

Space and Time, the Transcendental External World

16. THE STRUCTURE OF SPACE AND TIME IN THEIR PHYSICAL EFFECTIVENESS

THE possible space-time locations or *world-points* form a four-dimensional continuum. Only to spatio-temporal coincidence and immediate spatio-temporal proximity can we assign an intuitively evident meaning. A definite *structure* is already ascribed to the four-dimensional extensive medium of the external world if one believes in a severance of the universe in the sense that it is objectively significant

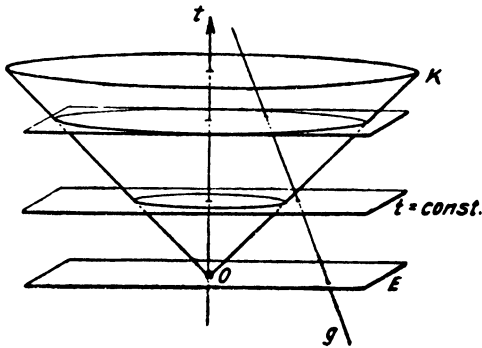


Figure 1. Graphic representation of stratification and fibration of the world. World line g of uniform translation. Light cone K .

to say of any two different events, narrowly confined in space-time, that they are happening at the same place (at different times) or at the same time (at different places). All simultaneous world-points form a three-dimensional *stratum*, all world-points of equal location a one-dimensional *fiber*. According to this view we may describe the structure of the world as possessing a stratification whose layers are traversed by fibers. (Through each world-point runs one stratum and one fiber; any one fiber intersects a stratum in but a single world-point.) Let us, for the sake of graphical representation, drop one of the spatial dimensions, thus concerning ourselves merely with the happenings on a surface, more particularly a plane. Let us represent the latter by a horizontal plane E and lay off the time t in the direction

perpendicular to it. Then we are able to draw a picture of the world in intuitive space; a picture in which the layers of simultaneous world-points all appear as horizontal planes while the fibers of equally located world-points are represented by vertical straight lines.

One attributes furthermore to time and space a *metrical structure* by assuming that equality of time intervals and congruence of spatial configurations have an objective meaning. The statements of Euclidean geometry describe the spatial structure in greater detail. If in our graphical image segments of equal length on the perpendicular time axis correspond to equal time intervals, then the graphical time table of the motion of a body travelling with uniform speed along a straight line will be an inclined straight line. On this *world-line* lie those, and only those, space-time places which are occupied by the body in the course of its history. The world-lines of bodies at rest are vertical straight lines. Two bodies will meet if their world-lines intersect in some space-time point.

The conceptual separation of its structure from the underlying amorphous continuum, the recognition that space as such is merely the medium of 'contact,' is already indicated in the Aristotelian idea of space. Lobatschewsky says (*Urkunden zur Geschichte der nicht-euklidischen Geometrie*, ed. by Engel and Stäckel, I, p. 83), "Contact forms the differentiating feature of bodies, and to it they owe the name of geometric bodies, inasmuch as we concern ourselves with this property alone to the exclusion of all others, be they essential or accidental." However, this thought is here expressed not for space-time but for space alone. Whatever the inner reason of the world structure may be, all laws of nature show that it influences in the most trenchant manner the course of physical events. Among its manifestations we find the behavior of rigid bodies and of clocks; the uniform straight-line motion of a body which is free from all outside influences; the straightness of a light ray in empty space (used when sighting); the propagation in concentric spheres or circles of a light or sound wave, or of a wave in water, etc. It is our task to recognize this structure through these its physical effects. How can we, so we must ask, ascertain objectively the equi-locality or the simultaneity of events, the equality of time intervals and the congruence of spatial configurations?

Concerning the first part of the question we note that the theory of the *relativity of motion* has always been opposed to the dogma of absolute space. Aristotle designates location (*τόπος*) as the relation of one body to the bodies of its vicinity. Descartes (*Principia*, Chap. II) defines motion as "transportation of a portion of matter or a body from the neighborhood of those bodies which are in immediate contact

with the former and which are considered at rest into the neighborhood of other bodies." A penetrating discussion of the relativity of location is given by Locke (*Enquiry concerning Human Understanding*, Book II, Chap. 13, Sections 7-10). Galileo illustrates it rather neatly with the example of the scribe who makes his notes aboard a moving vessel and who will therefore draw with his quill 'in reality,' i.e. relatively to the earth, a smooth slightly undulating line extending from Venice to Alexandrette (*Dialogo, Opere VII*, p. 198). In his controversy with Clarke (and Newton), Leibniz defends with all thoroughness, also in logico-epistemological respects, the relativity of location in space. On that occasion (Leibniz's fifth letter to Clarke, §47) he uses the happy illustration of positions in a family tree.

{Also of importance is the argumentation of Leibniz in his third letter, §5. "Under the assumption that space be something in itself, that it be more than merely the order of bodies among themselves, it is impossible to give a reason why God should have put the bodies (without tampering with their mutual distances and relative positions) just at this particular place and not somewhere else; for instance, why He should not have arranged everything in the opposite order by turning East and West about. If, on the other hand, space is nothing more than just the order and relation of things, if without the bodies it is nothing at all except the possibility of assigning locations to them, then the two states supposed above, the actual one and its transposition, are in no way different from each other. Their apparent difference is solely a consequence of our chimerical assumption of the reality of space in itself. In fact, however, each of them would be the same as the other since the two are completely indistinguishable, and therefore it is a quite inadmissible question to ask why one state was preferred to the other."¹ In contrast, Newton, the absolutist, considers motion a proof for the creation of the world out of God's arbitrary will; for otherwise it would be inexplicable why matter moves in this rather than in any other direction (Preface to the second edition of

¹ Compare with this the statement of Kant concerning left and right which was quoted at the end of Sec. 14, p. 80. Kant has been interpreted as follows: If the first creative act of God had been the forming of a left hand, then this hand, at the time even when it could be compared to nothing else, would already have possessed that definite character of the left one (in contrast to the right one) which can only intuitively but never conceptually be apprehended. This is incorrect, as Leibniz points out, if we intend this to mean that something else would have happened had God created a 'right' hand first, rather than a 'left' hand. One must follow the process of the world's genesis further in order to uncover a difference: Had God, rather than making first a left and then a right hand, begun by making a right one and proceeded to form another right one, then He would have changed the plan of the universe not in the first but in the second act, by bringing forth a hand which was equally rather than oppositely oriented to the first-created one.

Principia by Cotes, ed. Cajori, p. XXXII, and *Principia*, ed. Cajori, p. 546). Leibniz is prevented by his theology from burdening God with such decisions as lack 'sufficient reason.' }]

The body of reference upon which we rely with good reason most of the time in our daily lives when we speak of rest and motion is the "well-founded permanent earth."² For practical purposes this choice, suggested to us as a matter of course, is by far the most expedient. Only a sovereign imagination, breaking the bonds of sensuous appearance and freely constructing in space, could disengage itself from it. Thus Anaxagoras projected the conical shadow of the earth into space and deduced from the eclipses and the phases of the moon the correct spatial arrangement of the earth, the sun, the moon, and the stars; in the 'Moon's face' he recognized the effect of her mountains' shadows. Following the same method the Pythagoreans arrived at the hypothesis of the motion of the earth. In conscious opposition to the Pythagorean and Platonic spirit of *a priori* mathematical construction, Aristotle returned to the geocentric system. At the same time it is a definite religious attitude toward the universe that finds expression in reserving for the earth, the dwelling place of mankind, an absolute prerogative among all other bodies of reference. It is the attempt to uphold within the realm of objective reality the idealistic position, according to which I am the center of the world disclosed to me. But here where the recognition of the thou is required of the ego and the ego has to be extended so as to include the whole of mankind, the idealistic position of necessity takes on a historical and cosmological character. This is the reason why the book of Copernicus became a turning point of world conception; and in this direction Bruno drew the conclusions with stormy enthusiasm. The supreme act of redemption by the Son of God, crucifixion and resurrection, no longer the unique pivot of world history but the hurried small-town performance of a road show repeated from star to star — this blasphemy displays perhaps in the most pregnant manner the religiously precarious aspect of a theory which dislodges the earth from the center of the world. (Bruno had to pay for it at the stake.) "The statement, found equally with Kepler, Galileo, and Descartes, that it be foolish to think of the purpose of the universe as lying in man," says Dilthey (*Der entwicklungsgeschichtliche Pantheismus, Gesammelte Schriften*, II, third ed., 1923, p. 353), "consummates a complete change in the interpretation of the world. As these thinkers were led to an immanent teleology finding its expression in the harmony and beauty

² "Die wohlgegründete dauernde Erde," quotation from Goethe, *Grenzen der Menschheit*, verse 3.

of the universe, the character of the hitherto prevailing Christian religiosity was changed." And Goethe, in his *Geschichte der Farbenlehre* (3^{te} Abteilung, 2^{te} Zwischenbetrachtung), "Perhaps never before has a greater demand on mankind been made; for what did not go up in smoke with this acknowledgment: a second paradise, a world of innocence, of poetry and piety, the testimony of the senses, the conviction of a poetical-religious faith. Small wonder then that one did not want to let go of all this, that one opposed in every conceivable manner a theory which involved for him who accepted it the right and the challenge of a hitherto unknown, nay undreamed-of, freedom of thought and elevation of mind."

From the viewpoint of the relativity of motion there can be no quarrel as to the truth or falsity of the Copernican system. It is merely that the laws of planetary motion become much simpler if this motion is described as relative to the sun instead of relative to the earth.

[Newton bases the development of his mechanics in the *Principia* upon the ideas of absolute time, absolute space, and absolute motion. "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external. . . . Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. . . . Absolute motion is the translation of a body from one absolute place into another." (*Principia*, ed. Cajori, Scholium following the Definitions, I, II and IV, pp. 6-7.) As a kinematic differentiation of the various possible states of motion of a body is undeniably impossible, Newton strives to distinguish the state of rest among all possible states of motion *dynamically*, on the basis of phenomena such as the centrifugal forces. "The causes by which true and relative motion are distinguished, one from the other, are the forces impressed upon bodies to generate motion. . . . It is indeed a matter of great difficulty to discover, and effectually to distinguish, the true motions of particular bodies from the apparent; because the parts of that immovable space, in which those motions are performed, do by no means come under the observation of our senses. Yet the thing is not altogether desperate; for we have some arguments to guide us, partly from the apparent motions, which are the differences of the true motions; partly from the forces, which are the causes and effects of the true motions. For instance, if two globes, kept at a given distance one from the other by means of a cord that connects them, were revolved about their common center of gravity, we might, from the tension of the cord, discover the endeavor of the globes to recede from the axis of their motion, and

from thence we might compute the quantity of their circular motions. . . . But how we are to obtain the true motions from their causes, effects, and apparent differences, and the converse, shall be explained more at large in the following treatise. *For to this end it was that I composed it.*" (*Principia*, ed. Cajori, pp. 10 and 12.)

Newton's belief in absolute space is theologically influenced. Thus he says of God in his *Opticks* that "in infinite space, as it were in his Sensory, [He] sees the things themselves intimately, and thoroughly perceives them, and comprehends them wholly by their immediate presence to himself" (ed. Whittaker, p. 370). Newton adopts here the theology of Henry More. For More, space is the first and authentic witness for the verity and necessity of "immaterial natures"; in its properties he rediscovers the characteristics of the divine substance; space is the link between the latter and the individual objects. The nature of the world structure, that it consists of a fibration, is thus laid down by Newton in terms of an *a priori* metaphysical idea. But the actual course of the fibration in the real world has to be ascertained through its effects upon observable real phenomena. That is his scientific program. Incidentally, Newton does not succeed in mastering the problem completely. He accomplishes only the dynamic separation of uniform translation as the pure inertial motion of a body uninfluenced by external forces, from the other states of motion; but he does not succeed in isolating the state of rest among these translations. In this he *must* fail on account of the so-called *special relativity principle*, which is satisfied by the laws of Newtonian mechanics and whose validity for all natural phenomena has been confirmed today by a series of the most exact experiments: In the cabin of a ship sailing a straight course with uniform speed all events will take place in the same manner as if the vessel were at rest; given any event in nature, the one which arises from it by imparting to all participating bodies a uniform translation is equally possible. The principle has been developed by Galileo in his "Dialogo" (*Opere*, VII, pp. 212-214) in clear and lucid manner. Newton at this point resorts to a hypothesis unfounded in experience and a dialectical dodge which strike a discordant note in the midst of the magnificent and cogent inductive development of his system of the world in the third book of the *Principia*. The hypothesis states that the universe has a center and that this center is at rest. The common center of gravity of the solar system, like that of any system of bodies not subjected to external forces, moves uniformly along a straight line; thus he concludes correctly from the mechanical laws. And now we read (*Principia*, ed. Cajori, p. 419), ". . . but if that center moved, the center of the world would move also, against the hypothesis," no

attempt whatever being made to give a reason for this identification of the center of gravity of the planetary system with the hypothetical stationary center of the world (unless it be in the consideration that the stationary center ought to be a point, constructible on the basis of material events, whose motion according to the laws of mechanics is a uniform translation). }]

The experiences which prove the dynamic inequivalence of different states of motion teach us that the world bears a structure. But in the concept of absolute space this *inertial structure* is evidently not sized up correctly; the dividing line does not lie between rest and motion but between uniform translation and accelerated motion. Referring to the graphical representation described above we can say that it is in the world as it is in space: straight lines can be objectively distinguished from curves, but in the family of all straight lines one can single out the 'vertical' ones only by a convention based on individual exhibition.

And what about the stratification, the concept of simultaneity? The trust placed in its objective significance rests on the fact that everybody considers as a matter of course the events he observes as happening at the moment of their observation. In this manner I extend my time to the whole world which enters my field of vision. Although this naive opinion lost its basis through the discovery that light has a finite velocity of propagation, there yet remained (beyond the reluctance to abandon a prejudice once held) some reason for adhering to that belief. In our graphical representation the horizontal plane passing through a world point O separates past and future as seen from O . 'Past' and 'future,' what is the reality behind these words? By shooting bullets from O in all possible directions with all possible velocities I can only hit those world points which are later than O ; I cannot shoot into the past. Likewise any event happening at O has influence only upon the events at later world points; the past cannot be changed. That is to say, the stratification has a causal meaning; it determines the *causal connection of the world*. This was recognized by Leibniz, who explains in his "Initia rerum mathematicarum metaphysica" (*Math. Schriften* VII, p. 18), "If of two elements which are not simultaneous one comprehends the cause of the other, then the former is considered as *preceding*, the latter as *succeeding*." A simple method of instantaneously transferring time from one place A to another place B consists in giving a jerk to the end A of a rigid rod extending from A to B ; the jerk observed at B is simultaneous with the one given at A .

But in regard to the causal structure of the world the modern

development of physics has led to an essential correction. Let the segment representing one second on the t -axis of our graphical diagram be of the same length as the segment on the horizontal plane E representing the distance covered by a light ray in one second. A light signal issuing from O and spreading in all directions with the same velocity c will be received at all those world points which lie on the surface of a vertical circular cone with vertex at O and a vertex angle of 90° . According to Einstein's *special theory of relativity*, the 'light cone,' consisting of the above surface and its prolongation backward beyond O , rather than the horizontal plane through O , accomplishes the separation of the world into past and future. No effect is propagated at a greater speed than that of light (including the jerk given to a rigid rod for the purpose of transferring time); the velocity of any body remains of necessity below c . This is an inevitable consequence of the principle of special relativity and the fact that the light cone

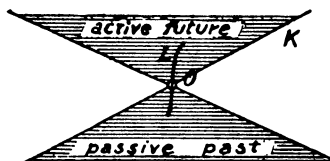


Figure 2. Causal structure. Light cone K , life line L .

issuing from O depends on O alone and not upon the state, in particular the state of motion, of the light source which emits the signal at O . (Unfortunately the somewhat inadequate phrase 'constancy of the velocity of light' has been chosen to describe the latter fact.)

If I am at O , then O will divide my life line, that is the world line of my body, into two parts, past and future; in this respect nothing has been changed. But the situation is different as far as my relation to the world is concerned. In the interior of the forward part of the cone are found all those world points upon which my doings at O are of influence, in its exterior all those events which lie closed behind me, about which nothing can be done any more; the front cone comprehends my *active future*. In the interior of the backward part of the cone, on the other hand, are located all those events of which I either was a witness or of which I might have received some message; only these events might possibly have influenced me at all; it is the domain of my *passive past*. The two regions, active future and passive past, do not border on each other without a gap as had been the case according to the older conception.

[It is our task, moreover, to describe in physical terms how to ascertain the equality of time intervals and the congruence of material

bodies. A clock is a closed material system which will return to exactly the same state S in which it found itself at some earlier instant. Let us assume the principle of causality, which asserts that the state of a system at any moment uniquely determines its entire history. Then the same process, the same cyclic sequence of states, leading from S to S will be repeated again and again, and each of these periods has by definition the same duration. What is measured in this way is the 'proper time' of the clock; it can be directly employed for all events occurring along the world line of the clock. Helmholtz says (*Zählen und Messen, Wissenschaftliche Abhandlungen*, III, p. 379), "Measurement of time presupposes that we have found physical processes, repeating themselves under equal conditions and in invariably the same manner such that if they are begun at the same instant (it would be more correct to say 'in contiguous space-time points') they also end simultaneously; such as days, the strokes of a pendulum, the running-down of sand- or water-clocks. The justification for the assumption of invariable duration rests on the circumstance that all different methods of measuring time, if carefully executed, always lead to concordant results." Concerning the empirical determination of spatial congruence he says on another occasion (*Wissenschaftliche Abhandlungen*, II, p. 648), "I call two spatial magnitudes physically equivalent if under equal conditions and in equal intervals of time the same physical events can occur within them. The process which, with appropriate caution, is employed most frequently to determine the physical equivalence of spatial magnitudes is the transfer of rigid bodies such as compasses and rulers from one place to another." The physical geometry founded on this concept of physically observable congruence is considered by Helmholtz to be an empirical science, in fact "the first and most perfect of the natural sciences." Speaking of this physical geometry in his inaugural lecture, Riemann points out what may conceivably become of major significance in the physics of the future, that "the empirical concepts upon which the spatial metric is based, the concepts of the rigid body and of the light ray, cease to be valid in the domain of the infinitely small." As a matter of fact it can be shown that the metrical structure of the world is already fully determined by its inertial and causal structure, that therefore mensuration need not depend on clocks and rigid bodies but that light signals and mass points moving under the influence of inertia alone will suffice.

A three-dimensional continuum when referred in some way to coordinates x_0, x_1, x_2 is thereby mapped upon the three-dimensional number space, i.e. upon the continuum of all number triples. Using a more familiar mode of expression, we shall replace the number space

by the three-dimensional intuitive space equipped with a Cartesian coordinate system. It does little harm that, in applying the procedure to the four-dimensional world, we shall have to deprive it in imagination of one of its dimensions. A two-dimensional example is provided by the planar geographical maps. On a Mercator map, for instance, I find that San Francisco, the southernmost point of Greenland, and the North Cape, lie on a straight line, but I am not surprised to discover that on an orthographic map of the northern hemisphere this fails to be the case. Likewise a certain mapping of the world serves as the basis for the application of the customary geometrical-kinematical terms, with the x_0 -axis being interpreted as the axis of time. (For instance, we shall say of a body that it is at rest if its world line is a vertical straight line, i.e. a line along which x_1, x_2, x_3 are constant and merely x_0 varies.) Only such relations will have objective meaning as are independent of the mapping chosen and therefore remain invariant under arbitrary deformations of the map. Such a relation is, for instance, the intersection of two world lines. If we wish to characterize a special mapping or a special class of mappings, we must do so in terms of the real physical events and of the structure revealed in them. That is the content of the *postulate of general relativity*. According to the *special theory of relativity*, it is possible in particular to construct a map of the world such that (1) the world line of each mass point which is subject to no external forces appears as a straight line, and (2) the light cone issuing from an arbitrary world point is represented by a circular cone with vertical axis and a vertex angle of 90° . In this theory the inertial and causal structure and hence also the metrical structure of the world have the character of rigidity, they are absolutely fixed once and for all. It is impossible objectively, without resorting to individual exhibition, to make a narrower selection from among the 'normal mappings' satisfying the above conditions (1) and (2).]

The discrepancy between the kinematical and the dynamical analyses of motion calls for a solution. Huyghens, as we know from his letters, endeavored to carry through the viewpoint of the equivalence of all states of motion even in their dynamical aspect; an attempt in this direction has been preserved in his posthumous papers (reprinted in *Jahresberichte der Deutschen Mathematiker-Vereinigung*, Vol. 29, 1920, p. 136). In our days Mach undertook the same thing in his *Mechanik* (seventh ed., 1912). He would see in the polar flattening of the earth an effect of its rotation relative to the fixed stars; the fixed stars are to hold and to carry with them the plane of Foucault's pendulum. Leibniz, on the other hand, however determinedly he

rejects Newton's metaphysics of space, holding firmly to the opinion that space is nothing more than "the mere order of things among themselves," evidently agrees with Newton's mechanical program to separate true from apparent motion by dynamical criteria. (Compare the letter to Huyghens dated June 12/22, 1694, *Math. Schriften*, II, p. 184, and the explanation in *In. rerum math. metaph.*, *Math. Schriften*, VII, p. 20: "We say that an object moves if it changes its position and if in addition the cause of this change lies within the object itself.") Euler (*Theoria motus*, 1765, especially §81) also is of the opinion that the principle of the relativity of motion, evident as it may be to our reason, has to be abandoned in the face of dynamical experiences. With some good will one may read into Kant's exposition in the *Metaphysische Anfangsgründe* a correct formulation of the problem, but they certainly throw no light on its solution.

Incidentally, without a world structure the concept of relative motion of several bodies has, as the postulate of general relativity shows, no more foundation than the concept of absolute motion of a single body. Let us imagine the four-dimensional world as a mass of plasticine traversed by individual fibers, the world lines of the material particles. Except for the condition that no two world lines intersect, their pattern may be arbitrarily given. The plasticine can then be continuously deformed so that not only one but all fibers become vertical straight lines. Thus no solution of the problem is possible as long as in adherence to the tendencies of Huyghens and Mach one disregards the structure of the world. But once the inertial structure of the world is accepted as the cause for the dynamical inequivalence of motions, we recognize clearly why the situation appeared so unsatisfactory. We were asked to believe that something producing such enormous effects as inertia — for instance, when in combat with the molecular forces it rends the cars of two colliding trains — is a rigid geometrical property of the world, fixed once and for all. Leibniz (opposing Descartes) has emphatically stressed the dynamic character of inertia as a tendency to resist deflecting forces; for instance, in a letter to de Volder (*Philosophische Schriften*, II, p. 170) he writes, "It is one thing if something merely retains its state until some event happens to change it — a circumstance which may occur if the subject is completely indifferent with respect to either state; it is another thing and signifies much more if the subject is not indifferent but possesses a power, an inclination as it were, to retain its state and to resist the causes of change." Hence the solution is attained as soon as we dare to *acknowledge the inertial structure as a real thing that not only exerts effects upon matter but in turn suffers such effects*. This step was taken by Riemann as early as the middle of the nineteenth century regarding

the metrical structure of space; for indeed the inertial and the metrical structures of the world are so intimately connected (the metric after all determines the straight lines) that the metrical field will of necessity become flexible as soon as the inertial field is deprived of its geometric rigidity.

Einstein rediscovered this idea independently of Riemann, completing it by an important insight that rendered it physically fruitful. From the *equality of inertial mass and weight* — before him an enigmatic fact well-established but not understood — he concluded that, in the dualism of force and inertia, *gravitation* has to be put on the side of inertia rather than on the side of force. The phenomena of gravitation thus divulge the flexibility of the field of inertia, or, as I prefer to call it, the ‘guiding field,’ and its dependence on matter. The splitting of the unified guiding field into a homogeneous part obeying Galileo’s law of inertia and a much weaker deviation called gravitation, which surrounds the individual stars, cannot be accomplished in an absolute manner but is relative to a system of coordinates. The laws replacing Newton’s law of attraction and governing the action of matter upon the field of inertia follow conclusively from this conception. Their consequences have been fully confirmed by experience.

[The guiding field is (very slightly) disturbed by matter, just as the surface of a lake is disturbed by the steamships cruising on it; it will go over into the undisturbed state described by the special theory of relativity when all matter disappears, as the surface of the lake becomes a smooth homogeneous plane when the ships ride at anchor. Although Einstein, too, flirts with that idea of Mach’s, it is impossible, according to an earlier remark, to eliminate the field of inertia, or the ‘ether,’ as an independent power from the natural phenomena. It is not the stars that guide the plane of Foucault’s pendulum, but the joint motion of both — of Foucault’s pendulum and of the star compass formed by the light rays reaching the terrestrial observer from the stars — is due to the overwhelming power of the ether in its interaction with matter. The old conception, separating inertia and gravitation in an absolute fashion, erred only in this point, that it saw in the actual position of all water particles of the lake, to return to our illustration, the resultant of a unique state of rest and a displacement caused by the cruising steamers. This is incorrect; indeed, as the water comes to rest at night when all ships ride at anchor, we shall undoubtedly have the same ‘qualitative state’ as in the morning before the ships got under way, the unruffled plane surface; but the ‘material state’ which is concealed behind this, i.e. the location of the various water particles, may have shifted com-

pletely. This does not contradict the principle of sufficient reason which calls for a uniquely determined state of equilibrium in the lake. For if all water particles are alike, the two states S and S' of the lake which arise from one another by having the particles interchange their locations in some arbitrary fashion are not different from each other if either is considered by itself. Only after 'a coordinate system has been introduced,' meaning in this case that numbers have been assigned to the particles, which introduce artificial differences among them and adhere to them during their motion, only then will it be meaningful to speak of the two material states S and S' as such (cf. Appendix B). In truth, however, it is not the individual material state S , the arrangement, which one can lay hands on, but only the permutation, that is the transition from the material state S to S' . This should be compared with the previously quoted remarks of Leibniz regarding the relativity of location (p. 97).

The group Δ_0 of the Euclidean rotations (see Section 15) in the three-dimensional space has now been replaced by the so-called Lorentz group. It consists of all homogeneous linear transformations

$$z'_i = \sum_k a_{ik} z_k \quad (i, k = 0, 1, 2, 3)$$

which leave the indefinite quadratic form $-z_0^2 + z_1^2 + z_2^2 + z_3^2$ invariant. For each such transformation the absolute values of the coefficient $a = a_{00}$ and of the determinant d of the 3×3 coefficients a_{ik} ($i, k = 1, 2, 3$) are ≥ 1 . We ascribe to the transformation the temporal signature $+$ or $-$ according to whether $a \geq 1$ or $a \leq -1$; in the same manner the sign of d determines the spatial signature. The Lorentz transformations of temporal signature $+$ form a subgroup of index 2 of the total group, and so do the transformations of spatial signature $+$. Their common part is contained in each of them again as a subgroup of index 2. The transformations of temporal signature $-$ interchange past and future, those of spatial signature $-$ interchange left and right. The most fundamental experiences of our life seem to indicate that Δ_0 should be limited to the Lorentz transformations with temporal signature $+$ (but include those of spatial signature $-$). But physics has found it rather hard to decide this question (cf. Section 23, C). A third signature, the topological signature, attaches to the coordinate transformation and is determined by the sign of its Jacobian.]

With the 'general theory of relativity' we may sum up the final result of the historical development of the structural problem of space

and time as follows: *The world is a four-dimensional Riemannian space.* There is associated with every line element, issuing from the point P with the coordinates x_0, x_1, x_2, x_3 and connecting it with the infinitely closely adjacent point $P' = (x_i + dx_i)$, a numerical measure

$$ds^2 = \sum_{i,k=0}^3 g_{ik} dx_i dx_k \quad (g_{ik} = g_{ki})$$

which is independent of the arbitrary coordinate system employed. The coefficients g_{ik} depend on the coordinates x_0, x_1, x_2, x_3 of P but not on the dx_i . The metric ground form on the right is not positive-definite but possesses one negative dimension; i.e. in an appropriate coordinate system at the point P it assumes the universal normal form

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.$$

In consequence of this circumstance, the 'light cone at P ,' containing all line elements emanating from P that make ds^2 equal to zero, separates a domain of active future for P from a domain of passive past. The metric ground form determines, in a manner readily describable in detail, the behavior of clocks and rulers, it defines the light cones in their entire extension, and it separates the world lines of purely inertial motion (traced, for instance, by the planets) from the totality of all possible world lines. Its coefficients, the continuous functions $g_{ik}(x_0, x_1, x_2, x_3)$, describe, in terms of the chosen coordinate system, the metrical field or the 'state of the ether,' which interacts with matter.

When raising the question about the *total extent of the universe* one must distinguish between the purely topological and the metrical aspects. The transition from the Aristotelian world system, enclosed in a crystal sphere and rotating about a center, to the indifferent expanse of the infinite Euclidean space, uncentered and populated by stars throughout, was welcomed by Bruno as a mighty emancipation. Nevertheless the Aristotelian space (the interior of the crystal sphere) differs only in its metrical relations, not topologically, from the infinite one. The infinite Euclidean space leads to absurdities if we assume that the masses are on the whole uniformly distributed throughout the universe and that Newton's law of attraction is valid. Even though the gravitational force of a constant mass decreases with the inverse square of the distance, the far-off masses would then be so predominant in the entire gravitational effect that the total force exerted upon any one star would remain completely indeterminate. It is possible, however, that space is finite and yet unbounded; indeed it may be a closed manifold, like the two-dimensional surface of a

sphere. It is an appealing interpretation of A. Speiser's (*Klassische Stücke der Mathematik*, 1925, p. 53) that Dante, without denying the validity of Aristotle's conception of perceptive space, assumes the real space of creation (of which the former is but an image) to be closed rather than bounded. The radii emanating from the center of the earth, the seat of Satan, converge toward an opposite pole, the source of divine force. The force of the personal God must radiate from a center, it cannot embrace the world sphere reposing in spatial quiescence like the "unmoved primal mover" of Aristotle (compare *Divina Comedia*, Paradiso, beginning with the 28th Canto). When Einstein tried, in the framework of his theory of gravitation, to carry through Mach's principle, he constructed a static universe U_a with a closed three-dimensional space in which matter is evenly distributed; the total mass in the world determines the volume of the space. Einstein's space, of course, in contrast to that of Dante, lacks a pair of distinguished opposite poles. It is as homogeneous as Euclid's space. U_a results as a possible solution of the laws of gravitation, provided they are made to include the so-called cosmological term which introduces a universal constant a of the dimension of a distance (and of the order of magnitude of the 'world radius').

Dropping two of its spatial dimensions we may picture U_a as the surface of a straight vertical cylinder of radius a and of infinite extent in both directions. This shows that U_a has two separate 'fringes,' that of infinitely remote past and that of infinitely remote future, and in this topological sense U_a extends from eternity to eternity. With the same reduction of dimensions the map of the universe U_∞ of the common Euclid-Bruno conception, i.e. of an empty world whose metrical structure is described by special relativity, is a vertical plane, and it therefore has but one connected fringe. It is this topological difference between U_a and U_∞ (two fringes as opposed to one fringe) to which in the last analysis the terms closed and open space allude.

[In Einstein's cosmology the metrical relations are such that the light cone issuing from a world point is folded back upon itself an infinite number of times. An observer should therefore see infinitely many images of a star (unless they are washed out by rarefied clouding media in interstellar space or by diffraction), showing him the star in states between which an 'eon' has elapsed, the time needed by the light to travel around the sphere of the world. The present would be shot through with the ghosts of the long ago. Moreover the solution is unstable. Yet de Sitter found that the laws of gravitation also admit the possibility of a mass-free world extending from eternity to eternity in which the domain of the future emanating from

a world point does not overlap with itself. The systematic shift of the spectral lines of the most remote celestial objects, the spiral nebulas, to the red side of the spectrum has been interpreted in terms of an expanding universe, of which de Sitter's construction provides the simplest model (Weyl, Friedmann, Lemaitre, H. P. Robertson, and others). For a one thus obtains a value $\sim 10^{27}$ cm. Incidentally the behavior of every world satisfying certain natural homogeneity conditions in the large (whether it is void or carries mass) follows this model asymptotically when, in the process of expansion, the world radius becomes essentially larger than a . (Compare also Section 23 C.) The postulate that for each world point O the two world domains of active future and passive past are disjoint (not only in the immediate vicinity of O but in their entire extent) rules out the possibility of a world which is closed in its spatial as well as its temporal dimensions. In such a world, that which happened once would, to the tradition handed down from generation to generation, appear as an eternal recurrence of the same events (Nietzsche's 'ewige Wiederkunft').]

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17. SUBJECT AND OBJECT (THE SCIENTIFIC IMPLICATIONS OF EPISTEMOLOGY)

The doctrine of the subjectivity of sense qualities has been intimately connected with the progress of science ever since Democritus laid down the principle, "Sweet and bitter, cold and warm, as well as the colors, all these things exist but in opinion and not in reality ($\nu\omicron\mu\omega$, $\acute{o}\nu\ \acute{\varphi}\acute{\upsilon}\sigma\epsilon\iota$)"; what really exists are unchangeable particles, atoms, which move in empty space. Also Plato (*Theætetus*, 156e) holds that "properties such as hard, warm, and whatever their names may be, are nothing in themselves," but arise in the encounter of "motions" originating in the subject and in the object. Reality is pure activity; only in the "image," in the consciousness suspended between the

motions is suffering. Galileo may be mentioned as another witness, "White or red, bitter or sweet, noisy or silent, fragrant or malodorous, are names for certain effects upon the sense organs." He holds that they can no more be ascribed to the external objects than the titillation or the pain which might be felt when things are touched. A detailed discussion of this is given in the final sections of Descartes' *Principia* and in his *Traité de la Lumière* (the theory of optical perception is indebted to him for important advances), likewise in Locke's *Enquiry Concerning Human Understanding* (Book II, Chap. 8, §§15-22). The subjectivity of sense qualities must be maintained in two regards, one philosophical, the other scientific. In the first place, such a quality by its very nature can only be given in our consciousness through sensation. One sees in it either an inherent attribute of sensation itself or, upon deeper analysis, an entity belonging to the intentional object which the act of consciousness puts before me. But it remains manifestly incomprehensible how quality disjoint from consciousness can be attributed as a property as such to a thing as such. This is the fundamental tenet of epistemological idealism. In the second place, the qualities in which the objects of the outer world garb themselves for me do not depend on the objects alone. They also depend quite essentially upon the concomitant physical circumstances, for instance, in the case of color, on illumination and on the nature of the medium between the object and my eye, and furthermore upon myself, on my own psycho-physical organization. My sense of vision does not grasp the objects where they are; rather, what I see is determined by the condition of the optical field in its zone of contact with my sensuous body (the retina). These are scientific facts which even the realist cannot deny. How differently the world would appear to our vision if the human eye were sensitive to other wave lengths or if the physiological processes on the retina were to transform the infinite-dimensional realm of composite physically different colors not merely into a two-dimensional but into a three- or four-dimensional manifold!

[To Locke we are indebted for the classical distinction of 'secondary' and 'primary' qualities; the primary ones are the spatio-temporal properties of bodies—extension, shape, and motion. Democritus, Descartes, and Locke held them to be objective. Locke expresses himself as follows: "The ideas of primary qualities of bodies are resemblances of them, and their patterns do really exist in the bodies themselves; but the ideas produced in us by the secondary qualities have no resemblance of them at all" (*op. cit.*, Book II, Chap. 8, beginning of §15). Although Descartes teaches that between an actual occurrence and its perception (sound wave and tone, for

instance) there is no more resemblance than between a thing and its name, he yet maintains that the ideas concerning space have objective validity because in contrast to the qualities we recognize them clearly and distinctly. And a fundamental principle of his epistemology claims that whatever we comprehend in such a way is true. In support of this principle, however, he has to appeal to the veracity of God, who does not want to deceive us. Obviously one cannot do without the idea of such a God who guarantees truth, once one has grasped the principle of idealism and yet insists on building up the real world out of certain elements of consciousness that for one reason or another seem particularly trustworthy. "He is the bridge . . . between the lonely, wayward and isolated thinking, which is certain only to its own selfawareness, and the external world. The attempt turned out somewhat naive, but still one sees how keenly Cartesius measured out the grave of philosophy. It is strange, though, how he uses the dear God as the ladder to climb out of it. Yet even his contemporaries did not let him get over the edge" (quotation from Georg Büchner's philosophical notes, G. Büchner, *Werke*, Inselverlag Leipzig, 1922, pp. 268-269). Hobbes in his treatise *De Corpore* starts with a fictitious annihilation of the universe (similar to Husserl's "epoché") in order to let it rise again by a step-by-step construction from reason. But even he uses as building material the general notions which form the residue of experience, in particular those of space and time. This viewpoint has its counterpart in the physics of Galileo, Newton, and Huyghens; for here all occurrences in the world are constructed as intuitively conceived motions of particles in intuitive space. Hence an absolute Euclidean space is needed as a standing medium into which the orbits of motion are traced. Well-known is Galileo's pronouncement in the "Saggiatore" (*Opere*, VI, p. 232), "The true philosophy is written in that great book of nature (*questo grandissimo libro, io dico l'universo*) which lies ever open before our eyes but which no one can read unless he has first learned to understand the language and to know the characters in which it is written. It is written in mathematical language, and the characters are triangles, circles, and other geometric figures." }

Leibniz seems to have been the first to push forward to a more radical conception: "Concerning the bodies I am able to prove that not only light, color, heat, and the like, but motion, shape, and extension too are only apparent qualities" (*Philos. Schriften*, VII, p. 322). Also Berkeley and Hume are to be named here. For d'Alembert, the justification for using the "residue of experience" in the construction of the objective world no longer lies in the clarity and

distinctness of the ideas involved as it did for Descartes, but exclusively in the practical success of this method. According to Kant, space and time are merely forms of our intuition. Stumpf (*Über den psychologischen Ursprung der Raumvorstellung*, 1873, p. 22) finds it impossible to imagine the atoms as spatial bodies without color, whose play of motion only engenders those oscillations of the ether which are the carriers of color by virtue of their wave lengths; for no more than color without spatial extension could space (according to Berkeley's and Hume's doctrine) be imagined without the raiment of some quality of color. Intuitive space and intuitive time are thus hardly the adequate medium in which physics is to construct the external world. No less than the sense qualities must the intuitions of space and time be relinquished as its building material; they must be replaced by a four-dimensional continuum in the abstract arithmetical sense. Whereas for Huyghens colors were 'in reality' oscillations of the ether, they now appear merely as mathematical functions of periodic character depending on four variables that as coordinates represent the medium of space-time. What remains is ultimately a *symbolic construction* of exactly the same kind as that which Hilbert carries through in mathematics.

The distillation of this objective world, capable only of representation by symbols, from what is immediately given to my intuition, takes place in different steps, the progression from level to level being enforced by the fact that what exists on one level will reveal itself as the mere apparition of a higher reality, the reality of the next level. A typical example of this is furnished by a body whose solid shape constitutes itself as the common source of its various perspective views. This would not happen unless the point from which it is viewed could be varied and unless the different viewpoints actually taken present themselves as instances of an infinite continuum of possibilities laid out within us. We shall return to this in the next section. A systematic scientific explanation, however, will reverse the order; it will erect the world of symbols as a realm by itself and then, skipping all intermediate levels, attempt to describe the relation that holds between the symbols representing objective conditions on the one hand and the corresponding data of consciousness on the other.

{Thus *perspective* teaches us to derive the optical image from the solid shape of a body and from the observer's location relative to the body. A physical example, taken from among the upper levels, is the constitution of the concepts 'electric field' and 'electric field strength.' We find that in the space between charged conductors a

weakly charged 'test particle' experiences a certain force $\vec{F} = \vec{F}(P)$ when put at a given place P . Well determined as to size and direction, the force turns out to be the same whenever the test particle is brought back to the same place P . Employing various test particles we find that the force depends on the latter, yet in such a manner that $\vec{F}(P)$ may be split up into two factors:

$$\vec{F}(P) = e \cdot \vec{E}(P),$$

where the vectorial factor $\vec{E}(P)$, the 'electric field strength,' is a point function independent of the state of the test particle, while the scalar factor e , the 'charge' of the test particle, is determined exclusively by the inner state of the particle, depending neither on its position nor on the conductors, and is thus found to be the same no matter into what electric field we may place the particle. Here we start from the force as the given thing; but the facts outlined lead us to conceive of an electric field, mathematically described by the vectorial point function $\vec{E}(P)$, which surrounds the conductors and which *exists, no matter whether the force it exerts on a test particle be ascertained or not.* The test particle serves merely to render the field accessible to observation and measurement. The complete analogy with the case of perspective is obvious. The field \vec{E} here corresponds to the object there, the test particle to the observer, its charge to his position; the force exerted by the field upon the test particle and changing according to the charge of the particle corresponds to the two-dimensional aspect offered by the solid object to the observer and depending on the observer's standpoint. Now the equation $\vec{F} = e \cdot \vec{E}$ is no longer to be looked upon as a definition of \vec{E} but as a *law of nature* (to be corrected if circumstances warrant it) determining the ponderomotoric force which an electric field \vec{E} exerts on a point charge e . Light, according to Maxwell's theory, is nothing but a rapidly alternating electromagnetic field; in our eyes, therefore, we have a sense organ capable of apprehending electric fields in another manner than by their ponderomotoric effects. A systematic presentation will introduce \vec{E} , the electric field strength, in a purely 'symbolic' way without explanations and then lay down the laws it satisfies (for instance, that the line integral of \vec{E} extended over a closed curve in space is zero) as well as the laws according to which ponderomotoric forces are connected with it. If forces are considered observable, the link between our symbols and experience will thus have been established.}]

One may say that only in the general theory of relativity did physics succeed in emancipating itself completely from intuitive space and time as means for the construction of the objective world. In the framework of this theory (which by the way includes all previously adopted standpoints either as particular or as limiting cases), the relation of subject and object may be illustrated by means of a typical example, the observation of two or more stars. By way of simplification we assume the apprehending consciousness to be a *point eye* whose world line may be called B . Let the observation take place at the moment O of its life. The construction is to be carried out in the four-dimensional number space, only for the sake of readier intelligibility we shall use a geometrical diagram instead. Let Σ be the world lines of two stars. The rearward light cone K issuing from O will meet each of the two star lines Σ in a single point, and the world lines

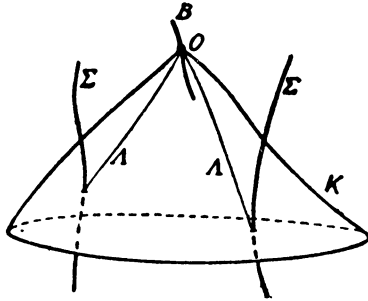


Figure 3. Data on which observation of angular distance of two stars depends.

λ of the light signals which arrive at O from the stars join these points to O on the cone K . With the help of a construction, describable in purely arithmetical terms, it is possible to determine from these data the numerical measure of the angle ϑ under which the stars will appear to the observer at O . This construction is invariant, that is, of such a kind as to lead to the same numerical measure ϑ if, after an arbitrary deformation of the entire picture, it is carried out anew on the deformed image according to the same prescription. And everything is contained in it — the dependence of the angle on the stars themselves, on the metrical field extending between the stars, on the observer's position in the world (spatial perspective), and on his state of motion (i.e. on the direction with which the line B passes through O ; this is the velocity perspective, known under the name of aberration). The angles ϑ between any two stars of a constellation determine the objectively indescribable, only intuitively experienced, visual shape of the constellation, which appears under the equally indescribable assumption that I myself am the point eye at O . If they coincide with those

of a second constellation, then both constellations at the moment O appear to be of equal, otherwise of different, shapes.

The objective world simply *is*, it does not *happen*. Only to the gaze of my consciousness, crawling upward along the life line of my body, does a section of this world come to life as a fleeting image in space which continuously changes in time.

[An important role in the construction of the angles ϑ is played by the 'splitting' of the world into space and time carried out at every moment O of my consciousness. Objectively this is to be described as follows: If e_0, e_1, e_2, e_3 are the components of a vector indicating the direction of B at O , then my immediate spatial vicinity will be spanned by the totality of all line elements (dx_0, dx_1, dx_2, dx_3) issuing from O which are orthogonal to e , i.e. which satisfy the equation

$$\sum_{i,k=0}^3 g_{ik} dx_i \cdot e_k = 0, \quad g_{ik} = g_{ik}(O).]$$

Thus the objective state of affairs contains all that is necessary to account for the subjective appearances. There is no difference in our experiences to which there does not correspond a difference in the underlying objective situation (a difference, moreover, which is invariant under arbitrary coordinate transformations). It comprises as a matter of course the body of the ego as a physical object. The immediate experience is *subjective and absolute*. However hazy it may be, it is given in its very haziness thus and not otherwise. The objective world, on the other hand, with which we reckon continually in our daily lives and which the natural sciences attempt to crystallize by methods representing the consistent development of those criteria by which we experience reality in our natural everyday attitude — this *objective* world is of necessity *relative*; it can be represented by definite things (numbers or other symbols) only after a system of coordinates has been arbitrarily carried into the world. It seems to me that this pair of opposites, subjective-absolute and objective-relative, contains one of the most fundamental epistemological insights which can be gleaned from science. Whoever desires the absolute must take the subjectivity and egocentricity into the bargain; whoever feels drawn toward the objective faces the problem of relativity. This thought is vividly and beautifully developed in the introduction of Born's book on relativity theory, quoted earlier.

Within the natural sciences the conflicting philosophies of idealism and realism signify principles of method which do not contradict each other. We construct through science an objective world which, in

order to explain the sense data, must satisfy the following fundamental principle that was already mentioned on p. 26: *A difference in the perceptions offering themselves to us is always founded on a difference in the real conditions* (Helmholtz). Lambert, in his *Photometria* (1760), enunciates as an axiom the following special case: "An appearance is the same whenever the same eye is affected in the same way." Here the natural sciences proceed realistically.

[For as long as I do not go beyond what is given, or more exactly, what is given at the moment, there is no need for the substructure of an objective world. Even if I include memory and in principle acknowledge it as valid testimony, if I furthermore accept as data the contents of the consciousness of others on equal terms with my own, thus opening myself to the mystery of intersubjective communication, I would still not have to proceed as we actually do, but might ask instead for the 'transformations' which mediate between the images of the several consciousnesses. Such a presentation would fit in with Leibniz's monadology. Instead of constructing the perspective view which a given body offers from a given point of observation, or conversely constructing the body from several perspective images, as it is done in photogrammetry, we might eliminate the body and formulate the problem directly as follows: let *A*, *B*, *C* each represent a consciousness bound to a point body, and let *K* be a solid contained in their field of vision. The task is to describe the lawful geometrical connections between the three images which each one of the three persons *A*, *B*, *C* receives of *K* and of the locations of the other two persons. This procedure would be more unwieldy; in fact, it would be bound to fail on account of the limitations and gaps in any single consciousness as compared to the complete real world. At any rate, there can be no doubt that in this respect science proceeds in tune with a realistic attitude.]

On the other hand science concedes to idealism that its objective reality is not given but to be constructed (*nicht gegeben, sondern aufgegeben*), and that it cannot be constructed absolutely but only in relation to an arbitrarily assumed coordinate system and in mere symbols. Above all the central thought of idealism comes into its own in the converse of the above fundamental principle: *the objective image of the world may not admit of any diversities which cannot manifest themselves in some diversity of perceptions*; an existence which as a matter of principle is entirely inaccessible to perception is not admitted. Leibniz says concerning the fiction of absolute motion (Leibniz's fifth letter to Clarke, §52): "I reply that motion is indeed independent of

actual observation, but not of the possibility of observation altogether. Motion exists only where a change accessible to observation takes place. If this change is not ascertainable by any observation then it does not exist." To be sure, many physically different colors will produce the same sensation of red; but if one sends all these various reds through the same prism, then the physical differences will manifest themselves in the perceptible differences between the streaks of colored light emerging from the prism. The prism, so to speak, unfolds the hidden differences to our senses. A difference which can in no way be broken down for our perception is non-existent. This is of great importance as a methodical principle of theoretical construction.

{The formula customarily given (Schwarzschild formula) for the metrical field surrounding a mass, such as the sun, can be interpreted as follows, if the coordinates occurring in it are taken to stand

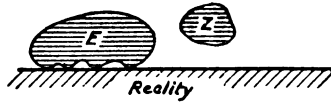


Figure 4. Schematic representation of a theory with a redundant part *Z*.

for a mapping of the real space into a Euclidean one: '(I) In reality Euclidean geometry holds. But the spherically symmetric field of gravity surrounding the mass center *O* acts upon rigid bodies in such a fashion that a radially directed rod at *P* is foreshortened in the ratio $\sqrt{1 - 2\alpha/r}:1$ (where $r = \overline{OP}$ and α is a constant determined by the mass), while a rod perpendicular to *OP* remains unchanged.' Rods after all are known to change their length with changing temperatures, why should they not react in a similar way to a gravitational field? But in making use of a certain other coordinate system one arrives at the following description: '(II) In reality Euclidean geometry holds. But the rod at *P*, no matter what its direction, will be changed in the ratio $(1 + \alpha/2r)^2:1$ by the field of gravitation.' Both descriptions express the same factual situation. To every possible coordinate system there corresponds a corrective prescription salvaging Euclidean geometry. Yet one is as good as the other. Each introduces into the factual state of affairs an arbitrary element which has no perceptually confirmable consequences and which therefore must be eliminated. And it can be eliminated by employing, with Einstein, none but the physical geometry as it is defined by the direct comparison of measuring rods. (That geometry is, of course, not the Euclidean one.) Each of the two theories can, if properly formulated, be split into two parts:

the theory (E) of Einstein, and an addition (Z) which is neither connected with (E) nor touching on reality and which must therefore be shed (compare the schematic diagram in Figure 4).

In Bohr's model of the hydrogen atom the period of the emitted light has nothing to do with the time in which the electron completes a revolution around the nucleus. Though it explains the spectrum as satisfactorily as could be wished, this lack of observable data corresponding to the period of revolution of the electron is felt as a disturbing feature which ought to be removed. In order to clarify the idea of relativity, Poincaré once set up the fiction that overnight, while all consciousness was asleep, the world with all its bodies in it, including my own, had been magnified in a definite ratio; awake again, neither I nor anyone else would notice the change in any way. In the face of such an event science makes common cause with the idealist; for what on earth could be meant under such circumstances by the statement that the world was magnified? A difference may be posited only where the assumption of equality would conflict with the principle that equals under equal conditions (especially equal objective characteristics, location, and motion, of the observer) will be perceived as equal³ (and with the principle of causality).

Between the real world and the given there is a correspondence, a mapping in the mathematical sense. Yet, while on one side there is the one, quantitatively determined, objective world, we have on the other not only what is actually given at the moment but also the *possible* perceptions (perhaps remembered, or expected in response to definite intentions of will) of an ego; and further there enter into this correspondence, besides the unique objective state of the world, the *possible* objective states of this perceiving ego (world line of its body, etc.). Helmholtz sets up the principle of the "empiristic view" (*Physiologische Optik*, III, p. 433): "The sensations are signs to our consciousness, and it is the task of our intelligence to learn to understand their meaning." In this one may agree with Helmholtz as he means it and yet be of the opinion with Husserl that the spatial object which I see, notwithstanding all its transcendency, is perceived as such bodily in its concreteness (Husserl, *Ideen zu einer reinen Phänomenologie*, in his *Jahrbuch für Philosophie*, Vol. 1, 1913, pp. 75, 79); for within the concrete unity of perception the data of sensation are animated by 'interpretations,' and only in union with them do they perform the 'function of representation' and help to constitute what we call 'the appearance of' color, shape, etc. A dog approaching

³ This principle cannot be taken as a definition of objective equality but only as an implicit requirement, because the concept of equality occurs in it twice: *equals* under *equal* conditions. . . .

another dog will see and smell 'a fellow dog,' an integrated whole that is more than a 'bundle of sensations.' We merely describe here one of the levels through which the process of constitution of the external world will pass. And there is no denying that the definite manner in which a thing is bodily put before me by means of those animating functions will be directed by a multitude of previous experiences. For how else should we describe this than by saying that "we always imagine such objects to be present in our field of vision as must be present in order to bring about the given impression under ordinary normal conditions for the use of our eyes" (Helmholtz, *Physiologische Optik*, III, p. 4). Helmholtz speaks here of "unconscious inferences." This sounds somewhat questionable; yet he stresses explicitly that only in their result do they resemble inferences, more accurately inferences by analogy, although the underlying psychic acts probably are quite different from acts of conscious inference and although their effects cannot be annulled by better knowledge. The sense impression of a mirror image, or of a broken rod immersed in water, or of the rainbow, does not deceive; only the bodily object which, as Husserl says, is put before me by this impression is an error. What truly exists can be ascertained only by taking into consideration all sensuous signs, which in the examples adduced above will soon reveal the prevalence of 'abnormal' conditions. Only imagine our eyes to be sensitive to light whose wave length is of the order of magnitude of the atomic distances in solids; how difficult it would then become for us to interpret the optical 'signs' (the Laue interference patterns)! In the ultimate description of the connection between appearance and reality one therefore does better to ignore all intermediary levels of constitution.]

And what significance does this objective world, representable only in symbols, have for the everyday life of man, taking place as it does in the sphere of integrated data of perceptions? Helmholtz answers (*op. cit.*, p. 18), "Once we have learned to read those symbols correctly we shall be able with their help to design our actions so that they yield the desired result, namely, that the expected new sensations will arise. A different comparison between conceptions and things not only does not exist in reality — all schools agree on this point — but a different manner of comparison is inconceivable and would be devoid of meaning. . . . Thus such a presentation (*Vorstellung*) of a single individual body is indeed already a concept (*Begriff*) which comprises an infinite number of intuitions in temporal sequence all of which can be derived from it.⁴ The presentation of a single

⁴ In agreement with a number of philosophers writing in English the term *presentation* has been chosen here as the equivalent of Kant's and Helmholtz's *Vorstellung* and Locke's *idea*. [Translator's note.]

individual table that I carry within me is correct and accurate if I am able to derive from it correctly and accurately the sensations I shall experience when I bring my eyes or my hands into this or that definite position with respect to the table. What other kind of similarity there can subsist between such a presentation and the object represented by it I cannot comprehend" (*op. cit.*, p. 26). In the same sense, Leibniz remarks concerning the Cartesian principles (*Philosophische Schriften*, IV, p. 356), "Of the sense data we cannot know more, nor do we have to require more, than that they are in agreement with each other as well as with the indisputable dictates of reason and that thus to a certain extent the future may be predicted from the past. To search for a truth or reality other than thus vouched for would be futile—the sceptic may not demand, the dogmatist not promise more." Or Husserl (*Ideen*, p. 311), "To the essence of a thing-noema there belong ideal possibilities of unlimited development of concordant intuitions that follow, moreover, prescribed directions of determinate type." But in the erection of empirical reality discrepancies will occur which will force us to make "corrections." Owing to its empirical character cognition of reality must of necessity pass through errors. "What is given never implies material existence as certain and necessary but merely as presumptive reality. This means that it can always happen that the further course of experience will force one to abandon what with good empirical justification had earlier been posited." (Husserl, *Ideen*, p. 86.) It might well be within the range of possibility that in the moving picture of perceptions every beginning of concordance would irreparably "explode." In that case the attempt to harmonize them according to principles of reason would fail, and no real world would be constituted.

The requirements which emerge from our discussion for a correct theory of the course of the world may be formulated as follows:

1. *Concordance.* The definite value which a quantity occurring in the theory assumes in a certain individual case will be determined from the empirical data on the basis of the theoretically posited connections. *Every such determination has to yield the same result.* Thus all determinations of the electronic charge e , that follow from observations in combination with the laws established by physical theory, lead to the same value of e (within the accuracy of the observations). Not infrequently a (relatively) direct observation of the quantity in question (for instance, of the location of a comet among the stars at a certain moment) is compared with a computation on the basis of other observations (for instance, the location at the desired moment computed according to Newton's theory from the locations on previous

days). The demand of concordance implies consistency,⁵ yet transcends the latter in that it brings the theory in contact with experience.

2. It must in principle always be possible to determine on the basis of observational data the definite value which a quantity occurring in the theory will have in a given individual case. This expresses the postulate that the theory, in its explanation of the phenomena, must not contain redundant parts.

Hume attempted to uphold with inexorable consistency the viewpoint that the given is the whole of reality. Since it became apparent through him that this viewpoint fails completely in the explanation of those cognitive positions which play a basic role in everyday life and in science, he was indeed the first to reveal the problem of reality in its full difficulty. Reason in its function of constituting reality is described by him as the faculty of imagination. With complete sincerity he confesses the irreconcilable conflict between thought and life, into which he finds himself thrown. To carry his approach through is as impossible as to found arithmetic on nothing but the concretely existing numerals. The positivism of a Mach or Avenarius appears to me merely as a less consistent renewal of Hume's attempt; for in their systems theoretical hypostases, strictly avoided by Hume, play once more a considerable role. But then we are back in the midst of theoretical construction, which supplements the given in the interest of totality, and we are no longer forced to use sense data as our building material. Kant's transcendental idealism reestablished the insights already gained by Leibniz. The content of this Section may be considered as an elucidation of Kant's concept of reality as "that which is connected with perception according to laws." He advances beyond Leibniz in transmuting the old metaphysical ontological concepts of substance and causality into methodical principles for the construction of empirical reality.

{In the part on logic we had insisted that existence could not be stated about something exhibited, that the logical symbol Σ_x carries an index x which refers to a blank. This seems to be contradicted by a proposition such as 'this chair is real.' But the assertion of real existence contains either, idealistically interpreted, the prediction of a multitude of concordant impressions expected in response to certain intentions of will, or, realistically interpreted, the statement that a thing x exists which stands in a certain metaphysical relation to the given chair phenomenon.

⁵ Indeed in an inconsistent theory the formula $e = 2e$ would be deducible, and hence the actual value e as well as $2e$ for the electronic charge could be derived from such a theory in combination with the observational data.

Concerning the problem of realism versus idealism we find a striking analogue in geometry, which has a factual connection with it in so far as in the objective world the coordinate system is, as it were, the residue of the annihilation of the ego. As in Section 12, we consider vectors \vec{x} in a plane and represent them in terms of a basis consisting of two linearly independent vectors \vec{e}_1, \vec{e}_2 , thus: $\vec{x} = \xi_1 \vec{e}_1 + \xi_2 \vec{e}_2$ (compare p. 69). The numbers ξ_1, ξ_2 , which are uniquely determined by \vec{x} and \vec{e}_1, \vec{e}_2 , are called the coordinates of \vec{x} with respect to the basis (\vec{e}_1, \vec{e}_2) . We construe the vectors in our plane as analogues of the objects in the real world, bases as analogues of real observers, the numbers as analogues of subjective phenomena, and thus speak in our analogy of the pair of coordinates (ξ_1, ξ_2) as the 'appearance of the object \vec{x} for the observer (\vec{e}_1, \vec{e}_2) .' For the geometric vector plane we can construct an algebraic model, defining a vector \vec{x} as a pair of numbers (x_1, x_2) and the operations of adding two vectors \vec{x}, \vec{y} and of multiplying a vector \vec{x} by a number α as follows:

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad \alpha(x_1, x_2) = (\alpha x_1, \alpha x_2).$$

Calling the numbers x_1, x_2 the absolute coordinates of $\vec{x} = (x_1, x_2)$ and $\vec{i}_1 = (1, 0), \vec{i}_2 = (0, 1)$ the absolute basis in our model, we realize at once that the absolute coordinates of a vector are its relative coordinates with respect to the absolute basis. Transition from the geometric vector space, in which all bases are equally admissible, to its algebraic model is effected by assigning, as it were, to an arbitrarily chosen basis the role of absolute basis. On the other hand, the individual character of the various bases in the model can be extinguished and all bases put on an equal footing by ascribing objective significance only to such properties and relations as are definable in terms of the two fundamental operations, or, what is the same, as are invariant under arbitrary linear coordinate transformations. The relation of a vector to its absolute coordinates is not objective, but is a special case of the objective relation prevailing between a vector \vec{x} , a basis (\vec{e}_1, \vec{e}_2) , and the coordinates ξ_1, ξ_2 of \vec{x} relative to (\vec{e}_1, \vec{e}_2) . Our analogy assumes that only the realm of numbers (the appearances) but not the geometrical space (the things themselves) is open to our intuition. Hence the model is the world of my phenomena and the absolute basis is that distinguished observer 'I' who claims that all phenomena are as they appear to him: on this level, object, observer and appearance all belong to the same world of phenomena, linked however by rela-

tions among which we can distinguish the 'objective' or invariant ones. Real observer and real object, I, thou, and the external world arise, so to speak, in unison and correlation with one another by subjecting the sphere of 'algebraic appearances' to the viewpoint of invariance. On this issue our theory bears out Leibniz (compare, for instance, *Nouveaux Essais*, Livre IV, Chap. 11) as opposed to Descartes, who through his "cogito ergo sum" assigns to the reality of the ego a precedence in principle over the reality of the external world. The analogy renders the fact readily intelligible that the unique 'I' of pure consciousness, the source of meaning, appears under the viewpoint of objectivity as but a single subject among many of its kind.

Yet in truth, the absolute subject, I, remains forever unique, notwithstanding the *objective* equivalence of the various subjects. This is in agreement with the facts as I find them. On purely cognitive grounds consciencism is irrefutable, it can be carried through completely. But for all this the recognition of the 'thou' is demanded of me not only in the sense that in my thinking I yield to the abstract norm of 'objectivity,' but in an absolute sense: Thou art for thyself once more what I am for myself, conscious-existing carrier of the world of phenomena. This step can be taken in our analogy only if we pass from the algebraic model of affine vector geometry to its axiomatic description, where the concepts of a vector and of the two fundamental operations enter as undefined terms. In the axiomatic system it is no longer necessary to enforce the equivalence of all coordinates by abstraction, for in it a definite pair of vectors (\vec{e}_1, \vec{e}_2) can be distinguished only by an individual act of exhibition. Pattern and source of any such demonstrative act is the word 'I.' Thus axiomatics reveals itself once again (compare p. 66) as the method of a purified realism which posits a transcendental world but is content to recreate it in symbols.

The postulation of the ego, of the 'thou,' and of the external world is without influence upon the cognitive treatment of reality. It is a matter of metaphysics, not a judgment but an act of acknowledgment or belief (as Fichte emphasizes in his treatise *Über die Bestimmung des Menschen*). Yet this belief is the soul of all knowledge. From the metaphysico-realistic viewpoint, however, egohood remains an enigma. Leibniz (*Metaphysische Abhandlung, Philosophische Schriften*, IV, pp. 454-455) believed that he had resolved the conflict of human freedom and divine predestination by letting God (for sufficient reasons) assign existence to certain of the infinitely many possibilities, for instance to the beings Judas and Peter, whose substantial nature determines their entire fate. This solution may objectively be sufficient, but it is shattered by the desperate outcry of Judas,

“Why did *I* have to be Judas?” The impossibility of an objective formulation of this question is apparent. Therefore no answer in the form of an objective insight can ensue. Knowledge is incapable of harmonizing the luminous ego with the dark erring human being that is cast out into an individual fate.

Postulation of the external world does not guarantee that such a world will rise from the phenomena through the cognitive work of reason which attempts to create concordance. For this to take place it is necessary that the world be governed throughout by simple elementary laws. Thus the mere positing of the external world does not really explain what it was meant to explain, the question of the reality of the world mingles inseparably with the question of the reason for its lawful mathematical harmony. The latter clearly points in another direction of transcendency than that of a transcendental world; towards the origin rather than the product. Thus the ultimate answer lies beyond all knowledge, in God alone; emanating from him, the light of consciousness, its own origin hidden from it, grasps itself in self-penetration, divided and suspended between subject and object, between meaning and being.]

18. THE PROBLEM OF SPACE

[A. ORIGIN OF THE PRESENTATION OF SPACE. A detailed investigation into the psychological origin of the presentation of space was not undertaken until the 19th century. The sense regions which contribute above all to the constitution of space are the visual and tactile impressions. Bain added to these the sensations of motion and the muscular feelings.

A single eye sees the qualities spread out in a two-dimensional field of vision. The latter is two-dimensional because it is dissected by any one-dimensional curve which runs through it. It is a basic physiological fact that the place in the field of vision at which we localize a visual impression is determined by the portion of the retina that is stimulated. We have here a one-to-one continuous ‘mapping’ in the mathematical sense. The places in the field of vision are continuously connected in the same way as the places on the retina to which they correspond. J. Müller, the originator of the law of the specific sense energies, even says (*Zur vergleichenden Physiologie des Gesichtssinnes*, p. 54), “In any field of vision, the retina sees only itself in its spatial extension during a state of affection. It perceives itself as spatially dark when the eye is at rest and completely closed.” A great step forward is marked by Helmholtz’s *Physiologische Optik*, as he no longer speaks of identity but of correspondence. The same

remark clarifies the famous problem as to why things are seen upright (in the objective space, the image on the retina is turned through 180° as compared to the original). If we confront the 'objective' space on one side and my intuitive space on the other, and if we assume both to bear a Euclidean metrical structure, then the utmost in faithfulness that could be demanded of the correspondence between objective thing and its image given in my intuition is an isomorphic (or similar) mapping in the sense defined in Section 4. Such an isomorphism would mean that all geometrical characteristics of the thing, describable in terms of the metrical concepts of objective space, are reflected in geometrical characteristics of the image expressed in terms of the synonymous metrical concepts of intuitive space. But it is nonsensical to ask questions which would be meaningful only if the thing as well as its image were located in the same space.

The field of vision has indeed a metrical structure; the resting eye undoubtedly is capable of apprehending something like shape, which here appears as a quality of what is seen, and of distinguishing different shapes. Such shape, however, is similar neither to the thing seen nor to the objective image produced on the retina. (The deformation with respect to the configuration formed by the rays of vision is described by Helmholtz in *Physiologische Optik*, III, pp. 151–153.) A more detailed shaping and partial correction of this metrical structure is achieved through the movements of the eyes; if a shift in the direction of my glance has the effect of changing image I into image II (i.e., objectively, if the same portion of the retina stimulated prior to the movement by the visual impression I is stimulated after the movement by the visual impression II), then I and II are mutually congruent. Consequently, in the domain of ocular movements, shape is no longer a given quality but a concept obtained by abstraction from the relation of congruence (compare Section 2).

Lotze demands the existence of 'local signs' on the basis of his physiological principle "that only the qualities of sensation may be considered as directly perceptible and intellectually differentiable" (*Wagners Handwörterbuch der Physiologie*, III, Section 1, 1846, p. 183) and that from these qualities the mind has to build up the presentation of spatial extension. The local signs are sensations whose qualitative gradations form the basis of the different locations in the field of vision. He has attempted to characterize them more precisely as impulses to move the eyes so as to bring the place in question into the center of the field of vision, and the feelings in the ocular muscles that would accompany such a movement. But here that is taken as basic which itself calls for a basis. For the eye at rest has its continuously spread field of vision, independently of the ocular movements, which belong to the

next higher level in the constitution of space. An attempt by Wundt to stamp color gradations as local signs is better passed over in silence. Helmholtz, though accepting Lotze's thesis, admits that the local signs are *qualitates occultae*. In view of the indissoluble connection between color and extension in the field of vision, Lotze's thesis leads to the problem (cf. Poincaré, *La valeur de la science*, ed. Flammarion, p. 91) of explaining how the one sensation can split up into the two components of color and extension — how, in other words, it is possible that two sensations of the same red produced at two different points *P*, *Q* of the retina should have a close affinity which is absent in the case of a red at *P* and a green at *Q*. But if one understands well the punctiform character of a simple sensation, one will hardly be inclined to consider that which gives the red its extensiveness again as a continuously graded sensation, but will acknowledge with Kant and Fichte (*Bestimmung des Menschen*, ed. Medicus, III, p. 326): "I am originally not only sentient but also intuiting." There is something for me only inasmuch as a continuum of quality covers a (temporal or spatio-temporal) continuum of extension. This conceded, sensations as local signs become redundant.

How about the visual impressions of rest and motion? When I glance up and down I have the impression that the things in my field of vision stay at rest, although their images are produced at varying places of the retina. But this is true only when the ocular movement is produced voluntarily by the motor apparatus of the eye. Thus it is here not a question of the muscular feelings connected with the ocular movement but of the voluntary intentions. There exists an original impression of rest and motion (change). *A thing gives the impression of being at rest, if its retina image does not shift about and if at the same time no ocular movements are intended.* Between the displacements of the retina image and the voluntary intentions directing the ocular movements there exists a system of compensations which experience apparently has developed to considerable refinement. The task is simplified by a circumstance known as Listing's law. It states that the eye cannot be voluntarily rotated while fixing a definite point in the field of vision, but that any one such point determines (except for minute fluctuations) only one corresponding position of the eye. Thereby the three degrees of freedom of the eye ball are reduced to two. The possibility of turning freely the direction of sight in response to the will is made use of when certain changes in the field of vision are 'interpreted' as motions.

The immediate impression of rest may not be invoked as 'testimony of the senses' for a refutation of the physical theory of relativity. For we saw that the objective 'explanation' of this phe-

nomenon (which after all can only consist in exhibiting an objective difference where a difference exists among intuitive data) takes recourse only to the idea of relative motion of physical entities covering each other (displacement of the image on the retina) and to the dynamical concept of motion (voluntary intentions, which, through muscular forces, cause the eye ball to deviate from its natural movement as it is conditioned by the field of inertia and by the eye's imbeddedness into the human body). The same holds for the motory sensations of our body; they do not tell of 'absolute motion' but are invariably sensations of acceleration, indicating that the body or part of it is torn out of its natural inertial motion and registering the ensuing dynamical disturbances.

The optical perception of depth, as Wheatstone has shown strikingly by the stereoscope, is closely tied up with binocular vision. (In addition, sensations produced by the accommodation effort come into play.) The positions in the fields of vision of the two eyes are in one-to-one correspondence, with the effect that the images formed at corresponding places are seen as one. The stereoscopic apperception of depth depends on the disagreement between the two images, which results when the same color quality and in particular the same contours do not appear at corresponding places of the two fields of vision. The details form a matter of dispute between two rival theories, a "nativistic" theory, represented especially by Hering, and the "empiristic" theory of Helmholtz. The former places all responsibility on the sensations, maintains that the stimulation of corresponding points on the two retinas, e.g. of the two retinal foveas, produces a simple sensation, and ascribes to the places on the retina, in addition to local signs indicating direction, a depth value modifying the sensation. Helmholtz's theory, on the other hand, considers optical depth as the result of a constitutive process. Only the latter theory is easily reconcilable with the facts. Yet it must be added, in the sense of the nativistic theory, that with the dimension of depth something new and original emerges. With its help the material of the two preceding levels — the two-dimensional purely visual field and the field of ocular movement — serves to constitute the centered three-dimensional space in which the body of the ego finds its position, though still the distinguished position of the center. (On the two previous levels we evidently do not yet have such a body-ego.) In the case of the (involuntary or voluntary) 'reversal' of the perspective interpretation of a plane figure (compare, for instance, Helmholtz, *Physiologische Optik*, III, p. 239), the 'animating' or 'integrating' function which converts the figure in the field of vision into the appearance of an object hit by the visual ray in centered space is felt particularly

clearly. It is on this level, too, that the tie-up with the localization field of the sense of touch and of the movements of limbs occurs. The grasping for the seen object is constantly used as a control in the pertinent psychological experiments on vision. Husserl emphasizes that "all these facts, allegedly mere contingencies of spatial intuition that are alien to the 'true,' 'objective' space, reveal themselves, except for minor empirical particularities, as essential necessities" (*Ideen*, p. 315); and in this sense O. Becker has given a more detailed phenomenological description of the constitutive levels of spatiality.

By walking toward the indefinitely far horizon of the centered space and by the displacements connected therewith, by the feeling of the free possibility of bodily movement in response to voluntary intentions, the homogeneous space arises from the centered one. Only at this stage the body becomes an equal among other spatial objects, and we become capable of adopting in imagination someone else's standpoint. Only this space can be conceived as being one and the same for several subjects; it is the presupposition for the construction of the intersubjective world. And thus the ascertainment of the orientation of objects in it is capable of intersubjective control and correction.

As opposed to Aristotle, who held that space is an *αἰσθητὸν κοινόν*, Berkeley has taken the view that there are only distinct sense spaces. Stumpf (*op. cit.*, p. 287) objects to this by asking, "Are we to believe that also the duration of a tactile sensation and that of a visual sensation are heterogeneous contents?" Berkeley may be right in that the pre-spatial localization fields (of the first and second levels) are separate ones for the senses of touch and vision. But beginning with the third level it can only be a question of one space, which comprehends the sense data of touch as well as vision. Thus space becomes the connecting link between the various sense domains. Bain's association theory of space aims at bringing out this function of it. In more precise form, such a theory has been developed by Poincaré. He first distinguishes qualitative changes and motions by pointing out that the latter can be reversed by a movement of the ego-body, which betrays itself by voluntary intentions and accompanying kinesthetic sensations (*La valeur de la science*, Chap. IV, §§1-4). He then attempts to set up criteria for the coincidence of two points in space arrived at by different series of kinesthetic sensations and voluntary intentions; and finally he investigates the 'mapping' upon one another — usually interpreted as identification — of the spaces appertaining to different sense organs (for instance, to the two finger-tips, or to the visual sense of the left eye and the tactile sense of the right thumb). According to this view, the statement that the sense of vision, but not

that of touch, reaches into the distance merely brings out the fact that two places in the space of any sense organ must be coincident if they correspond to two coincident places in the space of a tactile organ; while to two non-coincident places in a tactile space there may correspond two coincident places in the visual space. J. S. Mill accepted Bain's view, except that his presentation of space is not made up of Bain's sensations and their associations but emerges from them by creative synthesis ('psychic chemistry'). All these theories ignore the undeniable data on the lowest levels of constitution that do not possess the character of sensations, such as juxtaposition in the pure field of vision. }]

B. THE ESSENCE OF SPACE. The penetration of the This (here-now) and the Thus is the general form of consciousness. A thing exists only in the indissoluble unity of intuition and sensation, through the superimposition of continuous extension and continuous quality. Phenomenologically it is impossible to go beyond this. If, metaphysically, with Plato, one lets the passive consciousness spring from the encounter of two 'motions,' one originating with the ego, the other with the object, then one will tend to relegate quality to the sphere of the object, extension to that of the ego (and not vice versa, since extension is the qualitatively undifferentiated field of free possibilities, while the concrete variety resides in the qualities). "Translucent penetrable space pervious to sight and thrust, that purest image of my knowledge (*Wissen*)," so Fichte says (*Werke*, ed. Medicus, III, p. 325), "is not seen but intuited, and in it my seeing itself is intuited. The light is not without but within me, and I myself am the light." But the manner in which this intuition as an integrating force penetrates the sense data and utilizes their material is largely conditioned by experience.

The fact that both constituents, extension and quality, are bound to each other is the root of Aristotle's thesis of the impossibility of empty space. Thus Hume interprets it (*Treatise*, Book I, Part II) (in which connection it must be remembered that the spatial — or, more exactly, the spatio-temporal — separation is a fact as immediately ascertainable as spatio-temporal contact). But only through a *metabasis eis allo genos* can this essential epistemological fact be turned about into an assertion concerning substantial-physical events, leading to such conclusions as Descartes drew (and which were ridiculed by Hume), namely, that the walls of a box would have to touch if the latter were pumped empty. Leibniz denies emptiness on grounds connected with the perfection of the world and the principle of sufficient reason. He explains the fact that space is bound to the sensuous qualities by

denoting space, together with time, as the order of the phenomena. Stumpf (*op. cit.*, pp. 15, 26) objects, "When we distinguish different orders, we have to acknowledge in each case a specific absolute content with respect to which the order takes place," and consequently he asserts that "space denotes, rather, that positive absolute content upon which order is based." He demands that positional relations between points in space must be founded in a 'position' of the individual points severally, and by adopting this logical principle of the self-insufficiency of relations (which he may have taken over from F. Brentano, *Zur Lehre von Raum und Zeit, Kant-Studien*, XXVI) he bars himself from an understanding of the relativity of position.

Since the mere Here is nothing by itself that might differ from any other Here, space is the *principium individuationis*. It makes the existence of numerically different things possible which are equal in every respect. That is why Kant contradistinguishes it as the *form of intuition* from "the matter of phenomena, i.e. that which corresponds to sensation." Here lies the root of the concepts of similarity and congruence. Leibniz infers from this the ideality of space and time; for they violate the principle of the identity of indiscernibles, which — along with Spinoza — he postulates as necessary in the domain of substances (namely as a consequence of the principle of sufficient reason).

The dual nature of reality accounts for the fact that we cannot design a theoretical image of being except upon the background of the *possible*. Thus the four-dimensional continuum of space and time is the field of the *a priori* existing possibilities of coincidences. That is why Leibniz calls the "abstract space the order of all positions assumed to be possible" and adds that "consequently it is something ideal" (Leibniz's fifth letter to Clarke, §104).

[If we state the distance of the earth from the sun in yards, this statement acquires a meaning verifiable through what is given only if a rigid ladder, on which a scale has been marked off by means of a movable yardstick, is placed with one end upon the earth and with the other against the sun. The physically clearest realization of a rigid body is the crystal. If coordinates are to have an immediately ascertainable meaning, we must imagine the whole world to be filled out by a crystal. Among the motions of the crystal lattice that carry it into itself (covering motions) we can distinguish the translations by their peculiar properties; the covering translations can be used (by actually carrying out the translative motions) to introduce number triples as coordinates for the atoms of the lattice, and these can then be employed as position marks in the entire space. But that ladder

joining the earth and the sun is non-existent, its mensuration by a rigid yard stick is not actually carried out. Similarly the 'coordinate crystal' fails to exist and the covering translations are not carried out. Indeed their ideality is essential, for their existence would produce real forces which would influence the course of world events. As to the structure discussed in Section 16, we may assert only the *possibility* of ascertaining it from events producible by the experimenter's free interference. The geometrical statements, therefore, are merely ideal determinations, which taken in individual isolation lack any meaning verifiable by what is given. Only here and there does the entire network of ideal determinations touch upon experienced reality, and at these points of contact it must 'check.' That, expressed in the most general terms, may well be called *the geometrical method*. "It must be admitted that he who undertakes to deal with questions of natural sciences without the help of geometry is attempting the unfeasible," Galileo says (*Dialogo, Opere*, VII, p. 229). Enemies of this method are, on the one hand, the empiricists, because any aprioristic construction is a thorn in their flesh; they fondly imagine it to be possible to grasp reality as a thing of one stratum, as it were, without aprioristic ingredients, by a purely descriptive approach (Bacon versus Galileo, Hume versus Kant, Mach versus Einstein). On the other hand, out of hatred for the freedom, the open field of geometrical construction, those metaphysicians oppose the method who build up a rigid dialectical world of concepts as the true reality (Hegel versus Newton). From both angles Aristotle (versus Archytas-Plato) is the great anti-mathematician.]

C. A PRIORI OR A POSTERIORI? The belief in the aprioristic character of geometrical cognition, in particular of Euclidean geometry, had taken deep roots in former times. Thus Kepler says (in his famous letter to Galileo, April 1610; Galileo, *Opere*, X, p. 338), "The science of space is unique and eternal and is reflected out of the spirit of God. That men may partake of it is one of the reasons why man is called the image of God." Leibniz has tried to show that the geometrical truths are analytic. With respect to geometry Kant raises the problem of the Critique of Pure Reason: How are synthetic judgments *a priori* possible? And he believes that he has answered this question for geometry by his thesis that space is pure non-empirical intuition. "That in which sensations are merely arranged, and by which they are susceptible of assuming a certain order, cannot itself be sensation; hence indeed the matter of all phenomena is given to us *a posteriori* only, while its form must lie ready *a priori* in the mind and therefore must be capable of investigation independently of all

sensation. . . . Hence our explanation alone renders comprehensible the possibility of geometry as synthetic knowledge *a priori*." This certainty is shaken by the development of non-Euclidean geometry.

[Proclus already, in his commentary on Euclid, sounded a warning in connection with the axiom of parallels not to make undue use of intuitive evidence. Gauss writes to Olbers (1817, *Werke*, VIII, p. 177), "I am coming more and more to the conviction that the necessity of our geometry cannot be demonstrated, at least neither by, nor for, the human intellect. Perhaps in some other life we may arrive at other insights into the nature of space that are at present inaccessible to us. Until such time geometry should be ranked, not with arithmetic, which is purely aprioristic, but with mechanics." Or, in 1830, to Bessel (*op. cit.*, p. 201), "We must admit humbly that, while the number is a product of our intellect alone, space has a reality beyond our mind whose rules we cannot completely prescribe."]

Helmholtz shows that the two parts of the Kantian doctrine of space, namely, (i) that space is pure form of intuition, and (ii) that the science of space, Euclidean geometry, holds *a priori*, are not so closely connected that (ii) follows from (i). He is willing to accept (i) as a correct expression of the state of affairs; but nothing can be inferred from that, according to him, beyond the fact that all things of the external world have spatial extension. In accord with Riemann he points out the empirical physical content of geometry and refers to Newton, who in the introduction to *Principia* had declared, "Therefore geometry is founded in mechanical practice, and is nothing but that part of universal mechanics which accurately proposes and demonstrates the art of measuring." If there were, aside from the "physical equivalence" of spatial quantities (cf. p. 103), an equality given by immediate transcendental intuition, then the agreement of the two concepts could after all be only a matter of experience, while in the case of conflict the transcendental equality "would be degraded to the level of a sense illusion, i.e. an objectively false semblance" (Helmholtz, *Wissenschaftliche Abhandlungen*, II, p. 654). Against the argument that non-Euclidean geometry is devoid of intuitivity (*Anschaulichkeit*), he sets up a definition of intuitivity. The latter consists, he says, in "the complete imaginability of those sense impressions which the object would produce in us according to the known laws of our sense organs under any conceivable observational conditions and by which it would differ from other similar objects." We may refer to the description given in Section 17 of the relation between the objective world and its subjective image as conceived by

the point eye moving along a world line. Against the argument that an attempted experimental test of geometry always involves physical statements about the behavior of rigid bodies and light rays it may be pointed out that the individual laws of physics no more than those of geometry admit of an experiential check if each is considered by itself, but that a constructive theory can only be put to the test as a whole.

Under the influence of modern mathematical axiomatic investigations one has come to distinguish the 'mathematical space,' whose laws are logical consequences of arbitrarily assumed axioms, from the 'physical space,' the ordering scheme of the real things, which enters as an integral component into the theoretical construction of the world. With regard to this distinction Einstein says (*Geometrie und Erfahrung*, p. 3), "As far as the propositions of mathematics refer to reality they are not certain, and in so far as they are certain they do not refer to reality." The general philosophical development, on the other hand, has since taken a course that led to a split of Kant's judgments *a priori* into two directions. On the one hand, there are the non-empirical laws (*Wesensgesetze*), which express the manner in which data and strata of consciousness are founded upon each other, but do not claim to involve statements of fact; this line of pursuit culminated in Husserl's phenomenology, in which the *a priori* is much richer than in the Kantian system. On the other hand, principles of theoretical construction are formulated, which according to the most extreme point of view (Poincaré) rest on pure convention.

After what has been said in Part I we need not enter here into a detailed discussion of the general mathematics of continua and of the more important structures with which they can be endowed. In the case of physical space it is possible to counterdistinguish aprioristic and aposterioristic features in a certain objective sense without, like Kant, referring to their cognitive source or their cognitive character. In fact, according to the Riemann-Einstein view, we may contrast the one absolutely given Euclidean-Pythagorean *nature* of the metric, which does not participate in the irradicable vagueness of that which occupies a variable place on a continuous scale, with the mutual *orientation* of the metrics in the various points, i.e. the quantitative course of the metrical field; the latter is accidental, dependent on the distribution of matter, ever-changing, and ascertainable only approximately and with the help of immediate intuitive reference to reality. Thus the general theory of relativity does not altogether deny that there is in this sense something aprioristic to the structure of the extensive medium of the external world, but the line between *a priori* and *a posteriori* is drawn at a different place. (To be exact, this

juxtaposition, or separation, must be understood as meaning — as always in cases of this kind — that the aprioristic factor can be isolated from the whole without thereby exhausting the latter; there is no residue of purely *a posteriori* character, however, that would be left after the first part has been 'subtracted' from the whole.) Among the aprioristic features of the world, beside and above the one nature of the metrical field, there is the topological connectivity, which is fixed once and for all, especially the dimension number 4. The quantitative course of the metrical field obeys exact natural laws, namely, the Einstein laws of gravitation, which resemble the Maxwell laws of the electromagnetic field. Within the *a posteriori* one has thus to make yet another distinction, between what is necessitated by natural law and what even under their rule remains free and thus appears as contingent. The binary gradation is replaced by a ternary one.

[In addition to the physical space one may acknowledge the existence of a *space of intuition* and maintain that its metrical structure of necessity satisfies Euclidean geometry. This view does not contradict physics, in so far as physics adheres to the Euclidean quality of the infinitely small neighborhood of a point *O* (at which the ego happens to be at the moment). For the angles which are formed by the spatial directions of the light beams issuing from the various stars and striking the point eye do indeed fulfil the laws of spherical trigonometry in Euclidean space. But then it must be admitted that the relation of the intuitive to the physical space becomes the vaguer the farther one departs from the ego center. The intuitive space may be likened to a tangent plane touching a curved surface (the physical space) at a point *O*; in the immediate vicinity of *O* the two coincide, but the larger the distance from *O* the more arbitrary will the one-to-one correspondence between plane and surface become that one tries to establish by continuing the relation of coincidence near *O*. This does not mean that the intuitive space as such must necessarily be of a vague character. The intuitive space after all does not overcome the discrepancy created by binocular sight by vacillation or compromise (provided extreme circumstances, or attention directed toward the visual perceptions as such, do not cause a contest between the fields of vision to break out) but is intuitively of unobscured clarity, though in the objective construction the state of affairs can only be represented as a compromise.]

Regarding the aprioristic features of space the task arises to understand on rational grounds the peculiarities that give them their distinctive position within the range of the more general possibilities

revealed by formalized mathematics. Thus there are three different possibilities as to the nature of a four-dimensional Riemann manifold, according as its fundamental metrical form possesses 0, 1, or 2 negative dimensions. If the world corresponded to the case 0, no propagation of effects from a world point O would be possible, while in the case of 2, past and future would be melted into one world domain. Thus it can be argued that the middle case of 1 negative dimension is realized by the metrical field of the real world because of the necessity of a causal structure by virtue of which an ego may be actively and passively connected with the world in such a manner as to separate past from future, what is known from what is planned. Likewise it must be asked in connection with n -dimensional Euclidean or Riemannian geometry, which resulted by cogent formalization from the three-dimensional one (Section 12), what inner reasons there are for the distinction of the case $n = 3$ realized by the actual space. Aristotle gave several answers to this, which still move in the sphere of mythical thought. Galileo discusses and rejects them at the beginning of his "*Dialogo*." The solution which he himself proposes is merely a clearer formulation of the problem but is no answer. The best chances for success seem to me to lie in theoretical physical construction.⁶ Thus it can be shown by means of the wave equation of light (which can be immediately extended to n dimensions) that only in a space of an odd number of dimensions is the extinction of a candle followed by complete darkness about the candle (within a radius that increases as rapidly as light travels). This, at least, shows up an important inner difference regarding the propagation of effects between *even* and *odd* numbers of dimensions. Those particularly simple and harmonious laws which Maxwell had developed for the electromagnetic field in empty space are invariant with respect to an arbitrary change of the standard unit length at every world point, provided the world is four-dimensional. This principle of 'gauge invariance' holds for no other number of dimensions.

{The group structure of the Euclidean group of rotations (which still dominates the metrical nature of the world even if the Riemann-Einstein infinitesimal geometry is adopted) is decidedly different for the various numbers of dimensions. This circumstance suggests that the mathematical and physical laws may cease to be indifferent to the number of dimensions on some deeper level that has hardly been touched by the physics of today. There is thus good reason

⁶ One blushes at the thought of the naive geometrical blunders committed again and again in an attempt to solve this deep problem. A recent example of this can be found in Natorp's *Logische Grundlagen der exakten Wissenschaften*, pp. 303 ff.

to hope that our problem will one day find a cogent solution along such lines. An attempt to make the three-dimensionality of space comprehensible through its role in the constitution of the external world for the consciousness was made by Bolzano (*Abhandlungen der Böhmisches Gesellschaft der Wissenschaften*, 1843). A more recent attack by O. Becker in the same direction is less absurd, but still far from satisfactory.

A way to understand the Pythagorean nature of the metric (which finds its expression in the Euclidean group of rotations) exactly through the separation of *a priori* and *a posteriori* has been pointed out by the author. Only in the case of this particular group does the contingent quantitative distribution of the metrical field, however that distribution is chosen within the framework of its *a priori* fixed nature, uniquely determine the infinitesimal translation, the non-rotational progression from a point into the world. This assertion involves a rather deep group-theoretical theorem which was proved by the author. The space problem, thus solved, plays a similar part within the Riemann-Einstein theory as the Helmholtz-Lie problem (Section 14) plays for the rigid Euclidean space. It may be that the postulate of the unique determination of 'straight progression' can be justified on the basis of the requirements posed by the phenomenological constitution of space. Becker persists in attempting to base the significance for intuitive space of the Euclidean group of rotations upon Helmholtz's postulate of free mobility. If in agreement with a remark made in Section 15 the transformation group Δ_0 in 3 or 4 dimensions is considered as representation of an abstract group, then more emphasis should be placed on the distinctive features of the structure of this abstract group than on the special concrete representation Δ_0 .]

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