

ARE THERE REALLY INSTANTANEOUS VELOCITIES?

Frank Arntzenius
Dept. of Philosophy
Rutgers University

ABSTRACT

Zeno argued that since at any instant an arrow does not change its location, the arrow does not move at any time, and hence motion is impossible. I discuss the following three views that one could take in view of Zeno's argument:

i) the 'at-at' theory, according to which there is no such thing as instantaneous velocity, while motion in the sense of the occupation of different locations at different times is possible,

ii) the 'impetus' theory, according to which instantaneous velocities do exist but these are only contingently and causally related to the temporal developments of positions,

iii) the 'no instants' theory, according to which instants in time do not exist, and hence instantaneous velocities do not exist, while motion, in the sense of different areas occupied during different time intervals, is possible.

I argue that, despite the fact that there have been interesting and relevant developments in mathematics and physics since the time of Zeno, each of these views still has serious drawbacks.

ARE THERE REALLY INSTANTANEOUS VELOCITIES?¹

1. Zeno's arrow argument and three possible responses to it.

Zeno argued that, appearances notwithstanding, motion was not possible. He had several arguments, but I shall only be interested in one of them; the 'arrow' argument, which for convenience, I will convert into a 'ball' argument. It is an extremely simple argument, and its resolution is apparently simple. However, I think it contains the seeds of further problems that can still enlighten us as to the sense in which velocity can be said to exist.

Consider the smallest units of time, instants. During an instant a ball will occupy a region in space, and it will not move. For there are no smaller units of time that afford the ball the opportunity to change location. So it does not move at any time. But if it does not move at any time then it does not move at all.

Aristotle responded to Zeno's argument by denying the sensibility of the notion of instantaneous velocity, the notion of a definite velocity at an instant of time.² Any motion occurs over a period of time. Thus the only notion of velocity that made any sense according to Aristotle was that of average velocity during a time interval. Average velocity of course, simply is the distance traversed in such a time interval divided by the length of the time period. And throughout his works he only uses such average velocities to discuss his physics. One can in fact distinguish two separate lines of thought in Aristotle's discussion of Zeno.

The first and more radical idea is that there are no such things as instants in time, no 0-sized temporal 'atoms'. Let us call this the 'no instants' view. On this view there are no smallest time intervals, there are only finite sized, smaller and larger, intervals in time. On this view therefore

there is no such thing as instantaneous velocity, nor any such thing as instantaneous position. This is a very radical view. For instance, the development of the position of an object over a period of time can not be represented by a point-valued function $X(t)$ from the real numbers to the real numbers. Indeed it is not immediately clear how one could do mathematical physics at all in anything like the manner that we are accustomed to. Aristotle himself certainly did not develop the mathematical tools to implement this view at any level of rigor. Such tools were not developed until the 20th Century.

The second line of thought occurring in Aristotle is that instants and positions at instants do exist, but velocities at instants do not exist, for exactly the reason that Zeno gives, namely that there is no change of position during an instant, and hence there can be no velocity at an instant. This idea was elaborated into the so-called ‘at-at’ theory of motion during the Middle Ages.³ For instance, William of Ockham maintained that motion should be identified with the various positions occupied by the body that is in motion. One has said all there is to be said about the state of motion of an object when one has given its positions at all times. To be in motion is just to be at different places at different times. There is no question of one's motion, or velocity, at a given time. But that does not mean that motion over time is not possible; it obviously is. Motion is nothing over and above the locations occupied at the various times. Motion is no more than a word which designates the various positions that an object occupies, and there is “no single reality outside our minds” that corresponds to it. Motion, or velocity, is not an intrinsic quantity; it is a relational quantity constructed out of the fundamental quantities of location and time.

During the Middle Ages a third view about instantaneous velocities was developed. It was not developed as a response to Zeno’s puzzle, rather it was developed in response to problems with

Aristotle's theory of projectiles. Aristotle had maintained that unnatural motions, motions other than down for heavy objects and up for light ones in the sublunary sphere, required an external applied force. A problem for this view was the fact that, for instance, an object that is thrown away does not fall straight down as soon as the hand that threw it is no longer applying an external force. Instead it continues in a projectile motion even though apparently there is no applied force to make it deviate from its natural motion, which is straight down. In the Middle Ages the so-called 'impetus' theory was supposed to deal with this problem. It claimed that a body in motion has some kind of internal property, called 'impetus', that provides, or is, the driving force to keep the object going in the direction that it is already going, whether that is a natural motion or not. There were different and somewhat unclear theories of impetus, but it was broadly agreed that the amount of impetus that a body has is proportional both to the amount of matter that a body has and the velocity that it has. But that is not to say that impetus was identified with some aspect of its motion. In so far as such issues were clearly discussed it seems that impetus was supposed to cause and explain motion, but was not to be identified with some feature of its motion. If one now divides (the value of) such an impetus by (the value of its) mass, one gets a quantity, let us call it 'intrinsic velocity', which has a definite value at each instant, and causes change of location. Since this 'intrinsic velocity' is not defined in terms of changes of locations, however, it is not prey to Zeno's argument that it has to be zero at any instant.

Now let me discuss the viability of each of these three view in turn.

2. The 'at-at' theory of motion.

The 'at-at' theory resolves Zeno's puzzle by claiming that there is nothing more to motion

that the occupation of different locations at different times, and by claiming (at least on the initial version of the ‘at-at’ theory) that there is indeed no sense in which any object can be said to be in motion at an instant, that there is no sense to be made of the notion of instantaneous velocity. This view does indeed straightforwardly allow for the possibility of motion, so conceived, in the face of Zeno’s argument. There is however something that is somewhat disquieting about this stripped down sense of motion. Is there really no sense to be made of the claim that this car is moving right now, at this instant? Doesn’t the complete state of the car at an instant not include the fact that it is moving? Aren’t cars that are moving in different directions in different instantaneous states? Let me try to be more precise about this worry about the ‘at-at’ theory.

Consider a ball moving from right to left through some region in space, and a qualitatively identical ball, perhaps the same one, some time later, moving from left to right through that very same region. If the full state at a time of a ball does not include an instantaneous velocity then the full state of the two balls is exactly the same when they occupy the same region. So how does the one ball ‘know’ to keep moving right and the other ball ‘know’ to keep moving to the left. Or, removing the anthropomorphism: why does the one ball subsequently move to the left and the other ball subsequently move to the right? Surely it is conservation of velocity, or something like that, which determines that the one ball will keep moving to the left and the other will keep moving to the right. But if there is no such thing as instantaneous velocity, as there is not in the ‘at-at’ theory, then why do the balls continue their motions in different directions?

Another way to put the problem is as follows: the development of the states of the balls is not deterministic if we deny the existence of instantaneous velocities. Now of course we should have no a priori commitment to determinism. But surely determinism shouldn’t be bound to fail just

because of Zeno's arrow argument.

The idea that the state of an isolated system at a time, or of the world, determines its state at all times is an idea that many philosophers and scientists have had regarding various types of systems throughout history. With the emergence of Newtonian mechanics it became rigorously provable in a wide range of cases. Laplace stated that the state of the (Newtonian) universe at any given time completely determined the state at all other times. Now it is obvious that determinism so formulated, i.e. in terms of the state at a time, could not possibly hold for a Newtonian system unless the state at a time includes the velocity at that time. This is presumably the main reason why velocities are normally regarded as part of the physical state of a system at a time, and why accelerations and higher order derivatives are typically not regarded as part of the physical state of a system at a time. But on the 'at-at' view, as developed so far, there is no such thing as instantaneous velocity. Hence e.g. Newtonian mechanics is not deterministic. Surely this is wrong.

Before discussing this worry about the 'at-at' theory in more detail I should note that determinism is not really what drives this worry. For even if one had an indeterministic theory, it would still be puzzling if the states at a time did not fix the probabilities of future developments of states. What drives the puzzle is our idea that Newtonian mechanics, and a whole host of other theories, are Markovian. We take it that these theories are such that the full state at a time renders prior histories of states probabilistically irrelevant w.r.t. future developments of states. Indeed most of us surely believe that the actual world is Markovian: all influences that the past has upon the future run through the present. What is crucial for our purposes is that we do not think Markovianism must fail in all cases in which the probabilities of future developments depend on instantaneous velocities. And surely the 'at-at' theory is wrong if it entails that.

Now one obvious reaction to this worry is to adopt the impetus view. I will discuss that view later. But another reaction is that the development of the calculus makes it clear that one can, within the basic approach of the ‘at-at’ theory, define a concept of instantaneous velocity that allows one to recover determinism. Let me now discuss that idea.

3. Calculus to the rescue of the ‘at-at’ theory?

Dealing with certain problems, requiring integration and differentiation, many people since Greek times had, one way or another, appealed to intuitive notions of infinitesimals, and intuitive notions of limits. Leibniz and Newton were the first to synthesize and systematize these ideas into an almost algorithmic general method for differentiating and integrating functions. Leibniz and Newton had various, not completely clear, ways of understanding the metaphysical underpinnings of the calculus that they developed. In the first place they had the idea that there are infinitesimal changes in the values of quantities during infinitesimal periods of time. Velocity, and in general derivatives, were then defined as ratios of such infinitesimals. An infinitesimal was supposed be smaller than anything finite yet larger than zero. The coherence of this idea was ridiculed by Berkeley and others. And in places both Leibniz and Newton indeed agree that, literally speaking, there are no such infinitesimal quantities. Instead, they appealed to the notion of a limit, a value that is the result of some limiting process. But they did not define the notion of limit rigorously.

How do these two conceptions of derivatives and hence of velocity affect our problem? One might either adopt the limit conception of velocity as rigorised by Cauchy and Weierstrass, or the infinitesimal conception as rigorised by Robinson⁴, or as rigorised by Lawvere and others⁵. On this view velocity at a time is defined in terms of the development of positions in finite or infinitesimal

neighborhoods of that time. But it seems that on either view velocity is still some kind of relationally defined, non-intrinsic, quantity that can not be used to explain why the balls continue to move in the directions that they do. Let me argue for this claim by giving an analogous case.

Suppose that one defined an object to have property X at time t iff it is blue at time $t+1$. Suppose one sees a ball that turns from red to blue between t and $t+1$, and one asks: “why did it turn blue during that period?” It seems clear that the answer “because it had property X at t ”, is not to be regarded as a satisfactory answer. Property X is not the kind of intrinsic property that could cause it to turn blue.

Similarly suppose that velocity at instant t is defined in terms of limits of position developments in a (finite or infinitesimal) neighborhood of instant t . Even though such a velocity is well-defined at an instant t , it would seem that it can not be used to explain why the object after time t moved in the direction that it did. For the only (relevant) intrinsic property that the object has at instant t is its position: its kinematic state at t is its position. Its velocity at t is defined in terms of developments of positions in a neighborhood around t . When one specifies the velocity of an object at a time t , one is specifying an aspect of the development of the object in a finite, or infinitesimal, neighborhood time t , one is not specifying a feature that is determined by its state at time t alone. Therefore it seems that such a velocity is just as much non-explanatory with respect to the direction of its subsequent motion as property X is w.r.t. to subsequent color developments.

One might respond that there is a crucial difference between property X and velocity. Indeed the intuition surely is that things like velocities and accelerations at times, in contrast to properties such as X, are perfectly acceptable properties that can be non-trivially used in explanations and predictions of future behavior, even though they are not part of the state of the

system at that time. The intuition is that something like neighborhood properties are acceptable for purposes of explanation, prediction, causation and determinism, whereas non-neighborhood ones like X are not. The obvious question then is: what exactly are the neighborhood properties?

The natural suggestion is that any property whose truth value (or quantity whose value) is determined by the position development alone in any finite, or infinitesimal, neighborhood of t , is an acceptable property (quantity) for purposes of explanation, causation and determinism.⁶ For instance, the velocity of an object at time t , defined in the Weierstrass limit manner, is determined by the position development in any finite neighborhood of t . And so all are the higher temporal derivatives (if they are well-defined). More generally, let us call the set of all neighborhood properties and quantities of an object, the ‘neighborhood state’ of that object.⁷

Let us now concentrate on the issue of determinism, in order to avoid the complex and muddy issues as to when it is that events can be said to cause or explain other events. A natural suggestion then for the defender of the ‘at at’ theory is that one has determinism iff the neighborhood state at a time determines the state at all times (and hence the neighborhood state at all times). Newtonian particle mechanics will then indeed be deterministic.⁸

However, the neighborhood state is not the same as the state of the system at time t . The neighborhood state at time t is defined in terms of arbitrary size finite, or infinitesimal, developments around time t . Indeed, in addition to all the temporal derivatives, neighborhood states include some properties which may, at first sight, seem unacceptable as properties which are to be used when assessing determinism (and perhaps also unacceptable as causal or explanatory properties). Consider for instance a particle that is at position $x=0$ at all times prior to and including $t=0$, which then jumps to $x=5$ and remains there. The fact that immediately after $t=0$ it makes a discontinuous jump to $x=5$

is a neighborhood property of the particle at $t=0$. To be precise: the fact that the limit of positions equals $x=5$ as one approaches $t=0$ from above is determined by the positions in any finite, or infinitesimal, neighborhood of $t=0$, and thus is a neighborhood property at $t=0$. It may therefore seem that neighborhood properties are worryingly like property X in that they encode information about future developments.

Indeed one can give an example of a neighborhood property that is perhaps even more worrying and surprising. Suppose that a particle P exists at all times except $t=1$, and that it is located at $x=1$ at all times that it exists, i.e. at all times other than $t=1$. This particle, which is a timeless entity, would still have the neighborhood property that it has a left and right limit of its position that equals $x=1$ (i.e. as you approach $t=1$ from above or below the limiting position is $x=1$), even though it does not exist at $t=1$, i.e. even though its world line has a gap at $t=1$! It does seem bizarre indeed to claim that a particle that does not exist at $t=1$ has certain properties at $t=1$, e.g. the indicated limiting properties. Unfortunately, in the most natural and plausible way of formulating the notion of a neighborhood property, such properties at t will be included even for particles that do not exist at t (have a gap in their world line at t).⁹

A defender of neighborhood properties might respond that these neighborhood properties nonetheless are rather innocuous properties, since any neighborhood state at t is compatible with any other neighborhood state at any time t' other than t . For instance, even if it is a neighborhood property at $t=1$ that the left and right limit of its position is $x=1$, this by itself does not entail anything about its position for any other time t , nor does it entail anything about its positions throughout any set of times T , as long as that set is not dense around $t=1$. Neighborhood information for f at a point t is quite a distinct type of information. Although it contains more information than

just the value of f at t , nonetheless it does not imply any particular value for f at any time t' other than t .

This also allows us to see that we will still get a non-trivial sense of determinism even if we allow neighborhood properties. If one allowed arbitrary non-intrinsic properties such as X , determinism would be trivially true no matter what the laws of development are. But allowing neighborhood properties does not by itself settle anything regarding determinism. Neighborhood properties at t by themselves do not determine the state at any time other than t . The question as to whether a theory is deterministic is essentially the same one that one always asks, and whether a theory is deterministic or not depends on the laws of development in the usual way. Thus neighborhood properties are not at all analogous to properties like X , and one could plausibly argue that our initial worries about allowing neighborhood properties are misplaced.

Let me try to draw some conclusions from this discussion. Neighborhood properties are exactly what the name suggests: they are not intrinsic fundamental properties had at a time, they are properties of finite, or infinitesimal, developments of states. There are, after all, definitional relations between the neighborhood states at different times. And surely our notion of a physical state is such that being in a particular physical state at some time does not by definition and logic alone put any constraints on what physical states the system can be in at other times. Physics may impose constraints on the possible developments of the physical states of systems, but surely logic and definition by itself should not do so. And that implies that neighborhood properties and neighborhood states are not physical states, they are features of finite developments of physical states.

Therefore, if one wishes to reconcile the 'at-at' view with determinism one should not

broaden the set of (intrinsic) properties that one claims an object can have or fail to have at a time. Instead one should try to understand determinism as the question whether the state of a system throughout any finite, or infinitesimal, time interval determines the state at all times. And one should understand Markovianism as the question whether the chances of states to the future of any finite, or infinitesimal, time interval are independent of states to the past of that interval, given the state during that interval. This does not make the question of determinism trivial. Nor does it show that explanations that use neighborhood properties in order to explain later states of the system are not bona fide explanations, or that these neighborhood properties are not, in some sense, causally relevant. But it does mean that one will have to give up on the idea that the truly instantaneous features of a system, or the world, can determine the future developments or the chances of future developments if there is a velocity dependence of developments.

On the ‘at at’ view, despite the calculus, there simply are no such things as instantaneous velocities. Therefore the worry about this view is still the puzzle that I started with. Even though the development of the balls may be deterministic in some non-trivial sense, the state of a ball at a time does not determine the direction in which it continues its motion. And this implies that it is not the case that all influences from the past upon the future are represented in the present. Other things being equal, I, for one, would like to avoid that conclusion. Let us see whether other things are equal.

4. The impetus theory.

One could claim that the reason that a ball at any instant keeps moving in one direction rather than another, is that at each instant it has an impetus of a certain magnitude and direction at each

instant. Of course, this requires that one does not have an ‘at-at’ theory of impetus, that impetus exists at each instant, and that it causes motions rather than is defined in terms of its motion. One could also put it this way. On this view there is a kinematic quantity in addition to position, which one could call ‘intrinsic velocity’, which equals impetus divided by mass, which is part of the intrinsic state of an object at time t . This quantity is not defined in terms of position developments, but it is a law of nature that ‘intrinsic velocities’ always equal the temporal derivatives of position developments. Thus, compared to the ‘at-at’ view, one has additional ontology (one has a larger state space) and one needs an additional law to forbid developments in this state-space in which position developments of objects in a neighborhood of time t do not correspond to their ‘intrinsic velocities at t ’.

A more modern version of this view would be that the full kinematical state of an object at a time is given by its Hamiltonian canonical position and momentum. Hamilton's equations will then determine how this state will develop in time, and will also determine what relation there is between a canonical momentum and position. The relation between momentum and position can vary for different Hamiltonians and is therefore to be understood as a natural law governing how position development is causally related to momentum rather than as an identification of the momentum with mass times the temporal derivative of position¹⁰. Let me now discuss two problems with this view.

One problem is that the impetus view assumes the existence of a quantity in addition to the usual ontology, namely ‘intrinsic velocity’, which is causally, and not definitionally, related to the temporal development of position. By Ockham’s razor, one would like to be able to get rid of this quantity. Surely, other things being equal, it is implausible that in addition to the rate of change of a particle’s location, there is a local intrinsic vector quantity that by natural law always has a

magnitude equal to the particle's rate of change.

The other problem is that one could argue that if intrinsic velocity exists, then theories that we would typically regard as time reversible should in fact not be regarded as time reversible. I will spend most of this section presenting this argument, which I think has more intuitive appeal than one might initially think, and has not received the attention it deserves, before dismissing it.

In the late 19th century Boltzmann attempted to reduce thermodynamics to statistical mechanics. In attacking Boltzmann Loschmidt, Zermelo and Poincare pointed out that classical physics was time reversible and that this was a problem for Boltzmann's account. The notion of time reversibility has played a significant role in the foundations of physics ever since. Roughly speaking a theory is said to be time reversible if a reverse development of an allowed development of states is also allowed. However, if one looks more closely one will find that what typically is demanded is not that the reversed sequence of states is allowed by the theory whenever that sequence is allowed. Instead one allows a time-reversal operation on states, and one demands that a time reversed sequence of time-reversed states is allowed when the sequence is allowed. For instance, for Newtonian mechanics the time reverse of the kinematic state of an object is the state with the same position but with inverse velocity in that frame of reference, and in electromagnetism the time reverse of an electromagnetic field is a field with the same electric field but inverse magnetic field.

Now prima facie it seems very worrying that such time reversal operations on states are allowed. Isn't the only interesting sense of time reversibility be the demand that the very same states can occur in inverse order? Isn't that sense of time reversibility the one that lets us know whether time has an objective arrow, whether space-time includes an objective temporal orientation? It may be of some interest to know that when you have a development of states at times that is physically

possible, and you then modify them in some way, the inverse development of modified states is also physically possible. But this seems to have no direct bearing on the question whether time has an objective direction. Nor does it appear to answer the intuitive question whether the reverse of any possible development is possible. So let us start by not allowing time reversal operations when considering whether a theory is time reversible.

This presents immediate difficulties for the impetus view. For instance Newtonian particle mechanics would not be time reversible if one assumes that 'intrinsic velocities' or 'momenta' are part of the state of a system at a time. For the inverse of a sequence of positions and 'intrinsic velocities' is not physically possible, since in the inverse sequence the 'intrinsic velocities' will be pointing in the wrong direction. But this seems crazy. There is nothing about the Newtonian particle phenomena that suggests in any way that time has an objective direction. There is nothing that suggests that we should have a theory of this that in the relevant sense is not time reversible. There are two rather obvious reactions that one can have to this problem. The first reaction is that Newtonian mechanics is obviously time reversible, so the impetus theory is obviously wrong. The second reaction is that we were obviously mistaken when we ruled out time reversal operations on intrinsic instantaneous states. I will start by discussing the first reaction. This discussion will lead to the conclusion that the second reaction is the correct one.

If we do not allow time reversal operations the obvious reaction is that consideration of the case of Newtonian mechanics shows that the impetus theory is fundamentally misguided. Of course you get yourself into trouble with respect to time reversibility if you suppose that the intrinsic state of an object at a time includes some kind of directed velocity that is only causally related to position developments. Of course such a weird entity will point in the 'wrong' direction when you consider

the reversed sequence of states. But it was crazy to suppose that there was such an entity present in any sense other than as the temporal derivative of the position development in the first place. The conclusion is that the ‘at-at’ view is the correct view. The kinematic state at a time of an object is simply its position. Velocity is not part of the state, it is simply a feature of temporal developments of states. And of course velocities are inverted in the time reversed sequence of states. This is so because of the way that velocities are defined in terms of finite temporal developments of states. Newtonian mechanics then is time reversible in the most straightforward sense: if a sequence of states is allowed then the inverse sequence of those very same states is also allowed. As nice and plausible as this sounds, I think there are good reasons to reject this line of reasoning.

In the first place notice that if one does not allow time reversal operations and if one holds a theory to be deterministic if the state at a time determines the state at all times, then there are no non-trivial theories that are both deterministic and time reversible: it is impossible to have any non-trivial theory which both implies that the state at a time fully determines all future and past states, and implies that any reverse of any allowed sequence of states at times is also allowed. For this would imply a mirror symmetry of developments of states in both directions of time around any point in time. And that is impossible unless there is no change of state ever. Thus a theory that is deterministic, in the sense that a state at a time determines the state at all times, and is non-trivial in that allows for changes of state to occur, just has to be non-time reversible. But surely this is implausible. Surely theories can be deterministic, time reversible and non-trivial. So this would seem to be a good argument for allowing time reversal operations.

However, one might respond that all theories that we typically consider to be time reversible, when properly stated, are not deterministic in this sense, but are deterministic in the ‘neighborhood’

sense that finite or infinitesimal histories of states determine states at all times. That is to say, rather than accept that one has to allow time reversal operations, one might argue that the proper notion of determinism is ‘neighborhood’ determinism, and that it is not worrying in the least that ‘instant determinism’, together with ‘non-triviality’, entails lack of time reversibility.

It is not immediately clear how to resolve this deadlock. So let me now give an argument in favor of allowing time reversal operations that does not depend on what sense of determinism one thinks should be compatible with time reversibility.

Electromagnetism is invariably said to be time reversible in textbooks, and it is taken as obvious that electromagnetism does not imply that time has an objective direction. However, it is not true that the inverse sequence of a possible sequence of electromagnetic field configurations is also possible. What is true is that if one inverts an allowed sequence of electromagnetic field configurations, and one flips over the magnetic fields, then one obtains another allowed sequence of electromagnetic field configurations. But, magnetic fields are not the velocities of anything, so there is no justification for this inversion on our current approach. So, on the approach that does not allow time reversal operations on states electromagnetism, contrary to standard practice, should not be counted as time reversible. This seems implausible.

Now, one might dig in one’s heels and claim that the standard view of electromagnetism is simply wrong: electromagnetism, contrary to popular belief and standard textbook wisdom, just isn’t time reversible. (This is the line that David Albert takes in his forthcoming book on time and chance¹¹.) And one might claim that therefore indeed electromagnetism gives one a reason to believe that time has an objective direction.¹² Could this be right? Could it be that indeed we have been misreading the signs?

I do not believe so. Let me explain why. Consider the view that the only reason to transform velocities under time reversals is because of the way in which they are defined in terms of histories of position developments, and hence that no aspect of the intrinsic state of an object should change under a time reversal. The basic reasoning behind this view surely is the following. In addition to the positions of particles there are local intrinsic quantities (particle properties, field amplitudes,...) defined at points in space-time. We can specify the state of these local quantities by real numbers. Of course these numbers are gauge dependent: one use different coordinatisations of the space of possible states of these (local) intrinsic quantities. However, the coordinates that we use to give those values have nothing to do with space-time coordinates: they are coordinates for a local intrinsic quantity space, not of space-time. Thus under an active or passive time-reversal transformation the coordinate values of the local intrinsic quantities should not change at all. Of course the coordinate dependent values of velocities should. This is because velocity is not a local intrinsic quantity. Velocities are defined as tangents to world lines, they are non-local (neighborhood) space-time quantities. Thus, how they transform under an active or passive time reversal is determined by how they are defined in terms of worldlines.

Let me now argue that this view is extremely implausible. If this view were correct then, e.g., the values of magnetic field vectors should remain the same under any space-time transformation, e.g. under a velocity boost (active or passive). And that would mean that just about all the theories that we have taken seriously since Newton actually are not invariant under velocity boosts, contrary to popular belief. Indeed we would have reason not only to suppose that time has an objective orientation on the grounds of electromagnetism, we would have reason to believe that there is an objective preferred rest-frame on grounds of the fact that electromagnetism is not invariant under

velocity boosts.

But this is crazy. What we really should infer is that electromagnetic fields are geometric objects and that the values of the electromagnetic field strengths are dependent on the space-time coordinates that we use. Field strengths are the same kind of geometric object as velocities are, they ‘live’ in the same space, they transform in the same way under all space-time transformations, whether velocity boosts or time reversals. Differential geometry is the theory describing such geometric objects. Thus there is no problem about allowing time reversal transformations on field vectors, nor is there in allowing it on ‘intrinsic velocities’ even though they are not defined in terms of position developments. Thus considerations of time reversibility are not an objection to the impetus theory.

In the end then, the main worry about the impetus view is that, other things being equal, one would not wish to add a further quantity, ‘intrinsic velocity’, to one’s ontology, and add a natural law which equates the value of this quantity with the temporal derivative of its position development, thereby denying that this relation holds by definition. It may only be one worry, and it may be very briefly stateable, but it does seem a very significant objection to the impetus view.

Finally let me turn to the no-instants view.

5 Pointless geometry

In the 1920's Caratheodory showed a way of rigorising Aristotle’s idea that there are no instants in time (See Caratheodory 1963). His primary inspiration was Euclid rather than Aristotle. In Euclid’s way of doing geometry one can cut figures exactly into two completely symmetrical halves. One can for instance cut a square into two exactly symmetrical rectangles. In standard

modern geometry one can not do this. For in standard modern geometry one has to decide which of the two rectangles contains the cutting line: in standard modern geometry one has to decide which region is open and which is closed. Caratheodory thought that such a distinction between closed and open regions was an artifact of the formulation, and did not correspond to a true feature of the structure of space. To put it another way, on his view space is not atomic, it does not consist of points, it consists of regions of various sizes, where all possible sizes are finite. There are arbitrarily small regions, but regions do not consist of ultimate zero-sized points. Following Brian Skyrms (for more detail see Skyrms 1993), let me give some indication of how one can think of such a geometry devoid of points.

Let us start with the collection of open intervals of the real line. One can form the so-called Borel algebra of regions by closing this collection up under complementation and countable intersection and union. And then one can form a new ‘coarser’, atomless, algebra, by identifying regions that differ by Lebesgue measure 0, and identifying all measure 0 regions with the null element of the algebra. The remaining algebra of regions will behave exactly as Caratheodory wanted: there is no difference between open and closed regions, and there are no regions of measure 0. A bonus of this approach is that one gets rid of the measure theoretic paradoxes: Zeno’s problem of making finite sized regions out of 0-sized points disappears and the problem of the existence of non-measurable regions disappears.

But if space and time are atomless, what of functions on space or functions of time? They obviously can no longer be thought of as functions from points to points (from point values to point values). However, one can still have maps from non-atomic regions to (non-atomic) regions. One can indeed show that there is a nice correspondence between the possible point functions from the

reals to the reals and the possible maps from non-atomic regions to non-atomic regions. Consider a point function f from real line A to real line B . Form, as outlined above, the algebra of atomless regions of A and the algebra of atomless regions of B . Clearly, point function f will induce a unique mapping from the atomless algebra of regions of A to the atomless algebra of regions of B . Moreover point function f will induce a unique (σ) homomorphism h from the atomless algebra of regions of B to the atomless algebra of regions of A . But more importantly, and less obviously, any (σ) homomorphism h from the atomless algebra of regions in B to the atomless algebra of regions in A corresponds to a unique equivalence class of point functions from A to B , where all point functions in this equivalence differ in values at most in a set of Lebesgue measure 0. In other words, if one supposes that space and time (and perhaps other physical quantities) do not consist of points (do not have point values), but form atomless algebras as outlined above, it is just as if one is working with equivalence classes of point functions from the reals to the reals that differ at most on sets of points of measure 0. This theory allows one to rigorously implement Aristotle's idea that time does not consist of 0-sized instant.

Now let us turn to a two issues that were problems for the 'at-at' view and the impetus view, namely determinism and time reversibility: how can these be handled in the no instants view?

In this view, determinism of course will have to be formulated in the same way as it is for the 'at at' view: the state throughout any finite or infinitesimal period of time determines the states at all times. In so far as one has worries about the fact that determinism can at best obtain in this sense one has worries about the no-instants view.

The notion of time reversibility is hard to formulate in a natural way within the no instants view. Intuitively a theory (or world) is time reversible if whenever a history of states is possible the

inverse history of states is possible. The problem with this formulation in the context of the no-instants view is that one is speaking of reversing a history of states at times. What the inverse sequence is during some interval is determined by what states it occupies at all of the instants in an interval (plus possibly a time reversal operation on these states at times.) But on the no instants view all one has is mappings between non-atomic regions of time and regions in state space. What to do?

Well, one can use the noted correspondence between such mappings and equivalence classes of point functions to define the time reverse of such mappings. The time reverse of a mapping is simply the mapping that corresponds to the time reverse of the equivalence class of point functions that corresponds to the original mapping.

This kind of suggestion brings out a worry about the no-instants view. It seems to be a view whose useability and intuitive appeal to a large extent rides piggy back on the view it is supposed to replace: the one which allows instants and points to exist, the one which allows one to truly speak of the inverse history of states at times. I do not mean to suggest that one can not have a more direct characterization of the notion of time reversibility within the no-instants view. One can¹³. But one is still translating the intuitive concept of time reversibility in the setting in which there are instants into a principle which formulates the consequences of such a requirement in a setting in which there are no instants, rather than that there is some simple notion of time reversibility that arises in a natural way in the 'no-instants' setting, which moreover in that setting plausibly relates to the issue as to whether time has an objective direction.

Let me bring a similar worry out in another way. Consider the motions of extended objects. You would think that a 'no points' view would have an easier time dealing with extended objects than with point particles. But in fact it is not so easy. How would you indicate the motion of such

an extended object over a period of time? You might think: all you need is a mapping from intervals in time to regions in space which indicates which regions it occupies during each finite time interval. But this is not good enough to fully specify the motion of the object. For instance a completely homogenous disk might be rotating on its own axis. It will then always occupy the same region in space. So the suggested mapping which indicates the regions occupied by the disk over finite time intervals would be the identity whether the disk was rotating or not. So the mappings do not really specify the motion of the disk.

This kind of problem of course also occurs in standard geometry. But in that case one would respond as follows. In order to fully specify the state of motion of the disk one needs to specify the internal motions of the parts of the disk. One can easily do this by assigning each point part of the disk an instantaneous velocity at all times¹⁴. But in pointless geometry there are no points and no point parts. So instead, what one must do is assign motions to each finitely extended part of the disk. And one does this by assigning finite time mappings from all the finite parts of the sphere to the finite regions occupied in all the finite time intervals.

However, this procedure appears rather unintuitive. In the first place note that one can not specify a part of the disk by specifying the region in space that it occupies at some instant in time. For there are no instants. So for instance if the disk is rotating, than a part of the disk that we would like to call a half of the disk will during any finite interval occupy more than half of the region in space that is occupied by the disk during that time interval. So there is a part of the disk that consist of half the mass of the disk, yet in the only way that one can talk in pointless geometry of the region in space that it occupies it always occupies more than half the region occupied by the whole disk. Of course the limit for smaller and smaller time intervals will be half the region occupied. But the

limit of smaller and smaller regions is not part of the temporal continuum.

Let me put this another way. If someone asks you ‘what do you mean by the left half of this hockey puck?’, you can not say: ‘the part that occupied this region of space at this instant’. Parts of extended objects are primitive entities that can not be identified by referring to the region that they occupy at some time. And in pointless geometry such parts must come equipped with mappings from time intervals, and their subintervals, to regions in space which, together with those mappings for all their subparts specify their motions during those time intervals. But by far the easiest and most natural way to summarize such sets of mappings is by specifying velocities for point parts. You could subsequently eliminate those very point parts from your ontology by forming equivalence classes of the Borel regions (in time and space). But it seems clear that your thinking in pointless geometry is quite parasitic on your thinking about standard geometry. In a clear sense pointless geometry is not a natural or simple way to think about space-time or physics more generally.

Consider also equations of motion: one can not state differential equations or any of the usual equations in the usual manner in the ‘no points’ setting. One has to translate this into claims about homomorphisms. The simplest way to do physics is obviously to do it within standard geometry and then at the end say “ignore measure 0 differences, they do not really exist”. If simplicity and naturalness of physical theory is any epistemological guide the ‘no-instants’ view does not appear plausible.

6 Conclusions

I have discussed three possible views regarding the existence of velocities. On the ‘at at’ view velocities are not intrinsic properties of objects at times, and are not part of the states of objects

at times. On this view velocities are defined in terms of features of finite (or infinitesimal) temporal developments of positions. If one believes that any influence that past states have upon future states must be mediated through states at all intermediate times, then one must disbelieve this theory. To be more precise: unless one thinks that this mediation is trivially ruled out as soon as different velocities lead to different developments, one has to reject the ‘at at’ view.

The ‘impetus’ view rescues intrinsic instantaneous velocities. On the ‘impetus’ view a particle has two different types of velocity: an intrinsic velocity, and a rate of change of position. The impetus view thus assumes a larger ontology than seems plausible or necessary. The impetus view also assumes a law of nature which states that intrinsic velocity always equals the rate of change of position. But it is conceptually possible that they would not be equal. These are possibilities that, other things being equal, one surely would prefer not to countenance.

On the ‘no instants’ view there are no instants, so there are no velocities at times. In addition to all the ills of the ‘at at’ view, it carries with it an unwieldy and unintuitive formalism, that is best understood by representing it within the normal machinery and the ‘at at’ view. However it has its compensations: a loss of the measure theoretic paradoxes.

I wish I could end with a snappy and devastating argument for one of these three views. But I can’t, so the reader has some work left. Indeed there is quite a lot of interesting work left, as I have also not considered what the significance of quantum mechanics is for the issues discussed in this paper. Bon chance.

REFERENCES

Albert, D. (Manuscript): Time and Chance

Aristotle (1996): Physics I-VIII, translated by R. Waterfield, Oxford University Press, Oxford and New York.

Bell, J. L. (1988): "Infinitesimals", Synthese 75, 285-315.

Caratheodory, C. (1963): Algebraic Theory of Measure and Integration, Chelsea Publishing Company, New York.

Dijksterhuis, E.J. (1986): The Mechanisation of the World Picture, Princeton University Press, Princeton

Earman, J. (1986): A primer on determinism, Reidel, Dordrecht.

Grant, E. (1981): Physical Science in the Middle Ages, Wiley, New York and London.

McClarty, C. (1988): "Defining sets as sets of points of spaces", Journal of Philosophical Logic 17, 95-80.

Robinson, A. (1996): Non-standard Analysis, Princeton University Press, Princeton.

Skyrms, B. (1993): "Logical atoms and combinatorial possibility", The Journal of Philosophy,

Vol 90, No 5, pp 219-232.

NOTES

1. I would like to thank David Albert for making me worry about time reversal operations, John Norton for giving me the example of a particle that ceases to exist for an instant, Brian Skyrms for trying to explain non-atomic algebras to me, and Marc Wilson for making me think about neighborhood properties. They are to blame for all remaining errors.

2. See e.g. his Physics VI, 234b8.

3. See, for instance, Grant 1981 and Dijksterhuis 1986.

4. See Robinson 1996.

5. See e.g. Bell 1988 and McClarty 1988 and references therein.

6. If one believes that there are truly infinitesimal intervals, one could define neighborhood properties as those that are determined by any infinitesimal development. If one has Robinson's view of infinitesimals this does not alter the subsequent discussion in the main text in any essential way. If one has Lawvere's view of infinitesimals it does. For such a theory does not allow discontinuous trajectories, and hence does not allow the examples of strange neighborhood properties that I discuss further on in the main text. Indeed on a natural use of Lawvere's theory of infinitesimals the only neighborhood property that an object has is the single temporal derivative of its state, which must always be well-defined. I would nonetheless still maintain the conclusion that I reach in this section: a neighborhood property is a property of a neighborhood, it is not an intrinsic property had at an instant. However, a full discussion that does justice to this view of infinitesimals would take us too far afield. For more detail see Bell 1988 and McClarty 1988.

7. It is easy to see that $g(t)$ and $h(t)$ have the same neighborhood state at $t=1$ iff there is a finite neighborhood of $t=1$ on which g and h are identical. For consider the following neighborhood property P : Function f has property P at $t=1$ iff $(\exists \delta \forall x \ |x| < \delta \rightarrow f(1+x) = h(1+x))$. Obviously function h has this neighborhood property (h equals h everywhere so trivially there is a δ around $t=1$ such that h equals h everywhere in that δ -region). If g has the same neighborhood state at $t=1$ as h , it must have all the same neighborhood properties, so it must also have property P . So there must be a δ such that g agrees with h in a region δ around $t=1$. Conversely, if g agrees with h in a finite neighborhood δ around $t=1$, it obviously must have all the same neighborhood properties at $t=1$, since these are all determined by the behavior of the function in any arbitrarily small neighborhood of $t=1$.

This does not mean that a neighborhood state determines the function in a finite neighborhood, for there is no lower bound on the size of the neighborhoods that functions with identical neighborhood states must agree on. For each pair of such functions there is a finite neighborhood upon which they must be identical, but there is no finite neighborhood upon which all functions that share a neighborhood state will agree.

8. Modulo some space-invader problems. See Earman 1986.

9. If one does not like to speak of particles as timeless entities one can phrase the property in question as a neighborhood property of the space-time point $\langle x=1, t=1 \rangle$, namely the neighborhood property that in any temporal neighborhood of that space-time point there is a particle located at $x=1$.

10. To be even more precise, and more radical, one might suggest that the state of a system at a time simply consists of a point in the phase-space of the system, and that the only structure the phase-space has is symplectic (canonical) structure. A dynamics is then specified by specifying the Hamiltonian. This view is radical in that it denies definitional significance to the way in which phase space is normally constructed: namely out of a configuration space by adding the cotangent vector space at each point of the configuration space.

11. See Albert (manuscript).

12. If one did not claim that I could not care whether one allow time reversal operations or not. One is of course free to define notions of time reversibility as one pleases. If the time reversibility of theories is to have no consequences regarding the objective structure of space-time, if time reversal invariance is not to play a role analogous to the other invariances, Lorentz boosts e.g., that exist in modern physics, then one can define time reversibility as one pleases. I just loose interest.

13. For instance as follows. Let us call a doubly infinite sequence of equal sized time intervals $\dots I_{-2}, I_{-1}, I_1, I_2, \dots$ an 'equal division of time' D . We can then say that a mapping M from time intervals I_j to regions of some state space S is the time reverse of mapping N from the I_{-j} to regions of S iff $M(I_n) = N(I_{-n})$. One can more generally say that a mapping M from all the non-atomic regions of time to regions of state space S is the time reverse of mapping N iff for all equal divisions D one has $M(I_n) = N(I_{-n})$. And one can then say that a theory is time reversible iff the time reverse of any allowed mapping is an allowed mapping.

14. For rigid objects of course there is far less that needs to be specified than the velocities of all the parts.