# The Gamma function

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## 1 Gamma function

The Euler Gamma function is defined as

\[ \Gamma(z) \equiv \int_0^\infty dt \, e^{-t} \, t^{z-1} . \] (1)

It is easy to show that \( \Gamma(z) \) satisfy the recursion relation

\[ \Gamma(z + 1) = z \Gamma(z) ; \] (2)

indeed, integrating by parts,

\[
\begin{align*}
\Gamma(z + 1) &= \int_0^\infty dt \, e^{-t} \, t^z \\
&= -e^{-t} \bigg|_0^\infty + z \int_0^\infty dt \, e^{-t} \, t^{z-1} \\
&= z \Gamma(z) .
\end{align*}
\] (3)

For \( z = n \) integer eq. (2) is the recursion relation of the factorial, and thus we have

\[ \Gamma(n + 1) \propto n! ; \] (4)

because in addition \( \Gamma(1) = 1 \) (easily derived from the definition), we have the identification

\[ \Gamma(n + 1) = n! , \] (5)

and in this sense the Gamma function is a complex extension of the factorial.

The sequence of Gamma function computed in all half-integers can be obtained using subsequently the recursion relation (2) and knowing that

\[ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \] (6)

that is easy to compute:

\[
\begin{align*}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty dt \, e^{-t} \, t^{-\frac{1}{2}} \\
&= 2 \int_0^\infty dx \, e^{-x^2} \\
&= 2 \frac{\sqrt{\pi}}{2} = \sqrt{\pi} .
\end{align*}
\] (7)
1.1 Analytical structure

First, from the definition (1), we see that

\[ \Gamma(z) = \overline{\Gamma(z)}, \] (8)

that is to say that Gamma is a real function, and in particular Im \( \Gamma(x) = 0 \) for \( x \in \mathbb{R} \).

Inverting eq. (2) we have

\[ \Gamma(z) = \frac{1}{z} \Gamma(z + 1), \] (9)

and when \( z = 0 \) it diverges (because \( \Gamma(1) = 1 \) is finite). It is evident that \( z = 0 \) corresponds to a simple pole of order 1 and residue 1: indeed

\[ \text{Res } z = 0 \Gamma(z) = \lim_{z \to 0} z \Gamma(z) = \lim_{z \to 0} \Gamma(z + 1) = 1. \] (10)

Formally, we can discover that the Gamma function has simple poles in all negative integers simply iterating the recursion (2):

\[ \Gamma(-n) = \frac{1}{-n} \Gamma(-n + 1) \]
\[ = \ldots \]
\[ = \frac{(-1)^n}{n!} \Gamma(0) \quad (n \in \mathbb{N}) . \] (11)

This brings to the conclusion that \( \Gamma(-n) \) has a simple pole of order 1 and residue \( \frac{(-1)^n}{n!} \). More rigorously, this can be shown in the usual way:

\[ \text{Res } z = 0 \Gamma(z - n) = \lim_{z \to 0} z \Gamma(z - n) \]
\[ = \lim_{z \to 0} \frac{z}{z - n} \Gamma(z - n + 1) \]
\[ = \ldots \]
\[
\begin{align*}
\Gamma(z) &= \lim_{\varepsilon \to 0} \frac{z}{(z-n)(z-n+1)\cdots(z-1)} \Gamma(z) \\
&= \frac{(-1)^n}{n!} \quad (n \in \mathbb{N}). 
\end{align*}
\] (12)

In all the complex plane (except the negative real axis) the Gamma function is well defined. For non-integer negative real values the Gamma function can be analytically continued (as we have seen for example for half-integers, positives and negatives).

We can prove these results in a simpler way. Starting from the definition (1) we rewrite

\[
\begin{align*}
\Gamma(z) &= \int_1^{\infty} dt \ e^{-t} t^{z-1} + \int_0^1 dt \ e^{-t} t^{z-1} \\
&= \int_1^{\infty} dt \ e^{-t} t^{z-1} + \int_0^1 dt \ t^{z-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n \\
&= \int_1^{\infty} dt \ e^{-t} t^{z-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n} 
\end{align*}
\] (13)

where the first integral is an analytic function of \( z \) and the second term gives the position of the poles and their residues.

Moreover, the Gamma function has an essential singularity to complex infinity, because \( \Gamma \left( \frac{1}{z} \right) \) has a non-defined limit for \( z \to 0 \). This is to say that the Gamma function is not well defined in the compactified complex plane.

Now we want to show that near a pole in \( -n \) one has the expansion

\[
\Gamma(z-n) = \frac{(-1)^n}{n!} \left[ \frac{1}{z} + \psi(n+1) + O(z) \right] 
\] (14)

where \( \psi(z) \) is the logarithmic derivative of the Gamma function, defined in (41); note that \( \psi(n+1) \) has an explicit expression given in eq. (44). First, note that

\[
\begin{align*}
\Gamma(z-n) &= \frac{1}{z-n} \Gamma(z-n+1) \\
&= \cdots \\
&= \frac{1}{(z-n)(z-n+1)\cdots(z-1) z(z+1)\cdots(z+n)} \Gamma(z+n+1) \\
&= \frac{1}{z} G(z,n) 
\end{align*}
\] (15)

where we have defined

\[
G(z,n) = \frac{\Gamma(z+n+1)}{(z^2-n^2)\cdots(z^2-1)} = \Gamma(z+n+1) \frac{(-1)^n}{(n!)^2} \left[ 1 + O(z^2) \right]. 
\] (16)

By Taylor expanding \( G(z,n) \) around \( z = 0 \) in (15) we get

\[
\begin{align*}
\Gamma(z-n) &= \frac{1}{z} \left[ G(0,n) + z G'(0,n) + O(z^2) \right] \\
&= \frac{(-1)^n}{n!} \left[ \frac{1}{z} + \frac{1}{n!} \Gamma'(n+1) + O(z) \right], 
\end{align*}
\] (17)

that is the claimed result.
1.2 Alternative definitions

There are some alternative definitions of the Gamma function. One, due to Euler, is

\[ \Gamma(z) = \frac{1}{z \prod_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^z}{1 + \frac{z}{n}}} = \lim_{n \to \infty} \frac{n! (n+1)^z}{z(z+1) \cdots (z+n)} \tag{18} \]

It is easy to prove (using the expression with the limit) that with this definition the recursion relation (2) is satisfied, and also that for \( z = 1 \) gives \( \Gamma(1) = 1 \).

Another one, due to Weierstrass, is

\[ \Gamma(z) = e^{-\gamma z} \frac{1}{z \prod_{n=1}^{\infty} \frac{e^{z/n}}{1 + \frac{z}{n}}} = e^{-\gamma z} \lim_{n \to \infty} \frac{n! \exp[z (1 + \frac{1}{2} + \ldots + \frac{1}{n})]}{z(z+1) \cdots (z+n)} , \tag{19} \]

where \( \gamma \) is the Euler-Mascheroni number defined in Section 2.2. The equivalence to the Euler definition can be seen by taking the logarithm of this two forms: the first gives

\[ \log \Gamma(z) = -\log z + \sum_{n=0}^{\infty} \left[ z \log \left(1 + \frac{1}{n}\right) - \log \left(1 + \frac{z}{n}\right) \right] \] \tag{20} \]

and the second gives

\[ \log \Gamma(z) = -\log z - \gamma z + \sum_{n=0}^{\infty} \left[ \frac{z}{n} - \log \left(1 + \frac{z}{n}\right) \right] . \tag{21} \]

Comparing this two expressions we note that they are equal because of eq. (48). Note that this two expression are useful in order to numerically approximate \( \log \Gamma(z) \), by truncating the series to a certain value of \( n \).

1.3 Euler reflection formula

A useful formula is

\[ \Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \] \tag{22} \]

called the Euler reflection formula. To demonstrate it, consider the alternative definition (18) and note that

\[ \frac{1}{\Gamma(z) \Gamma(1-z)} = \frac{1}{-z \Gamma(z) \Gamma(-z)} \]

\[ = \frac{z(-z)}{-z} \prod_{n=1}^{\infty} \frac{(1 + \frac{z}{n}) (1 - \frac{z}{n})}{(1 + \frac{1}{n})^2 (1 + \frac{1}{n})^z} \]

\[ = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \]

\[ = \frac{\sin(\pi z)}{\pi} \] \tag{23} \]

where we have used

\[ \sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) . \] \tag{24} \]
1.4 Stirling approximation

The Stirling approximation for the factorial is

\[ n! = \sqrt{2\pi n} e^{-n} n^n \]  \hspace{1cm} (25)

which is valid for large \( n \). To demonstrate it, we perform a saddle point approximation to the integral (1):

\[ \Gamma(z + 1) = \int_0^\infty dt \, e^{-t} t^z \]
\[ = \int_0^\infty dt \, e^{-t+z \log t} \]

now we expand the exponent near its maximum and get

\[ \approx \int_0^\infty dt \, \exp \left( -z + z \log z - \frac{1}{2z} t^2 \right) \]
\[ = e^{-z} z^z \int_0^\infty dt \, e^{-\frac{1}{2z} t^2} \]
\[ = \sqrt{2\pi z} e^{-z} z^z \]  \hspace{1cm} (26)

that is the claimed result.

1.5 Derivative recursion relation

Consider the Taylor expansion near \( z = 0 \) of \( \Gamma(z + z_0 + 1) \):

\[ \Gamma(z + z_0 + 1) = \sum_{k=0}^{\infty} \frac{\Gamma(k)(z_0 + 1)}{k!} z^k ; \]  \hspace{1cm} (27)

using eq. (2) this is equal to \( (z + z_0) \Gamma(z + z_0) \), that expanded is

\[ (z + z_0) \Gamma(z + z_0) = (z + z_0) \sum_{k=0}^{\infty} \frac{\Gamma(k)(z_0)}{k!} z^k \]
\[ = z_0 \Gamma(z_0) + \sum_{k=1}^{\infty} \frac{z^k}{k!} \left[ z_0 \Gamma^{(k)}(z_0) + k \Gamma^{(k-1)}(z_0) \right] . \]  \hspace{1cm} (28)

Comparing the two expansions, we get the equality \( k > 0 \)

\[ \Gamma^{(k)}(z_0 + 1) = z_0 \Gamma^{(k)}(z_0) + k \Gamma^{(k-1)}(z_0) \].  \hspace{1cm} (29)

1.6 Inverse Gamma function

We can define an inverse Gamma function

\[ \Delta(z) \equiv \frac{1}{\Gamma(z)} \]  \hspace{1cm} (30)
which is an entire function (analytical in all the complex plane). The recursion relation (2) becomes

$$\Delta(z) = z \Delta(z + 1).$$  \hfill (31)

We are able to produce an integral representation of the $\Delta$ function: consider the Laplace transform of the function $t^{z-1}$ with respect to $t$:

$$\int_0^\infty \! \! dt \ e^{-st} t^{z-1} = s^{-z} \Gamma(z)$$ \hfill (32)

and then we have that the inverse Laplace tranform of $s^{-z}$ is

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \! \! ds \ e^s \ s^{-z} = \Delta(z) t^{z-1}$$ \hfill (33)

where $c$ has to be choosen to the right of all the singularities of $s^{-z}$, that is $c > 0$. Now, by choosing $t = 1$, we get

$$\Delta(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \! \! ds \ e^s \ s^{-z} ,$$ \hfill (34)

where the integration path can be deformed to be a contour of the negative real axis. Note that for positive integers $z = n + 1$ the branch cut vanishes and only a pole in $s = 0$ remains, and then the path can be closed around $s = 0$ to obtain

$$\frac{1}{n!} = \frac{1}{2\pi i} \oint \! \! ds \ e^s \ s^{-n-1}$$ \hfill (35)

that can be also easily demonstrated using the residue theorem.

Now consider the Taylor expansion near $z = 0$ of $\Delta(z + z_0)$:

$$\Delta(z + z_0) = \sum_{k=0}^{\infty} \frac{\Delta^{(k)}(z_0)}{k!} z_0^k ;$$ \hfill (36)

using eq. (31) this is equal to $(z + z_0) \Delta(z + z_0 + 1)$, that expanded is

$$(z + z_0) \Delta(z + z_0 + 1) = (z + z_0) \sum_{k=0}^{\infty} \frac{\Delta^{(k)}(z_0 + 1)}{k!} z_0^k$$
\[ z_0 \Delta(z_0 + 1) + \sum_{k=1}^{\infty} \frac{z_0}{k!} \left[ z_0 \Delta^{(k)}(z_0 + 1) + k \Delta^{(k-1)}(z_0 + 1) \right]. \]  

(37)

Comparing the two expansions, we get the equality \((k \geq 1)\)

\[ \Delta^{(k)}(z_0) = z_0 \Delta^{(k)}(z_0 + 1) + k \Delta^{(k-1)}(z_0 + 1) \]  

(38)

and, for \(z_0 = 0\),

\[ \Delta^{(k)}(0) = k \Delta^{(k-1)}(1). \]  

(39)

1.6.1 Saddle point approximation

The analogous of the Stirling approximation can be done for the \(\Delta(z)\) function. The result is:

\[ \Delta(z) \approx \sqrt{\frac{z}{2\pi}} e^{\frac{z}{2}} z^{-\frac{1}{2}}. \]  

(40)
2 Gamma related functions

2.1 Log derivative $\psi(z)$

Define the logarithmic derivative of the Gamma function as follows

$$
\psi(z) \equiv \frac{d}{dz} \log \Gamma(z) .
$$

(41)

Form the recursion relation (2) it follows that

$$
\psi(z + 1) = \psi(z) + \frac{1}{z} ,
$$

(42)

as we can easily demonstrate:

$$
\psi(z + 1) = \frac{d}{dz} \log \Gamma(z + 1)
= \frac{d}{dz} \log [z \Gamma(z)]
= \frac{1}{z} + \psi(z)
$$

(43)

(note that the same result can be obtained from eq. (29) for $k = 1$). For a positive integer $n$, we can iterate the recursion (42) to obtain

$$
\psi(n + 1) = \psi(1) + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} ,
$$

(44)

and the value of $\psi(1)$ is

$$
\psi(1) = -\gamma
$$

(45)

where $\gamma$ is the Euler-Mascheroni number, as demonstrated in Section 2.2.

The functions

$$
\psi_n(z) \equiv \psi^{(n)}(z) = \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z)
$$

(46)

are also defined. Iterating the derivative in (42) we obtain the recursion relation

$$
\psi_n(z + 1) = \psi_n(z) + (-1)^n n! \frac{1}{z^{n+1}} .
$$

(47)

2.2 Euler-Mascheroni number

The Euler-Mascheroni number is defined as

$$
\gamma \equiv \sum_{n=1}^{\infty} \left[ \frac{1}{n} - \log \left( 1 + \frac{1}{n} \right) \right] ,
$$

(48)

and its value is

$$
\gamma = 0.577216 \ldots .
$$

(49)
Note that
\[
\sum_{n=1}^{\infty} \log \left(1 + \frac{1}{n}\right) = \lim_{N \to \infty} \sum_{n=1}^{N} \log \left(1 + \frac{1}{n}\right)
= \lim_{N \to \infty} \sum_{n=1}^{N} [\log(1 + n) - \log n]
= \lim_{N \to \infty} \log(1 + N)
= \lim_{N \to \infty} \log \frac{N}{n},
\]
(50)
where we noted that the sum is a telescopic sum and that in the limit \( N \to \infty \) \( \log(1 + N) \) and \( \log N \) are equivalent. Thus we can rewrite eq. (48) as
\[
\gamma = \lim_{N \to \infty} \left[ \sum_{n=1}^{N} \frac{1}{n} - \log N \right].
\]
(51)

Now we prove that \( \psi(1) = -\gamma \). To do this, let us consider eq. (20), and take its derivative:
\[
\psi(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} \left[ \log \left(1 + \frac{1}{n}\right) - \frac{1}{n} \frac{1}{1 + \frac{1}{z}} \right].
\]
(52)
For \( z = 1 \) we get
\[
\psi(1) = -1 + \sum_{n=1}^{\infty} \left[ \log \left(1 + \frac{1}{n}\right) - \frac{1}{1 + n} \right]
= -\sum_{n=1}^{\infty} \left[ \frac{1}{n} - \log \left(1 + \frac{1}{n}\right) \right],
\]
(53)
in which we recognise the definition (48).

### 2.3 Generalized Gamma functions

You can also define an incomplete Gamma function (or plica function)
\[
\Gamma(z, a) \equiv \int_a^{\infty} dt e^{-t} t^{z-1}
\]
(54)
and a truncated Gamma function
\[
\gamma(z, a) \equiv \int_0^a dt e^{-t} t^{z-1},
\]
(55)
that have a branch cut on the negative real axis in the \( a \) complex plane. Obviously
\[
\Gamma(z) = \Gamma(z, a) + \gamma(z, a).
\]
(56)
For integer $z = k + 1$, integrating repeatedly by parts we get

$$\Gamma(k + 1, a) = k! e^{-a} \sum_{n=0}^{k} \frac{a^n}{n!}$$  \hspace{1cm} (57)$$

$$\gamma(k + 1, a) = k! \left( 1 - e^{-a} \sum_{n=0}^{k} \frac{a^n}{n!} \right).$$  \hspace{1cm} (58)$$

2.4 Beta function

The Beta function is defined as

$$B(a, b) \equiv \int_{0}^{1} dx \ x^{a-1} (1-x)^{b-1} = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}.$$  \hspace{1cm} (59)$$

To show the last equality, consider the product

$$\Gamma(a) \Gamma(b) = \int_{0}^{\infty} dt \ e^{-t} t^{a-1} \int_{0}^{\infty} du \ e^{-u} u^{b-1}$$

$$= 4 \int_{0}^{\infty} dx \ e^{-x^2} x^{2a-1} \int_{0}^{\infty} dy \ e^{-y^2} y^{2b-1}$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ e^{-(x^2+y^2)} |x|^{2a-1} |y|^{2b-1}$$

$$= \int_{0}^{2\pi} d\theta \ |\cos^{2a-1} \theta \ \sin^{2b-1} \theta| \int_{0}^{\infty} dr \ e^{-r^2} r^{2a+2b-1}$$

$$= 2 \Gamma(a+b) \int_{0}^{\pi/2} d\theta \ \cos^{2a-1} \theta \ \sin^{2b-1} \theta$$

$$= \Gamma(a+b) \int_{0}^{1} dx \ x^{a-1} (1-x)^{b-1}.$$  \hspace{1cm} (60)$$