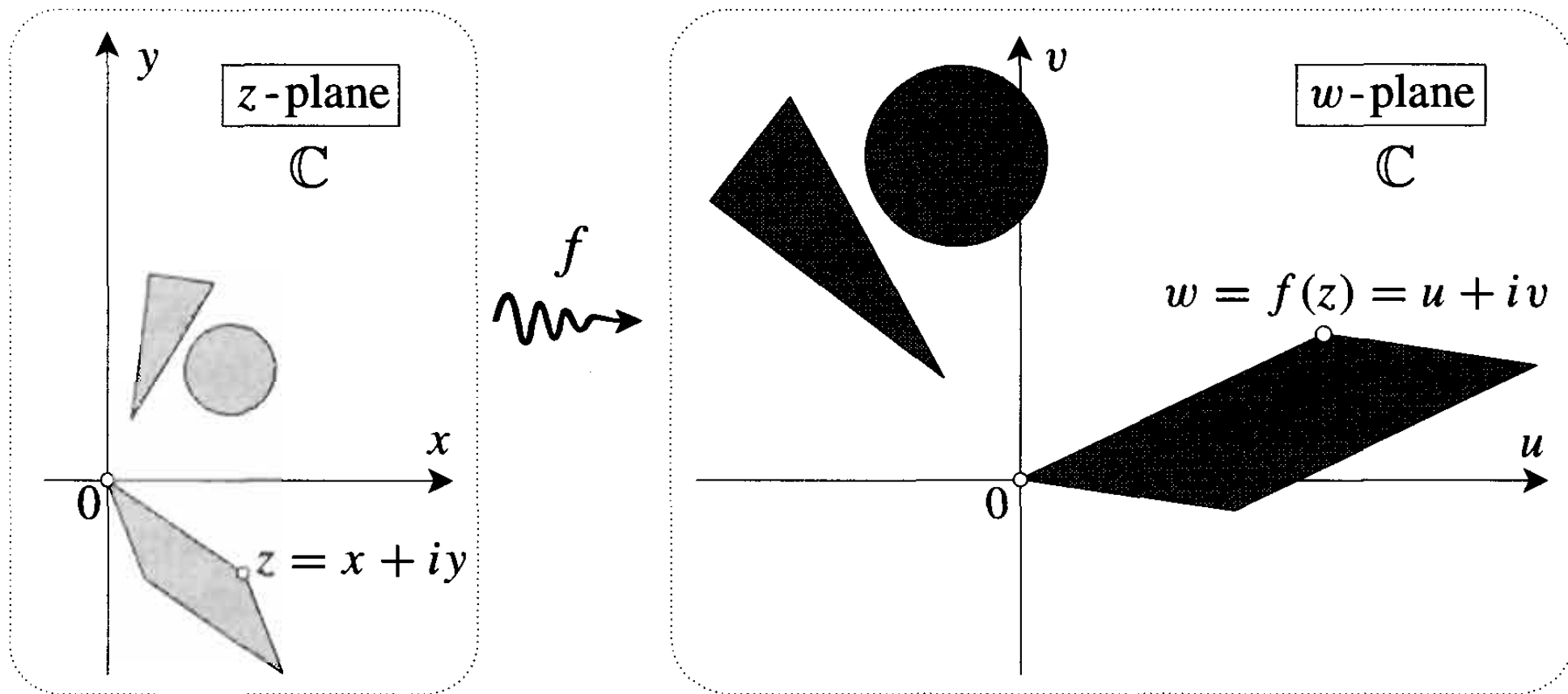
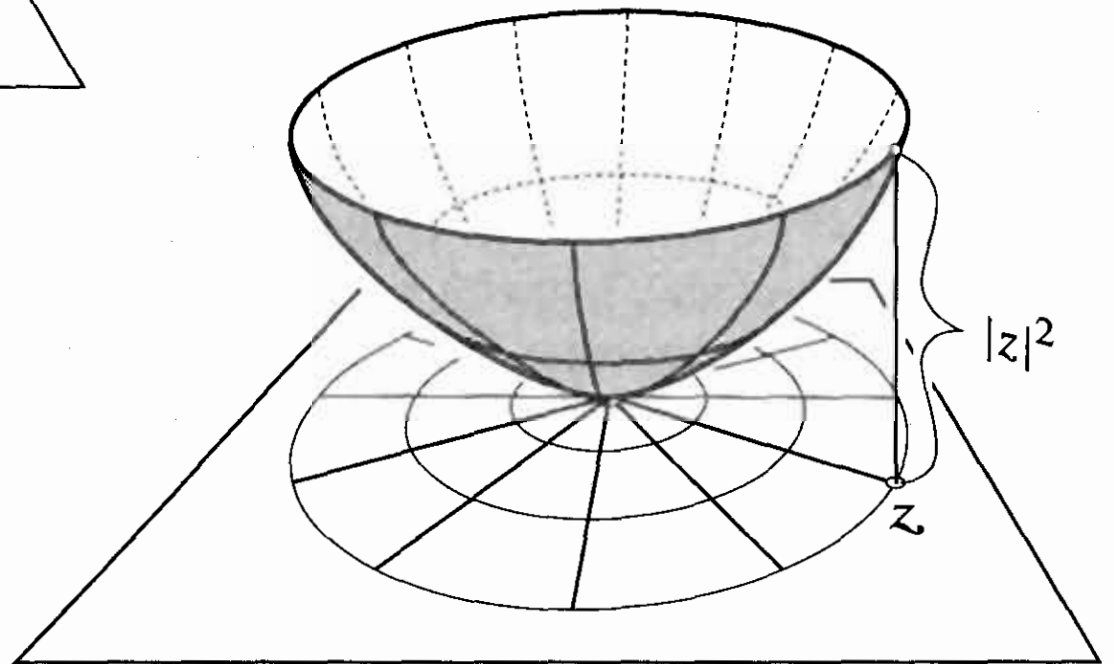
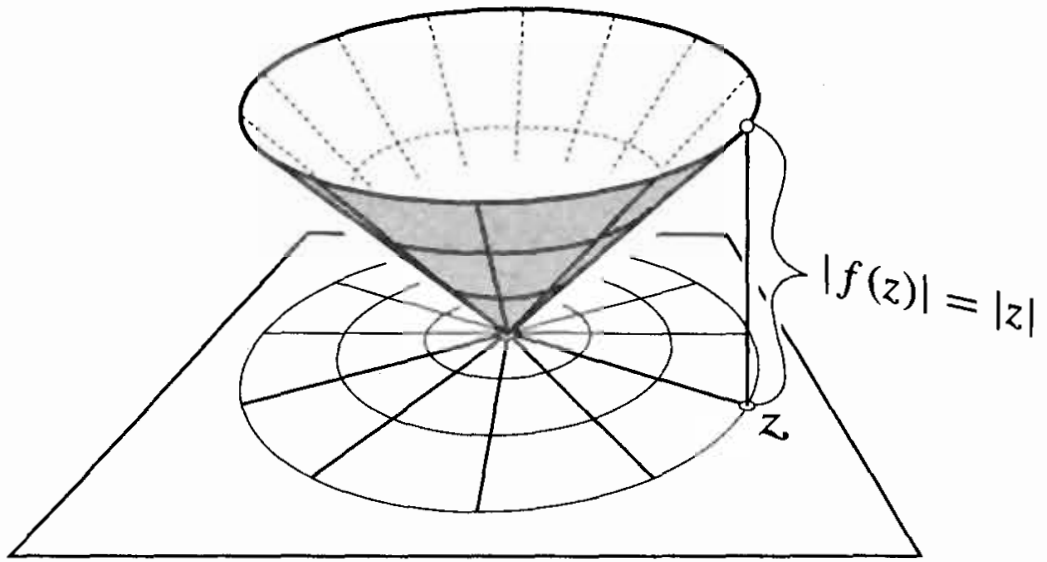


# Funzioni complesse come trasformazioni

## I. Introduzione

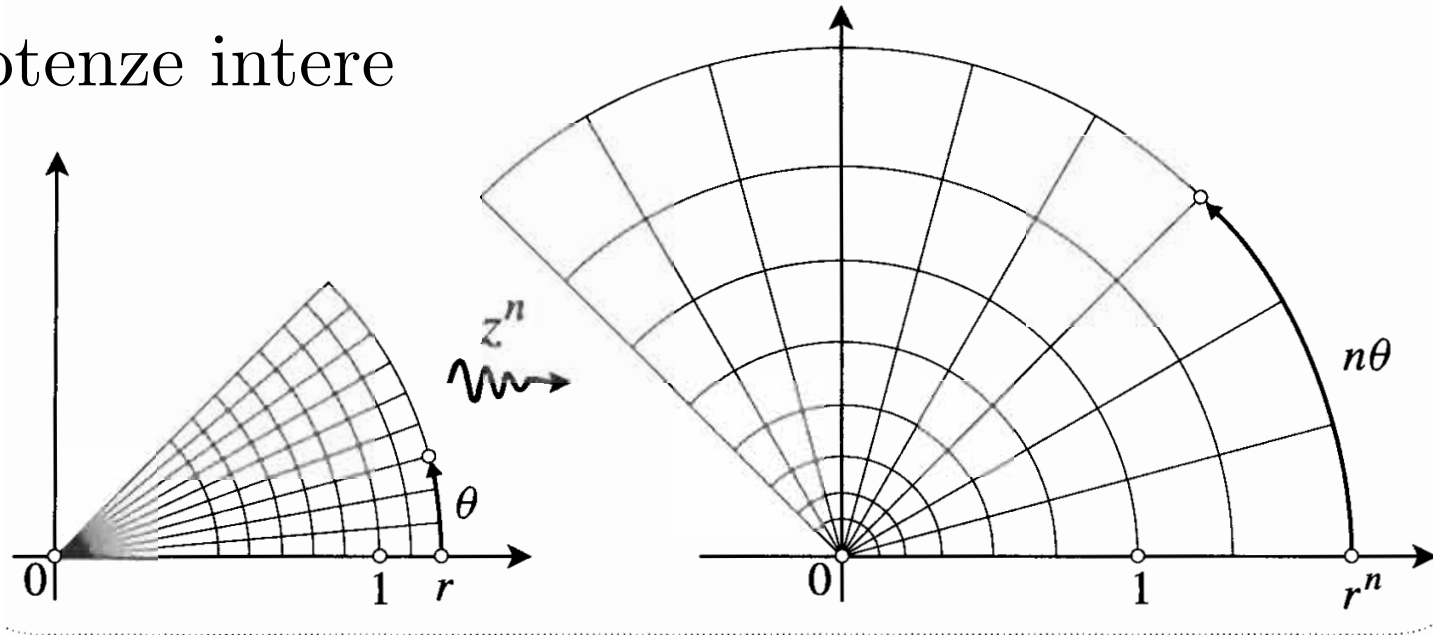


$$z \mapsto w = f(z) = (1 + i\sqrt{3})z$$

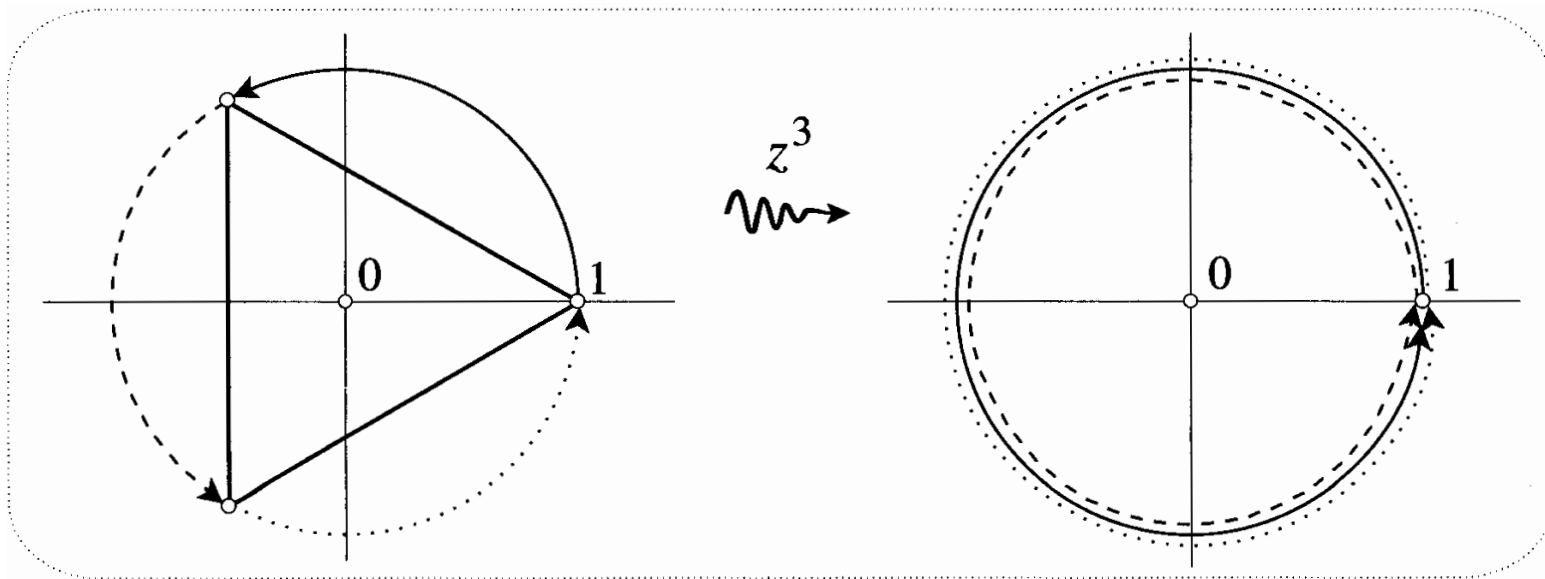


|

potenze intere

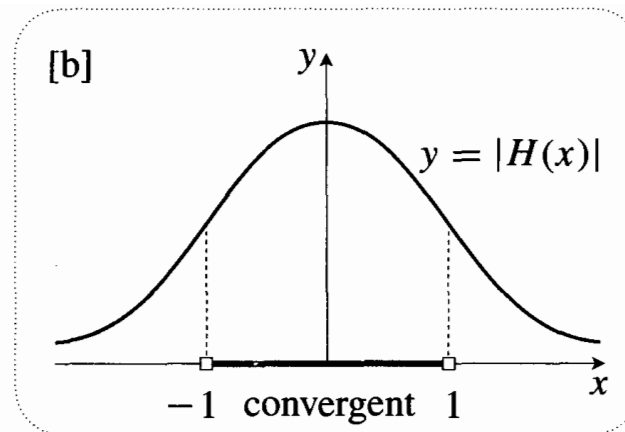
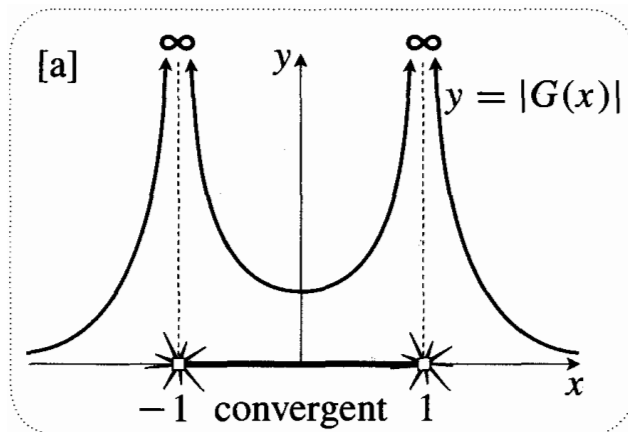


Soluzioni di  $z^n = 1$



# III. Serie di potenze

## Il mistero delle serie di potenze reali



$$G(x) = \frac{1}{1-x^2}$$

$$H(x) = \frac{1}{1+x^2}$$

$$\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j = 1 + x + x^2 + x^3 + \dots \quad \text{if and only if } -1 < x < 1,$$

$$\longrightarrow \quad G(x) = \sum_{j=0}^{\infty} x^{2j} \quad \text{and} \quad H(x) = \sum_{j=0}^{\infty} (-1)^j x^{2j}$$

$$\frac{1}{a-x} = \frac{1}{a-(X+k)} = \frac{1}{(a-k)} \frac{1}{\left[1 - \left(\frac{X}{a-k}\right)\right]},$$

$$\frac{1}{a-x} = \sum_{j=0}^{\infty} \frac{X^j}{(a-k)^{j+1}}, \quad \text{if and only if } |X| < |a-k|$$

$$\frac{1}{1-x^2} = \frac{1}{2} \left[ \frac{1}{1-x} - \frac{1}{-1-x} \right] = \frac{1}{2} \sum_{j=0}^{\infty} \left[ \frac{1}{(1-k)^{j+1}} - \frac{1}{(-1-k)^{j+1}} \right] X^j$$

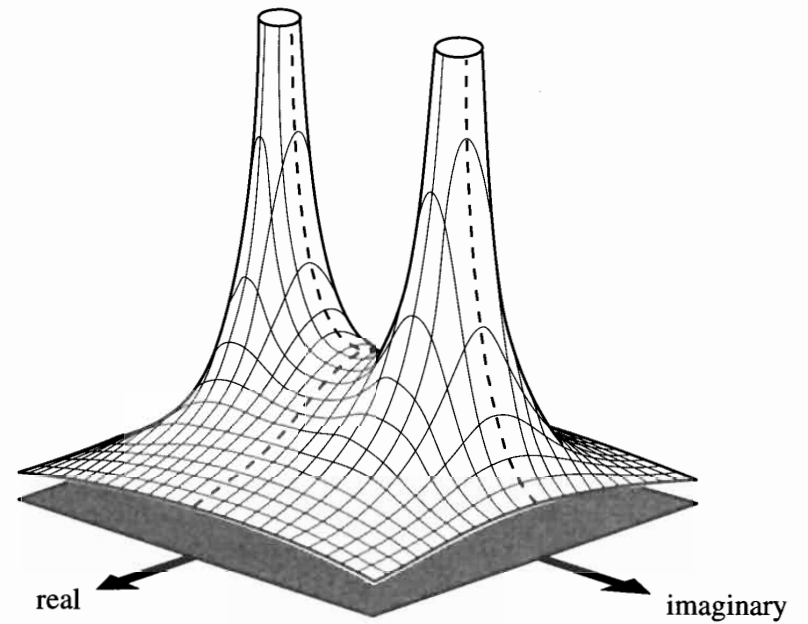
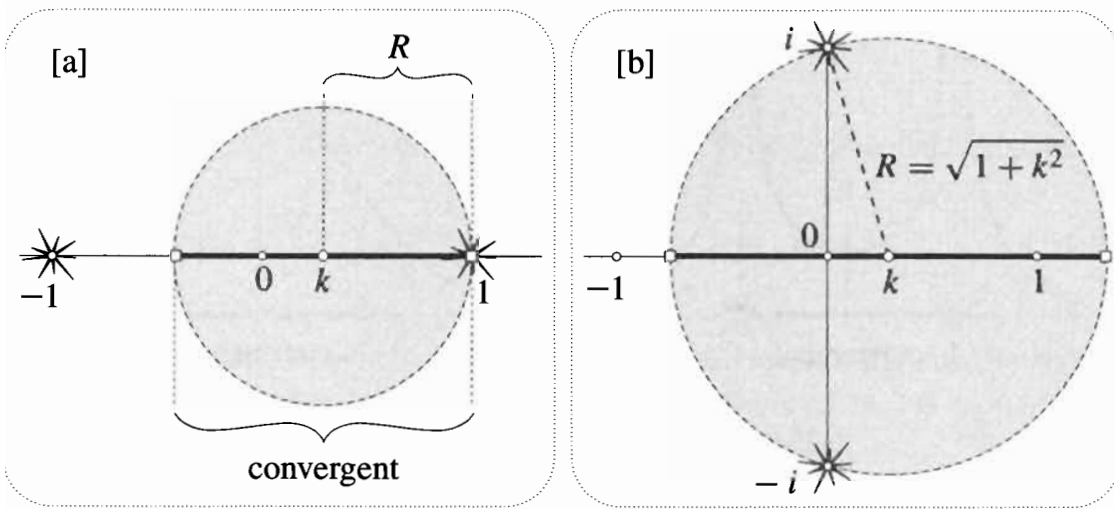
$R = \min \{|1-k|, |1+k|\}$  = (distance from  $k$  to the nearest singularity of  $G$ )

$$H(x) = \frac{1}{1+x^2}$$

$$R = \sqrt{1+k^2}.$$

$$G(x) = \frac{1}{1-x^2}$$

$$H(x) = \frac{1}{1+x^2}$$



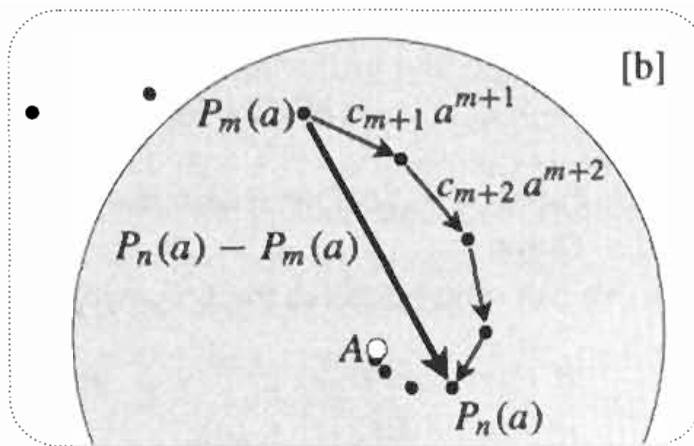
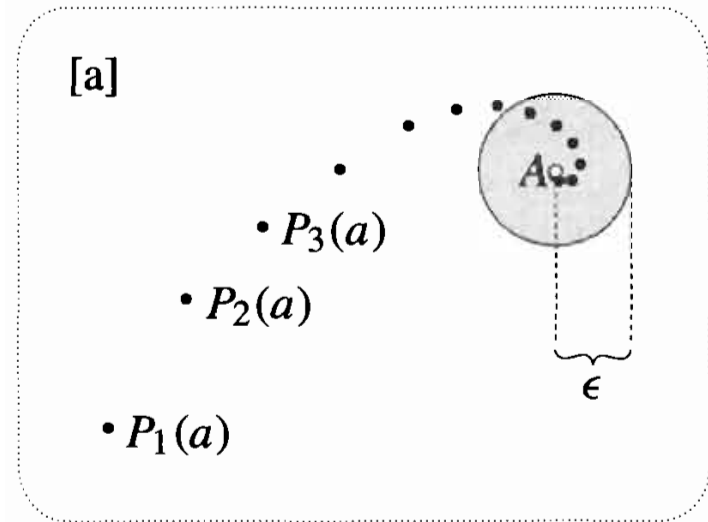
# Il disco di convergenza

$$P(z) = \sum_{j=0}^{\infty} c_j z^j = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \dots$$

$$P_n(z) = \sum_{j=0}^n c_j z^j = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \dots + c_n z^n$$

$\{P_n(z)\}$  converge ad  $A$  nel punto  $z = a$  sse

$$\forall \epsilon \exists N : |A - P_n(a)| < \epsilon, \forall n > N$$



$$|c_{m+1} a^{m+1} + c_{m+2} a^{m+2} + \dots + c_n a^n| = |P_n(a) - P_m(a)|$$

Criterio di Cauchy:  $P(a)$  converge sse  $\exists N : |P_n(a) - P_m(a)| < 2\epsilon, n, m > N$

$P(z)$  è detta **assolutamente convergente** per  $z = a$  se la serie reale  $\tilde{P}(z) \equiv \sum_{j=0}^{\infty} |c_j z^j| = |c_0| + |c_1 z| + |c_2 z^2| + |c_3 z^3| + \dots$  converge per  $z = a$

1. Se  $P(z)$  è assolutamente convergente in qualche punto, allora è anche convergente in quel punto.

$$\tilde{P}_n(a) - \tilde{P}_m(a) = |c_{m+1} a^{m+1}| + |c_{m+2} a^{m+2}| + \dots + |c_n a^n|$$

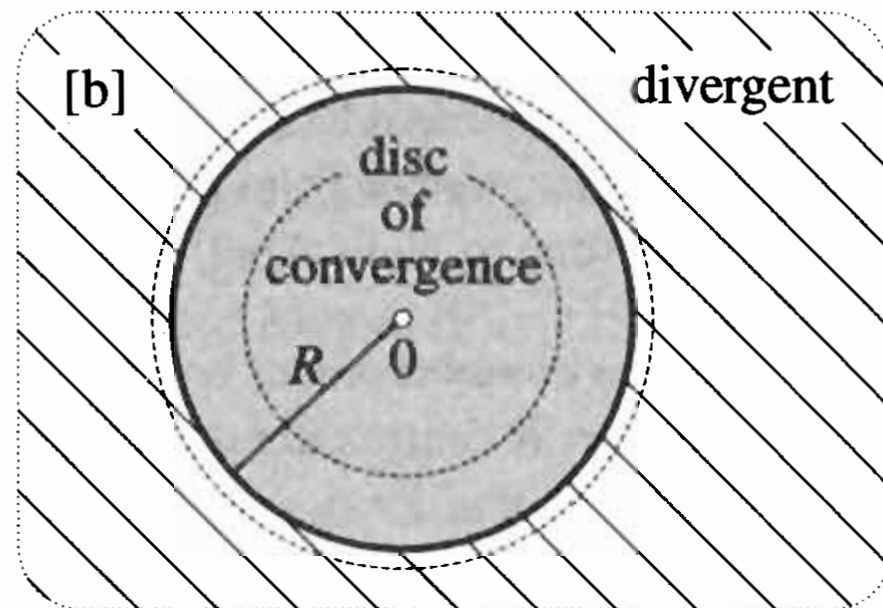
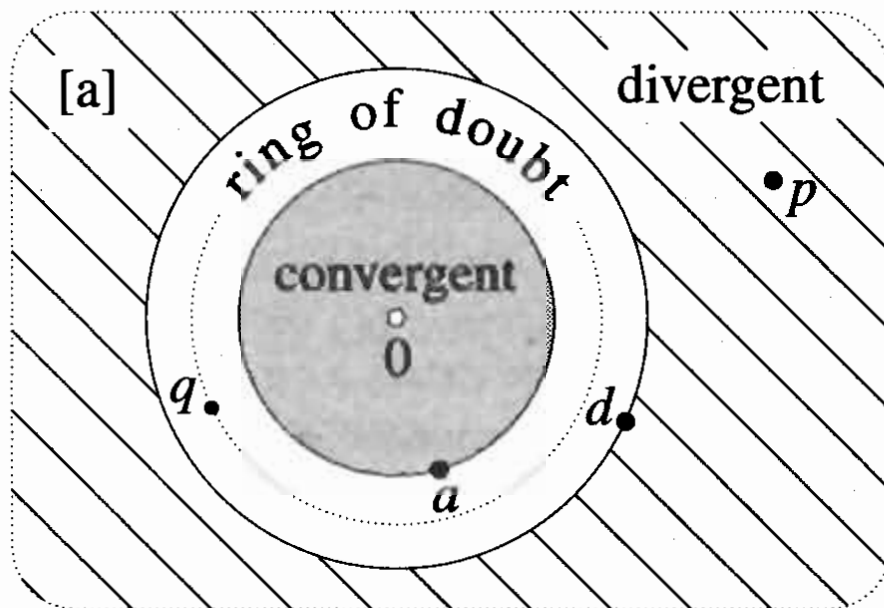
$$|P_n(a) - P_m(a)| \leq \tilde{P}_n(a) - \tilde{P}_m(a)$$

**2.** Se  $P(z)$  converge in  $z = a$ , allora converge (assolutamente) dappertutto nel disco  $|z| < a$ .

Se  $P(a)$  converge  $|c_n a^n| \rightarrow 0$ . Allora  $\exists M : |c_n a^n| < M, \forall n$  Se  $|z| < a$  allora  $\rho = |z|/|a| < 1$  e quindi  $|c_n z^n| < M \rho^n$ . Perciò

$$\tilde{P}_n(z) - \tilde{P}_m(z) \leq M(\rho^{m+1} + \rho^{m+2} + \dots + \rho^n) = \frac{M}{1-\rho} [\rho^{m+1} - \rho^{n+1}]$$

**3.** Se  $P(z)$  diverge in  $z = d$ , allora diverge dappertutto fuori dal cerchio  $|z| = d$ .



**4. Teorema di Abel.** *Data una serie di potenze complessa  $P(z)$  centrata in  $k$ , esiste un cerchio  $|z - k| = R$  centrato in  $k$  tale che  $P(z)$  converge dappertutto dentro al cerchio e  $P(z)$  diverge dappertutto fuori dal cerchio.*

## Approssimazione di una serie di potenze con un polinomio

**5.** *In ogni punto  $z$  del suo disco di convergenza,  $P(z)$  può essere approssimata con precisione arbitrariamente grande da un polinomio  $P_m(z)$  di grado sufficientemente elevato.*

errore  $E_m(z) = |P(z) - P_m(z)|$

**6.** *Se  $P(z)$  ha disco di convergenza  $|z| < R$ , allora  $P(z)$  è uniformemente convergente nel disco chiuso  $|z| \leq r$ , dove  $r < R$ .*



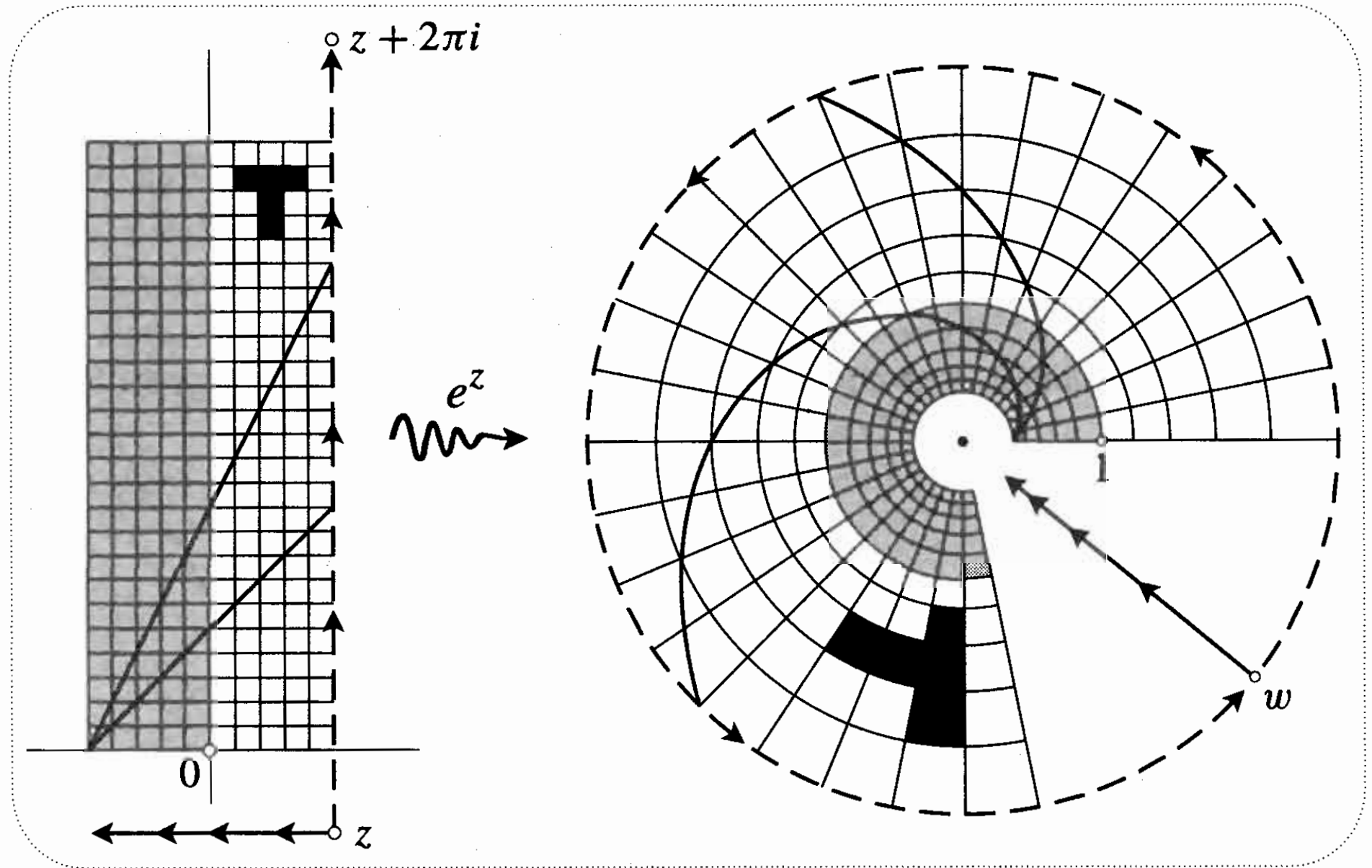
Trovare il raggio di convergenza

$$R = \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right|$$

$$R = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{|c_n|}}$$

$$R = \frac{1}{\limsup \sqrt[n]{|c_n|}}$$

# La funzione esponenziale



# Seni e coseni

