

Introducing Pietarinen expansion method into single-channel **pole extraction problem**

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Introducing the Pietarinen expansion method into the single-channel pole extraction problem

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Motivation and justification

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Poles are finally established as the resonance criterion

1. Conclusions of ATHOS 2012, ATHOS2013

2. Recent change in PDG attitude

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Immediate problem:

It is a common knowledge how to extract Breit-Wigner parameters from experimental data,

However, it is rather obscure how to do it with poles

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We know how to extract Breit-Wigner parameters from experiment because they are defined on the real axes.

(see Camogli Michel)

But, how do we extract pole parameters from experiment because we have to go to the complex energy plane?

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The usual answer was:

1. Do it globally

One first has to **make a model** which fits the data, **SOLVE IT**, and obtain an **explicit analytic function in the full complex energy plane**. Second, one has to look for the **complex poles** of the obtained **analytic functions**.

2. Do it locally

Speed plot, expansions in power series

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Global methods

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Features:

- **Many models**
- **Complicated *and different* analytic structure**
- **Elaborated method for solving the problem**
- **SINGLE USER RESULTS**

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Local and „QUASI-local” methods

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Taylor expansions

1. Padé approximants

Padé Approximants and Resonance Poles

Pere Masjuan^{1,*} and Juan José Sanz-Cillero^{2,†}

Let us consider a function $F(x)$, analytical in a disk $B_\delta(x_0)$. Then, the Taylor expansion

$$\mathcal{P}_N(x, x_0) = \sum_{n=0}^N a_n (x - x_0)^n, \quad (1)$$

converges to $F(x)$ in $B_\delta(x_0)$ for $N \rightarrow \infty$, with derivatives given by $a_n = F^{(n)}(x_0)/n!$.

The scenario changes, however, when the function $F(x)$ is not analytical anymore, for example when it has a single pole at $x = x_p$ inside the disk $B_\delta(x_0)$. In this case, the Taylor series does not converge any more, so we need a different procedure to extract information about the function and its derivatives.



A special case of interest for the present work is Montessus de Ballore's theorem [6, 17, 18]. Montessus' theorem states that when the amplitude $F(x)$ is analytical inside the disk $B_\delta(x_0)$ except for a single pole at $x = x_p$ the sequence of one-pole Padé Approximants $P_1^N(x, x_0)$ around x_0 ,

$$P_1^N(x, x_0) = \sum_{k=0}^{N-1} a_k(x - x_0)^k + \frac{a_N(x - x_0)^N}{1 - \frac{a_{N+1}}{a_N}(x - x_0)}, \quad (3)$$

converges to $F(x)$ in any compact subset of the disk excluding the pole x_p , i.e,

$$\lim_{N \rightarrow \infty} P_1^N(x, x_0) = F(x). \quad (4)$$



2. Regularization method

PHYSICAL REVIEW D 77, 116007 (2008)

Resolution of the multichannel anomaly in the extraction of S-matrix resonance-pole parameters

Saša Ceci,^{1,2,*} Jugoslav Stahov,^{3,4} Alfred Švarc,¹ Shon Watson,³ and Branimir Zauner¹

The function $T(z)$ with a simple pole at μ is regularized by multiplying it with a simple zero at μ

$$f(z) = (\mu - z)T(z). \quad (8)$$

From this definition and Eq. (7), it is evident that the value of $f(\mu)$ is equal to the residue r of $T(z)$ at point μ . As we have the access to the function values on real axis only, the Taylor expansion of f is performed about some real x to give the value (residue) at the pole μ (where background is highly suppressed)

$$f(\mu) = \sum_{n=0}^N \frac{f^{(n)}(x)}{n!} (\mu - x)^n + R_N(x, \mu). \quad (9)$$



The expansion is explicitly written to the order N and the remainder is designated by $R_N(x, \mu)$. Using the mathematical induction one can show that the N th derivative of $f(x)$, given by Eq. (8), is

$$f^{(n)}(x) = (\mu - x)T^{(n)}(x) - nT^{(n-1)}(x). \quad (10)$$

Insertion of this derivative into the Taylor expansion conveniently cancels all consecutive terms in the sum, except the last one

$$f(\mu) = \frac{T^{(N)}(x)}{N!}(\mu - x)^{(N+1)} + R_N(x, \mu), \quad (11)$$

$$\mu = a + ib$$

function residue $|f(\mu)|$

$$\frac{(a - x)^2 + b^2}{\sqrt{N+1}|f(\mu)|^2} = \sqrt{N+1} \frac{(N!)^2}{|T^{(N)}(x)|^2}. \quad (13)$$



In both cases we have n -TH DERIVATIVE of the function

PROBLEMS !

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In Camogli 2012, during „coffee-break conversation” I have claimed that extracting poles from theoretical and even from experimental data should in principle be possible, and I have promised to try to propose a simple method.

Now I am fulfilling this promise.

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Is it possible to create universal approach, usable for everyone, and above all REPRODUCIBLE?

I have tried to do it starting from very general principles:

1. Analyticity

2. Unitarity

Idea:

TRADING ADVANTAGES

GLOBALITY FOR SIMPLICITY

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THEORETICAL MODELS

*If you create a model, the advantage is that your solution is absolutely **global**, valid in the full complex energy plane (all Riemann sheets). The drawback is that the solution is **complicated**, pole positions are usually energy dependent otherwise you cannot ensure simple physical requirements like absence of the poles on the first, physical Riemann sheet, Schwartz reflection principle, etc. It is complicated and demanding to solve it.*

WE PROPOSE

*Construct an analytic function **NOT** in the full complex energy plane, but **CLOSE** to the real axes in the area of dominant nucleon resonances, which is fitting the data by using*

LAURENT EXPANSION.

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Why Laurent's decomposition?

- It is a **unique** representation of the complex analytic function on a dense set in terms of pole parts and regular background
- It explicitly separates pole terms from regular part
- It has constant pole parameters
- It is not a representation in the full complex energy plane, but has its well defined **area of convergence**

IMPORTANT TO UNDERSTAND:

It **is not** a representation of the full function in terms of pole positions with **constant coefficients** (as some referees reproached), because it is defined only in a part of the complex energy plane.



Expansion of the T-matrix in terms of constant coefficients

$$T(\omega) \approx \sum_{i=1}^k \frac{x_i \Gamma_i/2}{\omega - M_i - i\Gamma_i/2} + B(\omega)$$

$x_i, M_i, \Gamma_i, \omega \in \mathbb{R}. \quad (4)$

cannot be valid in the full complex energy plane **in principle**.

Namely, poles with **constant coefficients** have poles on **ALL** physical sheets, and that violates common sense because only **bound states** are allowed to be located on **the physical sheet**.



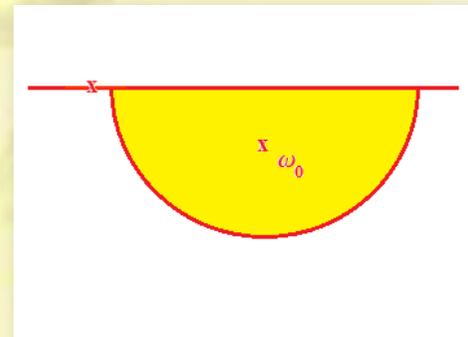
The only way how to accomodate both, requirements of absence of poles on the physical sheet, and Schwartz principle requires that pole positions become energy dependent:

$$\frac{\Gamma(\omega)e^{i\phi}}{\omega - M(\omega) - iK(\omega)\Gamma(\omega)}$$

However, even this function has its Laurent decomposition

$$\frac{\Gamma(\omega)e^{i\phi}}{\omega - M(\omega) - iK(\omega)\Gamma(\omega)} \equiv \frac{a_{-1}}{\omega - \omega_0} + \sum_{n=0}^{\infty} a_n (\omega - \omega_0)^n. \quad (5)$$

But it is valid only in the part of the complex energy plane



1. Analyticity

**Analyticity is introduced via generalized Laurent's decomposition
(Mittag-Leffler theorem)**

However, the functions we meet and analyze in reality may and do contain more than one pole for $\omega \neq \omega_0$. So if we iterate this procedure using **Mittag-Leffler theorem** [4] which says that a meromorphic function can be expressed in terms of its poles and associated residues combined with additional entire function, we can without loss of generality write down the generalized Laurent expansion for the function with k poles:

Assumption:

- **We are working with first order poles so all negative powers in Laurent's expansion lower than $n < -1$ are suppressed**

Now, we have two parts of Laurent's decomposition:

- 1. Poles**
- 2. Regular part**

Idea: TO MIMICK THE PROCEDURE FOR BREIT-WIGNER CASE

Bw:

$$T = \frac{x \frac{\Gamma(w)}{2} e^{i\phi}}{M(w) - w - i \frac{\Gamma(w)}{2}} + Bg(w)$$

With Laurent's decompositions for simple poles

$$T(\omega) = \frac{(a_R + i a_I)_{-1}}{\omega_0 - \omega} + \sum_{n=0}^{\infty} a_n (\omega_0 - \omega)^n$$

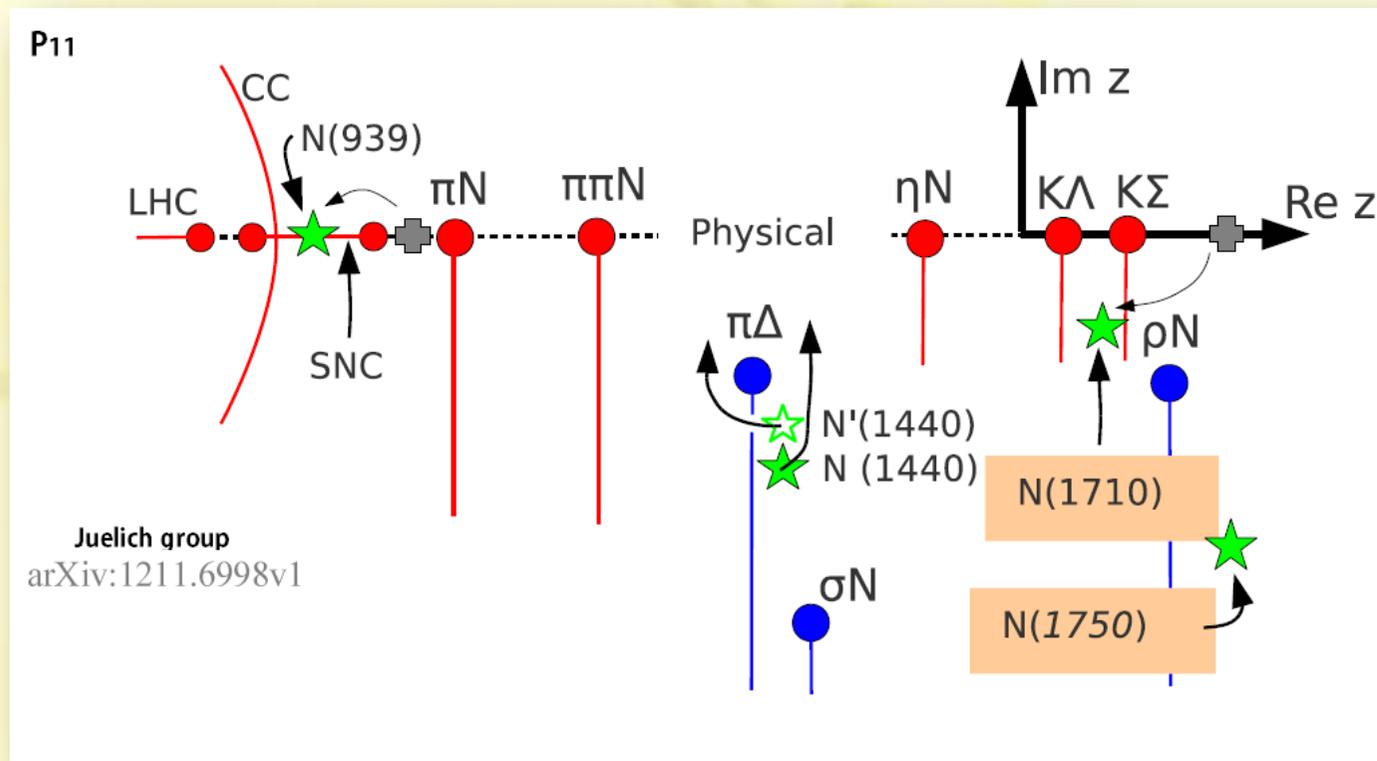
where

$$\sum_{n=0}^{\infty} a_n (\omega_0 - \omega)^n = \tilde{B}(\omega) \quad \text{regular function}$$

The problem is how to determine regular function $B(w)$.

What do we know about it?

We know it's analytic structure for each partial wave!



We **do not** know its **EXPLICIT** analytic form!

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So, instead of „guessing” its exact form by using model assumptions we

EXPAND IT IN FASTLY CONVERGENT POWER SERIES OF PIETARINEN („Z”) FUNCTIONS WITH WELL KNOWN BRANCH-POINTS!

Original idea:

1. S. Ciulli and J. Fischer in *Nucl. Phys.* 24, 465 (1961)
2. I. Ciulli, S. Ciulli, and J. Fisher, *Nuovo Cimento* 23, 1129 (1962).

Convergence proven in:

1. S. Ciulli and J. Fischer in *Nucl. Phys.* 24, 465 (1961)
2. Detailed proof in I. Caprini and J. Fischer: "Expansion functions in perturbative QCD and the determination of α_s ", *Phys.Rev. D*84 (2011) 054019,

Applied in πN scattering on the level of invariant

1. E. Pietarinen, *Nuovo Cimento Soc. Ital. Fis.* 12A, 522 (1972).
2. Hoehler – Landolt Boernstein *RIBI F” (198?)*

i) Fixed- t analysis

The analysis would be too complicated if one would insist on working with dispersion integrals. It is possible only by the use of PIETARINEN's expansion of the invariant amplitudes in terms of functions which have the correct analytic properties (Sect. A.6.3.4). The coefficients are determined from fits to the data, using the fixed- s solution as a constraint.

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What is Pitarinen's expansion?

In principle, in mathematical language, it is " ...a conformal mapping which maps the physical sheet of the ω -plane onto the interior of the unit circle in the Z -plane..."

In practice this means:

If $F(\omega)$ is a general, unknown analytic function having a cut starting at $\omega = x_P$, then it can be represented in a power series of Pietarinen functions in the following way:

$$F(\omega) = \sum_{n=0}^N c_n Z(\omega)^n, \quad \omega \in \mathbb{C}$$
$$Z(\omega) = \frac{\alpha - \sqrt{x_P - \omega}}{\alpha + \sqrt{x_P - \omega}}, \quad c_n, x_P, \alpha \in \mathbb{R}, \quad (3)$$

with the α and c_n being tuning parameter and coefficients of Pietarinen function $Z(\omega)$ respectively.

Or in another words, Pietarinen functions $Z(\omega)$ are **a complet set of functions** for an arbitrary function $F(\omega)$ which **HAS A BRANCH POINT AT x_P !**

Observe:

Pietarinen functions do not form a complete set of functions *for any function*, but only for the function having a well defined branch point.

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Illustration:

$$\text{Powers series for } Z(\omega) = \frac{3.3 - \sqrt{2-\omega}}{3.3 + \sqrt{2-\omega}}$$

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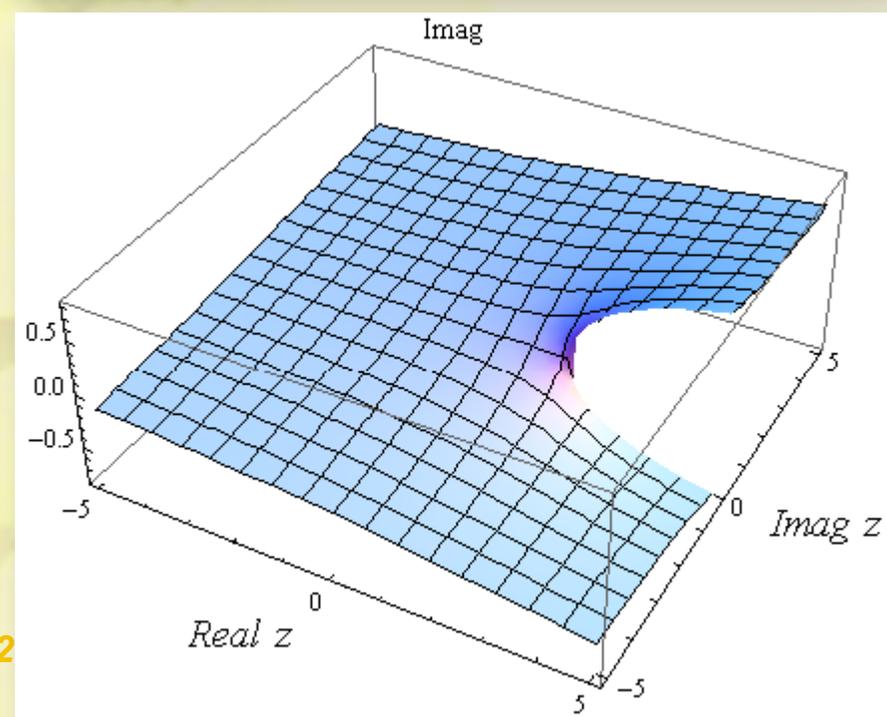
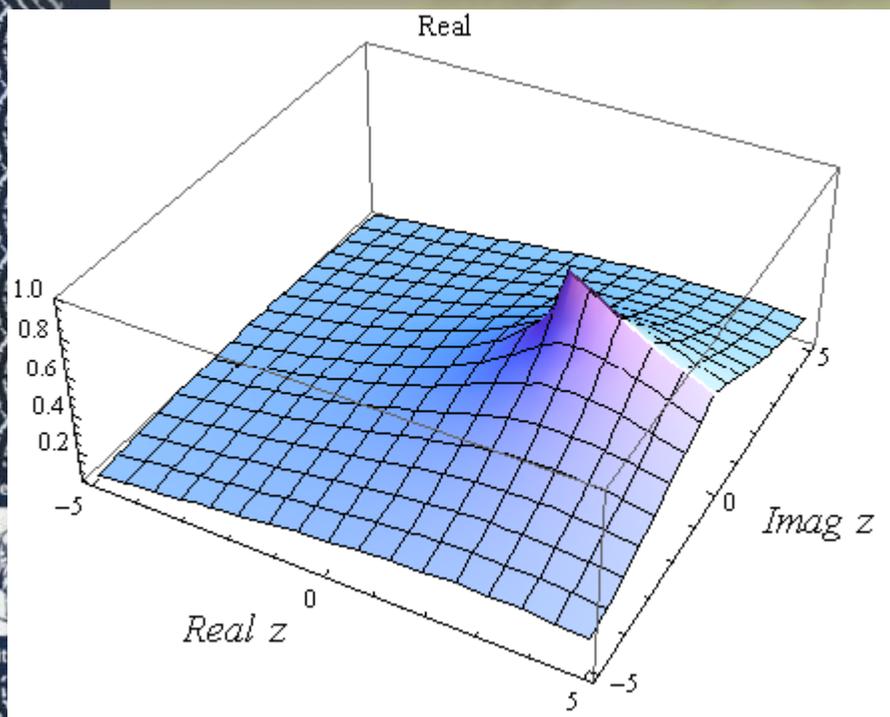
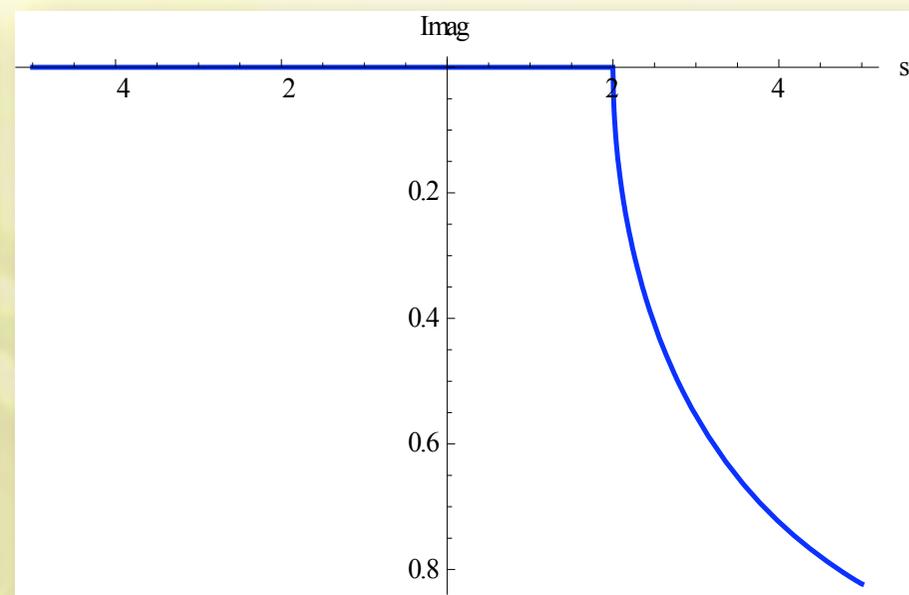
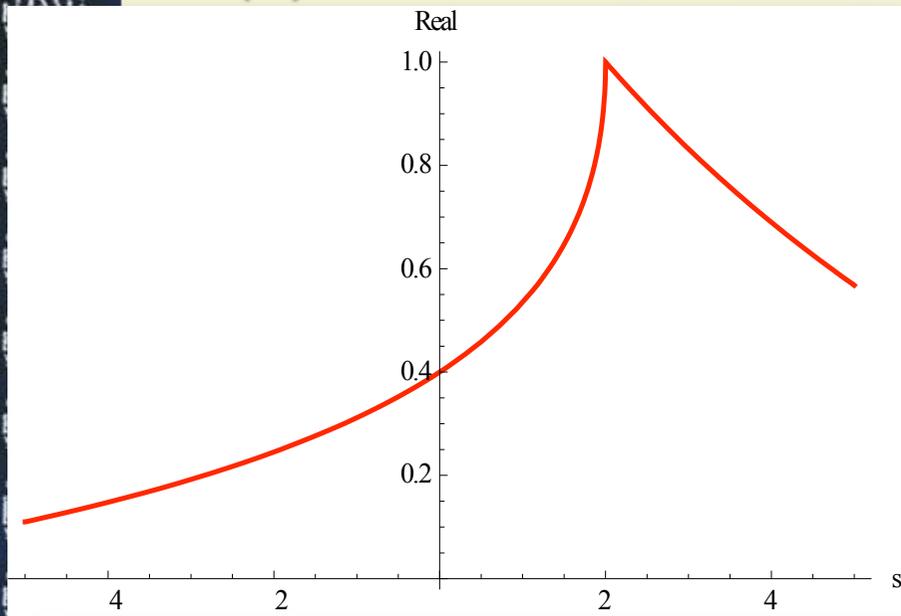


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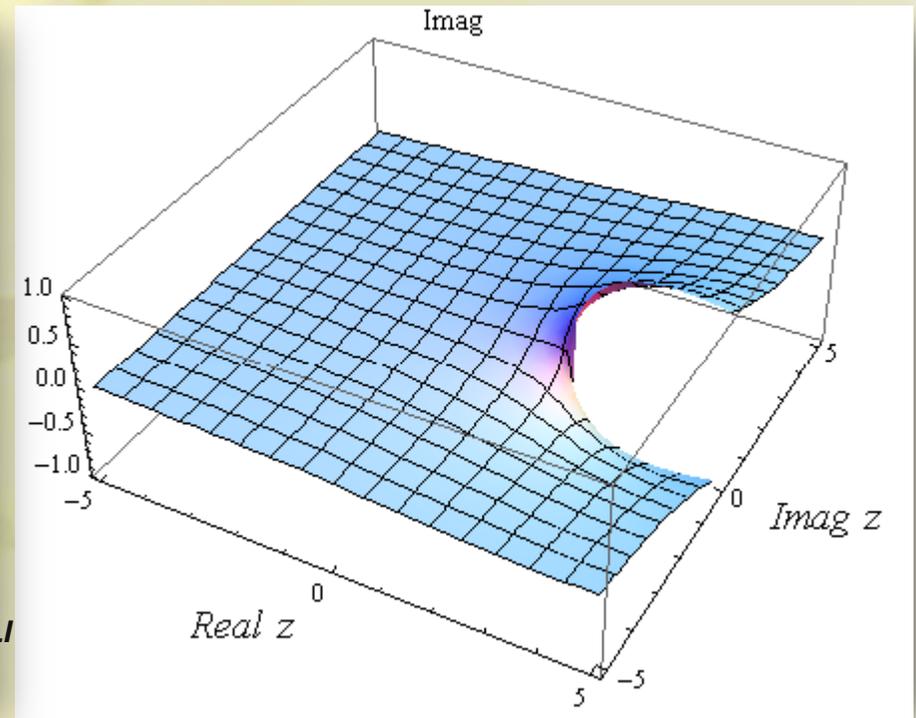
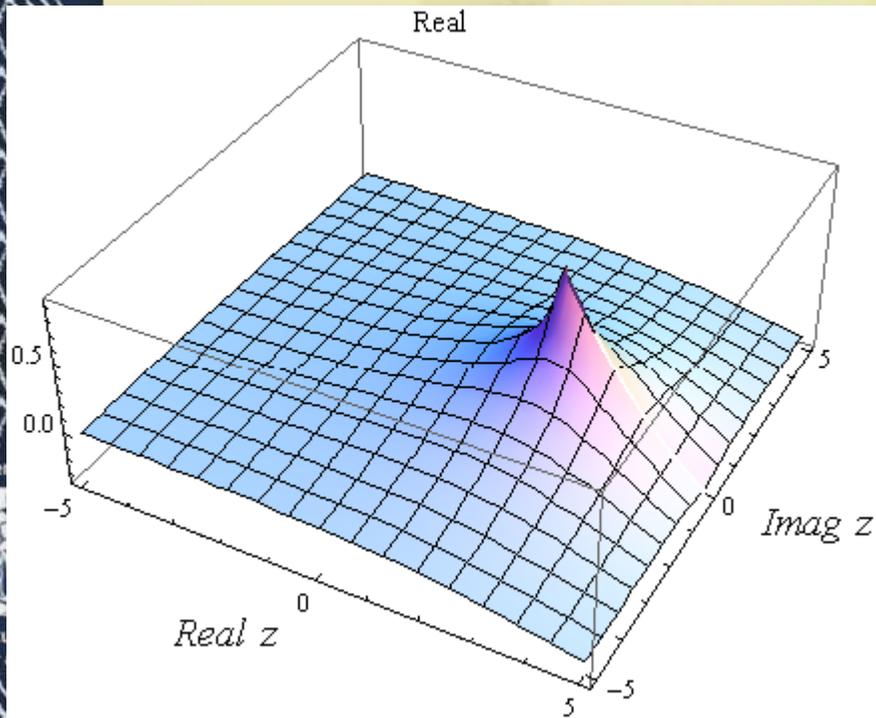
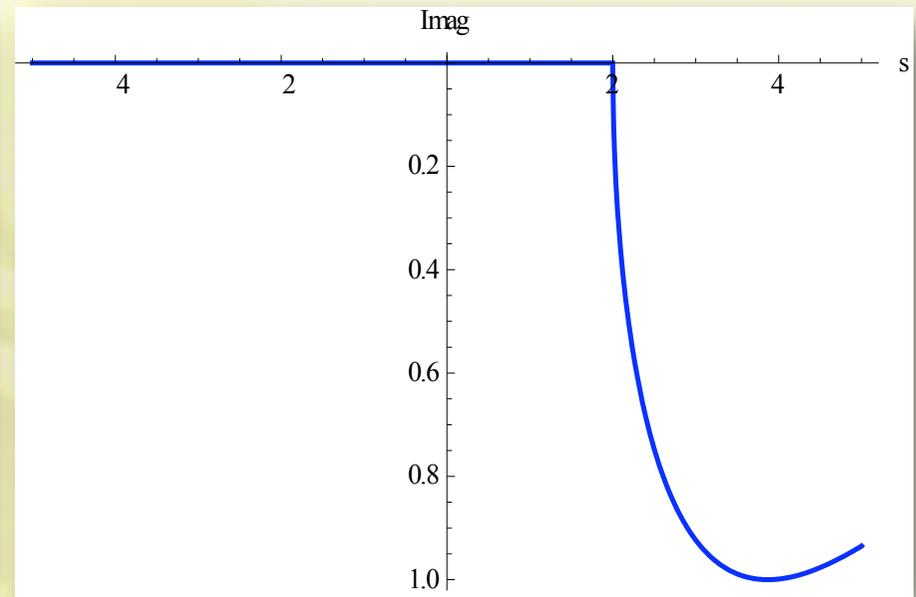
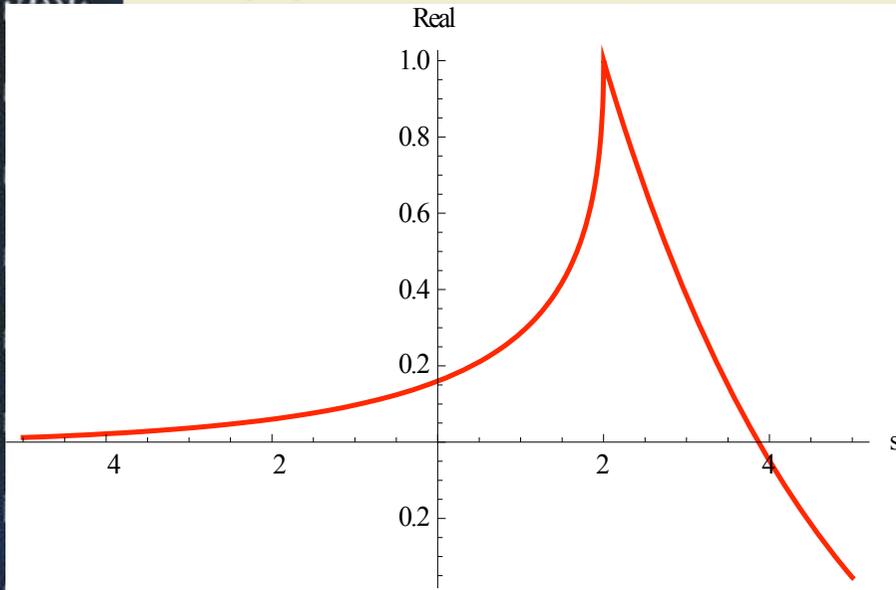
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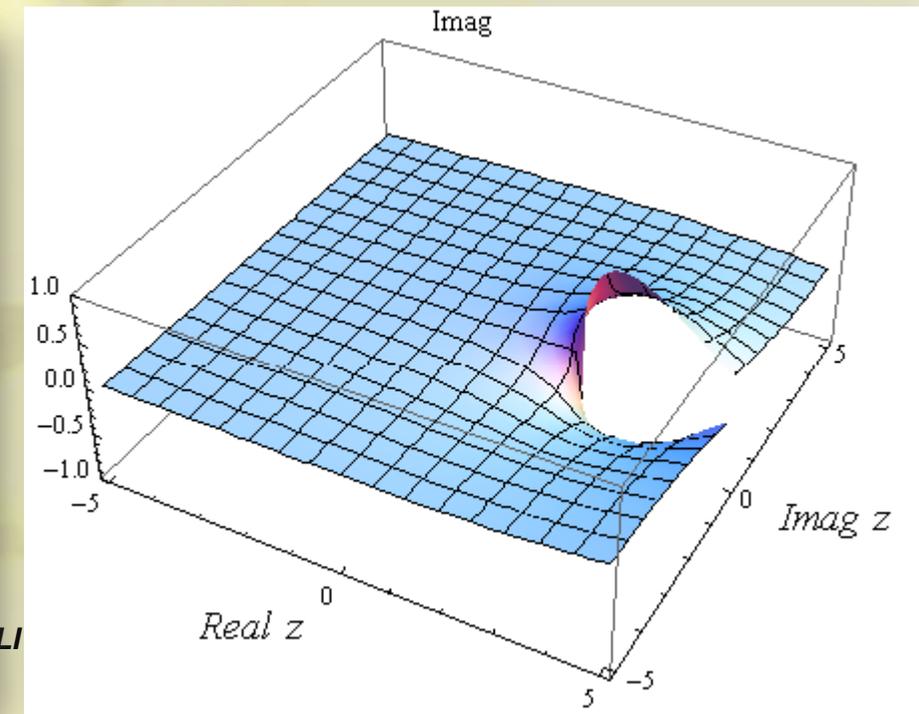
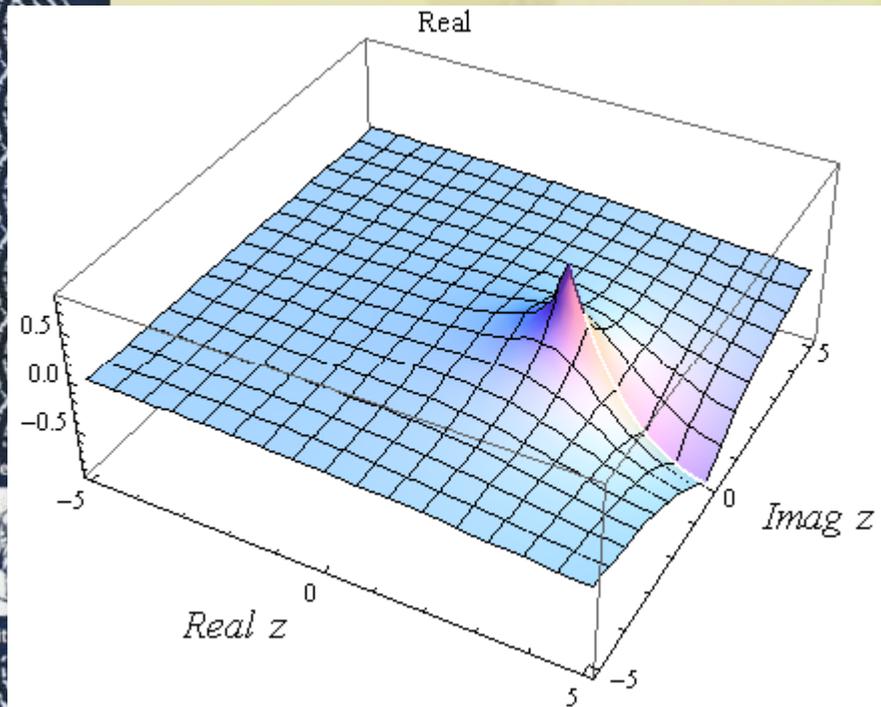
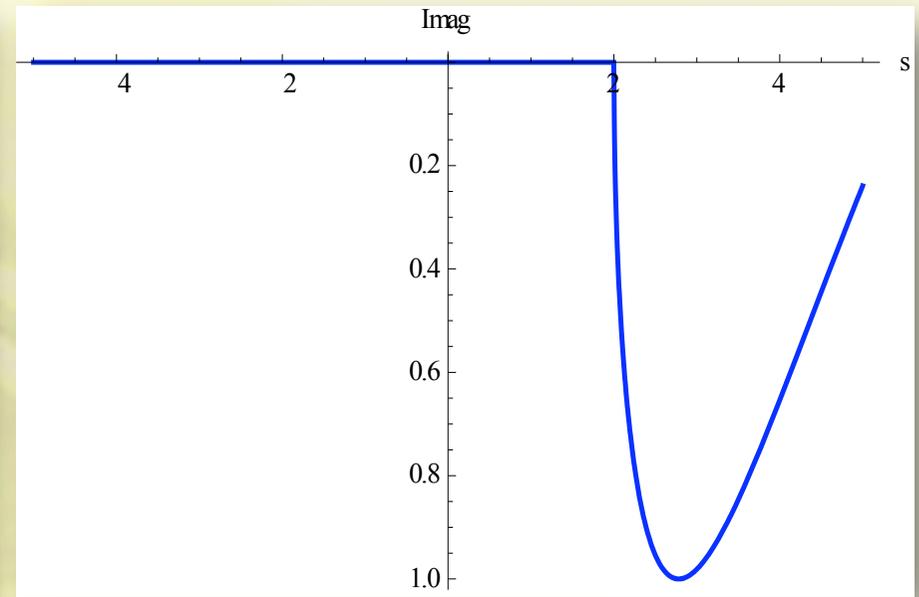
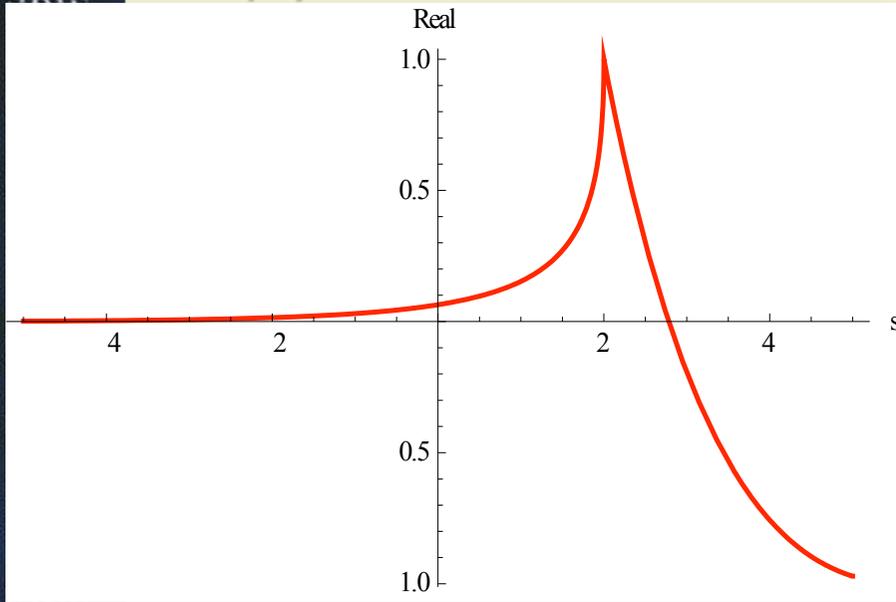
$Z(\omega)$



$Z(\omega)^2$



$Z(\omega)^3$



Important!

A resonance CANNOT be well described by Pietarinen series.

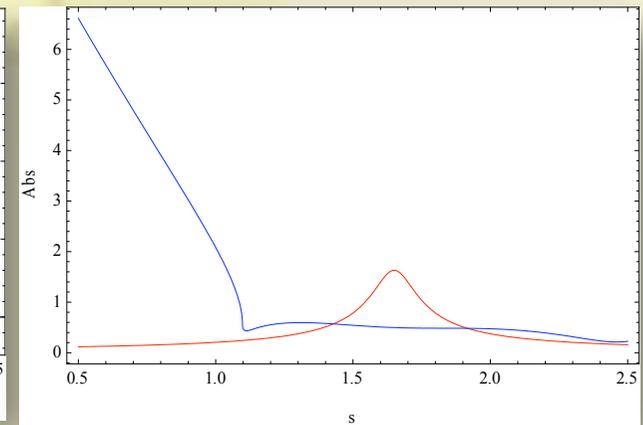
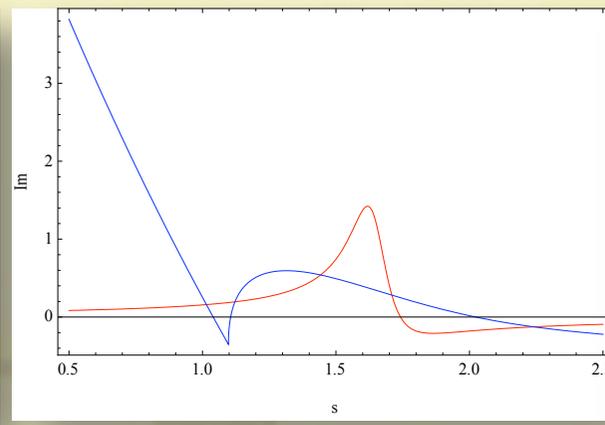
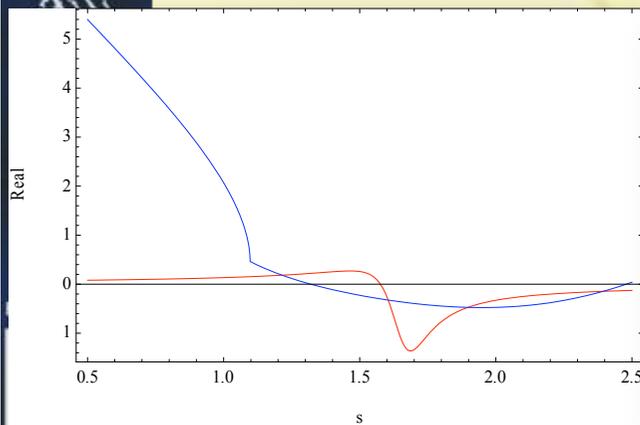
$$BW[s_] := r1 / \left(M1 - s - i \frac{\Gamma1}{2} \right)$$

$$ZI[x_] := \left(\alpha - \sqrt{xP - x} \right) / \left(\alpha + \sqrt{xP - x} \right);$$

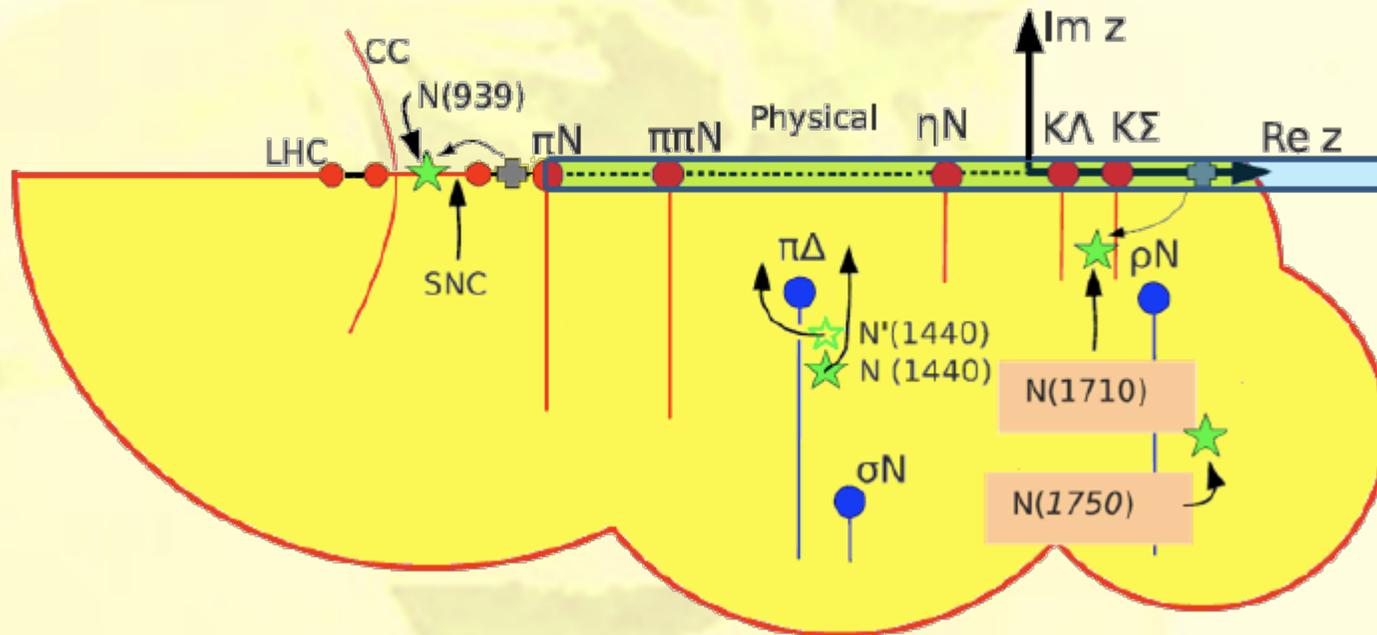
$$WI[x_] := \left(\beta - \sqrt{xQ - x} \right) / \left(\beta + \sqrt{xQ - x} \right);$$

$$FPietfit[x_] := c0 * ZI[x]^0 + c1 * ZI[x]^1 + c2 * ZI[x]^2 + c3 * ZI[x]^3 + c4 * ZI[x]^4 + \\ c5 * ZI[x]^5 - d0 * WI[x]^0 - d1 * WI[x]^1 - d2 * WI[x]^2 - d3 * WI[x]^3 - d4 * WI[x]^4 - \\ d5 * WI[x]^5;$$

$$xP \stackrel{23}{\parallel} \Rightarrow 4.93028, \quad xQ \stackrel{23}{\parallel} 1.09731$$



Finally, the area of convergence for Laurent expansion of P11 partial wave



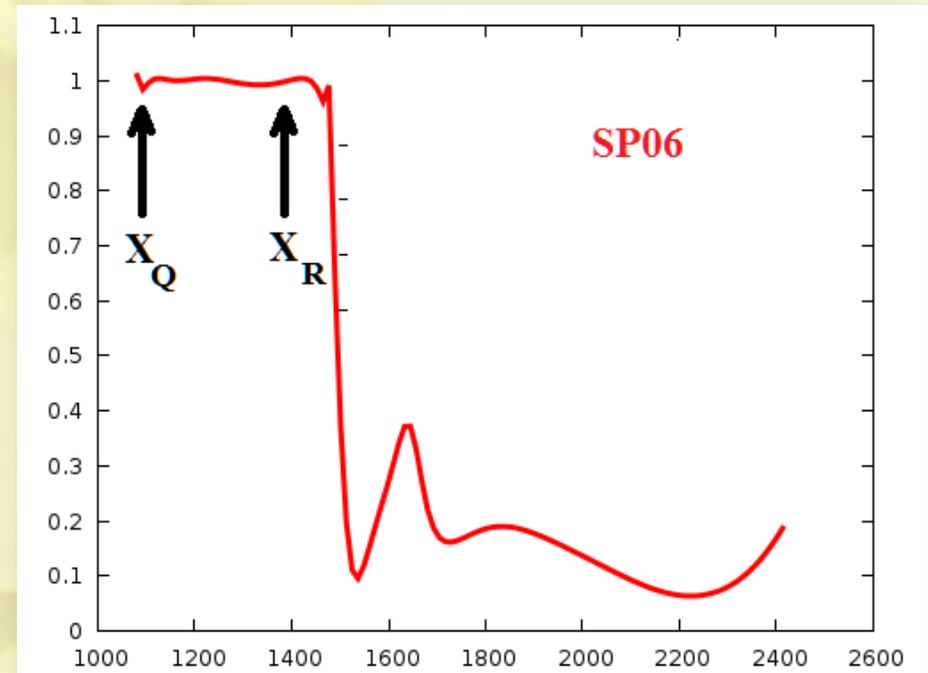
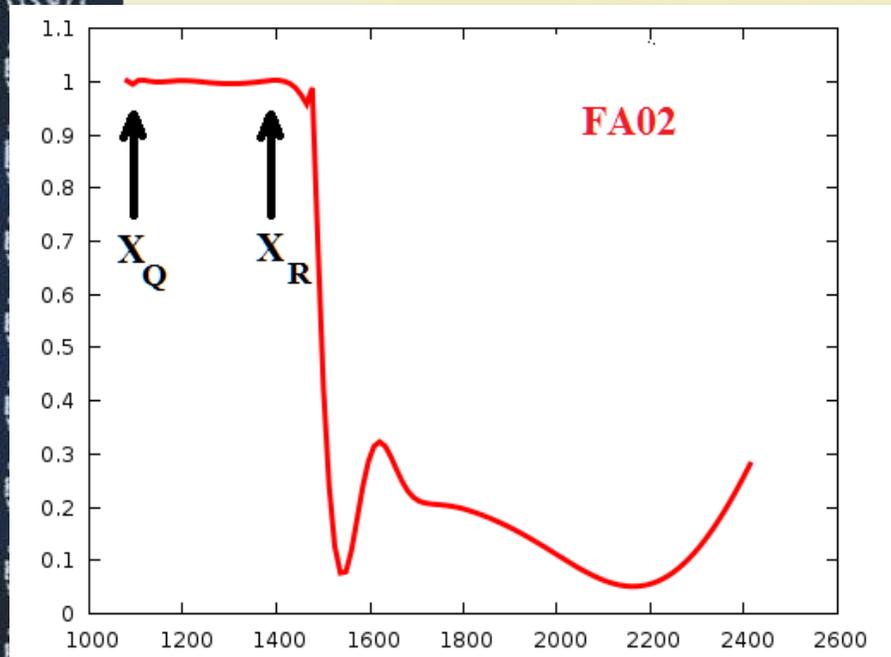
2. Unitarity

Elastic unitarity is introduced via penalty function

$$\chi^2 = \sum_{j=1}^{N_{pts}} |T^{inp}(\omega_j) - T(\omega_j)|^2 / w_j^2 + \sum_{j=1}^3 \lambda^j \chi_{Pen}^j + \beta \sum_{j=1}^{N_{pts}^{el}} (1 - \mathbf{S}(\omega_j))\mathbf{S}(\omega_j)^\dagger.$$



Unitarity test



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The model

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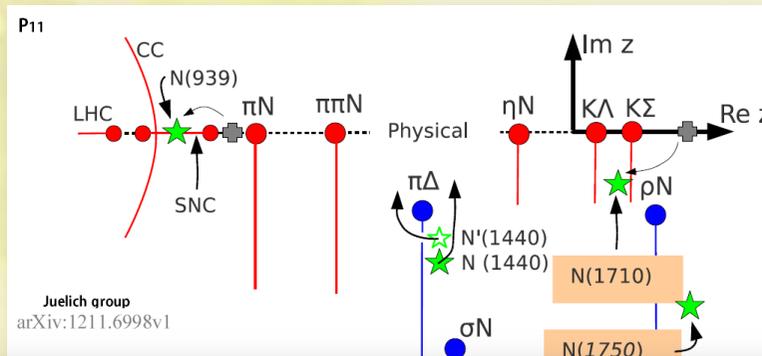
We use Mittag-Leffler decomposition of „analyzed” function:

$$T(\omega) = \sum_{i=1}^k \frac{a_{-1}^{(i)}}{\omega_i - \omega} + B^L(\omega)$$

k - simple poles

regular background

We know analytic properties (number and position of cuts) of analyzed function



$$B^L(\omega) = \sum_{n=0}^M c_n Z(\omega)^n + \sum_{n=0}^N d_n W(\omega)^n + \dots$$

$$Z(\omega) = \frac{\alpha - \sqrt{x_P - \omega}}{\alpha + \sqrt{x_P - \omega}}; \quad W(\omega) = \frac{\beta - \sqrt{x_Q - \omega}}{\beta + \sqrt{x_Q - \omega}} + \dots$$

$$a_{-1}^{(i)}, \omega_i, \omega \in \mathbb{C}$$

$$c_n, x_P, d_n, x_Q, \alpha, \beta \dots \in \mathbb{R}$$

$$\text{and } k, M, N \dots \in \mathbb{N}.$$

(4)

ONE
Pietarinen
power
series
per cut

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Method has problems, and the one of them definitely is:

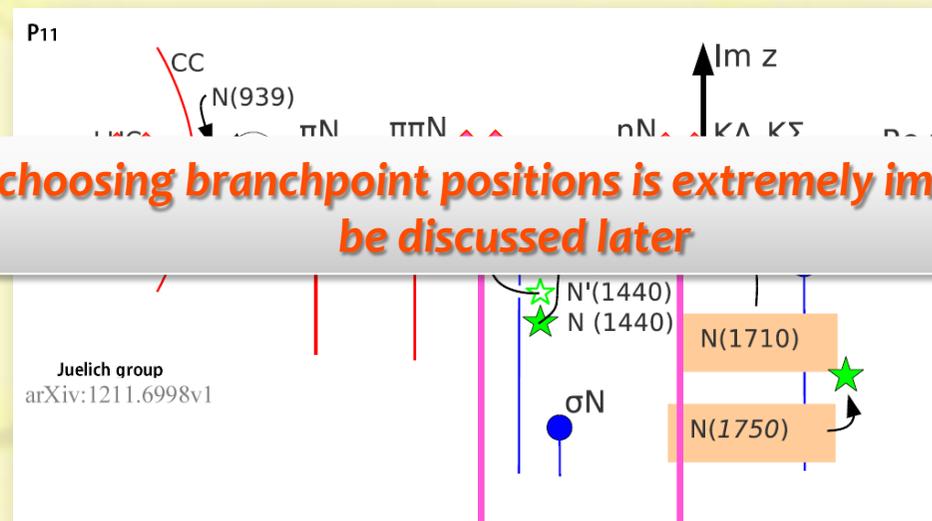
There is a lot of cuts, so it is difficult to imagine that we shall be able to represent each cut with one Pietarinen series (too many possibly interfering terms).

Answer:

We shall use „effective” cuts to represent dominant effects.

We use three Pietarinen series:

- **One to represent subthreshold, unphysical contributions**
- **Two in physical region to represent all inelastic channel openings**



Strategy of choosing branchpoint positions is extremely important and will be discussed later

Advantage:

The method is „self-checking” !

It might not work.

But, if it works, and if we obtain a good χ^2 , then we have obtained

**AN ANALYTIC FUNCTION WITH WELL KNOWN POLES AND CUTS WHICH
DEFINITELY DESCRIBES THE INPUT!**

So, if we have disagreements with other methods, then we are looking at two different analytic functions which are almost identical on a discrete set, so we may discuss the general stability of the problem.

However, our solution definitely IS A SOLUTION!

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What can we do with this model?

1. We may analyze **various** kinds of inputs

a. Theoretical curves coming from **ANY** model

but also

b. Information coming directly from experiment
(**partial wave data**)

Observe: **Partial wave data are much more convenient to analyze!**

To fit „**theoretical input**” we have to „guess” both:
pole position AND exact analyticity structure of the background
imposed by the analyzed model

To fit „**experimental input**” we have to „guess” only:
pole position AND the simplest analyticity structure of the
background as no information about functional type is imposed

Does it work?

Testing is a very simple procedure. It comes to:



Does it work

TESTING

a. Testing on a toy model: [arXiv nucl-th 1212.1295](#)

b. Testing and application on realistic amplitudes

i. πN elastic scattering

a. ED PW amplitudes (some solutions from GWU/SAID)

b. ED PW amplitudes (some solutions from Dubna-Mainz-Taipei)

ii. Photo – and electroproduction on nucleon

a. ED multipoles (all solutions from MAID and SAID)

b. SES multipoles (all solutions from MAID and SAID)



a. Toy model

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We have constructed a toy model using two poles and two cuts, used it to construct the input data set, attributed error bars of 5%, and tried to use L+P method to extract pole parameters under different conditions.

$$T^{ty}(\omega) = \sum_{i=1}^2 \frac{r_i^{ty} + i g_i^{ty}}{M_i^{ty} - \omega - i W_i^{ty}} + \quad (5)$$

$$+ C_1 \Phi(\omega, 0.25) + C_2 \Phi(\omega, 1.) + B^{ty}(\omega),$$

$$\Phi(\omega, a) = \frac{\sqrt{\omega(-4a + \omega)}}{2\pi\omega} \ln \frac{2a - \omega - \sqrt{\omega(-4a + \omega)}}{2a}$$

$$B^{ty}(\omega) = B_1 \frac{10.}{-10. - \omega - i 5.} + B_2 \frac{10.}{-6. - \omega - i 4.},$$

where

$$r_i^{ty}, g_i^{ty}, M_i^{ty}, W_i^{ty} \in \mathbb{R}.$$

$$\Phi(x, a) = \frac{x-x_0}{\pi} \int_{x_0}^{\infty} \frac{\Re e(x', a)}{(x'-x)(x'-x_0)} dx'$$

where

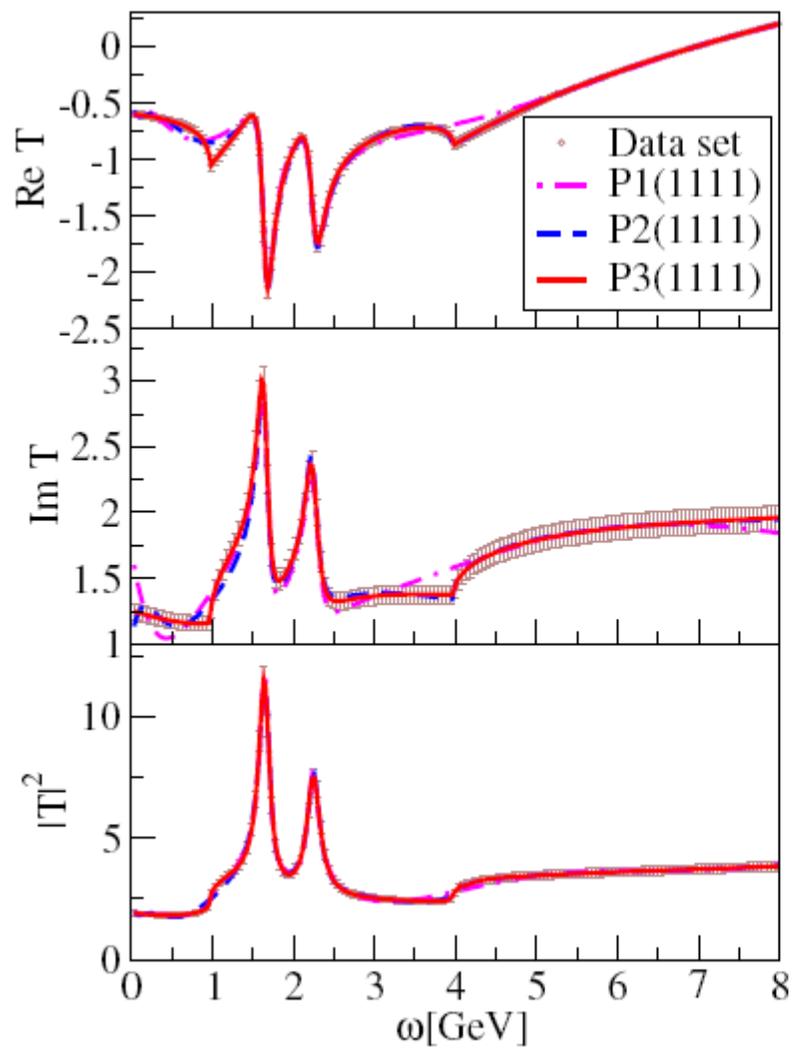
$$\Re e(x, a) = \sqrt{x^2 - 4ax}/2x$$

cut is at $x_0 = 4a$

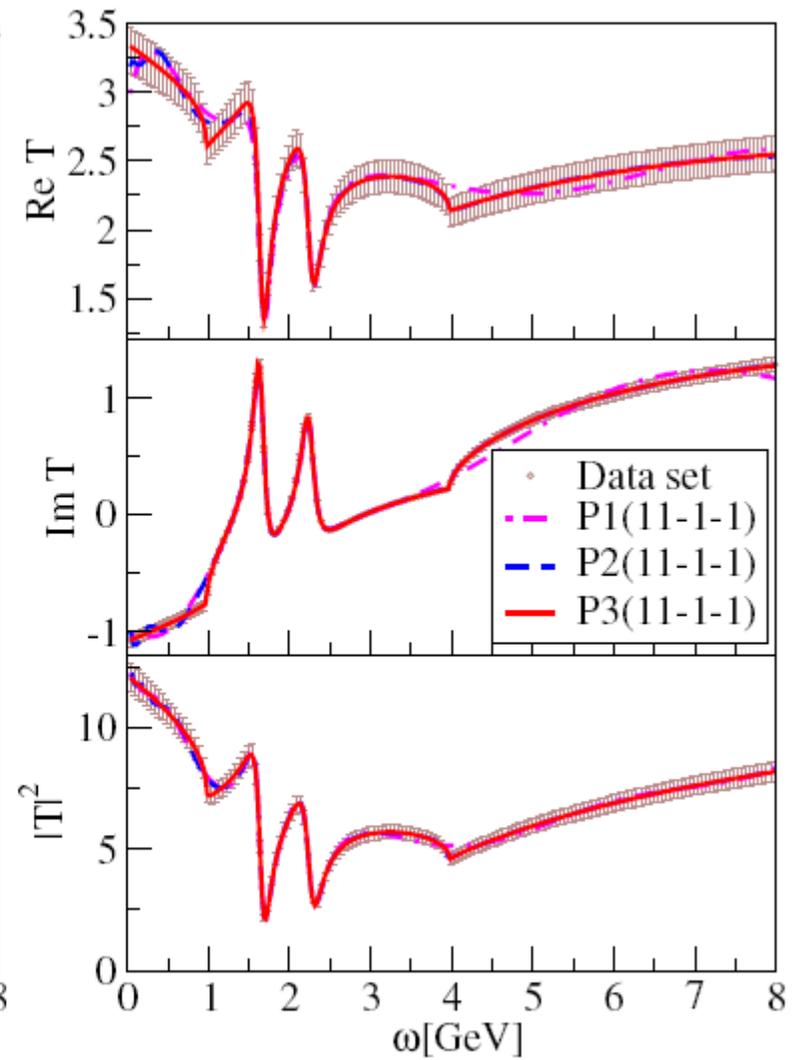
$$\Gamma_i = -2 W_i.$$

$$C_1, C_2, B_1, B_2 = -1, 0, 1$$

r_1	g_1	M_1	Γ_1	r_2	g_2	M_2	Γ_2
0.1	0.09	1.65	0.165	0.09	0.06	2.25	0.2



(a)



(b)

$$B_1 = 1, B_2 = 1$$

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$$B_1 = -1, B_2 = -1$$



C_1	C_2	B_1	B_2	r_1	g_1	M_1	Γ_1	r_2	g_2	M_2	Γ_2	α	x_P	N_1	β	x_Q	N_2	γ	x_R	N_3	$10^2 \chi_R^2$	
Toy-model																						
				0.1	0.09	1.65	0.165	0.09	0.06	2.25	0.2											
Fitted results																						
Strategy a.																						
1	0	0	0	0.100	0.089	1.649	0.165	0.090	0.060	2.249	0.200	2.48	0.97	5								0.03
0	1	0	0	0.099	0.090	1.650	0.165	0.090	0.060	2.249	0.199	3.97	3.97	5								0.01
0	0	1	1	0.098	0.091	1.650	0.165	0.090	0.060	2.250	0.200	1.19	-14.94	7								0.2
0	0	-1	-1	0.099	0.089	1.649	0.1649	0.089	0.059	2.249	0.199	0.99	-9.63	7								0.01
1	0	1	1	0.103	0.100	1.653	0.171	0.101	0.067	2.249	0.221	0.71	-0.23	11								28
1	0	1	1	0.099	0.090	1.650	0.164	0.089	0.060	2.250	0.199	-2.04	-17.58	5	4.27	0.97	5					0.28
1	0	-1	-1	0.097	0.087	1.651	0.161	0.090	0.060	2.250	0.201	0.90	-0.39	20								22.0
1	0	-1	-1	0.099	0.089	1.649	0.164	0.090	0.059	2.249	0.199	2.96	-8.97	6	1.56	0.97	6					1.00
0	1	1	1	0.107	0.088	1.646	0.166	0.093	0.048	2.239	0.197	2.06	-0.89	10								114.79
0	1	1	1	0.099	0.090	1.650	0.165	0.090	0.060	2.250	0.200	1.94	-16.33	5	6.42	3.97	5					0.02
0	1	-1	-1	0.090	0.086	1.651	0.156	0.095	0.058	2.248	0.202	0.969	-0.37	12								238.38
0	1	-1	-1	0.099	0.090	1.650	0.165	0.090	0.060	2.250	0.200	0.81	-7.89	8	1.24	3.97	8					0.06
1	1	1	1	0.085	0.102	1.663	0.171	0.087	0.075	2.262	0.216	1.09	-2.64	10								328.19
1	1	1	1	0.098	0.086	1.650	0.161	0.095	0.058	2.247	0.199	0.44	-0.47	9	1.95	3.97	8					70.37
1	1	1	1	0.099	0.090	1.650	0.164	0.089	0.061	2.251	0.200	4.19	-22.99	5	2.22	3.98	5	1.67	0.97	3		0.24
1	1	-1	-1	0.090	0.105	1.657	0.182	0.078	0.061	2.260	0.189	1.38	-3.12	10								467.54
1	1	-1	-1	0.095	0.098	1.654	0.173	0.086	0.061	2.254	0.198	0.61	-0.20	9	25.91	3.98	8					60.94
1	1	-1	-1	0.100	0.090	1.650	0.165	0.090	0.060	2.250	0.200	1.85	-6.25	3	16.36	3.97	3	1.32	0.98	3		0.72



b. Testing on realistic amplitude

- ***πN elastic***
 - ***GWU/SAID FA02***
 - ***GWU/SAID SP06***
 - ***GWU/SAID WI08***
 - ***DMT***

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Quality of the fit

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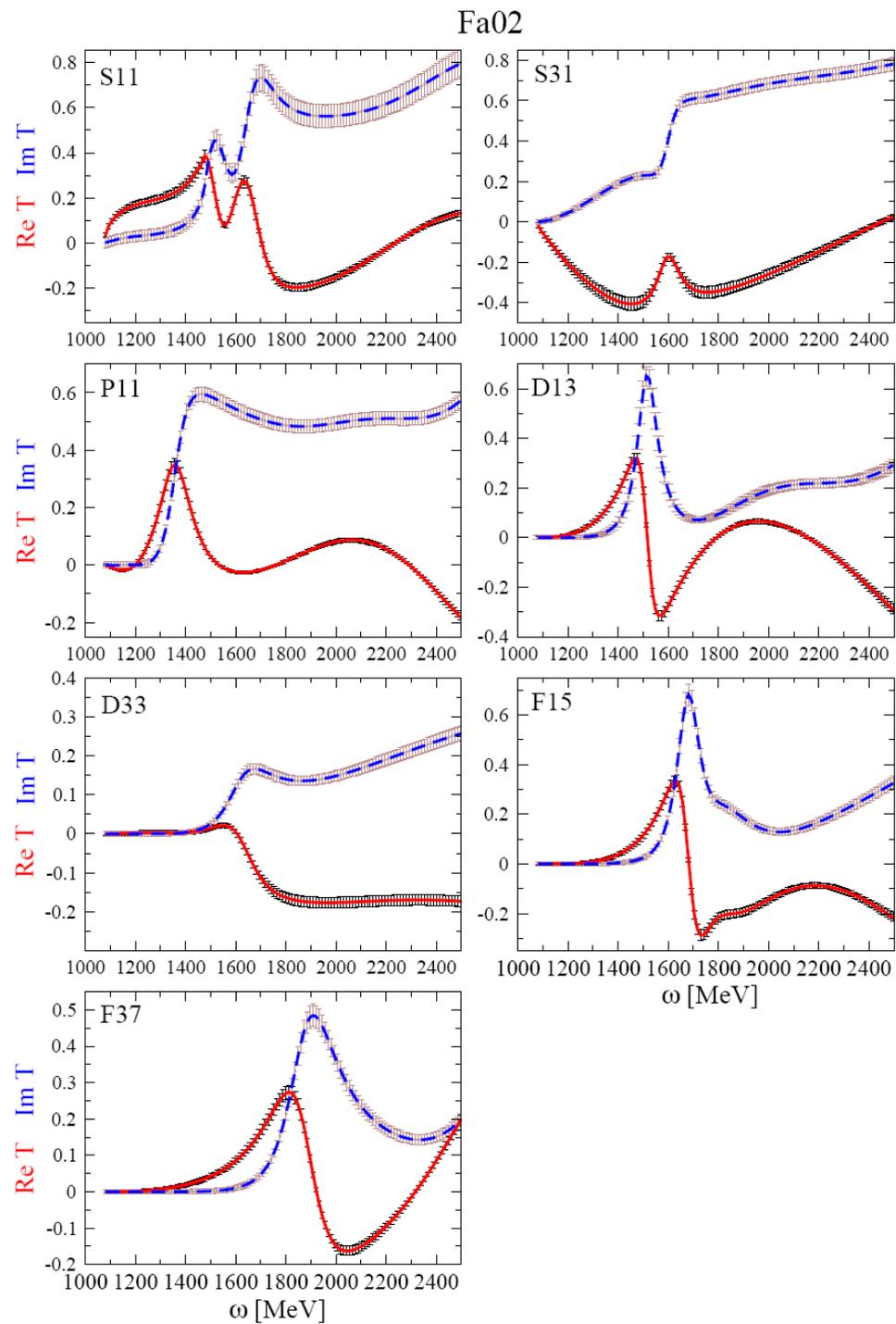
Camogli 2013

**πN elastic scattering
SAID FA02 ED**

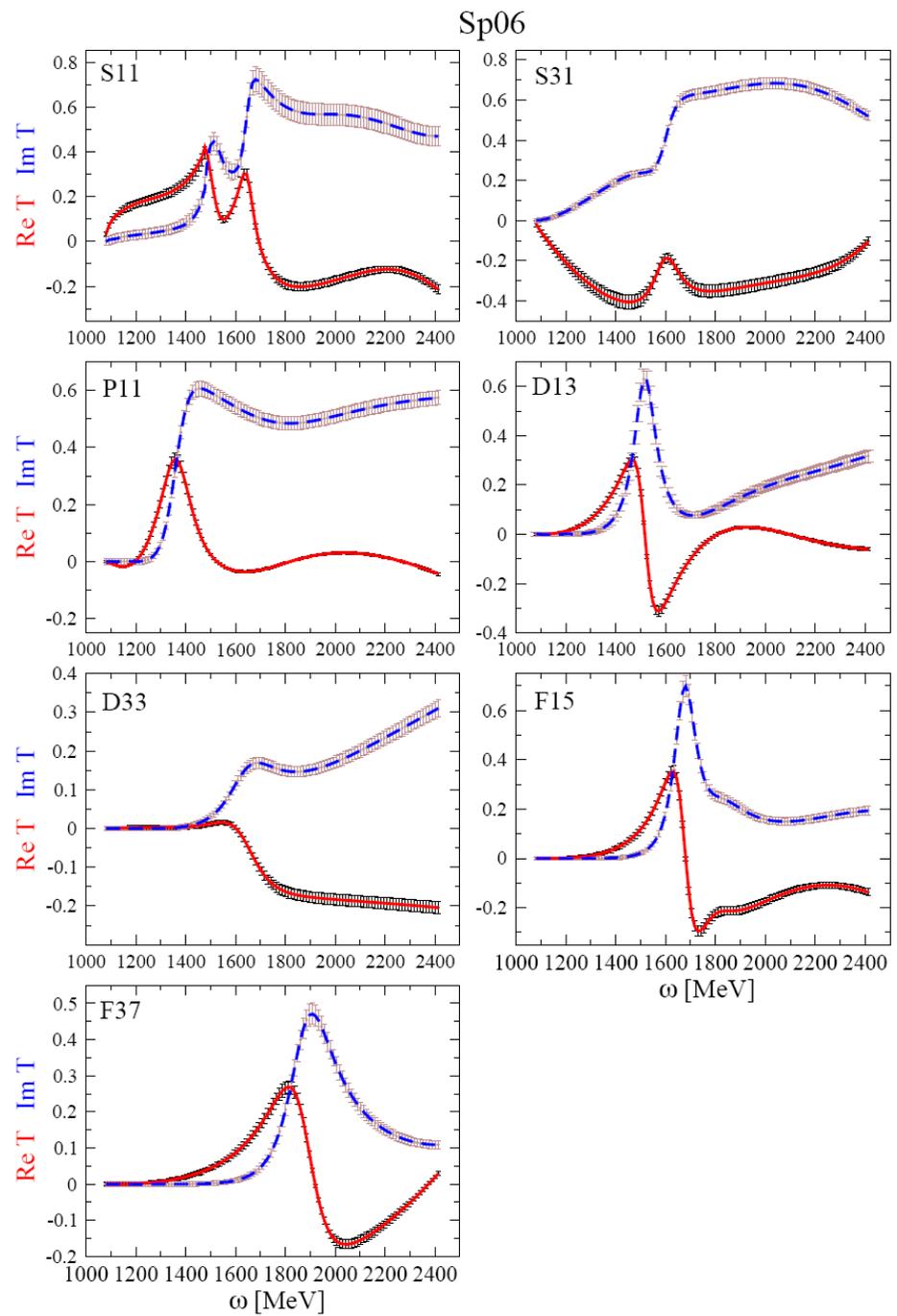
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**πN elastic scattering
SAID SP06 ED**

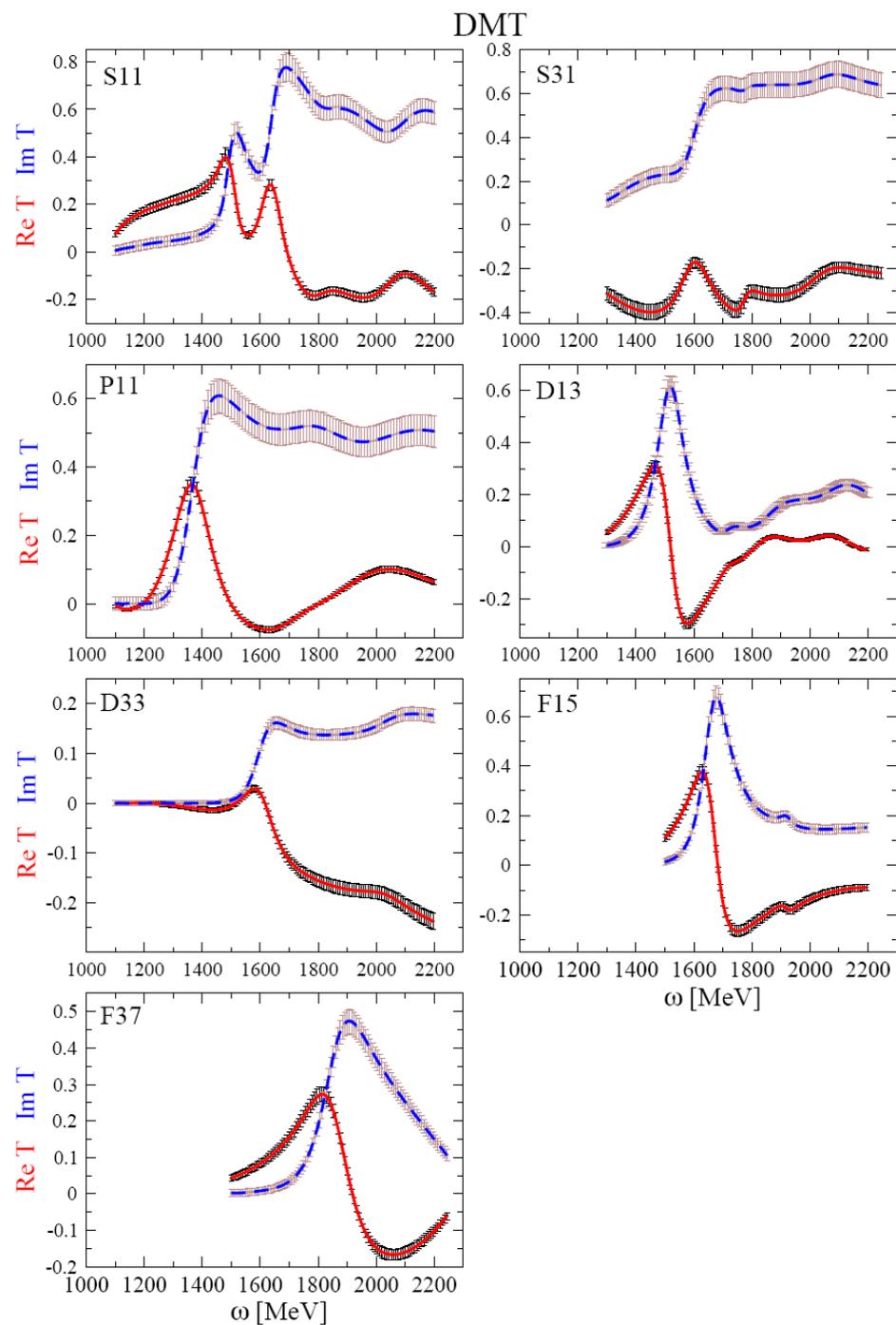


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πN elastic scattering DMT



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PW	Solution	M_1	Γ_1	$ a_1 $	θ_1^0	M_{2-4}	Γ_{2-4}	$ a_{2-4} $	θ_{2-4}^0	$x_P/10^3$	$x_Q/10^3$	$x_R/10^3$	$10^2\chi_R^2$
S ₁₁	DMT [3, 4]	1499	78	14	-45	1631	120	35	-83				
	DMT [22]					1733	180	16	-29				
	DMT L+P	1500	76	13.4	-46	1636	99	22	-94				
						1810	164	9.6	-176				
						2077	220	22.5	-122	1.0	1.077	1.486	0.6
S ₃₁	DMT [3, 4]	1598	148	23	-98	1774	72	3.8	-181				
						1984	254	26	-170				
	DMT L+P	1597	140	21	-104	1771	69	2.2	-172				
						2040	195	7	-109	-11.476	1.077	1.739	0.2
P ₁₁	DMT [3, 4]	1371	190	50	-79	1746	368	11	-54				
	DMT [22]					1997	458	56	-145				
	DMT L+P	1370	190	50	-81	1763	235	5	-56				
						2015	467	36	-99	0.699	1.077	1.537	0.05
D ₁₃	DMT [3, 4]	1515	120	40	-7	1718	96	2.8	-91				
						1854	214	16	-96				
	DMT [22]					2099	216	13	-58				
	DMT L+P	1517	120	40	-5	1721	89	2.1	-76				
						1858	228	15	-87				
						2101	231	14	-49	1.00	1.077	1.266	0.32
D ₃₃	DMT [3, 4, 22]	1604	142	9.4	-63	2042	254	4.84	-75				
	DMT L+P	1605	141	9.3	-63	2023	241	4	-93	0.623	1.077	1.324	0.06
F ₁₅	DMT [3, 4]	1664	114	38	-26	1919	52	1.0	15				
	DMT L+P	1664	114	38	-26	1920	52	1.0	16	0.7	1.077	1.225	0.02
F ₃₇	DMT [3, 4]	1858	208	43	-48								
	DMT L+P	1858	207	43	-49					-3.999	1.077	1.223	0.48





PW	Solution	M_1	Γ_1	$ a_1 $	θ_1^0	M_{2-4}	Γ_{2-4}	$ a_{2-4} $	θ_{2-4}^0	$x_P/10^3$	$x_Q/10^3$	$x_R/10^3$	$10^2 \chi_R^2$
S ₁₁	FA02 [1]	1526	130	33	14	1653	182	69	-55				
	FA02 L+P	1518	121	17	-32	1656	182	68	-39	-60.1	1.077	1.471	0.32
	SP06 [1]	1502	95	16	-16	1648	80	14	-69				
	SP06 L+P	1509	96	15	-21	1645	80	14	-80	-29.5	1.077	1.479	2.90
	WI08 [2]	1499	98	-	-	1647	84	-	-				
	WI08 L+P	1504	78	11	-60	1666	520	-	-				
						1669	517	419	-74	-0.339	1.077	1.483	3.57
S ₃₁	FA02 [1]	1594	118	17	-104								
	FA02 L+P	1596	112	15	-101					-59.3	1.077	1.183	0.48
	SP06 [1]	1595	135	15	-92								
	SP06 L+P	1596	133	18	-105					-16.7	1.077	1.309	0.35
	WI08 [2]	1594	136	-	-								
	WI08 L+P	1598	130	18	-104					-92.7	1.077	1.589	0.57
P ₁₁	FA02 [1]	1357	160	36	-102								
	FA02 L+P	1354	169	38	-98					-100	1.077	1.202	0.66
	SP06 [1]	1359	162	38	-98								
	SP06 L+P	1358	183	53	-92					-62.1	1.077	1.215	0.09
	WI08 [2]	1358	160	-	-								
	WI08 L+P	1357	177	47	-93					-98.9	1.077	1.202	0.07
D ₁₃	FA02 [1]	1514	102	35	-6								
	FA02 L+P	1513	101	34	-9					-67.4	1.077	1.222	0.85
	SP06 [1]	1515	113	38	-5								
	SP06 L+P	1515	113	38	-6					-50.1	1.077	1.216	0.57
	WI08 [2]	1515	110	-	-								
	WI08 L+P	1515	111	38	-5					-81.1	1.077	1.169	0.15
D ₃₃	FA02 [1]	1617	226	16	-47								
	FA02 L+P	1618	227	16	-47					-27.3	1.077	1.204	0.008
	SP06 [1]	1632	253	18	-48								
	SP06 L+P	1635	251	18	-37					-54.1	1.077	1.198	0.009
	WI08 [2]	no results											
	WI08 L+P												
F ₁₅	FA02 [1]	1678	120	43	1	1779	248	47	-61				
	FA02 L+P	1679	118	42	-5	1779	245	31	-84	1.032	1.077	1.549	0.64
	SP06 [1]	1674	115	42	-4	1785^a	244^b	60	-67				
	SP06 L+P	1673	116	43	-11	1776	226	24	-98	-8.99	1.077	1.301	0.08
	WI08 [2]	1674	114	-	-	1779	276	-	-				
	WI08 L+P	1675	115	44	-8	1776	233	34	-99	-51.7	1.077	1.726	0.28
F ₃₇	FA02 [1]	1874	236	57	-34								
	FA02 L+P	1874	236	55	-35					-14.9	1.077	1.739	0.04
	SP06 [1]	1876	227	53	-31								
	SP06 L+P	1876	226	53	-31					-32.8	1.077	1.137	0.06
	WI08 [2]	1883	230	-	-								
	WI08 L+P	1874	227	55	-35					-37.8	1.077	1.736	0.03

N(1860) POLE POSITION

REAL PART

VALUE (MeV)

~~1830~~⁺¹²⁰₋₆₀

1785

DOCUMENT ID TECN COMMENT

ANISOVICH 12A DPWA Multichannel

ARNDT 06 DPWA $\pi N \rightarrow \pi N, \eta N$

-2xIMAGINARY PART

VALUE (MeV)

~~250~~⁺¹⁵⁰₋₅₀

244

DOCUMENT ID TECN COMMENT

ANISOVICH 12A DPWA Multichannel

ARNDT 06 DPWA $\pi N \rightarrow \pi N, \eta N$

N(1860) ELASTIC POLE RESIDUE

MODULUS |r|

VALUE (MeV)

~~50~~ ± 20

43

DOCUMENT ID TECN COMMENT

ANISOVICH 12A DPWA Multichannel

ARNDT 06 DPWA $\pi N \rightarrow \pi N, \eta N$

PHASE θ

VALUE (°)

~~-80~~ ± 40

-64

DOCUMENT ID TECN COMMENT

ANISOVICH 12A DPWA Multichannel

ARNDT 06 DPWA $\pi N \rightarrow \pi N, \eta N$



Error analysis

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The only problem in the model are thresholds. Their number is definitely at this moment insufficient, so we must propose a strategy.

Namely, if we fail to reproduce background exactly (and that we certainly do as soon as number of thresholds is insufficient), the pole terms try to compensate for the approximation made.


$$T(\omega) = \sum_{i=1}^k \frac{a_{-1}^{(i)}}{\omega_i - \omega} + B^L(\omega)$$

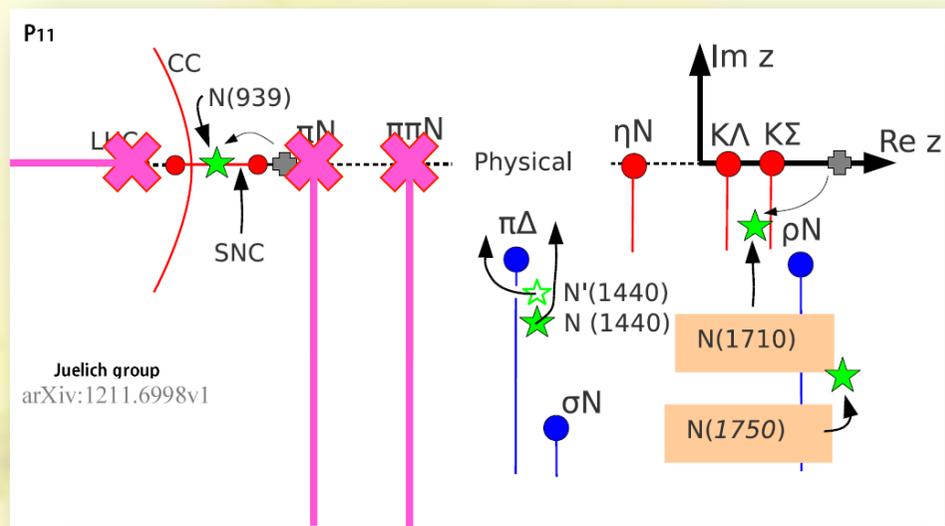
We propose two strategies:

- 1. To fix the pole at the values expected to dominate for a chosen channel**
- 2. To allow poles to vary as a fitting parameter and allow the fit to find optimal choice of two effective thresholds which will replace the exact values**

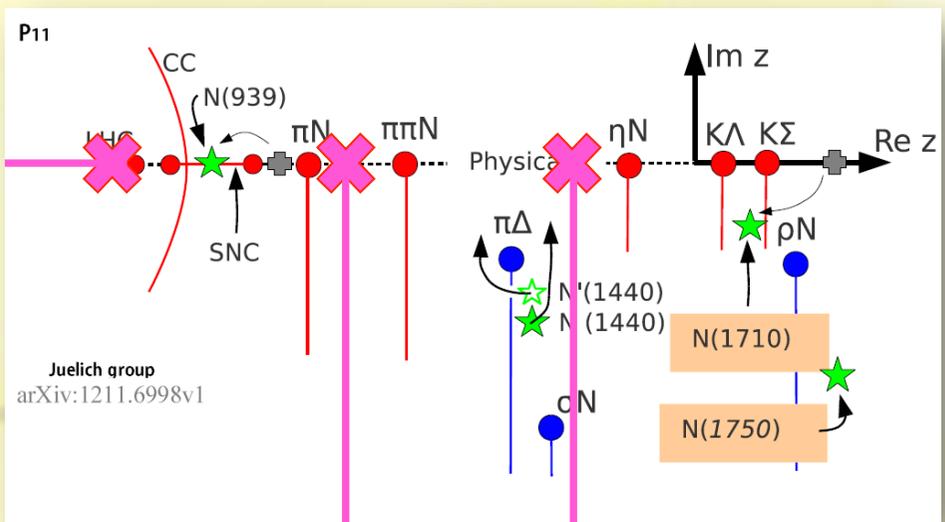


In practice this looks like that:

Option 1:



Option 2:



Conclusion

The L+P method defined as:

$$\begin{aligned}T(\omega) &= \sum_{i=1}^k \frac{a_{-1}^{(i)}}{\omega_i - \omega} + B^L(\omega) \\B^L(\omega) &= \sum_{n=0}^M c_n Z(\omega)^n + \sum_{n=0}^N d_n W(\omega)^n + \dots \\Z(\omega) &= \frac{\alpha - \sqrt{x_P - \omega}}{\alpha + \sqrt{x_P - \omega}}; \quad W(\omega) = \frac{\beta - \sqrt{x_Q - \omega}}{\beta + \sqrt{x_Q - \omega}} + \dots \\a_{-1}^{(i)}, \omega_i, \omega &\in \mathbb{C} \\c_n, x_P, d_n, x_Q, \alpha, \beta \dots &\in \mathbb{R} \\ \text{and } k, M, N \dots &\in \mathbb{N}.\end{aligned}\tag{4}$$

WORKS

