

Generalized Orthogonality Relations and $SU(1, 1)$ -Quantum Tomography

C. Carmeli*, G. Cassinelli †, F. Zizzi ‡

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Abstract

We present a mathematically precise derivation of some generalized orthogonality relations for the discrete series representations of $SU(1, 1)$. These orthogonality relations are applied to derive tomographical reconstruction formulas. Their physical interpretation is also discussed.

1 Introduction

The aim of the present paper is proving generalized orthogonality relations for certain representations of the group $SU(1, 1)$. The motivation is that orthogonality relations can be used to deduce reconstruction formulae for the states of a quantum system.

A reconstruction formula is an integral representation of the matrix elements of the state density operator of a quantum system, of the form

$$(e_i, T e_j)_{\mathcal{H}} = \int_{\Omega} dP(\omega) \int_{\mathbb{R}} d\alpha_T^{A_\omega}(\lambda) f_{ij}^\omega(\lambda)$$

where

- T is the density operator of the quantum system, whose Hilbert space is \mathcal{H}
- $\{e_i\}$ is a Hilbert basis of \mathcal{H} ,
- (Ω, P) is a probability space,
- $\{A_\omega, \omega \in \Omega\}$ is a family of self-adjoint operators on \mathcal{H} , representing physical quantities of the quantum system.

*C. Carmeli, Dipartimento di Fisica, Università di Genova, and I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy. e-mail: carmeli@ge.infn.it

†G. Cassinelli, Dipartimento di Fisica, Università di Genova, and I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy. e-mail: cassinelli@ge.infn.it

‡F. Zizzi, Dipartimento di Matematica, Università di Genova and I.N.F.N., Sezione di Genova, Via Dodecaneso 35, 16146 Genova, Italy. e-mail: zizzi@dima.unige.it

- for each $\omega \in \Omega$, $\alpha_T^{A_\omega}$ is the probability distribution of the physical quantity (represented by the self-adjoint operator) A_ω in the state T ; explicitly $\alpha_T^{A_\omega}$ is the probability measure on \mathbb{R} defined by

$$E \mapsto \text{Tr}(TP_A(E)), \quad E \in \mathcal{B}(\mathbb{R})$$

(here $\mathcal{B}(\mathbb{R})$ denotes the σ -algebra of the Borel sets of \mathbb{R} , and $P_A(\cdot)$ is the projection valued measure associated with the self-adjoint operator A by the spectral theorem).

- for all i, j, ω f_{ij}^ω is a function defined on \mathbb{R} (we stress that it doesn't depend on T).

A variant of the reconstruction formula, that will be used in this paper, is obtained if there is a family of unitary operators $\{U(\omega) \mid \omega \in \Omega\}$ and one self-adjoint operator A , such that

$$A_\omega = U(\omega)^*AU(\omega), \quad \forall \omega \in \Omega$$

In this case, we have

$$\begin{aligned} \alpha_T^{A_\omega}(E) &= \text{Tr}(P_{A_\omega}(E)T) \\ &= \text{Tr}(P_A(E)U(\omega)TU(\omega)^*) \\ &= \alpha_{T(\omega)}^A(E) \end{aligned}$$

where $T(\omega) := U(\omega)TU(\omega)^*$. In this way, the probability distribution $\alpha_T^{A_\omega} = \alpha_{T(\omega)}^A$ is the distribution of the physical quantity A in the “transformed state” $U(\omega)TU(\omega)^*$. In this case the reconstruction formula is

$$(e_i, Te_j)_{\mathcal{H}} = \int_{\Omega} dP(\omega) \int_{\mathbb{R}} d\alpha_{T(\omega)}^A(\lambda) f_{ij}^\omega(\lambda)$$

A reconstruction formula is an equality that expresses the matrix elements $(e_i, Te_j)_{\mathcal{H}}$ in another (perhaps, more complicated) way. However it can be read “backward” as a way to determine the state T , and, indeed, this is the way it is used in Quantum Tomography. Suppose we do not know the state of a quantum system, but we are able to determine (by means of experimental observations) the probability distributions of the physical quantities A_ω (in the unknown state) along with the functions f_{ij}^ω . Then, by plugging these distributions into the reconstruction formula, it is possible to compute the matrix elements $(e_i, Te_j)_{\mathcal{H}}$, which, in turn, determine uniquely the unknown state T . In this way T is “reconstructed” by means of experimental observations.

A possible way of proving a reconstruction formula is by using the orthogonality relations that hold for the square-integrable representations of a (unimodular) Lie group. We briefly recall that if (π, \mathcal{H}) is a unitary irreducible representation of a unimodular Lie group G (we denote by dg a Haar measure of G , arbitrarily fixed), we say that π is square-integrable if there is a non zero element u of \mathcal{H} , such that

$$\int_G |(u, \pi(g)u)_{\mathcal{H}}|^2 dg < \infty$$

It can be proved that, in this case, all the matrix elements $(u, \pi(\cdot)v)_{\mathcal{H}}$, define functions in $L^2(G)$, and there exists a strictly positive number d_{π} (depending only on the normalization of the Haar measure dg), such that

$$\int_G \overline{(u, \pi(g)v)_{\mathcal{H}}}(w, \pi(g)y)_{\mathcal{H}} dg = \frac{1}{d_{\pi}}(w, u)_{\mathcal{H}} \cdot (v, y)_{\mathcal{H}}$$

for all $u, v, w, y \in \mathcal{H}$. The number d_{π} is called the formal degree of π and the equalities are called “orthogonality relations” for the representation π (see, for more details, [1]).

An example of how a reconstruction formula can be deduced from orthogonality relations is the homodyne quantum tomography. In this case, the quantum system of interest is the single mode radiation field, and the set of physical quantities, whose probability distributions are needed in order to reconstruct the state, are the “quadratures” of the field that can be measured by a homodyne detector. The reconstruction formula used in homodyne quantum tomography is derived from the orthogonality relations that hold for the Schrödinger representation of the Heisenberg group (see [2]).

Our aim is getting a reconstruction formula that contains the probability distribution of the physical quantity “number of photons” of the radiation field, that can be measured by a photon counter. It turns out that this can be achieved by writing down orthogonality relations for some representations of $SU(1, 1)$ (namely, those belonging to the so called “discrete series”). The interesting and, to a certain extent, surprising feature is that we need new orthogonality relations, different from the ones that hold for square integrable representations. Hence, the main content of the paper is proving generalized orthogonality relations for the representations of the discrete series of $SU(1, 1)$. We stress that these generalized orthogonality relations hold also for the representation limit of the discrete series, which is not square-integrable.

The idea of using generalized orthogonality relations (different from the ones that hold for square-integrable representations) appears for the first time (to the best of our knowledge) in the papers [3, 4], and there is some overlapping between these papers and the present one.

The orthogonality relations of section 4.2, and the reconstruction formula of section 4.3 are new, although our reconstruction formula has a structure similar to the one of paper [4]. Our orthogonality relations of section 4.4 are essentially the same as the ones in paper [3]. However, we present a mathematically precise and self contained proof of them. The reconstruction formula of section 4.5 is new.

The outline of the paper is the following: in section 2 the group $SU(1, 1)$ and its Lie algebra are introduced, the representations of the discrete series are described as well. In section 3 an explicit realization of all the representations of the discrete series, which has a transparent physical interpretation, is described. In section 4, the generalized orthogonality relations are proved and the reconstruction formulae are deduced from them.

Dedication. We are very pleased to dedicate this paper to Pekka Lahti, for his sixtieth birthday. In particular, one of us (G. C.) would like to remember a long lasting fraternal friendship that goes well beyond the (rather narrow) borders of Quantum Mechanics

2 The group $SU(1, 1)$ and its Lie algebra. The representations (D_n^+, H_n)

In this section we briefly recall some basic facts about the group $SU(1, 1)$ and its Lie algebra. We describe, also, a family of unitary irreducible representations of $SU(1, 1)$, that form the so-called “discrete series”. All statements and facts contained in this section are well known since the path breaking paper of Bargmann [5]; they are recalled here for the reader’s convenience and to fix our notations. In this section, and in the paper, we shall use freely some very elementary and basic properties of unitary representations of simple Lie groups and their Lie algebras. General references for these topics and for the properties of the representations of $SU(1, 1)$ are (other than Bargmann’s paper) the monographs [6, 7, 8, 9, 10]; each of them gives a modern and very pedagogical explanations of these matters.

The group $SU(1, 1)$ is defined by

$$SU(1, 1) := \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}$$

it is a linear Lie group (a closed subgroup of $SL(2, \mathbb{C})$).

We define its closed subgroup

$$T := \left\{ \begin{pmatrix} \alpha & \\ & \bar{\alpha} \end{pmatrix} \mid \alpha \in \mathbb{C}, |\alpha|^2 = 1 \right\};$$

it is a maximal compact (and abelian) subgroup of $SU(1, 1)$. Topologically, $SU(1, 1)$ is homeomorphic to the product

$$\mathbb{R}^2 \times T$$

hence, it is connected but not simply connected. T is a one-parameter subgroup of $SU(1, 1)$: its elements can be written in terms of a single real parameter θ as

$$t_\theta := \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$$

The three-dimensional real manifold $SU(1, 1)$ can be coordinatized in different ways, according to the problems one has to deal with. A possible parametrization of the elements of $SU(1, 1)$ is

$$SU(1, 1) = \left\{ \begin{pmatrix} e^{i\theta} \cosh r & e^{i\phi} \sinh r \\ e^{-i\phi} \sinh r & e^{-i\theta} \cosh r \end{pmatrix} \mid 0 \leq \theta, \phi < 2\pi, 0 \leq r < +\infty \right\}$$

Another coordinatization can be obtained by using the decompositions of the elements g of $SU(1, 1)$ given by

$$g = t_\theta \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix} t_\phi;$$

(this a special case of the “ KAK ” decomposition that holds for all semisimple Lie groups), in terms of this decomposition, we have

$$SU(1, 1) = \{t_\theta a_r t_\phi \mid 0 \leq \theta, \phi < 2\pi, 0 \leq r < +\infty\}$$

where we have introduced the notation

$$a_r = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix}$$

Another decomposition of the elements of $SU(1, 1)$ (called the ‘‘Cartan decomposition’’) is described in paragraph 4.4.

We denote by $\mathfrak{su}(1, 1)$ the Lie algebra of $SU(1, 1)$; explicitly

$$\mathfrak{su}(1, 1) = \left\{ \begin{pmatrix} ia & b - ic \\ b + ic & -ia \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\};$$

The Lie algebra $\mathfrak{su}(1, 1)$ is a real vector space of dimension three; it has a ‘‘natural’’ basis (over \mathbb{R})

$$\begin{pmatrix} i & \\ & -i \end{pmatrix}, \quad \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} -i & \\ & i \end{pmatrix},;$$

we have the commutation relations

$$\begin{aligned} \left[\begin{pmatrix} i & \\ & -i \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right] &= -2 \begin{pmatrix} -i & \\ & i \end{pmatrix} \\ \left[\begin{pmatrix} i & \\ & -i \end{pmatrix}, \begin{pmatrix} -i & \\ & i \end{pmatrix} \right] &= 2 \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \\ \left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -i & \\ & i \end{pmatrix} \right] &= \begin{pmatrix} i & \\ & -i \end{pmatrix} \end{aligned}$$

The one-parameter subgroups (of $SU(1, 1)$) generated by the elements of this basis of $\mathfrak{su}(1, 1)$ are

$$\begin{aligned} \exp \left(\theta \begin{pmatrix} i & \\ & -i \end{pmatrix} \right) &= \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \\ \exp \left(t \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right) &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \\ \exp \left(t \begin{pmatrix} -i & \\ & i \end{pmatrix} \right) &= \begin{pmatrix} \cosh t & -i \sinh t \\ i \sinh t & \cosh t \end{pmatrix} \end{aligned}$$

The complexification, $\mathfrak{su}(1, 1)_{\mathbb{C}}$ of the real Lie algebra $\mathfrak{su}(1, 1)$ can be identified with $\mathfrak{sl}(2, \mathbb{C}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbb{C} \right\}$. A basis (over \mathbb{C}) of $\mathfrak{sl}(2, \mathbb{C}) \simeq \mathfrak{su}(1, 1)_{\mathbb{C}}$ is given by

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \begin{pmatrix} 01 & \\ & 00 \end{pmatrix}, \quad \begin{pmatrix} 00 & \\ & 10 \end{pmatrix}$$

with commutation relations

$$\begin{aligned} \left[\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} 01 & \\ & 00 \end{pmatrix} \right] &= 2 \begin{pmatrix} 01 & \\ & 00 \end{pmatrix} \\ \left[\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} 00 & \\ & 10 \end{pmatrix} \right] &= 2 \begin{pmatrix} 00 & \\ & 10 \end{pmatrix} \\ \left[\begin{pmatrix} 01 & \\ & 00 \end{pmatrix}, \begin{pmatrix} 00 & \\ & 10 \end{pmatrix} \right] &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \end{aligned}$$

The group $SU(1, 1)$ is a simple Lie group: hence, it is unimodular. We wish to fix the normalization of the Haar measure of $SU(1, 1)$ that will be used in the rest of the paper. In the coordinatization of $SU(1, 1)$ given by

$$g = \begin{pmatrix} e^{i\theta} \cosh r & e^{i\phi} \sinh r \\ e^{-i\phi} \sinh r & e^{-i\theta} \cosh r \end{pmatrix}, \quad g \in SU(1, 1),$$

we fix the Haar measure μ as

$$d\mu = \sinh(2r) dr \frac{d\theta}{2\pi} \frac{d\phi}{2\pi}$$

Accordingly, in the coordinatization $g = t_\theta a_r t_\phi$, μ is given by

$$d\mu = \sinh(2r) dr \frac{d\theta}{2\pi} \frac{d\phi}{2\pi}$$

This means, precisely, that, if f is any function μ -integrable on $SU(1, 1)$, we can compute

$$\int_{SU(1,1)} f(g) d\mu(g)$$

both as

$$\int_0^\infty dr \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} f \left(\begin{pmatrix} e^{i\theta} \cosh r & e^{i\phi} \sinh r \\ e^{-i\phi} \sinh r & e^{-i\theta} \cosh r \end{pmatrix} \right) \sinh(2r)$$

and as

$$\int_0^\infty dr \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} f(t_\theta a_r t_\phi) \sinh(2r)$$

We shall always use this normalization of μ , in the rest of the paper.

The group $SU(1, 1)$ has (among others) a family of unitary irreducible representations that form the “discrete series”. We shall use these representations and we shall describe them briefly. We denote

$$\begin{aligned} \mathcal{D} &:= \{z \in \mathbb{C} \mid |z| < 1\}, \\ d^2z &:= dx dy, \quad \text{if } z = x + iy, \\ dm_n(z) &:= (1 - |z|^2)^{n-2} d^2z \quad \text{for each } n \geq 2 \end{aligned}$$

and we define, for each integer $n \geq 2$, the space H_n as

$$H_n = \{f \text{ holomorphic on } \mathcal{D}, f \in L^2(\mathcal{D}, m_n)\}$$

H_n is a (complex, separable) Hilbert space. If $\begin{pmatrix} \alpha\beta \\ \beta\bar{\alpha} \end{pmatrix} \in SU(1, 1)$, we define $D_n^+ \begin{pmatrix} \alpha\beta \\ \beta\bar{\alpha} \end{pmatrix}$ by

$$D_n^+ \begin{pmatrix} \alpha\beta \\ \beta\bar{\alpha} \end{pmatrix} f(z) = \frac{1}{(-\beta z + \bar{\alpha})^n} f \left(\frac{\alpha z - \bar{\beta}}{-\beta z + \bar{\alpha}} \right)$$

Then, for each $n \geq 2$, (D_n^+, H_n) is a unitary, irreducible representation of $SU(1, 1)$.

We denote by H_1 the space of holomorphic functions on \mathcal{D} such that

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < +\infty$$

The space H_1 is a Hilbert space (it is the ‘‘Hardy space’’ on \mathcal{D}). We define D_1^+ , on H_1 , by

$$D_1^+ \begin{pmatrix} \alpha\beta \\ \beta\bar{\alpha} \end{pmatrix} f(z) = \frac{1}{-\beta z + \bar{\alpha}} f\left(\frac{\alpha z - \bar{\beta}}{-\beta z + \bar{\alpha}}\right)$$

then (D_1^+, H_1) is a unitary irreducible representation of $SU(1, 1)$.

We shall refer to the family (D_n^+, H_n) , $n \geq 1$ as the ‘‘discrete series’’ of $SU(1, 1)$ (sometimes D_1^+ is called the ‘‘limit of the discrete series’’). It is well known that the representations (D_n^+, H_n) , $n \geq 2$ are square-integrable; but (D_1^+, H_1) is not square-integrable. In this paper, we shall not use the properties of square-integrable representations.

The algebraic structure of the representation D_n^+ is very clear. We denote

$$v_k^n := \frac{z^k}{\|z^k\|_{H_n}}, \quad n \geq 1, k \geq 0;$$

then $\{v_k^n\}_{k \geq 0}$ is a Hilbert (ordered) basis of H_n ; we have

$$D_n^+(t_\theta)v_k^n = D_n^+ \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \frac{z^k}{\|z^k\|_{H_n}} = e^{i(n+2k)\theta} v_k^n$$

This implies that the vectors $setv_{k \geq 0}^n$ are smooth (even analytic) vectors for the representation D_n^+ ($n \geq 1$).

The derived action of the element $\begin{pmatrix} i & \\ & -i \end{pmatrix}$ of $\mathfrak{su}(1, 1)$ (which is the generator of the maximal compact subgroup T) is given by

$$dD_n^+ \begin{pmatrix} i & \\ & -i \end{pmatrix} v_k^n = \frac{d}{d\theta} \Big|_{\theta=0} D_n^+ \begin{pmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{pmatrix} v_k^n = i(n+2k)v_k^n$$

The algebraic span of the v_k^n 's (the set of finite linear combinations) is a representation of the Lie algebra $\mathfrak{su}(1, 1)_{\mathbb{C}}$, which has the structure of a lowest weight module. This means, precisely, that the vectors $\{v_k^n\}_{k \geq 0}$ (that are an algebraic basis of this vector space) are eigenvectors of the operator $dD_n^+ \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$:

$$dD_n^+ \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} v_k^n = (n+2k)v_k^n;$$

moreover, v_0^n is a lowest-weight vector, which is annihilated by the action of $dD_n^+ \begin{pmatrix} 00 \\ 10 \end{pmatrix}$:

$$dD_n^+ \begin{pmatrix} 00 \\ 10 \end{pmatrix} v_0^n = 0$$

and has the property

$$dD_n^+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v_k^n = v_{k+1}^n, \quad k \geq 0$$

The unitary irreducible representations of $SU(1, 1)$ are completely classified. An important consequence of this classification, that will be used later, is the following property: if (π, V) is any irreducible unitary representation of $SU(1, 1)$, with the property

$$\text{spc } d\pi \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \{n, n+2, n+4, \dots\}$$

with n integer ≥ 1 , then, necessarily

$$(\pi, V) \simeq (D_n^+, H_n);$$

in particular, V has a Hilbert basis, that will be denoted by $\{v_k^n\}_{k \geq 0}$ that diagonalizes the action of the maximal compact subgroup T :

$$\pi \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} v_k^n = e^{i\theta(n+2k)} v_k^n, \quad n \geq 1, k \geq 0,$$

or, equivalently

$$d\pi \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} v_k^n = (n+2k)v_k^n$$

3 The discrete series of $SU(1, 1)$ embedded into the quantum states space of the two modes radiation field

We denote by $\mathcal{H}^{(2)}$ the complex vector space of all holomorphic functions $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ such that

$$z_1, z_2 \mapsto e^{-(|z_1|^2 + |z_2|^2)} |f(z_1, z_2)|^2$$

is integrable over \mathbb{C}^2 with respect to the Lebesgue measure. The vector space $\mathcal{H}^{(2)}$ endowed with the scalar product

$$(f_1, f_2)_{\mathcal{H}^{(2)}} = \frac{1}{\pi^2} \int_{\mathbb{C}^2} e^{-(|z_1|^2 + |z_2|^2)} \overline{f_1(z_1, z_2)} f_2(z_1, z_2) d^2 z_1 d^2 z_2$$

is a Hilbert space of “true” functions (as opposed to “equivalence classes” of functions), called the *Bargmann-Fock space* in two variables.

For each pair of non negative integers k_1, k_2 , we define

$$w_{k_1, k_2} = \sqrt{\frac{1}{k_1! k_2!}} z_1^{k_1} z_2^{k_2}$$

The family of elements of $\mathcal{H}^{(2)}$

$$\{ w_{k_1, k_2} \mid k_1, k_2 \text{ integers } \geq 0 \}$$

is a Hilbert basis of $\mathcal{H}^{(2)}$. We denote by

$$\mathcal{P}^{(2)} := \mathbb{C}[z_1, z_2] := \text{span} \{ w_{k_1, k_2} \}$$

$\mathcal{P}^{(2)}$ is a dense linear subspace of $\mathcal{H}^{(2)}$.

We define the (unbounded) operators a_i ($i = 1, 2$), on $\mathcal{H}^{(2)}$, by

$$\begin{aligned} \text{dom } a_i &= \left\{ f \in \mathcal{H}^{(2)} \mid \frac{\partial}{\partial z_i} f \in \mathcal{H}^{(2)} \right\} \supset \mathcal{P}^{(2)} \\ a_i f &= \frac{\partial}{\partial z_i} f, \quad f \in \text{dom } a_i \end{aligned}$$

It is well known that the a_i 's are closed operators, and that

$$\begin{aligned} \text{dom } a_i^* &= \{ f \in \mathcal{H}^{(2)} \mid z_i f \in \mathcal{H}^{(2)} \} \supset \mathcal{P}^{(2)} \\ a_i^* f &= z_i f, \quad f \in \text{dom } a_i^* \end{aligned}$$

moreover we have the commutation relations

$$[a_i, a_j^*] \subset \delta_{ij} I_{\mathcal{H}^{(2)}}, \quad i, j = 1, 2$$

The Bargmann space $\mathcal{H}^{(2)}$ can be interpreted as the quantum states space of the two modes radiation field. The unit vectors w_{k_1, k_2} of the Hilbert basis of $\mathcal{H}^{(2)}$ represent the states of the field where k_1 photons of frequency (labeled by) 1 and k_2 photons of frequency (labeled by) 2 are present. In particular the state $w_{0,0}$ plays the role of the vacuum state, and we have

$$w_{k_1, k_2} = \frac{(a_1^*)^{k_1} (a_2^*)^{k_2}}{\sqrt{k_1! k_2!}} w_{0,0}$$

The operators a_i^* are interpreted as creation operators for the photon states; correspondingly, the a_i are annihilation operators. We can also define the operators $N_1 := a_1^* a_1$ and $N_2 := a_2^* a_2$, whose action on the basis vectors is given by

$$a_i^* a_i w_{k_1, k_2} = k_i w_{k_1, k_2} \quad i = 1, 2$$

Hence the operator N_i multiply the state vector w_{k_1, k_2} by the number of photons in the i -th mode.

We want to define on $\mathcal{H}^{(2)}$ two representations

- i) a unitary representation ρ of the torus T
- ii) a unitary representation Φ of $SU(1, 1)$

in such a way that their actions on $\mathcal{H}^{(2)}$ commute.

We define the unitary representation ρ of T on $\mathcal{H}^{(2)}$ by

$$\left[\rho \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix} f \right] (z_1, z_2) = f(e^{-i\theta} z_1, e^{i\theta} z_2), \quad f \in \mathcal{H}^{(2)}$$

It is a simple check that ρ is a continuous unitary representation of T on $\mathcal{H}^{(2)}$. The Hilbert basis $\{w_{k_1, k_2}\}$ of $\mathcal{H}^{(2)}$ diagonalizes ρ ; indeed

$$\rho \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix} w_{k_1, k_2}(z_1, z_2) = e^{-i\theta(k_1 - k_2)} w_{k_1, k_2}(z_1, z_2)$$

The vectors w_{k_1, k_2} are analytic vectors for the representation ρ ; hence $\mathcal{P}^{(2)}$ is a dense subspace of $\mathcal{H}^{(2)}$ of analytic vectors for ρ . The derived representation $d\rho$ of the (one-dimensional) Lie algebra of T acts on $\mathcal{P}^{(2)}$ as

$$\left[d\rho \begin{pmatrix} i \\ -i \end{pmatrix} p \right] (z_1, z_2) = -iz_1 \frac{\partial}{\partial z_1} p(z_1, z_2) + iz_2 \frac{\partial}{\partial z_2} p(z_1, z_2)$$

so that, on $\mathcal{P}^{(2)}$ we have

$$d\rho \begin{pmatrix} i \\ -i \end{pmatrix} = iA = i(N_2 - N_1)$$

The self-adjoint operator A has the physical meaning of representing the quantity ‘‘difference of the number of photons of frequency 1 and the number of photons of frequency 2’’. A is diagonalized by the basis $\{w_{k_1, k_2}\}$

$$Aw_{k_1, k_2} = (k_2 - k_1)w_{k_1, k_2}$$

Since T is a compact Lie group, we can decompose canonically $\mathcal{H}^{(2)}$, with respect to ρ ; we have

$$\mathcal{H}^{(2)} = \hat{\bigoplus}_{k \in \mathbb{Z}} (V_k \mid k \in \mathbb{Z})$$

where $\hat{\bigoplus}$ denotes the Hilbert sum. The index k parametrizes the irreducible representations of T ; the isotypical subspace V_k can be characterized equivalently as

- i) the closed subspace of $\mathcal{H}^{(2)}$ where $\rho \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix}$ acts through multiplication by $e^{ik\theta}$,
- ii) the closed subspace of $\mathcal{H}^{(2)}$ where A acts through multiplication by k ;

hence

$$\mathcal{H}^{(2)} = \hat{\bigoplus}_{k \in \mathbb{Z}} \overline{\text{span}} \{w_{k_1, k_2} \mid k_2 - k_1 = k\}$$

where $\overline{\text{span}}$ denotes, as usual, the closure of the span.

We now define a unitary representation of $SU(1,1)$ on $\mathcal{H}^{(2)}$, that commutes with the representation ρ of T . This representation, denoted by Φ , will be first defined at the infinitesimal level.

We define on $\mathcal{P}^{(2)}$ a representation $d\Phi$ of the Lie algebra $\mathfrak{su}(1,1)$, by skew-adjoint operators. The notation $d\Phi$ anticipates that it is the derived representation of a unitary representation of $SU(1,1)$. We define $d\Phi$ on $\mathcal{P}^{(2)}$ according to

$$\begin{aligned} d\Phi \begin{pmatrix} i \\ -i \end{pmatrix} &:= i \left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + 1 \right) = i(N_1 + N_2 + 1) \\ d\Phi \begin{pmatrix} 1 \\ 1 \end{pmatrix} &:= z_1 z_2 - \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} = a_1^* a_2^* - a_1 a_2 \\ d\Phi \begin{pmatrix} -i \\ i \end{pmatrix} &:= -i z_1 z_2 - i \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} = -i(a_1^* a_2^* + a_1 a_2) \end{aligned}$$

the operators

$$d\Phi \begin{pmatrix} i \\ -i \end{pmatrix} \quad d\Phi \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad d\Phi \begin{pmatrix} -i \\ i \end{pmatrix}$$

are skew-symmetric (with respect to the scalar product of $\mathcal{H}^{(2)}$ restricted to $\mathcal{P}^{(2)}$). $d\Phi$ can be extended to $\mathfrak{su}(1,1)$ by \mathbb{R} -linearity, and to $\mathfrak{su}(1,1)_{\mathbb{C}} \simeq \mathfrak{sl}(2, \mathbb{C})$ by \mathbb{C} -linearity.

Choosing the basis

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

of $\mathfrak{sl}(2, \mathbb{C})$. we have

$$d\Phi \begin{pmatrix} 1 \\ -1 \end{pmatrix} = N_1 + N_2 + 1 \quad d\Phi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a_1^* a_2^* \quad d\Phi \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -a_1 a_2$$

The symmetry properties of these operators are

$$\begin{aligned} d\Phi \begin{pmatrix} 1 \\ -1 \end{pmatrix}^* &= d\Phi \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad d\Phi \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* = -d\Phi \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ d\Phi \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* &= -d\Phi \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

It is easy to check that the representation $d\rho$ and $d\Phi$ commute among them.

It is possible to prove (we omit here the proof, and we refer for example to [11]) that the representation $d\Phi$ of $\mathfrak{su}(1,1)$ is unitarily equivalent to the tensor square of the metaplectic (or ‘‘oscillator’’, or Segal-Shale-Weil) representation of $\mathfrak{su}(1,1)$. It is known that the metaplectic representation of $\mathfrak{su}(1,1)$ is the derived representation of a unitary representation of the universal covering of $SU(1,1)$ (indeed, of the twofold cover of $SU(1,1)$, but we need only the weaker property). Hence, $d\Phi$ is the derived representation of a unitary representation, Φ , of

the universal covering of $SU(1, 1)$. It is easy to see that Φ is a representation of $SU(1, 1)$ itself. Indeed, the kernel of the covering mapping, from the universal covering to $SU(1, 1)$, is

$$\left\{ \exp \left(2\pi k \begin{pmatrix} i \\ -i \end{pmatrix} \right) \mid k \in \mathbb{Z} \right\};$$

hence

$$\Phi \left(\exp \left(2\pi k \begin{pmatrix} i \\ -i \end{pmatrix} \right) \right) = \exp(2\pi k i(N + 1))$$

where $N := N_1 + N_2$. We know that N is diagonalized by the basis $\{w_{k_1, k_2}\}$, with a spectrum of integer eigenvalues, hence

$$\exp(2\pi k i(N + 1)) w_{k_1, k_2} = w_{k_1, k_2}$$

This shows that

$$\Phi \left(\exp \left(2\pi k \begin{pmatrix} i \\ -i \end{pmatrix} \right) \right) = I \quad \text{on } \mathcal{H}^{(2)}$$

and that Φ is a representation of $SU(1, 1)$.

Since the representation $d\rho$, of the Lie algebra of T , and $d\Phi$, of $\mathfrak{su}(1, 1)$ commute, then also ρ and Φ commute. In particular, the invariant subspaces V_k , $k \in \mathbb{Z}$, that appear in the decomposition of ρ are invariant with respect to Φ , as well.

The essential point is that we can prove that the representations (ϕ, V_k) , $k \in \mathbb{Z}$ are irreducible. Again we omit the proof, and we refer to [11]. Since the representations (ϕ, V_k) are unitary irreducible representations of $SU(1, 1)$, we can determine whether they are equivalent to the representations of the discrete series, described in the previous paragraph, by computing the spectrum of the operator

$$d\Phi \begin{pmatrix} 1 \\ -1 \end{pmatrix} = N_1 + N_2 + 1$$

restricted to V_k . Since

$$V_k = \overline{\text{span}} \{w_{k_1, k_2} \mid k_2 - k_1 = k\}$$

and

$$d\Phi \begin{pmatrix} 1 \\ -1 \end{pmatrix} w_{k_1, k_2} = k_1 + k_2 + 1,$$

we have that

$$\text{spc } d\Phi|_{V_k} = |k| + 1, |k| + 3, \dots$$

hence

$$\Phi|_{V_k} \simeq D_{|k|+1}^+, \quad k \in \mathbb{Z},$$

as representations of $SU(1, 1)$.

4 Generalized orthogonality relations and reconstruction formulas

4.1 The Jacobi polynomials and the matrix elements of the representations D_n^+ ($n \geq 1$).

In this section we briefly recall the properties of the Jacobi polynomials; we follow the definition and the notations of the classical [12].

Fix an integer $r \geq 0$, and two integers p, q such that $p \geq q > 0$. The *Jacobi polynomial* $G_r(p, q; x)$ with $x \in \mathbb{R}$ is uniquely determined by the following two properties

- i) $G_r(p, q; x) = x^r +$ terms of lower degree in x .
- ii) $\int_0^1 (1-x)^{p-q} x^{q-1} G_r(p, q; x) G_s(p, q; x) dx = 0$, if $r \neq s$.

In other words, the Jacobi polynomials are functions on $[0, 1]$ orthogonal with respect to the weight function $(1-x)^{p-q} x^{q-1}$. They have the normalization property

$$\int_0^1 (1-x)^{p-q} x^{q-1} G_r^2(p, q; x) dx = \frac{r!(r+q-1)!(r+p-1)!(r-p-q)!}{(2r+p)((2r+p-1)!)^2} \quad (1)$$

We fix a (unitary, irreducible) representation (D_n^+, H_n) , with $n \geq 1$ of $SU(1, 1)$. We recall that we denote by $\{v_k^n\}_{k \geq 0}$, $n \geq 1$ the Hilbert basis of H_n that diagonalizes the action of T

$$D_n^+(t_\theta) = D_n^+ \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} v_k^n = e^{i(n+2k)\theta} v_k^n$$

In his seminal work [5], Bargmann derived an explicit formula for the matrix elements

$$(v_k^n, D_n^+(g)v_j^n)_{H_n}$$

in terms of hypergeometric functions. We shall need the matrix elements expressed in terms of Jacobi polynomials. For this we recall that a generic element g in $SU(1, 1)$ can be represented in matrix form as

$$g = \begin{pmatrix} e^{i\theta} \cosh r & e^{i\phi} \sinh r \\ e^{-i\phi} \sinh r e^{-i\theta} & \cosh r \end{pmatrix} \quad \theta, \phi \in [0, 2\pi), r \in [0, +\infty)$$

It is then a straightforward, though tedious, exercise getting the following formula for $k \geq j$

$$\begin{aligned} & (v_k^n, D_n^+ \begin{pmatrix} e^{i\theta} \cosh r & e^{i\phi} \sinh r \\ e^{-i\phi} \sinh r e^{-i\theta} & \cosh r \end{pmatrix} v_j^n)_{H_n} = \\ & = e^{i\theta(n+k+j)} e^{i\phi(k-j)} (-1)^j \sqrt{\frac{k!(k+n-1)!(n+k+j-1)!}{j!(j+n-1)! k!(k+n-1)!}} \times \\ & \times (\tanh^2 r)^{\frac{k-j}{2}} (1 - \tanh^2 r)^{\frac{n}{2}} G_j(n+k-j, 1+k-j; \tanh^2 r) \end{aligned}$$

In the sequel, we shall use the previous formula mainly in the case

$$g = a_r = \begin{pmatrix} \cosh r \sinh r \\ \sinh r \cosh r \end{pmatrix};$$

and, in this case, we can readily get the following explicit expression, that holds for all $k, j \geq 0$:

$$\begin{aligned} (v_k^n, D_n^+(a_r)v_j^n)_{H_n} &= (-1)^j \frac{(n+k+j-1)!}{\sqrt{j!(j+n-1)!k!(n+k-1)!}} (\tanh^2 r)^{\frac{|k-j|}{2}} \times \\ &\times (1 - \tanh^2 r)^{\frac{n}{2}} G_{\min(k,j)}(n+|k-j|, 1+|k-j|; \tanh^2 r) \end{aligned}$$

for $k, j \geq 0$. From the normalization condition 1, we obtain

$$\begin{aligned} \int_0^1 (1-x)^{n-1} x^{|k-j|} G_{\min(k,j)}^2(n+|k-j|, 1+|k-j|; x) dx &= \\ \frac{k!j!(n+j-1)!(n+k-1)!}{(n+k+j)((n+k+j-1)!)^2} \end{aligned}$$

From this, we obtain the following equality

$$\int_0^\infty |(v_k^n, D_n^+(a_r)v_j^n)_{H_n}|^2 2 \tanh r dr = \frac{1}{n+k+j}$$

that will be used in the sequel.

4.2 Generalized orthogonality relations. Integration on $SU(1, 1)$

We state, in the following proposition, the orthogonality relations which generalize in the most direct way the ones that hold for square-integrable representations. They have the structure of an integration over the group $SU(1, 1)$, like the usual ones. However, we stress, once more, that they hold for all the representations, D_n^+ of the discrete series, including the case $n = 1$.

Proposition 1 *Let h be a function defined on $SU(1, 1)$, with the properties*

- i) h is continuous and non negative*
- ii) h is T -biinvariant, i. e. $h(g) = h(t_\theta g t_\phi)$ for all $g \in SU(1, 1)$ and all $t_\theta, t_\phi \in T$*
- iii) for any $n \geq 1$, the functions*

$$g \mapsto h(g) \overline{(v_k^n, D_n^+(g)v_j^n)_{H_n}} \cdot (v_l^n, D_n^+(g)v_m^n)_{H_n}$$

are integrable with respect to the Haar measure of $SU(1, 1)$, for all indices k, j, l, m .

(We shall show in a moment that functions with these properties exist), then we have:

$$\int_{SU(1,1)} h(g) \overline{(v_k^n, D_n^+(g)v_j^n)_{H_n}} \cdot (v_l^n, D_n^+(g)v_m^n)_{H_n} d\mu(g) = K \delta_{k,l} \delta_{j,m} \quad (2)$$

where K depends on the choice of the function h , and on k, j , and n .

Proof. We use the parametrization of the elements g of $SU(1, 1)$ given by the decomposition $SU(1, 1) = TA_+^*T$:

$$g = t_\theta a_r t_\phi$$

where

$$t_\theta = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}, \quad 0 \leq \theta < 2\pi \quad a_r = \begin{pmatrix} \cosh r \sinh r & \\ \sinh r \cosh r & \end{pmatrix}, \quad 0 \leq r < +\infty$$

In this coordinatization the Haar measure of $SU(1, 1)$ is given by

$$d\mu = \sinh(2r) dr \frac{d\theta d\phi}{2\pi 2\pi} \quad (3)$$

The integral $\int_{SU(1,1)} h(g) \overline{(v_k^n, D_n^+(g)v_j^n)_{H_n}} \cdot (v_l^n, D_n^+(g)v_m^n)_{H_n} d\mu(g)$ becomes

$$\begin{aligned} & \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{+\infty} dr \sinh(2r) \cosh r h(t_\theta a_r t_\phi) e^{-i\theta(n+2k)} e^{-i\phi(n+2j)} \times \\ & \quad \times \overline{(v_k^n, D_n^+(a_r)v_j^n)_{H_n}} e^{-i\theta(n+2l)} e^{-i\phi(n+2m)} (v_l^n, D_n^+(a_r)v_m^n)_{H_n} \\ & = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(2l-2k)} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi(2m-2j)} \int_0^{+\infty} dr \sinh 2r \cosh r \times \\ & \quad \times h(a_r) \overline{(v_k^n, D_n^+(a_r)v_j^n)_{H_n}} \cdot (v_l^n, D_n^+(a_r)v_m^n)_{H_n} \\ & = \delta_{l,k} \delta_{j,m} \int_0^{+\infty} 2 \sinh r \cosh r h(a_r) |(v_k^n, D_n^+(a_r)v_j^n)_{H_n}|^2 dr \end{aligned}$$

■

It must be emphasized that, if we consider the case $n \geq 2$, the representations D_n^+ are square integrable and the function h can be chosen as the constant function 1 on $SU(1, 1)$. With this choiche the constant K coincides with the formal degree of the representation D_n^+ (see [1]), and depends only on the integer n , and on the choice of the normalization of the Haar measure over $SU(1, 1)$. Moreover, in this case, the orthogonality relations are the known ones.

The proposition shows that, if the function h decays fast enough in the A_+^* direction of the group $SU(1, 1)$, more general orthogonality relations hold, that include the case $n = 1$.

To make our orthogonality relations more precise, we can choose the function h so that the constant K is the simplest possible. At the same time, an explicit example of a function h (with the properties required in the proposition) shows the existence of such functions.

For each integer $p \geq 0$, we define a function h_p on $SU(1, 1)$, by

$$h_p(g) := |(v_0^1, D_1^+(g)v_0^1)_{V_1}|^p$$

Obviously, we have

1. h_p is continuous on $SU(1, 1)$, and $h_p \geq 0$;
2. $h_p(t_\theta g t_\phi) = h_p(g)$, $\forall t_\theta, t_\phi \in T, g \in SU(1, 1)$

Hence, we have to show that all functions

$$g \mapsto h_p(g) \overline{(v_k^n, D_n^+(g)v_j^n)_{H_n}} \cdot (v_l^n, D_n^+(g)v_m^n)_{H_n}$$

are integrable for all $k, j, l, m \geq 0$ with respect to the Haar measure μ on $SU(1, 1)$. Using again the decomposition $g = t_\theta a_r t_\phi$ we obtain immediately that the function

$$\left| h_p(\cdot) \overline{(v_k^n, D_n^+(\cdot)v_j^n)_{H_n}} \cdot (v_l^n, D_n^+(\cdot)v_m^n)_{H_n} \right|$$

is constant with respect to θ and ϕ . Hence, due to Tonelli theorem, we are led to consider the integrability of the function

$$\sinh(2r) \frac{1}{(\cosh r)^p} \left| \overline{(v_k^n, D_n^+(a_r)v_j^n)_{H_n}} \cdot (v_l^n, D_n^+(a_r)v_m^n)_{H_n} \right| \quad (4)$$

on $[0, +\infty[$. Using the explicit formulae of the matrix elements of the representation D_n^+ , recalled in section 3, we have

$$h_p(g) = \left| (v_0^n, D_n^+ \begin{pmatrix} \alpha\beta \\ \beta\bar{\alpha} \end{pmatrix} v_0^n)_{H_1} \right|^p = \frac{1}{|\alpha|^p} \\ \left| \overline{(v_k^n, D_n^+(a_r)v_j^n)_{H_n}} \right| = F_{nkj}(r) \cdot \frac{1}{(\cosh r)^n}$$

where F_{nkj} is a continuous bounded function of r (clearly depending on n, k, j). We thus have that formula 4 is equal to

$$F_{nkj}(r) F_{nlm}(r) \sinh(2r) \frac{1}{(\cosh r)^p} \frac{1}{(\cosh r)^{2n}}$$

Clearly such a function is integrable on $[0, +\infty)$ for every $n \geq 1$ provided $p \geq 1$.

We thus have that all functions h_p with $p \geq 1$ fulfill the conditions of the Proposition 1. We shall make the choice $p = 2$:

$$h_2(g) = h_2(a_r) = \frac{1}{\cosh^2 r}$$

With this choice the constant K is given by

$$K = \int_0^{+\infty} \sinh(2r) \frac{1}{\cosh^2 r} \left| (v_k^n, D_n^+(a_r)v_j^n)_{H_n} \right|^2 dr = \frac{1}{n+k+j}$$

In conclusion we have the following generalized orthogonality relations, valid for all $n \geq 1$

$$\int_{SU(1,1)} d\mu(g) h_2(g) \overline{(v_k^n, D_n^+(g)v_j^n)_{H_n}} \cdot (v_l^n, D_n^+(g)v_m^n)_{H_n} = \\ = N(\mu) \delta_{k,l} \delta_{j,m} \frac{1}{n+k+j}$$

where the constant $N(\mu)$ depends only on the normalization of the Haar measure μ . With our choice of μ , $N(\mu) = 1$.

4.3 A possible reconstruction formula. Integration on $SU(1,1)$.

We want to use the orthogonality relations of the previous paragraph to obtain a tomographic reconstruction formula. We have (with $w = \sum_{k=0}^{\infty} \alpha_k v_k^n$)

$$\begin{aligned}
& (v_k^n, P[w]v_j^n)_{H_n} = (v_k^n, w)_{H_n} (w, v_j^n)_{H_n} \\
& = \sum_{l,m} \alpha_m \bar{\alpha}_l (v_k^n, v_m^n) (v_l^n, v_j^n) \\
& = \sum_{l,m} \alpha_m \bar{\alpha}_l \delta_{k,m} \delta_{l,j} \\
& = \sum_{l,m} \alpha_m \bar{\alpha}_l (n+k+j) \int_{SU(1,1)} d\mu(g) h(g) \overline{(v_k^n, D_n^+(g)v_j^n)_{H_n}} \times \\
& \quad \times (v_l^n, D_n^+(g)v_m^n)_{H_n} \\
& \text{(using the orthogonality relations)} \\
& = (n+k+j) \int_{SU(1,1)} d\mu(g) h_2(g) (v_k^n, D_n^+(g)v_j^n)_{H_n} (w, D_n^+(g)w)_{H_n}
\end{aligned}$$

We wish to shift the integration from the group $SU(1,1)$ to the its Lie algebra $\mathfrak{su}(1,1)$. This operation is complicated by the fact that the exponential map $\exp : \mathfrak{su}(1,1) \rightarrow SU(1,1)$ is neither injective nor surjective. We define the following subsets of $\mathfrak{su}(1,1)$:

$$\begin{aligned}
W_+ & := \left\{ \begin{pmatrix} in_3 & n_1 - in_2 \\ n_1 + in_2 & -in_3 \end{pmatrix} \in \mathfrak{su}(1,1) \mid n_3 > 0, n_3^2 - n_1^2 - n_2^2 = 1 \right\} \\
W_- & := \left\{ \begin{pmatrix} in_3 & n_1 - in_2 \\ n_1 + in_2 & -in_3 \end{pmatrix} \in \mathfrak{su}(1,1) \mid n_3^2 - n_1^2 - n_2^2 = -1 \right\} \\
V_+ & := \{ \theta X \mid 0 \leq \theta < 2\pi, X \in W_+ \} \\
V_- & := \{ rX \mid r \geq 0, X \in W_- \}
\end{aligned}$$

We have the properties:

1. The subsets of $SU(1,1)$

$$\exp V_+, \quad \exp V_-, \quad \text{and} \quad \left\{ \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} g \mid g \in \exp V_- \right\}$$

are disjoint and their union covers $SU(1,1)$, up to a μ -negligible set. In particular, if f is any μ -integrable function on $SU(1,1)$, the following formula holds true

$$\begin{aligned}
\int_{SU(1,1)} f(g) d\mu(g) & = \int_{\exp(V_+)} f(g) d\mu(g) + \int_{\exp(V_-)} f(g) d\mu(g) + \\
& \quad + \int_{\exp(V_-)} f\left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \cdot g\right) d\mu(g)
\end{aligned}$$

2. The restrictions of the exponential mappings to both V_+ and V_- are diffeomorphisms:

$$\begin{aligned}\exp : V_+ &\rightarrow \exp V_+ \text{ is a diffeomorphism} \\ \exp : V_- &\rightarrow \exp V_- \text{ is a diffeomorphism}\end{aligned}$$

We have to integrate over $SU(1, 1)$ the function

$$f(g) = h_2(g) \overline{(v_k^n, D_n^+(g)v_j^n)}_{H_n} (w, D_n^+(g)w)_{H_n}$$

The element $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ is in the center of $SU(1, 1)$, then $D_n^+\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ is as a scalar of modulus one, since the representation D_n^+ is irreducible; moreover $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in T$, hence

$$h_2\left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} g\right) = h_2(g)$$

This shows that the integration formula

$$\int_{SU(1,1)} f(g) d\mu(g) = \int_{\exp(V_+)} f(g) d\mu(g) + 2 \int_{\exp(V_-)} f(g) d\mu(g)$$

holds.

Since the exponential map is bijective, on V_+ and V_- , we can transport the integration to V_+ and V_- . We describe the formulae that we obtain, without giving the details of the calculations. It is possible (see, for example, [11]) to coordinatize W_+ by

$$n_3 = \cosh s, \quad n_1 = e^{-i\phi} \sinh s, \quad n_2 = e^{i\phi} \sinh s$$

with $0 \leq \phi < 2\pi$, $0 \leq s < \infty$. If we compute the Jacobian of the exponential map, with these coordinates, we get

$$\begin{aligned}\int_{\exp V_+} f(g) d\mu(g) &= \\ \int_0^\infty ds \sinh s \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \sin^2 \theta f\left(\exp \begin{pmatrix} i\theta \cosh s & \theta e^{-i\phi} \sinh s \\ \theta e^{i\phi} \sinh s & -i\theta \cosh s \end{pmatrix}\right)\end{aligned}$$

The other integral can be treated in the same way. We can coordinatize W_- by

$$n_3 = \sinh s, \quad n_1 = e^{-i\phi} \cosh s, \quad n_2 = e^{i\phi} \cosh s$$

with $0 \leq \phi < 2\pi$, $0 \leq s < \infty$. Then

$$\begin{aligned}\int_{\exp V_-} f(g) d\mu(g) &= \\ \int_0^\infty ds \cosh s \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \sinh^2 \theta f\left(\exp \begin{pmatrix} ir \sinh s & re^{-i\phi} \cosh s \\ re^{i\phi} \cosh s & -ir \sinh r \end{pmatrix}\right)\end{aligned}$$

The function f that we have to integrate is a product of three functions

$$f(g) = h_2(g) \overline{(v_k^n, D_n^+(g)v_j^n)}_{H_n} (w, D_n^+(g)w)_{H_n}$$

We have:

1. if $g \in \exp V_+$,

$$\begin{aligned} h_2(g) &= h_2\left(\exp \begin{pmatrix} i\theta \cosh s & \theta e^{-i\phi} \sinh s \\ \theta e^{+i\phi} \sinh s & -i\theta \cosh s \end{pmatrix}\right) \\ &= \left| \frac{1}{\cos \theta + i \sin \theta \cosh s} \right|^2 = \frac{1}{\cos^2 \theta + \sin^2 \theta \cosh^2 s} \end{aligned}$$

2. if $g \in \exp V_-$,

$$\begin{aligned} h_2(g) &= h_2\left(\exp \begin{pmatrix} ir \sinh s & re^{-i\phi} \cosh s \\ re^{+i\phi} \cosh s & -ir \sinh s \end{pmatrix}\right) \\ &= \left| \frac{1}{\cosh r + i \sinh r \sinh s} \right|^2 = \frac{1}{\cosh^2 r + i \sinh^2 r \sinh^2 s} \end{aligned}$$

Moreover, we have quite explicit expressions for the matrix element

$$(v_k^n, D_n^+(g)v_j^n)_{H_n};$$

so we are left with the discussion of the matrix element

$$(w, D_n^+(g)w)_{H_n}.$$

We start considering the adjoint action of $SU(1, 1)$ on its Lie algebra $\mathfrak{su}(1, 1)$:

$$gXg^{-1}, \quad X \in \mathfrak{su}(1, 1), \quad g \in SU(1, 1).$$

We can prove (see, for example, [11]) that this action is transitive, both on W_+ and on W_- . This means that:

1. if we fix $X_0 \in W_+$, then, for each $X \in W_+$, there is a (non unique) $T(X) \in SU(1, 1)$, such that:

$$X = T(X)X_0T(X)^{-1};$$

2. if we fix $Y_0 \in W_-$, then, for each $Y \in W_-$, there is a (non unique) $U(Y) \in SU(1, 1)$, such that:

$$Y = U(Y)Y_0U(Y)^{-1}.$$

To fix ideas, we choose $X_0 = \begin{pmatrix} i & \\ & -i \end{pmatrix}$ and $Y_0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, but we stress that the following discussion doesn't depend on this choice. In this way, we have:

1. if $g = \exp \begin{pmatrix} i\theta \cosh s & \theta e^{-i\phi} \sinh s \\ \theta e^{i\phi} \sinh s & -i\theta \cosh s \end{pmatrix} \in \exp V_+$, then

$$g = T(s, \phi) \exp \begin{pmatrix} i\theta & \\ & -i\theta \end{pmatrix} T(s, \phi)^{-1};$$

so that

$$(w, D_n^+(g)w)_{H_n} = \left(D_n^+(T)^{-1}w, D_n^+(\exp \begin{pmatrix} i\theta & \\ & -i\theta \end{pmatrix}) D_n^+(T)^{-1}w \right)_{H_n}$$

2. if $g \in \exp V_-$, in the same way,

$$(w, D_n^+(g)w)_{H_n} = \left(D_n^+(U)^{-1}w, D_n^+(\exp \begin{pmatrix} r \\ r \end{pmatrix}) D_n^+(U)^{-1}w \right)_{H_n}$$

We define

$$\frac{1}{i} dD_n^+ \begin{pmatrix} i \\ -i \end{pmatrix} = A_3, \quad \frac{1}{i} dD_n^+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} = A_1;$$

then A_1 and A_3 are self-adjoint operators on H_n , and we have

$$D_n^+(\exp \begin{pmatrix} i\theta \\ -i\theta \end{pmatrix}) = e^{i\theta A_3}$$

$$D_n^+(\exp \begin{pmatrix} r \\ r \end{pmatrix}) = e^{ir A_1}.$$

We know that A_3 is diagonalized by the basis $\{v_k^n\}_{k \geq 0}$:

$$A_3 v_k^n = (n + 2k) v_k^n$$

then, by using the Spectral Theorem, we get (we introduce, for the sake of simplicity, the shortened notation $T^{-1}w = D_n^+(T(s, \phi))^{-1}w$):

if $g \in \exp V_+$

$$(w, D_n^+(g)w)_{H_n} = \sum_{m=0}^{\infty} e^{i\theta(n+2m)} (T^{-1}w, P[v_m^n] T w)_{H_n}$$

$$= \sum_{m=0}^{\infty} e^{i\theta(n+2m)} \left| (v_m^n, T^{-1}w)_{H_n} \right|^2$$

We recall that $|(v_m^n, T^{-1}w)_{H_n}|^2 = |(v_m^n, D_n^+(T(s, \phi))^{-1}w)_{H_n}|^2$ is the probability that the physical quantity (represented by the self-adjoint operator) A_3 has value $(n + 2m)$, when the system in the state $T^{-1}w$.

We denote by $P_{A_1}(\cdot)$ the spectral PVM of the self-adjoint operator A_1 , and we define the measure

$$\alpha_v^{A_1}(\cdot) := (v, P_{A_1}(\cdot)v)_{H_n}, \quad v \in H_n$$

then, by the Spectral Theorem:

if $g \in \exp V_-$

$$(w, D_n^+(g)w)_{H_n} = \int_{\mathbb{R}} e^{i\lambda r} d\alpha_{U^{-1}w}^{A_1}(\lambda)$$

We recall that $\alpha_{U^{-1}w}^{A_1}(E)$ with $E \in \mathcal{B}(\mathbb{R})$ represents the probability that the physical quantity (represented by the self-adjoint operator) A_1 has value in the Borel set E , when the system is in the state $U^{-1}w = D_n^+(U(s, \phi))^{-1}w$.

If we put the expressions that we have obtained into the integrals, we obtain a reconstruction formula of the form

$$\begin{aligned} (v_k^n, P[w]v_j^n)_{H_n} = & \\ & \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \int_0^{+\infty} ds P_3(\phi, \theta, s) \sum_{m=0}^{\infty} |(v_m^n, T^{-1}w)|^2 f_{k,j}(m) + \\ & + \int_0^{2\pi} d\phi \int_0^{\infty} dr \int_0^{\infty} ds P_1(\phi, r, s) \int_{\mathbb{R}} d\alpha_{U^{-1}w}^{A_1} f_{k,j}(\lambda) \end{aligned}$$

where

$$P_3(\phi, \theta, s) d\phi d\theta ds$$

is a probability measure on the set $[0, 2\pi] \times [0, 2\pi] \times [0, +\infty[$; and

$$P_1(\phi, r, s) d\phi dr ds$$

is a probability measure on the set $[0, 2\pi] \times [0, +\infty[\times [0, +\infty[$ (we do not give the details of the formula, in view of the following discussion).

It is clear that the previous reconstruction formula can be used in a realistic tomographic experiment only if we are able to measure the probability distributions of A_1 and A_3 . At this stage a serious difficulty, regarding the physical meaning of the observables associated with the operators A_1 and A_3 , arises. Of course, this physical interpretations depend on the concrete realization of the Hilbert space, H_n , where the irreducible representation D_n^+ acts. We have, in the realization described in section 3,

$$A_3 = N_1 + N_2 + 1$$

This represents the total number of photons and it can be measured by photon counters.

We have also

$$A_1 = \frac{1}{i}(a_1^* a_2^* - a_1 a_2) :$$

it seems (to the best of our knowledge) that there is no realistic measuring set up for the observable represented by this elf-adjoint operator. This is a flaw of the reconstruction formula described in this paragraph. Although it has a legitimate mathematical status, it seems that it has not a physical implementation. In the next paragraphs we shall obtain another reconstruction formula without this flaw.

4.4 Generalized orthogonality relations. Integration on \mathfrak{p}

We define

$$\mathfrak{p} := \{ X \in \mathfrak{su}(1, 1) \mid X^* = X \} = \left\{ \begin{pmatrix} 0 & b - ic \\ b + ic & 0 \end{pmatrix} \mid b, c \in \mathbb{R} \right\}$$

Notice that \mathfrak{p} is a real vector subspace of dimension two in $\mathfrak{su}(1, 1)$ (it is not a Lie subalgebra). The following are well known properties:

1. the restriction of the exponential map to \mathfrak{p} is a diffeomorphism onto its image
2. each element g in $SU(1, 1)$ can be decomposed in a unique way as a product

$$g = e^X t_\theta, \quad X \in \mathfrak{p}, t_\theta \in T$$

The decomposition of the elements of $SU(1, 1)$, described in point 2 above, is known as ‘‘Cartan decomposition’’.

In particular, properties 1. and 2. allow us to identify $SU(1, 1)/T$ with \mathfrak{p} . By this identification, we can transfer any measure on $SU(1, 1)/T$ to a measure on \mathfrak{p} . Since $SU(1, 1)$ is unimodular and T is a compact subgroup, there exists a unique (up to positive scalars) $SU(1, 1)$ -invariant Borel measure on $SU(1, 1)/T$. We denote by α the measure on \mathfrak{p} which is the image of such $SU(1, 1)$ -invariant Borel measure on $SU(1, 1)/T$, normalized so that

$$d\alpha \frac{d\phi}{2\pi} = \mu$$

where μ is the Haar measure on $SU(1, 1)$ we have already fixed.

If we parametrize \mathfrak{p} by polar coordinates

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & re^{i\theta} \\ re^{-i\theta} & 0 \end{pmatrix} \mid r \geq 0, \theta \in [0, 2\pi[\right\}$$

we can readily compute

$$d\alpha \begin{pmatrix} 0 & re^{i\theta} \\ re^{-i\theta} & 0 \end{pmatrix} = \frac{d\theta}{2\pi} dr \sinh(2r)$$

Let h be one of the functions defined in paragraph 4.2. It is immediate that the restrictions of these functions to $\exp(\mathfrak{p})$ satisfy the following property: for any $n \geq 1$, the functions

$$h(e^X) \overline{(v_k^n, D_n^+(e^X)v_j^n)_{H_n}} \cdot (v_l^n, D_n^+(e^X)v_m^n)_{H_n}$$

are integrable with respect to the invariant measure α , for all indices k, j, l, m .

We have the following proposition.

Proposition 2 *For all k, j, l, m , we have the following orthogonality relations*

$$\int_{\mathfrak{p}} d\alpha(X) h_2(e^X) \overline{(v_k^n, D_n^+(e^X)v_j^n)_{H_n}} (v_l^n, D_n^+(e^X)v_m^n)_{H_n} = \frac{1}{n+k+j} \delta_{k,l} \cdot \delta_{m,j} \quad (5)$$

Proof. Denote temporarily by h a generic function over $SU(1, 1)$ satisfying the same properties as in prop. 1. We wish to study integrals of the form

$$\int_{\mathfrak{p}} d\alpha(X) h(e^X) \overline{(v_k^n, D_n^+(e^X)v_j^n)_{H_n}} \cdot (v_l^n, D_n^+(e^X)v_m^n)_{H_n}$$

Using the polar coordinates for \mathfrak{p} , we have

a)

$$\begin{aligned} \exp \begin{pmatrix} 0 & r e^{i\theta} \\ r e^{-i\theta} & 0 \end{pmatrix} &= \begin{pmatrix} \cosh r & e^{i\theta} \sinh r \\ e^{-i\theta} \sinh r & \cosh r \end{pmatrix} \\ &= \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \begin{pmatrix} \cosh r \sinh r & \\ \sinh r \cosh r & \end{pmatrix} \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix} = t_{\frac{\theta}{2}} a_r t_{-\frac{\theta}{2}} \end{aligned}$$

b)

$$h(e^X) = h(a_r)$$

c)

$$\begin{aligned} (v_k^n, D_n^+(e^X)v_j^n)_{H_n} &= (D_n^+(t_{-\frac{\theta}{2}})v_k^n, D_n^+(a_r)D_n^+(t_{-\frac{\theta}{2}})v_j^n)_{H_n} \\ &= e^{i\frac{\theta}{2}(n+2k)} e^{-i\frac{\theta}{2}(n+2j)} (v_k^n, D_n^+(a_r)v_j^n)_{H_n} \end{aligned}$$

Hence we have that the integral

$$\int_{\mathfrak{p}} d\alpha(X) h(e^X) \overline{(v_k^n, D_n^+(e^X)v_j^n)_{H_n}} (v_l^n, D_n^+(e^X)v_m^n)_{H_n}$$

equals

$$\begin{aligned} &\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta(k-j)} e^{i\theta(l-m)} \int_0^{+\infty} dr \sinh(2r) h(a_r) \overline{(v_k^n, D_n^+(a_r)v_j^n)_{H_n}} \times \\ &\times (v_l^n, D_n^+(a_r)v_m^n)_{H_n} = \\ &\delta_{k-j, l-m} \int_0^{+\infty} dr \sinh(2r) h(a_r) \overline{(v_k^n, D_n^+(a_r)v_j^n)_{H_n}} (v_l^n, D_n^+(a_r)v_m^n)_{H_n} = \\ &(\text{defining } k-j = l-m = p, \quad p \in \mathbb{Z}) \\ &\delta_{k-j, l-m} \int_0^{+\infty} dr \sinh(2r) h(a_r) \overline{(v_{j+p}^n, D_n^+(a_r)v_j^n)_{H_n}} (v_{m+p}^n, D_n^+(a_r)v_m^n)_{H_n} \end{aligned}$$

This equality hold for all $n \geq 1$ and for all possible h functions.

We choose (as in paragraph 4.2) $h = h_2$; then, by using the explicit formulas for the matrix coefficients and by grouping together the constants that appear in this formulas, we get that the integral

$$\int_{\mathfrak{p}} d\alpha(X) h_2(e^X) \overline{(v_k^n, D_n^+(e^X)v_j^n)_{H_n}} (v_l^n, D_n^+(e^X)v_m^n)_{H_n}$$

is equal to

$$\begin{aligned}
& K \delta_{k-j, l-m} \int_0^{+\infty} dr 2 \tanh r (\tanh^2 r)^{|p|} (1 - \tanh^2 r)^n \times \\
& \times G_{\min(j+p), j}(n+|p|, 1+|p|; \tanh^2 r) G_{\min(m+p), m}(n+|p|, 1+|p|; \tanh^2 r) \\
& \text{(with the substitution } \tanh^2 r = \rho \text{)} \\
& = K \delta_{k-j, l-m} \int_0^1 d\rho (1 - \rho)^{n-1} \rho^{|p|} G_{\min(j+p), j}(n+|p|, 1+|p|; \rho) \times \\
& \times G_{\min(m+p), m}(n+|p|, 1+|p|; \rho) \\
& = K \delta_{k, l} \delta_{m, j}
\end{aligned}$$

The constant K can be computed by

$$K := \int_0^{+\infty} dr 2 \tanh r | (v_k^n, D_n^+(a_r)v_j^n) |^2 = \frac{1}{n+k+j}$$

In conclusion we have the generalized orthogonality relations

$$\int_{\mathfrak{p}} d\alpha(X) h_2(e^X) \overline{(v_k^n, D_n^+(e^X)v_j^n)_{H_n}} (v_l^n, D_n^+(e^X)v_m^n)_{H_n} = K(\alpha) \delta_{k, l} \cdot \delta_{m, j}$$

where

1. $h_2(g) = |(v_0^1, D_1^+(g)v_0^1)_{V_1}|^2 = |\frac{1}{\alpha}|^2$ (with $g = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$)
2. the constant $K(\alpha)$ depends only on the normalization of α . With our choice of α , $K(\alpha) := \int_0^{+\infty} dr 2 \tanh r | (v_k^n, D_n^+(a_r)v_j^n) |^2 = \frac{1}{n+k+j}$.

■

We stress the following facts:

- i) these orthogonality relations are independent of those (very similar) obtained for $SU(1, 1)$ (no one of them can be deduced from the other)
- ii) these relations are valid for all $n \geq 1$, independently of the integrability properties of the representations
- iii) the problem arises of clarifying their geometrical meaning

4.5 A reconstruction formula. Integration over \mathfrak{p} .

We wish to obtain a reconstruction formula for the matrix element

$$(v_{j+k}^n, P[w]v_j^n)_{H_n}, \quad \|w\|_{H_n} = 1 \quad j, k \geq 0$$

using the generalized orthogonality relations.

We shall use some properties of the element $\begin{pmatrix} i & \\ & -i \end{pmatrix} = \begin{pmatrix} e^{i\frac{\pi}{2}} & \\ & e^{-i\frac{\pi}{2}} \end{pmatrix} = t_{\frac{\pi}{2}} \in SU(1, 1)$, The unitary operator $D_n^+(t_{\frac{\pi}{2}})$ is diagonalized by the basis $\{v_j^n\}_{j \geq 0}$

$$D_n^+ \begin{pmatrix} i & \\ & -i \end{pmatrix} v_j^n = e^{i\frac{\pi}{2}(n+2j)} v_j^n = (i)^n (-1)^j v_j^n$$

We can evaluate the matrix coefficient $(v_{k+j}^n, P[w]v_j^n)_{H_n}$ as follows

$$\begin{aligned} (v_{k+j}^n, P[w]v_j^n)_{H_n} &= (v_{j+k}^n, w)_{H_n} (w, v_j^n)_{H_n} \\ (\text{with } w &= \sum_{m=0}^{\infty} \alpha_m v_m^n) = \sum_{m,l=0}^{\infty} \alpha_m \bar{\alpha}_l (v_{j+k}^n, v_m^n)_{H_n} \cdot (v_j^n, v_l^n)_{H_n} \\ &= \sum_{m,l=0}^{\infty} \alpha_m \bar{\alpha}_l \delta_{m,j+k} \delta_{l,j} \\ &= \sum_{m,l=0}^{\infty} \alpha_m \bar{\alpha}_l (2j+k+n) \int_{\mathfrak{p}} d\alpha(X) h(e^X) \times \\ &\quad \times (v_{j+k}^n, D_n^+(e^X)v_j^n)_{H_n} \overline{(v_m^n, D_n^+(e^X)D_n^+ \begin{pmatrix} i & \\ & -i \end{pmatrix} v_l^n)_{H_n}} \\ &= (i)^n (-1)^j (2j+k+n) \int_{\mathfrak{p}} d\alpha(X) h(e^X) \times \\ &\quad \times (v_{j+k}^n, D_n^+(e^X)v_j^n)_{H_n} \overline{(v_m^n, D_n^+(e^X)D_n^+ \begin{pmatrix} i & \\ & -i \end{pmatrix} v_l^n)_{H_n}} \end{aligned}$$

We consider first the matrix element

$$(w, D_n^+(e^X)D_n^+ \begin{pmatrix} i & \\ & -i \end{pmatrix} w)_{H_n}$$

We have the following simple calculations. If $X \in \mathfrak{p}$, we have

$$e^{\frac{X}{2}} \begin{pmatrix} i & \\ & -i \end{pmatrix} e^{-\frac{X}{2}} \begin{pmatrix} i & \\ & -i \end{pmatrix}^{-1} = e^{\frac{X}{2}} e^{-\frac{1}{2} \begin{pmatrix} i & \\ & -i \end{pmatrix} X} \begin{pmatrix} i & \\ & -i \end{pmatrix}^{-1}$$

Since $X = \begin{pmatrix} \gamma & \\ & \bar{\gamma} \end{pmatrix}$, we get

$$e^{\frac{X}{2}} \begin{pmatrix} i & \\ & -i \end{pmatrix} e^{-\frac{X}{2}} \begin{pmatrix} i & \\ & -i \end{pmatrix}^{-1} = e^X$$

or, equivalently

$$e^X \begin{pmatrix} i & \\ & -i \end{pmatrix} = e^{\frac{X}{2}} \begin{pmatrix} i & \\ & -i \end{pmatrix} e^{-\frac{X}{2}}$$

so that

$$(w, D_n^+(e^X)D_n^+ \begin{pmatrix} i \\ -i \end{pmatrix} w)_{H_n} = (D_n^+(e^{-\frac{X}{2}})w, D_n^+ \begin{pmatrix} i \\ -i \end{pmatrix} D_n^+(e^{-\frac{X}{2}})D_n^+w)_{H_n}$$

Using the spectral decomposition of $D_n^+ \begin{pmatrix} i \\ -i \end{pmatrix}$, we obtain

$$\begin{aligned} (D_n^+(e^{-\frac{X}{2}})w, D_n^+ \begin{pmatrix} i \\ -i \end{pmatrix} D_n^+(e^{-\frac{X}{2}})w)_{H_n} &= \\ \sum_{m=0}^{\infty} (i)^n (-1)^m (D_n^+(e^{-\frac{X}{2}})w, P[v_m^n]D_n^+(e^{-\frac{X}{2}})w)_{H_n} &= \\ \sum_{m=0}^{\infty} (i)^n (-1)^m \left| (v_m^n, D_n^+(e^{-\frac{X}{2}})w) \right|^2 & \end{aligned}$$

We have an explicit expression for the other matrix elements. Using the parametrization $X = \begin{pmatrix} 0 & re^{i\theta} \\ re^{-i\theta} & 0 \end{pmatrix}$ we get the following explicit formula

$$\begin{aligned} (v_{j+k}^n, D_n^+(e^X)v_j^n)_{H_n} &= e^{i\theta k} (-1)^j \left(\frac{(k+j)!(k+j+n-1)!}{j!(j+n-1)!} \right)^{\frac{1}{2}} \times \\ &\times \frac{(n+k+2j-1)!}{(k+j)!(k+j+n-1)!} (\tanh^2 r)^{\frac{k}{2}} \times \\ &\times (1 - \tanh^2 r)^{\frac{n}{2}} G_j(n+k, 1+k, \tanh^2 r) \end{aligned}$$

Hence we have

$$\begin{aligned} (v_{k+j}^n, P[w]v_j^n)_{H_n} &= \\ \frac{(2j+k+n)!}{\sqrt{[(k+j)! \cdot j!(k+j+n-1)! \cdot (j+n-1)!]}} \times & \\ \times \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^\infty dr \, 2r \tanh r (1 - \tanh^2 r)^{\frac{n}{2}} e^{ik\theta} (\tanh^2 r)^{\frac{k}{2}} \times & \\ \times G_j(n+k, 1+k; \tanh^2 r) \sum_{m=0}^{\infty} (-1)^m \left| (v_m^n, D_n^+ e^{-\frac{r}{2}} \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix} w)_{H_n} \right|^2 & \end{aligned}$$

We notice that

$$\int_0^\infty dr \, (2r \tanh r (1 - \tanh^2 r)^{\frac{n}{2}}) = \frac{2}{n} \quad n \geq 1$$

hence

$$\frac{n}{2} dP_n := 2r \tanh r (1 - \tanh^2 r)^{\frac{n}{2}} \frac{d\theta}{2\pi} dr$$

is a probability measure on

$$\mathbf{p} = \left\{ X = \begin{pmatrix} 0 & re^{i\theta} \\ re^{-i\theta} & 0 \end{pmatrix} \mid 0 \leq \theta < 2\pi, \quad 0 \leq r < \infty \right\}$$

Moreover, we define a family of functions

$$f_{k+j,j}^X(m) = f_{k+j,j}^{(\theta,r)}(m) := (-1)^m \frac{2}{n} \frac{(n+k+2j)!}{\sqrt{j!(j+n-1)!(k+j)!(k+j+n-1)!}} \times \\ \times e^{ik\theta} (\tanh^2 r)^{\frac{k}{2}} G_j(n+k, 1+k; \tanh^2 r)$$

With these notations we can write

$$(v_{k+j}^n, P[w]v_j^n)_{H_n} = \\ \int_{\mathbf{p}} dP_n(X) \sum_{m=0}^{\infty} |(v_m^n, D_n^+(e^{-\frac{X}{2}})w)_{H_n}|^2 f_{k+j,j}^{(\theta,r)}(m) = \\ \int_0^{2\pi} \int_0^{\infty} dP_n(r, \theta) \sum_{m=0}^{\infty} |(v_m^n, D_n^+ \left(\exp -\frac{r}{2} \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix} \right) w)|^2 f_{k+j,j}^{(\theta,r)}(m)$$

The self-adjoint operator $dD_n^+ \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ is diagonalized by the basis $\{v_j^n\}_{j \geq 0}$ of H_n

$$dD_n^+ \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} v_j^n = (n+2j)v_j^n \quad \forall n \geq 1, \forall j \geq 0$$

Accordingly

$$|(v_m^n, D_n^+(e^{-\frac{X}{2}})w)_{H_n}|^2$$

is the probability that the physical quantity, represented by the self-adjoint operator $dD_n^+ \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ has value $n+2m$ when the system is in the state represented by the unit vector $D_n^+(e^{-\frac{X}{2}})w$.

The physical interpretation of the operator $dD_1^+ \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ depends on the concrete realization of the Hilbert space H_n , where the representation D_n^+ acts. We have, in the realization described in section 3,

$$dD_1^+ \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = N_1 + N_2 + 1;$$

hence, its probability distribution can be measured by photon counters.

The probability $|(v_m^n, D_n^+(e^{-\frac{X}{2}})w)_{H_n}|^2$ can be really measured only if we are able to physically implement the unitary operator $D_n^+(e^{-\frac{X}{2}})$. We have, formally:

$$D_n^+(e^{-\frac{X}{2}}) = D_n^+ \left(\exp \left(-\frac{1}{2} r \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix} \right) \right) \\ = D_n^+ \left(\exp \left(-\frac{1}{2} r \left(e^{i\theta} \begin{pmatrix} 01 \\ 00 \end{pmatrix} + e^{-i\theta} \begin{pmatrix} 00 \\ 10 \end{pmatrix} \right) \right) \right) \\ = \exp \left(-\frac{1}{2} r \left(e^{i\theta} dD_n^+ \begin{pmatrix} 01 \\ 00 \end{pmatrix} + e^{-i\theta} dD_n^+ \begin{pmatrix} 00 \\ 10 \end{pmatrix} \right) \right)$$

In our explicit realization of D_n^+ , on a closed subspace of $\mathcal{H}^{(2)}$, we have

$$dD_n^+ \begin{pmatrix} 01 \\ 00 \end{pmatrix} = -a_1 a_2 \quad dD_n^+ \begin{pmatrix} 00 \\ 10 \end{pmatrix} = a_1^* a_2^*$$

hence

$$D_n^+(e^{-\frac{x}{2}}) = \exp\left(-\frac{r}{2}(e^{i\theta} a_1^* a_2^* - e^{-i\theta} a_1 a_2)\right)$$

This is a “squeezing operator” and it describes a non degenerate parametric amplifier (or phase insensitive amplifier), with r and θ squeezing amplitude and phase respectively (see, for example, [13]).

We add two final comments

1. It is easy to get a reconstruction formula for any density operator T (bounded, self-adjoint, trace-class operator of trace one). We readily get

$$(v_{j+k}^n, T v_j^n) = \int_{\mathfrak{p}} dP_n(X) \sum_{m=0}^{\infty} (v_m^n, D_n^+(e^{-\frac{x}{2}}) T D_n^+(e^{\frac{x}{2}}) v_m^n)_{H_n} f_{j+k,j}^X(m)$$

2. This reconstruction formula has the general form we have anticipated in the introduction:

- a) (\mathfrak{p}, P_n) is a probability space; it plays the role of the probability space Ω, P ;
- b) $D_n^+(e^{-\frac{x}{2}})$ is a family of unitary operators, parametrized by the points of \mathfrak{p} ; it plays the role of the family of operators $U(\omega)$;
- c) the self-adjoint operator $dD_n^+ \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ whose physical meaning is the total number of photons of the radiation field, plays the role of the operator A ;
- d) the integration over \mathbb{R} ($\int_{\mathbb{R}} d\alpha_{T(\omega)}^A(\lambda)$) is replaced by a sum, since the operator $dD_n^+ \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ has a pure point spectrum;
- e) $(v_m^n, D_n^+(e^{-\frac{x}{2}}) T D_n^+(e^{-\frac{x}{2}})^* v_m^n)_{H_n}$ is the probability that the physical quantity represented by $dD_n^+ \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ has value $(n+2m)$ when the quantum system is in the transformed state $D_n^+(e^{-\frac{x}{2}}) T D_n^+(e^{-\frac{x}{2}})^*$;
- f) the functions $f_{j+k,k}^X$ are defined quite explicitly.

References

- [1] A. Borel, *Représentations de groupes localement compacts*. Lecture Notes in Mathematics, Vol. 276 (Springer-Verlag, Berlin, 1972)
- [2] G. Cassinelli, G.M. D’Ariano, E. De Vito, A. Levrero, J. Math. Phys. **41**(12), 7940 (2000)

- [3] G.M. D'Ariano, E. De Vito, L. Maccone, Phys. Rev. A **64**(1-4) (2002)
- [4] P.P. G. Chiribella, G. M. D'Ariano, Laser Phys. **16**, 1572 (2006)
- [5] V. Bargmann, Ann. of Math. (2) **48**, 568 (1947)
- [6] S. Lang, $SL_2(\mathbf{R})$, *Graduate Texts in Mathematics*, vol. 105 (Springer-Verlag, New York, 1985). Reprint of the 1975 edition
- [7] V.S. Varadarajan, *An introduction to harmonic analysis on semisimple Lie groups*, *Cambridge Studies in Advanced Mathematics*, vol. 16 (Cambridge University Press, Cambridge, 1999). Corrected reprint of the 1989 original
- [8] R. Howe, E.C. Tan, *Nonabelian harmonic analysis*. Universitext (Springer-Verlag, New York, 1992). Applications of $SL(2, \mathbf{R})$
- [9] A.W. Knap, *Representation theory of semisimple groups*. Princeton Landmarks in Mathematics (Princeton University Press, Princeton, NJ, 2001). An overview based on examples, Reprint of the 1986 original
- [10] M.E. Taylor, *Noncommutative harmonic analysis*, *Mathematical Surveys and Monographs*, vol. 22 (American Mathematical Society, Providence, RI, 1986)
- [11] G. Cassinelli, Quantum tomography (2008). Lecture notes
- [12] M. Abramowitz, I.A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, *National Bureau of Standards Applied Mathematics Series*, vol. 55 (For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964)
- [13] R.R. Puri, *Mathematical methods for Quantum Optics* (Springer, Heidelberg, 2001)