The local functors of points of supermanifolds

L. Balduzzi, C. Carmeli, R. Fioresi

e-mail: luigi.balduzzi@ge.infn.it, claudio.carmeli@ge.infn.it, fioresi@dm.unibo.it

Abstract

We study the local functor of points (which we call the Weil–Berezin functor) for smooth supermanifolds, providing a characterization, representability theorems and applications to differential calculus.

1 Introduction

Since the 1970s the foundations of supergeometry have been investigated by several physicists and mathematicians. Most of the treatments (e.g. [BL75, Kost77, Ber87, Lei80, Man88, DM99, Var04]) present supermanifolds as classical manifolds where the structure sheaf is modified so that the sections are allowed to take values in \( \mathbb{Z}_2 \)-graded commutative algebras and the sheaf itself is assumed to be locally of the form \( C^\infty(\mathbb{R}^p) \otimes \Lambda_q \), with \( \Lambda_q \) denoting the Grassmann algebra in \( q \) generators. This approach is very much in the spirit of classical algebraic geometry and dates back to the seminal works of F. A. Berezin and D. A. Leites [BL75], and B. Kostant [Kost77].

It is nevertheless only later in [Man88, DM99], that the parallelism with classical algebraic geometry is fully worked out and the functorial language starts to be used systematically. In particular the functor of points approach becomes a powerful device allowing, among other things, one to recover some geometric intuition by giving a rigorous meaning to otherwise just formal expressions. In this approach, a supermanifold \( M \) is fully recovered by the knowledge of its functor of points, \( S \mapsto M(S) := \text{Hom}(S, M) \), which associates to a supermanifold \( M \), the set of its \( S \)-points for every supermanifold \( S \). The crucial result in this context is Yoneda’s lemma which establishes a bijective correspondence between morphisms of supermanifolds and natural transformations between their corresponding functors of points.

Other approaches to the theory of supermanifolds involving new local models and possibly non Hausdorff topologies were developed later ([Bat80,
Rog80, DeW84, Sch84]). For a detailed review of some of these approaches, that we do not pursue here, we refer the reader to [BBHR91, Rog87].

This paper is devoted to understanding the approach to supermanifolds theory via the \textit{local functor of points}, which associates to each smooth supermanifold $M$ the set of its $A$-points for all super Weil algebras $A$. These are finite dimensional commutative superalgebras of the form $A = R \oplus \hat{A}$ with $\hat{A}$ a nilpotent ideal. The set of the $A$-points of the smooth supermanifold $M$ is defined as $M_A = \text{Hom}_{\text{SA}}(\mathcal{O}(M), A)$, in striking analogy with the functor of points previously described. In fact, when $A$ is a finite dimensional Grassmann algebra, $M_A$ is indeed the set of the $R^{0|q}$-points of the supermanifold $M$ in the sense specified above, for suitable $q$. As we have defined it, the local functor of points does \textit{not} determine the supermanifold, unless we put an extra structure on $M_A$, in other words, unless we carefully define the category image for the functor $A \mapsto M_A$.

Our approach is a slight modification of the one in [Sch84, Vor84], by Schwarz and Voronov, the main difference being that they consider Grassmann algebras instead of super Weil algebras. In this sense our work is mainly providing additional insight into well known results and clarifies the representability issues often overlooked in most of the literature. Moreover the local functor of points that we examine in our work (\textit{Weil–Berezin functor}) has the advantage of being able to bring differential calculus naturally into the picture.

The paper is organized as follows.

In section 2 we review some basic definitions of supergeometry like the definition of superspace, supermanifold and its associated functor of points.

In section 3 we introduce super Weil algebras with their basic properties and we define the functor of the $A$-points of a supermanifold $M$, $A \mapsto M_A$ from the category of super Weil algebras to the category of sets. We show this functor does not characterize the supermanifold $M$. In order to obtain this, the image category needs to be suitably specialized by giving to each set $M_A$ an extra structure.

In section 4, we obtain a bijective correspondence between supermanifold morphisms and natural transformations between the functors of $A$-points, by endowing the set $M_A$ with the structure of an $A_0$-smooth manifold. For this new functor, called the \textit{Weil–Berezin functor of $M$} the analogue of Yoneda’s
lemma holds and, as a consequence, supermanifolds embed in a full and faithful way into the category of Weil–Berezin functors (Schwarz embedding) and we can prove a representability theorem. We end the section by giving a brief account of the functor of $\Lambda$-points originally described by Schwarz, which is the restriction of the Weil–Berezin functor to Grassmann algebras.

In section 5 we examine some aspects of super differential calculus on supermanifolds in the language of the Weil–Berezin functor, establishing a connection between our treatment and Kostant’s seminal approach to supergeometry and proving the Weil transitivity theorem.

**Acknowledgements.** We want to thank prof. G. Cassinelli, prof. M. Duflo, prof. P. Michor and prof. V. S. Varadarajan for helpful discussions.

## 2 Basic Definitions of Supergeometry

In this section we recall few basic definitions in supergeometry. Our main references are [Kost77, Man88, DM99, Var04].

Let $\mathbb{R}$ be our ground field.

A **super vector space** is a $\mathbb{Z}_2$-graded vector space, i.e. $V = V_0 \oplus V_1$; the elements in $V_0$ are called *even*, those in $V_1$ *odd*. An element $v \neq 0$ in $V_0 \cup V_1$ is said *homogeneous* and $p(v)$ denotes its parity: $p(v) = 0$ if $v \in V_0$, $p(v) = 1$ if $v \in V_1$. $\mathbb{R}^{p|q}$ denotes the super vector space $\mathbb{R}^p \oplus \mathbb{R}^q$. A **superalgebra** $A$ is an algebra that is also a super vector space, $A = A_0 \oplus A_1$, and such that $A_i A_j \subseteq A_{i+j}$ (mod 2). $A_0$ is an algebra, while $A_1$ is an $A_0$-module. $A$ is said to be **commutative** if for any two homogeneous elements $x$ and $y$, $xy = (-1)^{p(x)p(y)}yx$. The category of real commutative superalgebras is denoted by $\text{SAlg}$ and all our superalgebras are assumed to be in $\text{SAlg}$.

**Definition 2.1.** A **superspace** $S = (|S|, \mathcal{O}_S)$ is a topological space $|S|$, endowed with a sheaf of superalgebras $\mathcal{O}_S$ such that the stalk at each point $x \in |S|$, denoted by $\mathcal{O}_{S,x}$, is a local superalgebra (i.e. it has a unique graded maximal ideal). A **morphism** $\varphi : S \rightarrow T$ of superspaces is a pair $(|\varphi|, \varphi^*)$, where $|\varphi| : |S| \rightarrow |T|$ is a continuous map of topological spaces and $\varphi^* : \mathcal{O}_T \rightarrow |\varphi|^* \mathcal{O}_S$, called *pullback*, is such that $\varphi^*(\mathcal{M}_{|\varphi|(x)}) \subseteq \mathcal{M}_x$ where $\mathcal{M}_{|\varphi|(x)}$ and $\mathcal{M}_x$ denote the maximal ideals in the stalks $\mathcal{O}_{T,|\varphi|(x)}$ and $\mathcal{O}_{S,x}$ respectively.
Example 2.2 (The smooth local model). The superspace $\mathbb{R}^{p|q}$ is the topological space $\mathbb{R}^p$ endowed with the following sheaf of superalgebras. For any open set $U \subseteq \mathbb{R}^p$ define $\mathcal{O}_{\mathbb{R}^{p|q}}(U) := C_\infty^\text{c}(U) \otimes \Lambda(\vartheta_1, \ldots, \vartheta_q)$, where $\Lambda(\vartheta_1, \ldots, \vartheta_q)$ is the real exterior algebra (or Grassmann algebra) generated by the $q$ variables $\vartheta_1, \ldots, \vartheta_q$ and $C_\infty^\text{c}$ denotes the $C^\infty$ sheaf on $\mathbb{R}^p$.

Definition 2.3. A (smooth) supermanifold of dimension $p|q$ is a superspace $M = (|M|, \mathcal{O}_M)$ which is locally isomorphic to $\mathbb{R}^{p|q}$, i.e. for all $x \in |M|$ there exist open sets $x \in U \subseteq |M|$ and $U \subseteq \mathbb{R}^p$ such that: $\mathcal{O}_M|_U \cong \mathcal{O}_{\mathbb{R}^{p|q}}|_U$. In particular supermanifolds of the form $(U, \mathcal{O}_{\mathbb{R}^{p|q}}|_U)$ are called superdomains. A morphism of supermanifolds is simply a morphism of superspaces. $\text{SMan}$ denotes the category of supermanifolds. We shall denote with $\mathcal{O}(M)$ the superalgebra $\mathcal{O}_M(|M|)$ of global sections on the supermanifold $M$.

If $U$ is open in $|M|$, $(U, \mathcal{O}_M|_U)$ is also a supermanifold and it is called the open supermanifold associated with $U$. We shall often refer to it just by $U$, whenever no confusion is possible.

Suppose $M$ is a supermanifold and $U$ is an open subset of $|M|$. Let $J_M(U)$ be the ideal of the nilpotent elements of $\mathcal{O}_M(U)$. $\mathcal{O}_M/J_M$ defines a sheaf of purely even algebras over $|M|$ locally isomorphic to $C^\infty(\mathbb{R}^p)$. Therefore $\tilde{M} := (|M|, \mathcal{O}_M/J_M)$ defines a classical smooth manifold, called the reduced manifold associated with $M$. The projection $s \mapsto \tilde{s} := s + J_M(U)$, with $s \in \mathcal{O}_M(U)$, is the pullback of the embedding $\tilde{M} \rightarrow M$. If $\varphi$ is a supermanifold morphism, since $|\varphi|^*(\tilde{s}) = \varphi^*(s)$, the morphism $|\varphi|$ is automatically smooth.

There are several equivalent ways to assign a morphism between two supermanifolds. The following result can be found in [Man88, ch. 4].

Theorem 2.4 (Chart theorem). Let $U$ and $V$ two smooth superdomains, i.e. two open subsupermanifolds of $\mathbb{R}^{p|q}$ and $\mathbb{R}^m|n$ respectively. There is a bijective correspondence between
1. superspace morphisms $U \rightarrow V$;
2. superalgebra morphisms $\mathcal{O}(V) \rightarrow \mathcal{O}(U)$;
3. the set of pullbacks of a fixed coordinate system on $V$, i.e. $(m|n)$-uples
   
   $$(s_1, \ldots, s_m, t_1, \ldots, t_n) \in \mathcal{O}(U)^m_0 \times \mathcal{O}(U)^n_1$$

   such that $(\tilde{s}_1(x), \ldots, \tilde{s}_m(x)) \in |V|$ for each $x \in |U|$.
Any supermanifold morphism $M \to N$ is then uniquely determined by
a collection of local maps, once atlases on $M$ and $N$ have been fixed. A
morphism can hence be given by describing it in local coordinates.

Since we are considering the smooth category a further simplification
occurs: we can assign a morphism between supermanifolds by assigning the
pullbacks of the global sections (see [Kost77, § 2.15]), i.e.

$$\text{Hom}_{\text{SMan}}(M, N) \cong \text{Hom}_{\text{SAlg}}(\mathcal{O}(N), \mathcal{O}(M)).$$

The theory of supermanifolds resembles very closely the classical theory.
One can, for example, define tangent bundles, vector fields and the differential
of a morphism similarly to the classical case. For more details see [Kost77,
Lei80, Man88, DM99, Var04].

Due to the presence of nilpotent elements in the structure sheaf of a
supermanifold, supergeometry can also be equivalently and very effectively
studied using the language of functor of points, a very useful tool in algebraic
geometry.

Let us first fix some notation we will use throughout the paper. If $A$ and
$B$ are two categories, $[A, B]$ denotes the category of functors between $A$ and
$B$ (notice that in general $[A, B]$ won’t have small hom-sets). Clearly, the
morphisms in $[A, B]$ are the natural transformations. Moreover we denote by
$A^{\text{op}}$ the opposite category of $A$, so that the category of contravariant functors
between $A$ and $B$ is identified with $[A^{\text{op}}, B]$ (see [Mac71]).

**Definition 2.5.** Given a supermanifold $M$, we define its *functor of points*

$$M(\cdot) : \text{SMan}^{\text{op}} \longrightarrow \text{Set}, \quad S \mapsto M(S) := \text{Hom}(S, M)$$

as the functor from the opposite category of supermanifolds to the category
of sets defined on the morphisms as usual: $M(\varphi)f = f \circ \varphi$, where $\varphi : T \to S$,
$f \in M(S)$. The elements in $M(S)$ are also called the $S$-points of $M$.

Given two supermanifolds $M$ and $N$, Yoneda’s lemma (a general result
valid for all categories with small hom-sets) establishes a bijective correspon-
dence

$$\text{Hom}_{\text{SMan}}(M, N) \longleftrightarrow \text{Hom}_{\text{SMan}^{\text{op}}, \text{Set}}(M(\cdot), N(\cdot))$$

between the morphisms $M \to N$ and the natural transformations $M(\cdot) \to N(\cdot)$ (see [Mac71, ch. 3] or [EH00, ch. 6]). This allows us to view a morphism

5
of supermanifolds as a family of morphisms $M(S) \to N(S)$ depending functorially on the supermanifold $S$. In other words, Yoneda’s lemma provides a full and faithful immersion

$$\mathcal{Y}: \text{SMan} \longrightarrow [\text{SMan}^{\text{op}}, \text{Set}].$$

There are however objects in $[\text{SMan}^{\text{op}}, \text{Set}]$ that do not arise as the functors of points of a supermanifold. We say that a functor $\mathcal{F} \in [\text{SMan}^{\text{op}}, \text{Set}]$ is representable if it is isomorphic to the functor of points of a supermanifold.

We now want to recall a representability criterion, which allows to single out, among all the functors from the category of supermanifolds to sets, those that are representable, (see [DG70, ch. 1], [FLV07, A.13] for more details).

**Theorem 2.6 (Representability criterion).** A functor $\mathcal{F}: \text{SMan}^{\text{op}} \to \text{Set}$ is representable if and only if:

1. $\mathcal{F}$ is a sheaf, i.e. it has the sheaf property;
2. $\mathcal{F}$ is covered by open supermanifold subfunctors $\{ \mathcal{U}_i \}$.

### 3 Super Weil algebras and $A$-points

In this section we introduce the category $\text{SWA}$ of super Weil algebras. These are finite dimensional commutative superalgebras with a nilpotent graded ideal of codimension one. Super Weil algebras are the basic ingredient in the definition of the Weil–Berezin functor and the Schwarz embedding. The simplest examples of super Weil algebras are finite dimensional Grassmann algebras. These are the only super Weil algebras that can be interpreted as algebras of global sections of supermanifolds, namely $\mathbb{R}^{0|q}$.

We now define the category of super Weil algebras. The treatment follows closely that contained in [KMS93, § 35] for the classical case.

**Definition 3.1.** We say that $A$ is a (real) super Weil algebra if it is a commutative unital superalgebra over $\mathbb{R}$ and

1. $\dim A < \infty$,
2. $A = \mathbb{R} \oplus \hat{A}$, where $\hat{A} = \hat{A}_0 \oplus \hat{A}_1$ is a graded nilpotent ideal.

The category of super Weil algebras is denoted by $\text{SWA}$. The height of $A$ is the lowest $r$ such that $\hat{A}^{r+1} = 0$ and the width of $A$ is the dimension of $\hat{A}/\hat{A}^2$. Notice that super Weil algebras are local superalgebras, i.e. they contain a unique maximal graded ideal.
Remark 3.2. As a direct consequence of the definition, each super Weil algebra has an associated short exact sequence:

$$0 \rightarrow \mathbb{R} \xrightarrow{j_A} A = \mathbb{R} \oplus \hat{A} \xrightarrow{pr_A} A/\hat{A} \cong \mathbb{R} \rightarrow 0.$$ 

Clearly the sequence splits and each \( a \in A \) can be written uniquely as \( a = \tilde{a} + \hat{a} \) with \( \tilde{a} \in \mathbb{R} \) and \( \hat{a} \in \hat{A} \).

Example 3.3 (Dual numbers and super dual numbers). The simplest example of super Weil algebra in the classical setting is \( \mathbb{R}(x) = \mathbb{R}[x]/\langle x^2 \rangle \) the algebra of dual numbers. Here \( x \) is an even indeterminate. Similarly we have the super dual numbers: \( \mathbb{R}(x, \vartheta) = \mathbb{R}[x, \vartheta]/\langle x^2, x\vartheta, \vartheta^2 \rangle \) where \( x \) and \( \vartheta \) are respectively even and odd indeterminates.

Example 3.4 (Grassmann algebras). The polynomial algebra in \( q \) odd variables \( \Lambda(\vartheta_1, \ldots, \vartheta_q) \) is another example of super Weil algebra. Finite dimensional Grassmann algebras are actually a full subcategory of \( \text{SWA} \).

Lemma 3.5. Let \( R[k,l] := R[x_1, \ldots, x_k] \otimes \Lambda(\vartheta_1, \ldots, \vartheta_l) \) denote the superalgebra of real polynomials in \( k \) even and \( l \) odd variables. The following are equivalent:

1. \( A \) is a super Weil algebra;
2. \( A \cong O_{R^p,0}/J \) for suitable \( p, q \) and \( J \) graded ideal containing a power of the maximal ideal \( M_0 \) in the stalk \( O_{R^p,0} \);
3. \( A \cong R[k,l]/I \) for a suitable graded ideal \( I \supseteq \langle x_1, \ldots, x_k, \vartheta_1, \ldots, \vartheta_l \rangle^k \).

Proof. We leave this to the reader as an exercise. \( \square \)

Definition 3.6. Let \( M \) be a supermanifold and \( A \) a super Weil algebra. We define the set of \( A \)-points of \( M \),

\[ M_A := \text{Hom}_{\text{SAlg}}(O(M), A) \]

We can define the functor \( M(-) : \text{SWA} \rightarrow \text{Set} \), on the objects as \( A \mapsto M_A \) and on morphisms as \( \rho \mapsto \rho \) with \( \rho \in \text{Hom}_{\text{SAlg}}(A, B) \) and \( \rho : x_A \mapsto \rho \circ x_A \).

Remark 3.7. Observe that the only super Weil algebras which are equal to \( O(M) \) for some supermanifold \( M \) are those of the form \( \Lambda(\vartheta_1, \ldots, \vartheta_q) = O(R^{d|q}) \). In fact as soon as \( M \) has a nontrivial even part, the algebra \( O(M) \) becomes infinite dimensional. For this reason this functor is quite different from the functor of points introduced previously.
Let us recall a well known classical result.

**Lemma 3.8 ("Super" Milnor’s exercise).** Denote by $M$ a smooth supermanifold. The superalgebra maps $\mathcal{O}(M) \to \mathbb{R}$ are exactly the evaluations $ev_x: s \mapsto \tilde{s}(x)$ in the points $x \in |M|$. In other words there is a bijective correspondence between $M_{\mathbb{R}} = \text{Hom}_{SAlg}(\mathcal{O}(M),\mathbb{R})$ and $|M|$.

**Proof.** This is a simple consequence of the chart theorem 2.4 and eq. (2.1), considering that $\mathcal{O}(\mathbb{R}^0) = \mathbb{R}$ and the pullback of a morphism $\varphi: \mathbb{R}^0 \to M$ is the evaluation at $|\varphi|(\mathbb{R}^0)$. □

Let $x_A \in M_A$. Due to the previous lemma, there exists a unique point of $|M|$, that we denote by $\tilde{x}_A$, such that $pr_A \circ x_A = ev_{\tilde{x}_A}$, where $pr_A$ is the projection $A \to \mathbb{R}$. We thus have a map

$$\text{Hom}_{SAlg}(\mathcal{O}(M), A) \longrightarrow \text{Hom}_{SAlg}(\mathcal{O}(M), \mathbb{R}) \cong |M|$$

$$x_A \longmapsto pr_A \circ x_A = ev_{\tilde{x}_A}. \quad (3.1)$$

We say that $\tilde{x}_A$ is the base point of $x_A$ or that $x_A$ is an $A$-point near $\tilde{x}_A$. We denote with $\tilde{M}_{A,x}$ the set of $A$-points near $x \in |M|$.

The next proposition asserts the local nature of the functor of the $A$-points.

**Proposition 3.9.** Let $M$ be a smooth supermanifold. Let $s \in \mathcal{O}(M)$ and let $x_A \in \text{Hom}_{SAlg}(\mathcal{O}(M), A)$. Assume that $s$ is zero when restricted to a certain neighbourhood of $\tilde{x}_A$ (see eq. (3.1)). Then $x_A(s) = 0$.

**Proof.** Suppose $U \ni \tilde{x}_A$ is such that $s|_U = 0$. Let $t \in \mathcal{O}_M(U)$ be such that $\text{supp}(t) \subset U$ and $t|_V = 1$, where the closure of $V$ is contained in $U$. Then $0 = x_A(st) = x_A(s)x_A(t)$. So $x_A(s) = 0$, since $x_A(t)$ is invertible because of $ev_{\tilde{x}_A}(t) = 1$, where $ev_{\tilde{x}_A}$ denotes the evaluation at $\tilde{x}_A$. □

**Observation 3.10.** The above proposition shows that $x_A(s)$ depends only on the germ of $s$ in $\tilde{x}_A$, i.e. $x_A$ is also a superalgebra map from the stalk $\mathcal{O}_{M,\tilde{x}_A}$ of $\mathcal{O}_M$ in $\tilde{x}_A$ to $A$. Therefore it is possible to give a meaning to $x_A([s])$ for a germ $[s]$ in $\mathcal{O}_{M,\tilde{x}_A}$. It is not hard to show that $M_A \cong \bigsqcup_{x \in |M|} \text{Hom}_{SAlg}(\mathcal{O}_{M,x}, A)$. This identification allows to extend the definition of the local functor of points to the category of holomorphic or real analytic supermanifolds. Many of the results we prove extend relatively easily to the holomorphic (or real analytic) category, but we shall not pursue this point of view in the present paper.
Notation 3.11. Here we introduce a multi-index notation that we will use in the following. Let \( \{ x_1, \ldots, x_p, \vartheta_1, \ldots, \vartheta_q \} \) be a system of coordinates. If \( \nu = (\nu_1, \ldots, \nu_p) \in \mathbb{N}^p \), \( J = \{ j_1, \ldots, j_r \} \subseteq \{ 1, \ldots, q \} \), with \( 1 \leq j_1 < \cdots < j_r \leq q \), we define \( x^\nu := x_1^{\nu_1} x_2^{\nu_2} \cdots x_p^{\nu_p}, \vartheta^J := \vartheta_{j_1} \vartheta_{j_2} \cdots \vartheta_{j_r}. \) Moreover we set \( \nu! := \prod \nu_i! \), \( |\nu| := \sum \nu_i \) and \( |J| \) the cardinality of \( J \).

In order to obtain further information about the structure of \( M_A \) we need some preparation. Next lemma gives some insight on the structure of the stalk at a given point (for the proof see [Lei80, §2.1.8] or [Var04, ch. 4]).

Lemma 3.12 (Hadamard’s lemma). Let \( M \) be supermanifold, \( x \in |M| \) and \( \{ x_i, \vartheta_j \} \) is a system of coordinates in a neighborhood \( U \) of \( x \). Denote by \( M_{U,x} \) the ideal of the sections in \( \mathcal{O}_M(U) \) whose value at \( x \) is zero. For each \( s \in \mathcal{O}_M(U) \) and \( k \in \mathbb{N} \) there exists a polynomial \( P \) in \( x_i \) and \( \vartheta_j \) such that

As a consequence we have the following proposition.

Proposition 3.13. Each element \( x_A \) of \( M_A \) is determined by the images of a system of local coordinates around \( \tilde{x}_A \). Conversely, given \( x \in |M| \), a system of local coordinates \( \{ x_i \}_{i=1}^p \), \( \{ \vartheta_j \}_{j=1}^q \) around \( x \), and elements \( \{ x_i \}_{i=1}^p \), \( \{ \vartheta_j \} \) such that \( \tilde{x}_i = \tilde{x}_i(x) \), there exists a unique morphism \( x_A \in \text{Hom}_{\text{SAlg}}(\mathcal{O}(M), A) \) with \( x_A(x_i) = x_i, x_A(\vartheta_j) = \vartheta_j \).

Proof. Suppose that \( x_A \) is given. We want to show that \( x_A(x_i), x_A(\vartheta_j) \) determine \( x_A \) completely. This follows noticing that

1. the image of a polynomial section under \( x_A \) is determined,
2. there exists \( k \in \mathbb{N} \) such that the kernel of \( x_A \) contains \( M_{U,x} \) (see 3.5), and using previous lemma. We now come to existence. Suppose that the images of the coordinates are fixed as in the hypothesis and let \( s \) in \( \mathcal{O}_M(U) \). We define \( x_A(s) \) through a formal Taylor expansion. More precisely let \( s = \sum_{J \subseteq \{1, \ldots, q \}} s_J \vartheta^J \) where the \( s_J \) are smooth functions in \( x_1, \ldots, x_p \). Define

\[
x_A(s) = \sum_{\nu \in \mathbb{N}^p} \frac{1}{\nu!} \frac{\partial |s|}{\partial x^\nu} \bigg|_{(\tilde{x}_1, \ldots, \tilde{x}_p)} \tilde{x}^\nu \vartheta^J.
\]

This is the way in which the purely formal expression

\[
s(x_A) = s(\tilde{x}_1 + \tilde{x}_1, \ldots, \tilde{x}_p + \tilde{x}_p, \vartheta_1, \ldots, \vartheta_q)
\]

\( ^1 \)The reader should notice the difference between \( \{ x_i, \vartheta_j \} \) and \( \{ x_i, \vartheta_j \} \).
is usually understood. Eq. (3.2) has only a finite number of terms due to
the nilpotency of the $\hat{x}_i$ and $\theta_j$. $x_A$ is a superalgebra morphism as one can
readily check.

**Observation 3.14.** Let $U$ be a chart in a supermanifold $M$ with local
coordinates $\{x_i, \theta_j\}$. We have an injective map

$$U_A \longrightarrow A^p_0 \times A^q_1, \quad x_A \longmapsto (x_1, \ldots, x_p, \theta_1, \ldots, \theta_q) := (x_A(x_1), \ldots, x_A(\theta_q)).$$

We can think of it heuristically as the assignment of $A$-valued coordinates
$\{x_i, \theta_j\}$ on $U_A$. As we are going to see in theorem 4.2 the components of
the coordinates $\{x_i, \theta_j\}$, given by $\langle a^*_k, x_i \rangle$, $\langle a^*_k, \theta_j \rangle$ with respect to a basis
$\{a_k\}$ of $A$, are indeed the coordinates of a smooth manifold. The base point
$\tilde{x}_A \in U$ has coordinates $(\tilde{x}_1, \ldots, \tilde{x}_p)$. In this language, if $\rho: A \rightarrow B$ is a
super Weil algebra morphism, the corresponding morphism $\overline{\rho}: M_A \rightarrow M_B$ is
“locally” given by $\rho \times \cdots \times \rho: A^p_0 \times A^q_1 \rightarrow B^p_0 \times B^q_1$. This is well defined since $\rho$
does not change the base point.

If $M = R^{p|q}$ we can also consider the slightly different identification

$$R^p_A \longrightarrow (A \otimes R^{p|q})_0, \quad x_A \longmapsto \sum_i x_A(e_i^*) \otimes e_i$$

where $\{e_1, \ldots, e_{p+q}\}$ denotes a homogeneous basis of $R^{p|q}$ and $\{e_1^*, \ldots, e^*_{p+q}\}$
its dual basis. Here a little care is needed. In the literature the name $R^{p|q}$
is used for two different objects: it may indicate the super vector space
$R^{p|q} = R^p \oplus R^q$ or the superdomain $(R^p, C^\infty_{R^p} \otimes \Lambda_q)$. In the previous equation
the first $R^{p|q}$ is viewed as a superdomain, while the last as a super vector
space. Likewise the $\{e_i^*\}$ are interpreted both as vectors and sections of
$O(R^{p|q})$. As we shall see in section 4, the functor $A \mapsto (A \otimes R^{p|q})_0$ recaptures
all the information about the superdomain $R^{p|q}$, so that the two different
ways of looking at $R^{p|q}$ become identified naturally. In such identification,
the superdomain morphism $\rho: R^{p|q}_A \rightarrow R^{p|q}_B$ corresponds to the super vector
space morphism $\rho \otimes 1: (A \otimes R^{p|q})_0 \rightarrow (B \otimes R^{p|q})_0$.

As we have seen, we can associate to each supermanifold $M$ a functor
$M(\_): \text{SWA} \rightarrow \text{Set}$, $A \mapsto M_A$. Hence we have a functor: $B: \text{SMAN} \rightarrow
[\text{SWA}, \text{Set}]$. The natural question is whether $B$ is a full and faithful embedding or not. We are going to show that $B$ is not full, in other words, there
are many more natural transformations between $M(\_)$ and $N(\_)$ than those
coming from morphisms from $M$ to $N$. 

10
We first want to show that the natural transformations $M(\cdot) \to N(\cdot)$ arising from supermanifold morphisms $M \to N$ have a very peculiar form. Indeed, a morphism $\varphi: M \to N$ of supermanifolds induces a natural transformation between the corresponding functors of $A$-points given by

$$\varphi_A: M_A \longrightarrow N_A, \quad x_A \longmapsto x_A \circ \varphi^*$$

for all super Weil algebras $A$. Let $M = \mathbb{R}^{p|q}$ and $N = \mathbb{R}^{m|n}$, and denote respectively by $\{x_i, \theta_j\}$ and $\{x'_i, \theta'_l\}$ two systems of canonical coordinates over them. With these assumptions, $\varphi$ is determined by the pullbacks of the coordinates of $N$, while the $A$-point $\varphi_A(x_A)$ is determined by

$$(x'_1, \ldots, x'_m, \theta'_1, \ldots, \theta'_n) := (x_A \circ \varphi^*(x'_1), \ldots, x_A \circ \varphi^*(\theta'_n)) \in A_0^m \times A_1^n.$$

If $(x_1, \ldots, x_p, \theta_1, \ldots, \theta_q)$ denote the images of the coordinates of $M$ under $x_A$ ($x_1 = x_A(x_1)$, etc.) and $\varphi^*(x'_k) = \sum_J s_{k,J} \theta^J \in \mathcal{O}(\mathbb{R}^{p|q})_0$, where the $s_{k,J}$ are functions on $\mathbb{R}^p$, then we have

$$x'_k = x_A \circ \varphi^*(x'_k) = \sum_{J \subseteq \{1, \ldots, q\}} \frac{1}{\nu!} \frac{\partial^{\nu|s_{k,J}}}{\partial x^\nu} \bigg|_{(x_1, \ldots, x_p)} \xi^{\nu} \theta^J$$

and similarly for the odd coordinates (see prop. 3.13). Notice that if we pursue the point of view of observation 3.14, i.e. if we consider $\{x_i, \theta_j\}$ as $A$-valued coordinates of $\mathbb{R}^{p|q}$, this equation can be read as a coordinate expression for $\varphi_A$.

Not all the natural transformations $M(\cdot) \to N(\cdot)$ arise in this way. This happens also for purely even manifolds, as we see in the next example.

**Example 3.15.** Let $M$ and $N$ be two smooth manifolds and let $\varphi: M \to N$ be a map (smooth or not). The natural transformation $\alpha(\cdot): M(\cdot) \to N(\cdot)$, $\alpha_A(x_A) = \text{ev}_{\varphi(\overline{x}_A)}$, is not of the form seen above, even if $\varphi$ is assumed to be smooth, while we still have $\varphi = \alpha_{\mathbb{R}}$.

We end this section with a technical result, essentially due to A. A. Voronov (see [Vor84]), characterizing all possible natural transformations between the functors of $A$-points of two superdomains, hence also those not arising from supermanifold morphisms.
Definition 3.16. Let $U$ be an open subset of $\mathbb{R}^p$. We denote by $\mathfrak{A}_{plq}(U)$ the unital commutative superalgebra of formal series with $p$ even and $q$ odd generators and coefficients in the algebra $\mathcal{F}(U,\mathbb{R})$ of arbitrary functions on $U$, i.e. $\mathfrak{A}_{plq}(U) := \mathcal{F}(U,\mathbb{R})[[X_1,\ldots, X_p, \Theta_1,\ldots, \Theta_q]]$. An element $F \in \mathfrak{A}_{plq}(U)$ is of the form $F = \sum_{\nu \in \mathbb{N}^p} f_{\nu,J} X^\nu \Theta^J$, where $f_{\nu,J} \in \mathcal{F}(U,\mathbb{R})$ and $\{ X_i \}$ and $\{ \Theta_j \}$ are even and odd generators. $\mathfrak{A}_{plq}(U)$ is a graded algebra: $F$ is even (resp. odd) if $|J|$ is even (resp. odd) for each term of the sum.

Let us introduce a partial order between super Weil algebras by saying $A' \preceq A$ if and only if $A'$ is a quotient of $A$.

Lemma 3.17. The family of super Weil algebras is directed, i.e., if $A_1$ and $A_2$ are super Weil algebras, then there exists $A$ such that $A_1 \preceq A$.

Proof. In view of lemma 3.5, choosing carefully $k,l \in \mathbb{N}$ and $J_1$ and $J_2$ ideals of $\mathcal{O}_{\mathbb{R}^{p+0},0}$, we have $A_i \cong \mathcal{O}_{\mathbb{R}^{p+0},0}/J_i$. If $r$ is the maximum between the heights of $A_1$ and $A_2$, $\mathcal{M}_{r+1} \subseteq J_1 \cap J_2$. So $A \cong \mathcal{O}_{\mathbb{R}^{p+0},0}/(J_1 \cap J_2)$ and then it is a super Weil algebra. □

Proposition 3.18. Let $U$ and $V$ be two superdomains in $\mathbb{R}^{plq}$ and $\mathbb{R}^{mn}$ respectively. The set of natural transformations in $[\mathsf{SWA},\mathsf{Set}]$ between $U(\cdot)$ and $V(\cdot)$ is in bijective correspondence with the set of elements of the form

$F = (F_1,\ldots, F_{m+n}) \in (\mathfrak{A}_{plq}(|U|))_0^m \times (\mathfrak{A}_{plq}(|U|))_1^n$

such that, $F_k = \sum_{\nu,J} f_{\nu,J}^k X^\nu \Theta^J$, $(f_{0,0}^1(x),\ldots, f_{0,0}^m(x)) \subseteq |V|$, $\forall x \in |U|$.

Proof. As above, $\mathbb{R}_{A}^{plq}$ is identified with $A_0^p \times A_1^q$ and consequently a map $\mathbb{R}_{A}^{plq} \to \mathbb{R}_{A}^{mn}$ consists of a list of $m$ maps $A_0^p \times A_1^q \to A_0$ and $n$ maps $A_0^p \times A_1^q \to A_1$. In the same way, $U_A$ is identified with $|U| \times A_0^p \times A_1^q$.

Let $F = (F_1,\ldots, F_{m+n})$ be as in the hypothesis. A formal series $F_k$ determines a map $|U| \times A_0^p \times A_1^q \subseteq A_0^p \times A_1^q \to A$ in a natural way, defining

$F_k(x_1,\ldots, x_p, \theta_1,\ldots, \theta_q) := \sum_{\nu \in \mathbb{N}^p} f_{\nu,J}^k (\bar{x}_1,\ldots, \bar{x}_p) \bar{x}^\nu \theta^J$.

The parity of its image is the same as that of $F_k$. Then, in view of the restrictions imposed on the first $m$, $F_k$ given by the equation above, $F$ determines a
map \( U_A \to V_A \) and, varying \( A \in \text{SWA} \), a natural transformation \( U(\cdot) \to V(\cdot) \), as it is easily checked.

Let us now suppose now that \( \alpha(\cdot): U(\cdot) \to V(\cdot) \) is a natural transformation. We will see that it is determined by an unique \( F \) in the way just explained.

Let \( A \) be a super Weil algebra of height \( r \) and

\[
x_A = (\bar{x}_1 + \tilde{x}_1, \ldots, \bar{x}_p + \tilde{x}_p, \theta_1, \ldots, \theta_q) \in A^p_0 \times A^q_\theta \cong \mathbb{R}^{p/q}
\]

with \( \bar{x}_A \in |U| \). Let us consider the super Weil algebra

\[
\hat{A} := (\mathbb{R}[z_1, \ldots, z_p] \otimes \Lambda(\zeta_1, \ldots, \zeta_q))/\mathcal{M}^*
\]

(3.4)

with \( s > r \) (\( \mathcal{M} \) is as usual the maximal ideal of polynomials without constant term) and the \( \hat{A} \)-point \( y_{\bar{x}_A} := (\bar{x}_1 + z_1, \ldots, \bar{x}_1 + z_p, \zeta_1, \ldots, \zeta_q) \in \hat{A}_0^p \times \hat{A}_\zeta^q \cong \mathbb{R}^{p/q} \).

A homomorphism between two super Weil algebras is clearly fixed by the images of a set of generators, but this assignment must be compatible with the relations between the generators. The following assignment is possible due to the definition of \( \hat{A} \). If \( \rho_{x_A}: \hat{A} \to A \) denotes the map \( \rho_{x_A}(z_i) = \tilde{x}_i \), \( \rho_{x_A}(\zeta_j) = \theta_j \), then clearly \( \rho_{x_A}(y_{\bar{x}_A}) = x_A \).

Let \( (\alpha_{\hat{A}})_k \) with \( 1 \leq k \leq m + n \) be a component of \( \alpha_{\hat{A}} \), and let \( (\alpha_{\hat{A}})_k(y_{\bar{x}_A}) = \sum_{\nu} a_{\nu,J}^k (\bar{x}_A) z^\nu \zeta^J \) with \( a_{\nu,J}^k (\bar{x}_A) \in \mathbb{R} \) and \( (a_{\nu,J}^k (\bar{x}_A), \ldots, a_{\nu,J}^m (\bar{x}_A)) \in |V| \); the sum is on \( |J| \) even (resp. odd), if \( k \leq m \) (resp. \( k > m \)). Due to the functoriality of \( \alpha(\cdot) \)

\[
(\alpha_{\hat{A}})_k(x_A) = (\alpha_A)_k \circ \rho_{x_A}(y_{\bar{x}_A}) = \rho_{x_A} \circ (\alpha_A)_k(y_{\bar{x}_A}) = \sum_{\nu} a_{\nu,J}^k (\bar{x}_A) \tilde{x}^\nu \theta^J,
\]

so that there exists a non unique \( F \) such that \( F(x_A) = \alpha_A(x_A) \). Moreover \( F(x_{A'}) = \alpha_{A'}(x_{A'}) \) for each \( A' \preceq A \) and \( x_{A'} \in U_{A'} \) (it is sufficient to use the projection \( A \to A' \)). If \( F' \) is another list of formal series with this property, there exists a super Weil algebra \( A'' \) such that \( F(x_{A''}) \neq F'(x_{A''}) \) for some \( x_{A''} \in U_{A''} \). Indeed if a component \( F_k \) differs in \( f_{\nu,J}^k \), it is sufficient to consider \( A'':=\mathbb{R}[p|q]/\mathcal{M}^* \) with \( s > \max(|\nu|, q) \).

\[\square\]

4 The Weil–Berezin functor and the Schwarz embedding

In the previous section we saw that the functor \( B: \text{SMAN} \to [\text{SWA, Set}] \) does not define a full and faithful embedding of \( \text{SMAN} \) in \( [\text{SWA, Set}] \). Roughly
speaking, the root of such a difficulty can be traced to the fact that the functor $B(M): SWA \to \text{Set}$ looks only to the local structure of the supermanifold $M$, hence it loses all the global information. The following heuristic argument gives a hint on how we can overcome such problem.

It is well known (see, for example, [DM99, §1.7]) that if $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ are super vector spaces, there is a bijective correspondence between linear maps $V \to W$ and functorial families of $\Lambda_0$-linear maps between $(\Lambda \otimes V)_0$ and $(\Lambda \otimes W)_0$, for each Grassmann algebra $\Lambda$. This result goes under the name of even rule principle. Since vector spaces are local models for manifolds, the even rule principle seems to suggest that each $M_A$ should be endowed with a local structure of $A_0$-module. This vague idea is made precise with the introduction of the category $\mathcal{A}_0\text{Man}$ of $A_0$-smooth manifolds.

**Definition 4.1.** Fix an even commutative finite dimensional algebra $A_0$ and let $L$ be an $A_0$-module, finite dimensional as a real vector space. Let $M$ be a manifold. An $L$-chart on $M$ is a pair $(U, h)$ where $U$ is open in $M$ and $h: U \to L$ is a diffeomorphism onto its image. $M$ is an $A_0$-manifold if it admits an $L$-atlas. By this we mean a family $\{(U_i, h_i)\}_{i \in A}$ where $\{U_i\}$ is an open covering of $M$ and each $(U_i, h_i)$ is an $L$-chart, such that the differentials $\left. d(h_i \circ h_j^{-1})h_j(x) \right|_{h_j(x)}: T_{h_j(x)}(L) \cong L \to L \cong T_{h_i(x)}(L)$ are isomorphisms of $A_0$-modules for all $i$, $j$ and $x \in U_i \cap U_j$.

If $M$ and $N$ are $A_0$-manifolds, a morphism $\varphi: M \to N$ is a smooth map whose differential is $A_0$-linear at each point. We also say that such morphism is $A_0$-smooth. We denote by $\mathcal{A}_0\text{Man}$ the category of $A_0$-manifolds.

We define also the category $\mathcal{A}_0\text{Man}$ in the following way. The objects of $\mathcal{A}_0\text{Man}$ are manifolds over generic finite dimensional commutative algebras. The morphisms in the category are defined as follows. Denote by $A_0$ and $B_0$ two commutative finite dimensional algebras, and let $\rho: A_0 \to B_0$ be an algebra morphism. Suppose $M$ and $N$ are $A_0$ and $B_0$ manifolds respectively, we say that a morphism $\varphi: M \to N$ is $\rho$-smooth if $\varphi$ is smooth and $\left. (d\varphi)_x(av) = \rho(a)(d\varphi)_x(v) \right|$ for each $x \in M$, $v \in T_x(M)$, and $a \in A_0$ (see [Shu99] for more details).

The above definition is motivated by the following theorems. In order to ease the exposition we first give the statements of the results postponing their proofs to later.
Theorem 4.2. Let $M$ be a smooth supermanifold, and let $A \in \text{SWA}$.

1. $M_A$ can be endowed with a unique $A_0$-manifold structure such that, for each open subsupermanifold $U$ of $M$ and $s \in \mathcal{O}_M(U)$ the map defined by $\hat{s}: U_A \to A, x_A \mapsto x_A(s)$, is $A_0$-smooth.

2. If $\varphi: M \to N$ is a supermanifold morphism, then $\varphi_A: M_A \to N_A, x_A \mapsto x_A \circ \varphi^*$ is an $A_0$-smooth morphism.

3. If $B$ is another super Weil algebra and $\rho: A \to B$ is an algebra morphism, then $\rho: M_A \to M_B, x_A \mapsto \rho \circ x_A$ is a $\rho|_{A_0}$-smooth map.

The above theorem says that supermanifolds morphisms give rise to morphisms in the $\mathcal{A}_0\text{Man}$ category. From this point of view the next definition is quite natural.

Definition 4.3. We call $[[\text{SWA}, \mathcal{A}_0\text{Man}]]$ the subcategory of $[\text{SWA}, \mathcal{A}_0\text{Man}]$ whose objects are the same and whose morphisms $\alpha(\cdot)$ are the natural transformations $\mathcal{F} \to \mathcal{G}$, with $\mathcal{F}, \mathcal{G}: \text{SWA} \to \mathcal{A}_0\text{Man}$, such that $\alpha_A: \mathcal{F}(A) \to \mathcal{G}(A)$ is $A_0$-smooth for each $A \in \text{SWA}$.

Theorem 4.2 allows us to give more structure to the image category of the functor of $A$-points. More precisely we have the following definition, which is the central definition in our treatment of the local functor of points.

Definition 4.4. Let $M$ be a supermanifold. We define the Weil–Berezin functor of $M$ as

$$M(\cdot): \text{SWA} \to \mathcal{A}_0\text{Man}, \quad A \mapsto M_A$$

(4.1)

and the Schwarz embedding as

$$S: \text{SMan} \to [[\text{SWA}, \mathcal{A}_0\text{Man}]], \quad M \mapsto M(\cdot).$$

(4.2)

We can now state one of the main results in this paper.

Theorem 4.5. $S$ is a full and faithful embedding, i.e. if $M$ and $N$ are two supermanifolds, and $M(\cdot)$ and $N(\cdot)$ their Weil–Berezin functors, then

$$\text{Hom}_{\text{SMan}}(M, N) \cong \text{Hom}_{[[\text{SWA}, \mathcal{A}_0\text{Man}]]}(M(\cdot), N(\cdot)).$$

Observation 4.6. If we considered the bigger category $[\text{SWA}, \mathcal{A}_0\text{Man}]$ instead of $[[\text{SWA}, \mathcal{A}_0\text{Man}]]$, the above theorem is no longer true. In example 3.15 we examined a natural transformation between functors from $\text{SWA}$ to
Set, which does not come from a supermanifold morphism. If, in the same
example, \( \varphi \) is chosen to be smooth, we obtain a morphism in \([\text{SWA,}A_0\text{Man}]\)
that is not in \([\text{SWA,}A_0\text{Man}]\). Indeed, it is not difficult to check that if
\( \pi_A : A \to A \) is given by \( a \mapsto \tilde{a} \), then \( \alpha_A \) (in the example) is \( \pi_{A_0} \)-linear.

We now examine the proofs of theorems 4.2 and 4.5. First we need to
prove theorem 4.5 in the case of two superdomains \( U \) and \( V \) in \( \mathbb{R}^{p/q} \) and \( \mathbb{R}^{m/n} \)
respectively (lemma 4.7). As usual, if \( A \) is a super Weil algebra, \( U_A \) and
\( V_A \) are identified with \( |U| \times \mathbb{A}_p^q \times \mathbb{A}_1^1 \) and \( |V| \times \mathbb{A}_0^n \times \mathbb{A}_1^1 \) (see observation
3.14). Then they have a natural structure of open subsets of \( A_0 \)-modules.

Next lemma is due to A. A. Voronov in [Vor84] and it is the local version
of theorem 4.5.

**Lemma 4.7.** A natural transformation \( \alpha_{(-)} : U_{(-)} \to V_{(-)} \) comes from a su-
permanifold morphism \( U \to V \) if and only if \( \alpha_A : U_A \to V_A \) is \( A_0 \)-smooth for
each \( A \).

**Proof.** Due to prop. 3.18 we know that \( \alpha_{(-)} \) is determined by \( m \) even and
\( n \) odd formal series of the form \( F_k = \sum_{\nu, J} f_{\nu, J}^k \partial^\nu X^J \) with \( f_{\nu, J}^k \) arbitrary functions in \( p \) variables satisfying suitable conditions. Moreover as we have
seen in the discussion before example 3.15 a supermanifold morphism \( \varphi : U \to V \)
gives rise to a natural transformation \( \varphi_A : U_A \to V_A \) whose components are
of the form of eq. (3.3). Let us suppose that \( \alpha_A \) is \( A_0 \)-smooth. This clearly
happens if and only if all its components are \( A_0 \)-smooth and the smoothness
request for all \( A \) forces all coefficients \( f_{\nu, J}^k \) to be smooth. Let \( (\alpha_A)_k \) be the
\( k \)-th component of \( \alpha_A \) and let \( i \in \{1, \ldots, p\} \). We want to study \( \omega : A_0 \to A_j, \)
\( \omega(x_i) := (\alpha_A)_k(x_1, \ldots, x_i, \ldots, x_p, \theta_1, \ldots, \theta_q) \), supposing the other coordinates
fixed \( (j = 0 \text{ if } 1 \leq k \leq p \text{ or } j = 1 \text{ if } p < k \leq p + q) \). Since \( \tilde{x}_i \in A_0 \) commutes
with all elements of \( A \),

\[
\omega(x_i) = \sum_{t \geq 0} a_t(\tilde{x}_i)\tilde{x}_i^t, \quad a_t(\tilde{x}_i) := \sum_{\nu, J} f_{\nu, J}^k(\tilde{x}_1, \ldots, \tilde{x}_i, \ldots, \tilde{x}_p)\tilde{x}^{(\nu - t\delta_i)}(\theta^J) \quad (4.3)
\]

\((t\delta_i \text{ is the element of } \mathbb{N}^p \text{ with } t \text{ at the } i\text{-th component and } 0 \text{ elsewhere})\). If
\( y = \tilde{y} + \hat{y} \in A_0 \) and \( \omega \) is \( A_0 \)-smooth

\[
\omega(x_i + y) - \omega(x_i) = d\omega_{x_i}(y) + o(y) = (\tilde{y} + \hat{y})d\omega_{x_i}(1_A) + o(y) \quad (4.4)
\]

(where \( 1_A \) is the unit of \( A \)). On the other hand, from eq. (4.3) and defining

\[
da'_t(\tilde{x}_i) := \sum_{\nu, J} \partial_{\nu, J} f_{\nu, J}^k(\tilde{x}_1, \ldots, \tilde{x}_i, \ldots, \tilde{x}_p)\tilde{x}^{(\nu - t\delta_i)}(\theta^J) \quad (4.5)
\]

16
(\partial_i \text{ denotes the partial derivative respect to the } i\text{-th variable}), we have

\[
\omega(x_i + y) - \omega(x_i) = \sum_{t \geq 0} a_t(\tilde{x}_i + \tilde{y})\tilde{x}_i^t - \sum_{t \geq 0} a_t(\tilde{x}_i)\tilde{x}_i^t
\]

\[
= \sum_{t \geq 0} \left( a'_t(\tilde{x}_i)\tilde{y}\tilde{x}_i^t + a_t(\tilde{x}_i)t\tilde{x}_i^{t-1}\tilde{y} + o(y) \right)
\]

\[
= \tilde{y} \sum_{t \geq 0} a'_t(\tilde{x}_i)\tilde{x}_i^t + \tilde{y} \sum_{t \geq 0} (t + 1)a_{t+1}(\tilde{x}_i)\tilde{x}_i^t + o(y).
\]

Thus, comparing eq. (4.4) and (4.6), we get that the identity

\[
(\tilde{y} + \tilde{y})d\omega_x(1_A) = \tilde{y} \sum_{t \geq 0} a'_t(\tilde{x}_i)\tilde{x}_i^t + \tilde{y} \sum_{t \geq 0} (t + 1)a_{t+1}(\tilde{x}_i)\tilde{x}_i^t
\]

must hold and, consequently, also the following relations must be satisfied:

\[
\sum_{t \geq 0} a'_t(\tilde{x}_i)\tilde{x}_i^t = \sum_{t \geq 0} (t + 1)a_{t+1}(\tilde{x}_i)\tilde{x}_i^t
\]

and then, from eq. (4.3) and (4.5),

\[
\sum_{\nu,J} \partial_i f_{\nu,J}^k(\tilde{x}_1, \ldots, \tilde{x}_p)\tilde{x}_\nu^\theta^J = \sum_{\nu,J} (\nu_i + 1) f_{\nu+\delta_i,J}^k(\tilde{x}_1, \ldots, \tilde{x}_p)\tilde{x}_\nu^\theta^J.
\]

Let us fix \( \nu \in \mathbb{N}_p \) and \( J \subseteq \{1, \ldots, q\} \). If \( A = \mathbb{R}[p|q]/\mathcal{M}^s \) with \( s > \max(|\nu| + 1, q) \) (\( \mathcal{M} \) is as usual the maximal ideal of polynomials without constant term), we note that necessarily, due to the arbitrariness of \( (x_1, \ldots, \theta_q) \),

\[
\partial_i f_{\nu,J}^k = (\nu_i + 1)f_{\nu+\delta_i,J}^k
\]

and, by recursion, \( (\alpha_A)_k \) is of the form of (3.3) with \( s_{k,J} = f_{0,J}^k \).

Conversely, let \( (\alpha_A)_k \) be of the form of eq. (3.3). By linearity, it is \( A_0 \)-linear if and only if it is \( A_0 \)-linear in each variable. It is \( A_0 \)-linear in the even variables for what has been said above and in the odd variables since it is polynomial in them.

In particular the above discussion shows also that any superdiffeomorphism \( U \to U \) gives rise, for each \( A \), to an \( A_0 \)-smooth diffeomorphism \( U_A \to U_A \) and then each \( U_A \) admits a canonical structure of \( A_0 \)-manifold.

We now use the results obtained for superdomains in order to prove theorems 4.2 and 4.5 in the general supermanifold case.
of the coordinates. Let us suppose that \( U, M \) will give a morphism we obtain, in view of lemma 4.7, a family of local morphisms. Such a family of \( M \) which are to the same natural transformation on a superdomain are clearly equal. For all supermanifolds \( \text{Proposition 4.8.} \)

**Proof.** The fact that \( (M \times N)_A \cong M_A \times N_A \) for all \( A \) can be checked easily. Indeed, let \( z_A \in (M \times N)_A \) with \( z_A = (x, y) \), we have that \( \mathcal{O}(M) \) and \( \mathcal{O}(N) \) naturally inject in \( \mathcal{O}(M \times N) \). Hence \( z_A \) defines, by restriction, two \( A_0 \)-points \( x_A \in M_A \) and \( y_A \in N_A \). Using prop. 3.13 and rectangular coordinates over \( M \times N \) it is easy to check that such a correspondence is injective, and is also a natural transformation. Conversely, if \( x_A \in M_A \) is near \( x \) and \( y_A \in N_A \)

Next proposition states that the Schwarz embedding that the Schwarz embedding preserves products and, in consequence, group objects.

**Proposition 4.8.** For all supermanifolds \( M \) and \( N \),

\[
S(M \times N) \cong S(M) \times S(N).
\]

Moreover \( S(\mathbb{R}^{0|0}) \) is a terminal object in the category \([\text{SWA}, \mathcal{A}_0\text{Man}]\).

**Proof.** For theorem 4.2. Let \( \{(U_i, h_i)\} \) be an atlas over \( M \) and \( p|q \) the dimension of \( M \). Each chart \( (U_i, h_i) \) of such an atlas induces a chart \( \{(U_i)_A, (h_i)_A\} \), over \( M_A \) given by \( (h_i)_A: (U_i)_A \to \mathbb{R}^{p_iq}, x_A \mapsto x_A \circ h_i^* \). The coordinate changes are easily checked to be given, with some abuse of notation, by \( (h_i \circ h_i^{-1})_A \), which are \( A_0 \)-smooth due to lemma 4.7. The uniqueness of the \( A_0 \)-manifold structure is clear. This proves the first point. The other two points concern only the local behavior of the considered maps and are clear in view of lemma 4.7 and obs. 3.14.

**Proof of theorem 4.5.** Lemma 4.7 accounts for the case in which \( M \) and \( N \) are superdomains. For the general case, let us suppose we have \( \alpha \in \text{Hom}_\text{SWA,Man}_{A_0}(M_{(\_)}), N_{(\_)}\). Fixing a suitable atlas of both supermanifolds, we obtain, in view of lemma 4.7, a family of local morphisms. Such a family will give a morphism \( M \to N \) if and only if they do not depend on the choice of the coordinates. Let us suppose that \( U \) and \( V \) are open subsupermanifolds of \( M \) and \( N \) respectively, \( U \cong \mathbb{R}^{p|q}, V \cong \mathbb{R}^{m|n} \), such that \( \alpha^{\mathbb{R}([U])} \subseteq [V] \), and \( h_i: U \to \mathbb{R}^{p|q}, k_i: V \to \mathbb{R}^{m|n}, i = 1, 2 \) are two different choices of coordinates on \( U \) and \( V \) respectively. The natural transformations

\[
(\hat{\varphi}_i)(\cdot) := (k_i)(\cdot) \circ (\alpha(\cdot))_{U_i} \circ (h_i^{-1})(\cdot): \mathbb{R}^{p|q} \to \mathbb{R}^{m|n}
\]

give rise to two morphisms \( \hat{\varphi}_i: \mathbb{R}^{p|q} \to \mathbb{R}^{m|n} \). If \( \varphi_i := k_i^{-1} \circ \hat{\varphi}_i \circ h_i: U \to V \), we have \( \varphi_1 = \varphi_2 \) since \( (\varphi_i)(\cdot) = (\alpha(\cdot))_{U_i} \) and two morphisms that give rise to the same natural transformation on a superdomain are clearly equal. □
is near $y$ (see obs. 3.10), they define a map $z_A: \mathcal{O}(M \times N) \to A$ through $z_A(s_1 \otimes s_2) = x_A(s_1) \cdot y_A(s_2)$. Using again prop. 3.13, it is not difficult to check that this requirement uniquely determines a superalgebra morphism $\mathcal{O}(M \times N) \to A$ and that this correspondence defines an inverse for the morphism $(M \times N)_(-) \to M_(-) \times N_(-)$ defined above. Along the same lines we see that a similar condition for the morphisms holds. Finally $\mathcal{S}(\mathbb{R}^{0|0})$ is a terminal object, since $R^0_A = R^0$ for all $A$.

**Corollary 4.9.** The Weil–Berezin functor of a super Lie group (i.e. a group object in the category of supermanifolds) takes values in the category of $A_0$-smooth Lie groups.

We now turn to representability questions.

**Definition 4.10.** We say that a functor $F: \text{SWA} \to \mathcal{A}_0\text{Man}$ is representable if there exists a supermanifold $M_F$ such that $F \cong (M_F)_(-)$ in $[[\text{SWA}, \mathcal{A}_0\text{Man}]]$.

Notice that we are abusing the category terminology, that considers a functor $F$ to be representable if and only if $F$ is isomorphic to the Hom functor.

Due to theorem 4.5, if a functor $F$ is representable, then the supermanifold $M_F$ is unique up to isomorphism.

Since $\mathcal{F}(\mathbb{R})$ is a manifold, we can consider an open set $U \subseteq \mathcal{F}(\mathbb{R})$. If $A$ is a super Weil algebra and $\text{pr}_A := \mathcal{F}((\text{pr}_A)_(-))$, where $\text{pr}_A$ is the projection $A \to \mathbb{R}$, $\text{pr}_A^{-1}(U)$ is an open $A_0$-submanifold of $\mathcal{F}(A)$. Moreover, if $\rho: A \to B$ is a superalgebra map, since $\text{pr}_B \circ \rho = \text{pr}_A$, $\rho := \mathcal{F}(\rho)$ can be restricted to $\rho_{\text{pr}_A^{-1}(U)}: \text{pr}_A^{-1}(U) \to \text{pr}_B^{-1}(U)$. We can hence define the functor $\mathcal{F}_U: \text{SWA} \to \mathcal{A}_0\text{Man}$, $A \mapsto \text{pr}_A^{-1}(U)$, $\rho \mapsto \rho_{\text{pr}_A^{-1}(U)}$.

**Proposition 4.11 (Representability).** A functor $\mathcal{F}: \text{SWA} \to \mathcal{A}_0\text{Man}$ is representable if and only if there exists an open cover $\{U_i\}$ of $\mathcal{F}(\mathbb{R})$ such that $\mathcal{F}_{U_i} \cong (\tilde{V}_i)_(-)$ with $\tilde{V}_i$ superdomains in a fixed $\mathbb{R}^{p|q}$.

**Proof.** The necessity is clear due to the very definition of supermanifold. Let us prove sufficiency. We have to build a supermanifold structure on the topological space $|\mathcal{F}(\mathbb{R})|$. Let us denote by $(h_i)_(-): \mathcal{F}_{U_i} \to (\tilde{V}_i)_(-)$ the natural isomorphisms in the hypothesis. On each $U_i$, we can put a supermanifold structure $\tilde{U}_i$, defining the sheaf $\mathcal{O}_{\tilde{U}_i} := [(h_i^{-1})_R]_s \mathcal{O}_{\tilde{V}_i}$. Let $k_i$ be the
isomorphism \( U_i \rightarrow \tilde{V}_i \) and \((k_i)_{(\cdot)}\) the corresponding natural transformation. If \( U_{i,j} := U_i \cap U_j \), consider the natural transformation \((h_{i,j})_{(\cdot)}\) defined by the composition

\[
(k_i^{-1})_{(\cdot)} \circ (h_i)_{(\cdot)} \circ (k_j^{-1})_{(\cdot)} \circ (h_j)_{(\cdot)} : (U_{i,j}, \mathcal{O}_{\tilde{U}_{j|U_{i,j}}})_{(\cdot)} \rightarrow (U_{i,j}, \mathcal{O}_{\tilde{U}_{i|U_{i,j}}})_{(\cdot)}
\]

where in order to avoid heavy notations we didn’t explicitly indicate the appropriate restrictions. Each \((h_{i,j})_{(\cdot)}\) is a natural isomorphism in \([\text{SWA}, \mathcal{A}_0\text{Man}]\) and, due to lemma 4.7, it gives rise to a supermanifold isomorphism \( h_{i,j} : (U_{i,j}, \mathcal{O}_{\tilde{U}_{j|U_{i,j}}}) \rightarrow (U_{i,j}, \mathcal{O}_{\tilde{U}_{i|U_{i,j}}}) \). The \( h_{i,j} \) satisfy the cocycle conditions \( h_{i,i} = 1 \) and \( h_{i,j} \circ h_{j,k} = h_{i,k} \) (restricted to \( U_i \cap U_j \cap U_k \)). This follows from the analogous conditions satisfied by \((h_{i,j})_{A}\) for each \( A \in \text{SWA}\). The supermanifolds \( U_i \) can hence be glued (for more information about the construction of a supermanifold by gluing see for example [DM99, ch. 2] or [Var04, § 4.2]). Denote by \( M_F \) the manifold thus obtained. Moreover it is clear that \( F \) is represented by the supermanifold \( M_F \). Indeed, one can check that the various \((h_i)_{(\cdot)}\) glue together and give a natural isomorphism \( h_{(\cdot)} : F \rightarrow (M_F)_{(\cdot)} \).

Remark 4.12. The supermanifold \( M_F \) admits a more synthetic characterization. In fact it is easily seen that \(|M_F| := |F|(|\mathbb{R}|\text{ and } \mathcal{O}_{M_F}(U) := \text{Hom}_{([\text{SWA}, \mathcal{A}_0\text{Man}]\text{(F, }\mathbb{R}\text{)}\text{, 1})}((\mathcal{F}_U, \mathcal{R}_{1(\cdot)})\text{.)}}

We end this section giving a brief exposition of the original approach of A. S. Schwarz and A. A. Voronov (see [Sch84, Vor84]). In their work they considered only Grassmann algebras instead of all super Weil algebras. There are some advantages in doing so: Grassmann algebras are many fewer, moreover, as we noticed in remark 3.7, they are the sheaf of the super domains \( \mathbb{R}^{0|q} \) and so the restriction to Grassmann algebras of the local functors of points can be considered as a true restriction of the functor of points. Finally the use of Grassmann algebras is also used by A. S. Schwarz to formalize the language commonly used in physics.

On the other hand the use of super Weil algebras has the advantage that we can perform differential calculus on the Weil–Berezin functor as we shall see in section 5. Indeed prop. 5.3 is valid only for the Weil–Berezin functor approach, since not every point supported distribution can be obtained using only Grassmann algebras. Also theorem 5.5 and its consequences are valid only in this approach, since purely even Weil algebras are considered.

If \( M \) is a supermanifold and \( \mathcal{A} \) denotes the category of finite dimensional
Grassmann algebras, we can consider the two functors

\[ \Lambda \rightarrow \text{Set}, \quad \Lambda \mapsto M_\Lambda \quad \text{and} \quad \Lambda \rightarrow \mathcal{A}_0\text{Man}, \quad \Lambda \mapsto M_\Lambda \]

in place of those already introduced in the context of \(A\)-points. As in the case of \(A\)-points, with a slight abuse of notation we denote by \(M_\Lambda\) the \(\Lambda\)-points for each of the two different functors. What we have seen previously still remains valid in this setting, provided we substitute systematically \(\text{SWA}\) with \(\Lambda\); in particular theorems 4.2 and 4.5 still hold true. They are based on prop. 3.18 and lemma 4.7 that we state here in their original formulation as it is contained in [Vor84].

**Proposition 4.13.** The set of natural transformations between \(\Lambda \mapsto \mathbb{R}_{pq}^m\) and \(\Lambda \mapsto \mathbb{R}_{p|q}^m\) is in bijective correspondence with \((\mathbb{R}_{pq}^m)_{00} \times (\mathbb{R}_{pq}^m)_{01}^n\). A natural transformation comes from a supermanifold morphism \(\mathbb{R}_{pq}^m \rightarrow \mathbb{R}_{p|q}^m\) if and only if it is \(\Lambda_0\)-smooth for each Grassmann algebra \(\Lambda\).

**Proof.** See proofs of prop. 3.18 and lemma 4.7. The only difference is in the first proof. Indeed the algebra (3.4) is not a Grassmann algebra. So, if \(A = \Lambda_n = \Lambda(\varepsilon_1, \ldots, \varepsilon_n)\), we have to consider \(\hat{A} := \Lambda_{2p(n-1)+q} = \Lambda(\eta_{i,a}, \xi_{i,a}, \zeta_j)\) \((1 \leq i \leq p, 1 \leq j \leq q, 1 \leq a \leq n - 1)\). A \(\Lambda_n\)-point can be written as

\[
x_{\Lambda_n} = \left( u_1 + \sum_{a<b} \varepsilon_a \varepsilon_b k_{1,a,b}, \ldots, u_p + \sum_{a<b} \varepsilon_a \varepsilon_b k_{p,a,b}, \kappa_1, \ldots, \kappa_q \right)
\]

with \(u_i \in \mathbb{R}, \ k_{i,a,b} \in (\Lambda_n)_{00}\) and \(\kappa_j \in (\Lambda_n)_{11}\). Its image under a natural transformation can be obtained taking the image of the \(\Lambda_{2p(n-1)+q}\)-point

\[
y_{\hat{x}_{\Lambda_n}} := \left( u_1 + \sum_{a=1}^{n-1} \eta_{1,a} \xi_{1,a}, \ldots, u_p + \sum_{a=1}^{n-1} \eta_{p,a} \xi_{p,a}, \zeta_1, \ldots, \zeta_q \right)
\]

and applying the map \(\Lambda_{2p(n-1)+q} \rightarrow \Lambda_n, \ \eta_{i,a} \mapsto \varepsilon_a, \ \xi_{i,a} \mapsto \sum_{b>a} \varepsilon_b k_{i,a,b}, \ \zeta_j \mapsto \kappa_j\) to each component. The nilpotent part of each even component of \(y_{\hat{x}_{\Lambda_n}}\) can be viewed as a formal scalar product \((\eta_{1,1}, \ldots, \eta_{1,n-1}) \cdot (\xi_{1,1}, \ldots, \xi_{1,n-1}) = \sum_{a=1}^{n-1} \eta_{1,a} \xi_{1,a}\). This is stable under formal rotations and the same must be for its image. So \(\eta_{i,a}\) and \(\xi_{i,a}\) can occur in the image only as a polynomial in \(\sum_a \eta_{i,a} \xi_{i,a}\). In other words the image of \(y_{\hat{x}_{\Lambda_n}}\) (and then of \(x_{\Lambda_n}\)) is polynomial in the nilpotent part of the coordinates. \(\square\)
5 Applications to differential calculus

In this section we discuss some aspects of super differential calculus on supermanifolds using the language of the Weil–Berezin functor. In particular we establish a relation between the $A$-points of a supermanifold $M$ and the finite support distributions over it, which play a crucial role in Kostant’s seminal approach to supergeometry. We also prove the super version of the Weil transitivity theorem, which is a key tool for the study of the infinitesimal aspects of supermanifolds.

Let $(|M|, \mathcal{O}_M)$ be a supermanifold of dimension $p|q$ and $x \in |M|$. As in [Kost77, § 2.11], let us consider the distributions with support at $x$. In what follows we make a full use of obs. 3.10 which allows us to view any $x_A \in M_A$ as a map $x_A : \mathcal{O}_{M,x} \rightarrow A$.

**Definition 5.1.** Let $\mathcal{O}(M)^\ast$ be the algebraic dual of the superalgebra of global sections of $M$. The distributions with finite support over $M$ are defined as:

$$\mathcal{O}(M)^\ast := \{ v \in \mathcal{O}(M)^\prime \mid v(J) = 0, \text{ with } J \text{ ideal of finite codimension} \}$$

We define the distribution of order $k$, with support at $x \in \tilde{M}$ and the distributions with support at $x$ as follows:

$$\mathcal{O}_{M,x}^k := \{ v \in \mathcal{O}(M)^\prime \mid v(M^k_{M,x}) = 0 \}, \quad \mathcal{O}^\ast_{M,x} := \bigcup_{k=0}^{\infty} \mathcal{O}_{M,x}^k,$$

where $M_{M,x}$ denotes the maximal ideal of sections whose evaluation at $x$ is zero. Clearly $\mathcal{O}_{M,x}^k \subseteq \mathcal{O}_{M,x}^{k+1}$. 

**Observation 5.2.** If $x_1, \ldots, x_p, \vartheta_1, \ldots, \vartheta_q$ are coordinates in a neighbourhood of $x$, a distribution of order $k$ is of the form

$$v = \sum_{\nu \in \mathbb{N}^p, J \subseteq \{1, \ldots, q\}, |\nu| + |J| \leq k} a_{\nu, J} \text{ev}_x \partial^{[\nu]} \partial^{[J]}$$

with $a_{\nu, J} \in \mathbb{R}$. This is immediate since $\mathcal{O}_{M,x}^k \cong \mathcal{C}^\infty_{M,x} \otimes \Lambda(\vartheta_1, \ldots, \vartheta_q)^\ast$ and $\mathcal{C}^\infty_{M,x} = \sum a_{\nu, J} \text{ev}_x \partial^{[\nu]}$ because of the classical theory.

Moreover it is also possible to prove that for each element $v \in \mathcal{O}(M)^\ast$ there exists a finite number of points $x_i$ in $\tilde{M}$ such that $v = \sum_i v_{x_i}$ with $v_{x_i}$ denoting a nonzero distribution with support at $x_i$. 

22
Proposition 5.3. Let $A$ be a super Weil algebra and $A^*$ its dual. Let $x_A: \mathcal{O}_{M,x} \to A$ be an $A$-point near $x \in |M|$ (see obs. 3.10). If $\omega \in A^*$, then $\omega \circ x_A \in \mathcal{O}_{M,x}^k$. Moreover each element of $\mathcal{O}_{M,x}^k$ can be obtained in this way with $A = \mathcal{O}_{M,x}/\mathcal{M}_x^{k+1}$ (see lemma 3.5).

Proof. If $A$ has height $k$, since $x_A(\mathcal{M}_x) \subseteq \hat{A}$, $\omega \circ x_A \in \mathcal{O}_{M,x}^k$. If vice versa $v \in \mathcal{O}_{M,x}^k$, it factorizes through $\mathcal{O}_{M,x} \twoheadrightarrow \mathcal{O}_{M,x}/\mathcal{M}_x^{k+1} \simto \mathbb{R}$ with a suitable $\omega$.

In the next observation we relate the finite support distributions and their interpretation via the Weil–Berezin functor, to the tangent superspace.

Observation 5.4. Let us first recall that the tangent superspace to a smooth supermanifold $M$ at a point $x$ is the super vector space consisting of all the $ev_x$-derivations of $\mathcal{O}(M)$:

$$T_x(M) := \{ v: \mathcal{O}_M \longrightarrow \mathbb{R} \mid v(f \cdot g) = v(f)ev_x(g) + ev_x(f)v(g) \}.$$ 

As in the classical setting we can recover the tangent space by using the super Weil algebra of super dual numbers $A = \mathbb{R}(e, \varepsilon) = \mathbb{R}[e, \varepsilon]/\langle e^2, \varepsilon e, \varepsilon^2 \rangle$ (see example 3.3). If $x_A \in M_A$ is near $x$ and $s, t \in \mathcal{O}(M)$, we have $x_A(st) = ev_x(st) + x_e(st)e + x_\varepsilon(st)\varepsilon$ with $x_e, x_\varepsilon: \mathcal{O}(M) \to \mathbb{R}$. On the other hand

$$x_A(st) = x_A(s)x_A(t) = ev_x(s)ev_x(t) + (x_e(s)ev_x(t) + ev_x(s)x_e(t))e + (x_\varepsilon(s)ev_x(t) + ev_x(s)x_\varepsilon(t))\varepsilon.$$

Then $x_e$ (resp. $x_\varepsilon$) is a derivation that is zero on odd (resp. even) elements and so $x_e \in T_x(M)_0$ (resp. $x_\varepsilon \in T_x(M)_1$). The map

$$T(M) := \bigsqcup_{x \in |M|} T_x(M) \longrightarrow M_{\mathbb{R}(e, \varepsilon)}, \quad v_0 + v_1 \longmapsto ev_x + v_0e + v_1\varepsilon$$

(with $v_\varepsilon \in T_x(M)_1$) is an isomorphism of vector bundles over $\widetilde{M} \cong M_{\mathbb{R}}$, where $\widetilde{M}$ is the classical manifold associated with $M$, as in section 2 (see also [KMS93, ch. 8] for an exhaustive exposition in the classical case). The reader should not confuse $T(M)$, which is the classical bundle obtained by the union of all the tangent superspaces at the different points of $|M|$, with $T_M$, which is the super vector bundle of all the derivations of $\mathcal{O}_M$.
We now want to give a brief account on how we can perform differential calculus using the language of $A$-points. The essential ingredient is the super version of the transitivity theorem.

**Theorem 5.5 (Weil transitivity theorem).** Let $M$ be a smooth supermanifold, $A$ a super Weil algebra and $B_0$ a purely even Weil algebra, both real. Then $(M_A)_{B_0} \cong M_{A \otimes B_0}$ as $(A_0 \otimes B_0)$-manifolds.

**Proof.** Let $O_{M_A}$ and $O_{M_A}^A$ be the sheaves of smooth maps from the classical manifold $M_A$ to $\mathbb{R}$ and $A$ respectively. Clearly $O_{M_A}^A \cong A \otimes O_{M_A}$ through the map $f \mapsto \sum_i a_i \otimes \langle a_i^*, f \rangle$, where $\{a_i\}$ is a homogeneous basis of $A$.

Consider now the map $\tau: O(M) \rightarrow O(M_A)^A \cong A \otimes O(M_A)$, $\tau(s) = \hat{s}$, where, if $s \in O(M)$, $\hat{s}: y_A \mapsto y_A(s)$ for all $y_A \in M_A$.

Recalling that

$$(M_A)_{B_0} := \text{Hom}_{SAlg}(O(M_A), B_0) \quad M_{A \otimes B_0} := \text{Hom}_{SAlg}(O(M), A \otimes B_0),$$

we can define a map $\xi: (M_A)_{B_0} \rightarrow (M_{A \otimes B_0}), \xi(X): s \mapsto (1_A \otimes X)\tau(s)$. This definition is well-posed since $\xi(X)$ is a superalgebra map, as one can easily check. Fix now a chart $(U, h)$, $h: U \rightarrow \mathbb{R}^p$, in $M$ and denote by $(U_A, h_A)$, $(U_{A \otimes B_0}, (h_A)_{B_0})$ and $(U_{A \otimes B_0}, h_{A \otimes B_0})$ the corresponding charts lifted to $M_A$, $(M_A)_{B_0}$ and $M_{A \otimes B_0}$ respectively. If $\{e_1, \ldots, e_{p+q}\}$ is a homogeneous basis of $\mathbb{R}^p$, we have (here, according to observation 3.14, we tacitly use the identification $\mathbb{R}^p \cong (A \otimes \mathbb{R}^p)^0$):

$$(h_A)_{B_0} : (U_A)_{B_0} \rightarrow (A \otimes B_0 \otimes \mathbb{R}^p)^0, \quad X \mapsto \sum_i a_i \otimes X(h_A^*(a_i^* \otimes e_i^*)) \otimes e_j$$

Then, since $\xi(X)(h_A^*(e_i^*)) = (1 \otimes X)(h_A^*(e_i^*)) = (1 \otimes X)(\sum_i a_i \otimes h_A^*(a_i^* \otimes e_i^*))$, we have $h_{A \otimes B_0} \circ \xi \circ (h_A)_{B_0}^{-1} = 1_{(h_A)_{B_0}}((U_A)_{B_0})$. This entails in particular that $\xi$ is a local $(A_0 \otimes B_0)$-diffeomorphism. The fact that it is a global diffeomorphism follows noticing that it is fibered over the identity.

We want to briefly explain some applications of the Weil transitivity theorem.

**Definition 5.6.** If $x_A \in M_A$, we define the space of $x_A$-linear derivations of $M$ ($x_A$-derivations for short) as the $A$-module

$$\text{Der}_{x_A}(O(M), A) := \left\{ X \in \text{Hom}(O(M), A) \right| \forall s, t \in O(M),$$

$$X(st) = X(s)x_A(t) + (-1)^p(s)pX(t)x_A(s)x_AX(t) \right\}.$$
Proposition 5.7. The tangent superspace at \( x_A \) in \( M_A \) canonically identifies with \( \text{Der}_{x_A}(\mathcal{O}(M), A)_0 \).

Proof. If \( R(e) \) is the algebra of dual number (see example 3.3), \( (M_A)_{R(e)} \) is isomorphic, as a vector bundle, to the tangent bundle \( T(M_A) \), as we have seen in observation 5.4. Due to theorem 5.5, we thus have an isomorphism

\[
\xi: T(M_A) \cong (M_A)_{R(e)} \rightarrow M_A \otimes R(e).
\]

On the other hand, it is easy to see that \( x_A \otimes R(e) \in M_A \otimes R(e) \) can be written as \( x_A \otimes R(e) = x_A \otimes 1 + v_{x_A} \otimes e \), where \( x_A \in M_A \) and \( v_{x_A}: \mathcal{O}(M) \rightarrow A \) is a parity preserving map satisfying the following rule for all \( s, t \in \mathcal{O}(M) \):

\[
v_{x_A}(st) = v_{x_A}(s)x_A(t) + x_A(s)v_{x_A}(t).
\]

Then each tangent vector on \( M_A \) at \( x_A \) canonically identifies a even \( x_A \)-derivation and, vice versa, each such derivation canonically identifies a tangent vector at \( x_A \).

We conclude studying more closely the structure of \( \text{Der}_{x_A}(\mathcal{O}(M), A) \). The following proposition describes it explicitly.

Let \( K \) be a right \( A \)-module and let \( L \) be a left \( B \)-module for some algebras \( A \) and \( B \). Suppose moreover that an algebra morphism \( \rho: B \rightarrow A \) is given. One defines the \( \rho \)-tensor product \( K \otimes \rho L \) as the quotient of the vector space \( K \otimes L \) with respect to the equivalence relation \( k \otimes b \cdot l \sim k \cdot \rho(b) \otimes l \), for all \( k \in K, l \in L \) and \( b \in B \).

Moreover, if \( M \) is a supermanifold, we denote by \( T_M \) the super tangent bundle of \( M \), i.e., the sheaf defined by \( T_M := \text{Der}(\mathcal{O}_M) \).

Proposition 5.8. Let \( M \) be a smooth supermanifold and let \( x \in |M| \). Denote \( T_{M,x} \) the germs of vector fields at \( x \). One has the identification of left \( A \)-modules

\[
\text{Der}_{x_A}(\mathcal{O}(M), A) \cong A \otimes T_{x_A}(M) \cong A \otimes_{x_A} T_{M,\overline{x}_A}.
\]

This result is clearly local so that it is enough to prove it in the case \( M \) is a superdomain. Next lemma does this for the first identification. The second descends from eq. (5.1), since \( T_{M,\overline{x}_A} = \mathcal{O}_{M,\overline{x}_A} \otimes T_{\overline{x}_A}(M) \), where \( \mathcal{O}_{M,\overline{x}_A} \) denotes the stalk at \( \overline{x}_A \).
Lemma 5.9. Let $U$ be a superdomain in $\mathbb{R}^{p|q}$ with coordinate system $\{x_i, \vartheta_j\}$, $A$ a super Weil algebra, and $x_A \in U_A$. To any list of elements

$$f = (f_1, \ldots, f_p, F_1, \ldots, F_q) \quad f_i, F_j \in A$$

there corresponds a $x_A$-derivation $X_f: \mathcal{O}(U) \to A$ given by

$$X_f(s) = \sum_i f_i x_A \left( \frac{\partial s}{\partial x_i} \right) + \sum_j F_j x_A \left( \frac{\partial s}{\partial \vartheta_j} \right). \quad (5.1)$$

$X_f$ is even (resp. odd) if and only if the $f_i$ are even (resp. odd) and the $F_j$ are odd (resp. even). Moreover any $x_A$-derivation is of this form for a uniquely determined $f$.

Proof. That $X_f$ is a $x_A$-derivation is clear. That the family $f$ is uniquely determined is also immediate from the fact that they are the value of $X_f$ on the coordinate functions.

Let now $X$ be a generic $x_A$-derivation. Define $f_i = X(x_i)$, $F_j = X(\vartheta_j)$, and $X_f = f_i x_A \circ \frac{\partial}{\partial x_i} + F_j x_A \circ \frac{\partial}{\partial \vartheta_j}$. Let $D = X - X_f$. Clearly $D(x_i) = D(\vartheta_j) = 0$. We now show that this implies $D = 0$. Let $s \in \mathcal{O}(U)$. Due to lemma 3.12, for each $x \in U$ and for each integer $k \in \mathbb{N}$ there exists a polynomial $P$ in the coordinates such that $s - P \in \mathcal{M}_{k+1}^{U,x}$. Due to Leibniz rule $D(s - P) \in \hat{A}^k$ and, since clearly $D(P) = 0$, $D(s)$ is in $\hat{A}^k$ for arbitrary $k$. So we are done. \qed

Corollary 5.10. We have:

$$T_{x_A} M_A \cong (A \otimes T_{x_A}(M))_0 \cong (A \otimes_{x_A} T_M, \vartheta_A)_0.$$

References


