Seven Steps Towards the Classical World

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Abstract. Classical physics is about real objects, like apples falling from trees, whose motion is governed by Newtonian laws. In standard quantum mechanics only the wave function or the results of measurements exist, and to answer the question of how the classical world can be part of the quantum world is a rather formidable task. However, this is not the case for Bohmian mechanics, which, like classical mechanics, is a theory about real objects. In Bohmian terms, the problem of the classical limit becomes very simple: when do the Bohmian trajectories look Newtonian?

1 Introduction

The classical world, say the world of objects of familiar experience that obey Newtonian laws, seems far removed from the "wavy" world of quantum mechanics. In this paper we shall sketch what we believe are the basic steps to be taken in going from the quantum world to the classical world.

1. The first step is the crucial one: As is well known, and as Bell has emphasized [1], standard quantum mechanics is not a precise microscopic theory because the division between

the microscopic and the macroscopic world, which is essential to the very formulation of that theory, is not made precise by the theory [8]. In fact, the following conclusion seems inevitable: quantum mechanics does not contain the means for describing the classical world in any approximate sense and one needs to go beyond quantum mechanics in order to do so. There are two natural possibilities for amending ordinary quantum mechanics: either the wave function is not all there is, or Schrödinger's equation is wrong. In this paper we'll formulate the problem of the classical limit within the framework of Bohmian mechanics, a theory which follows the first path and in which the observer doesn't play any crucial role. It is a theory about reality, not about the result of measurements. A very short review of Bohmian mechanics is given in section 2 and the the relevant part of this theory, related to the classical limit, is discussed in section 4.

2. To get a handle on a problem, one should first simplify it as much as possible. The complex motion of a macroscopic body can be drastically simplified by making some rather standard approximations and reducing the problem to that of a "particle" moving in an external potential. This is what we shall do in section 3.

3. Good textbooks on quantum mechanics contain enlightening ideas. One of these ideas is the so called Ehrenfest theorem that we shall use in section 5 in order to obtain a necessary condition for the classicality for wave packets.

4. The structure of Bohmian mechanics contains the means for extending the condition for classicality to more general wave functions, namely wave functions which locally look like plane waves, as we shall see in section 6. In section 7 we shall then show how the problem of classical limit for general wave functions can be reduced to that for local plane waves.

5. Simplicity is good, but it has its limitations: the reduction of the motion of the center of mass to a one body problem doesn't explain the robustness and stability of classical behavior. This however can be explained by making the model a little more realistic, say by including in an effective way the external as well as the internal environment. We shall briefly touch this point in section 8.

6. This step is the crucial one from a mathematical point of view. In section 9 we shall put forward a mathematical conjecture on the emergence of classical behavior. Unfortunately we

cannot provide any rigorous mathematical justification for it. Mathematical work on it would be valuable since this conjecture goes beyond the standard mathematical work of semiclassical analysis (see, e.g., [13], [14]) or, in more modern terms, microlocal analysis (see, e.g., [12]).

7. This is the last step in what we believe is the main structure of the classical limit:

$$(\psi, X) \to (P, X)\,,$$

where on the two sides of the arrow are represented the complete state description of Bohmian mechanics, in terms of wave function and position, and of classical mechanics, in terms of momentum and position.

2 Bohmian Mechanics

In nonrelativistic Bohmian mechanics the world is described by particles which follow trajectories determined by a law of motion. The evolution of the positions of these particles is guided by the wave function which itself evolves according to Schrödinger's equation. In other words, in Bohmian mechanics the complete description of the state of an N-particle system is the pair (Ψ, Q), where $\Psi = \Psi(q) = \Psi(q_1, \ldots, q_N)$ and $Q = (Q_1, ..., Q_N)$ are respectively the wave function and the *actual* configuration of the system, with Q_k denoting the position of the k-th particle in ordinary three-dimensional space.

For non relativistic spinless particles the state (Ψ, Q) evolves according to the equations

$$\frac{dQ_k}{dt} = \frac{\hbar}{m_k}\mathrm{Im}\frac{\nabla_{q_k}\Psi(Q)}{\Psi(Q)} \tag{1}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\nabla_{q_k}^2\Psi + U(q)\Psi \tag{2}$$

Equations (1) and (2) form a complete specification of the theory. Agreement between Bohmian mechanics and quantum mechanics regarding the results of any experiment is guaranteed by what has been called [5] the *quantum equilibrium hypothesis*: when a system has a wave function ψ, its configuration Q is random with probability distribution

$$\rho(q) = |\psi(q)|^2. \tag{3}$$

While the meaning and justification of this hypothesis is a delicate matter, which has been discussed at length elsewhere [5] (see also [4] and [6]), we wish to underline here an important

property of (3): if the probability density for the configuration satisfies $\rho(q,t_0) = |\psi(q,t_0)|^2$ at some time t_0, then the density at any time t to which this is carried by the motion (1) is also given by $\rho(q,t) = |\psi(q,t)|^2$. This is an extremely important property of any Bohmian system, expressing a compatibility between the two equations of motion (1) and (2) defining the dynamics, which we call the *equivariance* of $|\psi|^2$.

3 Motion in an External Potential

Our goal is to study the classical behavior of a macroscopic body composed of N particles with $N \gg 1$ (one may think of an apple falling from a tree or a planet moving around the sun). It is rather clear that one expects classical behavior only for appropriate macroscopic functions of the particle configuration $(Q_1, ..., Q_N)$. The relevant macroscopic variable, whose classical behavior we wish to investigate here, is the center of mass of the body

$$X = \frac{\sum_i m_i Q_i}{m}\,,$$

where $m_1, \ldots, m_N$ are the masses of the particles composing the body and $m = \sum_i m_i$ is the total mass of the body.

We shall assume that the particles interact through internal forces as well as being subjected to an external potential, so that the potential energy in (2) is of the form

$$U(q) = \sum_{i<j} U(q_i, q_j) + \sum_i V_i(q_i)\,.$$

Let $y = (y_1, \ldots, y_{N-1})$ be a suitable set of coordinates[1] relative to the center of mass $x = \sum_i m_i q_i/m$. Then under the change of variables $q = (x, y)$ Schrödinger's equation (2) assumes the form

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(H^x + H^y + H^{(x,y)}\right) \Psi \tag{4}$$

where

$$H^x = \frac{\hbar^2}{2m} {\nabla_x}^2 + V(x)\,, \qquad V(x) \equiv \sum_i V_i(x)\,,$$

H^y is the free Hamiltonian associated with the relative coordinates y and the operator $H^{(x,y)}$ describes the interaction between the center of mass and the relative coordinates. If V_i are

[1] For sake of concreteness one may think, e.g., of the so called Jacobi coordinates.

slowly varying on the size of the body, $H^{(x,y)}$ can be treated as a small perturbation, and, in first approximation, neglected. Thus, if $\Psi = \psi(x)\phi(y)$ at some time, the time evolution of the center of mass decouples from that of the relative coordinates and we end up with a very simple one particle problem: the wave function ψ of the center of mass evolves according to one-particle Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m}{\nabla_x}^2\psi + V(x)\psi \tag{5}$$

and its position X evolves according to

$$\frac{dX}{dt} = \frac{\hbar}{m}\mathrm{Im}\frac{\nabla_x\psi(X)}{\psi(X)}\,. \tag{6}$$

From now on, whenever no ambiguity will arise, we shall treat the center of mass as a "particle" and we shall refer to X and ψ as the position and the wave function of such a particle.

4 The Classical Limit in Bohmian Mechanics

In order to investigate the conditions under which X evolves classically it is useful to write the wave function $\psi = \psi(x)$ in the polar form

$$\psi(x) = R(x)\mathrm{e}^{\frac{i}{\hbar}S(x)}, \tag{7}$$

From Schrödinger's equation (5) one obtains, following Bohm [2], the continuity equation for R^2,

$$\frac{\partial R^2}{\partial t} + \mathrm{div}\left[\left(\frac{\nabla_x S}{m}\right)R^2\right] = 0, \tag{8}$$

and the modified Hamilton-Jacobi equation for S

$$\frac{\partial S}{\partial t} + \frac{(\nabla_x S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{{\nabla_x}^2 R}{R} = 0. \tag{9}$$

Note that equation (9) is the usual classical Hamilton-Jacobi equation with an additional term

$$V_Q \equiv -\frac{\hbar^2}{2m}\frac{{\nabla_x}^2 R}{R}, \tag{10}$$

called the quantum potential. Since $\frac{\nabla_x S}{m}$ is the right hand side of (6), one then sees that the (size of the) quantum potential provides a rough measure of the deviation of Bohmian evolution from its classical approximation.

Analogously, consider the modified Newton equation associated with (9), and obtained by differentiating both sides of equation (6) with respect to time,

$$m\frac{d^2X}{d\,t^2} = F + F_Q, \tag{11}$$

where $F = -\nabla_x V(X)$ and $F_Q = -\nabla_x V_Q(X)$ are respectively the *classical* force and the "*quantum*" force. Equation (11) shows that all the *deviations* from classicality are embodied in the quantum force F_Q.

Thus, the *formulation* of the classical limit in Bohmian mechanics turns out to be rather simple: classical behavior emerges whenever the particle trajectory $X = X(t)$, satisfying (11), approximately satisfies the classical Newton equation, i.e.,

$$m\frac{d^2X}{d\,t^2} \simeq F\,. \tag{12}$$

The problem is to determine the physical conditions ensuring (12). Usually, physicists consider classical behavior as ensured by the limit $\hbar \to 0$, meaning by this

$$\hbar \ll A_0, \tag{13}$$

where A_0 is *some* characteristic action of the corresponding classical motion (see, e.g., [13],[15],[3])). Condition (13) is often regarded as equivalent to another standard condition of classicality which involves the length scales of the motion (see, e.g., [11]): if the de Broglie wave length λ is small with respect to the characteristic dimension L determined by the scale of variation of the potential V, the behavior of the system should be close to the classical behavior in the same potential V. This is very reminiscent of how geometrical optics can be deduced from wave optics. We regard this condition, i.e.,

$$\lambda \ll L, \tag{14}$$

as the most natural condition of classicality since it relates in a completely transparent way a property of the state, namely its de Broglie wave length λ, and a property of the dynamics, namely the scale of variation of the potential L. In the remainder of this paper we shall argue that (14) is indeed a necessary and sufficient condition for (12).

5 Wave Packets

To explain the physical content of (14) and its implications we shall consider first the case for which the wave function has a well-defined de Broglie wave length: we shall assume that

ψ is a wave packet with diameter σ, with mean wave vector k and associated wave length $\lambda = 2\pi/|k|$.

As we shall see, the analysis of this situation will allow us to find a precise characterization of the scale L of variation of the potential. Our analysis will be rather standard—it is basically the *Ehrenfest's Theorem*—and can be found in good textbooks (see, e.g., [10]). We reproduce it here both for the sake of completeness and because we believe that it attains, within the Bohmian framework, a deeper and much more general significance than within standard formulations of quantum mechanics.

From the equivariance of (3) we have that the mean particle position at time t is given by

$$\langle X \rangle = \int x |\psi_t(x)|^2 dx \, .$$

From (5) it follows that

$$m \frac{d^2}{dt^2} \langle X \rangle = - \int \nabla_x V(x) |\psi_t(x)|^2 dx \, .$$

By expanding $F(x) = -\nabla_x V(x)$ in Taylor series around $\langle X \rangle$ one obtains

$$m \frac{d^2}{dt^2} \langle X \rangle = F(\langle X \rangle) + \frac{1}{2} \sum_{j,k} \Delta_{j,k} \frac{\partial^2 F}{\partial x_j \partial x_k} (\langle X \rangle) + ..., \tag{15}$$

where

$$\Delta_{j,k} = \langle X_j X_k \rangle - \langle X_j \rangle \langle X_k \rangle$$

is of order σ^2, where σ is the diameter of the packet. Therefore, the mean particle position should satisfy the classical Newton equation whenever

$$\sigma^2 \left| \frac{\partial^3 V}{\partial x_i \partial x_j \partial x_k} \right| \ll \left| \frac{\partial V}{\partial x_i} \right| \, , \tag{16}$$

i.e.,

$$\sigma \ll \sqrt{\left| \frac{V'}{V'''} \right|} \tag{17}$$

where V' and V''' denote respectively suitable estimates of the first and third derivatives (e.g., by taking a sup over the partial derivatives).

The minimum value of the diameter of the packet σ is of order λ. Hence (17) becomes

$$\lambda \ll \sqrt{\left| \frac{V'}{V'''} \right|} \tag{18}$$

This last equation gives a *necessary* condition for the classicality of the particle motion and, by comparing it with (14), a precise definition of the notion of scale of variation of the potential, namely,

$$L = L(V) = \sqrt{\left|\frac{V'}{V'''}\right|}. \tag{19}$$

In the following we shall argue that (14), with L given by (19), is indeed also *sufficient* for classical behavior of Bohmian trajectories. For wave packets this follows easily from the equivariance of $|\psi|^2$: over the lapse of time for which the spreading of the packet can be neglected, the overwhelming majority[2] of trajectories $X = X(t)$ will stick around their mean value $\langle X \rangle$ and follow its classical time evolution. Thus we expect (12) to hold for the overwhelming majority of trajectories.

6 Local Plane Waves

Suppose now that ψ is not a packet but a wave function that locally looks like a packet. By this we mean, referring to the polar representation (7), that the amplitude $R(x)$ and the *local* wave vector

$$k = k(x) \equiv \nabla_x S(x)/\hbar \tag{20}$$

are slowly varying over distances of order $\lambda(x) \equiv h/|\nabla_x S(x)|$, the *local* de Broglie wave length. We may call such a ψ a "local plane wave".

At any given time the local plane wave can be thought as composed of a sum of wave packets: Consider a partition of physical space into a union of disjoint sets Δ_i chosen in such a way that the local wave vector $k(x)$ doesn't vary appreciably inside of each of them and denote by k_i the almost constant value $k(x)$ for $x \in \Delta_i$. Let χ_{Δ_i} be the characteristic function of the set Δ_i ($\chi_{\Delta_i}(x) = 1$ if $x \in \Delta_i$ and 0 otherwise). Since $\sum_i \chi_{\Delta_i} = 1$, we have

$$\psi(x) = \sum_i \chi_{\Delta_i}(x)\psi(x) = \sum_i \psi_i(x). \tag{21}$$

Note that this decomposition is somewhat arbitrary: provided that $k(x)$ is almost constant in Δ_i, the extent of these sets can be of the order of many wave lengths down to a minimal

[2] With respect to the equivariant measure $|\psi|^2$.

size $\sigma_i \simeq |\Delta_i|^{1/3}$ of the same order of λ_i.[3]

At any time, the position X of the particle will be in the support of one of the packets forming the decomposition (21), say in the support of ψ_i. If the condition (17) holds for σ_i, we may then proceed as in the previous section: the minimal size of the packet ψ_i can be taken of order $\sigma_i = \lambda(x)$ and the condition of classicality is again (18) for $\lambda = \lambda(x)$.

Note that this straightforward reduction of the classical limit for local plane waves to that for wave packets is possible only within Bohmian mechanics: since the particle has at any time a well-defined position X and the different components of the local plane wave (21) don't interfere, we may "collapse" ψ to the wave packet ψ_i relevant to the dynamics of X.

7 General Wave Functions

We wish now to investigate the physical content of (14) and its implications for a general wave function. The first issue to address is what notion of wave length should be appropriate for this case. A rough estimate of λ could be given in terms of mean kinetic energy associated with ψ,

$$E_{\text{kin}}(\psi) = \langle \psi, -\frac{\hbar^2}{2m} \nabla_x{}^2 \psi \rangle \,, \tag{22}$$

with associated wave length

$$\lambda = \lambda(\psi) = \frac{h}{\sqrt{2m E_{\text{kin}}(\psi)}} \,. \tag{23}$$

Suppose now that (18), with λ given by (23), is satisfied. We claim that in this case the Schrödinger evolution should "quickly" produce a local plane wave, that can be effectively regarded as built of pieces that are wave packets satisfying (18) for $\lambda = \lambda(x)$ and hence themselves evolving classically as we have seen in the previous section.

In fact, if $\lambda \ll L$ the kinetic energy dominates the potential energy and the free Schrödinger evolution provides a rough approximation of the dynamics up to the time needed for the potential to affect the evolution significantly. During this time, the Schrödinger evolution produces a spatial separation of the different wave vectors contained in ψ, more or less as Newton's prism separates white light into the different colors of the rainbow. In other words,

[3] The use of the characteristic function may introduce an undesirable lack of smoothness, but this can be easily taken care by replacing the χ_{Δ_i} with functions θ_i forming a smooth partition of unity.

the formation of a local plane wave originates in the "*dispersive*" character of free Schrödinger evolution.

So, in order to gain some appreciation of this phenomenon consider the free Schrödinger evolution

$$\psi_t(x) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{e}^{it\left[k\frac{x}{t} - \frac{\hbar k^2}{2m}\right]} \hat{\psi}(k) dk \,, \tag{24}$$

where $\hat{\psi}$ is the Fourier transform of the initial wave function ψ. The stationary phase method yields straightforwardly the long time asymptotics of ψ_t,

$$\psi_t(x) \sim \left(\frac{im}{\hbar t}\right)^{3/2} \mathrm{e}^{i\frac{m}{2\hbar}\frac{x^2}{t}} \hat{\psi}(k) \,, \quad \text{where} \quad k = \frac{m}{\hbar}\frac{x}{t} \,, \tag{25}$$

which is indeed a local plane wave with local wave vector $k = mx/(\hbar t)$.

We said above that the local plane wave is "quickly" produced. But how quickly? In order to estimate such a time, consider the simple example of an initial wave function ψ composed of two overlapping wave packets with the same position spread Δx and with opposite momenta p and $-p$. The time τ of formation of a local plane should be of the order of the time for separation of the packets, which is basically the time needed to cover a space equal to Δx. From $\Delta x \Delta p \sim \hbar$ and $\Delta p \sim p$ we obtain

$$\tau \sim \frac{\Delta x}{p/m} \sim \frac{\hbar}{p^2/m} \sim \frac{\hbar}{\langle E \rangle}, \tag{26}$$

where $\langle E \rangle$ is the mean kinetic energy of the particle. It is reasonable to suggest that (26), with $\langle E \rangle$ given by (22), could give a very rough estimate of the time of formation of a local plane wave for a general wave function ψ. Note that the time needed for the potential to produce significant effects on the evolution is of order

$$T = \frac{L}{v}, \qquad \text{where} \qquad v = \frac{h}{m\lambda}. \tag{27}$$

Thus, if $\lambda \ll L$ we have that $\tau \ll T$, which means that the local plane wave gets formed on a time scale much shorter than the time scale over which the potential affects the dynamics.

We arrive in this way at a sharp (or, at least, sharper than usually encountered) mathematical formulation of the classical limit for a general wave function ψ. First of all, consider the dimensionless parameter

$$\epsilon = \frac{\lambda(\psi)}{L(V)} \,. \tag{28}$$

Secondly, consider the Bohm motion X on the "macroscopic" length and time scales defined by ψ and V. By this we mean $X' = X'(t')$, where

$$X' = X/L \quad \text{and} \quad t' = t/T \tag{29}$$

with T given by (27). Finally, consider F_Q/m, the "quantum" contribution to the total acceleration in (11), on the macroscopic scales (29), namely

$$D = \frac{T^2}{L} F_Q(X'L, t'T) \tag{30}$$

Then the Bohm motion on the macroscopic length and time scales will be approximately classical, with deviation from classicality D tending to 0 as $\epsilon \to 0$.

We'd like to point out that the use of macroscopic coordinates (29) for the formulation of the classical limit is rather natural from a physical point of view. First of all, the scales L and T are the fundamental units of measure for the motion: L is the scale on which the potential varies and T provides an estimate of the time necessary for the particle to see its effects. More importantly, in the limit $\epsilon \to 0$ the nonclassical behavior—occurring during the time τ of formation of the local plane wave—disappears, since, as we have argued above, in this limit $\tau \ll T$. In other words, on the macroscopic scales on which we expect classical behavior the local plane wave has been formed.

8 Limitations of the Model: Interference and the Role of the Environment

Before commenting on the mathematics of the limit $\epsilon \to 0$ we should stress a physical caveat. For motion in unbounded space, the expanding character of the Schrödinger evolution makes the set of local plane waves an "attractor" for the dynamics—so that the local plane wave form is in this sense "typical". However, for motion in a bounded region (with wave functions which are superpositions of bound states) the "typical" wave function is composed by a sum of local plane waves, this being due to interference between the waves reflected by the "edges" of the confining potential. Consider for example an infinite potential well of size L in one dimension and initial wave function ψ, well localized in the center of the well which is the superposition of two packets with opposite momenta p and $-p$. Suppose that $\lambda(\psi) \ll L$. Then the two packets move classically and at a certain time, say t_r, are reflected from the

walls of the potential. At the time $t_c = 2t_r$, they interfere in the middle of the well. t_c is the "*first caustic time*," the time at which the classical action $S_{\rm cl}(x,t)$ becomes multivalued. In general, we should not expect classical behavior for times larger than the first caustic time t_c.

What is going on? The emergence of classical behavior should be robust and stable, which would not be the case if it were restricted to times smaller than t_c. However, if one remembers that the model we are investigating is a strong idealization, the problem evaporates. We are in fact dealing with the one-body problem defined by (5) and (6), an approximation to the complete dynamics defined by (4) in which the term $H^{(x,y)}$, describing the interaction between the center of mass and the relative coordinates, is neglected. Note than even (4) is an idealization since it does not include the unavoidable interaction of the body with its *external* environment: in a more realistic model $H^{(x,y)}$ would take into account both the internal and external environment of the center of mass (with y now including both the relative coordinates and the degrees of freedom of the external environment). These interactions—even for very small interaction energy—should produce *entanglement* between the center of mass x of the system and the other degrees of freedom y, so that their effective role is that of "measuring" the position X and suppressing superpositions of spatially separated wave functions. (Taking these interactions into account is what people nowadays call decoherence, see, e.g., [7] and the references therein). Referring to the above example, the effect of the environment should be to select (as relevant to the dynamics of X, see [5] and [9]) one of the two packets on a time scale much shorter than the first caustic time t_c.

9 Towards a Mathematical Conjecture

The mathematical content of sections 7 and 8 is summarized by the following (not yet sharply formulated) conjecture:

Conjecture. *Let ϵ be the dimensionless parameter defined by (28) and D be the quantity given by (30). Then there are environmental interactions such that $D \rightsquigarrow 0$ as $\epsilon \to 0$, uniformly in ψ and V.*

Concerning this conjecture, we'd like to make here just a few remarks.

1. "$D \rightsquigarrow 0$" means convergence to 0 in a "suitable" probabilistic sense since D is a random variable. D is a function of X, and X is random with probability distribution given by

$|\psi|^2$. To require almost sure convergence is probably too strong a demand. Convergence in probability, or L^2 convergence, would seem more appropriate. Moreover, λ is defined in (23) in terms of the *average* kinetic energy. This average could be large even when there is a significant probability for a very small kinetic energy. Thus it is probably necessary to regard λ as random (with randomness inherited from the kinetic energy) and to understand $\epsilon \to 0$ also in a probabilistic sense.

2. Uniformity of the limit in ψ and V could be expressed as follows: let (V_n, ψ_n) be any sequence for which $\epsilon_n = \frac{\lambda_n}{L_n} \to 0$, with $\lambda_n = \lambda(\psi_n)$ given by (23), and $L_n = L(V_n)$. Then $D \rightsquigarrow 0$ as $n \to +\infty$. Understanding $D \rightsquigarrow 0$ as convergence in probability, we could also express uniformity in the following way: for any $\eta > 0$ and for any $\delta > 0$, there exists an $\epsilon_0 > 0$ such that $\mathbf{P}(D > \delta)$ is smaller than η whenever $\epsilon < \epsilon_0$. Here $\mathbf{P}$ is the probability measure defined by $|\Psi|^2$, i.e., $\mathbf{P}(dq) = |\Psi(q)|^2 dq$, which includes randomness arising from the environment.

3. For quadratic potentials (including free motion and motion in a uniform force field) $L = \infty$ so that $\epsilon = 0$. In this case the conjecture should be modified as follows: let L_o be *any* length scale and T_o the corresponding time scale $T_o = \frac{L_o}{v}$. Then for the Bohm motion on the scales given by L_o and T_o, $D \rightsquigarrow 0$ uniformly in ψ and L_o whenever $\tilde{\epsilon} \equiv \frac{\lambda(\psi)}{L_o} \to 0$.

4. There is an enormous amount of mathematical work, called semiclassical analysis or, in more modern terms, microlocal analysis, in which the limit $\hbar \to 0$ of Schrödinger evolutions is rigorously studied. It should be stressed that the limit $\epsilon \to 0$ is *much more general* than the limit $\hbar \to 0$. In fact $\epsilon = \lambda/L = h/mvL$. So keeping L and the momentum mv fixed, the limit $\hbar \to 0$ implies $\epsilon \to 0$. But there are many ways in which ϵ could go to zero. The classical limit, as expressed by the above conjecture, is (at the very least) a two-parameters limit, involving λ and L, and $\hbar \to 0$ is just a very special case. Moreover, these two parameters themselves live on infinite dimensional spaces since $\lambda = \lambda(\psi)$, with ψ varying in the Hilbert space of the system's wave functions, and $L = L(V)$, with V varying in the class of admissible one particle potentials (that is, potentials leading to a self-adjoint Hamiltonian).

5. Exactly for the reason expressed in the previous remark, the conjecture is really *very* hard to prove: it require a lot of uniformity both in the wave function ψ and in the potential V.

Just to have an idea of the difficulties, one may think of the analogous problem in statistical mechanics, namely the problem of studying the deviations from thermodynamic behavior of a large but finite system about which not so much is known.

6. While the conjecture is difficult to prove, it is still not completely satisfactory from a physical point of view. The conjecture states only that D depends on ϵ in such a way that $D \rightsquigarrow 0$ as $\epsilon \to 0$, uniformly in ψ and V. A physically more relevant result would be to estimate how rapidly D is tending to 0 (e.g., like ϵ, or ϵ^2 or whatever). Note that only this last kind of result can be of practical value: given V and ψ, it provides an estimate for the deviation from classicality, while any other results do not quite do this.

7. With the conjecture, and even with the refinement proposed in the previous remark, there is a further difficulty to consider: even if $H^{(x,y)}$ is treated as a small perturbation in (4), the suggestion of section 8 might not be too realistic. In fact, the autonomous Schrödinger evolution, even of a very narrow wave function $\psi = \psi(x)$, could be destroyed in very short times. This is a serious difficulty; one resolution might be found in the notion of *conditional wave function* of the x-system, $\psi(x) = \Psi(x, Y)$, where Y is the actual configuration of the environment (this notion has been introduced and analyzed in [5]). We regard the extension of the conjecture to this more realistic framework as the most interesting open problem on the classical limit, which we leave for future work.

10 The Classical Limit in a Nutshell

The key ingredient in our analysis of the emergence of the classical world, is that as soon as the local plane wave has formed, each configuration X is attached to a guiding wave packet with a definite wave vector $k(x,t)$ that locally determines the particle dynamics according to the local de Broglie relation

$$p(x,t) = \hbar k(x,t),$$

which, for $\lambda \ll L$, evolves according to classical laws. This means that the classical limit can be symbolically expressed as

$$(\psi, X) \to (P, X),$$

where (ψ, X) is the complete quantum state description in terms of wave function and position, while (P, X) is the complete classical state description in terms of momentum and position.

All the relevant macroscopic information contained in the pair (ψ, X) is, in the classical limit, embodied in the pair (P, X)—the only robust, stable quantity. In other words, as far as the macroscopic dynamics of X is concerned, only the information carried by P is relevant.

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