

nificance — although our language lacks a suggestive word for it. The normalizer of the group of parallel displacements consists of all affine transformations; hence affine geometry may be based on the one relation \parallel between figures. Or, still more simple, the subgroup consisting of the identity only is an invariant subgroup, and indeed the relation of identity between two figures is of objective significance. (There is none that has a better claim to objectivity, owing to the fact that the identity is contained in every possible group Γ of transformations.) The smaller the group Δ the larger its normalizer, and thus the wider the gap between congruence and similarity; or more precisely, if Δ' is a subgroup of Δ then the normalizer Γ' of Δ' contains the normalizer Γ of Δ . The normalizer of an invariant subgroup Δ of Γ always comprises Γ . A geometry whose group of automorphisms is Γ can be based on the objective relation of Δ -equivalence alone, provided the normalizer of Δ is not larger than Γ but coincides with Γ .

[A last remark will conclude this analysis. Space is a continuum, and when we speak of any transformation in space it is reasonable to interpret this as meaning any continuous transformation. We indicate by Ω the group of those transformations that are taken into account at all; in the case of a continuum this would be the group of all continuous transformations. By putting this explicitly in evidence our definition of normalizer may be repeated as follows. Given a subgroup Δ of the group Ω ; those elements of Ω that commute with Δ constitute the normalizer Γ of Δ . In this form the notion of normalizer makes sense even for abstract groups Ω and Δ .

Kant speaks about the divergence between congruent and similar in *Prolegomena*, §13, and claims that "by no single concept, but only by pointing to our left and right hand, and thus depending directly on intuition [Anschauung] can we make comprehensible the difference between similar yet incongruent objects (such as oppositely wound snails)"; and in his opinion only transcendental idealism offers a solution for this riddle. No doubt the meaning of congruence is based on spatial intuition, but so is similarity. Kant seems to aim at some subtler point, but just this point is one which can be completely clarified by an analysis in terms of a group Γ and its invariant subgroups Δ , or of a group Δ and its normalizer Γ . Whenever Δ is a proper invariant subgroup of Γ , the notions of congruence = Δ -equivalence and similarity = Γ -equivalence do not coincide although the former is of objective significance (= Γ -invariant). The phenomenon about which Kant wonders can thus be most satisfactorily subsumed under general and abstract 'concepts.']

Whoever raises congruence to the rank of the only basic relation of geometry is obliged to develop geometry from this one notion. Several ways are open to accomplish this. A deeper insight than by the elementary approach in the style of Euclid's axioms would be gained if one succeeded in formulating the fundamental facts of geometry as simple axioms concerning the group Δ^+ of Euclidean motions. Following Ueberweg, Helmholtz first carried out this program with surprising success in his essay "Ueber die Tatsachen, die der Geometrie zugrunde liegen." Later S. Lie, who established a general theory of transformation groups, resumed the problem with his more powerful mathematical tools and generalized it from 3 to n dimensions. The Euclidean group of motions Δ^+ turns out to be almost completely characterized by the fact that it permits the rigid body that measure of free mobility with which we are familiar by experience. In more exact terms: it is possible by suitable congruent mappings to carry any point into any other, and, if a point is kept fixed, to carry any line direction at that point into any other at the same point; furthermore, if a point and line direction are kept fixed, it is possible to carry by congruent mapping any surface direction through them into any other such direction, and so forth, up to the $(n - 1)$ -dimensional direction elements. If, on the other hand, a point and a line direction through it, and a surface direction through the latter, and so forth, up to an $(n - 1)$ -dimensional direction element, are given, then there exists no congruent mapping besides the identity under which this system of incident elements remains fixed. We just said that this axiom almost completely characterizes the Euclidean group of motions. In fact one thus obtains the group of congruent transformations of a slightly more general space, namely of a projective space endowed with a Cayley metric. That group contains a numerically indeterminate parameter λ , the constant space curvature, of which nothing but the sign is essential. According as λ is positive, zero, or negative, the resulting space is of the elliptic, parabolic (i.e. Euclidean), or hyperbolic type. These then are the only homogeneous spaces, in which all points are equivalent, likewise all directions at a point, and so on.

[It is hard to talk intelligently about these problems without an exact description of the Euclidean groups Γ and Δ^+ before our eyes. A Cartesian frame of reference in three-dimensional Euclidean space consists of a point O , the origin, and three mutually perpendicular vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ of equal length. The coordinates x_1, x_2, x_3 of a point P are defined by

$$\vec{OP} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3.$$