

Aspects of Factorization from a Coordinate Viewpoint

Workshop on QCD Factorization

SUNY Buffalo, Nov. 2-3, 2015

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A look at perturbative long distance behavior of amplitudes and cross sections in coordinate space. Where to large corrections come from?

- 1. Coordinate space leading regions for fixed-angle scattering amplitudes
- 2. Approximations and factorization for amplitudes
- 3. Coordinate picture for cross sections
- 4. Cancellation: the largest time equation
- 5. What becomes of the “Coulomb-Glauber” region?

- I. Coordinate space leading regions..

[Analogues in momentum space for amplitudes and cross sections are well-known and continue to be studied (Collins, GS 1981; Sen, 1983; ... Feige & Schwartz 1403.6472; Caron-Huot 1501.0354)]

Here, look at coordinate space VEVs for all massless fields in configurations reflecting scattering. **The scalar propagator, for example, with $D = 4 - 2\varepsilon$:**

$$\Delta(y - x) = \frac{1}{4\pi^{2-\varepsilon}} \frac{1}{(-(y - x)^2 + i\epsilon)^{1-\varepsilon}}$$

Schematically, we have integrals over positions of internal vertices y_k :

$$\begin{aligned} G_N(x_1, \dots, x_N) &= \langle 0 | T(\phi_N(x_N) \cdots \phi_1(x_1)) | 0 \rangle \\ &= \sum_{\text{many terms}} \prod_{\text{some vertices } k} \int d^D y_k \prod_{\text{lines } j} \frac{\text{polynomial}(x_I, y_j)}{[-(\sum_{k'} \eta_{jk'} y_{k'} + \sum_l \eta_{jl} x_l)^2 + i\epsilon]^{p_j}} \end{aligned}$$

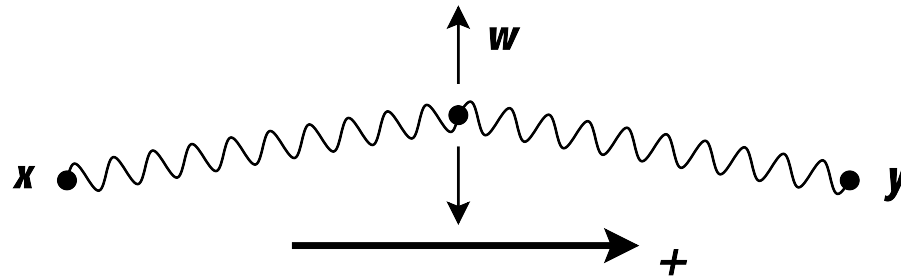
Powers $p_j = 1 - \varepsilon$ (boson) or $2 - \varepsilon$ (fermion, or derivative of boson).

Once UV renormalized, G_N is singular only at pinches in the complex integrals over positions of vertices, y_k between “incoming” and “outgoing” propagators (on the light cone).

(S. Date 1983; A.O. Erdogan 1312.0058, PRD).

Example:

$$\int d^D w \frac{1}{(-2(y-w)^+(y-w)^- - (y_\perp - w_\perp)^2 + i\epsilon)^{1-\epsilon}} \times \frac{1}{(-2(w-x)^+(w-x)^- - (w_\perp - x_\perp)^2 + i\epsilon)^{1-\epsilon}}$$



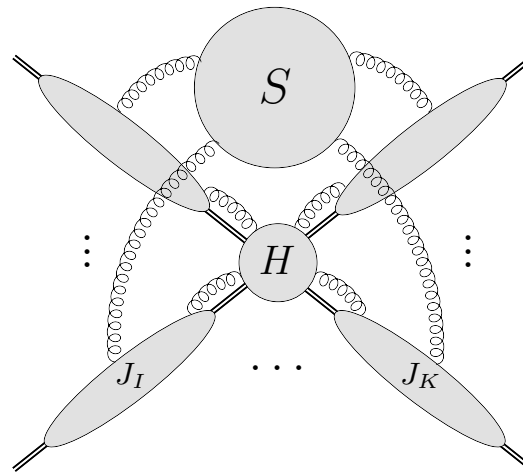
Corresponding to a pinch at $w^-, w_\perp = 0$ when x and y are lined up in the $+$ direction:

This generalizes to a set of Landau equations that determine the positions of pinch singularities in the integrations over vertex positions. For each line j , representing vector z_j^μ ,

$$z_j^2 = 0 \text{ or } \alpha_j = 0 \quad \text{and} \quad \sum_{\text{lines } j \text{ at vertex } k} \eta_{jk} \alpha_j z_j^\mu = 0$$

The result:

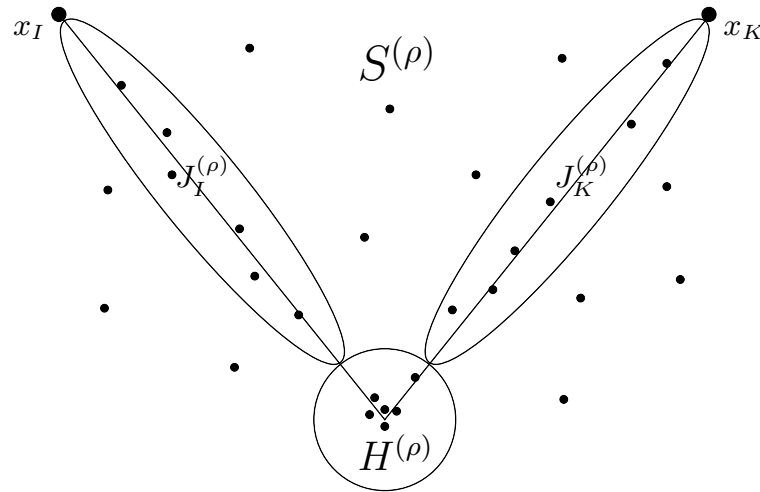
The same general “leading (-power) region”; as in momentum space, a “physical picture” (as “Coleman-Norton”):



General pinch surface (ρ) in coordinate space. Jets are in directions β_I , emerging from a localized hard scattering. Each $\beta_I^2 \rightarrow 0 = \bar{\beta}_I^2$, $\beta_I \cdot \bar{\beta}_I = 1$. Vertices group along the β_I , near the origin, or are at finite distances from these.

As in momentum space, find a natural power counting: $y_j^\mu \sim (1, \lambda, \sqrt{\lambda})$ for any $y_j \in J_i$, but $z_l \sim (1, 1, 1)$ for $z \in S$ or $(\lambda, \lambda, \lambda)$ for $z_l \in H$.

A picture of where the vertices are near pinch surface ρ :

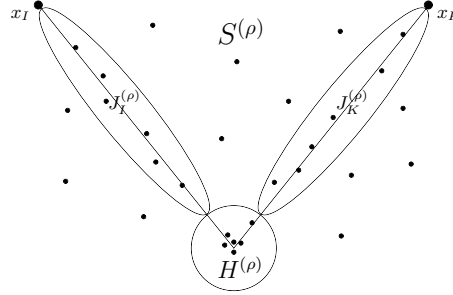


- Vertices in $H^{(\rho)}$ are near the origin
- Vertices in $J_I^{(\rho)}$ are near rays $\beta_I^\mu \propto x_I^\mu$ for $x_I^2 \rightarrow 0$
- Vertices in $S^{(\rho)}$ are separated from the origin and the rays.

This organizes large numbers of diagrams related by connecting vertices in all possible ways.

– 2. Approximations and factorization

In each such region (ρ), introduce an approximation operator acting on each diagram γ :
(GS, Erdogan 1411.4588)



$$\begin{aligned}
 t_\rho \gamma^{(n)} &\equiv \gamma^{(n)} \Big|_{\text{div } n[\rho]} \equiv \prod_I \int d\tau^{(I)} S_{\{\mu_I\}}^{(\rho)}(\{\tau^{(I)}\}) \beta_I^{\mu_I} \bar{\beta}_{I,\mu'_I} \\
 &\quad \times \int d\eta^{(I)} \int d^{D-1} z^{(I)} J_I^{(\rho)\mu'_I\nu'_I}(z^{(I)}, \eta^{(I)}) \bar{\beta}_{I,\nu'_I} \beta_I^{\nu_I} \\
 &\quad \times \int d^{D-1} y^{(I)} H_{\{\nu_I\}}^{(\rho)}(y^{(I)})
 \end{aligned}$$

For gluons attaching “soft” function to jet I in direction β_I at points z_I , keep only the $\bar{\beta}_I$ polarization and the coordinate $\tau_I = \bar{\beta}_i \cdot z_I$ along the β_I direction. Jets J_I attach to H through only polarization β_I and component $\beta_I \cdot y^{(I)}$. Note a difference from momentum space: the soft-collinear approximation acts on propagators in S , rather than on loops in J .

Soft-jet approximation recalls $A^\mu = A_c^\mu(x) + A_s^\mu(\bar{\beta}_I \cdot x)$ of SCET.

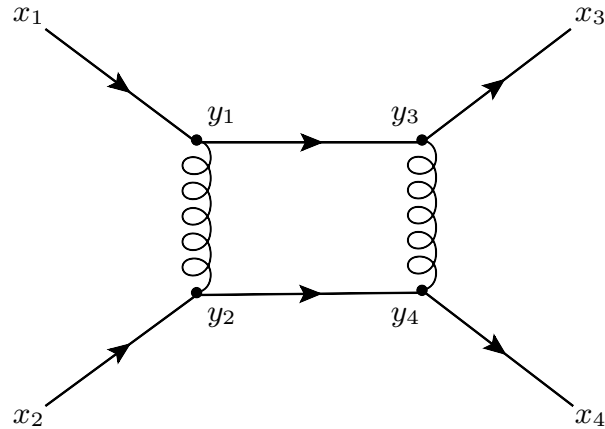
The operators t_ρ organize all divergences as external points approach the light cone relative to the hard scattering. Nested subtractions eliminates double counting:

$$R_P^{(n)} \gamma^{(n)} \Big|_{\text{div } \hat{n}[\rho]} = \left[\gamma^{(n)} + \sum_{N \in \mathcal{N}_P[\gamma^{(n)}]} \prod_{\rho \in N} (-t_\rho) \gamma^{(n)} \right] \Big|_{\text{div } \hat{n}[\rho]} = 0$$

Each t_ρ acts to approximate the integrand by its leading behavior near the singular surface. Following Collins and Soper (1983) in axial gauge and Collins (2013) in covariant gauge, the sum is over all possible nested regions, which **cancels overlapping divergences** . This appears to correspond to both a “strategy of regions” (Beneke Smirnov 1998, Jantzen 2011) and zero-bin subtractions. **Within each region ρ , only t_ρ approximation contributes, but each approximation extends over all coordinate space.**

This generalizes arguments given for Sudakov-related processes. The arguments are applicable to momentum space, and the relation to NNLO calculations is clear. Each subtraction corresponds to a leading region. Any application illustrates that the general elimination of double counting can be complicated even at low orders. Complicated or not, double counting can be avoided to all orders, even with jets in the final state.

A characteristic of QCD hard scattering:



Either gluon can carry the hard scattering, with the other gluon either soft, or (2 choices of) collinear or part of the hard scattering.

As $x_I^2 \rightarrow 0$ relative to the hard scattering (H): $x_I^\mu \propto \beta_I^\mu$, $\beta_I^2 = 0$, this allows the derivation of a factorized amplitude in coordinate space as above.

Already at one loop, nesting for fixed-angle scattering becomes nontrivial because in QCD, hard scatterings can be disjoint.

The subtractions imply soft-collinear and hard-collinear factorization, which follow from the action of the t_ρ . Look at the soft-collinear here ...

Jet-soft factorization begins with a soft-collinear approximation in t_ρ . For each propagator attaching S to J_K :

$$\begin{aligned} sc(K) \left[D^{\mu\nu}(x - z^{(K)}) \right] J_\nu(z^{(K)}) &= sc(K) \left[\frac{-g^{\mu\nu}}{[-(x - z)^2 + i\epsilon]^{1-\epsilon}} \right] J_\nu(z^{(K)}) \\ &= \beta_K^\mu \frac{-1}{[-2\bar{\beta}_K \cdot (x - z^{(K)}) \beta_K \cdot x + x_\perp^2 + i\epsilon]^{1-\epsilon}} \bar{\beta}_K^\nu J_\nu(z^{(K)}) \end{aligned}$$

with x the vertex at the other end of the soft line.

Break up the integrations for each $z^{(K)}$ where a soft gluon attaches,

$$\begin{aligned} d^D z^{(K)} &\equiv d\tau^{(K)} d^{D-1} z^{(K)}, \\ \tau^{(K)} &= \bar{\beta}_K \cdot z^{(K)} \end{aligned}$$

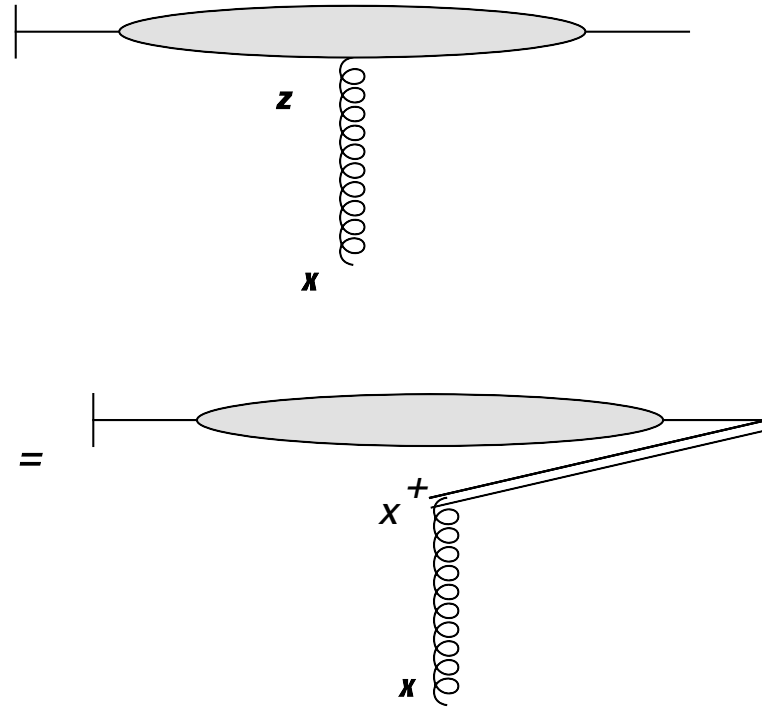
Then rewrite:

$$\begin{aligned} D^{\mu\nu}(x - \tau^{(K)}\beta_K) &= \frac{\partial}{\partial \tau^{(K)}} \int_{\pm\infty}^{\tau^{(K)}} d\tau_K D^{\mu\nu}(x - \tau_K \beta_K) \\ &= \frac{\partial}{\partial z^{(K)} \cdot \bar{\beta}_K} \int_{\pm\infty}^{z^{(K)} \cdot \bar{\beta}_K} d\tau_K D^{\mu\nu}(x - \tau_K \beta_K) \end{aligned}$$

After $sc(K)$, other derivatives acting on J are total derivatives, so we can replace $\frac{\partial}{\partial z^{(K)} \cdot \bar{\beta}_K}$ with a total divergence and integrate by parts ...

$$sc(K) [D^{\mu\nu}(x - z^{(K)})] J_\nu(z^{(K)}) \rightarrow \int_{\pm\infty}^{z^{(K)} \cdot \bar{\beta}_K} d\tau_K D^{\mu\nu'}(x - \tau_K \beta_K) \beta_{K\nu'} (-\partial^\nu J_\nu(z^{(K)}))$$

We can then apply Ward identities, and the result of $sc(K)$ factorizes, as:



And so on, with more gluons.

The result for VEVs with fixed-angle geometry:

$$G(\{x_I\}) = \prod_{I=1}^a \int d\eta_I j_I^{\text{part}}(x_I, \eta_I \bar{\beta}_I) S_{\text{ren}}(\{\beta_I \cdot \beta_J\}) \mathcal{H}(\{\eta_I \bar{\beta}_I\})$$

with appropriately-normalized jet functions

$$j_I^{\text{part}[f_\phi]}(x_I, \eta_I \bar{\beta}_I) = c_I^{\text{cusp}}(\beta_I, \bar{\beta}_I) \langle 0 | T(\phi(x_I) \phi^\dagger(\eta_I \bar{\beta}_I) \Phi_{\bar{\beta}_I}^{[f_\phi] \dagger}(\infty, \eta_I \bar{\beta}_I)) | 0 \rangle,$$

and a soft function constructed entirely from Wilson lines in the β_I directions.

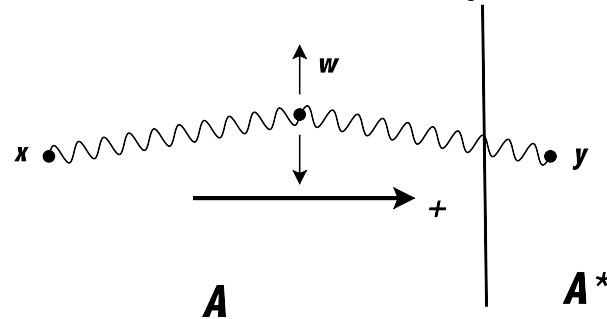
– 3. A coordinate picture for cross section: **Schematically:** $\prod_k \int d^D k A^*({k}) \prod_k \delta_+(k^2) A({k})$

Start with the “cut propagator” with momentum flowing out of w and into y :

$$\int \frac{d^D k}{(2\pi)^D} e^{-ik \cdot (y-w)} (2\pi) \delta_+(k^2) = \frac{1}{4\pi^{2-\varepsilon}} \frac{1}{(-(y-w)^2 + i\varepsilon(y-w)^0)^{1-\varepsilon}}$$

For $w^0 > y^0$ this is like a propagator in A^* , for $y^0 > w^0$, like a propagator in A .

This feature leads to pinch surfaces for cut diagrams, and hence in cross sections. Then when x and y go to the $+$ direction, w is also pinched in the plus direction.



For a pinch, we can assume that $w^0 > x^0$. Then

If $y^0 > w^0$, the “cut” propagator has a $+i\varepsilon$, and produces a pinch in w just like in the amplitude. If $w^0 > y^0$, the “cut” propagator has a $-i\varepsilon$, and it still produces a pinch because now $y^+ - w^+ < 0$.

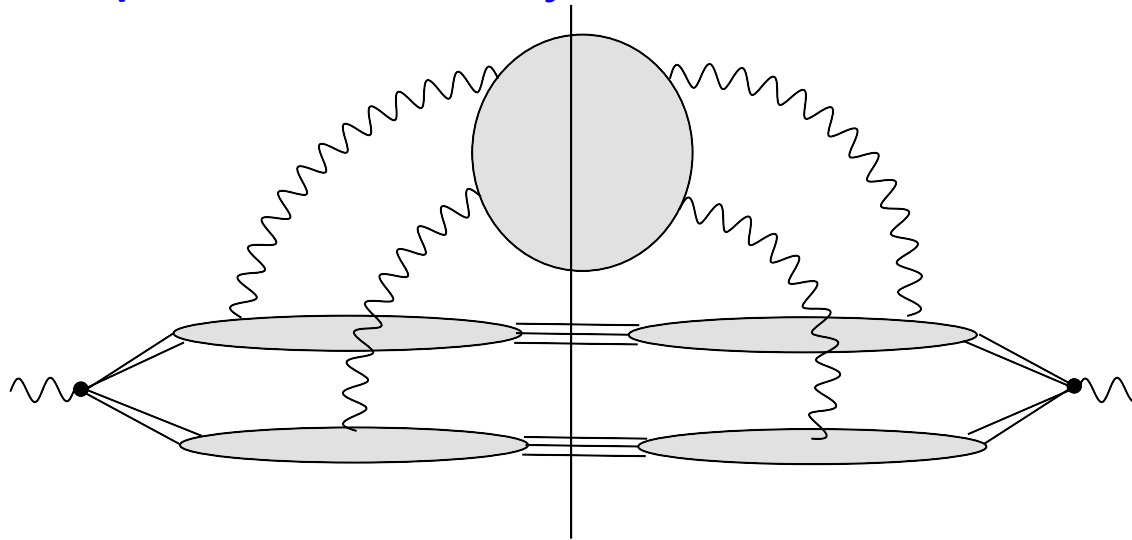
– 4. Cancellation: the largest time equation

All this means that cross sections have the same pinch surfaces as amplitudes, but with energy flow reversed in the complex conjugate. Just like time-ordered perturbation theory, but with the vertices everywhere in coordinate space.

Because vertices of both A and A^* are ordered, the vertex with the largest time is always adjacent to the final state, and connected across the cut.

As for example, in the simplest case of e^+e^- annihilation:

How cancellation takes place: the hermiticity of the interaction.

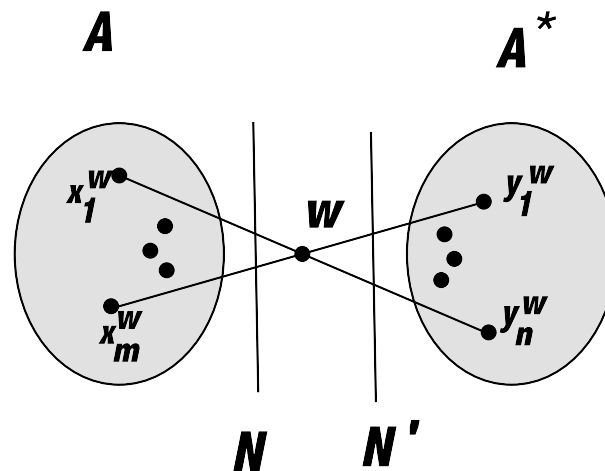


Cancellation of IR divergences: Let w^0 be the “largest time”.

(Veltman 1983 ... R. Akhoury 1992, Laenen, Larsen, Rietkerk 1505.02555)

$$w^0 > x^0 \quad \text{and} \quad w^0 > y^0$$

w may be in A or in A^* , and we must sum over both cases: N and N' with vertex at w in A^* and A :

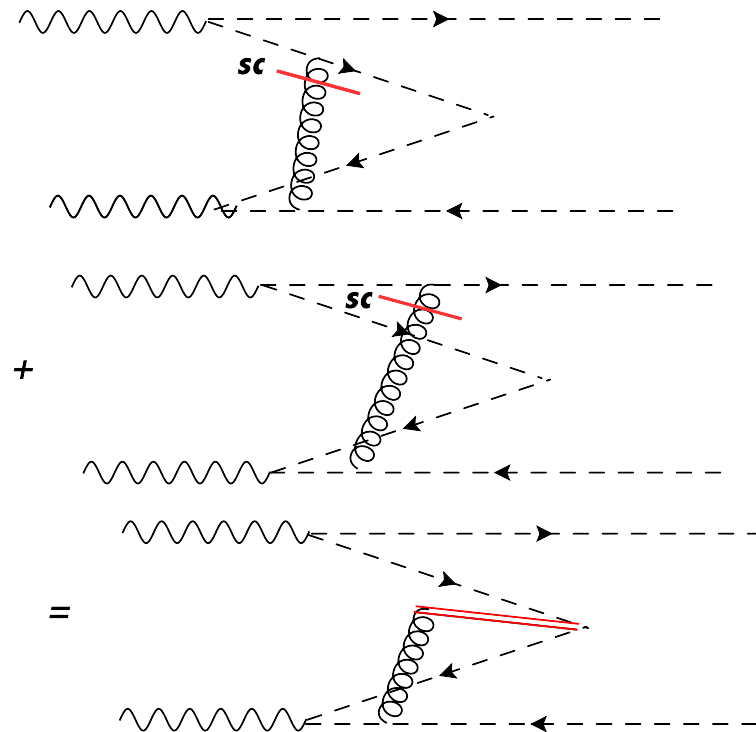


$$\int [d(PS)] A(PS) \int d^D w \left\{ \prod_j \frac{e^{-ip_j \cdot (y_j^w - w)}}{-(y_j^w - w)^2 + i\epsilon (y_j^w - w)^0} [iV(\partial_w)] \prod_i \frac{1}{-(w - x_i^w)^2 + i\epsilon} \right. \\ \left. + \prod_j \frac{1}{-(y_j^w - w)^2 - i\epsilon} [-iV(\partial_w)] \prod_i \frac{e^{-ip_i \cdot (w - x_i^w)}}{-(w - x_i^w)^2 + i\epsilon (w - x_i^w)^0} \right\}$$

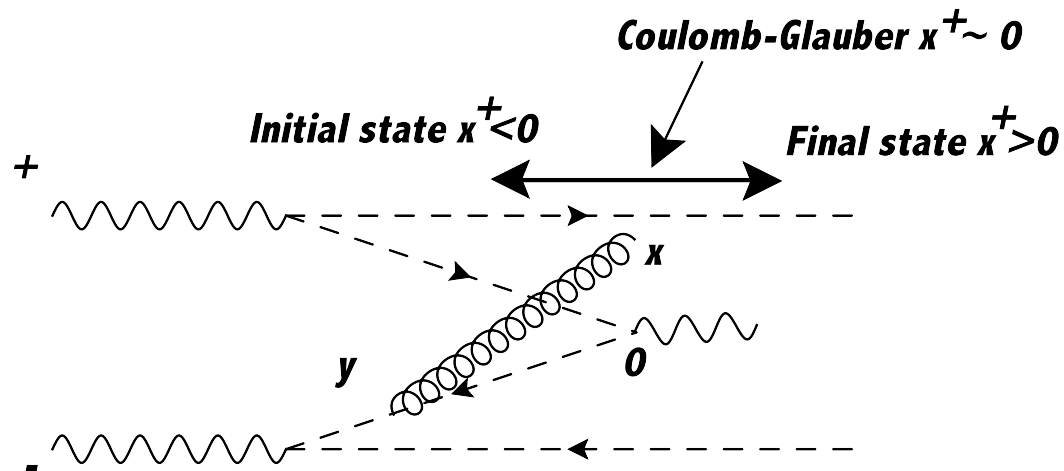
(Typically, $m, n = 1, 2$.) **If phases are small, the sum vanishes because $w^0 > x^0, y^0$, i.e., is the largest time. And they are small when all p_i, x_i, y_i are part of a single jet and/or some p_k are soft.**

– 5. Coulomb-Glauber analog in coordinate space

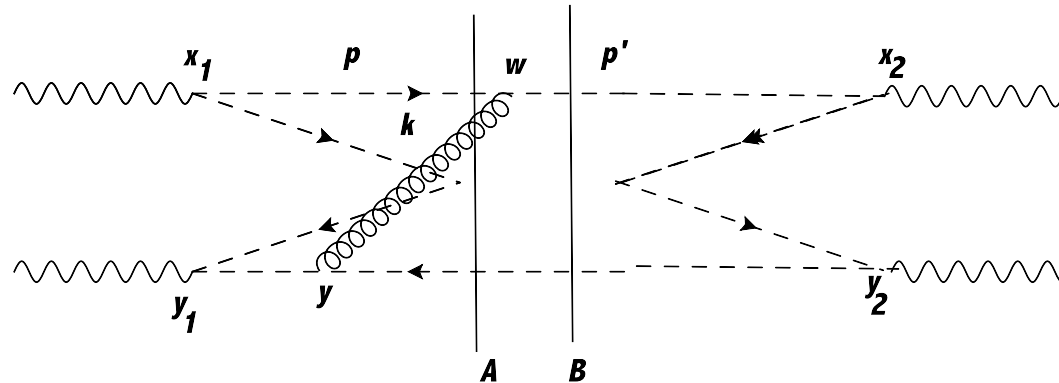
- A quite preliminary exploration.
- Whether it is good or not, the soft-collinear approximation factorizes soft from jet with spectators:



- We try to identify coordinate-space Coulomb-Glauber scattering with leading-power pinch surfaces for which incoming jets are connected by gluons for which the soft-jet approximation does not apply.



At the very lowest order:



If the phase of cut B is small, for final state A it's large, because there is no pinch near $w^+ = 0$ when w is in the "+" jet. Large $|y^-|$ forces $k^+ \sim \lambda$, and then $k_\perp \sim \lambda^{1/2}$ forces $k^- \sim 1$.

$$\text{Final state B : } e^{ip' \cdot (w - x_2)} \sim e^{i(1^+ \times \lambda^-)} \sim e^{i\lambda}$$

$$\text{Final state A : } e^{ip \cdot (x_1 - w)} e^{ik \cdot (y - w)} \sim e^{i(1^+ \lambda^-)} e^{i(1^- \times 1^+)}$$

In this case, we can't use the largest time equation. Rather, the cancellation is (as usual) with the complex conjugate.

Following this reasoning should be interesting ...

Some final thoughts

- **There is some way to go with this reasoning.**
- **My hope: the history associated with a given observable should correlate with the factorization(s) that apply.**