

Quotients in Supergeometry

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ABSTRACT. The purpose of this paper is to present the notion of quotient of supergroups in different categories using the unified treatment of the functor of points and to examine some physically interesting examples.

Introduction

The study of supergeometry was prompted by important physical questions linked to the symmetries of physical systems, which take into account the intrinsically different nature of the two fundamental types of particles: bosons and fermions.

While the bosons obey the Bose–Einstein statistics, the fermions are described by the Fermi one. These two types of particles have a fundamentally different behaviour: the bosons are described by *commuting* functions, while the fermions by *anticommuting* ones. Since these particles do transform into each other, it is necessary to consider symmetries which allow to mix these two types of functions.

From a purely mathematical point of view, we can view supergeometry as \mathbb{Z}_2 -graded geometry, where every ordinary geometric concept, as for example manifolds, varieties, vector fields and so on, has an \mathbb{Z}_2 -graded corresponding one. It is however important to stress that a supermanifold is not to be understood as an ordinary manifold with an associated \mathbb{Z}_2 -graded vector bundle, since in supergeometry we allow transformations which mix the even and the odd coordinates, as we shall see in Section 1.

Our treatment is organized as follows.

In Section 1 we quickly review some general facts on supergeometry including the functor of points approach to the study of superspaces.

In Section 2 we define what an action of a supergroup on a superspace is and the concept of homogeneous superspace.

In Section 3 we define the functor of points and the functor of A -points for homogeneous spaces. We also examine in detail the example of the superflag and its big cell, together with its physical interpretation as superconformal and super Minkowski spaces.

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1. Preliminaries

Let k be the ground field, $\text{char}(k) \neq 2, 3$.

For the basic definitions of superalgebra, supervector space and similar, refer to [14, ch. 4] and [13, ch. 3].

DEFINITION 1.1. A *superspace* $S = (|S|, \mathcal{O}_S)$ consists of a topological space $|S|$ together with a sheaf of commuting superalgebras \mathcal{O}_S , with the property that the stalk $\mathcal{O}_{S,x}$ is a local superalgebra for all $x \in |S|$. A morphism of superspaces $\varphi: S \rightarrow T$ is a continuous map $|\varphi|: |S| \rightarrow |T|$ together with a sheaf map $\varphi^*: \mathcal{O}_T \rightarrow \varphi_* \mathcal{O}_S$ so that $\varphi_x^*(\mathfrak{m}_{|\varphi|(x)}) \subseteq \mathfrak{m}_x$ where \mathfrak{m}_x is the maximal ideal in $\mathcal{O}_{S,x}$ and φ_x^* is the stalk map.

We shall denote with (sspaces) the category of superspaces.

Let's see some key examples of superspaces.

EXAMPLE 1.2. 1. $\mathbb{R}^{p|q}$. On the topological space \mathbb{R}^p we define the sheaf of commutative \mathbb{R} -superalgebras:

$$V \mapsto \mathcal{O}_{\mathbb{R}^{p|q}}(V) := C_{\mathbb{R}^p}^\infty(V)[\theta_1, \dots, \theta_q],$$

where $C_{\mathbb{R}^p}^\infty(V)[\theta^1, \dots, \theta^q] = C_{\mathbb{R}^p}^\infty(V) \otimes \wedge(\theta_1, \dots, \theta_q)$ and the θ_j have to be thought as odd (anti-commuting) indeterminates.

One can readily check that $\mathbb{R}^{p|q} := (\mathbb{R}^p, \mathcal{O}_{\mathbb{R}^{p|q}})$ is a superspace. Notice that the morphisms of superspaces are allowed to mix even and odd coordinates. For example we can define the morphism $\phi: \mathbb{R}^{1|2} \rightarrow \mathbb{R}^{1|2}$ on global section by: $\phi(x) = x + \theta_1 \theta_2$, $\phi(\theta_1) = \theta_1$, $\phi(\theta_2) = \theta_2$. This tells that $\mathbb{R}^{p|q}$ cannot be simply viewed as \mathbb{R} together with an exterior bundle.

2. $\mathbb{R}_h^{p|q}, \mathbb{C}_h^{p|q}$. Similarly define for $V \subseteq \mathbb{R}^p$ open, the sheaf of superalgebras:

$$V \mapsto \mathcal{H}_{\mathbb{R}^{p|q}}(V) := \mathcal{H}_{\mathbb{R}^p}(V)[\theta_1, \dots, \theta_q] := \mathcal{H}_{\mathbb{R}^p}(V) \otimes \wedge(\theta_1, \dots, \theta_q),$$

where $\mathcal{H}_{\mathbb{R}^p}$ denotes the sheaf of real analytic functions on V . Again one can check that $\mathbb{R}_h^{p|q} = (\mathbb{R}^p, \mathcal{H}_{\mathbb{R}^{p|q}})$ is a superspace. The definition of the superspace $\mathbb{C}_h^{p|q} = (\mathbb{C}^p, \mathcal{H}_{\mathbb{C}^{p|q}})$ goes along the same lines.

3. Spec A . Let A be a commutative superalgebra. Since A_0 is an algebra, we can consider the topological space

$$\text{Spec}(A_0) = \{\text{prime ideals } \mathfrak{p} \subset A_0\}.$$

The closed sets are $V(S) = \{\mathfrak{p} \in \text{Spec}(A_0) \mid \mathfrak{p} \supset S\}$. Classically we can define the structural sheaf \mathcal{O}_{A_0} on $\text{Spec}(A_0)$ by giving on an open cover of $\text{Spec}(A_0)$ by $U_i = \text{Spec}(A_0[f_i^{-1}])$ the sheaves $\mathcal{O}_{A_0}|_{U_i}(U_i) := A_0[f_i]$. The stalk of the structural sheaf at the prime $\mathfrak{p} \in \text{Spec}(A_0)$ is the localization of A_0 at \mathfrak{p} . We can replicate this construction in the super setting. As for any superalgebra, A is a module over A_0 , and we have indeed a sheaf \tilde{A} of \mathcal{O}_{A_0} -modules over $\text{Spec}(A_0)$ with stalk $A_{\mathfrak{p}}$, the localization of the A_0 -module A over each prime $\mathfrak{p} \in \text{Spec}(A_0)$. Spec $A := (\text{Spec } A_0, \tilde{A})$ is a superspace. As before Spec A is covered by open subsuperspaces

$U = \underline{\text{Spec}} A[f^{-1}]$, $f \in A_0$. (For more details concerning the construction of the sheaf \tilde{M} for a generic A_0 module M , see [10, § II.5] and [7, ch. 1].)

DEFINITION 1.3. We say that a superspace M is a *supermanifold* (resp. *real or complex analytic supermanifold*) if M is locally isomorphic to $\mathbb{R}^{p|q}$ (resp. $\mathbb{R}_h^{p|q}$ or $\mathbb{C}_h^{p|q}$). We also say that a superspace M is a *superscheme* if it is locally isomorphic to the spectrum of some superalgebra (of course the superalgebras may be different at different points).

DEFINITION 1.4. Given a superspace G , if we have three morphisms:

$$m: G \times G \longrightarrow G, \quad i: G \longrightarrow G, \quad 1: \{\bullet\} \longrightarrow G$$

satisfying the usual commutative diagrams for multiplication, inverse and identity in an abstract group, we say that G is a *supergroup*. If furtherly G is a supermanifold, (resp. complex or real analytic), we say G is a *Lie* (resp. *complex or real analytic supergroup*). If G is a superscheme, we say that G is a *supergroup scheme*.

The concept of functor of points allows us to recover some of the geometric intuition.

DEFINITION 1.5. We define the *functor of points* h_X of the superspace X as the representable functor

$$h_X: (\text{spaces}) \longrightarrow (\text{sets}), \quad T \mapsto h_X(T) = \text{Hom}(T, X).$$

In the same way, by the appropriate changes in the categories, we can define the functor of points of a supermanifold or a superscheme. Clearly if the superspace G is a supergroup, the functor is group-valued (and vice-versa).

The functor of points approach is so powerful because of Yoneda's Lemma, that we state in a special form of interest to us:

THEOREM 1.6. *Yoneda's Lemma.* We have a bijection between the set of morphisms of supermanifolds (supervarieties) $X \longrightarrow Y$ and the set of natural transformations $h_X \longrightarrow h_Y$.

OBSERVATION 1.7. By its very definition the functor of points h_S of a superspace S has the *presheaf property*, that is, when restricted to the open subsets of a superspace it is a presheaf of sets (recall that a presheaf is just a functor from the category of open sets of a topological space, where the morphisms are given by inclusions). However h_S has also the *sheaf property*; in other words if $\{T_i\}$ is a covering of the superspace T and we have a family $\alpha_i \in h_S(T_i)$, such that $\alpha_i|_{T_i \cap T_j} = \alpha_j|_{T_i \cap T_j}$, then there exists a unique $\alpha \in h_S(T)$ such that $\alpha|_{T_i} = \alpha_i$ ¹. We leave this verification as an exercise to the reader.

Any functor $F: (\text{spaces}) \longrightarrow (\text{sets})$ is a presheaf and as, for any presheaf, we can always build its *sheafification* $\tilde{F}: (\text{spaces}) \longrightarrow (\text{sets})$, which has the following properties:

- (1) \tilde{F} is a sheaf.
- (2) There is a canonically defined presheaf morphism $\psi: F \longrightarrow \tilde{F}$.

¹As customary we denote $\alpha|_{T_i}$ as the image of $\alpha \in h_S(T)$ under the map $h_S(\phi_i)$, where $\phi_i: T_i \hookrightarrow T$.

- (3) Any presheaf morphism $\phi: F \longrightarrow G$, with G sheaf, factors via ψ , i. e.
- $$\phi: F \xrightarrow{\psi} \tilde{F} \longrightarrow G.$$

Moreover \tilde{F} is locally isomorphic to F . For more details on this construction we refer the reader to [6] and [7].

Next, we want to introduce the concept of \mathcal{A}_0 -manifold and the functor of the A -points of a supermanifold X . This is substantially different from the functor of points h_X we have already described; in fact we can define it only in the differential and holomorphic categories. We are going to see that it characterizes the supermanifold and in many computational problems it allows to simplify significantly the notation. For a complete treatment see [2].

Let our ground field k be \mathbb{R} or \mathbb{C} .

DEFINITION 1.8. We call the commutative algebra A a *Weil algebra* if it is local, finite dimensional and $A = k \oplus J$, with the nilpotent maximal ideal J . We denote with (wa) the category of Weyl algebras (sometimes called *local algebras*) and with (swa) the category of Weyl superalgebras, defined in a similar way.

Let A_0 be a local algebra (the index 0 reminds us it has no odd elements). A manifold M is called an A_0 -manifold if there is an A_0 -module L and an open cover $\{U_i\}$ of M , such that $h_i: U_i \longrightarrow U'_i \subseteq L$ are diffeomorphisms (of C^∞ manifolds) and $d(h_i \cdot h_j^{-1})$ are isomorphisms of A_0 -modules. The set of all A_0 -manifolds for all $A_0 \in$ (wa) forms the objects of the category of \mathcal{A}_0 -manifolds that we denote with (\mathcal{A}_0 mflds). A morphism of two \mathcal{A}_0 -manifolds M and N , M being an A_0 -manifold, N a B_0 -manifold, consists of a pair (f, ϕ) , where $f: M \longrightarrow N$ is C^∞ morphism and $\phi: A_0 \longrightarrow B_0$ an algebra morphism such that $df(ax) = \phi(a)df(x)$.

We are ready to define the functor of the A -points of a supermanifold, through a definition-proposition (more details can be found in [2]).

DEFINITION-PROPOSITION 1.9. *Let M be a supermanifold. We define the set of A -points of M*

$$M_A := \bigsqcup_{x \in |M|} \text{Hom}_{(\text{salg})}(\mathcal{O}_{M,x}, A).$$

It has a natural structure of A_0 -manifold. We define the local functor of points of M the functorial assignment

$$M_{(\cdot)}: (\text{swa}) \longrightarrow (\mathcal{A}_0\text{mflds}), \quad A \mapsto M_A.$$

When M is smooth, we can write the functor M_A in a much simpler way.

PROPOSITION 1.10. *Let M be a smooth supermanifold, then:*

$$M_A \cong \text{Hom}(\mathcal{O}_M(|M|), A).$$

As it happens for the functor of points h_X , also in this case we can give an analogue of Yoneda's lemma. This means that the functor \mathcal{Y} , $\mathcal{Y}(M) := M_{(\cdot)}$ is a fully faithful embedding. As for the usual functor of points, \mathcal{Y} is not an equivalence of categories. In other words, not all the functors $h: (\text{swa}) \longrightarrow (\mathcal{A}_0\text{mflds})$ arise as the functors of A -points of a super manifold. If this is the case, in analogy with the functor of points notation, we say the functor is *representable*. In this frameworks it is possible to prove the following representability criterion, that we state for both the functor of A -points and the functor of points discussed in 1.5.

- PROPOSITION 1.11. (1) Let $F: (\text{smflds}) \longrightarrow (\text{sets})$ be a functor with the sheaf property. Suppose that F admits a cover by open subfunctors, i. e. there exist representable subfunctors of F , $U_i: (\text{smflds}) \longrightarrow (\text{sets})$, such that for any supermanifold M and any natural transformation $f: h_M \longrightarrow F$, $f^{-1}(U_i) = h_{V_i}$ and the V_i are open and cover M . Then F is representable, i. e. it is the functor of points of a supermanifold.
- (2) Let $h: (\text{swa}) \longrightarrow (\mathcal{A}_0\text{mflds})$ be a functor. Denote by $p_A: A \longrightarrow \mathbb{R}$ the canonical projection of an algebra $A \in (\text{swa})$ into $A/J \cong \mathbb{R}$. Suppose that an open cover $\{\tilde{U}_\alpha\}$ of $h(\mathbb{R}^{0|0})$ is given such that the functors

$$h_\alpha: (\text{swa}) \longrightarrow (\mathcal{A}_0\text{mflds}), \quad A \mapsto (h_{p_A})^{-1}(\tilde{U}_\alpha)$$

are representable by $\mathbb{R}^{n|m}$, for fixed n and m . Then h is representable, i. e. it is the functor of the A -points of a supermanifold.

PROOF. For (1) see [9], for (2) see [2]. □

As we shall see in the next sections, this is an important result that allows us to define properly the quotients of supergroups and their functor of points.

2. Actions of supergroups on superspaces

Let k be the ground field, $\text{char}(k) \neq 2, 3$

DEFINITION 2.1. Let G be a supergroup. We say that G acts on the superspace M if there exists a morphism $\phi: G \times M \longrightarrow M$ denoted as $(g, x) \mapsto g \cdot x$ for $g \in G(T)$ and $x \in M(T)$, such that for all superspaces T :

- (1) $1 \cdot x = x, \forall x \in M(T)$;
- (2) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x), \forall x \in M(T), \forall g_1, g_2 \in G(T)$.

We say that G acts *transitively* on M , or that M is an *homogeneous space* if there is $x_0 \in |M|$ such that the morphism $\phi_{x_0}: G \longrightarrow M, \phi_{x_0}(g) = g \cdot x_0$ is onto, i. e. the sheafification $\widetilde{\text{Im}(\phi_{x_0})}$ of the image presheaf coincides with M (see 1.7).

One can give in an obvious way this same definition in the categories of supermanifolds and superschemes.

When M is a supermanifold, our definition of homogeneous space is equivalent to the one appearing in [1] as the next proposition shows.

THEOREM 2.2. $\widetilde{\text{Im} \phi_{x_0}} = M$ if and only if ϕ_{x_0} is a surjective submersion.

PROOF. For brevity let $\phi = \phi_{x_0}$. Let us suppose that ϕ is a surjective submersion. Let $m \in |M|$ and $g \in |\phi|^{-1}(m)$ ($|\phi|$ is surjective, so it exists). Since ϕ is a submersion there exists $V \subseteq |G|$ with coordinates X_1, \dots, X_{p+q} ($\dim G = p|q$) and $W \subseteq |M|$ with coordinates Y_1, \dots, Y_{m+n} ($\dim M = m|n$) such that

$$\phi^*(Y_i) = X_i$$

Let $t \in U \subseteq |T|$ and $\alpha: U \longrightarrow M$ such that $m = |\alpha|(t)$. We can suppose $|\alpha|(U) \subseteq W$. If $\alpha^*(Y_i) = f_i \in \mathcal{O}_T(U)$, $\beta: U \longrightarrow V$ defined by

$$\beta^*(X_i) = \begin{cases} f_i & \text{if } i \leq m+n \\ 0 & \text{otherwise} \end{cases}$$

satisfies $\phi \circ \beta = \alpha$. Then $[\alpha] \in (Im \phi)_t$, hence $(Im \phi)_t = M_t$ and this gives one implication.

Vice-versa let us suppose that $\widetilde{Im \phi} = M$. Taking $T = \mathbb{R}^{0|0}$ we have that $|\phi|$ must be surjective. Let's now assume $T = M$ and $m \in |M|$. There exists $U \ni m$ and $\psi: U \rightarrow G$ such that $\phi \circ \psi = \mathbf{1}_U$. Then ϕ must be a submersion at $|\psi|(m)$ and this is true everywhere, since ϕ has constant rank. Indeed for all $g \in |G|$,

$$(d\phi)_g \circ (dl_g^G)_1 = (dl_g^M)_{x_0} \circ (d\phi)_1$$

where the isomorphisms l_g^G and l_g^M are the left actions of g on G and M respectively. \square

DEFINITION 2.3. Let's the notation be as above. The functor:

$$S_{x_0}(T) = \{g \in G(T) \mid g \cdot x_0 = x_0\}, \quad T \in (\text{sspaces})$$

is called the *stabilizer* of $x_0 \in |M|$.

We have given this definition in general, however we are especially interested in two cases:

- (1) G Lie supergroup, M a supermanifold.
- (2) G complex algebraic supergroup, M complex algebraic supervariety.

In each case the definitions above need to be suitably modified taking the superspaces in the appropriate category.

THEOREM 2.4. *Let G be a Lie or algebraic affine supergroup acting transitively on the supermanifold or supervariety M , $x_0 \in |M|$. Then*

- (1) $S_{x_0}^{\text{diff}}: (\text{smflds}) \rightarrow (\text{sets})$, $S_{x_0}^{\text{diff}}(T) = \{g \in G(T) \mid g \cdot x_0 = x_0\}$,
- (2) $S_{x_0}^{\text{alg}}: (\text{salg}) \rightarrow (\text{sets})$, $S_{x_0}^{\text{alg}}(A) = \{g \in G(A) \mid g \cdot x_0 = x_0\}$,

are the functor of points respectively of a Lie supergroup and of an algebraic supergroup. In other words the stabilizer supergroup functor is representable.

PROOF. For the differential category see [5] and [1], while for the algebraic category, see [8]. \square

There are many examples of actions of supergroups on superspaces, some of which are especially interesting. We now are going to see that Theorem 2.4 gives the representability for all the classical supergroups both in the categories of Lie and algebraic supergroups.

Let k be the field \mathbb{R} for the supermanifolds category and just a generic field, with $\text{char}(k) \neq 2, 3$ for the superschemes category.

1. $A(n)$ **series**. Let's first consider the algebraic setting. Let $A \in (\text{salg})$. Define $\text{GL}_{m|n}(A)$ as the set of all invertible morphisms $g: A^{m|n} \rightarrow A^{m|n}$. This is equivalent to ask that the *Berezinian* [3] or *superdeterminant*

$$\text{Ber}(g) = \text{Ber} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \det(p - qs^{-1}r) \det(s^{-1})$$

is invertible in A (where p and s are $m \times m$, $n \times n$ matrices of even elements in A , while q and r are $m \times n$, $n \times m$ matrices of odd elements in A). A necessary and sufficient condition for $g \in \text{GL}_{m|n}(A)$ to be invertible is that p and s are invertible. The group valued functor

$$\text{GL}_{m|n}: (\text{salg}) \rightarrow (\text{sets}), \quad A \mapsto \text{GL}_{m|n}(A)$$

is an affine supergroup called the *general linear supergroup* and it is represented by the algebra

$$k[\mathrm{GL}_{m|n}] := k[x_{ij}, y_{\alpha\beta}, \xi_{i\beta}, \gamma_{\alpha j}, z, w] / ((w \det(x) - 1, z \det(y) - 1), \\ i, j = 1, \dots, m, \quad \alpha, \beta = 1, \dots, n.$$

Consider the morphism

$$(2.1) \quad \rho: \mathrm{GL}_{m|n} \times k^{1|0} \longrightarrow k^{1|0}, \quad (g, c) \mapsto \mathrm{Ber}(g)c.$$

The stabilizer of the point $1 \in k^{1|0}$ coincides with all the matrices in $\mathrm{GL}_{m|n}(A)$ with Berezinian equal to 1, that is $\mathrm{SL}_{m|n}(A)$ the special linear supergroup. By the Theorem 2.4 we have immediately that $\mathrm{SL}_{m|n}$ is representable as an algebraic supergroup.

The supermanifold case is very similar. Define the functor (by an abuse of notation we use the same symbol) $\mathrm{GL}_{m|n}(T)$ as the invertible \mathcal{O}_T -module sheaf morphisms $\mathcal{O}_T^{m|n} \longrightarrow \mathcal{O}_T^{m|n}$. $\mathrm{GL}_{m|n}(T)$ can also be identified with the $m|n$ matrices with coefficients in $\mathcal{O}_T(|T|)$. In fact any morphism of supermanifold sheaves is determined once we know the morphism on the global sections $\mathcal{O}_T^{m|n}(|T|) \longrightarrow \mathcal{O}_T^{m|n}(|T|)$. Again we can define the Berezinian of a matrix and we can consider a morphism as in (2.1). The stabilizer of the point $1 \in k^{1|0}$ coincides with all the matrices in $\mathrm{GL}_{m|n}(T)$ with Berezinian equal to 1, that is $\mathrm{SL}_{m|n}(T)$ the special linear Lie supergroup. By the Theorem 2.4 we have that $\mathrm{SL}_{m|n}$ is representable as a Lie supergroup.

2. **$B(m, n)$, $C(n)$, $D(m, n)$ series.** Consider the morphism (both in the superscheme and supermanifold categories):

$$\rho: \mathrm{GL}_{m|2n} \times \mathcal{B} \longrightarrow \mathcal{B}, \quad (g, \psi(\cdot, \cdot)) \mapsto \psi(g, g),$$

where \mathcal{B} is the supervector space of all the symmetric bilinear forms on $k^{m|2n}$. We define $\mathrm{Osp}_{m|2n}$ as the stabilizer of the point Φ , the standard bilinear form on $k^{m|2n}$. Again this is an algebraic and Lie supergroup by Theorem 2.4.

3. **$P(n)$ series.** Define the algebraic and Lie supergroup $\pi\mathrm{Sp}_{n|n}$ as we did for $\mathrm{Osp}_{m|n}$, by taking antisymmetric bilinear forms instead of symmetric ones. Consider the action:

$$\pi\mathrm{Sp}_{n|n} \times k^{1|0} \longrightarrow k^{1|0}, \quad (g, c) \mapsto \mathrm{Ber}(g)c.$$

By Theorem 2.4 we have that Stab_1 is an affine algebraic supergroup, hence it is an algebraic and Lie supergroup. It is corresponding to the $P(n)$ series.

4. **$Q(n)$ series.** Let $D = k[\eta]/(\eta^2 + 1)$. This is a non commutative superalgebra. Define the supergroup functor $GL_n(D): (\mathrm{salg}) \longrightarrow (\mathrm{sets})$, with $GL_n(D)(A)$ the group of automorphisms of the left supermodule $A \otimes D$. In [5] is proven the existence of a morphism called the *odd determinant*

$$\mathrm{odet}: GL_n(D) \longrightarrow k^{0|1}.$$

Reasoning as before define:

$$GL_n(D) \times k^{0|1} \longrightarrow k^{0|1}, \quad g, c \mapsto \mathrm{odet}(g)c.$$

Then $G = \mathrm{Stab}_1$ is an affine algebraic supergroup and for $n \geq 2$ we define $Qg(n)$ as the quotient of G and the diagonal subgroup $GL_{1|0}$. This is an algebraic and Lie supergroup and its Lie superalgebra is $Q(n)$.

3. Homogeneous spaces via their functor of points

We now want to address the following question. Let G be a supergroup and H a closed subgroup, i. e. $|H|$ is closed in $|G|$. Consider the functor:

$$(\text{sspaces}) \longrightarrow (\text{sets}), \quad T \mapsto G(T)/H(T).$$

Is this functor representable? In this generality the answer is no, however we shall describe a representability result in the categories of supermanifolds and supervarieties.

THEOREM 3.1. *Let G be a Lie supergroup, H a closed Lie subgroup. Let $\widetilde{G/H}$ be the sheafification of the functor:*

$$T \mapsto G(T)/H(T).$$

Then $\widetilde{G/H}$ is the functor of points of a supermanifold that we denote with G/H . Moreover G/H is unique supermanifold with underlying topological space $|G|/|H|$ with respect to the following property:

The natural morphism $\pi: G \longrightarrow G/H$ is a submersion, moreover G acts on G/H and we have the commutative diagram:

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ \downarrow & & \downarrow \pi \\ G \times G/H & \longrightarrow & G/H \end{array}$$

PROOF. A complete proof of this statement can be found in [9]. □

REMARK 3.2. In the algebraic setting, Zubkov recently proved in [15] a similar result for G/H affine and in the case of $\text{char}(k) = 0$. In this setting one has to be more careful in taking the sheafification and more difficulties are present, since we don't have in general the local splitting of G as $H \times W$ at the identity.

We now turn to the formulation of the same problem for the functor of the A -points.

PROPOSITION 3.3. *Let G be a Lie supergroup and H be a closed subgroup. The functor*

$$(\text{swa}) \longrightarrow (\mathcal{A}_0\text{mflds}), \quad A \mapsto G_A/H_A$$

is representable.

PROOF. It is well known that there exists an open cover of G by tubular neighborhoods $U_\alpha \cong W_\alpha \times H$, where W_α are isomorphic to open sub superdomains in $\mathbb{R}^{p|q}$. Since the functor of A -points is product preserving we have that

$$(U_\alpha)_A/H_A \cong (W_\alpha)_A$$

and the result follows immediately from the Representability Theorem 2.2. □

As an example, we shall examine the construction of the superflag \mathcal{F} of $2|0$ and $2|1$ spaces in the $4|1$ dimensional complex super vector space $\mathbb{C}^{4|1}$. This is important in physics, since it gives the complexification of the super conformal space containing as big cell the Minkowski superspace (for more details on the physical interpretation see [9]).

Let \mathcal{F} be the functor $\mathcal{F}: (\text{smflds}) \longrightarrow (\text{sets})$, where $\mathcal{F}(T)$ is the set of $2|0$ and $2|1$ projective modules $Z_1 \subset Z_2$ inside $\mathcal{O}_T^{4|1} := \mathcal{O}_T \otimes \mathbb{C}^{4|1}$. \mathcal{F} is the functor of points

of a supermanifold called the *superflag* of 2|0 and 2|1 planes in $\mathbb{C}^{4|1}$, that we shall still denote by \mathcal{F} by an abuse of notation. Clearly $\mathcal{F} \subset \mathcal{G}_1 \times \mathcal{G}_2$, where \mathcal{G}_1 and \mathcal{G}_2 are respectively the supergrassmannians of 2|0 and 2|1 planes in $\mathbb{C}^{4|1}$ (for a direct proof of the non trivial fact that \mathcal{F} , \mathcal{G}_1 , \mathcal{G}_2 are supermanifolds see [13]).

We are now going to realize \mathcal{F} as the quotient of $\mathrm{SL}_{4|1}$ by a suitable parabolic subgroup.

The natural action of $G = \mathrm{SL}_{4|1}$ on $\mathcal{O}_T^{4|1}$ induces an action on \mathcal{G}_1 and \mathcal{G}_2 and also on \mathcal{F} :

$$G(T) \longrightarrow \mathcal{F}(T) \subset \mathcal{G}_1(T) \times \mathcal{G}_2, \quad g \mapsto g \cdot F.$$

Let us fix the element $F_0 = \{\mathcal{O}_T^{2|0} \subset \mathcal{O}_T^{2|1}\}$ in $\mathcal{F}(T)$. Then we can write the action as:

$$g \cdot F_0 = \left(\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \\ g_{31} & g_{32} \\ g_{41} & g_{42} \\ \gamma_{51} & \gamma_{52} \end{pmatrix}, \begin{pmatrix} g_{11} & g_{12} & \gamma_{15} \\ g_{21} & g_{22} & \gamma_{25} \\ g_{31} & g_{32} & \gamma_{35} \\ g_{41} & g_{42} & \gamma_{45} \\ \gamma_{51} & \gamma_{52} & g_{55} \end{pmatrix} \right) \in \mathcal{G}_1(T) \times \mathcal{G}_2(T).$$

The stabilizer subgroup functor at F_0 is given as the subgroup $H(T)$ of $G(T)$ consisting of all matrices in $G(T)$ of the form:

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} & \gamma_{15} \\ g_{21} & g_{22} & g_{23} & g_{24} & \gamma_{25} \\ 0 & 0 & g_{33} & g_{34} & 0 \\ 0 & 0 & g_{43} & g_{44} & 0 \\ 0 & 0 & \gamma_{53} & \gamma_{54} & g_{55} \end{pmatrix}.$$

H is clearly representable by a group supermanifold moreover we have that locally:

$$T \mapsto G(T)/H(T) = \mathcal{F}(T).$$

Hence $\widetilde{G/H} = \mathcal{F}$ and this is the functor of points of the superflag $\mathcal{F} = G/H$.

We wish now to describe explicitly G/H and its *big cell* U and to prove explicitly that the map $\pi: G \longrightarrow G/H$ is a submersion.

The big cell U in \mathcal{F} is defined as $\mathcal{F} \cap v_1 \times v_2$, where v_1 and v_2 are the big cells inside \mathcal{G}_1 and \mathcal{G}_2 . By definition $v_1(T)$ contains all the elements in $\mathcal{G}_1(T)$ having the determinant in the upper left corner invertible, while $v_2(T)$ contains all the elements in $\mathcal{G}_2(T)$ having the berezinian of rows 1, 2, 5 and columns 1, 2, 3 invertible. Hence we can write:

$$v_1(T) = \begin{pmatrix} I_2 \\ A \\ \alpha \end{pmatrix}, \quad v_2(T) = \begin{pmatrix} I_2 & 0 \\ B & \beta \\ 0 & 1 \end{pmatrix} \quad T \in (\text{smflds}),$$

where I_2 is the identity matrix, A and B are 2×2 matrices with even entries and $\alpha = (\alpha_1, \alpha_2)$, $\beta^t = (\beta_1, \beta_2)$ are rows with odd entries.

An element of $v_1(T)$ is inside $v_2(T)$ if and only if

$$(3.1) \quad A = B + \beta\alpha,$$

so we can take as coordinates for a flag in the big cell U the triplet (A, α, β) . We see then that U is an affine 4|4 superspace. Equation (3.1) is also known as *twistor relation*, in the physics literature.

In these coordinates, $F_0 = \left(\begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ is described by $(0, 0, 0)$.

We want to write the map π in these coordinates. In a suitable open subset near the identity of the group we can take an element $g \in G(T)$ as

$$g = \begin{pmatrix} g_{ij} & \gamma_{i5} \\ \gamma_{5j} & g_{55} \end{pmatrix}, \quad i, j = 1, \dots, 4.$$

Then, we can write an element $g \cdot \mathcal{F} \in \mathcal{G}_1 \times \mathcal{G}_2$ as:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \\ g_{31} & g_{32} \\ g_{41} & g_{42} \\ \gamma_{51} & \gamma_{52} \end{pmatrix}, \begin{pmatrix} g_{11} & g_{12} & \gamma_{15} \\ g_{21} & g_{22} & \gamma_{25} \\ g_{31} & g_{32} & \gamma_{35} \\ g_{41} & g_{42} & \gamma_{45} \\ \gamma_{51} & \gamma_{52} & g_{55} \end{pmatrix} \approx \begin{pmatrix} I \\ WZ^{-1} \\ \rho_1 Z^{-1} \end{pmatrix}, \begin{pmatrix} I & 0 \\ VY^{-1} & (\tau_2 - WZ^{-1}\tau_1)a \\ 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} \rho_1 &= (\gamma_{51} \quad \gamma_{52}), \quad W = \begin{pmatrix} g_{31} & g_{32} \\ g_{41} & g_{42} \end{pmatrix}, \quad Z = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \\ \tau_1 &= \begin{pmatrix} \gamma_{15} \\ \gamma_{25} \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} \gamma_{35} \\ \gamma_{45} \end{pmatrix}, \quad d = (g_{55} - \nu Z^{-1}\mu_1)^{-1}, \\ V &= W - g_{55}^{-1}\tau_2\rho_1, \quad Y = Z - g_{55}^{-1}\tau_1\rho_1. \end{aligned}$$

Finally the map π in these coordinates is given by:

$$g \mapsto (WZ^{-1}, \rho_1 Z^{-1}, (\tau_2 - WZ^{-1}\tau_1)d).$$

At this point one can compute the super Jacobian and verify that at the identity it is surjective.

Next, we are going to see how the big cell of the flag supermanifold \mathcal{F} can be interpreted as the complex super Minkowski space time, being the superflag its *superconformal compactification*.

The supergroup $G = \mathrm{SL}_{4|1}$ is the complexification of the real superconformal group. The subgroup of G that leaves the big cell invariant is the set of matrices in G of the form

$$\begin{pmatrix} L & 0 & 0 \\ NL & R & R\chi \\ d\varphi & 0 & d \end{pmatrix},$$

with L, N, R being 2×2 even matrices, χ and odd 1×2 matrix, φ a 2×1 odd matrix and d a scalar. This is the complex Poincaré supergroup and its action on the big cell can be written as

$$\begin{aligned} A &\mapsto R(A + \chi\alpha)L^{-1} + N, \\ \alpha &\mapsto d(\alpha + \varphi)L^{-1}, \\ \beta &\mapsto d^{-1}R(\beta + \chi). \end{aligned}$$

If the odd part is zero, then the action reduces to the one of the classical Poincaré group on the ordinary Minkowski space (for more details see [9]).

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