

Factorization

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Factorization

- For an observable \mathcal{A} we have

$$\begin{aligned} \sigma(\mathcal{A}) = & \sum_{a,b} \int_0^1 d\eta_a \int_0^1 d\eta_b f_{a/A}(\eta_a, \mu^2) f_{b/A}(\eta_b, \mu^2) \\ & \times \hat{\sigma}(a, b, \eta_a, \eta_b, \mathcal{A}, \mu^2) \\ & + \mathcal{O}(1 \text{ GeV}^2 / Q^2(\mathcal{A})) \end{aligned}$$

- μ^2 is an adjustable factorization scale.
- $Q^2(\mathcal{A})$ is a hard scale corresponding to the observable \mathcal{S} .
- Errors are power suppressed when $Q^2(\mathcal{A})$ is large.

The observable

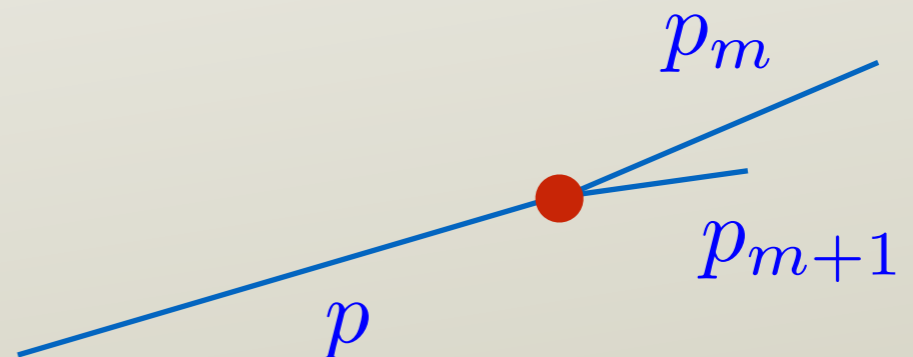
$$\begin{aligned} \hat{\sigma}(a, b, \eta_a, \eta_b, \mathcal{A}, \mu^2) &= \sum_m \frac{1}{m!} \int dy_1 \prod_{j=2}^m \int dp_{\perp,j} dy_j d\phi_j \\ &\times \frac{d\hat{\sigma}}{dy_1 dp_{\perp,2} dy_2 d\phi_2 \cdots dp_{\perp,m} dy_m d\phi_m} \\ &\times A_m(p_1, p_2, \dots, p_m) \end{aligned}$$

- \mathcal{A} defines, for instance, two jets with given minimum P_{\perp} values, plus anything else.
- By adding flavor indices, we could describe leptons, photons.
- We can choose $A_m(p_1, p_2, \dots, p_m)$ to be symmetric under interchange of its arguments.

Infrared safety

- For our discussion, \mathcal{A} needs to be infrared safe.
- We can be (a little) more precise by saying that \mathcal{A} is infrared safe at scale $Q^2(\mathcal{A})$.
- For partons m and $m + 1$ becoming collinear,

$$\begin{aligned} p_m &\rightarrow zp \\ p_{m+1} &\rightarrow (1-z)p \end{aligned}$$



when they are sufficiently collinear,

$$(p_m + p_{m+1})^2 < Q^2(\mathcal{A})$$

we ask that combining the partons leave \mathcal{A} unchanged:

$$A_{m+1}(p_1, \dots, p_{m-1}, p_m, p_{m+1}) \approx A_m(p_1, \dots, p_{m-1}, p)$$

- Also when one parton is becoming to aligned to the beam axis



$$p_{m+1,\perp}^2 < Q^2(\mathcal{A})$$

we ask that leaving it out leaves \mathcal{A} unchanged:

$$A_{m+1}(p_1, \dots, p_{m-1}, p_m, p_{m+1}) \approx A_m(p_1, \dots, p_{m-1}, p_m)$$

The parton distribution functions

- Consider quarks.

$$f_{j/A}(\xi, \mu^2) = \int \frac{dw^-}{2\pi} e^{-i\xi P^+ w^-} \\ \times \langle P | \bar{\psi}(0, w^-, \mathbf{0}) \frac{1}{2} \gamma^+ W(w^-, 0) \psi(0, 0, \mathbf{0}) | P \rangle$$

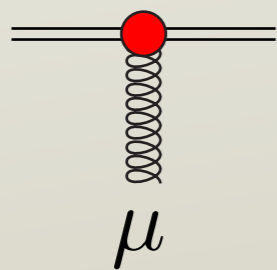
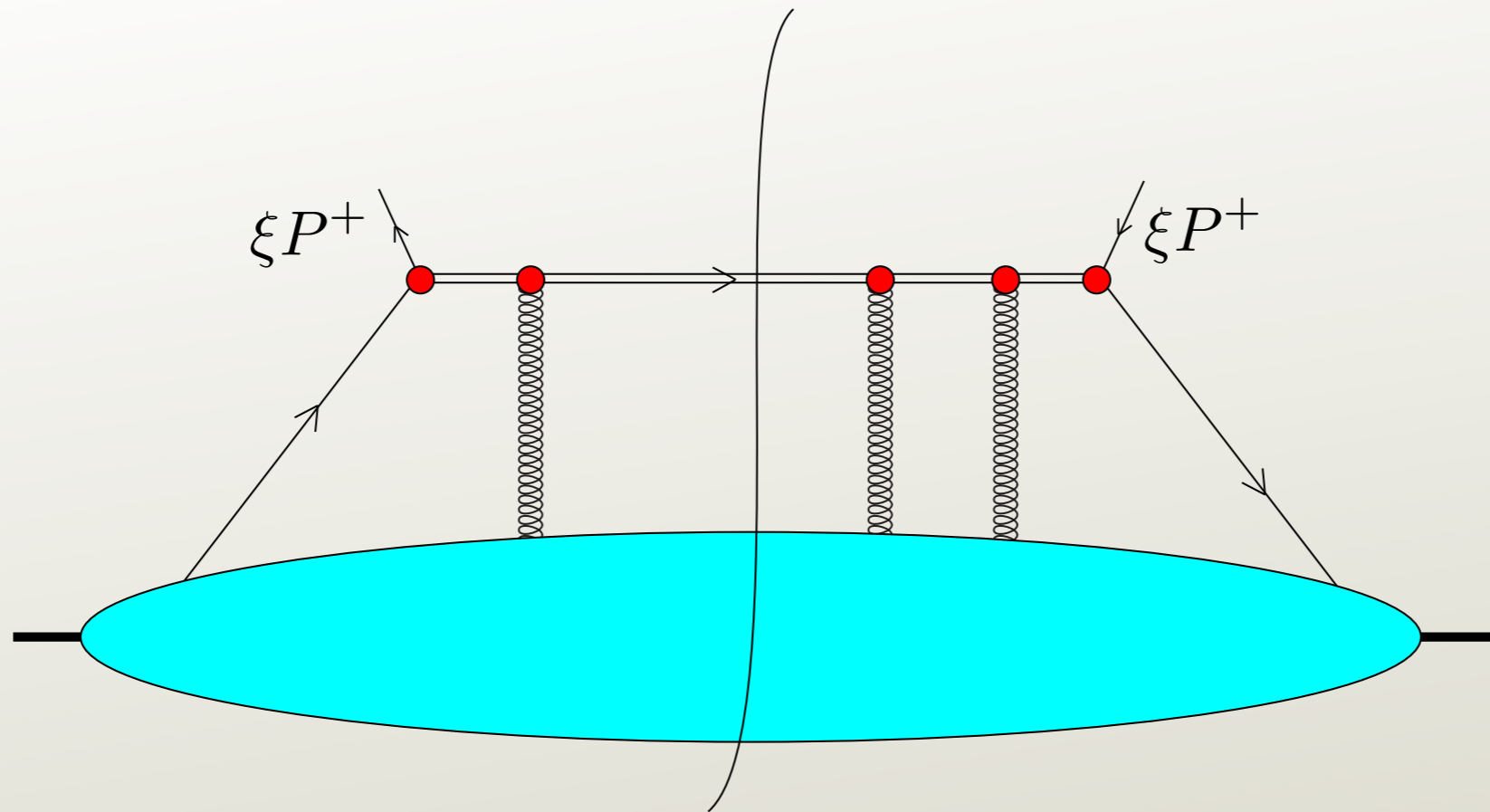
where

$$W(w^-, 0) = P \exp \left(-ig \int_0^{w^-} dy^- A_a^+(0, y^-, \mathbf{0}) t_a \right)$$

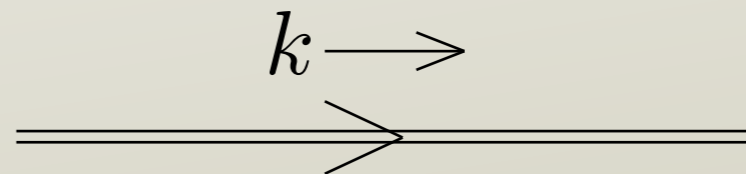
or

$$W(w^-, 0) = W(\infty, w^-)^\dagger W(\infty, 0) .$$

- Renormalize with $\overline{\text{MS}}$ subtractions.



$$-i g n^\mu t_a$$



$$\frac{i}{k \cdot n + i\epsilon}$$

Here n is a lightlike vector in the minus direction,
 $n = (0, 1, \mathbf{0})$ so that $n \cdot \varepsilon = \varepsilon^+$.

How can factorization be right?

- “Furthermore, we fully expect that the factorization breaking effects we have been discussing will cancel in inclusive observables...” Forsaw, Seymour, & Siódmok (2012)
- Is that right? Why?
- Consider the inclusive production of two high P_T jets in hadron collisions.

This analysis is mostly from J. C. Collins, D. E. Soper and G. F. Sterman,
“Soft Gluons and Factorization,” Nucl. Phys. B **308** (1988) 833.

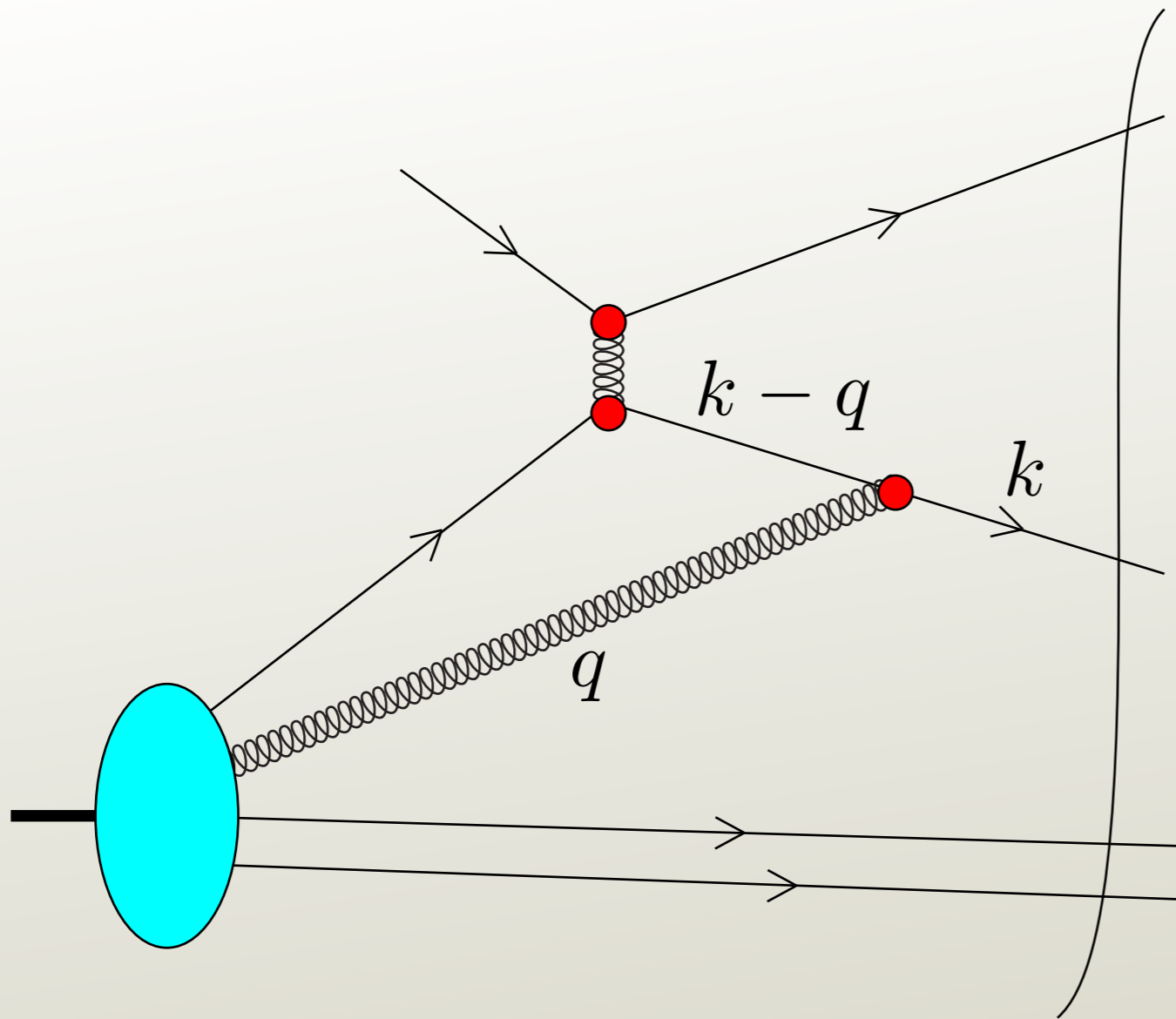
See also J. C. Collins, D. E. Soper and G. F. Sterman,
“Factorization for Short Distance Hadron-Hadron Scattering,”
Nucl. Phys. B **261** (1985) 104.

See also G. T. Bodwin,
“Factorization of the Drell-Yan Cross-Section in Perturbation Theory,”
Phys. Rev. D **31** (1985) 2616.

An explicit calculation was presented in W. W. Lindsay, D. A. Ross and C. T. Sachrajda,
“On the Cancellation of Long Distance, Nonfactorizing Contributions
to the Drell-Yan Cross-section,”
Nucl. Phys. B **214** (1983) 61.

A one loop argument was presented in J. C. Collins, D. E. Soper and G. F. Sterman,
“Factorization for One Loop Corrections in the Drell-Yan Process,”
Nucl. Phys. B **223** (1983) **381**.

What happens to collinear
gluons?



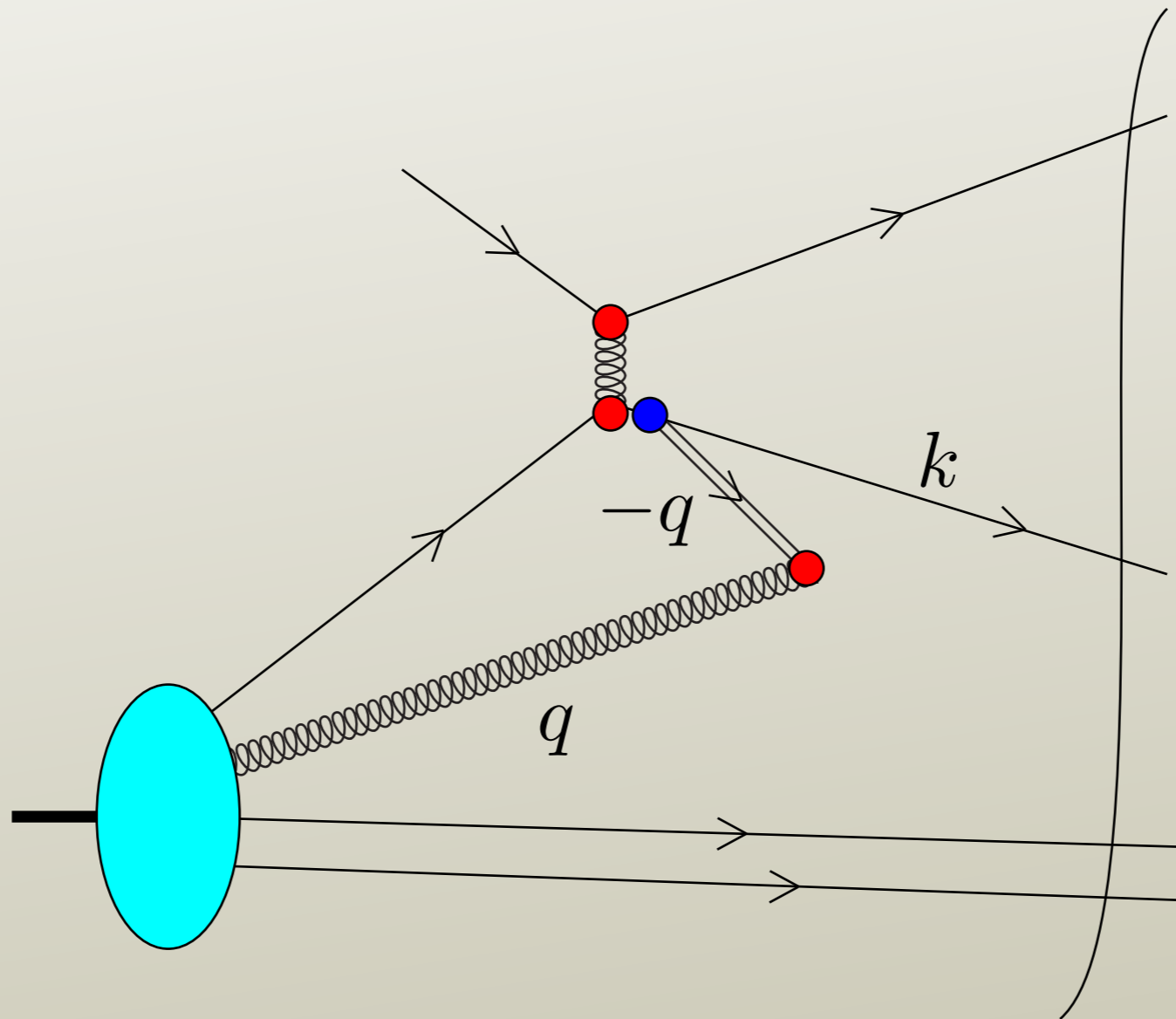
$$F = \frac{-gt_a J \cdot \varepsilon}{(k - q)^2 + i\epsilon}$$

- Both the polarization vector ε and the gluon momentum q are mainly in the plus direction.

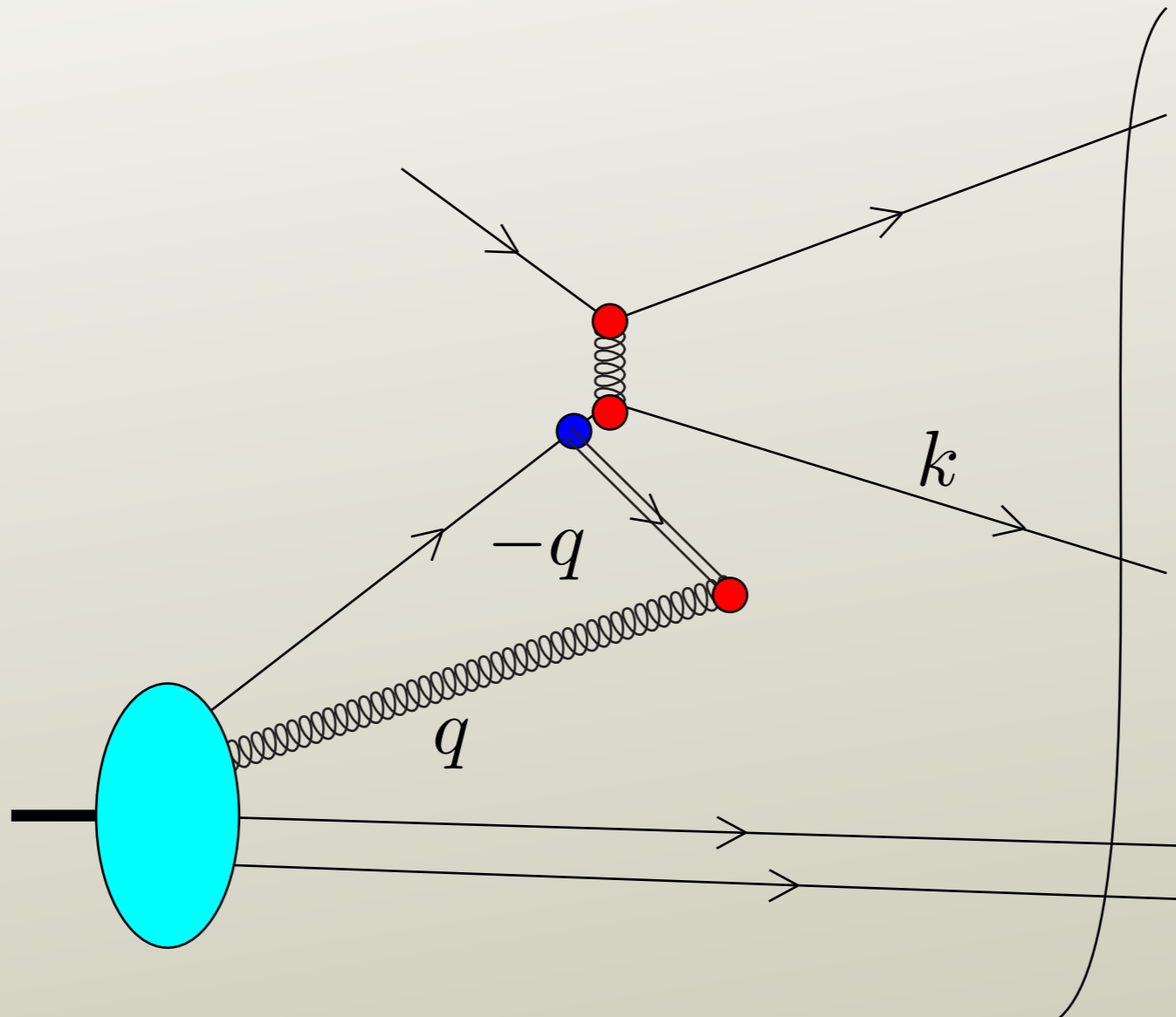
$$F \approx \frac{-gt_a 2k^- \varepsilon^+}{-2k^- q^+ + i\epsilon} = \frac{gt_a \varepsilon^+}{q^+}$$

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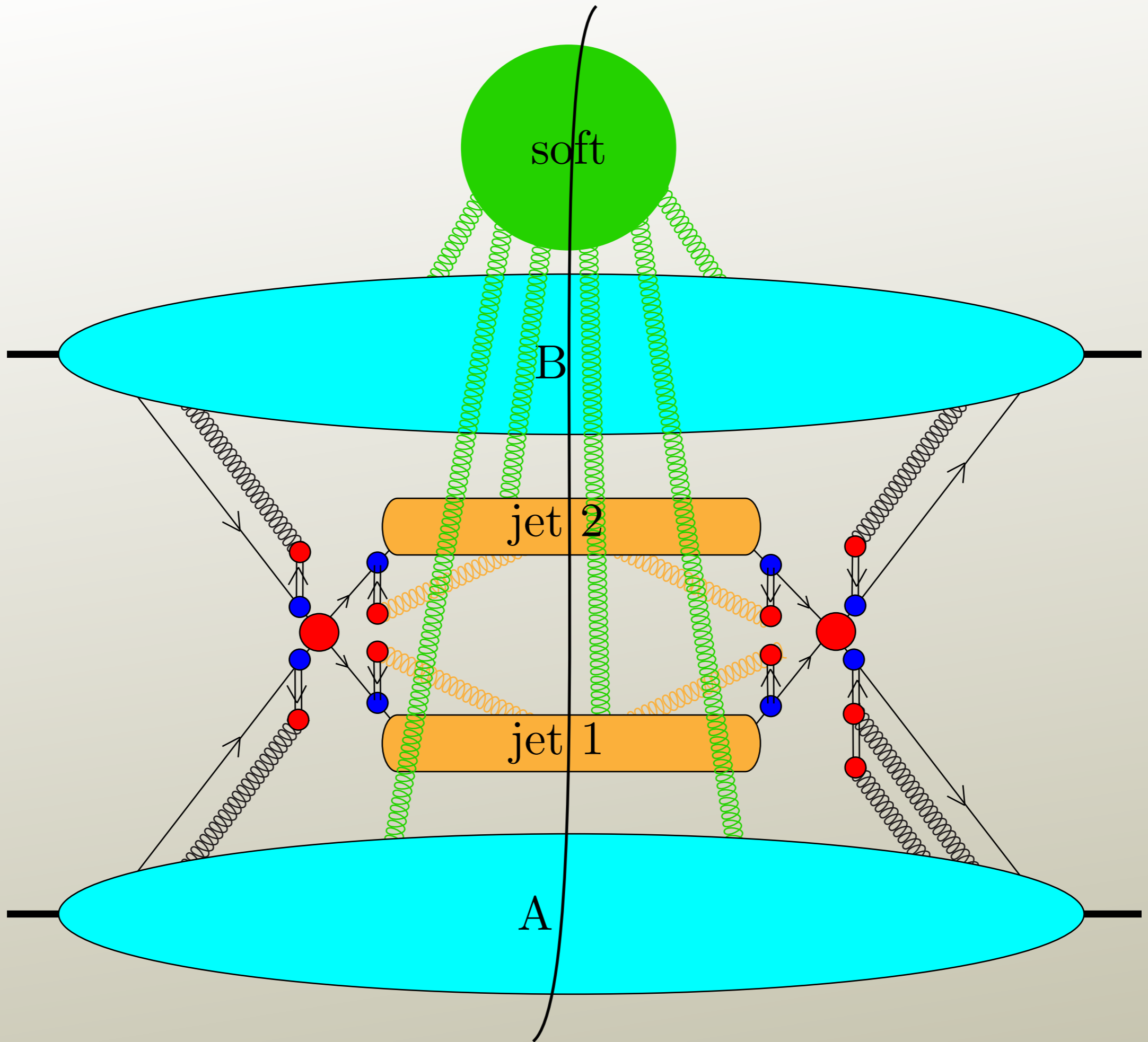
- This is the coupling to an eikonal line in the minus direction.



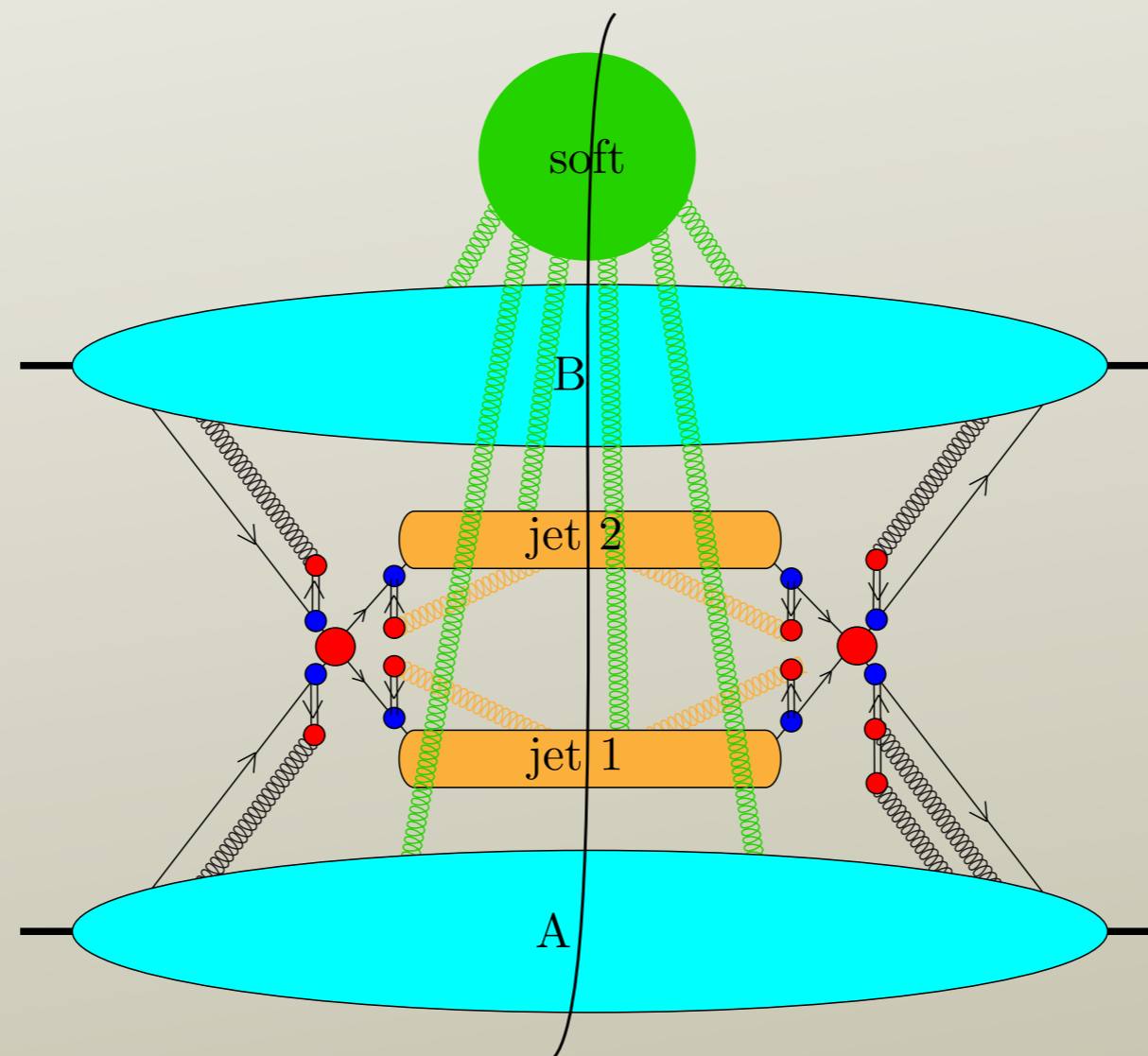
- Summing attachments to all of the lines gives



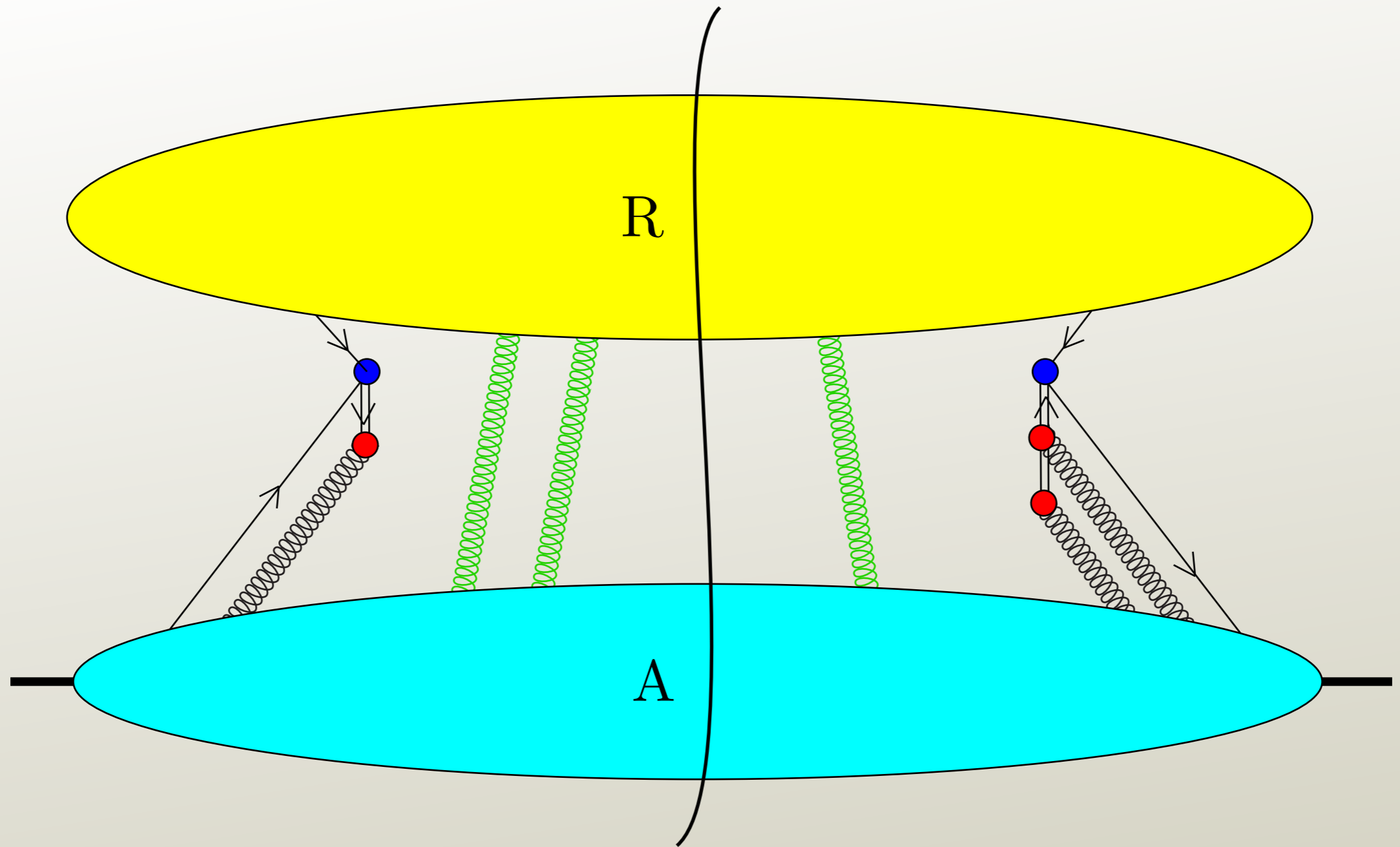
Integration regions



- We have taken care of the collinear gluons.
- Without the soft gluons, this factorizes.
- But soft gluons seem to ruin everything.



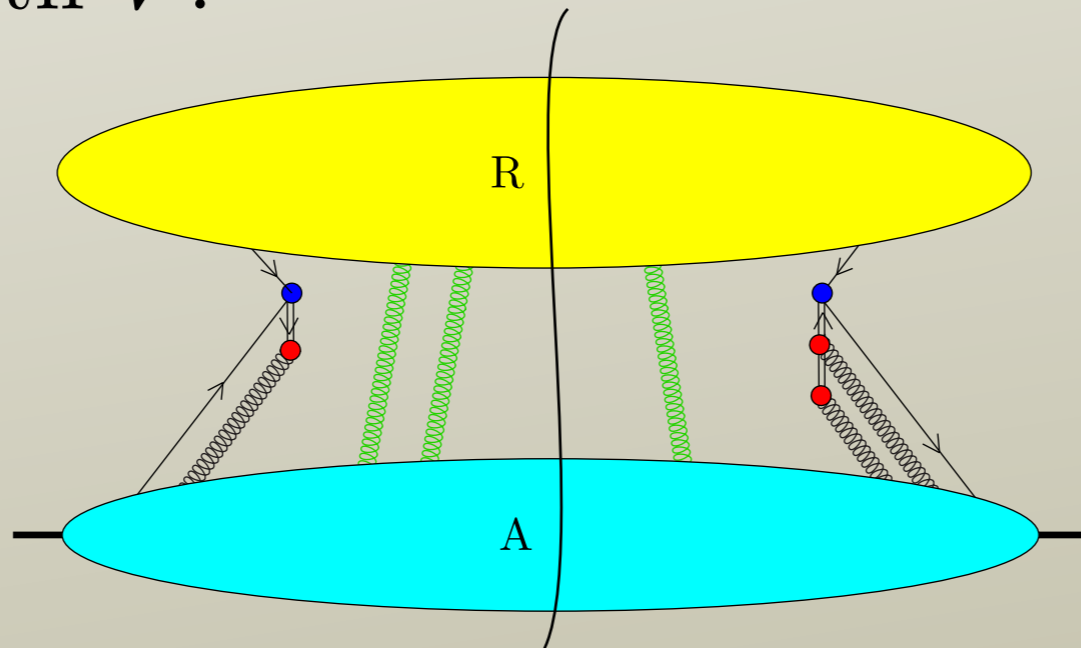
Decomposition into two parts



$$\begin{aligned}
 G_L = & \int \frac{d^4 k_A}{(2\pi)^4} \left(\prod_l \int \frac{d^4 q_l}{(2\pi)^4} \right) \sum_V \sum_{C_A \in \mathcal{A}(V)} \\
 & \times J_A^{(C_A)}(k_A, q_l^\alpha)_{\{\mu_1, \dots, \mu_n\}} \sum_{C_R \in \mathcal{R}(V)} R^{(C_R)}(k_A, q_l^\alpha)_{\{\mu_1, \dots, \mu_n\}}
 \end{aligned}$$

$$G_L = \int \frac{d^4 k_A}{(2\pi)^4} \left(\prod_l \int \frac{d^4 q_l}{(2\pi)^4} \right) \sum_V \sum_{C_A \in \mathcal{A}(V)} \times J_A^{(C_A)}(k_A, q_l^\alpha)_{\{\mu_1, \dots, \mu_n\}} \sum_{C_R \in \mathcal{R}(V)} R^{(C_R)}(k_A, q_l^\alpha)_{\{\mu_1, \dots, \mu_n\}}$$

- We sum over V , which designates which of the vertices connecting the soft gluons to J_A are to the left of the final state cut and which are to the right.
- Then we sum over the cuts of J_A and R that are consistent with V .



- Some simple approximations.

$$\begin{aligned}
G_L &\approx \int \frac{dk_A^- d\mathbf{k}_A}{(2\pi)^3} \left(\prod_l \int \frac{dq_l^- d\mathbf{q}_l}{(2\pi)^3} \right) \sum_V \sum_{C_A \in \mathcal{A}(V)} \\
&\times J_A^{(C_A)}(k_{\text{jets}}^+, k_A^-, \mathbf{k}_A, \hat{q}_l^\alpha) \{+, \dots, +\} \\
&\times \int \frac{dk_R^+}{2\pi} \left(\prod_l \int \frac{dq_l^+}{2\pi} \right) \sum_{C_R \in \mathcal{R}(V)} \\
&\times R^{(C_R)}(k_{\text{jets}}^+ + k_R^+, 0, \mathbf{0}, q_l^\alpha) \{+, \dots, +\}
\end{aligned}$$

- $\hat{q} = (0, q^-, \mathbf{q})$

- $k_A^+ = k_{\text{jets}}^+ + k_R^+$ but neglect k_R^+ in J_A .

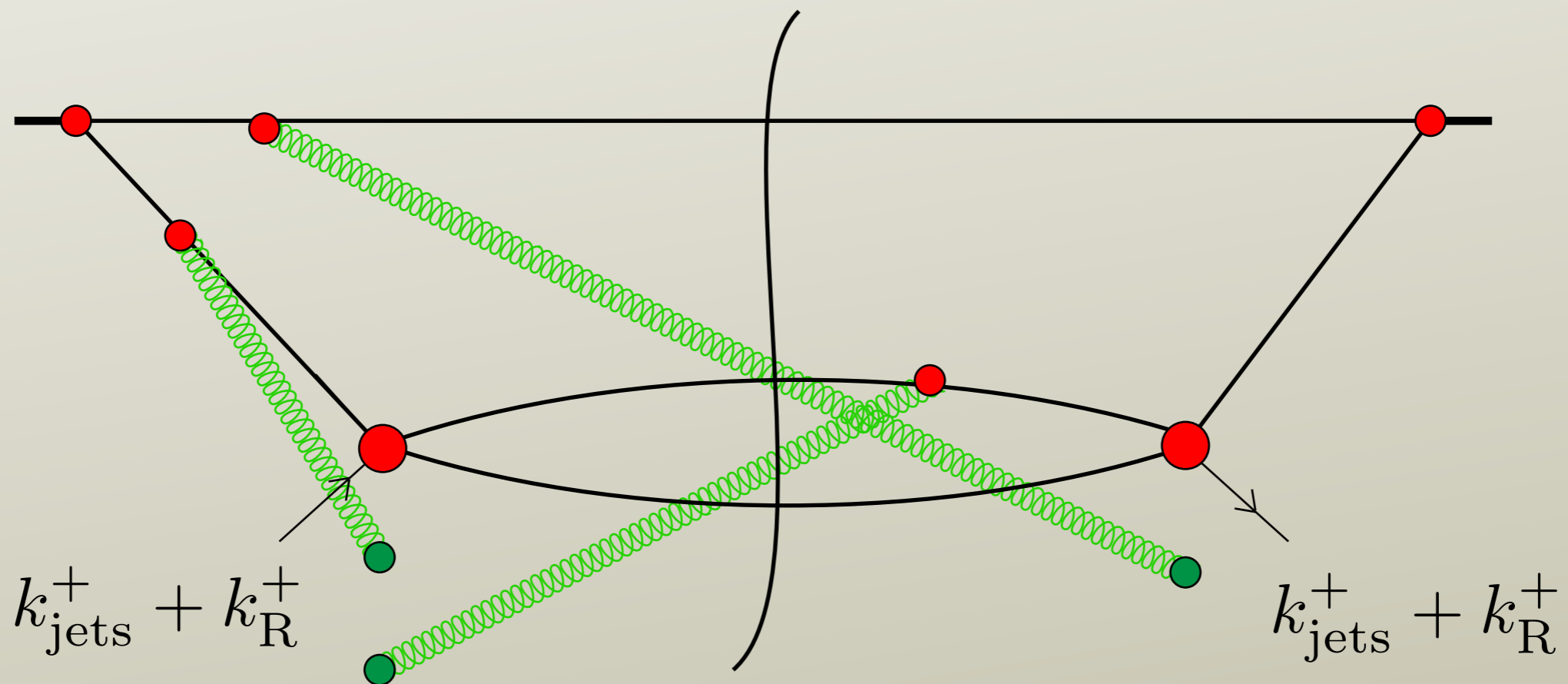
- Omit the $\{+, \dots, +\}$ from here on.

An unintuitive claim

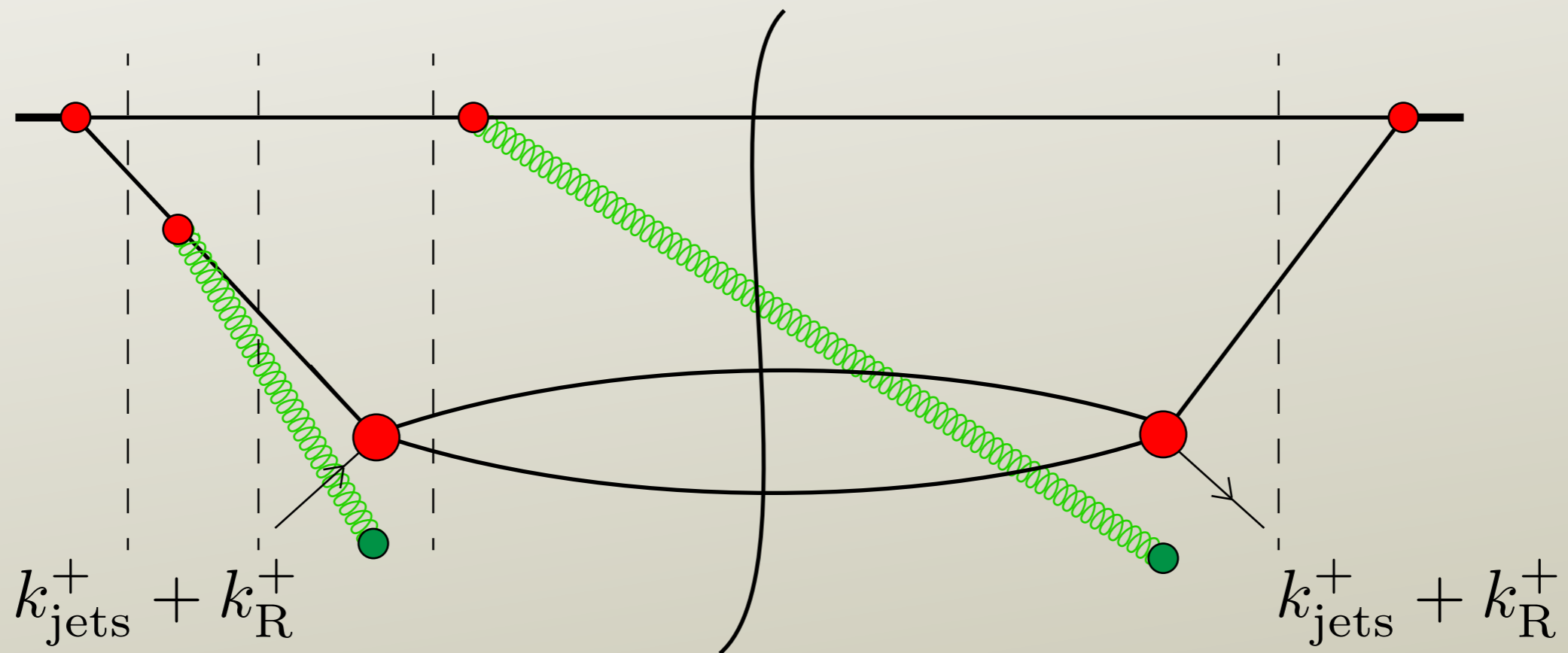
- Define

$$\bar{R}(\hat{q}_l^\alpha; V) = \int \frac{dk_{\mathbf{R}}^+}{2\pi} \left(\prod_l \int \frac{dq_l^+}{2\pi} \right) \sum_{C_{\mathbf{R}} \in \mathcal{R}(V)} R^{(C_{\mathbf{R}})}(k_{\text{jets}}^+ + k_{\mathbf{R}}^+, \mathbf{0}, \mathbf{0}, q_l^\alpha)$$

- Claim: $\bar{R}(\hat{q}_l^\alpha; V)$ is independent of V .



- In \overline{R} , we perform the integrations over loop momenta k^+ and rearrange the result with partial fractions so as to produce K^+ denominators, one for each intermediate state ξ .



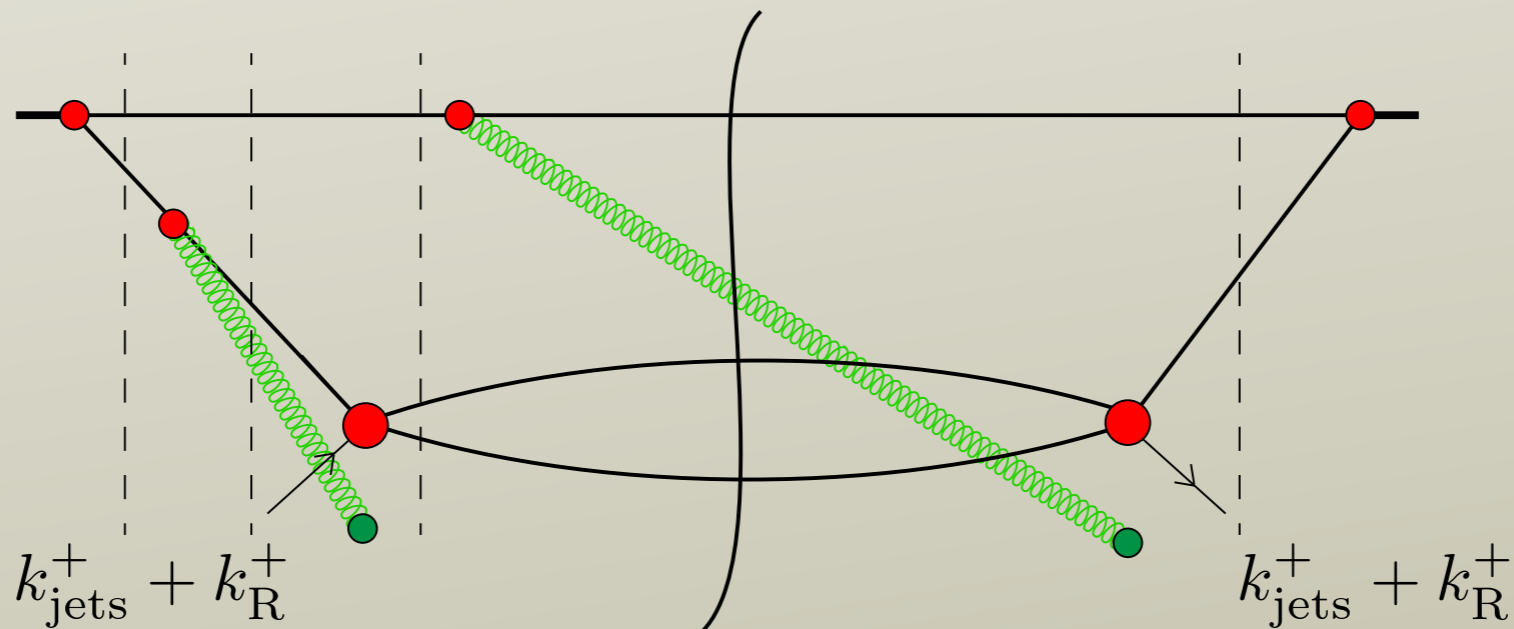
$$D_\xi = \sum_{j \in \xi} \frac{k_j^2}{2k_j^-} \quad \xi < H \text{ or } H' < \xi$$

$$D_\xi = \sum_{j \in \xi} \frac{k_j^2}{2k_j^-} - k_{\text{jets}}^+ \quad H < \xi < H'$$

$$f_\xi = \frac{1}{k_R^+ - D_\xi + i\epsilon}, \quad H < \xi < C$$

$$f_\xi = 2\pi\delta(k_R^+ - D_\xi), \quad \xi = C$$

$$f_\xi = \frac{1}{k_R^+ - D_\xi - i\epsilon}, \quad C < \xi < H'$$



- Use the unitarity identity.

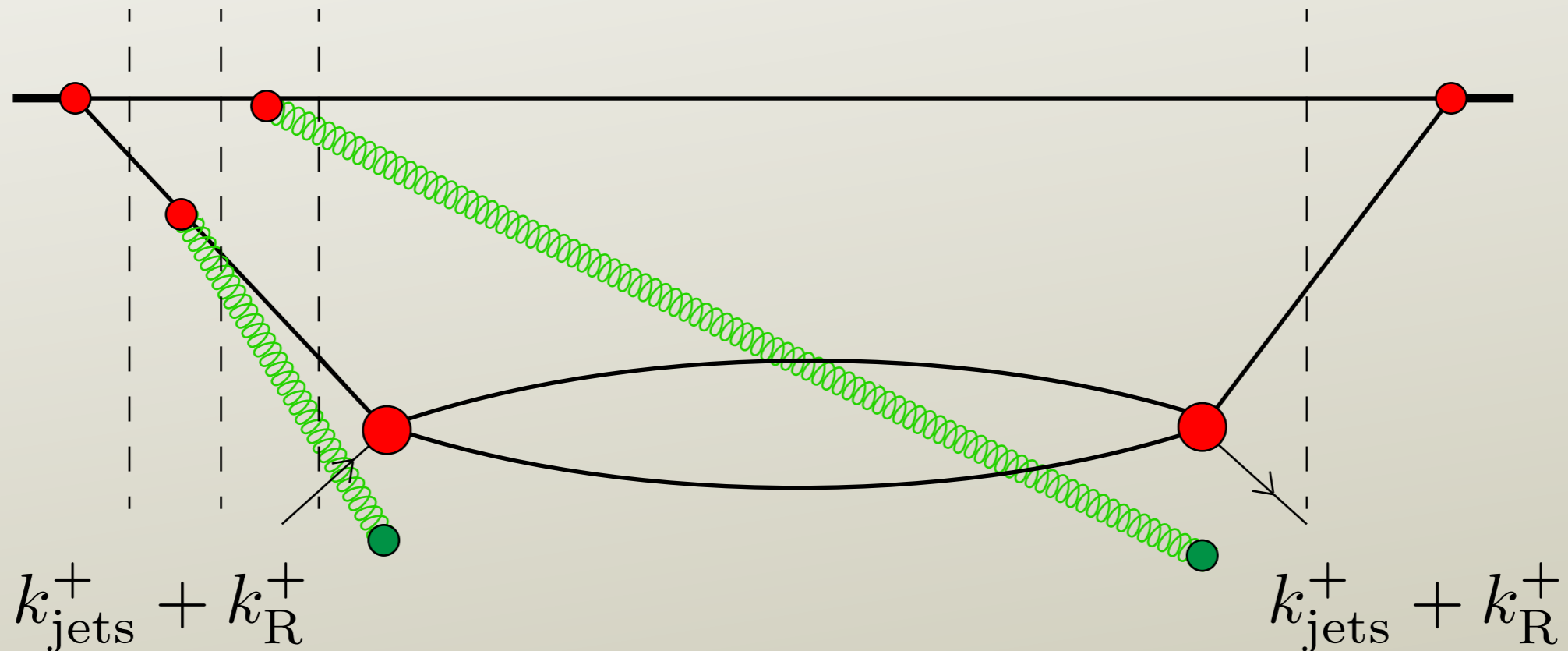
$$\sum_{C=1}^N \prod_{\xi=C+1}^N \frac{1}{k_{\mathbb{R}}^+ - D_{\xi} - i\epsilon} 2\pi\delta(k_{\mathbb{R}}^+ - D_C) \prod_{\xi=1}^{C-1} \frac{1}{k_{\mathbb{R}}^+ - D_{\xi} + i\epsilon}$$

$$= i \prod_{\xi=1}^N \frac{1}{k_{\mathbb{R}}^+ - D_{\xi} + i\epsilon} - i \prod_{\xi=1}^N \frac{1}{k_{\mathbb{R}}^+ - D_{\xi} - i\epsilon}$$

- If $N = 1$, so that there are no final state interactions, then the integral over $k_{\mathbb{R}}^+$ gives 1.
- If $N > 1$, integration over $k_{\mathbb{R}}^+$ gives ≈ 0 .
- The contribution from final state interactions is power suppressed.

- With no final state interactions, we need only look at the initial state factors

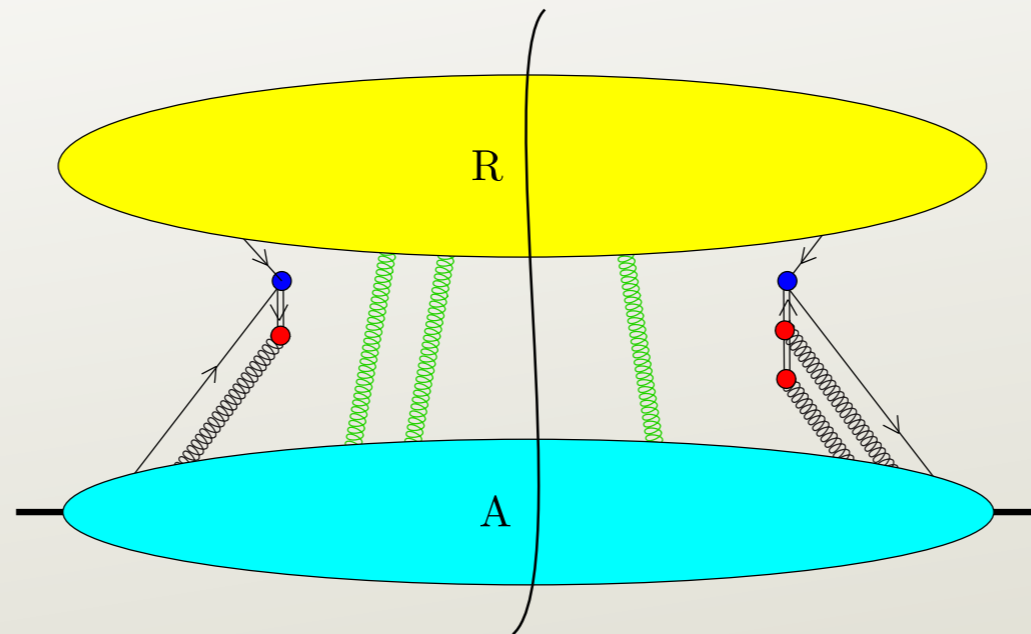
$$\prod_{\xi < H} \frac{1}{-D_{\xi} - i\epsilon} \prod_{\xi > H'} \frac{1}{-D_{\xi} + i\epsilon}$$



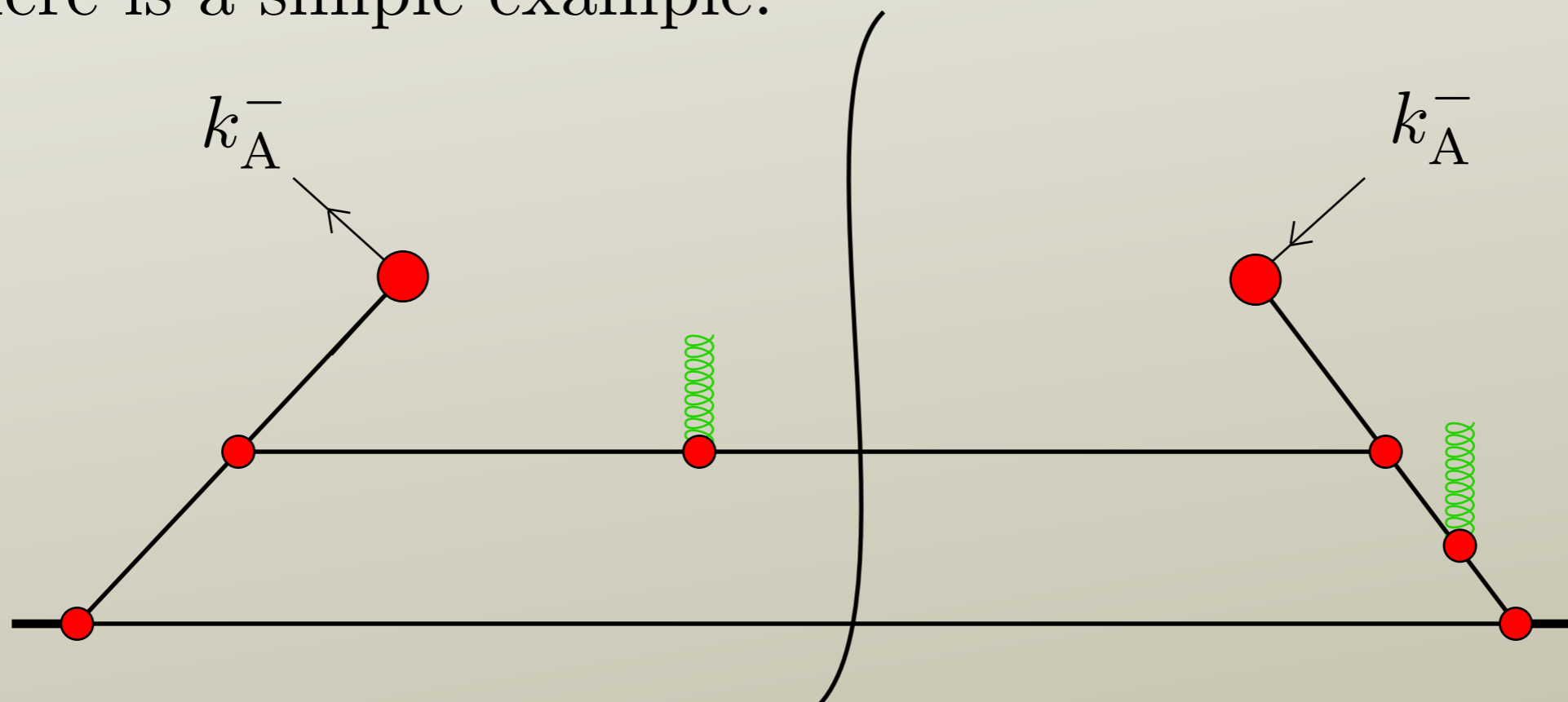
- These factors are not affected by the choice of V .
- Thus \bar{R} is independent of V .

Analysis of the A-jet

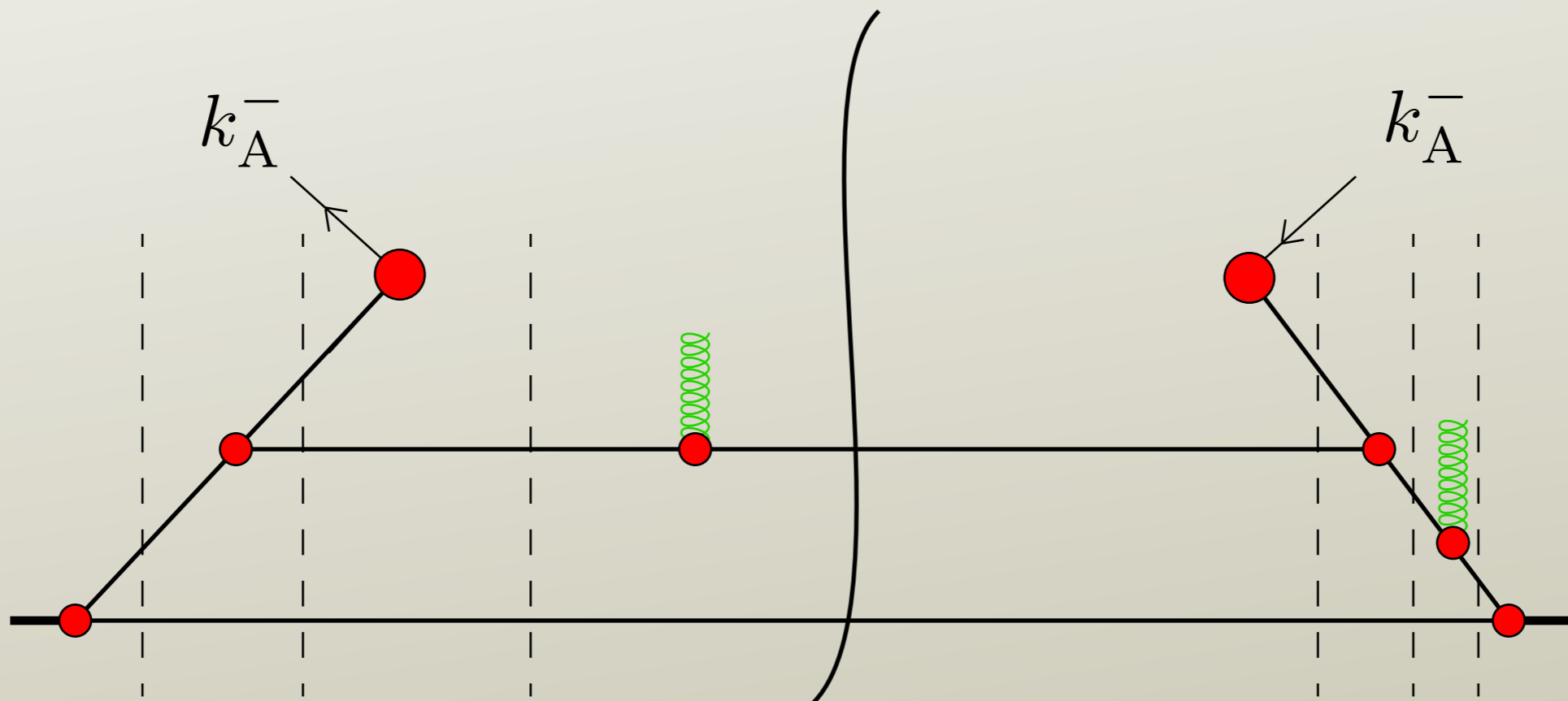
- The soft gluon connections seem to mess up factorization.



- Here is a simple example.



- In J_A , we perform the integrations over loop momenta k^- and rearrange the result with partial fractions so as to produce K^- denominators, one for each intermediate state ξ .

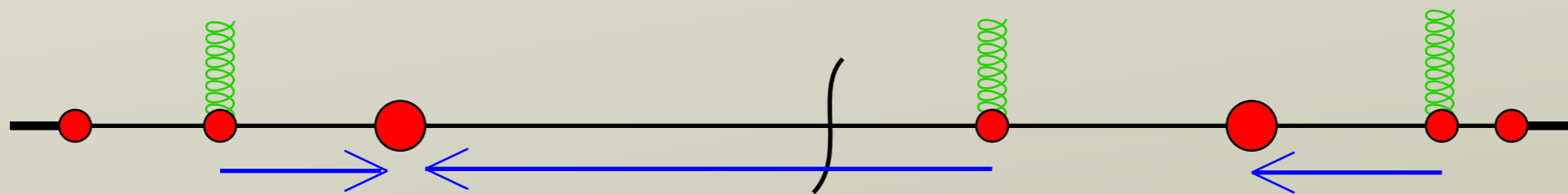


- Intermediate state P^- values:

$$D_\xi = - \sum_{l < \xi} q_l^- + \sum_{j \in \xi} \frac{k_j^2}{2k_j^+}, \quad \xi < H$$

$$D_\xi = \sum_{\xi < l < H'} q_l^- + \sum_{j \in \xi} \frac{k_j^2}{2k_j^+}, \quad H < \xi < H'$$

$$D_\xi = \sum_{\xi < l} q_l^- + \sum_{j \in \xi} \frac{k_j^2}{2k_j^+}. \quad H' < \xi$$



- Factors for final state intermediate states.

$$f_{\xi} = \frac{1}{-k_{\text{A}}^{-} - D_{\xi} + i\epsilon}, \quad H < \xi < C$$

$$f_{\xi} = 2\pi\delta(-k_{\text{A}}^{-} - D_{\xi}), \quad \xi = C$$

$$f_{\xi} = \frac{1}{-k_{\text{A}}^{-} - D_{\xi} - i\epsilon}, \quad C < \xi < H'$$

- Factors for initial state intermediate states.

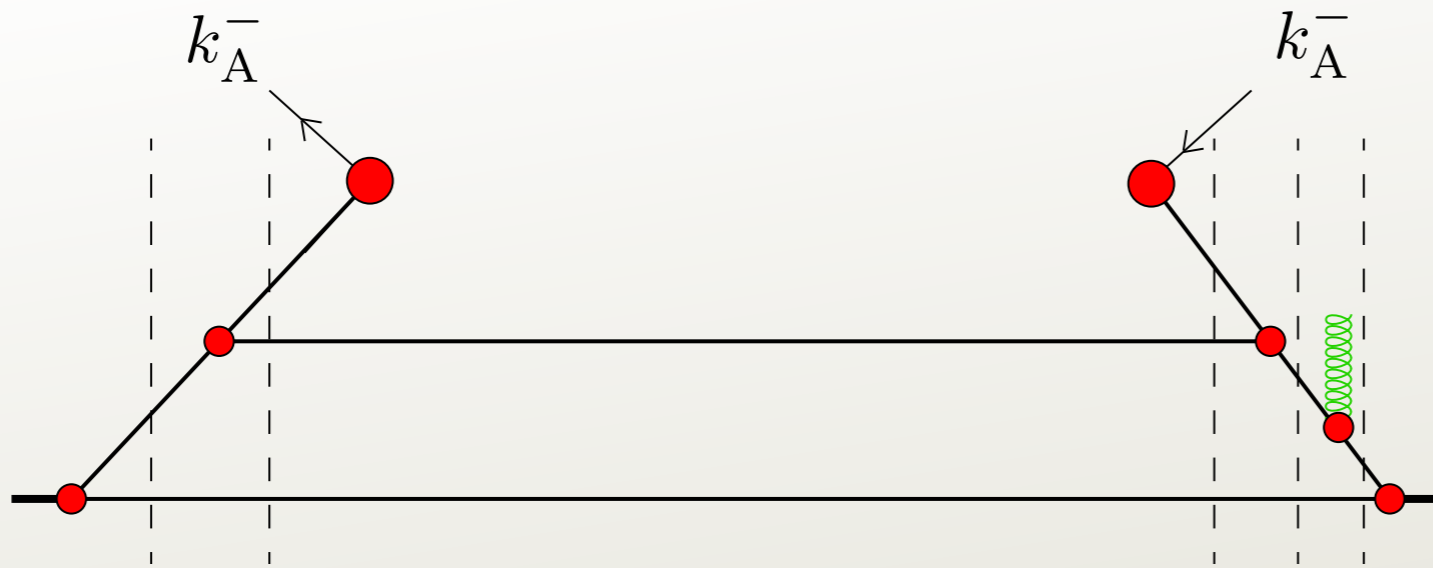
$$f_{\xi} = \frac{1}{-D_{\xi} + i\epsilon}, \quad \xi < H$$

$$f_{\xi} = \frac{1}{-D_{\xi} - i\epsilon}, \quad H' < \xi$$

- Use the unitarity identity:

$$\begin{aligned}
& \sum_{C=1}^N \prod_{\xi=C+1}^N \frac{1}{-k_A^- - D_\xi - i\epsilon} 2\pi\delta(-k_A^- - D_C) \prod_{\xi=1}^{C-1} \frac{1}{-k_A^- - D_\xi + i\epsilon} \\
&= i \prod_{\xi=1}^N \frac{1}{-k_A^- - D_\xi + i\epsilon} - i \prod_{\xi=1}^N \frac{1}{-k_A^- - D_\xi - i\epsilon}
\end{aligned}$$

- If $N = 1$, so that there are no final state interactions, then the integral over k_A^- gives 1.
- If $N > 1$, integration over k_A^- gives ≈ 0 .
- The contribution from final state interactions is power suppressed.



- With no final state interactions, we need only look at the initial state factors

$$f_\xi = \frac{1}{\sum_{l < \xi} q_l^- - \sum_{j \in \xi} \mathbf{k}_j^2 / (2k_j^+) + i\epsilon}, \quad \xi < H$$

$$f_\xi = \frac{1}{-\sum_{\xi < l} q_l^- - \sum_{j \in \xi} \mathbf{k}_j^2 / (2k_j^+) - i\epsilon}, \quad H' < \xi$$

- We have $q_l^- + i\epsilon$ in the denominators.
- Deform the integrations into the upper half q_l^- planes until $q_l^- \sim |\mathbf{q}|$.

- Deform the integrations into the upper half q_l^- planes until $q_l^- \sim |\mathbf{q}|$.
- Then look at the original Feynman representation for the integral on this deformed contour.
- q_l^- multiplies a large +-momentum.
- It is a good approximation to set \mathbf{q} to zero.
- Restore the integration contours to the real q_l^- axes.
- Contributions from crossing final state poles must cancel.

Apply Grammer-Yennie approximation

- We now have

$$J_A^{(C_A)}(k_{\text{jets}}^+, k_A^-, \mathbf{k}_A, \tilde{q}_l^\alpha)^{\{+, \dots, +\}}$$

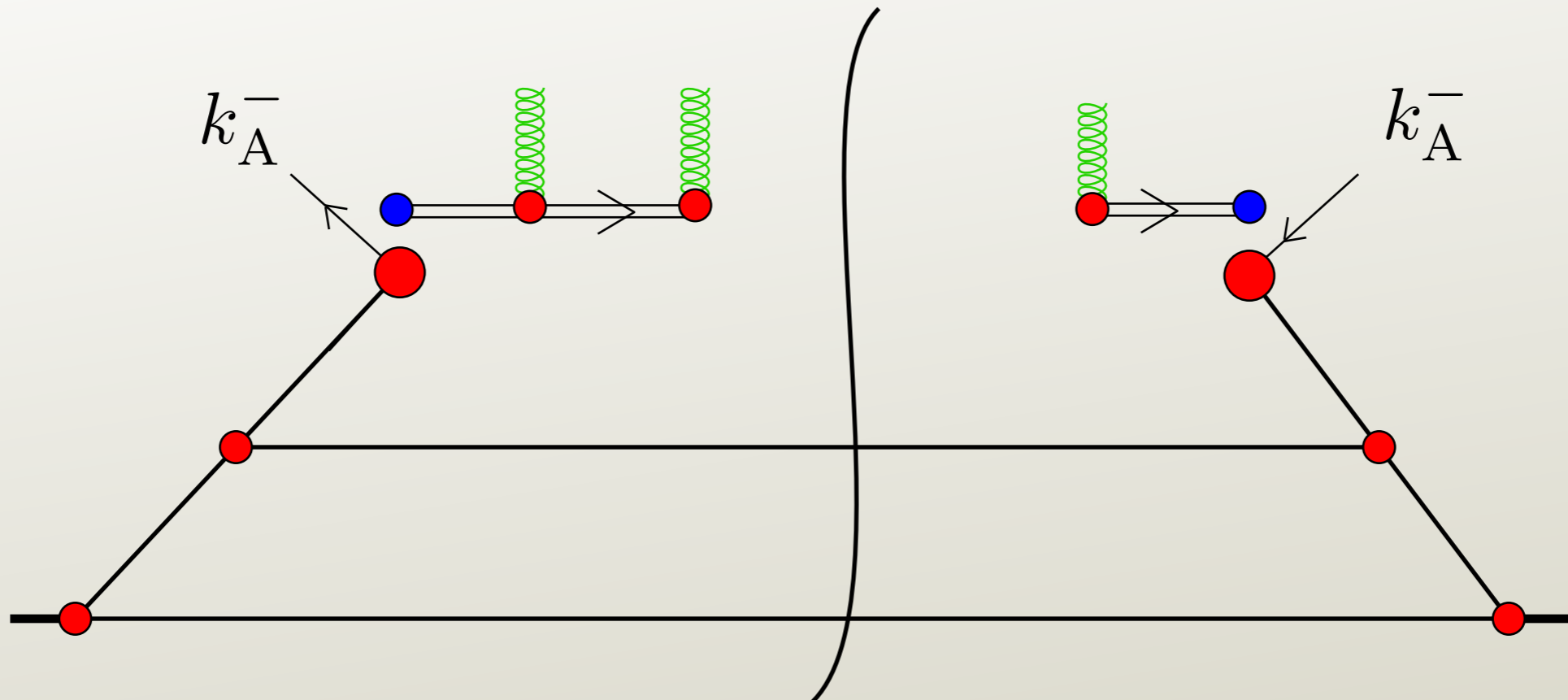
with

$$\tilde{q} = (0, q^-, \mathbf{0})$$

- We can apply the Grammer-Yennie soft gluon approximation

$$\begin{aligned} & J_A^{(C_A)}(k_{\text{jets}}^+, k_A^-, \mathbf{k}_A, \tilde{q}_l^\alpha)^{\{+, \dots, +\}} \\ &= J_A^{(C_A)}(k_{\text{jets}}^+, k_A^-, \mathbf{k}_A, \tilde{q}_l^\alpha)_{\mu_1, \dots, \mu_N} \frac{\tilde{q}_1^{\mu_1} \cdots \tilde{q}_N^{\mu_N}}{(q_1^- - i\epsilon) \cdots (q_N^- - i\epsilon)} \end{aligned}$$

- After some analysis, one gets



- The soft gluons are factored out.
- The soft gluons have zero transverse resolution.
- So they see a color singlet jet before the hard interaction.
- After the hard interaction, they see a color non-singlet jet.
- Actually, the soft gluon factor is 1.

Conclusion

$$\begin{aligned}\sigma(\mathcal{A}) = & \sum_{a,b} \int_0^1 d\eta_a \int_0^1 d\eta_b f_{a/A}(\eta_a, \mu^2) f_{b/A}(\eta_b, \mu^2) \\ & \times \hat{\sigma}(a, b, \eta_a, \eta_b, \mathcal{A}, \mu^2) \\ & + \mathcal{O}(1 \text{ GeV}^2 / Q^2(\mathcal{A}))\end{aligned}$$