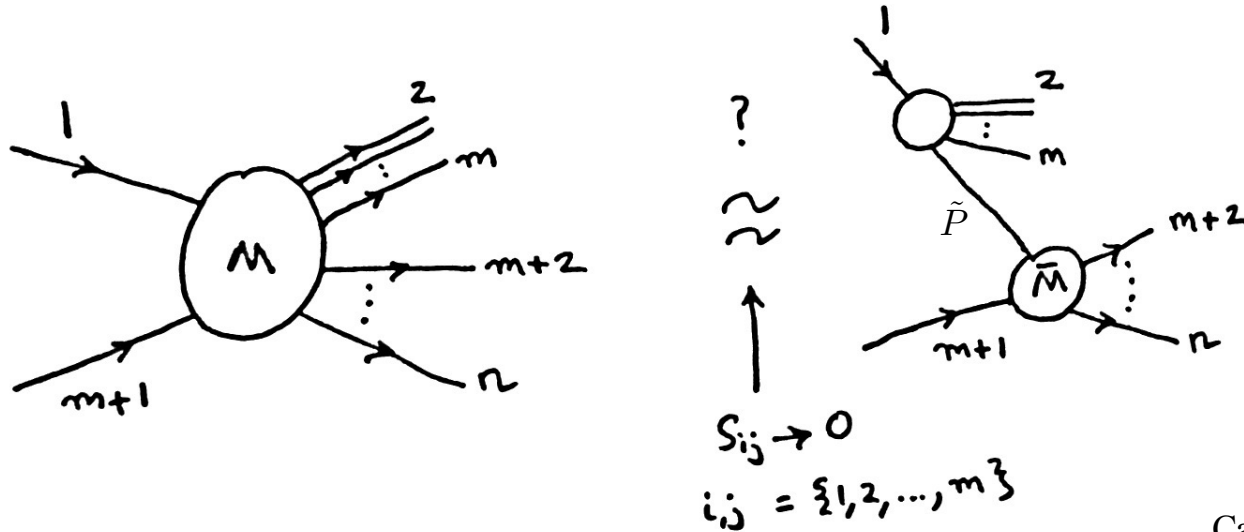


Soft gluons and the ordering variable

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1. Recap of infra-red behaviour of QCD and the way Coulomb/Glauber exchanges disrupt collinear factorization and induce “super-leading” logarithms.
2. What is the correct way to account for Coulomb exchanges to all orders? An explicit calculation in fixed-order (2 real emissions at 1 loop) indicates a dipole k_T ordering.



Catani, de Florian,
Rodrigo
[arXiv:1112.4405](https://arxiv.org/abs/1112.4405)

$$|M\rangle \approx \mathbf{Sp}(p_1, \dots, p_m; \tilde{P}; p_{m+1}, \dots, p_n) |\overline{M}\rangle$$

All orders.
Corrections are $\sqrt{s_{ij}}$ suppressed.

Even at one-loop it is not the case that

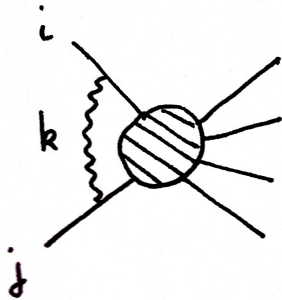
$$|M\rangle \approx \mathbf{Sp}(p_1, \dots, p_m; \tilde{P}) |\overline{M}\rangle$$

But the factorization breaking terms can cancel in $\langle M|M\rangle$.

This “generalized factorization” is consistent with the known factorization of all infra-red poles in QCD amplitudes.

Divergent structure of QCD: Loops

$$|M^{(1)}\rangle = \mathbf{I}^{(1)} |M^{(0)}\rangle + |M^{(1)\text{fin.}}\rangle$$



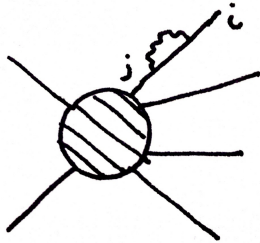
soft

$$\text{Re } I_{V,\text{soft}}^{(1)} = \sum_{i \neq j} \frac{\alpha_s}{4\pi} \mathbf{T}_i \cdot \mathbf{T}_j \int_0^{s_{ij}} \frac{dk_{\perp}^2}{(k_{\perp}^2)^{1+\epsilon}} \mu^{2\epsilon} \int_{\frac{k_{\perp}^2}{s_{ij}}}^1 \frac{dz}{z}$$

$$\text{Im } I_{V,\text{soft}}^{(1)} = \sum_{i \neq j} \frac{\alpha_s}{4\pi} \mathbf{T}_s^2 \int_0^{s_{ij}} \frac{dk_{\perp}^2}{(k_{\perp}^2)^{1+\epsilon}} \mu^{2\epsilon} 2\pi$$

Coulomb/Glauber

$$\mathbf{T}_s = \mathbf{T}_1 + \mathbf{T}_{m+1} = - \sum_{i \neq 1, m+1}^n \mathbf{T}_i$$



hard collinear

$$I_{V,\text{hard}}^{(1)} = \sum_i \frac{\alpha_s}{4\pi} \int_0^{\mu_i^2} \frac{dk_{\perp}^2}{(k_{\perp}^2)^{1+\epsilon}} \mu^{2\epsilon} \sum_j \int_0^1 dz P_{ij}(z)$$

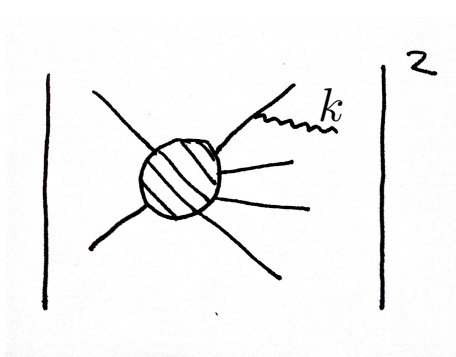
e.g. $P_{qq} = -\mathbf{T}_q^2 (1+z)$

$$\begin{aligned} P_{qq}^{\text{full}} &= \mathbf{T}_q^2 \left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right) \\ &= \mathbf{T}_q^2 \left(\frac{2}{(1-z)_+} + \frac{3}{2} \delta(1-z) + P_{qq} \right) \end{aligned}$$

$$P_{gq} = -\mathbf{T}_q^2 \left(\frac{1+(1-z)^2}{z} - \frac{2}{z} \right)$$

$$d\sigma_V^{(1)} = \langle M^{(0)} | I_V^{(1)} + I_V^{(1)\dagger} | M^{(0)} \rangle d(PS)_n$$

Divergent structure of QCD: Real emission



$$d\sigma_R^{(1)} = -\langle M^{(0)} | \mathbf{Sp}^{(0)\dagger} \mathbf{Sp}^{(0)} | M^{(0)} \rangle \Phi(\{p_i\}) d(P\mathcal{S})_{n+1}$$

$$\int \mathbf{Sp}^{(0)\dagger} \mathbf{Sp}^{(0)} d(P\mathcal{S})_k = I_V^{(1)} + I_V^{(1)\dagger}$$

$\Phi(\{p_i\}) = 1 \rightarrow$ perfect cancellation = KLN/Bloch-Nordsieck

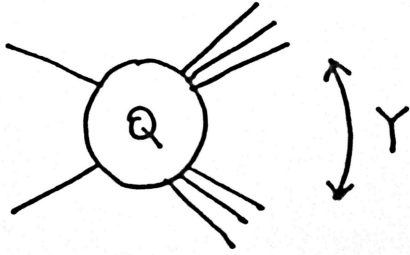
$$(I_V^{(1)} + I_V^{(1)\dagger}) \otimes \Phi(\{p_i\}) = \frac{\alpha_s}{4\pi} \mu^{2\epsilon} \int \frac{dk_{\perp}^2}{(k_{\perp}^2)^{1+\epsilon}} \frac{dz}{z} (\mathbf{T}_i \cdot \mathbf{T}_j + zP(z)) \Phi(\{p_i\})$$

For a general observable only the poles need to cancel (IRC safety).

Generally, the remnant of this cancellation is uncancelled logarithms.

Note: the imaginary part of the loops needs a different mechanism to cancel.

An example: dijet production with a veto



Veto in-gap real emission with $k_{\perp} > Q_0$

Emitted “out of the gap”

$$(I_V^{(1)} + I_V^{(1)\dagger}) \otimes \Phi(\{p_i\}) \sim 2\mu^{2\epsilon} \int_0^{Q^2} \frac{dk_{\perp}^2}{(k_{\perp}^2)^{1+\epsilon}} \int_{k_{\perp}^2/Q^2}^{e^{-Y/2}} \frac{dz}{z} (\mathbf{T}_i \cdot \mathbf{T}_j)$$

$$+ 2\mu^{2\epsilon} \int_0^{Q_0^2} \frac{dk_{\perp}^2}{(k_{\perp}^2)^{1+\epsilon}} \int_{e^{-Y/2}}^1 \frac{dz}{z} (\mathbf{T}_i \cdot \mathbf{T}_j) + \dots$$

Emitted “in the gap”

Putting $s_{ij} \sim Q^2 > Q_0^2$ for i and j on opposite sides of the gap.

Adding to the loop correction gives

$$d\sigma_V^{(1)} + d\sigma_R^{(1)} = \langle M^{(0)} | \mathbf{T}_L \cdot \mathbf{T}_R | M^{(0)} \rangle d(PS)_n \frac{\alpha_s}{2\pi} Y \log \frac{Q^2}{Q_0^2}$$

This is simply the loop correction integrated over the region of “phase-space” where the real emission is vetoed.

The singularities of the one-loop splitting matrix, defined by

$$\mathbf{Sp}^{(1)} = \mathbf{I}_C^{(1)} \mathbf{Sp}^{(0)} + \mathbf{Sp}^{(1)\text{fin.}},$$

can be extracted using

$$\mathbf{I}_C^{(1)} \mathbf{Sp}^{(0)} = \mathbf{I}^{(1)} \mathbf{Sp}^{(0)} - \mathbf{Sp}^{(0)} \bar{\mathbf{I}}^{(1)},$$

which we write as

$$\mathbf{I}_C^{(1)} = \mathbf{I}^{(1)} - \bar{\mathbf{I}}^{(1)}.$$

After some colour algebra:

$$\mathbf{I}_C^{(1)} = \frac{\alpha_s}{2\pi} \frac{1}{2} \left\{ \left(\frac{1}{\epsilon^2} C_{\tilde{P}} + \frac{1}{\epsilon} \gamma_{\tilde{P}} \right) - \sum_{i=1}^m \left(\frac{1}{\epsilon^2} C_i + \frac{1}{\epsilon} \gamma_i - \frac{2}{\epsilon} C_i \ln |z_i| \right) - \frac{i\pi}{\epsilon} \left(C_{\tilde{P}} - C_1 + \sum_{i=2}^m C_i \right) - \frac{1}{\epsilon} \sum_{\substack{i,\ell=1 \\ i \neq \ell}}^m \mathbf{T}_i \cdot \mathbf{T}_\ell \ln \frac{|s_{i\ell}|}{|z_i| |z_\ell| \mu^2} \right\} + \tilde{\Delta}_C^{(1)},$$

with

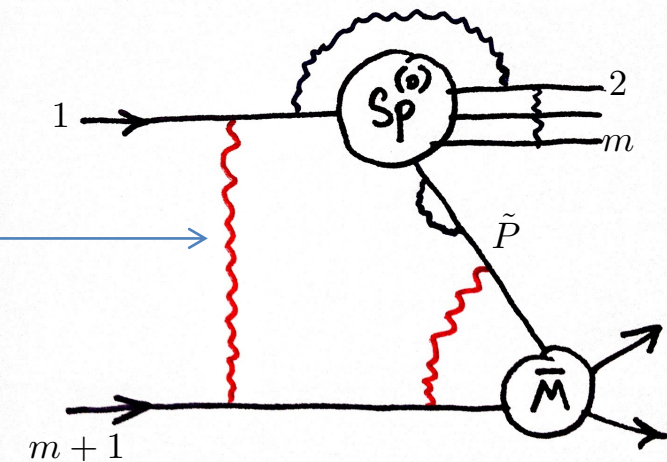
$$\tilde{\Delta}_C^{(1)} = \frac{\alpha_s}{2\pi} \left\{ 2 \times \frac{i\pi}{\epsilon} \mathbf{T}_{m+1} \cdot (\mathbf{T}_1 - \mathbf{T}_{\tilde{P}}) \right\}.$$

Coulomb exchange in the initial state breaks amplitude level factorization at one-loop level. It trivially **cancels** at the cross-section level.

Since $\tilde{\Delta}_C^{(1)}$ is anti-Hermitian

One loop

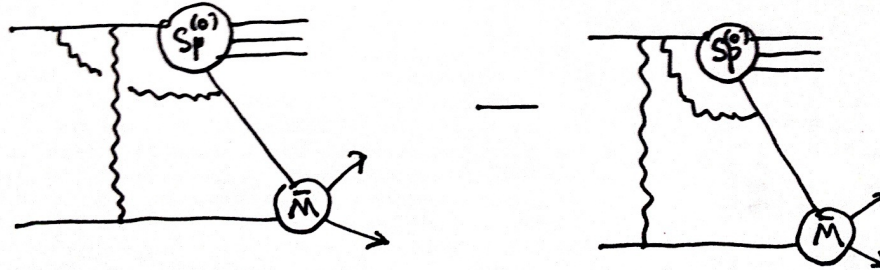
$$\begin{aligned} \langle M^{(0)} | M^{(1)} \rangle + \text{h.c.} &= \langle \bar{M}^{(0)} | \mathbf{Sp}^{(0)\dagger} \mathbf{Sp}^{(1)} | \bar{M}^{(0)} \rangle + \text{h.c.} \\ &+ \langle \bar{M}^{(0)} | \mathbf{Sp}^{(0)\dagger} \mathbf{Sp}^{(0)} | \bar{M}^{(1)} \rangle + \text{h.c.} \end{aligned}$$



Two loops

$$\langle \bar{M}^{(0)} | \mathbf{P}_{\text{n.f.}}^{(2)} | \bar{M}^{(0)} \rangle$$

$$\mathbf{P}_{\text{n.f.}}^{(2)} = \frac{1}{2} \mathbf{Sp}^{(0)\dagger} \left[\left(\bar{\mathbf{I}}^{(1)} + \bar{\mathbf{I}}^{(1)\dagger} + \mathbf{I}_C^{(1)\text{fact.}} + \mathbf{I}_C^{(1)\text{fact.}\dagger} \right), \tilde{\Delta}_C^{(1)} \right] \mathbf{Sp}^{(0)}$$



The factorization breaking still **cancels** at the cross-section level in pure QCD processes.

$$\text{Tr} \left[\left(|M^{(0)}\rangle \langle M^{(0)}| \right) [\mathbf{I}, \tilde{\Delta}_C^{(1)}] \right] = 0$$

$[\mathbf{I}, \tilde{\Delta}_C^{(1)}]$ is hermitian and a colour basis exists in which it is **anti-symmetric**.

$\mathbf{H} = |M^{(0)}\rangle \langle M^{(0)}|$ is real and **symmetric** in the same basis.

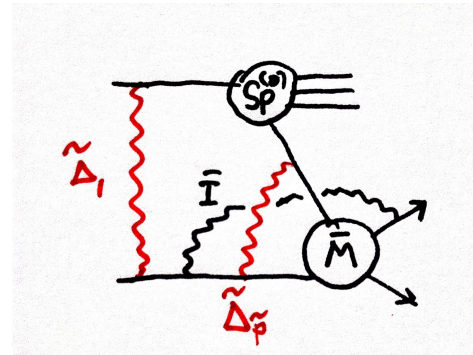
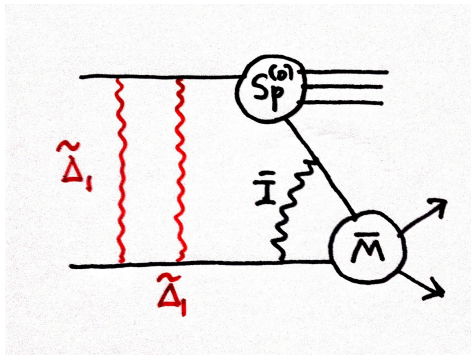
Seymour & Sjödal

[arXiv:0810.5756](https://arxiv.org/abs/0810.5756)

Three loops: no escape!

$$\langle \overline{M}^{(0)} | \mathbf{P}_{\text{n.f.}}^{(3)} | \overline{M}^{(0)} \rangle$$

$$\begin{aligned} \mathbf{P}_{\text{n.f.}}^{(3)} \sim & \frac{1}{6} \mathbf{Sp}^{(0)\dagger} \left(\left[\tilde{\Delta}_1^{(1)}, \left[\tilde{\Delta}_1^{(1)}, \bar{I}^{(1)} + \bar{I}^{(1)\dagger} \right] \right] - \left[\tilde{\Delta}_{\tilde{P}}^{(1)}, \left[\tilde{\Delta}_{\tilde{P}}^{(1)}, \bar{I}^{(1)} + \bar{I}^{(1)\dagger} \right] \right] \right) \mathbf{Sp}^{(0)} \\ & + \frac{1}{2} \mathbf{Sp}^{(0)\dagger} \left(\left[\tilde{\Delta}_{\tilde{P}}^{(1)}, \left[\tilde{\Delta}_{\tilde{P}}^{(1)}, \bar{I}^{(1)} + \bar{I}^{(1)\dagger} \right] \right] - \left[\tilde{\Delta}_1^{(1)}, \left[\tilde{\Delta}_1^{(1)}, \bar{I}^{(1)} + \bar{I}^{(1)\dagger} \right] \right] \right) \mathbf{Sp}^{(0)}. \end{aligned}$$



This gives a **non-zero** contribution to the cross-section in QCD.

It will cancel against real emission graphs for sufficiently inclusive observables, i.e. cuts through the “eikonal” gluon (in black).

These factorization breaking terms induce double-logarithmic corrections at fourth-order in “gaps between jets”:

$$\frac{\sigma_{\text{SLL}}(Q, Q_0)}{\sigma_0} \sim \alpha_s^4 \log^5 \left(\frac{Q}{Q_0} \right) N_c^2 \pi^2 Y$$

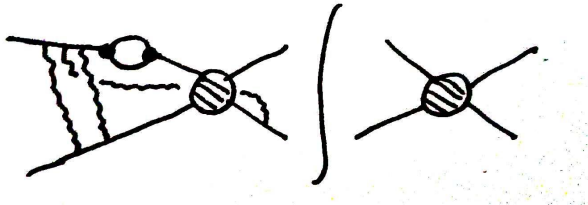
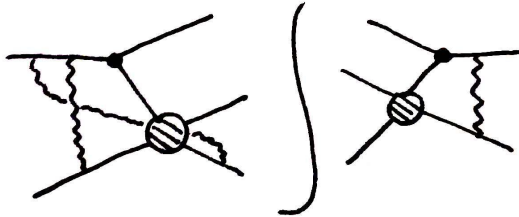
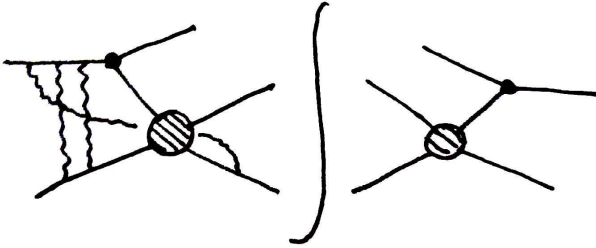
One collinear splitting contribution to gaps-between-jets

$$\sigma_1 = -\frac{2\alpha_s}{\pi} \int_{Q_0}^Q \frac{dk_T}{k_T} \left(2 \ln \frac{Q}{k_T} \right) \left\langle m_0 \left| e^{-\frac{2\alpha_s}{\pi} \int_{k_T}^Q \frac{dk'_T}{k'_T} \left(\frac{1}{2} Y t_t^2 - i\pi \mathbf{t}_1 \cdot \mathbf{t}_2 \right)} \right. \right.$$

$$\left. \left. \left\{ t_1^2 e^{-\frac{2\alpha_s}{\pi} \int_{Q_0}^{k_T} \frac{dk'_T}{k'_T} \left(\frac{1}{2} Y t_t^2 - i\pi \mathbf{t}_1 \cdot \mathbf{t}_2 \right)} e^{-\frac{2\alpha_s}{\pi} \int_{Q_0}^{k_T} \frac{dk'_T}{k'_T} \left(\frac{1}{2} Y t_t^2 + i\pi \mathbf{t}_1 \cdot \mathbf{t}_2 \right)} \right. \right.$$

$$\left. \left. - t_1^{a\dagger} e^{-\frac{2\alpha_s}{\pi} \int_{Q_0}^{k_T} \frac{dk'_T}{k'_T} \left(\frac{1}{2} Y \mathbf{T}_t^2 - i\pi \mathbf{T}_1 \cdot \mathbf{T}_2 \right)} e^{-\frac{2\alpha_s}{\pi} \int_{Q_0}^{k_T} \frac{dk'_T}{k'_T} \left(\frac{1}{2} Y \mathbf{T}_t^2 + i\pi \mathbf{T}_1 \cdot \mathbf{T}_2 \right)} t_1^a \right\} \right.$$

$$\left. \left. e^{-\frac{2\alpha_s}{\pi} \int_{k_T}^Q \frac{dk'_T}{k'_T} \left(\frac{1}{2} Y t_t^2 + i\pi \mathbf{t}_1 \cdot \mathbf{t}_2 \right)} \right| m_0 \right\rangle.$$



$$\left\{ \right\}_2 = \left(\frac{i\pi Y}{2} \right) \left(-\frac{2\alpha_s}{\pi} \int_{Q_0}^{k_T} \frac{dk'_T}{k'_T} \right)^2 \left\{ t_1^2 [t_t^2, \mathbf{t}_1 \cdot \mathbf{t}_2] - t_1^{a\dagger} [\mathbf{T}_t^2, \mathbf{T}_1 \cdot \mathbf{T}_2] t_1^a \right\}$$

$$\left\{ \right\}_3 \equiv -\frac{Y\pi^2}{6} \left(-\frac{2\alpha_s}{\pi} \int_{Q_0}^{k_T} \frac{dk'_T}{k'_T} \right)^3 \left\{ t_1^2 \left[[t_t^2, \mathbf{t}_1 \cdot \mathbf{t}_2], \mathbf{t}_1 \cdot \mathbf{t}_2 \right] \right.$$

$$\left. - t_1^{a\dagger} \left[[\mathbf{T}_t^2, \mathbf{T}_1 \cdot \mathbf{T}_2], \mathbf{T}_1 \cdot \mathbf{T}_2 \right] t_1^a \right\}.$$

Soft approximation:

$$\int dz \frac{1}{2} \left(\frac{1+z^2}{1-z} \right) \rightarrow \int dy$$

Real collinear emission:

$$\begin{aligned} \int_{y_{\max}}^{\infty} dy \left. \frac{d\sigma_R}{dy d^2 k_T} \right|_{\text{collinear}} &= \int_0^{1-\delta} dz \frac{1}{2} \left(\frac{1+z^2}{1-z} \right) \frac{q(x/z, \mu^2)}{q(x, \mu^2)} A_R \\ &= \int_0^{1-\delta} dz \frac{1}{2} \left(\frac{1+z^2}{1-z} \right) \left(\frac{q(x/z, \mu^2)}{q(x, \mu^2)} - 1 \right) A_R + \int_0^{1-\delta} dz \frac{1}{2} \frac{1+z^2}{1-z} A_R \end{aligned}$$

Virtual collinear emission:

$$\int_{y_{\max}}^{\infty} dy \left. \frac{d\sigma_V}{dy d^2 k_T} \right|_{\text{collinear}} = \int_0^{1-\delta} dz \frac{1}{2} \left(\frac{1+z^2}{1-z} \right) A_V$$

$$y > y_{\max}$$

implies

$$\delta \approx \frac{k_T}{Q} \exp \left(y_{\max} - \frac{\Delta y}{2} \right)$$

If $A_R + A_V = 0$ then the divergence would cancel leaving behind a regularized splitting, which would correspond to the DGLAP evolution of the incoming quark pdf.

The Coulomb gluons spoil this cancellation and we have

$$\int_0^{1-\delta} dz \frac{1}{2} \left(\frac{1+z^2}{1-z} \right) (A_R + A_V) = \ln \left(\frac{1}{\delta} \right) (A_R + A_V) + \text{subleading}$$

$$\approx \left(-y_{\max} + \frac{\Delta y}{2} + \ln \left(\frac{Q}{k_T} \right) \right) (A_R + A_V)$$

Hence

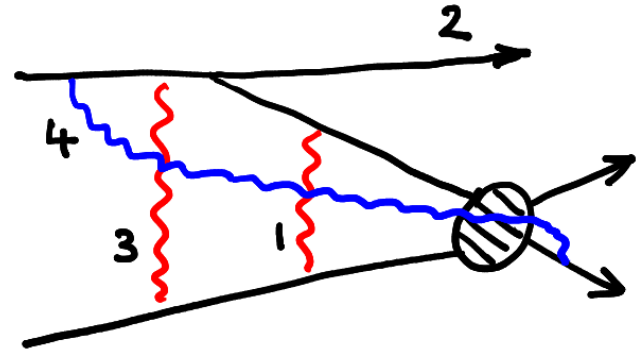
$$\int_{Q_0}^Q \frac{dk_T}{k_T} \int_{\text{out}} dy \rightarrow \int_{Q_0}^Q \frac{dk_T}{k_T} \left(\int^{y_{\max}} dy + (-y_{\max} + \ln \frac{Q}{k_T}) \right) = \frac{1}{2} \ln^2 \frac{Q}{Q_0}$$

The final result for the “one emission out-of-gap” cross-section is

$$\sigma_{1,\text{SLL}} = -\sigma_0 \left(\frac{2\alpha_s}{\pi} \right)^4 \ln^5 \left(\frac{Q}{Q_0} \right) \pi^2 Y \frac{(3N^2 - 4)}{480}$$

DGLAP evolution fails above the veto scale, whence the soft and collinear evolution can no longer be factorized – this is generic whenever real emission is not summed over inclusively.

Is k_t ordering correct?



k_t ordering: $k_{t1} \gg k_{t2} \gg k_{t3}$

$$\alpha_s^4 \int_{Q_0}^Q \frac{dk_{t1}}{k_{t1}} \int_{Q_0}^{k_{t1}} \frac{dk_{t2}}{k_{t2}} \int_{k_{t2}/Q}^1 \frac{d\theta_2}{\theta_2} \int_{Q_0}^{k_{t2}} \frac{dk_{t3}}{k_{t3}} \int_{Q_0}^{k_{t3}} \frac{dk_{t4}}{k_{t4}} = 2 \frac{\alpha_s^4}{5!} L^5$$

virtuality ordering: $k_{t1} \gg t_2 = k_{t2}\theta_2 \gg k_{t3}$

$$\alpha_s^4 \int_{Q_0}^Q \frac{dk_{t1}}{k_{t1}} \int_{Q_0}^{k_{t1}} \frac{dt_2}{t_2} \int_{\sqrt{t_2/Q}}^1 \frac{d\theta_2}{\theta_2} \int_{Q_0}^{t_2} \frac{dk_{t3}}{k_{t3}} \int_{Q_0}^{k_{t3}} \frac{dk_{t4}}{k_{t4}} = \frac{\alpha_s^4}{5!} L^5$$

energy ordering: $k_{t1} \gg E_2 = k_{t2}/\theta_2 \gg k_{t3}$

$$\alpha_s^4 \int_{Q_0}^Q \frac{dk_{t1}}{k_{t1}} \int_{Q_0}^{k_{t1}} \frac{dE_2}{E_2} \int_0^1 \frac{d\theta_2}{\theta_2} \int_{Q_0}^{E_2} \frac{dk_{t3}}{k_{t3}} \int_{Q_0}^{k_{t3}} \frac{dk_{t4}}{k_{t4}} = \infty$$

Soft gluon evolution - recap:

$$\sigma = \sum_{n=0}^{\infty} \int d\sigma_n F_n$$

$$d\sigma_0 = \langle M^{(0)} | \mathbf{V}_{0,Q}^\dagger \mathbf{V}_{0,Q} | M^{(0)} \rangle d\Pi_0$$

$$d\sigma_1 = \langle M^{(0)} | \mathbf{V}_{q_{1T},Q}^\dagger \mathbf{D}_{1\mu}^\dagger \mathbf{V}_{0,q_{1T}}^\dagger \mathbf{V}_{0,q_{1T}} \mathbf{D}_1^\mu \mathbf{V}_{q_{1T},Q} | M^{(0)} \rangle d\Pi_0 d\Pi_1$$

$$d\sigma_2 = \langle M^{(0)} | \mathbf{V}_{q_{1T},Q}^\dagger \mathbf{D}_{1\mu}^\dagger \mathbf{V}_{q_{2T},q_{1T}}^\dagger \mathbf{D}_{2\nu}^\dagger \mathbf{V}_{0,q_{2T}}^\dagger \mathbf{V}_{0,q_{2T}} \mathbf{D}_2^\nu \mathbf{V}_{q_{2T},q_{1T}} \mathbf{D}_1^\mu \mathbf{V}_{q_{1T},Q} | M^{(0)} \rangle d\Pi_0 d\Pi_1 d\Pi_2$$

(1.1)

etc.

$$\mathbf{V}_{a,b} = \exp \left[-\frac{2\alpha_s}{\pi} \int_a^b \frac{dk_T}{k_T} \sum_{i<j} (-\mathbf{T}_i \cdot \mathbf{T}_j) \frac{1}{2} \left\{ \int \frac{dy d\phi}{2\pi} \omega_{ij} - i\pi \Theta(ij = II \text{ or } FF) \right\} \right],$$

(1.2)

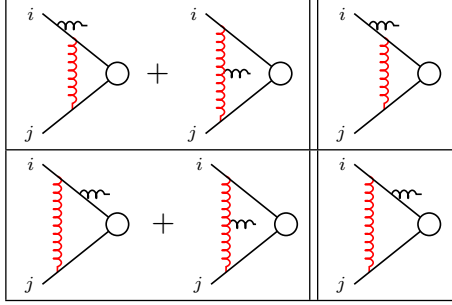
$$\mathbf{D}_i^\mu = \sum_j \mathbf{T}_j \frac{1}{2} q_{Ti} \frac{p_j^\mu}{p_j \cdot q_i},$$

$$\omega_{ij} = \frac{1}{2} k_T^2 \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)}$$

$$d\Pi_i = -\frac{2\alpha_s}{\pi} \frac{dq_{Ti}}{q_{Ti}} \frac{dy_i d\phi_i}{2\pi}.$$

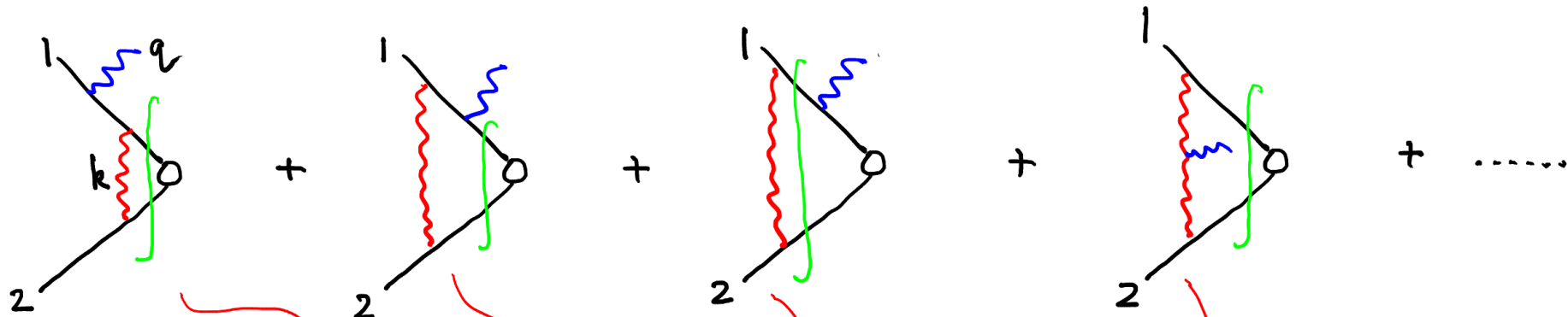
Aim is to test the correctness of

$$\mathbf{V}_{0,q_{1T}} \mathbf{D}_1^\mu \mathbf{V}_{q_{1T},Q} \quad \text{and} \quad \mathbf{V}_{0,q_{2T}} \mathbf{D}_2^\nu \mathbf{V}_{q_{2T},q_{1T}} \mathbf{D}_1^\mu \mathbf{V}_{q_{1T},Q}.$$



1		+		+		+		+		+			
2		+		+		+		+		+			
3		+		+		+		+					
4		+		+		+		+					
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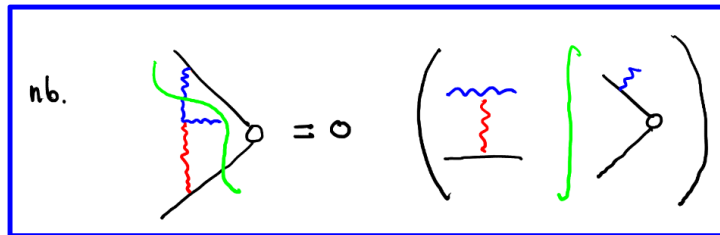
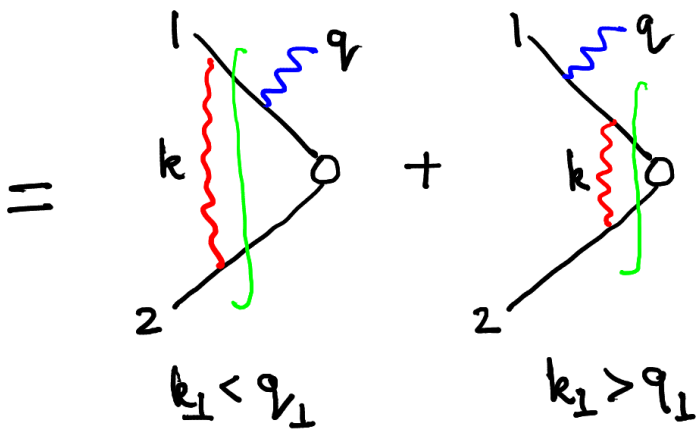
Start with one emission case



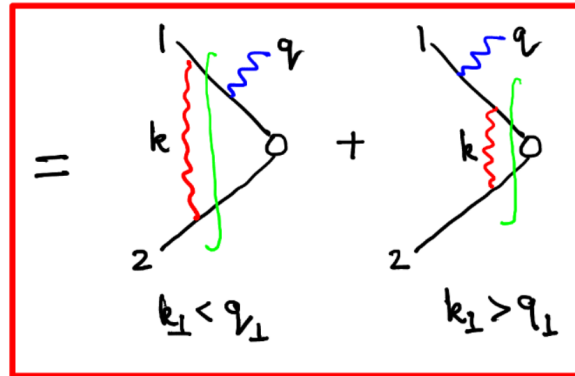
$$i\pi \int \frac{d^2 k_{\perp}}{k_{\perp}^2} \frac{p_i \cdot E^*}{p_i \cdot q} \left\{ t_1 (t_1 \cdot t_2) - T_1 \cdot T_2 t_1 + T_1 \cdot T_2 t_1 \right\}$$

$$+ i\pi \int \frac{d^2 k_{\perp}}{k_{\perp}^2} \frac{q_{\perp}^2}{k_{\perp}^2 + q_{\perp}^2} \frac{p_i \cdot E^*}{p_i \cdot q} \left\{ T_1 \cdot T_2 t_1 - t_1 t_1 \cdot t_2 \right\}$$

"switch"



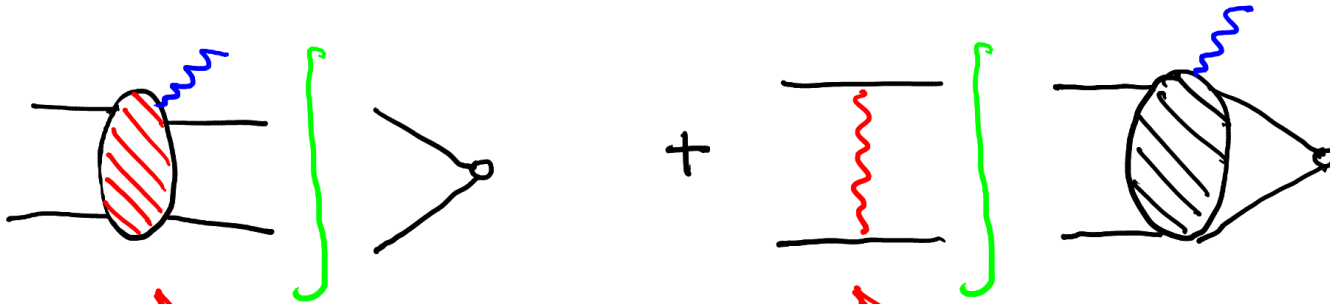
That was a non-trivial test of k_{\perp} ordering



$$\int \frac{dq_{\perp}}{q_{\perp}} \int \frac{dk_{\perp}}{k_{\perp}} + \int \frac{dk_{\perp}}{k_{\perp}} \int \frac{dq_{\perp}}{q_{\perp}}$$

$$= \left(C(0, q_{\perp}) J(q_{\perp}) + J(q_{\perp}) C(q_{\perp}, 0) \right) \times \left(\text{Vertex} \right)$$

Physical picture (gauge invariant)



$$\sim \int_{q_{\perp}}^Q \frac{dk_{\perp}}{k_{\perp}}$$

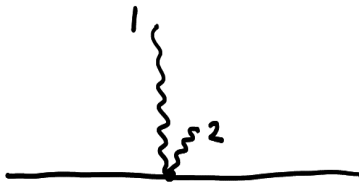
initial Coulomb exchange "knows nothing" of subsequent emission

$$\sim \int_0^Q \frac{dk_{\perp}}{k_{\perp}}$$

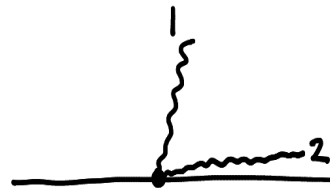
$$q_{1T} \gg q_{2T}$$

Double emission case

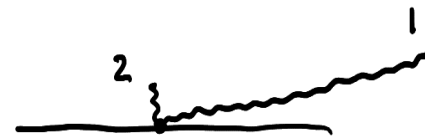
- Limit 1: Both emissions are at wide angle but one gluon is much softer than the other, i.e. $(q_1^\pm \sim q_{1T}) \gg (q_2^\pm \sim q_{2T})$. Specifically, we take $q_2 \rightarrow \lambda q_2$ and keep the leading term for small λ .
- Limit 2: One emission (q_2) collinear with p_i by virtue of its small transverse momentum and the other (q_1) at a wide angle, i.e. $q_2^+ \gg q_{2T}$ and $q_1^+ \sim q_{1T} \gg q_{2T}$. Specifically, we take $q_2 \rightarrow (q_2^+, \lambda^2 q_{2T}^2 / (2q_2^+), \lambda q_{2T})$ and keep the leading term for small λ .
- Limit 3: One emission (q_1) collinear with p_i by virtue of its high energy and the other (q_2) at a wide angle, i.e. $q_1^+ \gg q_{1T}$ and $q_{1T} \gg q_{2T} \sim q_2^+$. Specifically, we take³ $q_1 \rightarrow (q_1^+ / \lambda, \lambda q_{1T}^2 / (2q_1^+), q_{1T})$ and $q_2 \rightarrow \lambda q_2$, and keep the leading term for small λ .



Limit 1



Limit 2



Limit 3

Relevant for SLL

“Eikonal” cuts

e.g. 1st row of graphs

$$G_{11} = \frac{q_1^-}{(q_2^- + q_1^-)} \int_0^{Q^2} \frac{dk_T^2}{k_T^2} \quad G_{13} = \frac{q_2^-}{(q_1^- + q_2^-)} \int_0^{Q^2} \frac{dk_T^2}{k_T^2}$$

$$G_{11} + G_{13} = \int_0^{Q^2} \frac{dk_T^2}{k_T^2} \quad \text{as expected}$$

$$G_{12} = - \left[\int_0^{2q_1^- q_2^+} \frac{dk_T^2}{k_T^2} + \frac{q_2^- - q_1^-}{q_2^- + q_1^-} \int_0^{2(q_1^+ + q_2^-)^2 q_2^+ / q_1^-} \frac{dk_T^2}{k_T^2} \right]$$

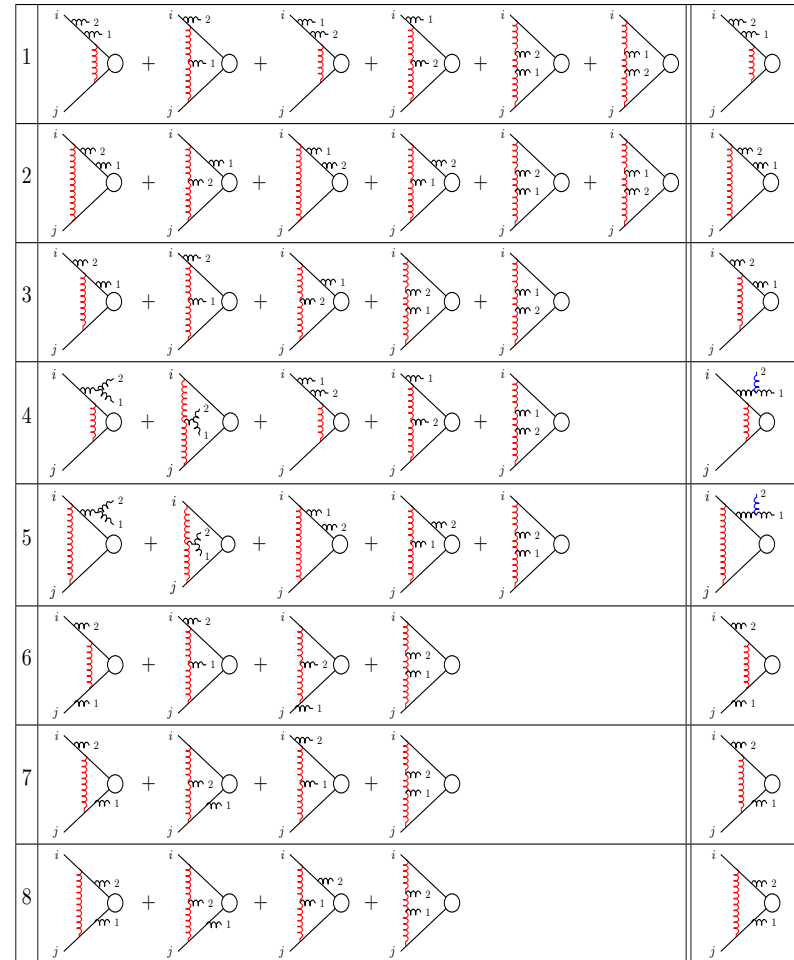
$$G_{12} + G_{14} = - \frac{1}{(q_1^- + q_2^-)} \left[q_2^- \int_0^{q_{2T}^2} \frac{dk_T^2}{k_T^2} + q_1^- \int_0^{q_{1T}^2} \frac{dk_T^2}{k_T^2} \right]$$

Subleading in limits 1 & 2

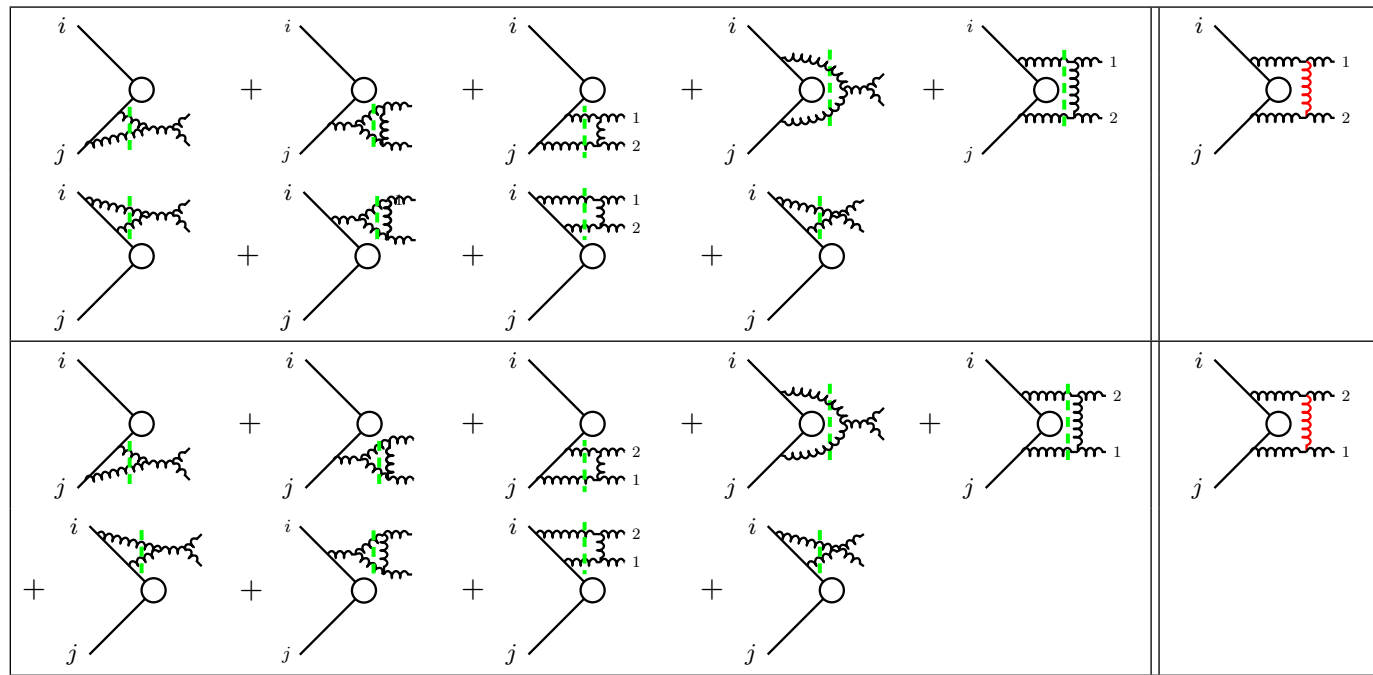
$$G_{15} + G_{16} \approx - \frac{q_2^-}{(q_1^- + q_2^-)} \int_{q_{2T}^2}^{q_{1T}^2} \frac{dk_T^2}{k_T^2} \quad \text{Only leading in limit 3}$$

In all 3 limits sum is

$$\int_{q_{1T}^2}^{Q^2} \frac{dk_T^2}{k_T^2}$$



“Soft gluon” cuts

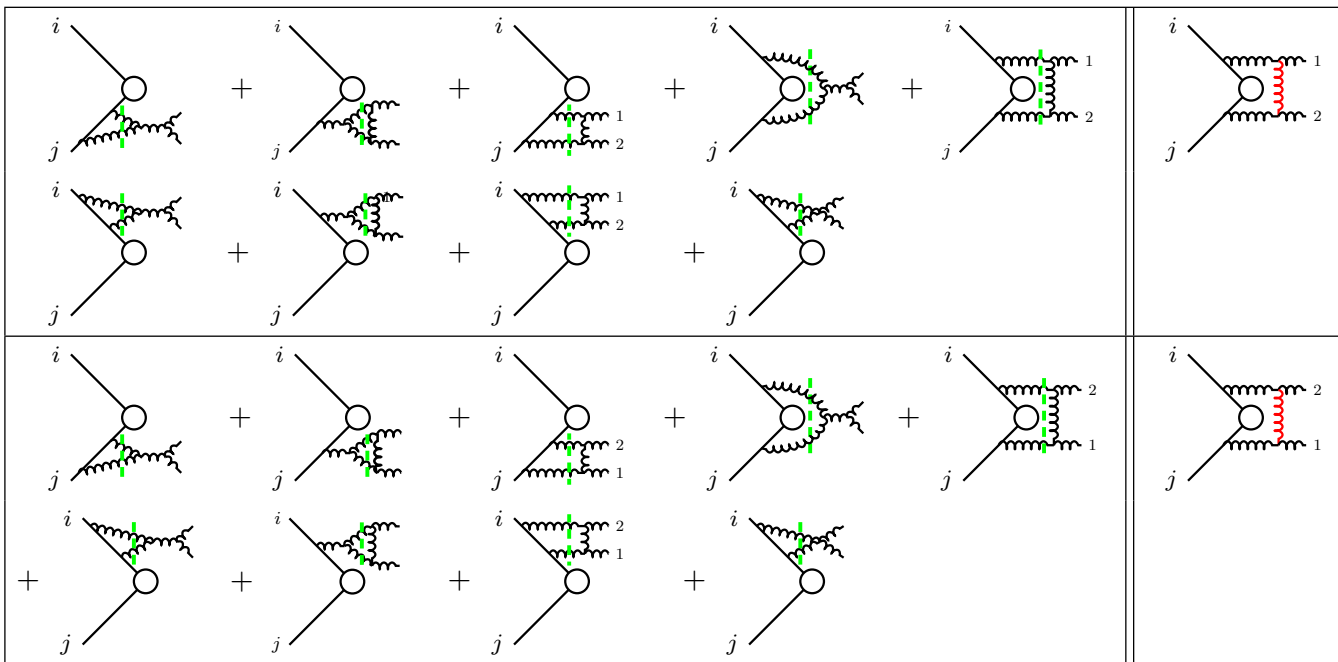


$$\begin{aligned}
 & - \frac{i\pi}{8\pi^2} \left[\frac{p_j \cdot \varepsilon_1}{p_j \cdot q_1} - \frac{p_i \cdot \varepsilon_1}{p_i \cdot q_1} \right] \left\{ \left[\frac{p_j \cdot \varepsilon_2}{p_j \cdot q_2} - \frac{q_1 \cdot \varepsilon_2}{q_1 \cdot q_2} \right] \left[-\frac{1}{\epsilon} + \ln \left(\frac{2q_2 \cdot q_1 p_j \cdot q_2}{q_1 \cdot p_j \mu^2} \right) \right] \mathbf{T}_j^d i f^{dc_2 b_i} f^{bc_1 a} \mathbf{T}_i^a \right. \\
 & - \left. \left[\frac{p_i \cdot \varepsilon_2}{p_i \cdot q_2} - \frac{q_1 \cdot \varepsilon_2}{q_1 \cdot q_2} \right] \left[-\frac{1}{\epsilon} + \ln \left(\frac{2q_2 \cdot q_1 p_i \cdot q_2}{q_1 \cdot p_i \mu^2} \right) \right] \mathbf{T}_j^d i f^{dc_1 b_i} f^{bc_2 a} \mathbf{T}_i^a \right\} |M^0\rangle .
 \end{aligned}$$

$$\ln \left(\frac{2q_2 \cdot q_1 p_i \cdot q_2}{q_1 \cdot p_i \mu^2} \right) = \ln \left(\frac{q_{2T(i1)}^2}{\mu^2} \right) = \ln \left(\frac{q_{2T}^2}{\mu^2} \right) + \ln \left(\frac{2q_1 \cdot q_2}{q_{1T} q_{2T}} \right) + y_1 - y_2,$$

$$\ln \left(\frac{2q_2 \cdot q_1 p_j \cdot q_2}{q_1 \cdot p_j \mu^2} \right) = \ln \left(\frac{q_{2T(j1)}^2}{\mu^2} \right) = \ln \left(\frac{q_{2T}^2}{\mu^2} \right) + \ln \left(\frac{2q_1 \cdot q_2}{q_{1T} q_{2T}} \right) + y_2 - y_1,$$

$$\left\{ -\frac{i\pi}{8\pi^2} i f^{c_1 e a_1} i f^{c_2 e a_2} \left[-\frac{1}{\epsilon} + \ln \left(\frac{q_{2T}^2}{\mu^2} \right) + \ln \left(\frac{2q_1 \cdot q_2}{q_{1T} q_{2T}} \right) \right] \right\} \mathbf{K}_2^{a_1 a_2}(q_1, q_2) |M^{(0)}\rangle$$



e.g. the $-\frac{i\pi}{8\pi^2} \frac{p_j \cdot \varepsilon_1 q_1 \cdot \varepsilon_2}{p_j \cdot q_1 q_1 \cdot q_2}$ terms in limit 1

$$G_{1c} = -\frac{3}{2} \int_{p_j \cdot q_1}^{p_j \cdot q_2} \frac{dl_T^2}{l_T^2} - \frac{3}{2} \int_0^{2q_1 \cdot q_2} \frac{dl_T^2}{l_T^2},$$

$$G_{1d} = \frac{3}{2} \int_0^{2q_1 \cdot q_2} \frac{dl_T^2}{l_T^2},$$

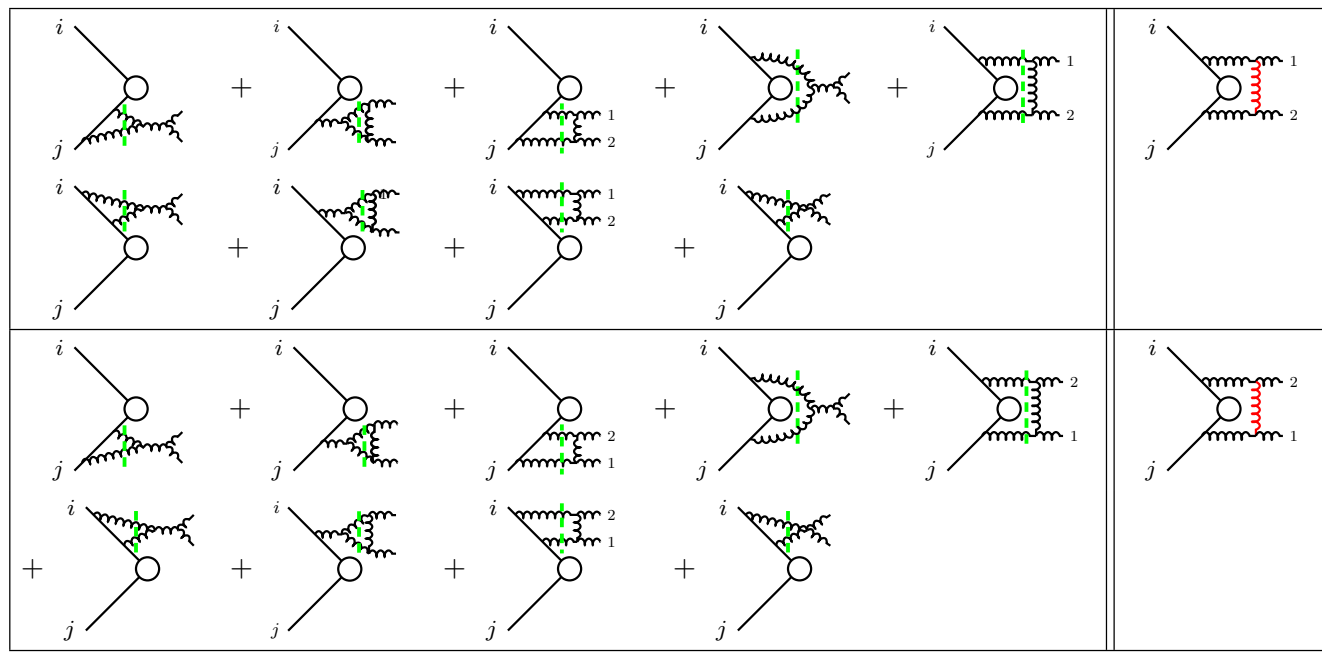
$$G_{1e} = -\int_0^{2q_1 \cdot q_2} \frac{dl_T^2}{l_T^2} + \frac{1}{2} \int_{p_j \cdot q_1}^{p_j \cdot q_2} \frac{dl_T^2}{l_T^2}.$$

$$G_{2c} = \frac{3}{4} \int_0^{2q_1 \cdot q_2} \frac{dl_T^2}{l_T^2},$$

$$G_{2d} = -\frac{3}{2} \int_0^{2q_1 \cdot q_2} \frac{dl_T^2}{l_T^2},$$

$$G_{2e} = \frac{7}{4} \int_0^{2q_1 \cdot q_2} \frac{dl_T^2}{l_T^2} + \int_{p_i \cdot q_1}^{p_i \cdot q_2} \frac{dl_T^2}{l_T^2}.$$

Limit 2
 q_2 collinear with p_i



Only graphs G_{2e} and G_{2h} are leading.

$$G_{2e} = -\frac{i\pi}{8\pi^2} \left[\frac{1}{2} \frac{p_i \cdot \varepsilon_1}{p_i \cdot q_1} \frac{p_i \cdot \varepsilon_2}{p_i \cdot q_2} \right] \int_0^{q_{2T}^2} \frac{dl_T^2}{l_T^2}$$

$$G_{2h} = -\frac{i\pi}{8\pi^2} \left[\frac{1}{2} \frac{p_i \cdot \varepsilon_1}{p_i \cdot q_1} \frac{p_i \cdot \varepsilon_2}{p_i \cdot q_2} - \frac{p_i \cdot \varepsilon_2}{p_i \cdot q_2} \frac{p_j \cdot \varepsilon_1}{p_j \cdot q_1} \right] \int_0^{q_{2T}^2} \frac{dl_T^2}{l_T^2}$$

Limit 3

q_1 collinear with p_i

The leading contributions to the second colour structure cancel.

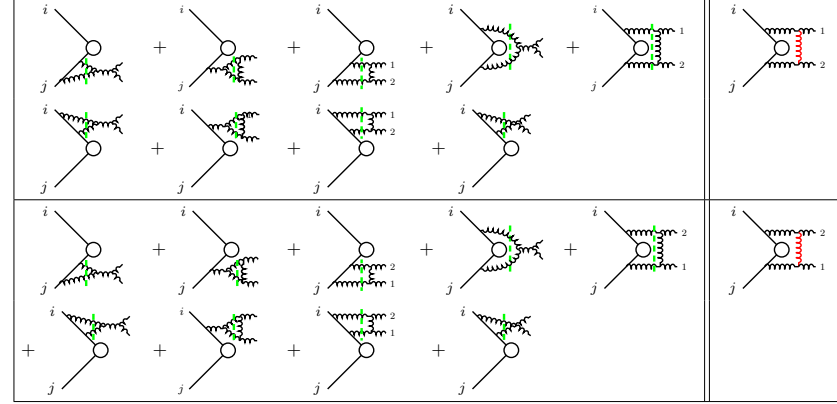
The first colour structure receives leading contributions to the following two Lorentz structures:

$$-\frac{i\pi}{8\pi^2} \left\{ \frac{\varepsilon_1^- \varepsilon_2^+}{q_1^- q_2^+}, \frac{\varepsilon_1^- \varepsilon_2^-}{q_1^- q_2^-} \right\}.$$

Only graph G_{1e} contributes to the first and it gives

$$-\int_0^{q_{2T}^2} \frac{dl_T^2}{l_T^2}.$$

Graphs $\{G_{1a}, G_{1b}, G_{1c}, G_{1d}, G_{1e}, G_{1i}\}$ contribute to the second. The contributions of graphs G_{1a}, G_{1b} cancel whilst



$$G_{1c} = \left[\frac{3(q_1^-)^2 + 3q_1^- q_2^- + 2(q_2^-)^2}{4(q_1^- + q_2^-)^2} \right] \int_0^{q_1^+ q_2^-} \frac{dl_T^2}{l_T^2} - \frac{1}{2} \int_{p_i \cdot q_1}^{p_i \cdot (q_1 + q_2)} \frac{dl_T^2}{l_T^2},$$

$$G_{1d} = -\frac{3q_1^- + 2q_2^-}{2(q_1^- + q_2^-)} \int_0^{q_1^+ q_2^-} \frac{dl_T^2}{l_T^2} + \int_{p_i \cdot q_1}^{p_i \cdot (q_1 + q_2)} \frac{dl_T^2}{l_T^2},$$

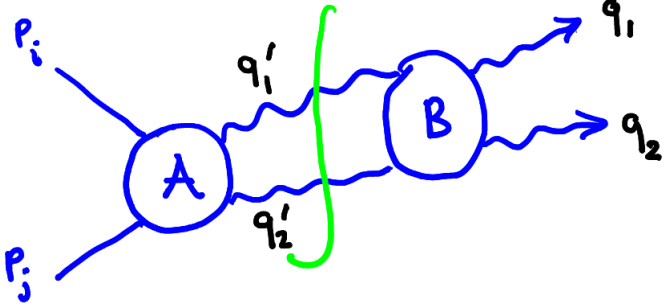
$$G_{1e} = \frac{7q_1^- + 6q_2^-}{4(q_1^- + q_2^-)} \int_0^{q_1^+ q_2^-} \frac{dl_T^2}{l_T^2} - \frac{1}{2} \int_{p_i \cdot q_1}^{p_i \cdot (q_1 + q_2)} \frac{dl_T^2}{l_T^2} - \int_{p_j \cdot q_2}^{p_j \cdot q_1} \frac{dl_T^2}{l_T^2}.$$

The sum of these three contributions is

$$\int_0^{q_{2T}^2} \frac{dl_T^2}{l_T^2} - \frac{q_1^- q_2^-}{2(q_1^- + q_2^-)^2} \int_0^{q_1^+ q_2^-} \frac{dl_T^2}{l_T^2}.$$

Finally, the four-gluon vertex graph G_{1i} exactly cancels the second term of this expression.

A more physical derivation of the same result



$$i\mathcal{M} = \int \frac{d^{d-2}k_T}{8q_1 \cdot q_2 (2\pi)^{d-2} \sqrt{1 - 2k_T^2/(q_1 \cdot q_2)}} \times \frac{1}{2} (i\mathcal{A}^{\mu\nu}) \left(\sum_{p'_1} \overset{\text{Sum over physical polarizations}}{\varepsilon_{1'\mu;p'_1}^* \varepsilon_{1'\sigma;p'_1}} \sum_{p'_2} \varepsilon_{2'\nu;p'_2}^* \varepsilon_{2'\lambda;p'_2} \right) (i\mathcal{B}^{\sigma\lambda}),$$

$$\varepsilon_{1'\sigma;p'_1} \varepsilon_{2'\lambda;p'_2} \mathcal{B}^{\sigma\lambda} \xrightarrow{k_T^2 \ll 2q_1 \cdot q_2} g_s^2 \mu^{2\epsilon} \frac{4q_1 \cdot q_2}{k_T^2} \varepsilon_{1';p'_1} \cdot \varepsilon_{1;p_1}^* \varepsilon_{2';p'_2} \cdot \varepsilon_{2;p_2}^*,$$

$$\mathcal{M} = \frac{ig_s^2 \mu^{2\epsilon}}{8\pi^2} \int \frac{d^{d-2}k_T}{k_T^2 (2\pi)^{d-4}} \sum_{p'_1 p'_2} \left(\mathcal{A}^{\mu\nu} \varepsilon_{1'\mu;p'_1}^* \varepsilon_{2'\nu;p'_2} \right) \varepsilon_{1';p'_1} \cdot \varepsilon_{1;p_1}^* \varepsilon_{2';p'_2} \cdot \varepsilon_{2;p_2}^*.$$

Gloun 1: momentum and polarization unchanged by the Coulomb scattering.

$$\sum_{p'_1} \mathcal{A}^\mu \varepsilon_{1'\mu;p'_1}^* \varepsilon_{1';p'_1} \cdot \varepsilon_{1;p_1}^* = -\mathcal{A}^\mu \varepsilon_{1\mu;p_1}^*.$$

$$\mathcal{M} = -\frac{ig_s^2 \mu^{2\epsilon}}{8\pi^2} \int \frac{d^{d-2}k_T}{k_T^2 (2\pi)^{d-4}} \sum_{p'_2} \left(\mathcal{A}^{\mu\nu} \varepsilon_{1\mu;p_1}^* \varepsilon_{2'\nu;p'_2} \right) \varepsilon_{2';p'_2} \cdot \varepsilon_{2;p_2}^*.$$

$$\mathcal{M} = \left(\mathcal{A}^{\mu\nu} \varepsilon_{1\mu;p_1}^* \varepsilon_{2\nu;p_2}^* \right) \frac{ig_s^2 \mu^{2\epsilon}}{8\pi^2} \int \frac{k_T^{d-3} dk_T d\phi \sin^{-2\epsilon} \phi d^{d-4}\Omega}{k_T^2 (2\pi)^{d-4}} \frac{1 + \frac{k_T}{q_{2T(i1)}} \cos \phi}{1 + 2 \frac{k_T}{q_{2T(i1)}} \cos \phi + \frac{k_T^2}{q_{2T(i1)}^2}}.$$

$$k_{T\mu} = k_T \sin \phi \varepsilon_{1\mu;\perp} + k_T \cos \phi \varepsilon_{1\mu;\parallel}$$

In four dimensions, the ϕ integral yields an exact Θ -function:

$$\mathcal{M} = \left(\mathcal{A}^{\mu\nu} \varepsilon_{1\mu;p_1}^* \varepsilon_{2\nu;p_2}^* \right) \frac{ig_s^2}{4\pi} \int_0^{q_{2T(i1)}} \frac{dk_T}{k_T}. \quad q_{2T(i1)}^2 = \frac{2q_2 \cdot q_1 p_i \cdot q_2}{q_1 \cdot p_i}$$

The k_T of the Coulomb gluon is exactly limited by the transverse momentum of the softer of the two gluons it is exchanged between, as measured in the rest frame of a dipole formed by the harder of the two gluons and one of the fast partons in the hard process.