

BOHMIAN MECHANICS AND QUANTUM EQUILIBRIUM

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1. INTRODUCTION

We report here on our work on Bohmian mechanics. A much more detailed presentation will be published elsewhere [7].

According to the quantum formalism, measurements performed on a quantum system with *definite* wave function typically yield *random* results. However, the quantum dynamics governing the evolution of the wave function over time, at least when no measurement is being performed, and given, say, by Schrödinger's equation, is completely deterministic. Thus if all physical processes, including those we call measurements, are governed by the same fundamental laws, and if one assumes the wave function to be the *complete* description of the physical state, one arrives at an inconsistency, often referred to as “the measurement problem” or the problem of the “collapse of the wave function.”

Debates over this were strongly influenced by “quantum epistemology”—by the obligatory reference to the observer in a measurement, whose act of “observation” somehow creates the random outcome. In recent years, however, the growing disenchantment with this irreducible reference to the observer has begun to assume the form of serious doubts about the validity and meaning of the quantum formalism.

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There have been several recent proposals—for example, by Wigner [23], by Leggett [16], by Stapp [20], by Weinberg [21] and by Penrose [18]—explicitly suggesting the abrogation of the (unitary) Schrödinger evolution. One of the most concrete of such proposals is due to Ghirardi, Rimini, and Weber [13].

There have also been proposals which, while leaving the evolution law intact, admit the need for an ontological basis for the formalism; i.e., they encourage the search for basic entities which have so far been left out of the description, but which are in fact the source of the appearance of “real events” like the outcomes of measurements. Here the wave function attains physical significance via its relation to the behavior of something else more concretely objective.

A most recent advance in this regard is by Gell-Mann and Hartle [11,12], for whom the reference to the observer in the context of cosmology is disturbingly metaphysical. They propose a program to extract from the quantum formalism a “quasiclassical domain of familiar experience,” which defines for them the basic ontology of quantum theory.

Now when the question of ontology is raised, it may seem appropriate that the ontology should be tightly bound to the quantum formalism—after all, the latter is to be explained by the (behavior of the) former. However, this leads naturally to a substantial decrease in flexibility. And considering that the quantum formalism has prompted the radical claim that we must abandon any pretension of describing nature in objective terms, it seems indeed unlikely that the formalism itself could yield an underlying objective theory. On the other hand, it may seem rather hopeless to conceive of even a completely flexible approach in which the quantum formalism indeed arises from a fully objective microscopic theory, as a phenomenological formalism describing macroscopic regularities—particularly in view of the many “theorems” to the contrary [1].

It is thus somewhat astonishing that this was done already in 1952 by David Bohm [3,4], who developed such a theory from the guiding field idea of de Broglie [6]. For Bohm it is particles, described by their positions in space, which form the basis of the ontology. The wave function choreographs their motion. Thus for Bohm the complete state of an N -particle quantum system is given by the configuration $\mathbf{Q}_1, \dots, \mathbf{Q}_N$ of its particles and by its wave function ψ .

Unfortunately, this theory was presented in close analogy with (the Hamilton-

Jacobi formulation of) Newtonian mechanics, in such a way that it seemed artificial—rather than as the radically non-Newtonian mechanics that it is. As a matter of fact, the theory was widely misunderstood, and often regarded as precluded by the various no-hidden-variables theorems, which are frequently misinterpreted as asserting the impossibility of any account of quantum randomness in terms of averaging over ignorance, an assertion to which Bohm’s theory is in fact a counterexample. We should add, however, that for Bell Bohm’s theory served as a paradigm for the notorious nonlocality of quantum physics, leading him to the formulation of “Bell’s inequalities,” which form the theoretical basis of the experimental verification of the nonlocal character of nature [1,2].

2. THE PROBLEM OF RANDOMNESS

Our formulation (see Section 3) of Bohm’s theory—which we call **Bohmian mechanics**—makes transparent that Bohmian mechanics is a new mechanics, a completely deterministic—but distinctly non-Newtonian—theory of particles in motion. And while its *formulation* does not involve the notion of quantum observables, as given by self-adjoint operators—so that its relationship to the quantum formalism may at first appear somewhat obscure—it can in fact be shown that Bohmian mechanics not only accounts for quantum phenomena [3,4,5,9], but also embodies the quantum formalism itself as the very expression of its empirical import [9].

That this is so is for the most part quite straightforward. It is based in part on what is now often referred to as decoherence—the ever decreasing possibility of interference between, e.g., macroscopically distinct wave functions due to (typically uncontrollable) interactions with, e.g., the environment [3,16,24,15].

But more fundamentally it involves a crucial subtlety, concerning the randomness of quantum phenomena. The predictions of Bohmian mechanics for the results of a quantum experiment on a system having wave function ψ are precisely those of the quantum formalism, provided it is assumed that prior to the experiment the particle configuration of the system is randomly distributed according to Born’s statistical law, i.e., according to the probability density $\rho = |\psi|^2$. But how, in this deterministic theory, does randomness enter? What is special about $\rho = |\psi|^2$? And precisely to which ensemble does this probability distribution refer?

To come to grips with these questions, we need first sharply to formulate and then

carefully to analyze Bohmian mechanics, the most striking feature of which being its nonseparability. As a consequence of this feature, the applicability of Bohmian mechanics to a subsystem of a larger system governed by Bohmian mechanics is not a matter of deliberate choice but rather one of factual possibility or impossibility. This character of “wholeness” has the virtue of forcing the *ultimate* answer to the problem of the origin of randomness in quantum theory.

In fact our difficulty is dissolved by careful scrutiny of the relevant concepts, most notably those of **quantum equilibrium** and of the **effective wave function** of a subsystem. We find that a Bohmian universe, though deterministic, evolves in such a manner that an appearance of randomness emerges, with empirical distributions in agreement with the predictions of the quantum formalism.

3. BOHMIAN MECHANICS AND QUANTUM EQUILIBRIUM

Consider a quantum system of N particles with masses m_1, \dots, m_N and generic configuration space variables $q = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ and wave function ψ .

Bohmian mechanics is the (simplest) theory of particles in motion determined by a first-order time-reversal and Galilean invariant guiding law. The state of the system is given by (Q, ψ) , where $Q = (\mathbf{Q}_1, \dots, \mathbf{Q}_N)$ is the actual configuration of the particles and ψ satisfies Schrödinger’s equation. The role of ψ is to generate a first-order evolution equation for the configuration Q , via the (velocity) vector field $v^\psi = (\mathbf{v}_1^\psi, \dots, \mathbf{v}_N^\psi)$ on configuration space with which it is associated. This vector field must be a homogeneous functional of ψ and \mathbf{v}_k^ψ must transform like a velocity under rotations and time-reversal. The simplest such vector field is easily seen to have components proportional to $\text{Im} \frac{\nabla_k \psi}{\psi}$, with ∇_k denoting the gradient with respect to \mathbf{q}_k . Moreover the constant of proportionality is easily determined by covariance under Galilean boosts.

We are thus led to the evolution law for **Bohmian mechanics**,

$$\frac{d\mathbf{Q}_k}{dt} = \frac{\hbar}{m_k} \text{Im} \frac{\nabla_k \psi}{\psi}(\mathbf{Q}_1, \dots, \mathbf{Q}_N) \quad (3.1)$$

$$i\hbar \frac{\partial \psi}{\partial t} = - \sum_{k=1}^N \frac{\hbar^2}{2m_k} \Delta_k \psi + V\psi. \quad (3.2)$$

Here (3.2) is Schrödinger’s equation with potential energy function V . (For a derivation of these equations from “first principles” see [10].)

The reader should be warned that for grasping the empirical import of this theory, integration of the differential equations (for simple models) is of less value than at first might be expected. It should also be noted that an (n -particle, $n < N$) subsystem need not in general be governed by Bohmian mechanics, since no wave function for the subsystem need exist. This will be so even with trivial interaction potential V , if the wave function does not properly factorize; and for nontrivial V the Schrödinger evolution would in any case quickly destroy such a factorization.

Therefore in a universe governed by Bohmian mechanics there is a priori only one wave function, namely that of the universe, as there is a priori only one system governed by Bohmian mechanics, namely the universe itself.

We shall return shortly to these considerations, but wish now to focus on the apparent need—since our primary concern is with the origin of randomness—for a qualitative statistical analysis, roughly analogous to that of statistical mechanics, where the stationary (Liouville) measure on phase space plays an important role. Note that since ψ will in general be time-dependent, the vector field on the right of (3.1) will be time-dependent as well, and we cannot expect the evolution on configuration space to possess a stationary distribution.

However, we may ask for an equivariant measure, stationary relative to ψ , playing essentially the same role as a stationary one: An ensemble density $\rho(q, t)$ on configuration space evolving under the solution flow of (3.1), i.e., satisfying the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v^\psi) = 0, \quad (3.3)$$

and given by a functional of ψ , is said to be **equivariant** if it retains its form as a functional of ψ .

An equivariant density is provided by $|\psi|^2$, as one easily checks by noting that $J^\psi = |\psi|^2 v^\psi$, where $J^\psi = (\mathbf{J}_1^\psi, \dots, \mathbf{J}_N^\psi)$ is the quantum probability current. In other words,

$$\text{if } \rho(q, t_0) = |\psi(q, t_0)|^2 \text{ at some time } t_0, \text{ then } \rho(q, t) = |\psi(q, t)|^2 \text{ for all } t.$$

We call the probability distribution on configuration space given by $\rho \equiv |\psi|^2$

the **quantum equilibrium** distribution, and say that a system is **in quantum equilibrium** if its configuration is so distributed. In accordance with an earlier observation, in a Bohmian universe *there can be a priori only one system in quantum equilibrium, namely, the universe itself*. And this statement, which to some may sound rather innocent, in fact embodies part of the subtlety to which we have earlier referred. Of universes, we have only one—*ours*—at our disposal. What possible physical significance can be attached to a quantum equilibrium *ensemble* of universes?

There is a rather simple answer: The quantum equilibrium distribution provides us with a natural notion of *typicality*: Statements valid for the overwhelming majority of configurations in the sense provided by the quantum equilibrium measure are true for a **typical** configuration. Note that by equivariance this notion of typicality is time-independent. However, it is best to regard it as applied to initial universal configurations, at the time of “creation” or shortly thereafter.

We may now reformulate our earlier questions concerning quantum randomness as seeking the clarification and justification of the **quantum equilibrium hypothesis**:

When a system has wave function ψ , the distribution ρ of its configuration satisfies

$$\rho = |\psi|^2. \tag{3.4}$$

The quantum formalism emerges rather straightforwardly from the quantum equilibrium hypothesis, the latter thus embodying the empirical import of the theory.

We establish the remarkable fact that the quantum equilibrium hypothesis, *properly understood*, is a simple manifestation of quantum equilibrium, in the sense of typicality. And if we are to make contact with physics, i.e., to arrive at a *proper understanding* of (3.4), we must relate ρ to the empirical distribution of configurations arising from repetitions of similar experiments, performed at different places or times within a single typical sample of the universe (such as ours). In order to do so, we must first come to grips with a problem hinted at earlier, that of the notion of the wave function of a subsystem.

4. THE EFFECTIVE WAVE FUNCTION

Consider a subsystem of a Bohmian universe with configuration variables x , i.e., suppose that $q = (x, y)$ where y represents the configuration of the *environment* of the x -system. The actual particle configurations are accordingly denoted by X and Y , i.e., $Q = (X, Y)$. Let Ψ denote the universal wave function.

How can one assign a wave function to the x -system? In orthodox quantum theory the question touches metaphysical ground, insofar as its answer would refer to an observer. How does Bohmian mechanics handle this question?

One obvious possibility is given by what we call the **conditional wave function**

$$\psi(x) = \Psi(x, Y). \tag{4.1}$$

(Nonvanishing scalar multiples of wave functions are identified). Note that by virtue of (3.1) the velocity vector field for the x -system is determined by its conditional wave function. This, however, does not in general have the usual dynamical significance of the wave function of a system, as it does not in general evolve according to Schrödinger's equation, even when the x -system is dynamically decoupled from its environment.

But one can go further, exploring the form of the universal wave function—a superposition of macroscopically distinct terms—which according to standard quantum measurement theory should arise from a measurement on the x -system. One then recognizes that the following definition captures all the desirable phenomenological aspects (through its reference to the “macroscopic”): Suppose that

$$\Psi(x, y) = \psi(x)\Phi(y) + \Psi^\perp(x, y) \tag{4.2}$$

where Φ and Ψ^\perp have macroscopically disjoint y -supports. If

$$Y \in \text{supp } \Phi \tag{4.3}$$

we say that ψ is the **effective wave function** of the x -system. The effective wave function is indeed a “collapsed” wave function, and agrees with the wave function assigned to the x -system by the quantum formalism (to the extent that the latter does assign a wave function), as well as with the conditional wave function.

Notice that this definition of the (conditional or effective) wave function is mathematically natural. Note also that the fact that this wave function depends critically

on the actual environment Y , in which the results of measurements and, indeed, all of our information must be grounded, is not unrelated to the epistemological component—the reference to the observer—with which the wave function has traditionally been entangled.

5. BORN'S STATISTICAL LAW

Consider, for the moment solely as a mathematical device, the probability space of initial universal configurations Q distributed according to the quantum equilibrium distribution $\mathbf{P}^{\Psi_0}(dQ) = |\Psi_0(Q)|^2 dQ$ —the only natural measure available—where Ψ_0 is the initial universal wave function. Note that by equivariance the distribution of $Q_t = (X_t, Y_t)$, the configuration of the universe at time t , is given by $|\Psi_t|^2$, where Ψ_t is the universal wave function at time t . It now follows at once from the definition of the (effective or conditional) wave function that the conditional distribution of the configuration of the x -system, given the configuration of its environment, is given by

$$\mathbf{P}^{\Psi_0}(X_t \in dx \mid Y_t) = |\psi_t(x)|^2 dx, \quad (5.1)$$

where ψ_t is the wave function of the x -system at time t .

Suppose now that at time t the x -system consists itself of many identical subsystems x_1, \dots, x_M , each one having effective wave function ψ (with respect to coordinates relative to suitable frames). Then the effective wave function of the x -system is the product wave function

$$\psi_t(x) = \psi(x_1) \cdots \psi(x_M). \quad (5.2)$$

(See [7] for a proof.) It follows immediately from (5.1) that, given the configuration Y_t of the environment of the x -system, the configurations X_1, \dots, X_M of the subsystems at time t form a collection of independent random variables, identically distributed, with common distribution $\rho_{qe} = |\psi|^2$. The weak law of large numbers yields then that for large M the empirical distribution of the configurations will be well approximated by this distribution for the $(\mathbf{P}_{Y_t}^{\Psi_0})$ overwhelming majority of initial universal configurations Q (where $\mathbf{P}_{Y_t}^{\Psi_0}(dQ) = \mathbf{P}^{\Psi_0}(dQ|Y_t)$). In other words, for typical initial configurations of the universe (consistent with the environment of the x -system at time t) the empirical distribution is given by Born's law.

This result is, however, too restrictive: Born’s statistical law is supposed to govern not only equal-time ensembles, but also ensembles referring to experiments performed on one or more (sub)systems at possibly different times. In order to deal with this general situation, we are forced to consider a **random system** $\sigma = (\pi, T)$, i.e., a random subsystem at a random time and satisfying a certain measurability condition.

We find that a random system version of (5.1)—analogous to a *strong Markov property*—remains valid, permitting a general random system analysis leading to empirical distributions in complete agreement with the predictions of the quantum formalism. However, unlike the case of equal-time distributions, for the general multi-time case it turns out that the “recording” of the “results of measurement” plays a crucial role.

The most remarkable and perhaps most surprising consequence of our analysis is what we have called **absolute uncertainty**: Notice that (5.1) involves conditioning on the detailed microscopic state of the environment, in which all our information must be grounded—and far more than we could ever reasonably have access to. It follows from (5.1) that in a universe governed by Bohmian mechanics, it is in principle impossible to know more about the configuration of a subsystem than what is expressed by the quantum equilibrium hypothesis (3.4).

When all is said and done, quantum randomness and quantum uncertainty are merely an expression of quantum equilibrium, a global configurational equilibrium subordinate to the universal (and, in fact, nonequilibrium) wave function Ψ (see [7]).

6. CONCLUDING REMARKS

For clarity we have restricted our attention to particles without spin. We remark that spin and the exclusion principle enter through the choice of appropriate wave function and corresponding guiding law [1,10,3,17,14].

Our analysis, though nonrelativistic, does not depend crucially on a particle ontology. With regard to relativistic generalizations, a major virtue of the kind of physics explored here is its flexibility, which may permit the opening of new paths towards an understanding of the content of relativistic quantum theory [8].

We believe that Bohmian mechanics reveals the essence of quantum theory. It

accounts for:

- (1) randomness
- (2) absolute uncertainty
- (3) the meaning of the wave function of a (sub)system
- (4) collapse of the wave packet
- (5) coherent—indeed, familiar—(macroscopic) reality

Moreover, it makes possible an appreciation of the basic significance of the universal wave function Ψ , as an embodiment of *law*, which cannot be clearly discerned without a coherent ontology to be governed by some law.

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