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**Mathematical
Methods of
Classical
Mechanics**

10: Introduction to perturbation theory

Definition. Two functions F_1 and F_2 on a symplectic manifold are *in involution* if their Poisson bracket is equal to zero.

Liouville proved that if, in a system with n degrees of freedom (i.e., with a $2n$ -dimensional phase space), n independent first integrals in involution are known, then the system is integrable by quadratures.

Here is the exact formulation of this theorem: Suppose that we are given n functions in involution on a symplectic $2n$ -dimensional manifold

$$F_1, \dots, F_n \quad (F_i, F_j) \equiv 0, \quad i, j = 1, 2, \dots, n.$$

Consider a level set of the functions F_i

$$M_f = \{x : F_i(x) = f_i, i = 1, \dots, n\}.$$

Assume that the n functions F_i are independent on M_f (i.e., the n 1-forms dF_i are linearly independent at each point of M_f). Then

1. M_f is a smooth manifold, invariant under the phase flow with hamiltonian function $H = F_1$.
2. If the manifold M_f is compact and connected, then it is diffeomorphic to the n -dimensional torus

$$T^n = \{(\varphi_1, \dots, \varphi_n) \bmod 2\pi\}.$$

3. The phase flow with hamiltonian function H determines a conditionally periodic motion on M_f , i.e., in angular coordinates $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$ we have

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{\omega}, \quad \boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{f}).$$

4. The canonical equations with hamiltonian function H can be integrated by quadratures.

Definition. Let T^n be the n -dimensional torus and $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n) \bmod 2\pi$ angular coordinates. Then by a *conditionally-periodic motion* we mean a one-parameter group of diffeomorphisms $T^n \rightarrow T^n$ given by the differential equations (Figure 221):

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{\omega}, \quad \boldsymbol{\omega} = (\omega_1, \dots, \omega_n) = \mathbf{const.}$$

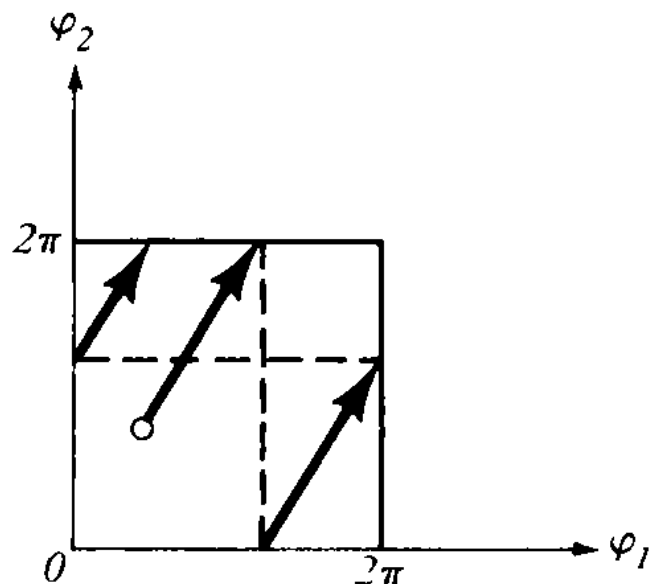


Figure 221 Conditionally-periodic motion

These differential equations are easily integrated:

$$\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}(0) + \boldsymbol{\omega}t.$$

Thus the trajectories in the chart $\{\boldsymbol{\varphi}\}$ are straight lines. A trajectory on the torus is called a *winding* of the torus.

The quantities $\omega_1, \dots, \omega_n$ are called the *frequencies* of the conditionally-periodic motion. The frequencies are called *independent* if they are linearly independent over the field of rational numbers: if $\mathbf{k} \in \mathbb{Z}^n$ and $(\mathbf{k}, \boldsymbol{\omega}) = 0$, then $\mathbf{k} = 0$.

B *Space average and time average*

Let $f(\boldsymbol{\varphi})$ be an integrable function on the torus T^n .

Definition. The *space average* of a function f on the torus T^n is the number

$$\bar{f} = (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\boldsymbol{\varphi}) d\varphi_1 \cdots d\varphi_n.$$

Consider the value of the function $f(\boldsymbol{\varphi})$ on the trajectory $\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}_0 + \boldsymbol{\omega}t$. This is a function of time, $f(\boldsymbol{\varphi}_0 + \boldsymbol{\omega}t)$. We consider its average.

Definition. The *time average* of the function f on the torus T^n is the function

$$f^*(\boldsymbol{\varphi}_0) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\boldsymbol{\varphi}_0 + \boldsymbol{\omega}t) dt$$

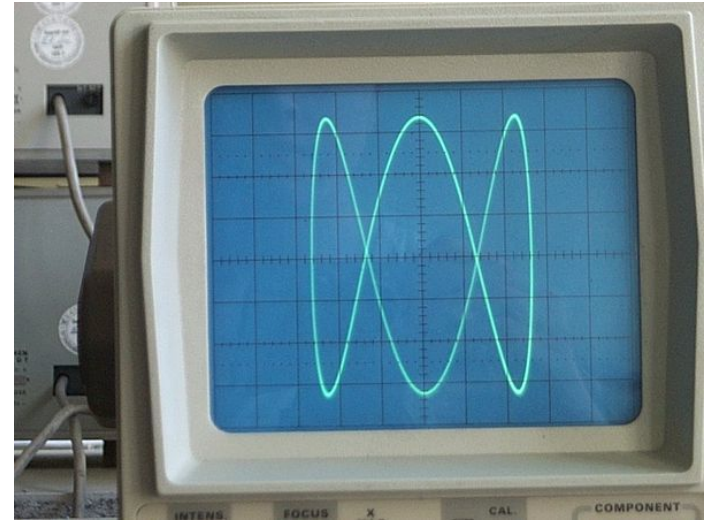
(defined where the limit exists).

Theorem on the averages. *The time average exists everywhere, and coincides with the space average if f is continuous (or merely Riemann integrable) and the frequencies ω_i are independent.*

Lissajous curve

$$x = \sin(t + \phi_1)$$

$$y = \sin(\omega t + \phi_2)$$



Lissajous figure on an oscilloscope, displaying a 1:3 relationship between the frequencies of the vertical and horizontal sinusoidal inputs, respectively.

2: Investigation of the equations of motion

PROBLEM. Show that if $\omega = m/n$, then the Lissajous figure is a closed algebraic curve; but if ω is irrational, then the Lissajous figure fills the rectangle everywhere densely.

Appendix 8: Theory of perturbations of conditionally periodic motion, and Kolmogorov's theorem

To better understand the whole picture, we consider the case of two degrees of freedom ($n = 2$). In this case, the phase space is four-dimensional so each energy level set is three-dimensional. We fix one such level set. This three-dimensional manifold, fibered by two-dimensional tori, can be represented in ordinary three-dimensional space as a family of concentric tori lying inside one another (Figure 242).

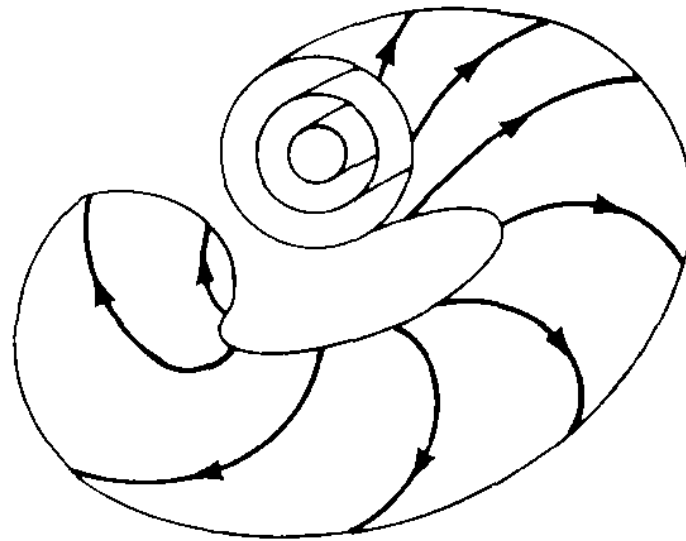


Figure 242 Invariant tori in a three-dimensional energy level manifold