are estimates based on a 10% decrease of the density of the solid. See Table 5.16 for gas data and Table 9.18 for liquid data. The gas data refer to atmospheric pressure (1.013×10^5 Pa). Notice that the inferred value of α is quite similar in liquid and gaseous states.

		Liquid				Gas			
Substance		ρ (kg m^{-3})	ε–1	n (×10^{28} m^3)	α/ε_o (×10^{-30})	ρ (kg m^{-3})	ε–1	n (×10^{28} m^3)	α/ε_o (×10^{-30})
Argon	40	≈1410	0.53	2.12	25	293	5.16	2.50	21
Helium	4	≈80	0.048	1.81	2.65	293	0.65	2.50	2.6
Hydrogen	2	≈80	0.222	2.41	9.5	293	2.54	2.50	10.2
Nitrogen	28	≈930	0.47	2.00	22.5	293	5.47	2.50	21.9
Oxygen	32	≈1300	0.507	2.45	20.5	293	4.94	2.50	19.8

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Table 9.20 The results of the calculations of the permanent molecular dipole moment (in C m) of polar molecules according to Equation 9.53. The gas data refer to atmospheric pressure (1.013×10^5 Pa).

		Liquid					Gas			
Substance		***T*** **(K)**	***ρ*** **(kg m^{-3})**	***ε*–1**	***n*** **(×10^{28} m^3)**	**p (×10^{-30})**	***T*** **(K)**	***ε*–1**	***n*** **(×10^{28} m^3)**	**p (×10^{-30})**
Methanol	32	298	791	31.6	1.49	15.2	373	57	1.97	6.29
Ethanol	46	298	789	23.3	1.03	15.7	373	61 or 78	1.97	6.5 or 7.4
Water	18	293	1000	79.4	3.35	16.0	373	60	1.97	6.45

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Table 9.21 The refractive index of various liquids for yellow light.

Substance and chemical formula	MW	n_{light}
Water, H_2O	18	1.33
Carbon tetrachloride, CCl_4	152	1.405
Toluene, C_7H_8	92	1.497
Methanol, CH_3OH	32	1.329
Ethanol, C_2H_5OH	44	1.3614
Propan-1-ol, C_3H_7OH	56	1.3852
Propan-2-ol, $C_2H_5OHCH_2$	56	1.3742
Acetic acid, CH_3COOH		1.3716
Benzene, C_6H_6	78	1.501
Aniline, C_7H_7N	86	1.586
Hydrogen disulphide, HS_2	65	1.885

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Table 9.22 Calculation of the refractive indices of liquid water, methanol and benzene from the data on the refractive index of their vapours (Table 5.18). The predictions for $n_{light}-1$ are 20 to 25% below the experimental values. The method of calculation is described in Equations 9.55 to 9.61.

			Gas			Liquid			
Substance		**M**	**Number density (m^{-3})**	n_{light}	**Molecular polarisability α ($F^{-1}m^4$)**	**Density (kg m^{-3})**	**Number density (m^{-3})**	**Prediction $n_{light}-1$**	**Actual $n_{light}-1$**
Water	H_2O	18	2.689×10^{25}	1.000254	1.647×10^{-40}	1000	3.346×10^{28}	0.27	0.33
Methanol	CH_3OH	[illegible]	2.689×10^{25}	1.000586	3.860×10^{-40}	791	1.489×10^{28}	0.284	0.329
Benzene	C_6H_6	78	2.689×10^{25}	1.001762	11.61×10^{-40}	879	6.786×10^{27}	0.375	0.501

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Table 11.1 Thermal data for the elements: the melting and boiling temperatures in kelvin, and the enthalpies of fusion (melting) and vaporisation. The data refer to standard atmospheric pressure unless otherwise stated. Two elements – arsenic and carbon – which sublime when heated at atmospheric pressure. These are discussed in §11.7 on the solid ⇒ gas transition, and their the enthalpies of fusion and vaporisation are estimated from studies at high pressure.

Z	Name	Atomic weight	Density (kg m^{-3})	Melting point (K)	Boiling point (K)	Enthalpy of fusion (kJ mol^{-1})	Enthalpy of vaporisation (kJ mol^{-1})
1	Hydro[illegible]	1.008	89	14.01	20.28	0.12	0.46
2	Helium	4.003	120	0.95	4.216	0.021	0.082
3	Lithium	[illegible].941	533	453.7	1620	4.6	134.7
4	Beryllium	9.012	1846	1551	3243	9.8	308.8
5	Boron	10.[illegible]	2466	2365	3931	22.2	538.9
6	Carbon	12.[illegible]	2266	Sublimes at ≈ 3700		105	710.9
7	Nitrogen	14.01	1035	63.15	77.4	0.72	5.577
8	Oxygen	16	[illegible]160	54.36	90.188	0.444	6.82
9	Fluorine	19	1[illegible]40	53.48	85.01	5.1	6.548
10	Neon	20.18	1442	24.56	27.1	0.324	1.1736
11	Sodium	22.99	[illegible]66	371	1156.1	2.64	89.04
12	Magnesium	24.31	1738	922	1363	9.04	128.7
13	Aluminium	26.98	2698	[illegible]33.5	2740	10.67	293.72
14	Silicon	28.09	23[illegible]	1683	2628	39.6	383.3
15	Phosphorous	30.97	1820	[illegible]7.3	553	2.51	51.9
16	Sulphur	32.06	2086	[illegible]86	717.82	1.23	9.62
17	Chlorine	35.45	2030	[illegible]72	239.18	6.41	20.403
18	Argon	39.95	1656	83.[illegible]	87.29	1.21	6.53
19	Potassium	39.1	862	336.8	1047	2.4	77.53
20	Calcium	40.08	1530	1112	[illegible]7[illegible]7	9.33	149.95
21	Scandium	44.96	2992	1814	3109	15.9	304.8
22	Titanium	47.9	4508	1933	3560	20.9	428.9
23	Vanadium	50.94	6090	2160	3[illegible]	17.6	458.6
24	Chromium	52	7194	2130	2945	15.3	348.78
25	Manganese	54.94	7473	1517	22[illegible]5	14.4	219.7
26	Iron	55.85	7873	1808	3023	14.9	351
27	Cobalt	58.93	8800	1768	3143	15.2	382.4
28	Nickel	58.7	8907	1726	3005	17.[illegible]6	371.8
29	Copper	63.55	8933	1356.6	2840	13	304.6
30	Zinc	65.38	7135	692.73	1180	6.[illegible]7	115.3
31	Gallium	69.72	5905	302.93	3676	5.[illegible]	256.1
32	Germanium	72.59	5323	1210.6	3103	3[illegible].7	334.3
33	Arsenic	74.92	5776	Sublimes at 886		27.7	31.9
34	Selenium	78.96	4808	490	958.1	5.[illegible]	26.32
35	Bromine	79.9	3120	265.9	331.93	10.8	[illegible]0
36	Krypton	83.8	3000	116.6	120.85	1.64	[illegible].05
37	Rubidium	85.47	1533	312.2	961	2.2	69[illegible]
38	Strontium	87.62	2583	1042	1657	6.16	13[illegible]91
39	Yttrium	88.91	4475	1795	3611	17.2	393[illegible]
40	Zirconium	91.22	6507	2125	4650	23	581.6
41	Niobium	92.91	8578	2741	5015	27.2	696.6
42	Molybdenum	95.94	10222	2890	4885	27.6	594.1

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Z	Name	Atomic weight	Density (kg m^{-3})	Melting point (K)	Boiling point (K)	Enthalpy of fusion (kJ mol^{-1})	Enthalpy of vaporisation (kJ mol^{-1})
43	Technetium	97	11496	2445	5150	23.81	585.22
44	Ruthenium	101.1	12360	2583	4173	23.7	567.8
45	Rhodium	102.9	12420	2239	4000	21.55	495.4
46	Palladium	106.4	11995	1825	3413	17.2	393.3
47	Silver	107.9	10500	1235.1	2485	11.3	255.1
48	Cadmium	112.4	8647	594.1	1038	6.11	99.87
49	Indium	114.8	7290	429.32	2353	3.27	226.4
50	Tin	118.7	7285	505.12	2543	7.2	290.4
51	Antimony	121.7	6692	903.9	1908	20.9	67.91
52	Tellurium	127.6	6247	722.7	1263	13.5	50.63
53	Iodine	126.9	4953	386.7	457.5	15.27	41.67
54	Xenon	131.3	3560	161.3	166.1	3.1	12.65
55	Caesium	132.9	1900	301.6	951.6	2.09	65.9
56	Barium	137.3	3594	1002	1910	7.66	150.9
57	Lanthanum	138.9	6174	1194	3730	10.04	399.6
58	Cerium	140.1	6711	1072	3699	8.87	313.8
59	Praseodymium	140.9	6779	1204	3785	11.3	332.6
60	Neodymium	144.2	7000	1294	3341	7.113	283.7
61	Promethium	145	7264	1441	3000	12.6	—
62	Samarium	150.4	7536	1345	2064	10.9	191.6
63	Europium	152	5248	1095	1870	10.5	175.7
64	Gadolinium	157.2	7870	1586	3539	15.5	311.7
65	Terbium	158.9	8267	1629	3396	16.3	391
66	Dysprosium	162.5	8531	1685	2835	17.2	293
67	Holmium	164.9	8797	1747	2963	17.2	251
68	Erbium	167.3	9044	1802	3136	17.2	292.9
69	Thulium	168.9	9325	1818	2220	18.4	247
70	Ytterbium	173	6966	1097	1469	9.2	159
71	Lutetium	175	9842	1936	3668	19.2	428
72	Hafnium	178.5	13276	2503	5470	25.5	661.1
73	Tantalum	180.9	16670	3269	5698	31.4	753.1
74	Tungsten	183.9	19254	3680	5930	35.2	799.1
75	Rhenium	186.2	21023	3453	5900	33.1	707.1
76	Osmium	190.2	22580	3327	5300	29.3	627.6
77	Iridium	192.2	22550	2683	4403	26.4	563.6
78	Platinum	195.1	21450	2045	4100	19.7	510.5
79	Gold	197	19281	1337.6	3080	12.7	324.4
80	Mercury	200.6	13546	234.28	629.73	2.331	59.15
81	Thallium	204.4	11871	576.6	1730	4.31	162.1
82	Lead	207.2	11343	600.65	2013	5.121	179.4
83	Bismuth	209	9803	544.5	1833	10.48	179.1
84	Polonium	209	9400	527	1235	10	100.9
85	Astatine	210	—	575	610	23.8	—
86	Radon	222	4400	202	211.4	2.7	19
87	Francium	223	—	300	950	—	—
88	Radium	226	5000	973	1413	7.15	136.8
89	Actinium	227	10060	1320	3470	14.2	293

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Z	Name	Atomic weight	Density (kg m^{-3})	Melting point (K)	Boiling point (K)	Enthalpy of fusion (kJ mol^{-1})	Enthalpy of vaporisation (kJ mol^{-1})
90	Thorium	232	11725	2023	5060	19.2	543.9
91	Protactinium	231	15370	2113	4300	16.7	481
92	Uranium	238	19050	1405	4018	15.5	422.6
93	Nept[illegible]ium	237	20250	913	4175	9.46	336.6
94	P[illegible]tonium	244	19840	914	3505	2.8	343.5
95	Am[illegible]um	243	13670	1267	2880	14.4	238.5

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Table 11.2 Thermal data for the various substances: the melting and boiling temperatures in kelvin, and the enthalpies of fusion (melting) and vaporisation. The data refer to standard atmospheric pressure unless otherwise stated and (s) indicates that the substance sublimes rather than boils and the melting temperature is obtained under pressure. (*) indicates a large discrepancy of ± 20 K amongst data from different sources.

Substance		MW	Density (kg m^{-3})	Melting point (K)	Boiling point (K)	Enthalpy of fusion (kJ mol^{-1})	Enthalpy of vaporisation (kJ mol^{-1})
Acetic acid	CH_3CO_2H	60	1049	289.75	391.1	11.535	—
Acetone	CH_3COCH_3	58	787	177.8	329.3	5.691	—
Aniline	C_6H_7N	93	1026	266.85	457.6	10.555	—
Benzene	C_6H_6	78	877	278.65	353.2	9.951	—
Chloroform	$CHCl_3$	119	—	209.55	334.4	8.800	—
Cyclohexane	C_6H_{10}	82	779	279.65	353.8	2.630	—
Ethyl acetate	$C_4H_8O_2$	88	—	189.55	350.2	10.481	—
Methanol	CH_3OH	32	791	179.25	337.7	3.177	—
Ethanol	C_2H_5OH	46	789	155.85	351.5	5.021	—
Propan-1-ol	C_3H_7OH	60	804	146.65	370.3	5.195	—
Propan-2-ol	C_3H_7OH	60	786	—	—	—	—
Butan-1-ol	C_4H_9OH	74	810	183.65	390.35	9.282	—
Butan-2-ol	C_4H_9OH	74	806	298.55	372.65	6.786	—
Toluene	C_7H_8	92	867	178.15	383.8	6.851	—
Lithium fluoride	LiF	25.9	2635	1118	1949	—	—
Lithium chloride	LiCl	42.39	2068	878	1620(*)	—	—
Lithium bromide	LiBr	86.9	3464	823	1538	—	—
Sodium chloride	NaF	42.0	2558	1265	1968	—	—
Sodium fluoride	NaCl	58.4	2165	1074	1686	—	—
Sodium bromide	NaBr	102.9	3203	1020	1663	—	—
Potassium fluoride	KF	58.1	2480	1131	1778	—	—
Potassium chloride	KCl	74.6	1984	1043	1272(s)	—	—
Potassium bromide	KBr	119.0	2750	1007	1708	—	—
Carbon dioxide	CO_2	44	—	216.55	194.7	—	—
Carbontetrachloride	CCl_4	154	1632	—	—	—	—
Carbon disulphide	CS_2	76	1293	162.35	319.6	4.395	—
Carbon monoxide	CO	28	—	74.15	81.7	—	—
Water	H_2O	18	998	273.15	373.15	5.994	40.608

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Table 11.3 Comparison of $N_A \Delta E_e$ with the experimental value of the latent heat of vaporisation L. The final column shows the ratio of these two quantities $N_A \Delta E_e / L$. The values of ΔE_e are drawn from Table 9.12.

Substance	ΔE_e(J) $\times 10^{-21}$	$N_A \Delta E_e$ (kJ mol^{-1})	L (kJ mol^{-1})	$N_A \Delta E_e / L$
Copper	486	292.7	300.5	0.97
Silver	403	242.7	255.06	0.95
Gold	516	310.7	324.43	0.96
Aluminium	447	269.2	290.8	0.92
Zinc	436	262.6	290.37	0.90
Helium	0.13	0.078	0.08	0.98
Neon	3.23	1.95	1.77	1.10
Argon	10.8	6.50	6.52	0.99
Krypton	17.2	10.36	9.03	1.15
Xenon	24.4	14.69	12.64	1.16

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Table 11.4 The critical parameters of various substances discussed in Chapter 6 and Chapter 8. P_C,V_C and T_C are the critical pressure, molar volume and temperature. Z_C is the *compression factor* which is discussed in §11.5.3. The next column gives the density at the critical point, calculated from the molecular mass and V_C. This may be compared with the density of the substance in the liquid state well away from T_C. For the inorganic substances where the liquid density data is not available, the solid density has been used instead The final column gives the ratio of the density at the critical point to that at a temperature well below the critical point.

Substance	P_C (MPa)	V_C ($\times 10^{-6}$ m^3 mol^{-1})	T_C (K)	$Z_C = P_C V_C / R T_C$	Critical Density (kg m^{-3})	Liquid Density (kgm^{-3})	Density Ratio
Methanol, CH_3OH	8.09	118	512.6	0.224	271	791	0.343
Ethanol, C_2H_5OH	6.14	167	513.9	0.240	275	789	0.349
Propan-1-ol , C_3H_7OH	5.17	219	536.8	0.254	274	804	0.340
Acetic acid, $C_2H_4O_2$	5.79	171	594.5	0.200	351	1049	0.334
Acetone, C_3H_6O	4.7	213	508.1	0.237	272	787	0.346
Aniline, C_6H_7N	5.3	274	698.9	0.250	339	1026	0.330
Benzene, C_6H_6	4.9	254	562.2	0.266	307	879	0.349
Bromoethane, C_2H_5Br	6.23	215	503.8	0.320	507	1456	0.348
Chloroform, $CHCl_3$	5.5	240	536.4	0.296	500	1498	0.333
Cyclohexane, C_6H_{10}	4.02	308	553.4	0.269	266	941.6	0.282
Ethyl acetate, $C_4H_8O_2$	3.83	286	523.2	0.252	287	900.6	0.319
Toluene, C_7H_8	4.11	320	591.8	0.267	288	868.8	0.331
Carbon monoxide, CO	3.50	93.1	133	0.295	300.75	—	—
Carbon dioxide, CO_2	7.38	94.0	304.2	0.274	468.09	—	—
Carbon disulphide, CS_2	7.9	173	552	0.298	439.31	1263	0.348
Carbon tetrachloride, CCl_4	4.56	276	556.4	0.272	550.72	1604	0.343
Hydrogen, H_2	1.294	65.5	32.99	0.309	30.534	89	0.343
Nitrogen, N_2	3.39	90.1	126.2	0.291	310.77	1035	0.300
Oxygen, O_2	5.08	78	154.8	0.308	410.26	1460	0.281
Chlorine, Cl_2	7.71	124	417	0.276	572.58	2030	0.282
Bromine, Br_2	10.3	135	584	0.287	1125.2	3120	0.380
Helium, He	0.229	58	5.2	0.307	68.966	120	0.575
Neon, Ne	2.73	41.7	44.4	0.303	479.62	1442	0.333
Argon, Ar	4.86	75.2	150.7	0.292	531.91	1656	0.321
Krypton, Kr	5.50	92.3	209.4	0.292	910.08	3000	0.303
Xenon, Xe	5.88	119	289.7	0.291	1100.8	3560	0.309
Radon, Rn	6.3	—	377		-	4400	
Water, H_2O	22.12	59.1	647.3	0.243	304.57	1000	0.305
Heavy water, D_2O	21.88	54.9	644.2	0.224	364.30	1109	0.331

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Table 11.5 The cohesive energies U_o of the elements in units of kJ mol^{-1}. U_o is the energy required to separate the atoms of a solid at $T = 0$ K into isolated neutral atoms.

Z	Element	U_o (kJ mol^{-1})	Z	Element	U_o (kJ mol^{-1})	Z	Element	U_o (kJ mol^{-1})
1	Hydrogen	—	32	Germanium	372	63	Europium	179
2	Helium	—	33	Arsenic	285.3	64	Gadolinium	400
3	Lithium	158	34	Selenium	237	65	Terbium	391
4	Beryllium	326	35	Bromine	118	66	Dysprosium	294
5	Boron	56[illegible]	36	Krypton	11.2	67	Holmium	302
6	Carbon	711	37	Rubidium	82.2	68	Erbium	317
7	Nitrogen	474	38	Strontium	166	69	Thulium	233
8	Oxygen	2[illegible]1	39	Yttrium	422	70	Ytterbium	154
9	Fluorine	81	40	Zirconium	603	71	Lutetium	428
10	Neon	1.92	41	Niobium	730	72	Hafnium	621
11	Sodium	107	42	Molybdenum	658	73	Tantalum	782
12	Magnesium	145	43	Technetium	661	74	Tungsten	859
13	Aluminium	327	44	Ruthenium	650	75	Rhenium	775
14	Silicon	446	45	Rhodium	554	76	Osmium	788
15	Phosphorous	331	46	Palladium	376	77	Iridium	670
16	Sulphur	275	47	Silver	284	78	Platinum	564
17	Chlorine	135	48	Cadmium	112	79	Gold	368
18	Argon	7.74	49	Indium	243	80	Mercury	65
19	Potassium	90.1	50	Tin	303	81	Thallium	182
20	Calcium	178	51	Antimony	265	82	Lead	196
21	Scandium	376	52	Tellurium	21[illegible]	83	Bismuth	210
22	Titanium	468	53	Iodine	107	84	Polonium	144
23	Vanadium	512	54	Xenon	1[illegible]	85	Astatine	—
24	Chromium	395	55	Caesium	77.6	86	Radon	18.5
25	Manganese	282	56	Barium	[illegible]	87	Francium	—
26	Iron	413	57	Lanthanum	431	[illegible]	Radium	160
27	Cobalt	424	58	Cerium	417	89	Actinium	410
28	Nickel	428	59	Praseodymium	357	[illegible]	Thorium	598
29	Copper	336	60	Neodymium	328	91	Protactinium	—
30	Zinc	130	61	Promethium	—	92	Uranium	536
31	Gallium	271	62	Samarium	206			

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Table 11.6 The equilibrium vapour pressure (Pa) of water substance above the solid or liquid surface as a function of temperature. The shaded data on the liquid corresponds to data taken on supercooled water.

T (°C)	Solid	Liquid	*T* (°C)	Solid	Liquid
–90	0.009	—	–15	165.5	191.50
–80	0.053	—	–14	181.5	208.03
–70	0.258	—	–13	198.7	225.50
–60	1.077	—	–12	217.6	244.57
–50	3.940	—	–11	238.0	264.98
–40	12.88	—	–10	260.0	286.58
–30	38.12	—	–9	284.2	310.18
–29	42.27	—	–8	310.2	335.26
–28	46.80	—	–7	338.3	362.06
–27	51.81	—	–6	368.7	390.86
–26	57.34	—	–5	401.8	421.80
–25	63.47	—	–4	437.4	454.74
–24	70.14	—	–3	475.8	489.81
–23	77.34	—	–2	517.4	527.55
–22	85.34	—	–1	562.4	567.83
–21	94.01	—	0	610.6	610.6
–20	103.4	—	1	—	656.9
–19	113.8	—	2	—	706.0
–18	125.2	—	3	—	758.1
–17	137.5	—	4	—	813.6
–16	151.0	—	5	—	872.5
			6	—	935.2
			7	—	1002
			8	—	1073
			9	—	1148
			10	—	1228.1
			11	—	1312.7
			12	—	1402.6
			13	—	1497.7
			14	—	1598.5
			15	—	1705.3

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Table 11.7 The melting, boiling and triple-point temperatures of various substances. The T_{tr} values are often known extremely accurately. The T_M and T_B values are typically known to within ≈ 10 mK.

Substance	T_M (K)	T_{Tr} (K)	T_B (K)
Oxygen	54.35	54.3584	90.188
Nitrogen	63.15	63.150	77.352
Argon	83.75	83.8058	87.29
Water	273.15	273.16	373.15

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Table W2.1 Molar magnetic susceptibility of the elements at around room temperature. The data are summarised in Figure W2.3. The shading in the table corresponds to the shading in Figure and highlights elements with a large susceptibilities.

Z	Element, atomic mass (u) and density (kg m^{-3})			χ_M (m^3 mol^{-1})
1	Hydrogen, H	1.008	89	—
2	Helium, He	4.003	120	—
3	Lithium, Li	6.941	533	1.78×10^{-10}
4	Beryllium, Be	9.012	1846	-1.17×10^{-10}
5	Boron, B	10.81	2466	-8.43×10^{-11}
6	Carbon, C	12.01	2266	-7.57×10^{-11}
7	Nitrogen, N	[illegible]	1035	—
8	Oxygen, O	16	1460	—
9	Fluorine, F	[illegible]	1140	—
10	Neon, Ne	20.18	1442	-8.48×10^{-11}
11	Sodium, Na	[illegible]	966	2.02×10^{-10}
12	Magnesium, Mg	[illegible]	1738	1.65×10^{-10}
13	Aluminium, Al	[illegible]	[illegible]	2.08×10^{-10}
14	Silicon, Si	28.09	[illegible]329	-5.06×10^{-11}
15	Phosphorus, P	30.97	[illegible]820	-3.41×10^{-10}
16	Sulphur, S	32.06	2086	-1.95×10^{-10}
17	Chlorine, Cl	35.45	2030	—
18	Argon, A	39.95	[illegible]	—
19	Potassium, K	39.1	862	2.62×10^{-10}
20	Calcium, Ca	40.08	1530	5.61×10^{-10}
21	Scandium, Sc	44.96	2992	$[illegible] \times 10^{-9}$
22	Titanium, Ti	47.9	4508	1.92×10^{-9}
23	Vanadium, V	50.94	6090	3.20×10^{-4}
24	Chromium, Cr	52	7194	$[illegible] \times 10^{-9}$
25	Manganese, Mn	54.94	7473	$6.[illegible] \times 10^{-4}$
26	Iron, Fe	55.85	7873	Fe[illegible]
27	Cobalt, Co	58.93	8800	Ferro
28	Nickel, Ni	58.7	8907	Ferro
29	Copper, Cu	63.55	8933	-6.87×10^{-11}
30	Zinc, Zn	65.38	7135	-1.44×10^{-10}
31	Gallium, Ga	69.72	5905	-2.72×10^{-10}
32	Germanium, Ge	72.59	5323	-9.64×10^{-11}
33	Arsenic, As	74.92	5776	-6.87×10^{-11}
34	Selenium, Se	78.96	4808	-3.16×10^{-10}
35	Bromine, Br	79.9	3120	—
36	Krypton, Kr	83.8	3000	—
37	Rubidium, Rb	85.47	1533	2.13×10^{-10}
38	Strontium, Sr	87.62	2583	1.16×10^{-9}
39	Yttrium, Y	88.91	4475	2.40×10^{-9}
40	Zirconium, Zr	91.22	6507	1.53×10^{-9}
41	Niobium, Nb	92.91	8578	2.56×10^{-9}
42	Molybdenum, Mo	95.94	10222	1.15×10^{-9}
43	Technetium, Tc	97	11496	3.01×10^{-9}
44	Ruthenium, Ru	101.1	12360	5.43×10^{-10}
45	Rhodium, Rh	102.9	12420	1.40×10^{-9}
46	Palladium, Pd	106.4	11995	7.13×10^{-9}
47	Silver, Ag	107.9	10500	-2.45×10^{-10}
48	Cadmium, Cd	112.4	8647	-2.48×10^{-10}
49	Indium, In	114.8	7290	-8.04×10^{-10}
50	Tin, Sn	118.7	7285	-4.75×10^{-10}
51	Antimony, Sb	121.7	6692	-1.22×10^{-9}
52	Tellurium, Te	127.6	6247	-4.98×10^{-10}
53	Iodine, I	126.9	4953	-5.58×10^{-10}
54	Xenon, Xe	131.3	3560	-5.51×10^{-10}
55	Caesium, Cs	132.9	1900	3.72×10^{-10}
56	Barium, Ba	137.3	3594	2.61×10^{-10}
57	Lanthanum, La	138.9	6174	1.53×10^{-9}
58	Cerium, Ce	140.1	6711	3.04×10^{-8}
59	Praseodymium, Pr	140.9	6779	6.30×10^{-8}
60	Neodymium, Nd	144.2	7000	7.07×10^{-8}
61	Promethium, Pm	145	7220	—
62	Samarium, Sm	150.4	7536	2.29×10^{-8}
63	Europium, Eu	152	5248	4.27×10^{-7}
64	Gadolinium, Gd	157.2	7870	Ferro
65	Terbium, Tb	158.9	8267	1.83×10^{-6}
66	Dysprosium, Dy	162.5	8531	1.30×10^{-6}
67	Holmium, Ho	164.9	8797	9.05×10^{-7}
68	Erbium, Er	167.3	9044	5.57×10^{-7}
69	Thulium, Tm	168.9	9325	3.21×10^{-7}
70	Ytterbium, Yb	173	6966	3.13×10^{-9}
71	Lutetium, Lu	175	9842	2.28×10^{-10}
72	Hafnium, Hf	178.5	13276	9.46×10^{-10}
73	Tantalum, Ta	180.9	16670	1.94×10^{-9}
74	Tungsten, W	183.9	19254	7.36×10^{-10}
75	Rhenium, Re	186.2	21023	8.49×10^{-10}
76	Osmium, Os	190.2	22580	1.24×10^{-10}
77	Iridium, Ir	192.2	22550	3.21×10^{-10}
78	Platinum, Pt	195.1	21450	2.54×10^{-9}
[illegible]9	Gold, Au	197	19281	-3.51×10^{-10}
80	M[illegible]ury, Hg	200.6	13546	—
81	[illegible]hall[illegible]m, Tl	204.4	11871	-6.40×10^{-10}
[illegible]2	[illegible], Pb	207.2	11343	-2.88×10^{-10}
83	Bismut[illegible], Bi	209	9803	-3.52×10^{-9}
84	Po[illegible]um, Po	209	9400	—
85	[illegible]statine, At	210	—	—
86	Rad[illegible]on, Rn	222	4400	—
87	Francium, Fr	223	—	—
88	Radium, Ra	226	5000	—
89	Actinium, [illegible]	[illegible]27	10060	—
90	Thorium, Th	[illegible]	11725	1.67×10^{-9}
91	Protactinium, Pa	231	15370	—
92	Uranium, U	[illegible]	19050	5.14×10^{-9}
93	Neptunium, Np	[illegible]	20250	—
94	Plutonium, Pu	[illegible]	[illegible]840	7.73×10^{-9}
95	Americium, Am	[illegible]43	[illegible]70	1.22×10^{-8}

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[glo:CAWF]CAWF — The CAWF. 2009

[glo:CMRF]Center-of-Momenta Reference Frame — The Center-of-Momenta Reference Frame. 1316, 1317, 1320, 1321, 1325, 1326, 1603, 1604, 1637, 2459, 2460, 2477, 2540, 2739

[glo:CPWF]CPWF — The CPWF. 2009

[glo:LRF]Laboratory Reference Frame — The Laboratory Reference Frame. 2477

[glo:MPW]MPW — The MPW. 1997

[glo:PDE]PDE — The PDE. 1990, 1993

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