

# LECTURES ON GROUP THEORETICAL QUANTUM TOMOGRAPHY

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## 1. INTRODUCTION AND AN ELEMENTARY EXAMPLE

**1.1. Introduction to the problem.** In the usual Hilbert space formalism of quantum mechanics, a (complex separable) Hilbert space is associated to each quantum physical system. We denote by  $\mathcal{H}$  this Hilbert space, that is called (sometimes) the “states space” of the quantum system. The physical quantities of the quantum system (position, momentum, energy, angular momentum, etc.) are described by self-adjoint operators acting in  $\mathcal{H}$ . Let  $A$  be one such operator. The spectral theorem associates to  $A$  a measure

$$P_A : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{P}(\mathcal{H}),$$

defined on the  $\sigma$ -algebra of Borel sets of  $\mathbb{R}$ , that takes values in the lattice of projection operators on  $\mathcal{H}$ .  $P_A$  is called the “spectral measure” of  $A$ .

The pure states of the quantum system are described by (and identified with) the projections onto the one-dimensional subspaces of  $\mathcal{H}$ . If  $M$  denotes any closed subspace of  $\mathcal{H}$ , we shall denote  $P[M]$  the projection operator onto  $M$ . Every one-dimensional subspace of  $\mathcal{H}$  is of the form

$$\mathbb{C}v, \quad v \in \mathcal{H}, \quad \|v\| = 1.$$

Then the projection operator onto  $\mathbb{C}v$  has the explicit form

$$P[\mathbb{C}v]w = \langle v|w\rangle v,$$

for all  $w \in \mathcal{H}$ . It has to be stressed that if we choose another base vector of the subspace  $\mathbb{C}v$ ,

$$\mathbb{C}v = \mathbb{C}u, \quad u \in \mathcal{H}, \quad \|u\| = 1,$$

we have necessarily

$$u = e^{i\varphi}v, \quad \varphi \in \mathbb{R}.$$

Hence

$$\langle u|w\rangle u = \langle v|w\rangle v,$$

for all  $w \in \mathcal{H}$ . This shows that the projection operator  $P[\mathbb{C}v]$  is intrinsic to the subspace  $\mathbb{C}v$ , and it does not depend on the basis, either  $\{u\}$  or  $\{v\}$ , chosen for this subspace. In the following we shall use, for simplicity, the notation

$$P[v], \quad \|v\| = 1,$$

instead of the complete one,  $P[\mathbb{C}v]$ .

The physical interpretation of the mathematical formalism briefly described above is encoded into the (generalized) Born rule, which can be stated as follows:

Suppose that the state of a quantum system is known to be (at a certain fixed time  $t$ )  $P[w]$ , then the physical quantity represented by the self-adjoint operator  $A$ , has values (at the time  $t$ ) distributed on the real line according to the probability distribution

$$E \mapsto \langle w|P_A(E)w\rangle, \quad E \in \mathcal{B}(\mathbb{R}).$$

In the sequel, we shall use also the notation  $\alpha_w^A(E)$  for this probability distribution. We shall always admit, in these notes, the idealization that the probability distributions of the physical quantities can be determined explicitly by measuring any quantity on a statistical ensemble of systems, all in the same state.

It is possible to consider non-pure states as well. That is, statistical mixtures of pure states. These non-pure states are described by (and identified with) operators of the form

$$T = \sum_{i=0}^{\infty} f_i P[w_i],$$

where  $P[w_i]$  is a pure state for each  $i \geq 1$ ,  $0 \leq f_i \leq 1$  for each  $i \geq 1$  and  $\sum_{i=1}^{\infty} f_i = 1$ . The coefficients  $f_i$  are the “statistical weights” of the pure states  $P[w_i]$  that are components of the state  $T$ . The “infinite weighted” sum of pure states has to be understood in the following, precise way:

The probability distribution of the values of  $A$  in the state  $T = \sum_{i=1}^{\infty} f_i P[w_i]$  is given by

$$E \mapsto \sum_{i=1}^{\infty} f_i \langle w_i | P_A(E) w_i \rangle, \quad E \in \mathcal{B}(\mathbb{R}).$$

A warning: the representation of a non-pure state as a weighted infinite sum of pure states, of the form

$$T = \sum_{i=1}^{\infty} f_i P[w_i]$$

is far from being unique. There are “many” different ways of expressing  $T$  as a different weighted sum

$$T = \sum_{i=1}^{\infty} \rho_i P[u_i],$$

where the sequences of weights and of pure states entering the two decompositions of  $T$  are different. This is the source of many of the difficulties of interpretation of quantum mechanics.

In the sequel, we shall limit ourselves to considering only pure states, the extension of the formalism to non-pure states is suggested by the previous formula.

After these (very sketchy) premises we can pose a problem whose discussion will lead us to the main problem of quantum tomography.

We consider a quantum system whose states space is  $\mathcal{H}$ . We know that the system is in some pure state, that we do not know. Is it possible “to measure” the (unknown) state of the system by measuring the probability distribution of some physical quantity? It is easy to see that the answer to this question is “no”, if we consider the distribution of one single physical quantity.

In fact, let us suppose that  $A$  is a self-adjoint operator on  $\mathcal{H}$  that describes some physical quantity of the system, we recall that  $P_A(\cdot)$  denotes the spectral measure of  $A$ . We are going to show that it is possible to find two different states such that  $A$  has the same distribution in both of them. It is clear that this fact makes it impossible to reconstruct the state from the (experimental) knowledge of the probability distribution of a single physical quantity.

By the spectral theorem, we have

$$\{f(A) | f \text{ a bounded Borel function on } \mathbb{R}\} = A'' \subset A'.$$

In particular, if  $f_1$  and  $f_2$  are two Borel functions,  $f_1(A)$  and  $f_2(A)$  commute among them. We fix a Borel function on  $\mathbb{R}$  that takes values in the set  $\{z \in \mathbb{C} | |z| = 1\}$ , and is not constant. Then

$f(A)$  is a unitary operator that is not trivial in the sense that it is not simply multiplication by a scalar of modulus 1. Let  $P[w]$  denote any state, then  $P[f(A)w]$  is another state, and our hypothesis on  $f$  assumes that  $P[w]$  and  $P[f(A)w]$  are different states. The probability distribution of  $A$  in the state  $P[f(A)w]$  is given by

$$\langle f(A)w | P_A(E) f(A)w \rangle = \langle w | f(A)^{-1} P_A(E) f(A)w \rangle.$$

We have

$$P_A(E) = \chi_E(A),$$

for all  $E \in \mathcal{B}(\mathbb{R})$ , then  $f(A)$  and  $P_A(E)$  commute for every  $E \in \mathcal{B}(\mathbb{R})$ . Hence

$$\langle w | f(A)^{-1} P_A(E) f(A)w \rangle = \langle w | P_A(E)w \rangle.$$

This shows that the probability distributions of  $A$  in the two (different) states  $P[w]$  and  $P[f(A)w]$  are identical.

At this stage there is a natural question. Is it possible to reconstruct the state from the knowledge of two (not necessarily compatible) physical quantities? A particular case of this problem is the so called ‘‘Pauli problem’’. In the next paragraph we shall discuss (very briefly) the wave mechanics of a particle in one dimension, and we shall give the precise formulation of the Pauli problem.

**1.2. Wave mechanics in one dimension and the Pauli problem.** We consider a spinless particle moving on a straight line. The quantum description of such a system is known under the name of ‘‘one dimensional wave mechanics’’, and it is based on the following points.

- (1) The Hilbert space associated to the system is  $L^2(\mathbb{R})$ .
- (2) The position and momentum of the particle are described by the operators  $Q$  and  $P$  (respectively), that are defined (only in a formal way) by

$$\begin{aligned} (Q\psi)(x) &= x\psi(x), \\ (P\psi)(x) &= \frac{1}{i} \frac{d}{dx} \psi(x). \end{aligned}$$

Some care is needed when handling these definitions:  $Q$  and  $P$  are unbounded operators whose domains are to be defined with precision.

- (3) The states of the system are identified with the projection operators onto the one-dimensional subspaces of  $L^2(\mathbb{R})$ . In turn, they are written by choosing an element of the subspace of  $L^2$ - norm one:

$$\psi \in L^2(\mathbb{R}), \quad \|\psi\| = 1.$$

If we choose another element  $\psi'$  (of the one-dimensional subspace spanned by  $\psi$ ) of  $L^2$ -norm one, we have necessarily  $\psi' = e^{i\theta}\psi$ ,  $\theta \in \mathbb{R}$ , and  $P[\psi'] = P[\psi]$ .

- (4) The physical interpretation is given by the ‘‘Born rule’’. Let  $E \in \mathcal{B}(\mathbb{R})$ , then

$$\int_E |\psi(x)|^2 dx$$

represents the probability that the position of the particle is contained in  $E$ , when its state is  $P[\psi]$ .

- (5) We have a Born rule for the momentum. We define the Fourier-Plancherel operator in  $L^2(\mathbb{R})$  by defining

$$\hat{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} \psi(x) dx,$$

if  $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ , and extending the definition to  $L^2(\mathbb{R})$  by continuity. The resulting map

$$\psi \mapsto \hat{\psi} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$$

is an unitary operator (this is the content of the Plancherel theorem). Then the Born rule for the momentum says that

$$\int_E |\hat{\psi}(k)|^2 dk$$

represents the probability that the momentum of the particle has value in  $E$ , if the state of the system is  $P[\psi]$ .

We are now ready to state the Pauli problem. W. Pauli in his classical exposition “Die Allgemeinen Prinzipien der Wellenmechanik” poses (in a foot note at page 17) the following problem: if the probability distributions of the position and momentum of the particle are known, is it possible to deduce from them what is the state of the particle? Due to the form of the probability distributions of position and momentum, it is clear that the Pauli problem can be reformulated in the following way: Let  $\psi \in L^2(\mathbb{R})$ ,  $\|\psi\| = 1$ . Is it possible to find another  $\varphi \in L^2(\mathbb{R})$ ,  $\|\varphi\| = 1$  such that  $|\psi(x)|^2 = |\varphi(x)|^2$  and  $|\hat{\psi}(k)|^2 = |\hat{\varphi}(k)|^2$ ? It is clear that the preceding relations are satisfied by choosing  $\varphi(x) = e^{i\theta} \psi(x)$ ,  $\theta \in \mathbb{R}$ . However, with this choice we have  $P[\varphi] = P[\psi]$ . Are there any other solutions?

Let us consider  $\psi \in L^2(\mathbb{R})$ , defined by

$$\begin{aligned} \psi(x) &= \left(\frac{2a}{\pi}\right)^{1/4} e^{-(a+ib)x^2} \\ &= \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2} (\cos bx^2 - i \sin bx^2), \end{aligned}$$

$a, b \in \mathbb{R}$ ,  $a > 0$ . We have  $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ ,  $\|\psi\| = 1$ . The first statement is obvious, regarding the second we have

$$\begin{aligned} \|\psi\| &= \int |\psi(x)|^2 dx \\ &= \sqrt{\frac{2a}{\pi}} \int e^{-2ax^2} dx \\ &= 1, \end{aligned}$$

where we have used the formula  $\int e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ .

We have to compute  $\hat{\psi}(k)$ . We shall describe explicitly this calculation. The starting point is the classical formula (the Fourier transform of the Gaussian)

$$(e^{-\frac{x^2}{2}})^\wedge = e^{-\frac{k^2}{2}}.$$

By using the formula  $(f(ax))^\wedge = \frac{1}{a} \hat{f}\left(\frac{k}{a}\right)$ ,  $a > 0$ , we get also

$$(e^{-ax^2})^\wedge = \frac{1}{\sqrt{2a}} e^{-\frac{k^2}{4a}}.$$

We claim that this equality remains true if we replace  $a$  with  $a + ib$ ,  $a > 0$ . To prove this claim, we notice that

$$e^{-(a+ib)x^2} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$

$a > 0$ , and hence

$$(e^{-(a+ib)x^2})^\wedge(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} e^{-(a+ib)x^2} dx.$$

We regard the integral at the right hand side as a function of the complex variable  $(a + ib)$ , defined on the half plane  $\{a + ib | a > 0\}$ . By the theorem of derivation under the integral we have that the function is analytic on its domain of definition.

We consider also the function

$$a + ib \mapsto \frac{1}{(2(a + ib))^{1/2}} e^{-\frac{k^2}{4(a+ib)}},$$

defined by choosing the branch of the square root so that on the half-line  $\{a + ib | a > 0, b = 0\}$  it is given by  $\sqrt{2a}$ . In this way, the function is analytic on the half-plane  $\{a + ib | a > 0\}$ .

Finally, we consider the two functions

$$\frac{1}{(2(a + ib))^{1/2}} e^{-\frac{k^2}{4(a+ib)}}$$

and

$$(e^{-(a+ib)x^2})^\wedge(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} e^{-(a+ib)x^2} dx.$$

They are both analytic on the half-plane and coincide on the half-line, so they coincide on the whole half-plane. This proves our claim. We get, in conclusion

$$\begin{aligned} \hat{\psi}(k) &= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{2(a+ib)}} e^{-\frac{k^2}{4(a+ib)}} \\ &= \left(\frac{a}{2\pi(a^2+b^2)}\right)^{1/4} e^{-\frac{i}{2} \arctan \frac{b}{a}} e^{-\frac{(a-ib)}{4(a^2+b^2)} k^2}. \end{aligned}$$

Then we have

$$\begin{aligned} |\psi(x)| &= \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}, \\ |\hat{\psi}(k)| &= \left(\frac{a}{2\pi(a^2+b^2)}\right)^{1/4} e^{-\frac{a}{4(a^2+b^2)} k^2}. \end{aligned}$$

On the other hand, we define

$$\varphi(x) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-(a-ib)x^2}.$$

It has to be stressed that  $\varphi$  and  $\psi$  are not proportional, they define different quantum states. We have

$$\begin{aligned}\hat{\varphi}(k) &= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{(2(a-ib))^{1/2}} e^{-\frac{k^2}{4(a-ib)}} \\ &= \left(\frac{a}{2\pi(a^2+b^2)}\right)^{1/4} e^{\frac{i}{2} \arctan \frac{b}{a}} e^{-\frac{(a+ib)}{4(a^2+b^2)} k^2},\end{aligned}$$

so that

$$\begin{aligned}|\varphi(x)| &= \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}, \\ |\hat{\varphi}(k)| &= \left(\frac{a}{2\pi(a^2+b^2)}\right)^{1/4} e^{-\frac{a}{4(a^2+b^2)} k^2}.\end{aligned}$$

In conclusion, we have  $|\varphi(x)| = |\psi(x)|$  and  $|\hat{\varphi}(k)| = |\hat{\psi}(k)|$ , but  $\varphi$  and  $\psi$  define different states.

**1.3. The structure of the reconstruction formula.** We recall some simple facts about bounded operators on a Hilbert space  $\mathcal{H}$ . We fix a Hilbert basis  $(e_i)_{i \in \mathcal{I}}$  in  $\mathcal{H}$ , where  $\mathcal{I}$  is some countable index set. A bounded operator  $A$  on  $\mathcal{H}$  is completely determined by the matrix elements  $\langle e_i | A e_j \rangle$ ,  $i, j \in \mathcal{I}$ . If  $A$  is Hermitian, we have  $\langle e_i | A e_j \rangle = \overline{\langle e_j | A e_i \rangle}$  for all  $i, j \in \mathcal{I}$ . Hence the operator is completely determined by the knowledge of the matrix elements  $\langle e_i | A e_j \rangle$ ,  $i \geq j$  (by the matrix elements  $\langle e_i | A e_j \rangle$ ,  $j \geq i$  as well).

We know that the operator that describes a (pure) state of a quantum system is a projection operator  $P[w]$  onto a one-dimensional subspace of  $\mathcal{H}$ , and is therefore a bounded Hermitian operator. As such,  $P[w]$  is completely determined by the matrix elements  $\langle e_i | P[w] e_j \rangle$ ,  $j \geq i$  (or  $\beta \geq i$ ). The point is that  $\langle e_i | P[w] e_j \rangle$  cannot be directly interpreted as a probability or a family of probability distributions. In fact, we have

$$\begin{aligned}\langle e_i | P[w] e_j \rangle &= \langle e_i | \langle w | e_j \rangle w \rangle \\ &= \langle e_i | w \rangle \langle w | e_j \rangle \\ &= \langle w | T_{e_i, e_j} w \rangle,\end{aligned}$$

where the (bounded, rank-one) operator  $T_{e_i, e_j}$  is defined by  $T_{e_i, e_j}(v) = \langle e_i | v \rangle e_j$  for all  $v \in \mathcal{H}$ . Since  $T_{e_i, e_j}$  is not self-adjoint,  $\langle w | T_{e_i, e_j} w \rangle$  cannot be interpreted as the mean value, in the state  $P[w]$ , of some physical quantity. In particular, it cannot be experimentally determined (even ideally).

The main, basic, idea of group theoretical quantum tomography is expressing

$$\langle e_i | P[w] e_j \rangle = \langle w | T_{e_i, e_j} w \rangle$$

by means of the probability distributions of a suitable set of physical quantities. We shall prove (and this is the main goal of these lectures) that it is possible to determine

- (1) a family  $\{A_\omega | \omega \in \Omega\}$  of self-adjoint operators (representing physical quantities of the quantum system whose states space is  $\mathcal{H}$ ), parametrized by a probability space  $(\omega, \Omega)$  (or more generally, by a measure space),
- (2) a Hilbert basis  $(e_i)_{i \in \mathcal{I}}$  of  $\mathcal{H}$ ,
- (3) for each  $\omega \in \Omega$ , and each pair  $i, j \in \mathcal{I}$ , a function  $f_{i,j}^\omega(\lambda)$ ,  $\lambda \in \mathbb{R}$ ,

in such a way that we have an integral representation of the matrix elements of  $P[w]$ :

$$\langle e_i | P[w] e_j \rangle = \int_{\Omega} d\mu(\omega) \int_{\mathbb{R}} d\alpha_w^\omega(\lambda) f_{i,j}^\omega(\lambda),$$

where we have used the simpler notation  $\alpha_w^\omega$  instead of  $\alpha_w^{A_\omega}$ . We call an equality, that has the form of the one above, a “reconstruction formula” for the state  $P[w]$ .

We stress once more that the explicit determination of the set  $\{A_\omega | \omega \in \Omega\}$  and of the functions  $f_{i,j}^\omega$  so that a reconstruction formula holds, is the central problem of our approach to quantum tomography.

A reconstruction formula is, as such, an equality that expresses the matrix element  $\langle e_i | P[w] e_j \rangle$  in another (perhaps more complicate) way. However, it can be interpreted “a posteriori” as a device to reconstruct an unknown (pure) state of a quantum system. If we are given a family of probability distributions. and we know that they are the distributions of the  $A_\omega$ 's in a (not known) pure state, then we can plug them into the reconstruction formula and compute the matrix elements  $\langle e_i | P[w] e_j \rangle$ . This determines uniquely the state. The essential step in this recipe is the fact that the functions  $f_{i,j}^\omega$  do not depend on  $P[w]$ .

From this point of view, we have obtained an answer to the problem we have raised in the first two paragraphs: essentially, to what extent do the probability distributions of a set of physical quantities of a quantum system determine the state of the system.

We end with a simple observation. In the examples that we describe in the sequel, the set  $\{A_\omega | \omega \in \Omega\}$  has the following particular structure: there is a unitary operator  $U(\omega)$ , for all  $\omega \in \Omega$ , and a self-adjoint operator  $A$ , such that

$$A_\omega = U(\omega)^* A U(\omega),$$

for all  $\omega \in \Omega$ . In this case, we have

$$\begin{aligned} \alpha_w^\omega(E) &= \langle w | P_{A_\omega}(E) w \rangle \\ &= \langle w | U(\omega)^* P_A(E) U(\omega) w \rangle \\ &= \langle U(\omega) w | P_A(E) U(\omega) w \rangle \\ &= \alpha_{U(\omega)w}^A(E). \end{aligned}$$

The probability distribution  $\alpha_w^\omega = \alpha_{U(\omega)w}^A$  can be interpreted also as the probability distribution of the fixed physical quantity (interpreted by)  $A$ , in the state  $P[U(\omega)w]$ , which is the state “evolved” (or “transformed”) with  $U(\omega)$  from the state  $P[w]$ .

In the next paragraph we shall derive a reconstructing formula in a simple case: when  $\mathcal{H}$  is finite dimensional.

**1.4. A simple example.  $SU(2)$  quantum tomography.** In this paragraph, we show how to write explicitly a reconstruction formula for a system whose states space  $\mathcal{H}$  is finite dimensional. We consider a “spin  $j$ ” quantum system (you might prefer calling it a “ $2j + 1$ -level” system). The Hilbert space of states  $\mathcal{H}$ , associated to it, has dimension  $2j + 1$ ,  $\mathcal{H} \simeq \mathbb{C}^{2j+1}$ .

We briefly recall some facts about unitary representations of a compact group  $G$  that will be used in the rest of this paragraph. Let  $G$  denote a compact (matrix Lie) group. Then

- (1)  $G$  is unimodular and its Haar measures are finite, we denote  $dg$  the Haar measure on  $G$  normalized to 1 (i.e.  $\int_G dg = 1$ ),
- (2) the (unitary) irreducible representations of  $G$  are finite dimensional,

(3) if  $(\Pi, V_\Pi)$  is an irreducible (unitary, finite dimensional) representation of  $G$ , we have the Shur's orthogonality relations

$$\int_G \overline{\langle u | \Pi(g)v \rangle} \langle w | \Pi(g)y \rangle dg = \frac{1}{\dim V_\Pi} \langle w | u \rangle \langle v | y \rangle,$$

for all  $u, v, w, y \in V_\Pi$ .

Let  $(\Pi, V_\pi)$  be any finite dimensional representation of a (linear) Lie group  $G$ , not necessarily compact, then there is a unique ( $\mathbb{R}$ -linear) representation of its Lie algebra  $\mathfrak{g}$ , acting in  $V_\Pi$ , that we denote  $d\Pi$ . It is defined by

$$\Pi(e^X) = e^{d\Pi(X)},$$

for all  $X \in \mathfrak{g}$ . It can be computed as

$$d\Pi(X) = \left. \frac{d}{dt} \Pi(e^{tX}) \right|_{t=0},$$

for all  $X \in \mathfrak{g}$ , and it has the property

$$d\Pi(AXA^{-1}) = \Pi(A)d\Pi(X)\Pi(A)^{-1},$$

for all  $X \in \mathfrak{g}$ ,  $A \in G$ .

In the sequel, we shall use the representation of  $SU(2)$ , so we recall the definition of this group, its structure, the structure of its Lie algebra and the classification of its irreducible (unitary) representations.

The group  $SU(2)$  is the group of  $2 \times 2$  unitary matrices (with complex entries) with determinant 1:

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

It is a compact linear Lie group. Its Lie algebra, denoted  $su(2)$ , is given by

$$su(2) = \left\{ \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

It is a real three-dimensional vector space. A basis of  $su(2)$  is given by

$$X_1 = \frac{1}{2}i\sigma_1, \quad X_2 = \frac{1}{2}i\sigma_2, \quad X_3 = \frac{1}{2}i\sigma_3,$$

where  $\sigma_1, \sigma_2$  and  $\sigma_3$  denote the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The basis  $\{X_1, X_2, X_3\}$  is orthonormal with respect to the scalar product  $2\text{Tr}[X^*Y]$ ,  $X, Y \in su(2)$ , and we have the Lie brackets

$$[X_1, X_2] = -X_3, \quad [X_3, X_1] = -X_2, \quad [X_2, X_3] = -X_1.$$

We can define a linear isomorphism between  $\mathbb{R}^3$  and  $su(2)$  by means of this basis. We denote

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, x_3) \in \mathbb{R}^3, \\ \sigma(\mathbf{x}) &= x_1X_1 + x_2X_2 + x_3X_3 \in su(2), \end{aligned}$$

then  $\sigma : \mathbb{R}^3 \rightarrow su(2)$  is a vector spaces isomorphism that preserves the scalar products of  $\mathbb{R}^3$  and  $su(2)$ . If we denote

$$\begin{aligned} S^2 &= \{\mathbf{u} \in \mathbb{R}^3 | u_1^2 + u_2^2 + u_3^2 = 1\} \\ r &= \sqrt{x_1^2 + x_2^2 + x_3^2}, \end{aligned}$$

we can also write

$$\sigma(\mathbf{x}) = \sigma\left(r \begin{pmatrix} \mathbf{x} \\ r \end{pmatrix}\right) = r\sigma(\mathbf{n}).$$

With these notations, we can state a few fundamental facts about the interplay between the Lie algebra  $su(2)$  and the group  $SU(2)$ :

- (1) any element  $g$  of  $SU(2)$  can be written in a unique way as

$$g = e^{r\sigma(\mathbf{n})},$$

$$\mathbf{n} \in S^2, 0 \leq r < 2\pi,$$

- (2) with this coordinatization of the elements of the group, the Haar measure  $dg$ , of  $SU(2)$  (we recall normalized to 1) is given by

$$dg = \frac{1}{\pi} \sin^2\left(\frac{r}{2}\right) dS(\mathbf{n}) dr,$$

where  $dS(\mathbf{n})$  denotes the normalized natural surface area element on the unit sphere  $S^2 \subset \mathbb{R}^3$ . Hence the integral of a function  $f$ , defined on  $SU(2)$ , can be written as

$$\int_{SU(2)} f(g) dg = \int_{S^2} dS(\mathbf{n}) \int_0^{2\pi} dr \sin^2\left(\frac{r}{2}\right) f(e^{r\sigma(\mathbf{n})}).$$

The group  $SU(2)$  has an infinite sequence of irreducible, unitary, finite dimensional representations that can be parametrized by an index  $j$ , ranging on the non-negative half-integers,  $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ . For each value of  $j$ , there is a unique irreducible unitary representation  $D^j$  acting on a vector space  $V_j$ ,  $\dim V_j = 2j + 1$ . These representations are usually described by means of the action, on  $V_j$ , of the derived representation  $dD^j$  of  $su(2)$ . We denote

$$\begin{aligned} J_k &= \frac{1}{i} dD^j(X_k), \\ \mathbf{n} \cdot \mathbf{J} &= \frac{1}{i} dD^j(\sigma(\mathbf{n})) \\ &= n_1 J_1 + n_2 J_2 + n_3 J_3, \\ J^2 &= J_1^2 + J_2^2 + J_3^2, \end{aligned}$$

where  $k = 1, 2, 3$  and  $\mathbf{n} \in S^2$ . The operators  $J^2$  and  $J_k$  ( $k = 1, 2, 3$ ) are Hermitian operators acting in  $V_j$ . Writing  $g = e^{r\sigma(\mathbf{n})}$ , we have

$$\begin{aligned} D^j(g) &= e^{dD^j(r\sigma(\mathbf{n}))} \\ &= e^{ir \frac{1}{i} dD^j(\sigma(\mathbf{n}))} \\ &= e^{ir \mathbf{n} \cdot \mathbf{J}}. \end{aligned}$$

The properties of the representations  $dD^j$  of  $su(2)$  can be stated as follows:

- (1)  $J^2$  acts in  $V_j$  as the scalar  $j(j+1)$ ,  
(2) for each  $\mathbf{n} \in S^2$ , there is an orthonormal basis  $(e_\lambda^\mathbf{n})_{\lambda=-j}^j$  of  $V_j$  such that the Hermitian operator  $\mathbf{n} \cdot \mathbf{J}$  is diagonal on this basis, that is  $(\mathbf{n} \cdot \mathbf{J})e_\lambda^\mathbf{n} = \lambda e_\lambda^\mathbf{n}$ ,  $\lambda = -j, -j+1, \dots, j-1, j$ .

The finite dimensional vector space  $V_j$  is the states space of the quantum system made by a single spin  $j$ . The operator  $\mathbf{n} \cdot \mathbf{J}$  represents the projection of the spin along the direction  $\mathbf{n}$ .

Having recalled these facts, we come back to the problem of proving a reconstruction formula for the spin  $j$  system. We fix an orthonormal basis of  $V_j$  by choosing a direction  $\mathbf{n}$ . The usual choice is  $\mathbf{n}_z = (0, 0, 1)$ . To simplify notations we shall write  $e_\lambda$  instead of the complete notation  $e_\lambda^{\mathbf{n}_z}$ .

On  $V_j$  we have the irreducible representation  $D^j$  of  $SU(2)$ . Using the orthogonality relations we have

$$\begin{aligned} \langle e_\mu | P[w] e_\nu \rangle &= \langle e_\mu | w \rangle \langle w | e_\nu \rangle \\ &= \dim V_j \int_{SU(2)} \langle e_\mu | D^j(g) e_\nu \rangle \langle D^j(g) w | w \rangle dg \\ &= \frac{2j+1}{\pi} \int_{S^2} dS(\mathbf{u}) \int_0^{2\pi} \sin^2\left(\frac{r}{2}\right) \langle e_\mu | e^{ir\mathbf{J} \cdot \mathbf{n}} e_\nu \rangle \langle e^{ir\mathbf{J} \cdot \mathbf{n}} w | w \rangle. \end{aligned}$$

Let us consider the matrix element  $\langle e^{ir\mathbf{n} \cdot \mathbf{J}} w | w \rangle$ . Since the basis  $(e_\lambda^{\mathbf{n}})_{\lambda=-j}^j$  diagonalizes  $\mathbf{n} \cdot \mathbf{J}$ , we have by the spectral theorem

$$\begin{aligned} \langle e^{ir\mathbf{n} \cdot \mathbf{J}} w | w \rangle &= \sum_{\lambda=-j}^j e^{-ir\lambda} \langle e | P[e_\lambda^{\mathbf{n}}] w \rangle \\ &= \sum_{\lambda=-j}^j e^{-ir\lambda} |\langle e_\lambda^{\mathbf{n}} | w \rangle|^2. \end{aligned}$$

The physical meaning of  $|\langle e_\lambda^{\mathbf{n}} | w \rangle|^2$  is the probability that the physical quantity  $\mathbf{n} \cdot \mathbf{J}$  (the projection of the spin along the direction  $\mathbf{n}$ ) has value  $\lambda$ , when the state of the spin is  $P[w]$ .

The other matrix element can be computed in the same way, by using the spectral theorem:

$$\begin{aligned} \langle e_\mu | e^{ir\mathbf{n} \cdot \mathbf{J}} e_\nu \rangle &= \sum_{\rho=-j}^j e^{i\rho r} \langle e_\mu | P[e_\rho^{\mathbf{n}}] e_\nu \rangle \\ &= \sum_{\rho=-j}^j e^{i\rho r} \langle e_\mu | e_\rho^{\mathbf{n}} \rangle \langle e_\rho^{\mathbf{n}} | e_\nu \rangle. \end{aligned}$$

Putting all pieces together, we get the reconstruction formula

$$\begin{aligned} \langle e_\mu | P[w] e_\nu \rangle &= \frac{2j+1}{\pi} \int_{S^2} dS(\mathbf{n}) \int_0^{2\pi} dr \sin^2\left(\frac{r}{2}\right) \left( \sum_{\lambda=-j}^j e^{-ir\lambda} |\langle e_\lambda^{\mathbf{n}} | w \rangle|^2 \right) \left( \sum_{\rho=-j}^j e^{i\rho r} \langle e_\mu | e_\rho^{\mathbf{n}} \rangle \langle e_\rho^{\mathbf{n}} | e_\nu \rangle \right) \\ &= \int_{S^2} dS(\mathbf{n}) \sum_{\lambda=-j}^j |\langle e_\lambda^{\mathbf{n}} | w \rangle|^2 f_{\mu,\nu}^{\mathbf{n}}(\lambda), \end{aligned}$$

where we have defined

$$f_{\mu,\nu}^{\mathbf{n}}(\lambda) = \frac{2j+1}{\pi} \int_0^{2\pi} \sin^2\left(\frac{r}{2}\right) e^{ir\lambda} \left( \sum_{\rho=-j}^j e^{i\rho r} \langle e_\mu | e_\rho^{\mathbf{n}} \rangle \langle e_\rho^{\mathbf{n}} | e_\nu \rangle \right) dr.$$

We stress that the functions  $f_{\mu,\nu}^{\mathbf{n}}(\lambda)$  can be computed explicitly, they do not depend on the state  $P[w]$  of the system. rather they depend on the set of physical quantities represented by the Hermitian operators  $\{\mathbf{n} \cdot \mathbf{J} | \mathbf{n} \in S^2\}$ .

Notice that the reconstruction formula

$$\langle e_\mu | P[w] e_\nu \rangle = \int_{S^2} dS(\mathbf{n}) \sum_{\lambda=-j}^j |\langle e_\lambda^{\mathbf{n}} | w \rangle|^2 f_{\mu,\nu}^{\mathbf{n}}(\lambda)$$

has the structure we have anticipated in the preceding paragraph:

- (1)  $(S^2, dS(\mathbf{n}))$  is the probability space that labels the physical quantities, it has the role of  $(\Omega, \mu)$  in the general presentation,
- (2)  $\{\mathbf{n} \cdot \mathbf{J} | \mathbf{n} \in S^2\}$  is the set of physical quantities whose probability distributions are needed, it has the role of  $\{A_\omega | \omega \in \Omega\}$ ,
- (3) the integration over  $\lambda$  is replaced by a finite sum, the probability distribution  $d\alpha_w^\omega(\lambda)$  is replaced by the discrete probability distribution that assigns to each  $\lambda$  in the set  $\{-j, -j+1, \dots, j-1, j\}$  the probability  $|\langle e_\lambda^{\mathbf{n}} | w \rangle|^2$ .

We end by describing another way of writing the scalar products  $|\langle e_\lambda^{\mathbf{n}} | w \rangle|^2$  and  $\langle e_\mu | e_\rho^{\mathbf{n}} \rangle \langle e_\rho^{\mathbf{n}} | e_\nu \rangle$ , that may lead to an alternative physical interpretation.

We need to recall the covering homomorphism of  $SU(2)$  onto  $SO(3)$ . If  $g$  is any element of  $SU(2)$ , there is a unique element,  $R(g)$ , of  $SO(3)$ , such that the equality

$$\sigma(R(g)\mathbf{n}) = g\sigma(\mathbf{n})g^{-1}$$

holds for all  $\mathbf{n} \in S^2$ . The surjective map  $\sigma : SU(2) \rightarrow SO(3)$  is called the "covering homomorphism" of  $SU(2)$  onto  $SO(3)$ .

Since  $SO(3)$  acts transitively on  $S^2$ , given any  $\mathbf{n} \in S^2$ , there exists a (non-unique)  $R \in SO(3)$ , such that  $R(\mathbf{n}_z) = \mathbf{n}$ . Let  $g$  be any element of  $SU(2)$  such that  $R(g) = R$ , and hence  $R(g)\mathbf{n}_z = \mathbf{n}$ . We have

$$\begin{aligned} \mathbf{n} \cdot \mathbf{J} &= \frac{1}{i} dD^j(\sigma(\mathbf{n})) \\ &= \frac{1}{i} dD^j(\sigma(R(g)\mathbf{n}_z)) \\ &= \frac{1}{i} dD^j(g\sigma(\mathbf{n})g^{-1}) \\ &= \frac{1}{i} D^j(g) dD^j(\sigma(\mathbf{n})) D^j(g)^{-1} \\ &= D^j(g) J_3 D^j(g)^{-1}. \end{aligned}$$

If we match this equality with  $(\mathbf{n} \cdot \mathbf{J})e_\lambda^{\mathbf{n}} = \lambda e_\lambda^{\mathbf{n}}$ , we get

$$J_3(D^j(g)^{-1}e_\lambda^{\mathbf{n}}) = \lambda(D^j(g)^{-1}e_\lambda^{\mathbf{n}}).$$

This means that  $D(g)^{-1}e_\lambda^{\mathbf{n}}$  is an eigenvector of  $J_3$ , with the eigenvalue  $\lambda$ . Since the eigenvalues of  $J_3$  are simple, we get  $D^j(g)^{-1}e_\lambda^{\mathbf{n}} = c_\lambda e_\lambda$ ,  $\lambda = -j, -j+1, \dots, j-1, j$ , where  $c_\lambda$  is a complex number of modulus 1. Then we have

$$|\langle e_\lambda^{\mathbf{n}} | w \rangle|^2 = |\langle e_\lambda | D^j(g)w \rangle|^2,$$

where  $g$  is any element of  $SU(2)$  such that  $R(g)\mathbf{n}_z = \mathbf{n}$ . In the same way

$$\langle e_\mu | e_\rho^{\mathbf{n}} \rangle \langle e_\rho^{\mathbf{n}} | e_\nu \rangle = \langle e_\mu | D^j(g)^{-1}e_\rho \rangle \langle D(g)^{-1}e_\rho | e_\nu \rangle,$$

where  $g$  is any element of  $SU(2)$  such that  $R(g)\mathbf{n}_z = \mathbf{n}$ .

Notice that  $|\langle e_\lambda | D^j(g)w \rangle|^2$  represents the probability that the component of the spin along the  $\mathbf{n}_z$  direction has value  $\lambda$ , when the system is in the transformed state  $P[D^j(g)w]$  (we stress that  $D^j(g)$  is a unitary operator).

**1.5. Orthogonality relations for noncompact groups.** If we recall the example of  $SU(2)$  quantum tomography, we see that the main step for getting a reconstruction formula is using Shur's orthogonality relations, that hold for all irreducible representations of compact groups. In this paragraph we briefly describe a particular family of representations of noncompact groups. Orthogonality relations, that are the exact analogue of Shur's ones, can be written for the representation of this family.

Let  $G$  be a noncompact, unimodular, linear Lie group (more generally, a locally compact, second countable group). We denote by  $dg$  an arbitrarily chosen Haar measure on  $G$ . Let  $(\Pi, V)$  be a unitary, irreducible representation of  $G$ . In general,  $V$  is not finite dimensional. We denote by  $Z$  the center of  $G$ . We recall

$$Z = \{z \in G | zg = gz \text{ for all } g \in G\}.$$

Then we have for any  $z \in Z$  and  $g \in G$

$$\Pi(z)\Pi(zg) = \Pi(zg) = \Pi(gz) = \Pi(g)\Pi(z).$$

Then, by Shur's lemma, since  $\Pi$  is irreducible,  $\Pi(z)\chi_\Pi(z)I$ , where  $\chi_\Pi(z)$  is a complex number of modulus 1. The function  $\chi_\Pi(z)$  defines a one-dimensional unitary representation of the abelian group  $Z$  (sometimes it is called the "central character of  $\Pi$ ").

Let  $u, v, w, y$  be elements of  $V$ . The function on  $G$ ,

$$g \mapsto \overline{\langle u | \Pi(g)v \rangle} \langle w | \Pi(g)y \rangle,$$

is constant on the cosets of  $Z$ . In fact, we have, for all  $z \in Z$

$$\begin{aligned} \overline{\langle u | \Pi(gz)v \rangle} \langle w | \Pi(gz)y \rangle &= \overline{\chi_\Pi(z) \langle u | \Pi(g)v \rangle} \chi_\Pi(z) \langle w | \Pi(z)y \rangle \\ &= \overline{\langle u | \Pi(g)v \rangle} \langle w | \Pi(g)y \rangle. \end{aligned}$$

This shows that the function  $\overline{\langle u | \Pi(g)v \rangle} \langle w | \Pi(g)y \rangle$  naturally defines also a function on  $G/Z$ .

We shall denote by  $\dot{g}$  the coset of  $Z$  defined by  $g$ ,

$$\dot{g} = \{gz | z \in Z\},$$

and by  $d\dot{g}$  the measure on  $G/Z$  (unique up to a multiplicative positive constant), invariant with respect to the action of  $G$  on  $G/Z$ . This measure coincides with the Haar measure on  $G/Z$ , which is a group by itself. We recall that the measures  $dg$ ,  $d\dot{g}$  and  $dz$  are linked together in such a way that, for any integrable function  $f$  on  $G$ , we have

$$\int_G f(g)dg = \int_{G/Z} d\dot{g} \int_Z f(gz)dz.$$

We are now ready to define the family of representations that admit orthogonality relations. We say that the irreducible representation  $(\Pi, V)$  of  $G$  is "square-integrable modulo  $Z$ " if there is a non-zero vector  $u$  in  $V$  such that the function

$$\dot{g} \mapsto \left| \int_{12} \langle u | \Pi(g)u \rangle \right|^2$$

is integrable on  $G/Z$ , with respect to the measure  $d\dot{g}$ . It can be proved that  $(\Pi, V)$  is square-integrable modulo  $Z$  if and only if, for all pairs  $u, v \in V$ , the function

$$\dot{g} \mapsto |\langle u | \Pi(g)v \rangle|^2$$

is integrable on  $G/Z$ , with respect to  $d\dot{g}$ .

We have the following fundamental property: Let  $(\Pi, V)$  be a (unitary, irreducible) square-integrable modulo  $Z$  representation of  $G$ . Then there exists a positive number  $d_\Pi$  (sometimes called "the formal degree of  $\Pi$ ") such that we have the orthogonality relations

$$\int_{G/Z} \overline{\langle u | \Pi(g)v \rangle} \langle w | \Pi(g)y \rangle d\dot{g} = \frac{1}{d_\Pi} \langle w | u \rangle \langle v | y \rangle$$

for all  $u, v, w, y \in V$ . The number  $d_\Pi$  depends only on the normalization chosen for the measure  $d\dot{g}$ .

There is an important particular case that is worth to be discussed, the case where  $Z$  is compact.

Let us suppose that  $(\Pi, V)$  is square-integrable modulo  $Z$ , and  $Z$  is compact. Let us fix the Haar measure,  $dz$ , on  $Z$  such that  $\int_Z dz = 1$ . Then we have

$$\begin{aligned} \int_Z |\langle u | \Pi(gz)v \rangle|^2 dz &= |\langle u | \Pi(g)v \rangle|^2 \int_Z dz \\ &= |\langle u | \Pi(g)v \rangle|^2. \end{aligned}$$

Hence,

$$\int_G |\langle u | \Pi(g)v \rangle|^2 = \int_{G/Z} |\langle u | \Pi(g)v \rangle|^2 dg.$$

This shows that the function

$$g \mapsto |\langle u | \Pi(g)v \rangle|^2$$

is  $dg$ -integrable on  $G$  itself. Conversely, if  $g \mapsto |\langle u | \Pi(g)v \rangle|^2$  is  $dg$ -integrable on  $G$ , the equalities

$$\begin{aligned} \int_G |\langle u | \Pi(g)v \rangle|^2 dg &= \int_{G/Z} d\dot{g} \int_Z |\langle u | \Pi(gz)v \rangle|^2 \\ &= \int_{G/Z} |\langle u | \Pi(g)v \rangle|^2 d\dot{g} \end{aligned}$$

show that the function

$$\dot{g} \mapsto |\langle u | \Pi(g)v \rangle|^2$$

is  $d\dot{g}$ -integrable on  $G/Z$ . So, in conclusion, when the center  $Z$  of  $G$  is compact, square-integrability on  $G/Z$  is equivalent to square-integrability on  $G$ . In this case, the orthogonality relations take the form

$$\int_G \overline{\langle u | \Pi(g)v \rangle} \langle w | \Pi(g)y \rangle dg = \frac{1}{d_\Pi} \langle w | u \rangle \langle v | y \rangle$$

for all  $u, v, w, y \in V$ .

## 2. THE HEISENBERG GROUP AND QUANTUM TOMOGRAPHY

**2.1. The Heisenberg group.** The Heisenberg group can be defined as the subgroup of the group of unitary operators on the Hilbert space  $L^2(\mathbb{R})$ , generated by the two one parameter groups of unitary operators,  $U(p)$  and  $V(q)$ , defined by

$$\begin{aligned}(U(p)f)(x) &= f(x+p), \\ (V(q)f)(x) &= e^{iqx}f(x),\end{aligned}$$

for all  $p, q \in \mathbb{R}$ . It is clear that both  $U(p)$  and  $V(q)$  preserve  $C_c^\infty(\mathbb{R})$ :

$$\begin{aligned}U(p)C_c^\infty(\mathbb{R}) &\subset C_c^\infty(\mathbb{R}), \\ V(q)C_c^\infty(\mathbb{R}) &\subset C_c^\infty(\mathbb{R}).\end{aligned}$$

We shall use, in the sequel, the following classical results:

- (1) the Stone theorem: if  $U(t)$  is a one-parameter unitary group, acting in a Hilbert space  $\mathcal{H}$ , then there is a unique (densely defined) self-adjoint operator  $A$ , such that

$$U(t) = e^{itA}$$

for all  $t \in \mathbb{R}$  (the right-hand member is to be understood in the sense of spectral theorem).

- (2) if  $U(t)$  is as above, and  $\mathcal{L} \subset \mathcal{H}$  is a dense linear subspace of  $\mathcal{H}$  which is stable under  $U(t)$ , that is

$$U(t)\mathcal{L} \subset \mathcal{L}$$

for all  $t \in \mathbb{R}$ , then the generator  $A$  of  $U(t)$  is the unique self-adjoint extension of its restriction to  $\mathcal{L}$ .

Using the previous results, we have that:

- (1) the generator  $P$  of  $U(p)$  is the unique selfadjoint extension of the operator, whose action on  $C_c^\infty(\mathbb{R})$  is  $\frac{1}{i}f'(x)$ ,  
(2) the generator  $Q$  of  $V(q)$  is the unique self-adjoint extension of the operator, whose action on  $C_c^\infty(\mathbb{R})$  is  $xf(x)$ .

For studying the group generated by  $U(p)$  and  $V(q)$ , we compute  $U(p)V(q)$  and  $V(q)U(p)$ . We have

$$(U(p)V(q)f)(x) = (V(q)f)(x+p) = e^{iq(x+p)}f(x+p) = e^{iqx}e^{ipq}f(x+p)$$

and

$$(V(q)U(p)f)(x) = e^{iqx}(U(p)f)(x) = e^{iqx}f(x+p).$$

Hence, we get

$$U(p)V(q) = e^{ipq}V(q)U(p),$$

for all  $p, q \in \mathbb{R}$ . These equalities (as  $p$  and  $q$  vary on  $\mathbb{R}$ ) are called "Weyl commutation relations". using the notation with the generators, the Weyl commutation relations are

$$e^{ipP}e^{iqQ} = e^{ipq}e^{iqQ}e^{ipP}.$$

Hence, the group generated by  $U(p)$  and  $V(q)$  can be described explicitly as the group of unitary operators on  $L^2(\mathbb{R})$  of the form

$$e^{it}V(q)U(p),$$

$t, p, q \in \mathbb{R}$ .

It is possible to write these operators in another (equivalent) way, which is more convenient for our goals. To achieve this, we need preparation.

For each triple  $(t, q, p) \in \mathbb{R}^3$  we define an operator  $U(t, q, p)$  on  $L^2(\mathbb{R})$  by

$$(U(t, q, p)f)(x) = e^{i(t + \frac{pq}{2} + qx)} f(x + p).$$

Then we have:

- (1)  $U(t, q, p)$  is a unitary operator (this is quite evident),
- (2) the map

$$(t, q, p) \mapsto U(t, q, p)f : \mathbb{R}^3 \rightarrow L^2(\mathbb{R})$$

is continuous for all  $f \in L^2(\mathbb{R})$ . We sketch the proof of this fact: It is well known that it is enough to check the continuity if  $f$  belongs to a dense subset of  $L^2(\mathbb{R})$ . Let  $f$  be in  $C_c^\infty(\mathbb{R})$ . If  $(t_j, q_j, p_j) \rightarrow (t, q, p)$  in  $\mathbb{R}^3$ , then  $(U(t_j, q_j, p_j)f)(x) \rightarrow (U(t, q, p)f)(x)$  for each fixed  $x \in \mathbb{R}$ . Moreover, the supports of all the functions  $U(t_j, q_j, p_j)f$  and  $U(t, q, p)f$  are contained in a fixed compact set, and all the functions are bounded by the same constant. Then, by the dominated convergence theorem

$$\int_{\mathbb{R}} |(U(t_j, q_j, p_j)f)(x) - (U(t, q, p)f)(x)|^2 dx \rightarrow 0,$$

if  $(t_j, q_j, p_j) \rightarrow (t, q, p)$  in  $\mathbb{R}^3$ . This shows the continuity.

Now, we fix a triple  $(t, q, p) \in \mathbb{R}^3$ , and we define  $U_s(t, q, p)$ , for all  $s \in \mathbb{R}$ , by

$$U_s(t, q, p) = U(st, sq, sp),$$

explicitly,

$$(U_s(t, q, p)f)(x) = e^{i(st + s^2 \frac{pq}{2} + sqx)} f(x + sp).$$

Then we have

- (1)  $U_{s_1}(t, q, p)U_{s_2}(t, q, p) = U_{s_1+s_2}(t, q, p)$  (this is an elementary verification),
- (2)  $s \mapsto U_s(t, q, p)$  is a one-parameter unitary group (this follows at once from the proved continuity of  $(t, q, p) \mapsto U(t, q, p)f$ , for all  $f \in L^2(\mathbb{R})$ ),
- (3)  $U_s(t, q, p)C_c^\infty(\mathbb{R}) \subset C_c^\infty(\mathbb{R})$  for all  $s \in \mathbb{R}$  (this is quite evident).

From the last property, we have that  $C_c^\infty(\mathbb{R})$  is a subspace of the domain of the generator of the unitary one-parameter group  $U_s(t, q, p)$ . We wish to have an explicit expression of the action of this generator, which we denote by  $A$ , on  $C_c^\infty(\mathbb{R})$ . If  $f \in \text{dom}A$ , we have

$$iAf = \lim_{s \rightarrow 0} \frac{U_s(t, q, p)f - f}{s},$$

where the limit is in the  $L^2(\mathbb{R})$ -norm. On the other hand, if  $f \in C_c^\infty(\mathbb{R})$ , we have

$$\lim_{s \rightarrow 0} \frac{(U_s(t, q, p)f)(x) - f(x)}{s} = \left( \frac{d}{ds}(U_s(t, q, p)f)(x) \right) \Big|_{s=0},$$

for each fixed  $x \in \mathbb{R}$ . By an easy calculation,

$$\left( \frac{d}{ds}(U_s(t, q, p)f)(x) \right) \Big|_{s=0} = (i(t + qx + p \frac{1}{i} \frac{d}{dx})f)(x).$$

Proving that this function is not only the pointwise limit but also the limit in  $L^2(\mathbb{R})$ , requires an argument, based on the dominated convergence theorem, exactly analogous to the one we

have sketched previously. Then we have proved that the action of the generator  $A$  of  $U_s(t, q, p)$  on  $C_c^\infty(\mathbb{R})$  is given by

$$\begin{aligned}(Af)(x) &= \left( (t + qx + p \frac{1}{i} \frac{d}{dx}) f \right)(x) \\ &= ((t + qQ + pP)f)(x).\end{aligned}$$

This shows that the operator  $qQ + pP$  with domain  $C_c^\infty(\mathbb{R})$  is essentially self-adjoint (it has a unique self-adjoint extension). Moreover

$$\begin{aligned}(U_s(t, q, p)f)(x) &= e^{i(st + s^2 \frac{pq}{2} + sqx)} f(x + sp) \\ &= (e^{is(t + qQ + pP)} f)(x),\end{aligned}$$

from which we get

$$\begin{aligned}(e^{i(t + qQ + pP)} f)(x) &= e^{i(t + \frac{pq}{2} + qx)} f(x + p) \\ &= (e^{i(t + \frac{pq}{2})} e^{iqQ} e^{ipP} f)(x).\end{aligned}$$

Equivalently,

$$e^{i(qQ + pP)} = e^{i\frac{pq}{2}} e^{iqQ} e^{ipP}.$$

We stress that the symbols  $e^{i(pP + qQ)}$  and  $e^{i(t + qQ + pP)}$ , that appear in these formulas, have to be understood in the precise meaning of the spectral theorem (functions of unbounded self-adjoint operators), since we have shown that the operator  $pP + qQ$  on  $C_c^\infty(\mathbb{R})$  has a unique self-adjoint extension.

From the previous discussion, we have that the operators of the form  $e^{it} e^{iqQ} e^{ipP}$ ,  $t, q, p \in \mathbb{R}$ , can be written also in the form  $e^{i(t + pP + qQ)}$ . Use of the equalities

$$\begin{aligned}e^{i(pP + qQ)} &= e^{i\frac{pq}{2}} e^{iqQ} e^{ipP} \\ e^{ipP} e^{iqQ} &= e^{ipq} e^{iqQ} e^{ipP}\end{aligned}$$

leads us to the multiplication rule

$$e^{i(t_1 + q_1 Q + p_1 P)} e^{i(t_2 + q_2 Q + p_2 P)} = e^{i[t_1 + t_2 + \frac{1}{2}(q_2 p_1 - q_1 p_2) + (q_1 + q_2)Q + (p_1 + p_2)P]}.$$

We use the last equality to define the Heisenberg group, which we denote by  $\mathbb{H}$ . As a set,  $\mathbb{H}$  is  $\mathbb{R}^3$ , with the composition rule

$$(t_1, q_1, p_1) \cdot (t_2, q_2, p_2) = (t_1 + t_2 + \frac{1}{2}(p_1 q_2 - q_1 p_2), q_1 + q_2, p_1 + p_2).$$

$\mathbb{H}$  is a connected, simply connected, noncompact, unimodular Lie group. The Lebesgue measure for  $\mathbb{R}^3$  is the Haar measure for  $\mathbb{H}$ . The subgroup  $\{(t, 0, 0) | t \in \mathbb{R}\}$ , is the center,  $Z(\mathbb{H})$ , of  $\mathbb{H}$ . It is clear that  $Z(\mathbb{H}) \simeq \mathbb{R}$  and  $\mathbb{H}/Z(\mathbb{H}) \simeq \mathbb{R}^2$  as Lie groups. The Lebesgue measure on  $\mathbb{R}$  is the Haar measure for  $Z(\mathbb{H})$ , the Lebesgue measure on  $\mathbb{R}^2$  is both the Haar measure for the group  $\mathbb{H}/Z(\mathbb{H}) \simeq \mathbb{R}^2$ , and an invariant measure for  $\mathbb{H}/Z(\mathbb{H})$ , considered as a transitive  $\mathbb{H}$ -space.

**2.2. The Schrödinger representation and the reconstruction formula for the Heisenberg group.** From the very construction of the Heisenberg group, outlined in the previous paragraph, it is clear that the operators on  $L^2(\mathbb{R})$ ,  $U(t, q, p) = e^{i(t + qQ + pP)}$ ,  $(t, q, p) \in \mathbb{H}$ , define a unitary representation of  $\mathbb{H}$  that is called the "Schrödinger representation". We recall the explicit form

$$(U(t, q, p)f)(x) = e^{i(t + \frac{pq}{2} + qx)} f(x + p).$$

We have the fundamental result:

**Proposition 1.** *The Schrödinger representation of  $\mathbb{H}$ ,  $(U, L^2(\mathbb{R}))$ , is irreducible.*

*Proof.* Suppose  $L^2(\mathbb{R})$  has a closed invariant subspace,  $M$ , that contains a function  $f$  with the property, that  $f(x) \neq 0$  on a subset of  $\mathbb{R}$  whose measure is non-zero. This means that  $M \neq \{0\}$ . Since  $U$  is unitary,  $M^\perp$  is also closed invariant subspace, and we have the orthogonal decomposition  $L^2(\mathbb{R}) = M \oplus M^\perp$ . Let  $h$  be an element of  $M^\perp$ . Then  $U(t, q, p)h \in M^\perp$  for all  $(t, q, p) \in \mathbb{H}$ . This implies

$$\begin{aligned} \langle U(t, q, p)h | f \rangle &= \int_{\mathbb{R}} e^{-i(t+\frac{pq}{2})} e^{-iqx} \overline{h(x+p)} f(x) dx = 0 \\ \Rightarrow \int_{\mathbb{R}} e^{-iqx} \overline{h(x+p)} f(x) dx &= 0, \end{aligned}$$

for all  $p, q \in \mathbb{R}$ . This shows that the Fourier transform of the function

$$(x \mapsto \overline{h(x+p)} f(x)) \in L^2(\mathbb{R})$$

is identically zero on  $\mathbb{R}$ . Then  $\overline{h(x+p)} f(x) = 0$  almost everywhere, for all  $p \in \mathbb{R}$ . Since  $f(x) \neq 0$  on a non-negligible set, we have  $h(x) = 0$  almost everywhere. Then  $M^\perp = \{0\}$  and  $M = L^2(\mathbb{R})$ . This shows the irreducibility.  $\square$

We wish to use the irreducible Schrödinger representation of  $\mathbb{H}$  for deducing a reconstruction formula for the states of  $L^2(\mathbb{R})$ . Since  $Z(\mathbb{H}) \simeq \mathbb{R}$  is not compact, we have to check the square-integrability modulo  $Z$  of the Schrödinger representation. Notice that the function

$$|\langle f | U(t, q, p) f \rangle|^2$$

doesn't depend on  $t$ , and defines on  $\mathbb{H}/Z(\mathbb{H}) \simeq \mathbb{R}^2$  the function

$$|\langle f | U(0, q, p) f \rangle|^2 = |\langle f | e^{i(qQ+pP)} f \rangle|^2.$$

We have to discuss the integrability of this function of  $p$  and  $q$  on  $\mathbb{R}^2$ .

We denote  $(h_n)_{n \geq 0}$  the Hilbert basis of  $L^2(\mathbb{R})$  whose elements are the normalized Hermite functions. There is a beautiful explicit form (due to Schwinger) for the matrix elements of the operators  $e^{i(pP+qQ)}$  in the basis  $(h_n)$ . We have

$$\begin{aligned} \langle h_n | U(0, q, p) h_{n+l} \rangle &= \langle h_n | e^{i(pP+qQ)} h_{n+l} \rangle \\ &= \sqrt{\frac{n!}{(n+l)!}} e^{-\frac{1}{4}(p^2+q^2)} (-1)^l \left( \frac{q+ip}{\sqrt{2}} \right)^l L_n^l \left( \frac{1}{2}(p^2+q^2) \right), \end{aligned}$$

where  $n, l \geq 0$ , and  $L_n^l$  denote the associated Laguerre polynomials.

In particular, for  $n = 0$ ,  $l = 0$ , we have

$$\langle h_0 | e^{i(pP+qQ)} h_0 \rangle = e^{-\frac{1}{4}(p^2+q^2)}.$$

The function  $e^{-\frac{1}{4}(p^2+q^2)}$  is obviously square-integrable on  $\mathbb{R}^2$ . Then the Schrödinger representation is square-integrable modulo the center.

We can use the last equality to compute the formal degree of  $U$ . Using the orthogonality relations, we have

$$\int_{\mathbb{R}^2} |\langle h_0 | U(0, q, p) h_0 \rangle|^2 dpdq = \frac{1}{d_U} \|h_0\|^4 = \frac{1}{d_U}.$$

On the other hand, using the explicit form of  $\langle h_0 | U(0, q, p) h_0 \rangle$ , we have

$$\begin{aligned}
\int_{\mathbb{R}^2} |\langle h_0 | U(0, q, p) h_0 \rangle|^2 dp dq &= \int_{\mathbb{R}^2} e^{-\frac{1}{2}(p^2+q^2)} dp dq \\
&= \int_0^{2\pi} d\theta \int_0^\infty dr r e^{-\frac{1}{2}r^2} \\
&= 2\pi.
\end{aligned}$$

Hence  $d_U = \frac{1}{2\pi}$ .

In this way, we can write down the orthogonality relations which will lead to the reconstruction formula. Let  $f$  be any element of  $L^2(\mathbb{R})$  of unit norm. We have

$$\begin{aligned}
\langle h_n | P[f] h_{n+l} \rangle &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \langle h_n | e^{i(qQ+pP)} h_{n+l} \rangle \langle e^{i(qQ+pP)} f | f \rangle dp dq \\
&= \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^\infty dr r \langle h_n | e^{ir(\cos\theta Q + \sin\theta P)} h_{n+l} \rangle \langle e^{ir(\cos\theta Q + \sin\theta P)} f | f \rangle.
\end{aligned}$$

Using the spectral theorem, we have (recall that  $\cos\theta Q + \sin\theta P$ , with domain  $C_c^\infty(\mathbb{R})$ , is essentially self-adjoint)

$$\langle e^{ir(\cos\theta Q + \sin\theta P)} f | f \rangle = \int_{\mathbb{R}} e^{-irx} \text{Prob}(\theta, x; f) dx,$$

where we denote by  $\text{Prob}(\theta, x; f) dx$  the integration with respect to the spectral measure

$$\alpha_f^{\cos\theta Q + \sin\theta P}(\cdot) = \langle f | P_{\cos\theta Q + \sin\theta P}(\cdot) f \rangle.$$

From the physical point of view,  $\text{Prob}(\theta, x; f)$  represents the probability that in the state  $P[f]$  the physical quantity represented by  $\cos\theta Q + \sin\theta P$  has value  $x$ .

Then

$$\langle h_n | P[f] h_{n+l} \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^\infty dr r \langle h_n | U(0, q, p) h_{n+l} \rangle \left( \int_{\mathbb{R}} e^{-irx} \text{Prob}(\theta, x; f) dx \right).$$

Let us consider the function

$$(r, x) \mapsto r e^{-irx} \text{Prob}(\theta, x; f) e^{-\frac{1}{4}r^2} r^l L_n^l\left(\frac{1}{2}r^2\right).$$

By Torelli's theorem, it is integrable in the two variables  $r$  and  $x$ . Then, by Fubini's theorem, the order of the integration can be interchanged. We have, in this way, the reconstruction formula

$$\langle h_n | P[f] h_{n+l} \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_{\mathbb{R}} \text{Prob}(\theta, x; f) f_{n,n+l}^\theta(x) dx,$$

where

$$f_{n,n+l}^\theta(x) = \sqrt{\frac{n!}{(n+l)!}} \frac{(-1)^l}{(\sqrt{2})^l} e^{il\theta} \int_0^\infty r^{l+1} e^{-\frac{1}{4}r^2 - irx} L_n^l\left(\frac{r^2}{2}\right) dr.$$

The reconstruction formula has (like in the  $SU(2)$  case) the structure anticipated in the general discussion:

- (1)  $([0, 2\pi], \frac{d\theta}{2\pi})$  is the probability space that labels the set of physical quantities, it has the role of  $(\Omega, \mu)$  in the general discussion of paragraph 1.3,
- (2)  $\{\cos\theta Q + \sin\theta P | \theta \in [0, 2\pi]\}$  is the set of physical quantities whose probability distributions are needed, it has the role of  $\{A_\omega | \omega \in \Omega\}$ ,

(3) the functions  $f_{n,n+l}^\theta$ ,  $\theta \in [0, 2\pi]$ ,  $n, l \geq 0$ , can be explicitly computed, they do not depend on  $P[f]$ .

To end our discussion of the reconstruction formula for the Heisenberg group, we wish to show that the matrix element

$$\langle e^{ir(\cos \theta Q + \sin \theta P)} f | f \rangle$$

can be written also in another way, with a slightly different physical interpretation. A similar discussion, for the group  $SU(2)$ , has been made at the end of paragraph 1.4.

We have to use a unicity property of the Schrödinger representation. It would follow easily from the classification of all unitary irreducible representations of  $\mathbb{H}$ . Since we have not extended our study of  $\mathbb{H}$  to include this classification, we shall simply state the unicity property without proof.

Notice that the restriction of the Schrödinger representation to  $Z(\mathbb{H})$  is given by  $U(t, 0, 0) = e^{it}$ . Suppose that  $U'$  is another unitary irreducible representation of  $\mathbb{H}$ , acting in a Hilbert space  $\mathcal{H}$ , and that the restriction of  $U'$  to  $Z(\mathbb{H})$  is  $e^{it}$  as well, that is,  $U'(t, 0, 0) = e^{it}$ . Then  $U$  and  $U'$  are unitarily equivalent, that is, there is a unitary operator  $W : \mathcal{H} \rightarrow L^2(\mathbb{R})$  such that  $U' = W^{-1}UW$ .

Let us consider the function  $\varphi^\theta : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by

$$\varphi^\theta(t, q, p) = (t, q \cos \theta - p \sin \theta, q \sin \theta + p \cos \theta).$$

It is obvious that  $\varphi^\theta$  is a smooth automorphism of  $\mathbb{H}$  onto itself. We define  $U^\theta = U \circ \varphi^\theta$ , explicitly

$$U^\theta(t, q, p) = U(t, q \cos \theta - p \sin \theta, q \sin \theta + p \cos \theta).$$

Then  $U^\theta$  is a continuous unitary irreducible representation of  $\mathbb{H}$ , acting in  $L^2(\mathbb{R})$ . We have  $U^\theta(t, 0, 0) = e^{it}$ , so  $U^\theta$  and  $U$  have the same restriction to the center. Then, by the unicity property of the Schrödinger representation there is a unitary operator  $W_\theta : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  such that  $U^\theta = W_\theta^{-1}UW_\theta$ . Hence, we have

$$U(0, r \cos \theta, r \sin \theta) = U^\theta(0, r, 0) = W_\theta^{-1}U(0, r, 0)W_\theta.$$

Moreover,  $U(0, r, 0) = e^{irQ}$  and  $U(0, r \cos \theta, r \sin \theta) = e^{ir(\cos \theta Q + \sin \theta P)}$ . Then,

$$e^{ir(\cos \theta Q + \sin \theta P)} = W_\theta^{-1}e^{irQ}W_\theta.$$

We can use this equality for the calculation of the matrix element. We have

$$\begin{aligned} \langle e^{ir(\cos \theta Q + \sin \theta P)} f | f \rangle &= \langle W_\theta^{-1}e^{irQ}W_\theta f | f \rangle \\ &= \langle e^{irQ}(W_\theta f) | (W_\theta f) \rangle. \end{aligned}$$

We wish to use the spectral theorem, we have the spectral measure

$$E \mapsto \langle W_\theta f | P_Q(E)W_\theta f \rangle = \int_E |(W_\theta f)(x)|^2 dx.$$

Hence

$$\langle e^{ir(\cos \theta Q + \sin \theta P)} f | f \rangle = \int_{\mathbb{R}} e^{-irx} |(W_\theta f)(x)|^2 dx$$

Notice that  $|(W_\theta f)(x)|^2$  is the probability that the position  $Q$ , has value  $x$ , when the state is  $P[W_\theta f]$ . We have shown that this probability is the same as the probability that  $\cos \theta Q + \sin \theta P$  has value  $x$ , when the state is  $P[f]$ .

The reconstruction formula can also be written as

$$\langle h_n | P[f] | h_{n+l} \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_{\mathbb{R}} |(W_\theta f)(x)|^2 f_{n,n+l}^\theta(x) dx.$$

It remains to find an explicit expression for  $W_\theta$ . We have that

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mapsto W_\theta : SO(2) \rightarrow \mathcal{U}(L^2(\mathbb{R}))$$

is the restriction to  $SO(2)$  of the metaplectic representation of  $SL(2, \mathbb{R})$ . Hence  $W_\theta = e^{i\theta H_{ho}}$ , where  $H_{ho}$  is the Hamiltonian of the harmonic oscillator.