

Axiomatic systems for Minkowski space–time

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Abstract

Axiomatic systems for Minkowski space–time are briefly reviewed and then an outline of a system of independent axioms is presented.

1. Introduction

The main purpose of describing a theory axiomatically is to clarify the basic concepts of the theory and the assumptions on which it is based. In a lecture delivered before the International Congress of Mathematicians in 1900, Hilbert [1] suggested:

“ . . . that wherever, from the side of the theory of knowledge or in geometry, or from the theories of natural or physical science, mathematical ideas come up, the problem arises for mathematical science to investigate the principles underlying these ideas and so to establish them upon a simple and complete system of axioms, . . . ”

and also stated several mathematical problems, the sixth problem being

“Investigations of the foundations of geometry suggest the problem: to treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; first of all, the theory of probability and mechanics.”

Ever since there has been considerable interest in axiomatic foundations of physical theories and, particularly, in Minkowski space–time which Penrose [2] categorises as one of the “superb” theories since it applies over a wide range of situations with high accuracy.

2. Axiomatic systems for geometries and space–times

Hilbert [3] demonstrated the independence of the five major groupings of axioms of Euclidean geometry so as to illustrate the significance of the conclusions that can be drawn from the individual axioms.

2.1 *Axiomatic systems for geometries*

Many axiomatic systems have been given for geometries [4][5]. The earliest approach due to Euclid has been further developed by Pasch [6][7], Peano [8][9] and Hilbert [3] who demonstrated the independence of the five major groupings of axioms of Euclidean geometry so as to illustrate the significance of the conclusions that can be drawn from the individual axioms. These approaches were further developed by Veblen [10] and Moore [11] who showed that Euclidean geometry could be developed from fifteen independent axioms.

A second tradition is to characterize geometries by properties of symmetry and free mobility: this is known as the Helmholtz–Lie space form problem and its solutions are discussed and reviewed by Busemann [12] and Freudenthal [13]. Most of the solutions are stated within a context of differentiable manifolds or spaces satisfying various topological and geometric axioms.

2.2 *Axiomatic systems for space–times*

Minkowski space–time has a more complicated structure than Euclidean geometry and this may explain the diversity of axiomatic approaches, which is in marked contrast to the two traditions for the Euclidean and non–Euclidean geometries. Minkowski space–time has three distinct types of interval — timelike, spacelike and lightlike — and the lightlike intervals enable comparisons to be made between other types of interval, so that “light signals” can be used to define a relation of congruence. This was done in one of the earliest axiomatic systems by Robb [14] whose axiomatic system was stated in terms of a single undefined relation of temporal order. However Robb’s aim of “mathematical simplicity” was achieved with a minimal number of primitive concepts rather than by the intuitive acceptability, minimal number, or independence of the axioms. Robb’s approach has been further developed by Mundy [15][16].

2.2.1 *The Helmholtz–Lie tradition*

The space-time problem analogous to the Helmholtz–Lie space form problem has been investigated by Busemann [17], Freudenthal [48] and Mayr [18] in wider contexts of space–time structures which are assumed to satisfy various topological axioms .

2.2.2 *The predecessors of the present axiomatic system*

Axiomatic systems using properties of symmetry together with properties of the paths of freely-moving observers, have been proposed by several authors. Walker [19][20] considered the foundations for an approach to relativistic cosmology which had been developed by Milne [21] and Whitrow [22] to describe space-times in terms of observations which could be made using direct and reflected light signals together with local time measurements. The undefined basis involved the concepts of “particles”, “light signals”, and “temporal order”. Walker’s axiom system was not developed sufficiently to describe

Minkowski space-time and was, in fact, restricted to sets of relatively stationary particles, but it succeeded in clarifying many kinematic concepts, especially those of “particle” (or “path” in the present system), “light signal”, “optical line” and “collinearity”. Walker [20] discusses cosmologies of “fundamental particles” which satisfy two axioms of symmetry, one of which is a property of isotropy similar to that of the present treatment. The axiomatic system of Szekeres [23] has an axiom of isotropy and two other symmetry axioms. Schutz [24][25] uses a property of isotropy similar to that of the present treatment, but characterizes Minkowski space-time indirectly by first demonstrating that its velocity space is hyperbolic. Of these axiomatic systems, only those of Schutz [24] and Szekeres [23] describe Minkowski space-time in terms of assumptions related to what one might describe as either “kinematic experience” or physical intuition.

2.2.3 Axiomatic systems for curved space-times

More general space-times, such as those of general relativity and other gravitational and unified field theories, have a considerable amount of structure which involves causal, topological, differentiable, conformal, projective and pseudo-metric properties. Early attempts at axiomatising general relativity were proposed by Reichenbach [26] and Weyl [27].

Castagnino [28] axiomatised space-time by developing the metric structure from consideration of the paths of particles and light signals with methods previously developed by Kundt and Hoffmann [29] and Marzke and Wheeler [30].

Ehlers, Pirani and Schild [31] have given a detailed axiomatic discussion of the topological, differentiable, conformal, projective, affine and metric structures. This has been further developed by Ehlers [32], Ehlers and Köhler [33], Ehlers and Schild [34], Pirani [35], Woodhouse [36] and Coleman and Korte [37].

Kronheimer and Penrose [38] discussed general cases of space-times which satisfy causal properties. This approach has been further developed by Kronheimer [39] and Carter [40]. Busemann [17], Pimenov [41] and Szekeres [42][43] have considered space-times which satisfy causal and various topological axioms.

3. A system of independent axioms for Minkowski space-time

I would now like to describe a system of axioms which is being published in a forthcoming monograph [47] as a system of independent axioms complete with independence models and proof of the isomorphism of all its models.

This axiomatic system may be considered to follow from the two traditional approaches to the absolute geometries. The first tradition of Euclid–Hilbert–Veblen–Moore can be clearly seen in most of the axioms, apart from the Axiom of Symmetry (Axiom S) which expresses a property of mobility and is more in the spirit of the Helmholtz–Lie tradition. The two traditions of geometry are combined for Minkowski space-time which is expressed in terms of two undefined sets of *events* and *paths* and a single three-point *betweenness* relation.

Many of the axioms of the present system bear a strong resemblance to those of Hilbert [3], Veblen [10][45] and other writers. The exceptions are Axioms I5—I7 (of incidence) which distinguish a space–time from a geometry and Axiom S (of symmetry) which replaces the congruence postulates for the geometries. Apart from these exceptions the axioms of the present system have counterparts in the axiom systems for the geometries.

There are several assumptions which we do not make. In particular, as in the previous axiomatic system [24][25], we do not assume the concept of a coordinate frame, we do not assume that the events of a path can be ordered by the real numbers, nor do we assume that paths and light signals move with constant speed and we do not assume the existence of a temporal order relation on the set of events.

Three properties of Minkowski space–time are of central importance to the subsequent development. Firstly, one–dimensional kinematics is in many ways analogous to plane absolute geometry except for one significant difference, namely the existence of “unreachable sets”. Secondly, the most important analogous property is parallelism and the corresponding question of uniqueness of parallelism is closely related to uniform motion along paths. Both Robb [53] and Szekeres [23] observed that uniform motion implies uniqueness of parallelism but, as in the previous system [24], we are able to prove the uniqueness of parallelism and then show that this implies uniform motion along paths, so that we do not need to assume Newton’s first law of motion explicitly. Thirdly, in contrast with the Euclidean velocity space of Newtonian kinematics, the velocity space associated with Minkowski space–time is hyperbolic, a property which is established in the present treatment using a characterisation of ellipsoids in affine spaces due to Busemann [12]. This characterisation is much more readily accessible than the results used in the previous axiomatic systems [24][25].

To clarify the meaning the axioms, I have relaxed the formal requirement of independence to provide a corresponding system of non–independent axioms which describe essentially the same properties but which do not permit the existence of some of the independence models. Also, for the purposes of conveying the general ideas and properties introduced by the axioms, some of the axioms will be described rather than formally stated.

3.1 A brief comparison with euclidean geometry

For Euclidean geometry the metric properties are a consequence [3] [11] [10] of further axioms of congruence and an axiom of parallels. Minkowski space–time has a richer structure than Euclidean geometry since there are some pairs of events which can be joined by single paths and other pairs of events which can not be joined by single paths. This additional structure is specified by the fifth, sixth and seventh axioms of incidence which could, perhaps, be called “axioms of ordered non-incidence”. As a consequence of its richer structure Minkowski space–time can be specified categorically without an axiom of uniqueness of parallelism and axioms of congruence.

3.2 Primitive undefined basis

Minkowski space time is

$$\mathcal{M} = \langle \mathcal{E}, \mathcal{P}, [\dots] \rangle$$

where \mathcal{E} is a set whose elements are called *events*, \mathcal{P} is a set of subsets of \mathcal{E} called *paths* and $[\dots]$ is a ternary relation on the set of events of \mathcal{E} called a *betweenness relation*.

Paths will be denoted by upper case symbols Q, R, S, \dots ; events will be denoted by lower case symbols a, b, c, d, \dots or by a path symbol with a subscript, such as for example Q_a, Q_1, Q_x, Q_α for events which belong to the path Q and R_b, R_3, R_w, R_γ for events which belong to the path R . Given a pair of distinct events $a, b \in S$ we say that a, b *belong to* S or *lie on* S or that they can *be connected by* S or that S *passes through* them.

We will presuppose logic, set theory and the arithmetic of real numbers.

3.3 Axioms of Order

The axioms of order resemble axioms or theorems of the geometric axiom systems of Hilbert [3][44], Veblen [45], Moore [11] and Veblen and Young [46]. The first five axioms of order (Axioms O1–O4, O5) ensure that each path is a linearly ordered set of events.

Axioms O1 – O5

The set of events of a path is linearly ordered by the three-event betweenness relation $[\dots]$.

3.4 The Axiom of Collinearity (Axiom O6)

This axiom is a kinematic analogue of the geometric axiom of plane order given by Veblen [10] [45] and Moore [11] and makes it possible to discuss “rectilinear motion” in terms of “collinear sets” of events and paths. To obtain an appreciation of the kinematic ideas expressed by the Axiom of Collinearity, the reader is encouraged to draw a diagram corresponding to the statement of the axiom. If a template is made by cutting a narrow slit in a sheet of paper, the paths may be observed “in motion” by moving the template gradually across the diagram. This axiom is an analogue of the geometric “Axiom of Pasch” [10] [11].

Axiom O6

If Q, R, S are distinct paths which meet at events $a \in Q \cap R, b \in Q \cap S, c \in R \cap S$ and if:

- (i) *there is an event $d \in S$ such that $[bcd]$, and*
- (ii) *there is an event $e \in R$ and a path T which passes through both d and e such that $[cea]$,*

then T meets Q in an event f such that $[afb]$.

As is the case with the analogous geometric axiom of plane order, this axiom makes statements about both incidence and order. The ensuing development of “order” properties on paths and properties of “collinear sets” has much in common with the treatment of order properties of points on lines and properties of planes as given by Veblen [10][45]. However the investigation of space-time is more complicated due to the existence of pairs of events which can not be connected by paths (see Axiom I5).

3.5 Axioms of Incidence

Of the seven axioms of incidence which follow, the first four have counterparts in the axiomatic systems of Hilbert [3], Veblen [10][45] and Moore (1908) for Euclidean geometry. The fifth, sixth and seventh axioms describe properties of the “unreachable set” which distinguishes a space-time from a geometry. These latter properties are essentially properties of “non-incidence” and involve both order properties and incidence properties in their formulations. Axiom I5 excludes Galilean space–time as a possible model, while Axiom I6 and Axiom I7 specify the “connectedness” and “boundedness” of the unreachable set.

Axiom I1 (Existence).

\mathcal{E} is not empty.

Axiom I2 (Connectedness).

For any two distinct events $a, b \in \mathcal{E}$ there are paths R, S such that $a \in R$, $b \in S$ and $R \cap S \neq \emptyset$.

Axiom I3 (Uniqueness).

For any two distinct events, there is at most one path which contains both of them.

In the subsequent development we will frequently be discussing the properties of sets of paths which meet at a given event. We will call any such set a *SPRAY of paths*, or more concisely a *SPRAY*, where the upper case letters indicate that we are referring to a set of paths rather than to a set of events: given any event x , we define

$$SPR[x] := \{R : R \ni x, R \in \mathcal{P}\} .$$

The corresponding set of events is called a *spray*, where the lower case letters indicate a set of events. We define

$$spr[x] := \{R_y : R_y \in R, R \in SPR[x]\} .$$

A subset of three paths $\{Q, R, S\}$ of a SPRAY is *dependent* if there is a path which does not belong to the SPRAY and which contains one event from each of the three paths : we also say that any one of the three paths is *dependent on* the other two. Otherwise the subset is *independent*.

We next give recursive definitions of dependence and independence which will be used to characterize the concept of dimension. A path T is *dependent on* the set of n paths (where $n \geq 3$)

$$\mathcal{S} = \{Q^{(i)} : i = 1, 2, \dots, n; Q^{(i)} \in SPR[x]\}$$

if it is dependent on two paths $S^{(1)}$ and $S^{(2)}$, where each of these two paths is dependent on some subset of $n - 1$ paths from the set \mathcal{S} . We also say that the set of $n + 1$ paths $\mathcal{S} \cup \{T\}$ is a *dependent set*. If the set of $n + 1$ paths has no dependent subset, we say that the set of $n + 1$ paths is an *independent set*.

We now make the following definition which enables us to specify the dimension of Minkowski space-time. A SPRAY is a *3-SPRAY* if:

- (i) it contains four independent paths, and
- (ii) all paths of the SPRAY are dependent on these four paths.

Axiom I4 (Dimension).

If \mathcal{E} is non-empty, then there is at least one 3-SPRAY.

Given a path Q and an event $b \notin Q$, we define the *unreachable subset of Q from b* to be

$$Q(b, \emptyset) := \{x : \text{there is no path which contains } b \text{ and } x, x \in Q\} .$$

That is, the unreachable subset of Q from b is the subset of events of Q which can not be joined to b by a single path. If two events can not be connected by any path, we say that each is *unreachable* from the other; otherwise each is *reachable* from the other.

Axiom I5

For any path Q and any event $b \notin Q$, the unreachable set $Q(b, \emptyset)$ contains (at least) two events.

Axiom I6 (Connectedness of the Unreachable Set).

Given any path Q , any event $b \notin Q$, and distinct events $Q_x, Q_z \in Q(b, \emptyset)$, then for all $Q_y \in Q$,

$$[Q_x Q_y Q_z] \implies Q_y \in Q(b, \emptyset) .$$

Axiom I7 (Boundedness of the Unreachable Set).

Given any path Q , any event $b \notin Q$ and events $Q_x \in Q \setminus Q(b, \emptyset)$ and $Q_y \in Q(b, \emptyset)$, there is an event $Q_z \in Q \setminus Q(b, \emptyset)$ such that

$$[Q_x Q_y Q_z] .$$

3.6 Axiom of Continuity

This axiom resembles the second-order geometric axiom of the same name [3] [10] [11] [45].

Before we state the axiom we make the definition: a sequence of events

$$Q_0, Q_1, Q_2, \dots$$

(of a path Q) is called a *chain* if:

- (i) it has two distinct events, or
- (ii) it has more than two distinct events and for each Q_i other than Q_0 and Q_1 ,

$$[Q_{i-2} Q_{i-1} Q_i] .$$

A *finite chain* is denoted by writing $[Q_0 Q_1 Q_2 \dots Q_n]$ and an *infinite chain* is denoted by writing $[Q_0 Q_1 Q_2 \dots]$ (note that the concept of a “chain” used by Veblen and Young [46] for the discussion of projective geometries is entirely different from the concept defined above).

Given a path $Q \in \mathcal{P}$ and an infinite chain $[Q_0 Q_1 \cdots]$ of events in Q , the set

$$\mathcal{B} = \{Q_b : \forall Q_i \neq Q_0, [Q_0 Q_i Q_b]; Q_b \in Q\}$$

is called the *set of bounds* of the chain: if \mathcal{B} is non-empty we say that the chain is *bounded*. If there is a bound $Q_b \in \mathcal{B}$ such that for all $Q_{b'} \in \mathcal{B} \setminus \{Q_b\}$,

$$[Q_0 Q_b Q_{b'}]$$

we say that Q_b is a *closest bound*.

Axiom C (Continuity).

Any bounded infinite chain has a closest bound.

Eventually after a large number of theorems this completeness property allows us to define the concepts of “signal relations” which are not specifically involved in the statement of the next axiom.

3.7 Axiom of Isotropy or Symmetry

Compared to the absolute geometries, Minkowski space-time has the additional structure provided by the existence and properties of unreachable sets (Axioms I5, I6, I7). These properties, together with the single property of isotropy of the following axiom, are sufficient to take the place of all the axioms of congruence and the axiom of uniqueness of parallels [3] [11] [10] [45] for Euclidean geometry.

For any two distinct paths Q, R which meet at an event x , we define the *unreachable subset of Q from Q_a via R* to be

$$Q(Q_a, R, x, \emptyset) := \{Q_y : [x Q_y Q_a] \text{ and } \exists R_w \in R \text{ such that } Q_a, Q_y \in Q(R_w, \emptyset)\} .$$

For a closest bound Q_b of $Q(Q_a, R, x, \emptyset)$ to x , it is as if a “light signal” could be emitted at Q_b and bounce back from the path R to return to the path Q at the event Q_a .

The statement of the Axiom of Isotropy can be expressed intuitively as: “if an observer Q observes that paths R and S appear to move in the same way, then there is an automorphism of \mathcal{M}^* which leaves the events of Q invariant and maps R onto S ”.

Axiom S (Symmetry or Isotropy).

If Q, R, S are distinct paths which meet at some event x and if $Q_a \in Q$ is an event distinct from x such that

$$Q(Q_a, R, x, \emptyset) = Q(Q_a, S, x, \emptyset)$$

then

- (i) *there is a mapping $\theta : \mathcal{E} \longrightarrow \mathcal{E}$*
- (ii) *which induces a bijection $\Theta : \mathcal{P} \longrightarrow \mathcal{P}$*

such that

- (iii) *the events of Q are invariant, and*
- (iv) *$\Theta : R \longrightarrow S$.*

The mapping θ is called a *symmetry mapping* or an *isotropy mapping*, with Q as the *invariant path* .

4. Conclusion

In the forthcoming monograph [47] it is shown that all models of this axiomatic system are isomorphic and that the axiomatic system is consistent relative to the reals.

The axiomatic system described in this talk has been stated in such a way as to make the meaning of each axiom as clear as possible. In order to establish a set of mutually independent axioms, some additional care is required in the detailed statements of some of the axioms (namely axioms O5–6, I6–7, C). A suitably detailed set of mutually independent axioms, together with independence models can be found in the forthcoming monograph [47].

Foundational studies often give rise to as many questions as they answer. Euclid's axiomatic system and the attempts to prove the parallel postulate led to the discovery by Bolyai and Lobachevsky of hyperbolic geometry. The present investigation has resulted in several open problems for future research [47].

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