

QUANTUM RELATIVISTIC DYNAMICS

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Abstract

The aim of these lectures is to describe a construction, as self-contained as possible, of the dynamics of quantum fields.

They are based on a short description of Haag-Ruelle scattering theory and of its relation with LSZ theory and on an introduction to renormalization theory based on Wilson-Polchinski renormalization group.

1 Introduction

Quantum field theories was born from a generalization of QED to other interactions. The main characters of the theory where:

1) a one-to-one correspondence between particles and fields, the vector potential for the photon and the Dirac field for the electron.

2) the Lorentz invariance of a finite number of dynamical field equations with the fields transforming according to finite dimensional representation of $SL(2C)$, the universal covering of the Lorentz group.

3) the locality of the field equations.

The most famous extensions were the Fermi theory of weak interactions, where the neutrino field was introduced and the Yukawa theory of strong interaction where a scalar field was introduced assuming the existence of a corresponding spin-less particle identified after the discovery of the π meson.

This gave origin to a new branch of Physics, that of Elementary Particles. More recently, even if the name survived, the idea of the identification of relativistic dynamics with an elementary particle theory based on point (1) above was abandoned due to the discovery of a great number of new particles, among which many were unstable, and after the construction of non abelian gauge theories and their application to strong interactions (QCD). In fact it was understood that, keeping the three points, one is obliged to introduce unphysical fields, and the concept of confinement excluded the identification of strong interacting fundamental fields with particle fields in the sense of QED.

The multitude of new particles generated the idea of abandoning a theory based on a finite number of field equations and on a limited set of *fundamental fields* extending the concept of particle to any discrete eigenvalue of the mass, thus including the bound states, and introducing on the same ground a local operator for each particle (*interpolating field*). The hope was the possibility of reducing the dynamics to direct relations among scattering amplitudes. This possibility was strongly supported by the successful formulation of a relativistic scattering theory by Haag and many others. This remains one of the basic foundation of relativistic quantum mechanics and will be the subject of next sections.

On the contrary the success of QCD and a better understanding of Renormalization Theory gave new strength to the original scheme where, however point (1) was replaced by the idea that the dynamics be constructed in terms

of fundamental fields and, in general, interpolating fields be composite operators.

The line of these lecture notes is coherent with this point of view. We start from the relativistic scattering theory. Then we come to the dynamical construction based on renormalized local field theory.

2 Difficulties with quantized fundamental fields

We consider the simplest case of a scalar field ϕ . It turns out that it is not an operator. This can be put into evidence using the Lehmann spectral representation for the two point vacuum correlator:

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \langle \Omega | \phi(x-y) \phi(0) | \Omega \rangle \equiv C(x-y) , \quad (1)$$

due to translation invariance. It is clear that C gives information of the properties of the vector state $\phi(x)|\Omega\rangle$. Inserting into the correlator a complete set of states labeled by their total momentum \vec{p} , their total mass M and further quantum numbers α , we do not consider the vacuum state since we assume here for simplicity $\langle \Omega | \phi | \Omega \rangle = 0$, one gets:

$$\begin{aligned} C(x) &= \sum_{M,\alpha} \int d^3p \langle \Omega | \phi(x) | \vec{p}, M, \alpha \rangle \langle \vec{p}, M, \alpha | \phi(0) | \Omega \rangle \\ &= \sum_{M,\alpha} \int d^3p e^{-ip_M x} |\langle \vec{p}, M, \alpha | \phi(0) | \Omega \rangle|^2 , \end{aligned} \quad (2)$$

where we have used translation invariance and defined by p_M the four vector with time and space components E_M, \vec{p} that is $\sqrt{p^2 + M^2}, \vec{p}$.

Among the above states we shall distinguish single particle states, corresponding to discrete eigenvalues of the mass. these states are assumed to be uniquely identified by their momentum, mass and helicity. The single particle masses satisfy $m_1 \leq m_2 \leq \dots \leq m_{N_p}$, $m_1 > m > 0$.

Eq.(2) can be further simplified using the fact that ϕ is a Lorentz scalar and hence:

$$\langle \vec{p}, M, \alpha | \phi(0) | \Omega \rangle = \sqrt{\frac{M}{E_M}} \langle \vec{0}, M, \alpha | \phi(0) | \Omega \rangle . \quad (3)$$

Introducing the mass density function: $\rho(M) = 2 \sum_{\alpha} M | \langle \vec{0}, M, \alpha | \phi(0) | \Omega \rangle |^2$ which is the sum of a finite number of Dirac deltas corresponding to the single particle, or bound, states and a continuous part corresponding to the scattering states, that is:

$$\rho(M) = \sum_{a=1}^{N_p} \rho_a \delta(M - m_a) + \theta(M - 2m_1) R(M) , \quad (4)$$

we get:

$$C(x) = \int_0^{\infty} dM \rho(M) \int \frac{d^3 p}{2E_M} e^{-ip_M x} \equiv \int_0^{\infty} dM \rho(M) \Delta_M^{(+)}(x) . \quad (5)$$

Now it is apparent that $\phi(x) | \Omega \rangle$ has infinite norm, due to the divergence of the momentum integral, and hence is not a state vector even if there is only a single discrete mass contribution. This difficulty is overcome noticing that the identification of a single space point needs an infinite amount of energy and hence the value of a field in a point must be necessarily ill defined. One can consider therefore the smeared field over a finite space region $\int d^3 r' \chi(\vec{r} - \vec{r}') \phi(\vec{r}', t) \equiv \phi_{\chi}(\vec{r}, t)$ assuming χ real and supported by a small space region and infinitely differentiable, we shall say C^{∞} as it is commonly written.

In this case one has:

$$\| \phi_{\chi}(\vec{0}, 0) | \Omega \rangle \|^2 = \int_0^{\infty} dM \rho(M) \int \frac{d^3 p}{2E_M} | \tilde{\chi}(\vec{p}) |^2 . \quad (6)$$

Now the momentum integral converges since $\tilde{\chi}(\vec{p})$ is a fast decreasing function at infinity due to the smoothness of χ . However there remains the problem of the convergence of the M integral which requires $\rho(M)$ to vanish at infinity since one has asymptotically $\int \frac{dM}{M} \rho(M)$. As a matter of fact for purely dimensional reasons one expects $\rho(M) M^{1-\epsilon}$ to vanish at infinity for any positive ϵ , since $\rho(M)$ has the dimension of an inverse mass, and hence the above norm is expected to be finite.

The above result is however not general enough. Had we considered the time derivative of the smeared field, which is of course an independent observable, we would have obtained an infinite norm for $\dot{\phi}_{\chi}(\vec{0}, 0) | \Omega \rangle$, same result, of course, for the D'Alembertian of the field.

Thus one must conclude that local operators must be smeared in space and time. We shall consider for example:

$$\int dt' \int d^3 r' \chi(|\vec{r} - \vec{r}'|) \chi(t - t') \phi(\vec{r}', t') \equiv \phi_{\chi}(\vec{r}, t) , \quad (7)$$

where χ can have either compact support, such as an interval, in which case the smeared operator is called *local*, or be rapidly vanishing, the resulting operator being called *almost-local*. Mathematically this means that the basic fields must be considered *distribution valued operators*.

As we shall see the recourse to these smeared operators solves the problem of the construction of a relativistic scattering theory, it does not solve, however, the problem of the construction of an interacting theory. Indeed the interacting theory must be based on strictly local field equations, that is, equations involving strictly local composite operators built in terms of products of fundamental fields at the same space-time point. We shall see in the following that to overcome this difficulty one must have recourse to a rather technical and sophisticated tool which is called *Renormalization Group* .

We now sketch the construction of scattering theory.

3 Construction of a relativistic scattering theory

We sketch Haag's construction. Haag considers generalized local, or almost-local, operator fields of the form:

$$Q(x) = \int dy f(x-y)\phi(y) + \int dy dz f(x-y, x-z)\phi(y)\phi(z) + \dots, \quad (8)$$

where ϕ is the fundamental field and the coefficient functions are assumed C^∞ and, either of compact support and labeled with \mathcal{D} , or rapidly vanishing at infinity and labeled with \mathcal{S} . The presence of terms non-linear in the fundamental field might be necessary to deal with bound states. There is however the possibility, which is consistent with renormalization theory, to introduce renormalized, strictly local, that is distribution valued, composite operators $\Phi(x)$, in analogy with the Wick ordered monomials of free field theory, that is with monomials of the fields and their derivatives at the same space-time point ordered shifting the destruction operator to the right-hand side of the creation ones; in this case the non linear terms in the above expression could be unnecessary. The explicit construction of these operators is postponed after the construction of Renormalization Theory. In general Φ is, either a fundamental, or a composite, distribution valued field transforming according to a finite dimensional and linear representation under the action of the Lorentz group.

Having in mind the construction of scattering amplitudes it is convenient to introduce in correspondence with the a -th single particle a local, and in general composite, field $\Phi_a(x)$ for any $1 \leq a \leq N_p$ such that the matrix element $\langle \vec{p}, a | \Phi_a(0) | \Omega \rangle \equiv \sqrt{\frac{m_a}{E_a}} \zeta_a$ does not vanish.

If we assume that the number of discrete mass eigenvalues be finite, without loss of generality, we can refine the choice of the $\Phi_a(x)$'s asking their vacuum expectation values to vanish together with the matrix elements $\langle \vec{p}, a | \Phi_b(0) | \Omega \rangle$ for $a \neq b$ and that $|\zeta_a| \equiv 1$. Then, if we assume time reversal invariance, $\zeta_a \equiv 1$. Therefore we have:

$$\langle \vec{p}, a | \Phi_b(0) | \Omega \rangle \equiv \sqrt{\frac{m_a}{E_a}} \delta_{a,b} \quad , \quad \langle \Omega | \Phi_a(0) | \Omega \rangle \equiv 0. \quad (9)$$

Notice that the second condition is trivially satisfied if the field is not scalar.

In the case of a scalar field the condition is implemented subtracting from the field operator a constant, which is a trivial scalar field.

Studying the scattering theory, we limit our discussion to almost local operators of the form:

$$Q(x)_a = \int dy f(x-y)\Phi_a(y) , \quad (10)$$

where f is a function of class \mathcal{D} or \mathcal{S} . For simplicity we shall consider only operators associated with scalar fields, and hence we shall deal only with scalar particles and bound states. However our analysis can be extended without major difficulties to particles with any spin. With our simple choice the Lehmann representation (5) for the two-field vacuum expectation value that we call, as it is commonly done, *2-point Wightman functions* holds true in general, however the expected asymptotic behavior of the spectral function $\rho(M)$ depends on the nature of the field. The commutator of two almost local operators is expected to vanish faster than any inverse power of their space distance if the time distance is kept fixed.

The physical Hilbert space is assumed to coincide with a Fock space of scattering states. The vacuum state $|\Omega\rangle$ is assumed to be an isolated eigenvector of the mass operator, which means that all the particles and bound states involved have positive mass larger than a given *mass gap* m .

The kernel of Haag's construction is the cluster property that we are going to describe and which is a natural consequence of the mass gap.

Let us consider the almost-local operators (10) and consider the long distance, fixed time, behavior of the 2-point Wightman function $\langle \Omega|Q_a(x)Q_b(0)|\Omega\rangle$. Using an obvious generalization of (5) we have:

$$\begin{aligned} \langle \Omega|Q_a(x)Q_b(0)|\Omega\rangle &= \int_m^\infty dM \rho_{a,b}(M) \int \frac{d^3p}{2E_M} \int dx' dy' f_a(x-x') \\ & f_b(-y') e^{ip_M(x'-y')} + \langle \Omega|Q_a(0)|\Omega\rangle \langle \Omega|Q_b(0)|\Omega\rangle \\ &= \int_m^\infty dM \rho_{a,b}(M) \int \frac{d^3p}{2E_M} e^{ip_M x} \tilde{f}_a(p_M) f_b(-p_M) \\ &+ \langle \Omega|Q_a(0)|\Omega\rangle \langle \Omega|Q_b(0)|\Omega\rangle \\ &\equiv \int d^3p \int_m^\infty \frac{dM}{E_M} \sigma_{a,b}(\vec{p}, M) e^{ip_M x} + \langle \Omega|Q_a(0)|\Omega\rangle \langle \Omega|Q_b(0)|\Omega\rangle \quad (11) \end{aligned}$$

where $\rho_{a,b}(M) = \sum_\alpha \langle \Omega|\Phi_a(0)|\vec{0}, M, \alpha\rangle \langle \vec{0}, M, \alpha|\Phi_b(0)|\Omega\rangle$ and we have introduced $\tilde{f}(p) \equiv \int dx e^{-ipx} f(x)$ that is assumed to be of class \mathcal{S} . It fol-

lows that $\sigma_{a,b}(\vec{p}, M)$ is C^∞ in \vec{p} and hence from the Riemann-Lebesgue lemma one has that for large $r \equiv |\vec{x}|$ the above 2-point function tends to $\langle \Omega | Q_a(0) | \Omega \rangle \langle \Omega | Q_b(0) | \Omega \rangle$ faster than any inverse power of r . This result suggests the introduction of the truncated, or *connected*, Wightman functions. Up to 2-points we define:

$$\begin{aligned} & \langle \Omega | Q_a(x) Q_b(0) | \Omega \rangle \equiv \langle \Omega | Q_a(x) Q_b(0) | \Omega \rangle_T \\ & + \langle \Omega | Q_a(x) | \Omega \rangle_T \langle \Omega | Q_b(0) | \Omega \rangle_T \\ & \langle \Omega | Q_a(x) | \Omega \rangle = \langle \Omega | Q_a(0) | \Omega \rangle \equiv \langle \Omega | Q_a(0) | \Omega \rangle_T = \langle \Omega | Q_a(x) | \Omega \rangle_T \end{aligned} \quad (12)$$

In the second line we have use the translation invariance of the vacuum state. Comparing (11) and (12) we see that the connected 2-point Wightman function vanishes at fixed time and infinite distance faster than any inverse power of the distance.

This result can be generalized to any n-point Wightman function. Let d be a purely space-like four-vector, $d^0 = 0$, $d^2 = -R^2$, consider a set of n almost-local operators Q_a , for $a = 1 \dots, n$. Let χ_i^σ be the characteristic function of a subset σ of the first n integers; $\chi_a^\sigma = 1$ if i belongs to σ , $\chi_a^\sigma = 0$ otherwise. If $R \rightarrow \infty$ the n-point function:

$$\langle \Omega | \prod_{i=1}^n Q_a(x_a + d \chi_a^\sigma) | \Omega \rangle \rightarrow \langle \Omega | \prod_{a \in \sigma} Q_a(x_a + d) \prod_{a \notin \sigma} Q_a(x_a) | \Omega \rangle, \quad (13)$$

faster than any inverse power of R since the operators at distance R commute up to negligible corrections. Notice that the operator products appearing in (13) and in the following formulae are ordered with the index increasing from the left to the right-hand side. Now we can treat the right-hand side of (13) as a 2-point function:

$$\begin{aligned} & \langle \Omega | \prod_{a \in \sigma} Q_a(x_a + d) \prod_{a \notin \sigma} Q_a(x_a) | \Omega \rangle \\ & = \sum_{\alpha} \int_m^{\infty} dM \int d^3 p \langle \Omega | \prod_{a \in \sigma} Q_a(x_a + d) | \vec{p}, M, \alpha \rangle \\ & \langle \vec{p}, M, \alpha | \prod_{a \notin \sigma} Q_a(x_a) | \Omega \rangle = \sum_{\alpha} \int_m^{\infty} dM \int d^3 p e^{-ipM d} \\ & \langle \Omega | \prod_{a \in \sigma} Q_a(x_a) | \vec{p}, M, \alpha \rangle \langle \vec{p}, M, \alpha | \prod_{a \notin \sigma} Q_a(x_a) | \Omega \rangle. \end{aligned} \quad (14)$$

Taking into account that the product of matrix elements in the last line of (14) is a C^∞ function of \vec{P} we conclude, much in the same way as for (11), that

$$\langle \Omega | \prod_{a=1}^n Q_a(x_a + d\chi_a^\sigma) | \Omega \rangle \rightarrow \langle \Omega | \prod_{a \in \sigma} Q_a(x_a + d) | \Omega \rangle \langle \Omega | \prod_{a \notin \sigma} Q_a(x_a) | \Omega \rangle . \quad (15)$$

Thus if we generalize (12) defining implicitly the connected n-point functions by the following *cluster decomposition* formula:

$$\begin{aligned} \langle \Omega | \prod_{a=1}^n Q_a(x_a) | \Omega \rangle &\equiv \sum_{\sigma_1} \langle \Omega | \prod_{a \in \sigma} Q_a(x_a) | \Omega \rangle_T \langle \Omega | \prod_{b \notin \sigma_1} Q_b(x_b) | \Omega \rangle \\ &= \sum_{k=1}^n \sum_{\{\Pi_j, j=1, \dots, k\}} \prod_{j=1}^k \langle \Omega | \prod_{a \in \Pi_j} Q_a(x_a) | \Omega \rangle_T \end{aligned} \quad (16)$$

where σ_1 runs over all the subsets of the first n integers containing the first one, as above, and the index sets Π_j for $j = 1, \dots, k$ are partitions of the first n integers and one has to sum over all such partitions, we have that the connected n-point functions vanish faster than any inverse power of the maximum space distance among the points if times are kept fixed.

This is the cluster property upon which Haag's theory is based. For future convenience we can translate our results into the functional language introducing a tool that will be very useful in the following.

We associate to every almost-local operator $Q_a(x)$ a *source* $J(x, \theta)$ of class \mathcal{S} where θ is a variable accounting for the position of the operator in the ordered products. Then we can define the *functional generator* of the Wightman functions:

$$\begin{aligned} W(J) &\equiv \langle \Omega | \Theta e^{\int d^4x d\theta J(x, \theta) Q(x)} | \Omega \rangle \\ &\equiv \sum_{n=0}^{\infty} \langle \Omega | \int d^4x_1 \cdots d^4x_n \int_{-\infty}^{\infty} d\theta_1 J(x_1, \theta_1) Q(x_1) \int_{-\infty}^{\theta_1} d\theta_2 J(x_2, \theta_2) Q(x_2) \\ &\quad \cdots \int_{-\infty}^{\theta_{n-1}} d\theta_n J(x_n, \theta_n) Q(x_n) | \Omega \rangle . \end{aligned} \quad (17)$$

If the connected n-point function generator $W(J)_C$ is defined in a completely analogous way, it is immediate to verify from (16) that:

$$W(J) = e^{W(J)_C} . \quad (18)$$

Coming back to Haag's construction we introduce positive/negative frequency solutions of the Klein-Gordon equation $(\partial^2 + m_a^2)\psi_a(x) = 0$

$$\psi_a^{(\pm)}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3p}{\sqrt{2E_{m_a}}} \tilde{\psi}(\vec{p}) e^{\pm i p_{m_a} x} \quad (19)$$

with $\tilde{\psi}(\vec{p})$ of class \mathcal{D} and support S_ψ and the corresponding *generalized destruction/creation operators*;

$$B_{\psi_a}^{(\pm)}(t) = \pm i \int_{x^0=t} d^3x \left[Q_a(x) \frac{\partial}{\partial x^0} \psi_a^{(\pm)}(x) - \psi_a^{(\pm)}(x) \frac{\partial}{\partial x^0} Q_a(x) \right], \quad (20)$$

and $Q_a(x) = \int dy f_a(x-y) \Phi_a(y)$, with $\tilde{f}_a(E_{m_a}, \vec{p}) = 1$ on S_ψ .

One can study the vacuum expectation value:

$$\langle \Omega | \prod_{k=1}^m B_{\psi_{a_k}}^{(s_k)}(t) | \Omega \rangle, \quad (21)$$

where s_k is either + or -.

Taking into account that $\langle \Omega | B_{\psi_b}^{(\pm)}(t) | \Omega \rangle \equiv 0$ for all b 's, due to our choice of the local fields, it turns out that in the limit $|t| \rightarrow \infty$ (21) vanishes unless it contains the same number of generalized creation and destruction operators. More precisely the asymptotic time limit of (21) reduces to the sum over all the possible pairings of $B^{(+)}$'s and $B^{(-)}$'s of the vacuum expectation values of the products.

This result can be proven applying the cluster decomposition (16) to (21) and using the fact that $\psi_a^{(\pm)}(x)$ in (19) and its time derivative satisfy the following inequalities:

$$\begin{aligned} |\psi_a^{(\pm)}(x)| &< A |x^0|^{-3/2}, \\ |\psi_a^{(\pm)}(x)| &< \frac{A_N}{[|x^0| + |\vec{x}|]^N}, \quad \text{for any } N \quad \text{if } S_\psi \ni \frac{|\vec{x}|}{|x^0|}. \end{aligned} \quad (22)$$

The contribution of a $k > 2$ point connected term in the cluster decomposition of (21) is:

$$\begin{aligned} &\left(\prod_{l=1}^k s_l i \right) \int_{x_i^0 \equiv t} d^3x_1 \cdots d^3x_k (\dot{\psi}_{a_1}^{(s_1)}(x_1) - \psi_{a_1}^{(s_1)}(x_1) \frac{\partial}{\partial x_1^0}) \cdots \\ &(\dot{\psi}_{a_k}^{(s_k)}(x_k) - \psi_{a_k}^{(s_k)}(x_k) \frac{\partial}{\partial x_k^0}) < \Omega | Q_{a_1}(x_1) Q_{a_2}(x_2) \cdots Q_{a_k}(x_k) | \Omega \rangle_C \end{aligned} \quad (23)$$

which can be decomposed in the sum of 2^k terms of the form:

$$\int_{x_i^0 \equiv t} \prod_{j=1}^k (d^3 x_j \hat{\psi}_{a_j}^{(s_j)}(x_j)) < \Omega | \hat{Q}_{a_1}(x_1) \hat{Q}_{a_2}(x_2) \cdots \hat{Q}_{a_k}(x_k) | \Omega >_C, \quad (24)$$

where the functions $\hat{\psi}$ are solution of the Klein Gordon equation and satisfy (22) and the operators \hat{Q} satisfy (10).

Now, changing integration variables (24) can be written:

$$\begin{aligned} & t^{3k} \int \prod_{j=1}^k (d^3 v_j \hat{\psi}_{a_j}^{(s_j)}(\vec{v}_j t, t)) < \Omega | \hat{Q}_{a_1}(\vec{v}_1 t, t) \hat{Q}_{a_2}(\vec{v}_2 t, t) \cdots \hat{Q}_{a_k}(\vec{v}_k t, t) | \Omega >_C \\ & = t^{3k} \int d^3 v_1 \hat{\psi}_{a_1}^{(s_1)}(\vec{v}_1 t, t) \prod_{j=2}^k (d^3 w_j \hat{\psi}_{a_j}^{(s_j)}((\vec{v}_1 + \vec{w}_j) t, t)) \\ & < \Omega | \hat{Q}_{a_1}(\vec{v}_1 t, t) \hat{Q}_{a_2}((\vec{v}_1 + \vec{w}_2) t, t) \cdots \hat{Q}_{a_k}((\vec{v}_1 + \vec{w}_k) t, t) | \Omega >_C. \end{aligned} \quad (25)$$

Due to the cluster property the connected vacuum expectation value vanishes faster than any inverse power of the the maximum distance among the points, that is of $\sqrt{t^2 \sum_{l=2}^k w_l^2}$, for example think of $e^{-t^2 \sum_{l=2}^k w_l^2}$. Thus, if the supports of $\tilde{\psi}_{a_i}$ overlap the w -integrals give a contribution proportional to $|t|^{-3(k-1)}$ times the absolute value of the product of $\hat{\psi}$'s, which, on account of the first inequality in (22) amounts to $|t|^{-3k/2}$. Since the integral with respect to \vec{v}_1 is limited by (22) to a sphere of radius one, we find that (25) vanishes as proportionally to $|t|^{-3(k-2)/2}$. Notice however that if the supports of $\tilde{\psi}$'s do not overlap (25) vanishes faster than any inverse power of $|t|$ due to the second inequality in (22). Notice that this is independent of the s_i 's, that is of the choice of generalized creation or destruction operators.

Therefore, in the cluster decomposition of (21) one is left with 2-point connected terms only.

Considering the 2-point cluster terms and referring for simplicity to (11) one has instead:

$$\begin{aligned} & (-s_1 s_2) < \Omega | B_{\psi_{a_1}}^{(s_1)}(t) B_{\psi_{a_2}}^{(s_2)}(t) | \Omega >_C \\ & = (-s_1 s_2) \int_{x_i^0 \equiv t} d^3 x_1 d^3 x_2 (\dot{\psi}_{a_1}^{(s_1)}(x_1) - \psi_{a_1}^{(s_1)}(x_1) \frac{\partial}{\partial x_1^0}) (\dot{\psi}_{a_2}^{(s_2)}(x_2) - \psi_{a_2}^{(s_2)}(x_2) \frac{\partial}{\partial x_2^0}) \\ & < \Omega | Q_{a_1}(x_1) Q_{a_2}(x_2) | \Omega >_C \end{aligned}$$

$$\begin{aligned}
&= (-s_1 s_2) \int_{x_i^0 \equiv t} d^3 x_1 d^3 x_2 (\dot{\psi}_{a_1}^{(s_1)}(x_1) - \psi_{a_1}^{(s_1)}(x_1) \frac{\partial}{\partial x_1^0}) (\dot{\psi}_{a_2}^{(s_2)}(x_2) - \psi_{a_2}^{(s_2)}(x_2) \frac{\partial}{\partial x_2^0}) \\
&\int d^3 p \int_0^\infty \frac{dM}{2E_M} \sigma_{a_1, a_2}(\vec{p}, M) e^{ip_M(x_2 - x_1)} = (-s_1 s_2) (2\pi)^3 \int d^3 p \int_0^\infty \frac{dM}{2E_M} \sigma_{a_1, a_2}(\vec{p}, M) \\
&\frac{(E_M + s_1 E_{m_{a_1}})(E_M - s_2 E_{m_{a_2}})}{2\sqrt{E_{m_{a_1}} E_{m_{a_2}}}} \tilde{\psi}_{a_1}(-s_1 \vec{p}) \tilde{\psi}_{a_2}(s_2 \vec{p}) e^{i(s_1 E_{m_{a_1}} + s_2 E_{m_{a_2}})t} . \tag{26}
\end{aligned}$$

From the C^∞ choice of the functions σ_{a_1, a_2} and $\tilde{\psi}_a$ and from the Riemann-Lebesgue lemma it turns out that (26) vanishes faster than any inverse power of $|t|$ unless $m_{a_1} = m_{a_2}$ and $s_1 = -s_2$ and the supports of the $\tilde{\psi}$'s overlap, in which case (26) is t -independent.

Thus we conclude that the n -point functions of generalized creation/destruction operators have non-trivial asymptotic limit only if n is even and the number of creation operators equals that of destruction ones. In this case one has:

$$\begin{aligned}
\lim_{|t| \rightarrow \infty} \langle \Omega | \prod_{j=1}^m B_{\psi_{a_j}}^{(s_j)}(t) | \Omega \rangle &= \delta_{m, 2n} \sum_{i=2}^{2n} \delta_{m_{a_1}, m_{a_i}} \delta_{s_1, -s_i} \langle \Omega | B_{\psi_{a_1}}^{(s_1)} B_{\psi_{a_i}}^{(s_i)} | \Omega \rangle \\
\lim_{|t| \rightarrow \infty} \langle \Omega | \prod_{j=2, j \neq i}^{2n} B_{\psi_{a_j}}^{(s_j)}(t) | \Omega \rangle &. \tag{27}
\end{aligned}$$

In the special case in which the destruction operators lie on the left-hand side and are in the same number of the creation ones one has:

$$\lim_{|t| \rightarrow \infty} \langle \Omega | \prod_{i=1}^n B_{\psi_{a_i}}^{(+)}(t) \prod_{j=1}^n B_{\psi_{a_j}}^{(-)}(t) | \Omega \rangle = \sum_{\pi_i} \prod_{i=1}^n \delta_{a_i, a_{\pi_i}} \langle \Omega | B_{\psi_{a_i}}^{(+)} B_{\psi_{a_{\pi_i}}}^{(-)} | \Omega \rangle , \tag{28}$$

where π_i means permutation of the corresponding indices.

This equation implies that in the state vector $\prod_{k=1}^m B_{\psi_{a_k}}^{(-)}(t) | \Omega \rangle$ and in the limit $|t| \rightarrow \infty$ the order of the $B^{(-)}$ operators is immaterial, indeed it is immediate to verify that the norm of the difference of two such vectors with different ordering of the same operators vanishes in the limit. If furthermore the supports of the $\tilde{\psi}$'s do not overlap the norm of the difference vanishes faster than any inverse power of $|t|$.

It is possible to get a stronger result with almost local operators of the form:

$$Q_{m_a}(x) \equiv \int dy f_{m_a}((x-y)^2) Q_a(y) , \tag{29}$$

where $\tilde{f}_{m_a}(p^2) \equiv \int dx f_{m_a}(x^2) e^{-ipx}$ satisfies $f_{m_a}(m_a^2) = 1$, and has compact support contained in the region in which $(p^2 - m_a^2)^2 < \epsilon$ and ϵ small enough to include only the four momenta of single particle states with mass m_a . This is due to the fact that each $Q_{m_a}(x)$ acting on the vacuum creates a single particle state of mass m_a :

$$Q_{m_a}(x)|\Omega\rangle = \sum_{M,\alpha} \int d^3p |\vec{p}, M, \alpha\rangle \langle \vec{p}, M, \alpha | \Phi_a(0) | \Omega \rangle e^{ip_M x}$$

$$\tilde{f}(p_M) \tilde{f}_{m_a}(M^2) = \int d^3p \sqrt{\frac{m_a}{E_a}} |\vec{p}, a\rangle e^{ip_{m_a} x} \tilde{f}(p_{m_a}) \quad (30)$$

We label the corresponding generalized creation/destruction operators by: $C_{\psi_a}^{(\mp)}(t)$. The crucial new point is that:

$$\frac{d}{dt} C_{\psi_a}^{(-)}(t) |\Omega\rangle = 0, \quad (31)$$

as it is easy to verify. This implies that;

$$\frac{d}{dt} \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}(t) \equiv \frac{d}{dt} \prod_{k=1}^m C_{\psi_{a_k}}^{(-)}(t) |\Omega\rangle \rightarrow_{|t| \rightarrow \infty} 0 \quad (32)$$

since, the derivative of each factor of the operator product can be shifted to the right up to asymptotically vanishing contributions. Furthermore in the present case (27) becomes:

$$\lim_{|t| \rightarrow \infty} \langle \Omega | \prod_{j=1}^m C_{\psi_{a_j}}^{(s_j)}(t) | \Omega \rangle = \delta_{m,2n} \sum_{i=2}^{2n} \delta_{m_{a_1}, m_{a_i}} \delta_{s_1, +} \delta_{s_i, -} \langle \Omega | C_{\psi_{a_1}}^{(s_1)} C_{\psi_{a_i}}^{(s_i)} | \Omega \rangle$$

$$\lim_{|t| \rightarrow \infty} \langle \Omega | \prod_{j=2, j \neq i}^{2n} C_{\psi_{a_j}}^{(s_j)}(t) | \Omega \rangle = \delta_{m,2n} \sum_{i=2}^{2n} \delta_{m_{a_1}, m_{a_i}} \delta_{s_1, +} \delta_{s_i, -}$$

$$\int d^3p \tilde{\psi}_{a_1}(-\vec{p}) \tilde{\psi}_{a_i}(\vec{p}) \lim_{|t| \rightarrow \infty} \langle \Omega | \prod_{j=2, j \neq i}^{2n} C_{\psi_{a_j}}^{(s_j)}(t) | \Omega \rangle. \quad (33)$$

Thus

$$\lim_{t \rightarrow \pm\infty} \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}(t) = \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)}, \quad (34)$$

and the convergence rate is faster than any inverse power of $|t|$ if the supports of the $\tilde{\psi}$'s do not overlap. Otherwise the rate is that of $|t|^{-3/2}$

From (33) one can easily verify that:

$$\langle \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)} | \Psi_{\psi'_{b_1}, \dots, \psi'_{b_n}}^{(out/in)} \rangle = \delta_{n,m} \sum_{\pi_k, k=1, \dots, n} \prod_{j=1}^n \delta_{a_j, b_{\pi_j}} \int d^3 p \tilde{\psi}_{a_j}^*(p) \tilde{\psi}'_{b_{\pi_j}}(\vec{p}) \quad (35)$$

which implies that the asymptotic states $\Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)}$ must be interpreted as scattering states and can be written in the form:

$$| \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)} \rangle = \prod_{j=1}^n a_{\psi_{a_j}}^{(out/in)\dagger} | \Omega \rangle, \quad (36)$$

where $a_{\psi}^{(out/in)\dagger}$ are the creation operators in the Fock space of the in/out scattering states and satisfy the commutation rules:

$$[a_{\psi_a}^{(out/in)}, a_{\psi_b}^{(out/in)\dagger}] = \delta_{a,b} \int d^3 p \tilde{\psi}_a^*(\vec{p}) \tilde{\psi}'_b(\vec{p}). \quad (37)$$

It is a natural hypothesis of relativistic scattering theory that the finite linear combinations of vectors (36) define a dense subset of the Hilbert space. This condition is called *asymptotic completeness*

What we have found until now leads to a relation between scattering amplitudes and Wightman functions. However the explicit calculations rely more on the time-ordered functions that we shall introduce in next section than on Wightman functions. The bridge between time-ordered functions and scattering amplitudes is given by the LSZ reduction formulae. These are based on a further important result of Haag's scattering theory which comes from the study of the asymptotic behavior of the matrix element:

$$\lim_{t \rightarrow \pm\infty} \langle \Psi_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out/in)} | B_{\psi_c}^{(s)}(t) | \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)} \rangle. \quad (38)$$

This matrix element can be decomposed according:

$$\begin{aligned} \langle \Psi_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out/in)} | B_{\psi_c}^{(s)}(t) | \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)} \rangle &= \langle \Psi_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out/in)}(t) B_{\psi_c}^{(s)}(t) \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)}(t) \rangle \\ &+ \langle \Psi_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out/in)}(t) | B_{\psi_c}^{(s)}(t) [\Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)} - \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)}(t)] \rangle \\ &+ \langle [\Psi_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out/in)} - \Psi_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out/in)}(t)] | B_{\psi_c}^{(s)}(t) | \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)} \rangle. \end{aligned} \quad (39)$$

Now, using again (26), (27) and (28), and on the basis of asymptotic completeness, one can verify that $B_{\psi_c}^{(s)}(t)$ is bounded by a positive power of E

in the subspace of the Hilbert space spanned by the scattering states with energy lower than E . It follows that the second and third terms in the right-hand side vanish in the asymptotic limit. Thus one finds that the limit (39) coincides with:

$$\lim_{t \rightarrow \pm\infty} \langle \Omega | \prod_{j=1}^n C_{\psi'_{a_j}}^{(+)}(t) B_{\psi''_c}^{(s)}(t) \prod_{i=1}^m C_{\psi_{b_i}}^{(-)}(t) | \Omega \rangle , \quad (40)$$

From (33) and (28) one sees that this limit is given by:

$$\begin{aligned} & \delta_{n,m+1} \delta_{s,-} \sum_{\pi_j} \delta_{c,a\pi_1} \langle \Omega | C_{\psi'_{a\pi_1}}^{(+)} B_{\psi''_c}^{(-)} | \Omega \rangle \prod_{i=2}^n \delta_{a\pi_i, b_i} \langle \Omega | C_{\psi'_{a\pi_i}}^{(+)} C_{\psi_{b_i}}^{(-)} | \Omega \rangle \\ & + \delta_{m,n+1} \delta_{s,+} \sum_{\pi_i} \delta_{c,b\pi_1} \langle \Omega | B_{\psi''_c}^{(+)} C_{\psi_{b\pi_1}}^{(-)} | \Omega \rangle \prod_{i=2}^m \delta_{a_i, b\pi_i} \langle \Omega | C_{\psi'_{a_i}}^{(+)} C_{\psi_{b\pi_i}}^{(-)} | \Omega \rangle . \end{aligned} \quad (41)$$

Therefore one has:

$$\begin{aligned} & \lim_{t \rightarrow \pm\infty} \langle \Psi'_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out/in)} | B_{\psi''_c}^{(s)}(t) | \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)} \rangle = \delta_{s,-} \langle \Psi'_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out/in)} | a_{\psi''_c}^{(out/in)\dagger} | \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)} \rangle \\ & + \delta_{s,+} \langle \Psi'_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out/in)} | a_{\psi''_c}^{(out/in)*} | \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)} \rangle . \end{aligned} \quad (42)$$

Taking into account asymptotic completeness we find that we can replace either $\Psi'_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out/in)}$, or $\Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(out/in)}$ with a generic vector of a finite energy subspace of the Hilbert space and hence we have:

$$\begin{aligned} & \lim_{t \rightarrow \infty} \langle \Psi'_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out)} | B_{\psi''_c}^{(s)}(t) | \Psi \rangle = \delta_{s,-} \langle \Psi'_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out)} | a_{\psi''_c}^{(out)\dagger} | \Psi \rangle \\ & + \delta_{s,+} \langle \Psi'_{\psi'_{a_1}, \dots, \psi'_{a_n}}^{(out)} | a_{\psi''_c}^{(out)*} | \Psi \rangle . \end{aligned} \quad (43)$$

and

$$\begin{aligned} & \lim_{t \rightarrow -\infty} \langle \Psi' | B_{\psi''_c}^{(s)}(t) | \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(in)} \rangle = \delta_{s,-} \langle \Psi' | a_{\psi''_c}^{(in)\dagger} | \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(in)} \rangle \\ & + \delta_{s,+} \langle \Psi' | a_{\psi''_c}^{(in)*} | \Psi_{\psi_{a_1}, \dots, \psi_{a_m}}^{(in)} \rangle , \end{aligned} \quad (44)$$

and the convergence is fast in the case of non-overlapping momentum wave functions.

This weak asymptotic limit result is the basis of the L.S.Z. construction of the scattering amplitudes that we shall describe in next sections.

4 Properties of the time-ordered functions

We have thus shown how asymptotic states can be built in field theory. This implicitly allows the construction of scattering amplitudes from Wightman functions. It turns out however that constructive field theory is better formulated in terms of time-ordered functions, even if, as we shall see in a moment, these functions are more difficult to define than Wightman functions.

A n -point time-ordered function is formally defined in terms of Wightman n -point functions by the following formula:

$$\langle \Omega | T \left(\prod_{a=1}^n \Phi_a(x_a) \right) | \Omega \rangle \equiv \sum_{\pi_i, i=1, \dots, n} \prod_{i < j} \theta(x_{\pi_i}^0 - x_{\pi_j}^0) \langle \Omega | \Phi_{\pi_1}(x_{\pi_1}) \cdots \Phi_{\pi_n}(x_{\pi_n}) | \Omega \rangle, \quad (45)$$

where, as above, π_i labels the permutations of indices. Since the Wightman functions of strictly local operators (fields) are distributions and θ 's are discontinuous, in general the above formula does not make sense even in the framework of distribution theory. We consider, for example the case of two points.

$$\begin{aligned} \tau_{a,b}(x) &\equiv \langle \Omega | T(\Phi_a(x)\Phi_b(0)) | \Omega \rangle = \theta(x^0) \langle \Omega | \Phi_a(x)\Phi_b(0) | \Omega \rangle \\ &+ \theta(-x^0) \langle \Omega | \Phi_b(0)\Phi_a(x) | \Omega \rangle = \int_m^\infty dM \int \frac{d^3p}{2E_M} \\ &[e^{-ip_M x} \theta(x^0) \rho_{a,b}(M) + e^{ip_M x} \theta(-x^0) \rho_{a,b}(M)] \equiv \int \frac{dq}{(2\pi)^4} e^{-iqx} \tilde{\tau}_{a,b}(q^2) \\ &= -i \int_m^\infty dM \rho_{a,b}(M) \int \frac{dq}{(2\pi)^4} \frac{e^{-iqx}}{M^2 - q^2 - i\epsilon}, \end{aligned} \quad (46)$$

where we have used (5) and the fact that $\rho_{a,b}(M)$ is a symmetric matrix due to T-invariance. Now the last equation to make sense as a distribution equation the mass integral must converge at infinity and this depends on the dimension of the fields. If e.g. they have dimension two it is expected that, for large M , $\rho(M) \sim M$ and hence the mass integral does not converge. This implies that (46) does not define a distribution. A deeper discussion of this point is in order here. In order that $\tau_{a,b}(x)$ be a distribution, the integral $\tau_{a,b}[f] \equiv \int dx f(x) \tau_{a,b}(x)$ with f of class \mathcal{D} should be well defined, this corresponds to the condition that the integrals in:

$$\tau_{a,b}[f] = \int dq \tilde{f}(q) \tilde{\tau}_{a,b}(q^2) = -i \int_m^\infty dM \rho_{a,b}(M) \int dq \frac{\tilde{f}(q)}{M^2 - q^2 - i\epsilon}, \quad (47)$$

be absolutely convergent.

In the situation we are presently discussing this is not true; however let us multiply f by x^μ , we have:

$$\tau_{a,b}[x^\mu f] = -2 \int_m^\infty dM \rho_{a,b}(M) \int dq q^\mu \frac{\tilde{f}(q)}{(M^2 - q^2 - i\epsilon)^2}, \quad (48)$$

which is well defined. Therefore we can conclude that $\tau_{a,b}$ is ill defined only in the origin. However one can give an alternative definition of the two point function substituting $\tilde{\tau}_{a,b}(q^2) = -i \int_m^\infty dM \frac{\rho_{a,b}(M)}{M^2 - q^2 - i\epsilon}$ with:

$$\begin{aligned} \tilde{\tau}_{R,a,b}(q^2) &= K - i \int_m^\infty dM \frac{\rho_{a,b}(M)}{M^2} \left[\frac{M^2}{M^2 - q^2 - i\epsilon} - 1 \right] \\ &= K - iq^2 \int_m^\infty dM \frac{\rho_{a,b}(M)}{M^2} \frac{1}{M^2 - q^2 - i\epsilon}, \end{aligned} \quad (49)$$

where K is an arbitrary constant. Now the mass integral converges and it is easy to verify that, whenever $\int_m^\infty dM \frac{\rho_{a,b}(M)}{M^2}$ converges the difference between $\tilde{\tau}_R$ and $\tilde{\tau}$ is just a constant and hence $\tau_{a,b}$ and $\tau_{R,a,b}$ coincide everywhere except in the origin. If we define

$$\tau_{S,a,b}(x) \equiv -i \int \frac{dq}{(2\pi)^4} e^{-iqx} \int_m^\infty dM \frac{\rho_{a,b}(M)}{M^2} \frac{1}{M^2 - q^2 - i\epsilon}, \quad (50)$$

it is apparent that $\tau_{S,a,b}(x)$ is a distribution and one has:

$$\tau_{R,a,b}(x) = K\delta(x) - \partial^2 \tau_{S,a,b}(x), \quad (51)$$

and hence it is also a distribution.

This example shows which are the difficulties related with the definition of T-ordered functions and in particular that these difficulties come from their lack of definition when two or more point coincide and finally it puts into evidence the need of supplementary conditions to identify the T-ordered functions completely as distributions in R^{4n} . It turns out that the construction of a consistent set of T-ordered functions in the case in which the fields are identified with the Wick monomials of some fundamental free field coincides with the perturbative renormalization program. We postpone any further discussion of the construction problem of T-functions, we assume to have a consistent definition of those involving the local fields defined in (9) and we proceed in the construction of the scattering amplitudes using the LSZ reduction formulae.

5 The L.S.Z. reduction formulae

Given a set of fields Φ_{a_i} we build a corresponding set of almost local operators $Q_{a_i}(x) = \int dy f_{a_i}(x-y)\Phi_{a_i}(y)$, where the functions $f_{a_i}(x)$ belong to the class \mathcal{D} and their support is contained in the sheet $|x^0| < \Delta$. Then we select a set $\psi_{a_i}^{s_i}$ of solutions of the Klein-Gordon equation as in (19) with $\tilde{\psi}_{a_i}(\vec{p})$ of class \mathcal{D} and non-overlapping support and we consider:

$$\begin{aligned} A_{\psi_{a_1}^{s_1}, \dots, \psi_{a_n}^{s_n}} &\equiv i^n \int \prod_j dy_j \int_{\Gamma} \prod_i (dx_i \psi_{a_i}^{s_i}(x_i) (\partial_{x_i}^2 + m_{a_i}^2) f_{a_i}(x_i - y_i)) \\ &< \Omega | T \left(\prod_k^n \Phi_{a_k}(y_k) \right) | \Omega > , \end{aligned} \quad (52)$$

where the integration domain Γ is defined by: $|x_k^0| \leq T_k$ with $T_i - T_l \geq 2(l-i)\Delta$. Noticing that:

$$\begin{aligned} &\psi_{a_i}^{s_i}(x_i) (\partial_{x_i}^2 + m_{a_i}^2) f_{a_i}(x_i - y_i) \\ &= \partial_{x_i^0} \left(\psi_{a_i}^{s_i}(x_i) \partial_{x_i^0} f_{a_i}(x_i - y_i) - f_{a_i}(x_i - y_i) \partial_{x_i^0} \psi_{a_i}^{s_i}(x_i) \right) , \end{aligned} \quad (53)$$

one identifies (52) with:

$$(-)^{\frac{n-\sum s_l}{2}} < \Omega | T \left(\prod_k^n [B_{\psi_{a_k}}^{(s_k)}(T_k) - B_{\psi_{a_k}}^{(s_k)}(-T_k)] \right) | \Omega > , \quad (54)$$

where the generalized creation and destruction operators $B^{(s_k)}$ are defined in (20) and the time order refers to the $\pm T_k$ variables. Now, using (43) and (44) and taking into account that the vectors

$$T \left(\prod_{k=2}^n [B_{\psi_{a_k}}^{(s_k)}(T_k) - B_{\psi_{a_k}}^{(s_k)}(-T_k)] \right) | \Omega >$$

and

$$\left[T \left(\prod_{k=2}^n [B_{\psi_{a_k}}^{(s_k)}(T_k) - B_{\psi_{a_k}}^{(s_k)}(-T_k)] \right) \right]^\dagger | \Omega >$$

belong to a finite energy subset of the Hilbert space, we have:

$$\begin{aligned}
& \lim_{T_1 \rightarrow \infty} A_{\psi_{a_1}^{s_1}, \dots, \psi_{a_n}^{s_n}} \\
&= \lim_{T_1 \rightarrow \infty} (-)^{\frac{n - \sum s_l}{2}} \langle \Omega | B_{\psi_{a_1}}^{(s_1)}(T_1) T \left(\prod_{k=2}^n [B_{\psi_{a_k}}^{(s_k)}(T_k) - B_{\psi_{a_k}}^{(s_k)}(-T_k)] \right) | \Omega \rangle \\
&\quad - \lim_{T_1 \rightarrow \infty} (-)^{\frac{n - \sum s_l}{2}} \langle \Omega | T \left(\prod_{k=2}^n [B_{\psi_{a_k}}^{(s_k)}(T_k) - B_{\psi_{a_k}}^{(s_k)}(-T_k)] \right) B_{\psi_{a_1}}^{(s_1)}(-T_1) | \Omega \rangle \\
&= \delta_{s_1, +} (-)^{\frac{n - \sum s_l}{2}} \langle \Omega | a_{\psi_{a_1}}^{(out)} T \left(\prod_{k=2}^n [B_{\psi_{a_k}}^{(s_k)}(T_k) - B_{\psi_{a_k}}^{(s_k)}(-T_k)] \right) | \Omega \rangle \\
&\quad - \delta_{s_1, -} (-)^{\frac{n - \sum s_l}{2}} \langle \Omega | T \left(\prod_{k=2}^n [B_{\psi_{a_k}}^{(s_k)}(T_k) - B_{\psi_{a_k}}^{(s_k)}(-T_k)] \right) a_{\psi_{a_1}}^{(in)\dagger} | \Omega \rangle . \quad (55)
\end{aligned}$$

Taking the asymptotic limits for all T_k we get:

$$\begin{aligned}
& \lim_{T_n \rightarrow \infty} \dots \lim_{T_1 \rightarrow \infty} A_{\psi_{a_1}^{s_1}, \dots, \psi_{a_n}^{s_n}} \\
&= \langle \Omega | \left(\prod_{i: s_i=+} a_{\psi_{a_i}^*}^{(out)} \right) \left(\prod_{j: s_j=-} a_{\psi_{a_j}}^{(in)\dagger} \right) | \Omega \rangle , \quad (56)
\end{aligned}$$

that is a transition amplitude.

Now it is apparent from this formula that the order of limits is immaterial. Taking into account the fast convergence rate we can write the identity:

$$\begin{aligned}
& i^n \int \prod_j dy_j \int \prod_i (dx_i \psi_{a_i}^{s_i}(x_i) (\partial_{x_i}^2 + m_{a_i}^2) f_{a_i}(x_i - y_i)) \langle \Omega | T \left(\prod_k \Phi_{a_k}(y_k) \right) | \Omega \rangle \\
&= \langle \Omega | \left(\prod_{i: s_i=+} a_{\psi_{a_i}^*}^{(out)} \right) \left(\prod_{j: s_j=-} a_{\psi_{a_j}}^{(in)\dagger} \right) | \Omega \rangle , \quad (57)
\end{aligned}$$

where the x_i integrals cover the whole R^{4n} . Finally we notice that, if, as chosen above, $\tilde{f}_{a_i}(E_{m_a}, \vec{p}) = 1$ on the support of the corresponding ψ_{a_i} one can get rid of the f_{a_i} 's in (57) replacing them by Dirac deltas.

Computing cross sections of a process with f particles in the final state, that is $n = f + 2$, one more comment is in order. Decomposing the time-ordered function in the left-hand side of (57) into connected parts one finds a momentum conservation constraint for each connected part; it follows that

the only contribution to a cross section comes from the connected $(f + 2)$ -point function with $s_1 = s_2 = -$ and $s_j = +$ for $j \geq 2$. In this case the transition amplitude is given by:

$$\begin{aligned}
i^{f+2} \int \prod_{j=1}^2 (dy_j \psi_{a_j}^{(-)}(x_j)) \prod_{i>2} (dx_i \psi_{a_i}^{(+)}(x_i)) &< \Omega | T \left(\prod_{k=1}^{f+2} (\partial_{x_k}^2 + m_{a_k}^2) \Phi_{a_k}(x_k) \right) | \Omega \rangle_C \\
\equiv A_{2 \rightarrow f} . & \tag{58}
\end{aligned}$$

6 The functional formalism and the Effective Action

Assuming that all needed time-ordered functions be defined in analogy with the Wightman function case one can define the functional generator:

$$\begin{aligned} Z[J] &\equiv \sum_{n=0}^{\infty} \frac{i^n}{n!} \sum_{a_1, \dots, a_n} \int \prod_{j=1}^n (dx_j J_{a_j}(x_j)) \langle \Omega | T(\prod_k^n \Phi_{a_k}(x_k)) | \Omega \rangle \\ &\equiv \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle \Omega | T(\sum_a \int dx J_a(x) \Phi_a(x))^n | \Omega \rangle, \end{aligned} \quad (59)$$

where it is understood that the ordering operator acts under integral sign. Notice that in the present case there is no operator ordering problem since the time-ordered functions are symmetric functions. The functional generator is a formal power series of the J 's that are chosen of class \mathcal{S} and (59) can be formally written;

$$Z[J] = \langle \Omega | T(e^{i \sum_a \int dx J_a(x) \Phi_a(x)}) | \Omega \rangle . \quad (60)$$

In analogy with (18) the cluster decomposition of the time-ordered functions can be described introducing the connected generator:

$$Z[J]_C \equiv \sum_{n=2}^{\infty} \frac{i^{n-1}}{n!} \langle \Omega | T(\sum_a \int dx J_a(x) \Phi_a(x))^n | \Omega \rangle_C , \quad (61)$$

where we have taken into account that the fields have null vacuum expectation value. From (59) we can show that:

$$Z[J] = e^{iZ_C[J]} . \quad (62)$$

Indeed, taking a functional derivative of (59) comparing it with that of (60) on account of (16) we have:

$$\begin{aligned} \frac{\delta Z_C[J]}{\delta J_a(x)} &= \sum_{n=0}^{\infty} \frac{i^{n+1}}{n!} \langle \Omega | T(\Phi_a(x) (\sum_a \int dx J_a(x) \Phi_a(x))^n) | \Omega \rangle \\ &= \sum_{n=0}^{\infty} \frac{i^{n+1}}{n!} \sum_{m=1}^n \frac{n!}{m!(n-m)!} \langle \Omega | T(\Phi_a(x) (\sum_a \int dx J_a(x) \Phi_a(x))^m) | \Omega \rangle_C \end{aligned}$$

$$\begin{aligned}
& \langle \Omega | T \left(\sum_b \int dy J_b(y) \Phi_b(y) \right)^{n-m} | \Omega \rangle \\
&= \sum_{m=1}^{\infty} \frac{i^{m+1}}{m!} \langle \Omega | T \left(\Phi_a(x) \left(\sum_a \int dx J_a(x) \Phi_a(x) \right)^m \right) | \Omega \rangle_C \\
& \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle \Omega | T \left(\sum_a \int dx J_a(x) \Phi_a(x) \right)^n | \Omega \rangle = i \frac{\delta Z_C[J]}{\delta J_a(x)} Z[J] . \tag{63}
\end{aligned}$$

This is a first order differential equation whose solution with initial conditions $Z[0] = 1$, $Z_C[0] = 0$ is given by (62).

Given the connected generator, which is a formal power series too, one defines its Legendre transform as follows. One defines the distribution valued functional:

$$\Phi_a(x, J) \equiv \frac{\delta Z_C[J]}{\delta J_a(x)} . \tag{64}$$

Then one considers the Fourier transformed connected two-point function:

$$\tau_{a,b}^{(2)}(p^2) \equiv \int dx e^{ipx} \langle \Omega | T(\Phi_a(x) \Phi_b(0)) | \Omega \rangle . \tag{65}$$

If $\tau_{a,b}^{(2)-1}(p^2)$ is a matrix valued tempered distribution one can define a further distribution valued functional $J_a(x, \varphi)$ such that;

$$\Phi_a(x, J(\cdot, \varphi)) \equiv \varphi_a(x) , \tag{66}$$

and the formal power series Legendre transform of Z_C ;

$$\Gamma[\varphi] \equiv Z_C[J(\cdot, \varphi)] - \int dx \sum_a \varphi_a(x) J_a(x, \varphi) . \tag{67}$$

The new functional is the effective action in the following sense. Taking its functional derivative one has:

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi_a(x)} = -J_a(x, \varphi) , \tag{68}$$

and hence

$$\frac{\delta \Gamma[\varphi(\cdot, J)]}{\delta \varphi_a(x)} = -J_a(x) , \tag{69}$$

thus $\Phi(x, J)$ is the solution of the classical field equation induced by Γ vanishing at $J = 0$.

The inverse equation to (67) is:

$$Z_C[J] = \Gamma[\Phi(\cdot, J)] + \int dx \sum_a \Phi_a(x, J) J_a(x) . \quad (70)$$

It is worth noticing now that (58) is equivalent to:

$$A_{2 \rightarrow f} = i \int \prod_{i=1}^{f+2} (dx_i \psi_{a_i}^{s_i}(x_i) (\partial_{x_i}^2 + m_{a_i}^2) \frac{\delta}{\delta J_{a_i}(x_i)}) Z_C[J]|_{J=0} . \quad (71)$$

It can be shown that in the limit of perfect resolution of the wave packets, defining:

$$\psi_a^{(as)}(x) \equiv \sum_{i=1}^{f+2} \delta_{a_i, a} \psi_{a_i}^{s_i}(x) , \quad (72)$$

one can write:

$$\begin{aligned} A_{2 \rightarrow f} &= i e^{\int dx \sum_{i=1}^{f+2} \psi_{a_i}^{s_i}(x) (\partial_x^2 + m_{a_i}^2) \frac{\delta}{\delta J_{a_i}(x)}} Z_C[J]|_{J=0} \\ &= i e^{\int dx \sum_a \psi_a^{(as)}(x) (\partial_x^2 + m_a^2) \frac{\delta}{\delta J_a(x)}} Z_C[J]|_{J=0} \\ &\equiv i e^{\Sigma} Z_C[J]|_{J=0} , \end{aligned} \quad (73)$$

since the terms which are not linear in all $\psi_{a_i}^{s_i}$ do not contribute in the perfect resolution limit.

If we define:

$$\hat{\Phi}_a(x) \equiv e^{\Sigma} \Phi_a(x, J)|_{J=0} , \quad (74)$$

we can write (73) in the form:

$$A_{2 \rightarrow f} = i [\Gamma[\hat{\Phi}] + \int dx \sum_a \psi_a^{(as)}(x) (\partial_x^2 + m_a^2) \hat{\Phi}_a(x)] . \quad (75)$$

Let us consider $\hat{\varphi}$ more closely.

$$\begin{aligned} e^{\Sigma} \frac{\delta \Gamma[\Phi(\cdot, J)]}{\delta \varphi_a(x)} \Big|_{J=0} &= \frac{\delta \Gamma[\hat{\Phi}]}{\delta \varphi_a(x)} \\ &= -e^{\Sigma} J_a(x) \Big|_{J=0} = - \int dy \sum_a \psi_a^{(as)}(y) (\partial_y^2 + m_a^2) \delta(x - y) \\ &= \sum_a (\partial_x^2 + m_a^2) \psi_a^{(as)}(x) = 0 . \end{aligned} \quad (76)$$

Thus $\hat{\varphi}$ satisfies the field equation induced by the effective action Γ .

It remains to discuss the asymptotic properties of $\hat{\varphi}$ that determine it. Since $\hat{\varphi}$ is a distribution we should better discuss the asymptotic properties of $\hat{Q}_a(x) = \int dy f_a(x-y)\hat{\varphi}_a(y)$. Taking into account the reduction formulae we have:

$$\begin{aligned} \hat{Q}_a(x) = & \langle \Omega | [1 + \sum_{m=1}^f i^{-m} \sum_{k_1 < \dots < k_m = 3, \dots, f+2} \left(\prod_j a_{\psi_{a_{k_j}}^*}^{(out)} \right)] | \Omega \rangle \\ & Q_a(x) | [1 - i \sum_{i=1}^2 a_{\psi_{a_i}}^{(in)\dagger} - a_{\psi_{a_1}}^{(in)\dagger} a_{\psi_{a_2}}^{(in)\dagger}] | \Omega \rangle >_C \\ & + R_a(x) , \end{aligned} \quad (77)$$

where $R_a(x)$ accounts for the terms at least quadratic in one of the ψ_a 's and hence not contributing to $A_{2 \rightarrow f}$ in the limit of perfect resolution. Considering the asymptotic limit of the first term of (77) we notice that Eq. (42) says that in this limit Q_a contributes as an asymptotic creation or destruction operator. In particular its contribution to the negative frequency part in the $t \rightarrow \infty$ limit corresponds to a $a^{(out)\dagger}$ operator, while that to the positive frequency part in the $t \rightarrow \infty$ limit corresponds to a $a^{(in)}$ operator.

Therefore, taking into account the energy-momentum conservation constraint, we find that the negative frequency part in the $t \rightarrow \infty$ limit is given by

$$\sum_{k=3}^f \langle \Omega | a_{\psi_{a_k}^*}^{(out)} Q_a(x) | \Omega \rangle \rightarrow_{t \rightarrow \infty} \sum_{k=3}^f \delta_{a, a_k} \int dy f_a(x-y) \psi_{a_k}^{(-)}(y) , \quad (78)$$

since all other terms in the first term in (77) which asymptotically correspond to a $n \rightarrow m-1$ transition amplitude violate energy momentum conservation. The positive frequency part in the $t \rightarrow \infty$ limit corresponds to a $n-1 \rightarrow m$ transition amplitude which violate energy momentum conservation with the exception of

$$\langle \Omega | Q_a(x) [a_{\psi_{a_1}}^{(in)\dagger} + a_{\psi_{a_2}}^{(in)\dagger}] | \Omega \rangle \rightarrow_{t \rightarrow -\infty} \sum_{k=1}^2 \delta_{a, a_k} \int dy f_a(x-y) \psi_{a_k}^{(+)}(y) . \quad (79)$$

Thus, through (77), (78) and (79) we have shown that, up to terms not contributing to the transition amplitude in the perfect resolution limit, $\hat{\varphi}_a(x)$

satisfies the classical field equation corresponding to the effective action Γ and satisfies the same boundary conditions as the classical field in the semi-classical approximation. If furthermore we consider the expression of the transition amplitude given by (75) in terms of Γ we conclude that the fully quantized scattering theory coincides with the semi-classical one after substitution of the action with the effective action.

7 The construction of the theory, the Euclidean Quantum Field Theory

What one has to do now is to give a construction procedure for the connected functional generator (61). Let us consider first of all a scalar field theory; it is an elementary exercise to prove that:

$$Z_C[J] = \frac{1}{2} \int dx dy \Delta_F(x-y) J(x) J(y) , \quad (80)$$

where

$$\Delta_F(x) = \int \frac{dx}{(2\pi)^4} \frac{e^{-ipx}}{m^2 - p^2 - i0^+} \quad (81)$$

coincides with the solution of the differential equation:

$$(m^2 + \partial^2) \frac{\delta Z_C[J]}{\delta J(x)} = J(x) , \quad (82)$$

with the condition that Z_C should vanish at $J = 0$ and that positive/negative frequency components of $\frac{\delta Z_C[J]}{\delta J(x)}$ vanish in the limit $x^0 \rightarrow \mp\infty$.

It was noticed by Symanzik that the above equation, and hence its generalization to non free theories is highly simplified if one turns to its Euclidean version. This is obtained extending analytically the Fourier transformed 2-point function Δ ;

$$\begin{aligned} \tilde{\Delta}(p) &\equiv \int dx e^{ipx} \Delta(x) = \frac{1}{m^2 - p^2 - i0^+} \\ &= \frac{1}{(E_m(\vec{p}) - i0^+ - p^0)(E_m(\vec{p}) - i0^+ + p^0)} , \end{aligned} \quad (83)$$

to the domain $p^0 = e^{i\theta} p^4$ with $0 \leq \theta < \pi$ and in particular to the points with $\theta = \pi/2$. If we call Schwinger 2-point function $S(x_1 - x_2)$ the Fourier transform of the analytic continuation in the Euclidean space:

$$S(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{i \sum_{j=1}^4 x^j p_j}}{m^2 + \sum_{j=1}^4 p_j^2} \quad (84)$$

and

$$F_C[J] = \frac{1}{2} \int dx dy S(x-y) J(x) J(y) , \quad (85)$$

the equation corresponding to (82) is

$$(m^2 - \partial^2) \frac{\delta F_C[J]}{\delta J(x)} = J(x) , \quad (86)$$

identifying $F_C[J]$ with its solution vanishing at $J = 0$ and at infinity of R^4 .

It turns also out that for a general interacting theory the connected Euclidean n-point functions S_n obtained by the same analytical continuation from the Fourier transform of the time-ordered functions are also obtained, up to contact terms, that is terms supported at coinciding points, by analytical continuation of n-point Wightman functions W_n at imaginary times. that is:

$$\begin{aligned} S_n(x_1, \dots, x_n) &= W(x'_1, \dots, x'_n) \quad \text{for } x_1^4 > x_2^4 > \dots > x_n^4 \\ \text{and } \vec{x}_i &\equiv \vec{x}'_i, \quad -ix_i^4 \equiv x_i'^0 . \end{aligned} \quad (87)$$

Notice that S_n is a symmetric function of its arguments while W_n is not.

In the case of the free theory two-point functions one has, on account of (84) and (5) and for $x^4 > 0$:

$$S(\vec{x}, x^4) = \int \frac{d\vec{p}}{(2\pi)^4} e^{i\vec{p}\cdot\vec{x}} \int dp_4 \frac{e^{ip_4 x^4}}{E^2 + p_4^2} = \int \frac{d\vec{p}}{(2\pi)^4} e^{i\vec{p}\cdot\vec{x}} \frac{2\pi i}{2iE} e^{-Ex^4} = W(\vec{x}, -ix^4) . \quad (88)$$

It turns out furthermore that a local Poincaré invariant theory goes into an Euclidean invariant one.

A non-trivial advantage of the Euclidean theory is that one can choose the smearing functions appearing in (8) and in (10) Euclidean invariant as e.g. the Gaussian

$$\frac{\Lambda_0^4}{(2\pi)^2} e^{-\frac{\Lambda_0^2(x-y)^2}{2}} \equiv g_{\Lambda_0}(x-y) . \quad (89)$$

Notice that whenever treating Euclidean functions we shall use the Euclidean scalar product as in (84).

8 The Functional Integral in Euclidean Quantum Field Theory

We now consider how interacting Euclidean Quantum Field Theory can be built starting from the field equations.

In principle one should consider the equations of the fundamental fields, for simplicity for the moment we limit our discussion to the case of a single scalar field ϕ . These equations should relate a certain combination of derivatives of ϕ to a suitable local composite operator $\mathcal{J}(\phi(x), \partial\phi(x), \dots) \equiv \mathcal{J}[\phi(x)]$, to fix the ideas we can choose

$$-\partial^2\phi(x) = \mathcal{J}[\phi](x) . \quad (90)$$

The problem is however that I does not make sense since ϕ is distribution valued. Translating (90) as a functional differential equation heuristically one should replace (86) with :

$$-\partial^2 \frac{\delta F_C[J]}{\delta J(x)} = J(x) + \mathcal{J}\left[\frac{\delta}{\delta J(x)}\right] F_C[J] , , \quad (91)$$

where the last term should insert the operator \mathcal{J} into the Schwinger function, this however does not make sense since Schwinger functions are distributions.

To define composite operators we should rather introduce the smeared field

$$q_{\Lambda_0}(x) \equiv \int dy g_{\Lambda_0}(x-y)\phi(y) \equiv (g_{\Lambda_0} * \phi)(x) , \quad (92)$$

where we have introduced the convolution symbol $*$ and define \mathcal{J} as a function of q . After this substitution (91) should be written

$$-\partial^2 \frac{\delta F_C[J]}{\delta J(x)} = J(x) + \mathcal{J}\left[(g_{\Lambda_0} * \frac{\delta}{\delta J(y)})(x)\right] F_C[J] , , \quad (93)$$

However replacing as above $\mathcal{J}(\phi(x), \partial\phi(x), \dots)$ with $\mathcal{J}(q(x), \partial q(x), \dots) \equiv \mathcal{J}_{\Lambda_0}(x)$ destroys the locality of (90) and hence one should find a method to recover the locality of the theory. This is the Renormalization Group (RG) method.

Roughly the idea is that if one limits the range of momenta to a suitably bounded region it should be possible to find an interaction built with the q_{Λ_0} operators which, however complicated, has the same effect as a local operator.

In other and more precise words, if one considers test functions J whose Fourier transform has support bounded by $p < \Lambda \ll \Lambda_0$ it should be possible to choose non-local, Λ_0 -dependent \mathcal{J}_{Λ_0} and define a suitable $F_C[J, \Lambda_0]$ such that (93) be true independently of Λ_0 and such that \mathcal{J}_{Λ_0} become local in the limit $\Lambda_0 \rightarrow \infty$. In these conditions \mathcal{J}_{Λ_0} plays the role of an *effective term* in the field equation.

To realize this idea one considers a sufficiently wide class of positive functionals $I_{\Lambda_0}[\phi]$, that we call *effective interaction*, with a range of non-locality of the order of $\frac{1}{\Lambda_0}$ and choose $\mathcal{J}_{\Lambda_0}[\phi](x) = - \int dy g_{\Lambda_0}(x-y) \frac{\delta I_{\Lambda_0}[\phi]}{\delta \phi(x)}$. The restriction of the effective term in the field equation to the functional derivative of some effective interaction is "a priori" arbitrary in contrast with the scattering effective action whose existence is guaranteed by the reduction formulae. However to my knowledge this restriction is a price to pay for renormalization.

With the mentioned choice it is apparent that (90) is written:

$$\frac{\delta}{\delta \phi(x)} \left(\int dy \frac{(\partial \phi)^2}{2} + I_{\Lambda_0}[q_{\Lambda_0}] \right) = J(x) , \quad (94)$$

and (93) :

$$-\partial^2 \frac{\delta F_C[J, \Lambda_0]}{\delta J(x)} + (g_{\Lambda_0} * \frac{\delta I_{\Lambda_0}[g_{\Lambda_0} * \frac{\delta}{\delta J}]}{\delta \phi})(x) F_C[J, \Lambda_0] = J(x) , \quad (95)$$

whose solution positive for J real and equal to one for J vanishing is:

$$e^{F_C[J, \Lambda_0]} = \int d\mu[\phi] e^{\int dy (J(y)\phi(y) - \frac{(\partial \phi)^2}{2}) - I_{\Lambda_0}[q_{\Lambda_0}]} , \quad (96)$$

and $d\mu[\phi]$ is a positive and translation invariant probability measure normalized according: $\int d\mu[\phi] e^{-\int dy \frac{(\partial \phi)^2}{2}} = 1$.

The "solution" (96) deserves a long list of comments that we are obliged to shorten quite drastically. Let us only mention that the condition at $J = 0$ fixes the field independent term in I_{Λ_0} . Concerning the appearance of the functional integral, consider the ordinary differential equation:

$$\frac{d}{dx} f(x) + a \left(\frac{d}{dx} \right)^{2n+1} f(x) = x , \quad (97)$$

with a real and positive, the solution:

$$f(x) = \int_{-\infty}^{\infty} dy e^{-[\frac{y^2}{2} + a\frac{y^{2n+2}}{2n+2} + c+xy]} , \quad (98)$$

with c fixed by the condition $f(0) = 1$ is unique if we ask f to be real and positive. (96) is the functional version of (98).

Now the RG condition one wants to ask is F_C to have a regular limit for $\Lambda_0 \rightarrow \infty$ and the field equation to become local in this limit that we shall call ultra-violet (UV). This is, of course, a condition on I_{Λ_0} . However this does not require that I_{Λ_0} remain regular in the limit. The requirement is that F_C have a regular limit and $\frac{\delta I_{\Lambda_0}}{\delta \phi}$, however singular, to become local. This means that $\frac{\delta^2 I_{\Lambda_0}[g_{\Lambda_0}]}{\delta \phi(x)\delta \phi(y)} \rightarrow 0$ when $\Lambda_0 \rightarrow \infty$ for $(x-y)^2 = d^2 > 0$ fixed.

In order to proceed along this line one has to get free of I_{Λ_0} . With this purpose let us write (96) in the alternative form:

$$e^{F_C[J,\Lambda_0]} = e^{-I_{\Lambda_0}[g_{\Lambda_0} * \frac{\delta}{\delta J}]} e^{\frac{1}{2} \int dx J(S*J)} = e^{F_C[J,\Lambda_0]} = e^{-I_{\Lambda_0}[g_{\Lambda_0} * \frac{\delta}{\delta J}]} e^{\frac{1}{2}(JS*J)} , \quad (99)$$

where we have further simplified our notation inserting (fg) for $\int dx f(x)g(x)$ and

$$S(x) = \int \frac{dp}{(2\pi)^4} \frac{e^{ipx}}{p^2} = \frac{1}{(2\pi)^2 x^2} , \quad (100)$$

It is an exercise left to the reader to verify that F_C is the generator of connected Feynman diagrams corresponding to the interaction I_{Λ_0} and to the propagator (100).

9 The Wilson Effective Action in Euclidean Quantum Field Theory

Introducing $\hat{g} = g_{\Lambda_0} - g_\Lambda$, we are going to show that there exists a functional I_{Λ, Λ_0} such that

$$e^{F_C[J, \Lambda_0]} =: e^{-I_{\Lambda, \Lambda_0}[g_{\Lambda_0} * \frac{\delta}{\delta J} + \hat{g} * S * J]} : e^{\frac{1}{2}(J S * J)} , \quad (101)$$

where the symbol $: X[J, \frac{\delta}{\delta J}] :$ means the ordering prescription that functional derivatives should be placed on the right-hand side of J 's. The proof of (101) is given in Appendix A.

A further exercise left to the reader is to prove that $-I_{\Lambda, \Lambda_0}$ is the functional generator of connected amputated diagrams corresponding to the interaction I_{Λ_0} and to the propagator:

$$\hat{S} \equiv g_{\Lambda_0} * S * g_{\Lambda_0} - g_\Lambda * S * g_\Lambda . \quad (102)$$

Notice that one has:

$$e^{F_C[J]} = \int d\mu[\phi] e^{\int dy (J(y)\phi(y) - \frac{(\partial\phi)^2}{2}) - I_{\Lambda, \Lambda_0}[g_{\Lambda_0} * \phi + \hat{g} * S * J]} , \quad (103)$$

Now that we have introduced I_{Λ, Λ_0} we notice that this functional could perfectly well have a regular UV limit ($\Lambda_0 \rightarrow \infty$) I_Λ (this is the point that will take few pages to be verified). If this happens one has

$$e^{F_C[J]} = \int d\mu[\phi] e^{\int dy (J(y)\phi(y) - \frac{(\partial\phi)^2}{2}) - I_\Lambda[g_\Lambda * \phi + \hat{g} * S * J]} , \quad (104)$$

which must be Λ independent. One can derive the field equation requiring, to first order in ϵ , the invariance of F_C under the change of functional integration variables $\phi \rightarrow \phi + \epsilon$:

$$\begin{aligned} & (\epsilon(J + \partial^2 \frac{\delta}{\delta J})) e^{F_C[J]} \\ &= \int d\mu[\phi] (\epsilon \frac{\delta I_\Lambda}{\delta \phi})|_{\phi = g_\Lambda * \phi + \hat{g} * S * J} e^{\int dy (J(y)\phi(y) - \frac{(\partial\phi)^2}{2}) - I_\Lambda[g_\Lambda * \phi + \hat{g} * S * J]} \end{aligned} \quad (105)$$

If the support of $J(p)$ kept bounded and the limit $\Lambda \rightarrow \infty$ is taken it is clear that the right-hand side of (105) tends to $(\epsilon \frac{\delta I_\Lambda}{\delta \phi} [\frac{\delta}{\delta J}]) e^{F_C[J]}$ since $\hat{g} * J$ vanishes

in the limit. Therefore we have obtained an equation strictly analogous to (95). Furthermore, due to the Λ independence of (105), the right-hand side of this equation corresponds to a local operator if I_Λ tends to a local $\Lambda \rightarrow \infty$ limit.

In conclusion in order that our theory be local we must require I_{Λ, Λ_0} to have a regular UV limit I_Λ and I_Λ to become local in the $\Lambda \rightarrow \infty$ limit.

It is then apparent that we have to study I_{Λ, Λ_0} and its dependence on Λ in particular.

The crucial point is that $e^{FC[J]}$ does not depend on Λ therefore, taking the derivative of (101) with respect of Λ that we label by a dot, that is putting:

$$\dot{F}(\Lambda) \equiv \Lambda^2 \frac{\delta F(\Lambda)}{\delta \Lambda^2}, \quad (106)$$

one has:

$$\begin{aligned} & : \left(-\dot{I}_{\Lambda, \Lambda_0}[\phi] - \left(\frac{\delta I_{\Lambda, \Lambda_0}}{\delta \phi} \dot{g}_\Lambda * \left[\frac{\delta}{\delta J} - S * J \right] \right) e^{-I_{\Lambda, \Lambda_0}[\phi]} \Big|_{\phi = g_{\Lambda_0} * \frac{\delta}{\delta J} + \hat{g} * S * J} : \right. \\ & \left. e^{\frac{1}{2} \int dx dy (J * S * J)} \right) \end{aligned} \quad (107)$$

With the chosen ordering prescription the functional derivative $\frac{\delta}{\delta J}$ in the second term of this expression lies on the right-hand side and hence it can be replaced by $S * J$. Thus at a first sight the two terms in brackets should sum to zero. This is however not true since the existing $S * J$ in (104) lies on the left-hand side of the ordered expression. What remains of the two contributions is what comes re-ordering $S * J$ the left-hand side of the J functional derivative appearing in ϕ . This is given by:

$$\int dx dy (\dot{g}_\Lambda * S * g_\Lambda)(x - y) \left[\frac{\delta I_{\Lambda, \Lambda_0}}{\delta \phi(x)} \frac{\delta I_{\Lambda, \Lambda_0}}{\delta \phi(y)} - \frac{\delta^2 I_{\Lambda, \Lambda_0}}{\delta \phi(x) \delta \phi(y)} \right]. \quad (108)$$

Thus, asking the resulting ordered expression to vanish, which is a sufficient condition for (107) be satisfied one has the evolution equation for the effective interaction I_{Λ, Λ_0} :

$$\begin{aligned} \dot{I}_{\Lambda, \Lambda_0}[\phi] &= \int dx dy (\dot{g}_\Lambda * S * g_\Lambda)(x - y) \left[\frac{\delta I_{\Lambda, \Lambda_0}}{\delta \phi(x)} \frac{\delta I_{\Lambda, \Lambda_0}}{\delta \phi(y)} - \frac{\delta^2 I_{\Lambda, \Lambda_0}}{\delta \phi(x) \delta \phi(y)} \right] \\ &= \frac{1}{2} \int dx dy \dot{S}(x - y) \left[\frac{\delta^2 I_{\Lambda, \Lambda_0}}{\delta \phi(x) \delta \phi(y)} - \frac{\delta I_{\Lambda, \Lambda_0}}{\delta \phi(x)} \frac{\delta I_{\Lambda, \Lambda_0}}{\delta \phi(y)} \right]. \end{aligned} \quad (109)$$

Considering the Fourier transforms, that is in momentum space, in which the propagator (100) is $S = \frac{1}{p^2}$ and choosing the gaussian smearing (89) one has:

$$\hat{S}(p) = \frac{e^{-\frac{p^2}{\Lambda_0^2}} - e^{-\frac{p^2}{\Lambda^2}}}{p^2} \quad (110)$$

and

$$\dot{\hat{S}}(p) = -\frac{e^{-\frac{p^2}{\Lambda^2}}}{\Lambda^2}, \quad (111)$$

thus the evolution equation is written:

$$\dot{I}_{\Lambda, \Lambda_0}[\phi] = \frac{1}{2} \int \frac{dp}{(2\pi)^4} \frac{e^{-\frac{p^2}{\Lambda^2}}}{\Lambda^2} \left[\frac{\delta I_{\Lambda, \Lambda_0}}{\delta \phi(p)} \frac{\delta I_{\Lambda, \Lambda_0}}{\delta \phi(-p)} - \frac{\delta^2 I_{\Lambda, \Lambda_0}}{\delta \phi(p) \delta \phi(-p)} \right]. \quad (112)$$

Equation (112) can be further elaborated introducing the scaled field variables $\phi(p) \equiv \Lambda^{-3} \varphi(\frac{p}{\Lambda})$ and the series expansion of I_{Λ, Λ_0} ;

$$I_{\Lambda, \Lambda_0}[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^n (dp_i \phi(p_i)) \delta(\sum_{j=1}^n p_j) I_n(\Lambda, \Lambda_0, p). \quad (113)$$

Setting $I_n(\Lambda, \Lambda_0, p) \equiv \Lambda^{4-n} i_n(\Lambda, \Lambda_0, \frac{p}{\Lambda})$ one can write I_{Λ, Λ_0} as a functional of φ through:

$$I_{\Lambda, \Lambda_0}[\phi] = \hat{I}_{\Lambda, \Lambda_0}[\varphi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^n (dp_i \varphi(p_i)) \delta(\sum_{j=1}^n p_j) i_n(\Lambda, \Lambda_0, p). \quad (114)$$

One has furthermore:

$$\begin{aligned} \frac{\delta I_{\Lambda, \Lambda_0}[\phi]}{\delta \phi(p)} &= \frac{1}{\Lambda} \frac{\delta \hat{I}_{\Lambda, \Lambda_0}[\varphi]}{\delta \varphi(\frac{p}{\Lambda})} \\ \dot{I}_{\Lambda, \Lambda_0}[\phi] &= \dot{\hat{I}}_{\Lambda, \Lambda_0}[\varphi] - \frac{1}{2} \int dp (3\varphi(p)) + p \cdot \partial_p \varphi(p) \frac{\delta \hat{I}_{\Lambda, \Lambda_0}[\varphi]}{\delta \varphi(p)}, \end{aligned} \quad (115)$$

and hence the evolution equation (112) becomes:

$$\begin{aligned} &\dot{\hat{I}}_{\Lambda, \Lambda_0}[\varphi] - \frac{1}{2} \int dp (3\varphi(p)) + p \cdot \partial_p \varphi(p) \frac{\delta \hat{I}_{\Lambda, \Lambda_0}[\varphi]}{\delta \varphi(p)} \\ &= \frac{1}{2} \int \frac{dp}{(2\pi)^4} e^{-p^2} \left[\frac{\delta \hat{I}_{\Lambda, \Lambda_0}}{\delta \varphi(p)} \frac{\delta \hat{I}_{\Lambda, \Lambda_0}}{\delta \varphi(-p)} - \frac{\delta^2 \hat{I}_{\Lambda, \Lambda_0}}{\delta \varphi(p) \delta \varphi(-p)} \right]. \end{aligned} \quad (116)$$

and the equation for $e^{-\hat{I}_{\Lambda, \Lambda_0}[\varphi]}$ is:

$$\begin{aligned} & [\Lambda^2 \partial_{\Lambda^2} - \frac{1}{2} \int dp (3\varphi(p)) + p \cdot \partial_p \varphi(p)] \frac{\delta}{\delta \varphi(p)} e^{-\hat{I}_{\Lambda, \Lambda_0}[\varphi]} \\ &= -\frac{1}{2} \int \frac{dp}{(2\pi)^4} e^{-p^2} \frac{\delta^2}{\delta \varphi(p) \delta \varphi(-p)} e^{-\hat{I}_{\Lambda, \Lambda_0}[\varphi]} . \end{aligned} \quad (117)$$

which looks very much like a diffusion equation in the functional space whose asymptotic solutions ($\Lambda \rightarrow 0$), whenever the second order differential operator is elliptic, are Gaussian solutions, that is, trivially free theories.

A more formal approach which is however less dependent of the particular nature of the field and of the structure of the propagator S is the iterative approach which generates perturbative renormalization.

One can perfectly continue the analysis studying the solutions to (116), however many applications are simplified if one considers an other functional which coincides with I_{Λ, Λ_0} in the $\Lambda \rightarrow \infty$ limit. We call this new functional the *effective proper generator*

10 The Effective Proper Generator

To construct this functional it is convenient to introduce the *effective connected generator*:

$$\mathcal{Z}[J] \equiv \frac{1}{2}(J\hat{S} * J) - I_{\Lambda, \Lambda_0}[\hat{S} * J] , \quad (118)$$

that is the generator of the connected diagrams corresponding to the interaction I_{Λ_0} and to the propagator \hat{S} and its formal power series Legendre transform \mathcal{V} defined much in the same way as in section 6 (67) .

Assuming for simplicity that $I_{\Lambda, \Lambda_0}[\phi]$ does not contain a linear term in its functional variable ϕ , we put

$$\phi \equiv \hat{S} * (J - \frac{\delta I_{\Lambda, \Lambda_0}}{\delta \phi}[\hat{S} * J]) , \quad (119)$$

with ϕ is of class \mathcal{D} in momentum space. The inverse functional $J[\phi]$ is defined as the formal power series in ϕ corresponding to the solution of:

$$J = \hat{C}\phi + \frac{\delta I_{\Lambda, \Lambda_0}}{\delta \phi}[\phi] , \quad (120)$$

where

$$\hat{C}(p) \equiv (\hat{S})^{-1}(p) = \frac{p^2}{e^{-\frac{p^2}{\Lambda_0^2}} - e^{-\frac{p^2}{\Lambda^2}}} . \quad (121)$$

Then we define:

$$\mathcal{V}[\phi] \equiv (\phi J[\phi]) - \mathcal{Z}[J[\phi]] = (\phi J[\phi]) - \frac{1}{2}(J[\phi]\hat{S} * J[\phi]) + I_{\Lambda, \Lambda_0}[J[\phi]] , \quad (122)$$

It is a direct consequence of its definition that:

$$\begin{aligned} \frac{\delta^2 \mathcal{Z}}{\delta J(x)\delta J(y)} &= \left(\frac{\delta^2 \mathcal{V}}{\delta \phi(x)\delta \phi(y)} \right)^{-1} = \hat{S}(x-y) - (\hat{S} * \frac{\delta^2 I_{\Lambda, \Lambda_0}}{\delta \phi^2} * \hat{S})(x, y) \\ \dot{\mathcal{V}} &= -\dot{\mathcal{Z}} \quad , \quad \frac{\delta \mathcal{V}}{\delta \phi} = J \end{aligned} \quad (123)$$

Furthermore from (109) one has:

$$\begin{aligned}
\dot{Z} &= \frac{1}{2}(J\dot{S} * J) - \dot{I}_{\Lambda\Lambda_0} - (J\dot{S} * \frac{\delta I_{\Lambda\Lambda_0}}{\delta\phi}) \\
&= \frac{1}{2} \left[\left((J - \frac{\delta I_{\Lambda\Lambda_0}}{\delta\phi})\dot{S} * (J - \frac{\delta I_{\Lambda\Lambda_0}}{\delta\phi}) \right) - \left(\frac{\delta}{\delta\phi}\dot{S} * \frac{\delta}{\delta\phi} \right) I_{\Lambda\Lambda_0} \right] \\
&= -\dot{\mathcal{V}} \\
&= \frac{1}{2} \left[-(\phi\dot{C} * \phi) + Tr(\dot{C}[\hat{S} - \left(\frac{\delta^2 \mathcal{V}}{\delta\phi^2} \right)^{-1}]) \right], \tag{124}
\end{aligned}$$

where we have introduced: $Tr(AB) \equiv \int dx dy A(x, y)B(y, x)$. Notice that on the basis of the first equation in (123) one can see that the above trace is well defined in spite of the singular behavior of \hat{C} at infinite momentum. Defining the *Effective Proper Generator*:

$$\mathcal{V}[\phi] \equiv \frac{1}{2}(\phi\hat{C} * \phi) + V[\phi], \tag{125}$$

where the sign of V has been chosen in order that V be positive, one has

$$\begin{aligned}
\left(\frac{\delta^2 \mathcal{V}}{\delta\phi^2} \right)^{-1} &= \left(\hat{C} + \frac{\delta^2 V}{\delta\phi^2} \right)^{-1} \\
&= \left((1 + \frac{\delta^2 V}{\delta\phi^2} * \hat{S}) * \hat{C} \right)^{-1} = \hat{S} * \left(\sum_{n=0}^{\infty} (- * \frac{\delta^2 V}{\delta\phi^2} * \hat{S})^n \right), \tag{126}
\end{aligned}$$

and the evolution equation for $V[\phi]$:

$$\begin{aligned}
\dot{V}[\phi] &= -\frac{1}{2}(\phi\dot{C} * \phi) + \dot{\mathcal{V}}[\phi] \\
&= -\frac{1}{2}Tr(\dot{C}[\hat{S} - \left(\frac{\delta^2 \mathcal{V}}{\delta\phi^2} \right)^{-1}]) \\
&= \frac{1}{2}Tr(\dot{C}\hat{S} \sum_{n=1}^{\infty} (- * \frac{\delta^2 V}{\delta\phi^2} * \hat{S})^n) \\
&= \frac{1}{2}Tr \left(\dot{S} \frac{\delta^2 V}{\delta\phi^2} \sum_{n=0}^{\infty} (- * \hat{S} * \frac{\delta^2 V}{\delta\phi^2})^n \right) \\
&= \frac{1}{2}Tr \left(\dot{S} \left(\frac{\delta^2 V}{\delta\phi^2} - \frac{\delta^2 V}{\delta\phi^2} * \hat{S} * \frac{\delta^2 V}{\delta\phi^2} + \dots \right) \right) \\
&\equiv R[\phi]. \tag{127}
\end{aligned}$$

It is left to the reader to verify that $V[\phi]$ has the same $\Lambda \rightarrow \infty$ limit as $I_{\Lambda, \Lambda_0}[\phi]$. For this reason we shall analyze the solution of (127) in some detail.

Once again it is convenient to use scaled fields as above setting:

$$\begin{aligned} V[\phi] &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^n (dp_i \varphi(p_i)) \delta(\sum_{j=1}^n p_j) v_n(p_1, \dots, p_n, \Lambda, \Lambda_0) \\ &= \hat{V}[\varphi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^n (dp_i \varphi(p_i)) \delta(\sum_{j=1}^n p_j) v_n(p_1, \dots, p_n, \Lambda, \Lambda_0) . \end{aligned} \quad (128)$$

one finds for the coefficient functions the evolution equations:

$$\begin{aligned} & \int \prod_{i=1}^n (dq_i \varphi(q_i)) \delta(\sum_{j=1}^n q_j) [\Lambda \partial_{\Lambda} + 4 - n - q \partial_q] v_n(q_1, \dots, q_n, \Lambda, \Lambda_0) \\ &= - \int \prod_{i=1}^n (dq_i \varphi(q_i)) \delta(\sum_{j=1}^n q_j) \left[\int \frac{dp}{(2\pi)^4} e^{-p^2} v_{n+2}(q_1, \dots, q_n, p, -p, \Lambda, \Lambda_0) \right. \\ &+ \int \frac{dp}{(2\pi)^4} e^{-p^2} \int dq \frac{e^{-\alpha q^2} - e^{-q^2}}{q^2} \delta(\sum_{j=1}^m q_j + p - q) \sum_{m=1}^n \frac{n!}{m!(n-m)!} \\ & \left. v_{m+2}(q_1, \dots, q_m, p, -q, \Lambda, \Lambda_0) v_{n-m+2}(q_{m+1}, \dots, q_n, q, -p, \Lambda, \Lambda_0) + \dots \right] \\ &\equiv \int \prod_{i=1}^n (dq_i \varphi(q_i)) \delta(\sum_{j=1}^n q_j) r_n(q_1, \dots, q_n, \Lambda, \Lambda_0) , \end{aligned} \quad (129)$$

where we have set $\alpha = \frac{\Lambda^2}{\Lambda_0^2}$. Notice that if the $|v_n|$'s are bounded and φ is of class \mathcal{S} all integrals in the right-hand side of (129) are absolutely convergent uniformly in α due to momentum conservation limiting the range of the integration variables (e.g. $p - q$ in the second term in brackets). Thus the absolute value of r_n is also bounded. To analyze the solutions of (129) we notice, first of all that the coefficients v_n and r_n in the corresponding expansion of R are analytic functions of the momenta due to the presence of the cut-off Λ which plays the role of an infra-red cut-off. Then we can consider first the evolution of v_0 and r_0 , of the coefficients of the Taylor expansion of v_2 and r_2 up to the second order in the momenta, that is $v_2(0, 0, \Lambda, \Lambda_0) \equiv v_2(0)$ and $\partial_{q^2} v_2'(0, 0, \Lambda, \Lambda_0) \equiv v_2(0)$ and analogous coefficients for r_2 and of the value of v_4 and r_4 at zero momenta. Then we complete the analysis studying $\partial_{q^2}^2 v_2(q, -q, \Lambda, \Lambda_0)$, the first derivatives $\partial_q v_4(q_1, \dots, q_4, \Lambda, \Lambda_0)$ and v_n for $n \geq 6$.

The Λ dependence of the q -independent coefficients is controlled by the evolution equations

$$\begin{aligned}
[\Lambda\partial_\Lambda + 4] v_0(\Lambda, \Lambda_0) &= r_0(\Lambda, \Lambda_0) \\
[\Lambda\partial_\Lambda + 2] v_2(0, 0, \Lambda, \Lambda_0) &= r_2(0, 0, \Lambda, \Lambda_0) \\
\Lambda\partial_\Lambda v'_2(0, 0, \Lambda, \Lambda_0) &= r'_2(0, 0, \Lambda, \Lambda_0) \\
\Lambda\partial_\Lambda v_4(0, 0, 0, 0, \Lambda, \Lambda_0) &= r_2(0, 0, 0, 0, \Lambda, \Lambda_0) ,
\end{aligned} \tag{130}$$

which are solved by:

$$\begin{aligned}
v_0(\Lambda) &= \left(\frac{\Lambda}{\Lambda_R}\right)^{-4} \left[c_0 + \int_1^{\left(\frac{\Lambda}{\Lambda_R}\right)} d\lambda \lambda^3 r_0(\Lambda_R \lambda) \right] \\
v_2(\Lambda) &= \left(\frac{\Lambda}{\Lambda_R}\right)^{-2} \left[c_2 + \int_1^{\left(\frac{\Lambda}{\Lambda_R}\right)} d\lambda r_2(\Lambda_R \lambda) \right] \\
v'_2(\Lambda) &= \left[c'_2 + \int_1^{\left(\frac{\Lambda}{\Lambda_R}\right)} \frac{d\lambda}{\lambda} r'_2(\Lambda_R \lambda) \right] \\
v_4(\Lambda) &= \left[c_4 + \int_1^{\left(\frac{\Lambda}{\Lambda_R}\right)} \frac{d\lambda}{\lambda} r_4(\Lambda_R \lambda) \right] ,
\end{aligned} \tag{131}$$

for some Λ_R .

Then we see that, taking a generic k -th momentum derivative $\partial_q^k v_n(q, \Lambda)$ of the amplitude v_n , one has evolution equations completely analogous to (129); that is :

$$[\Lambda\partial_\Lambda + 4 - n - k - q\partial_q] \partial_q^k v_n(q_1, \dots, q_{2n}, \Lambda, \Lambda_0) = \partial_q^k r_n(q_1, \dots, q_{2n}, \Lambda, \Lambda_0) \tag{132}$$

In the cases we are studying, in which $n + k > 5$ it is convenient to consider the solution:

$$\begin{aligned}
\partial_q^k v_n(q, \Lambda, \Lambda_0) &= \left(\frac{\Lambda}{\Lambda_0}\right)^{n+k-4} \int_1^{\left(\frac{\Lambda}{\Lambda_0}\right)} d\lambda \lambda^{3-n-k} \partial_q^k r_n\left(\frac{\Lambda q}{\Lambda_0 \lambda}, \Lambda_0 \lambda, \Lambda_0\right) \\
&= \Lambda^{n+k-4} \int_{\Lambda_0}^\Lambda dx x^{3-n-k} \partial_q^k r_n\left(\frac{\Lambda q}{x}, x, \Lambda_0\right) .
\end{aligned} \tag{133}$$

Computing iteratively r_n from (129) and taking into account (133) and (131) one finds coefficient functions $v_n(q, \Lambda)$ that are power series of the input

data c_2, c'_2 and c_4 and the terms of order ν in these series satisfy, uniformly in $\Lambda_0 > \Lambda, \Lambda_R$ and q , the inequality:

$$\sup |\partial_q^k v_n^{(\nu)}(q, \Lambda)| \leq P_{n,k,\nu} \left(\log \left(\frac{\Lambda}{\Lambda_R} \right) \right) \quad (134)$$

where $P_{n,k,\nu}$ is a suitable polynomial of degree lower or equal to ν . A completely analogous inequality holds true for $\sup |\partial_q^k r_n^{(\nu)}(q, \Lambda)|$. Therefore the UV limit can be taken on the found solutions and it appears harmless since the x -integral in (133) and those on (129) converge uniformly in the limit.

Furthermore, taking into account that the coefficient functions I_n in the series expansion of I_{Λ, Λ_0} (113) are strictly related and of the same order of magnitude of the V_n 's and the same is true for i_n and v_n , and by recalling that:

$$V_n(p, \Lambda, \Lambda_0) = \Lambda^{4-n} v_n \left(\frac{p}{\Lambda}, \Lambda, \Lambda_0 \right) \quad (135)$$

one verifies immediately that the functional $V[\phi]$ and hence $I_{\Lambda, \Lambda_0}[\phi]$ have a local, however singular $\Lambda \rightarrow \infty$ limit and hence the theory is local.

The result we have reached is based on the construction of the solution in power series of c_2, c'_2 and c_4 , As a matter of fact it is not a problem to sum the series expansion in c_2 that plays the role of mass parameter and c'_2 that fixes the scale of the field, for simplicity one can chose both constants vanishing and remain with c_4 playing the role of a coupling constant g which must be positive for stability, that is otherwise the functional integral (96) does not make sense. One gets in this way a perturbation theory which should be accurate for small values of $c_4 \equiv g$.

There is however a different and, potentially, more interesting construction strategy which is based on the smallness of $V_4(\Lambda)$ for some value of Λ rather than g . In particular this strategy is crucial for theories with $V_4(\Lambda)$ vanishing at infinite Λ , the limit we are interested in to implement locality of the theory. Indeed, if this situation that is called *asymptotic freedom* were actual, one would have a guarantee that the theory be consistent, independently of the possibility of producing accurate results at low energy.

In order to give a more precise idea of this possibility let us analyze our scalar model. If v_4 is small it is not difficult to verify that $r_{2n} \sim O((v_4)^n)$ this implies a strong correction to v_2 that, by the way, should apply to the Higgs particle mass and hence is the basic reason to invoke super-symmetric extensions of the Standard Model.

However the crucial point of our argument concerns the evolution equation of v_4 itself. Considering the evolution equation of v_4 at vanishing momenta and up to the first non-trivial order in v_4 one has from (129) and (130):

$$\begin{aligned}\Lambda\partial_\Lambda v_4(0, \Lambda) &= \frac{4!}{(2!)^2} \int \frac{dp}{(2\pi)^4} e^{-p^2} \frac{1 - e^{-p^2}}{p^2} v_4^2(0, \Lambda) \\ &= \frac{6\pi^2}{(2\pi)^4} \int_0^\infty dx (e^{-x} - e^{-2x}) v_4^2(0, \Lambda) = \frac{3}{(4\pi)^2} v_4^2(0, \Lambda),\end{aligned}\quad (136)$$

where we have disregarded the term in v_6 which is $O((v_4)^3)$ and any momentum dependence of v_4 itself which would induce further higher order corrections in v_4 . Integrating this differential equation between Λ_R and Λ with $v_4(0, \Lambda_R) = g > 0$ one has:

$$v_4(0, \Lambda) = \frac{g}{1 - \frac{3}{(4\pi)^2} g \ln \frac{\Lambda}{\Lambda_R}}. \quad (137)$$

This result is apparently deceiving since v_4 increases with Λ reaching a singular point in $\Lambda = e^{\frac{(4\pi)^2}{3g}} \Lambda_R$ which is called the Landau singularity.

It is clear that this result confirms that the only consistent scalar theory is the free one.

It has been a surprise the result found about 20 year ago that the same calculation in the case of the Yang-Mills gauge theory leads to the opposite result.

A The Wilson Action

To prove (101) we introduce a simplified notation understanding space-time integrals and variables and introducing the symbol ∂_1 for $g_\Lambda * \frac{\delta}{\delta J}$ and ∂_2 for $\hat{g}_\Lambda * \frac{\delta}{\delta J}$. It is then apparent that

$$g_{\Lambda_0} * \frac{\delta}{\delta J} = g_\Lambda * \frac{\delta}{\delta J} + \hat{g} * \frac{\delta}{\delta J} \equiv \partial_1 + \partial_2. \quad (138)$$

Thus we write (99) as:

$$e^{F_C[J, \Lambda_0]} = e^{-I_{\Lambda_0}[\partial_1 + \partial_2]} e^{\frac{(JS^*J)}{2}} \equiv F[\partial_1 + \partial_2] e^{\frac{(JS^*J)}{2}} \equiv \sum_{n,m=0}^{\infty} \frac{F_{n+m}}{n!m!} \partial_1^n \partial_2^m e^{\frac{(JS^*J)}{2}}. \quad (139)$$

Then we notice that:

$$\begin{aligned}
\partial_2^m e^{\frac{(JS^*J)}{2}} &= e^{\frac{(JS^*J)}{2}} \sum_{k=0}^m \frac{m!}{k!(m-k)!} \partial_2^k (S^*J)_2^{m-k} \\
&= e^{\frac{(JS^*J)}{2}} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{k!(m-2k)!} (SJ^*)_2^{m-2k} S_{2,2}^k, \tag{140}
\end{aligned}$$

where we have introduced $(SJ)_2 \equiv \hat{g} * S * J$, and $S_{2,2} \equiv \hat{g} * S * \hat{g}$ and by $\lfloor \frac{n}{2} \rfloor$ we mean the integer part of $\frac{n}{2}$. Now from (139) and (140) we get:

$$\begin{aligned}
e^{F_C[J, \Lambda_0]} &= \sum_{n,m=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{F_{n+m}}{k!m!(n-2k)!} \partial_1^m e^{\frac{(JS^*J)}{2}} (S^*J)_2^{n-2k} S_{2,2}^k \\
&= \sum_{n,m,k=0}^{\infty} \frac{F_{n+m+2k} S_{2,2}^k}{k!m!n!} \partial_1^m (S^*J)_2^n e^{\frac{(JS^*J)}{2}} \\
&= \sum_{n,m,k,q=0}^{\infty} \frac{F_{n+m+2q+2k} S_{2,2}^k S_{1,2}^q}{q!k!m!n!} (S^*J)_2^n \partial_1^m e^{\frac{(JS^*J)}{2}} \\
&\equiv: e^{-I_{\Lambda, \Lambda_0} [g_{\Lambda} * \frac{\delta}{\delta J} + \hat{g} * S^* J]} : e^{\frac{1}{2}(JS^*J)}, \tag{141}
\end{aligned}$$

where we have introduced $S_{1,2} \equiv g_{\Lambda} * S * \hat{g}$.

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