Characterization of decoherence-free subsystems

Emanuela Sasso

Dipartimento di Matematica dell’Università di Genova

Three days in Quantum Mechanics, Genova 6-8 June 2018

Joint work with Veronica Umanità
The evolution of an (isolated) closed quantum system is driven by the Hamiltonian $H$, according to

$$\frac{d}{dt} \rho(t) = -i[H, \rho(t)]$$

In particular, $\rho(t) = U_t \rho(0) U_t^*$, where $U_t = e^{-iHt}$. 
The evolution of an (isolated) closed quantum system is driven by the Hamiltonian $H$, according to

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)]$$

In particular, $\rho(t) = U_t \rho(0) U_t^*$, where $U_t = e^{-iHt}$.

A relevant class of open quantum systems obeys Markovian dynamics. The state change is described by a TPCP-map $\mathcal{T}_*$ on $\mathcal{H}$ trace class operators, i.e.

1. $\mathcal{T}_{*0} = \text{identity map}$
2. $\mathcal{T}_{*(t+s)} = \mathcal{T}_t \mathcal{T}_s$
3. $\text{tr}(\mathcal{T}_t(\rho) x)$ is a continuous function of $t$.

$$\frac{d}{dt}\rho(t) = \mathcal{L}_*(\rho(t))$$
Settings (in Heisenberg’s picture)

- $S$ open quantum system $(\mathcal{B}(h), \mathcal{T})$
- $\mathcal{M} = \mathcal{B}(h)$ von Neumann algebra acting on $h$
- $\mathcal{T} = \{\mathcal{T}_t\}_{t \geq 0}$ QMS on $\mathcal{B}(h)$ (irreversible dynamics)
Settings (in Heisenberg’s picture)

- $S$ open quantum system $(\mathcal{B}(h), \mathcal{T})$
- $\mathcal{M}(= \mathcal{B}(h))$ von Neumann algebra acting on $h$
- $\mathcal{T} = \{\mathcal{T}_t\}_{t \geq 0}$ QMS on $\mathcal{B}(h)$ (irreversible dynamics)

- $\mathcal{T}$ is a $w^*$—continuous semigroup on $\mathcal{B}(h)$;
- $\mathcal{T}_t(1) = (1)$ for all $t \geq 0$ ($\mathcal{T}$ is Markovian);
- $\mathcal{T}_t$ is completely positive ($\sum_{i,j=1}^{n} b_i^* \mathcal{T}_t(a_i^*a_j)b_j \geq 0$);
- $\mathcal{T}_t$ is $w^*$—continuous ($\mathcal{T}_t$ is normal).

$\mathcal{T}^* = \{\mathcal{T}^*_t\}_{t \geq 0}$ predual semigroup of $\mathcal{T}$:

$\mathcal{T}^*_t : \mathcal{S}(h) \to \mathcal{S}(h), \quad (\mathcal{T}^*_t\rho)(x) := tr(\rho \mathcal{T}_t(x))$
In the following we will restrict to uniformly continuous semigroups, i.e.

\[
\lim_{t \to 0} \sup_{\|x\| \leq 1} \|T_t(x) - x\| = 0
\]
Uniformly continuous semigroups

In the following we will restrict to uniformly continuous semigroups, i.e.

\[
\lim_{t \to 0} \sup_{\|x\| \leq 1} \|T_t(x) - x\| = 0
\]

\[
T_t(x) = e^{t\mathcal{L}}(x)
\]

\[
\mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_{k \geq 1} (L_k^*L_kx - 2L_k^*xL_k + xL_k^*L_k)
\]

where \( H = H^* \), \( \sum_k L_k^*L_k \) is strongly convergent.

In this case

\[
\mathcal{L}_*(\rho) = -i[H, \rho] - \frac{1}{2} \sum_{k \geq 1} (L_kL_k^*\rho - 2L_k\rho L_k^* + \rho L_k^*L_k)
\]
The most general structure which can faithfully embody quantum information is a subsystem of $h$.

We call **quantum subsystems** of a system on $h$ a system whose Hilbert space is a tensor factor $h_s$ of a subspace $h_{sf}$ of $h$, such that

$$h = h_{sf} \oplus h_r = (h_s \otimes h_f) \oplus h_r$$

(1)

for some factor $h_f$ and remainder space $h_r$. 

Notation: If we fix an onb for each of the three spaces $h_{sf}, h_f, h_r$, 

$$x = (x_{sf} x_p x_q x_r),$$

where $x_{sf} \in B(h_{sf}), x_q \in B(h_{sf}, h_r), x_p \in B(h_r, h_{sf})$ and $x_r \in B(h_r)$; in general, $x_{sf} \neq x_s \otimes x_f$. 
The most general structure which can faithfully embody quantum information is a subsystem of $h$.

We call *quantum subsystems* of a system on $h$ a system whose Hilbert space is a tensor factor $h_s$ of a subspace $h_{sf}$ of $h$, such that

$$h = h_{sf} \oplus h_r = (h_s \otimes h_f) \oplus h_r$$

for some factor $h_f$ and remainder space $h_r$.

**Notation:** If we fix an onb for each of the three spaces $h_s$, $h_f$ and $h_r$, 

$$x = \begin{pmatrix} x_{sf} \\ x_q \\ x_p \\ x_r \end{pmatrix},$$

where $x_{sf} \in \mathcal{B}(h_{sf})$, $x_q \in \mathcal{B}(h_{sf}, h_r)$, $x_p \in \mathcal{B}(h_r, h_{sf})$ and $x_r \in \mathcal{B}(h_r)$; in general, $x_{sf} \neq x_s \otimes x_f$. 
Decomposition of the Hilbert space $h$: Decoherence free-subsystems DFS

$\textbf{DFS}$

$h_s$ supports (or simply, $h_s$ is) a \textit{decoherence-free (or a noiseless) subsystem (DFS)} for $\mathcal{T}$ if the evolution of $\rho = \rho_s \otimes \rho_f$, with $\rho_s$ state on $h_s$ and $\rho_f$ state on $h_f$, is given by

$$\mathcal{T}_{*t}(\rho) = U_t \rho_s U^*_t \otimes \mathcal{T}^f_{*t}(\rho_f)$$

(2)

for $t \geq 0$, where $(U_t)_t$ is a unitary group on $h_f$ and $\mathcal{T}^f$ is a QMS on $h_f$.

If $h_f$ is one-dimensional that $h_s$ supports a \textit{decoherence-free subspace} if $h_s$ is one-dimensional, a DFS supported by $h_s$ is \textit{trivial}.
Theorem [Ticozzi, Viola (2008), S., Umanità (2017)]

\( h_s \) supports a DFS if and only if (up to a unitary transformation)

\[
L_k = \begin{pmatrix} \mathbb{1}_s \otimes L_{f,k} & L_{p,k} \\ 0 & L_{r,k} \end{pmatrix} \quad H = \begin{pmatrix} H_s \otimes \mathbb{1}_f + \mathbb{1}_s \otimes H_f \\ H_p \end{pmatrix}
\]

(3)

with

\[
iH_p = \frac{1}{2} \sum_k \left( \mathbb{1}_s \otimes L_{f,k}^* \right) L_{p,k},
\]

(4)

where \( H_q = H_p^* \). In particular, \( H_s \) is the generator of the unitary group \( (U_t)_t \) in equation (2).

Moreover, in this case, the following facts are equivalent:

1. the GKSL operators \( (L_k) \) and \( H \) have the diagonal form

\[
L_k = \mathbb{1}_s \otimes L_{f,k} + L_{r,k} \quad H = H_s \otimes \mathbb{1}_f + \mathbb{1}_s \otimes H_f + H_r,
\]

(5)

2. \( p_{sf} \) and \( p_r \) are fixed points of \( \mathcal{T} \).

If \( \mathcal{T} \) possesses a faithful normal invariant state, then equation (5) holds.
The **decoherence-free (DF) subalgebra** of $\mathcal{T}$ is defined by

$$\mathcal{N}(\mathcal{T}) = \{x : \mathcal{T}_t(x^* x) = \mathcal{T}_t(x)^* \mathcal{T}_t(x), \mathcal{T}_t(xx^*) = \mathcal{T}_t(x) \mathcal{T}_t(x)^* \ \forall \ t \geq 0\}.$$ 

**Proposition**

Consider operators $H$ and $(L_k)_k$ in a GKSL representation. Then

1. For all $x \in \mathcal{N}(\mathcal{T})$ we have $\mathcal{T}_t(x) = e^{itH}x e^{-itH}$, $t \geq 0$,
2. $\mathcal{N}(\mathcal{T})$ is the biggest von Neumann subalgebra on which every map $\mathcal{T}_t$ acts as a $\ast$-automorphism,
3. $\mathcal{N}(\mathcal{T}) = \{\delta^{(n)}_H(L_k), \delta^{(n)}_H(L_k^*) | n \geq 0, k \geq 1\}'$ with $\delta_H(x) := [H, x]$.
Theorem [Deschamps, Fagnola, S., Umanità]:

$\mathcal{N}(T)$ is an atomic algebra if and only if there exist two countable sequences of separable Hilbert spaces $(k_i)_i$, $(m_i)_i$, and a unitary operator $U : h \to \bigoplus_{i \in I} (k_i \otimes m_i)$ such that $U\mathcal{N}(T)U^* = \bigoplus_{i \in I} (\mathcal{B}(k_i) \otimes 1_{m_i})$.
Atomic $\mathcal{N}(\mathcal{T})$

**Theorem [Deschamps, Fagnola, S., Umanità]:**

$\mathcal{N}(\mathcal{T})$ is an atomic algebra if and only if there exist two countable sequences of separable Hilbert spaces $(k_i)_i$, $(m_i)_i$, and a unitary operator $U : h \to \bigoplus_{i \in I} (k_i \otimes m_i)$ such that $\mathcal{N}(\mathcal{T}) = \bigoplus_{i \in I} (\mathcal{B}(k_i) \otimes \mathbb{1}_{m_i})$. 
Atomic $\mathcal{N}(\mathcal{T})$

Theorem [Deschamps, Fagnola, S., Umanità]:

$\mathcal{N}(\mathcal{T})$ is an atomic algebra if and only if there exist two countable sequences of separable Hilbert spaces $(k_i)_i$, $(m_i)_i$, and a unitary operator $U : h \rightarrow \bigoplus_{i \in I} (k_i \otimes m_i)$ such that $\mathcal{N}(\mathcal{T}) = \bigoplus_{i \in I} (\mathcal{B}(k_i) \otimes \mathbb{1}_{m_i})$.

Moreover

- $T_t(x_{k_i} \otimes x_{m_i}) = e^{itK_i} x_{k_i} e^{-itK_i} \otimes T_t^{m_i}(x_{m_i})$

- For every GSKL representation of $\mathcal{L}$ by means of operators $H, (L_\ell)_{\ell \geq 1}$, we have

$$UL_\ell U^* = \bigoplus_{i \in I} \left( 1_{k_i} \otimes M_\ell^{(i)} \right)$$

and

$$UHU^* = \bigoplus_{i \in I} \left( K_i \otimes \mathbb{1}_{m_i} + 1_{k_i} \otimes M_0^{(i)} \right)$$
Preliminary remarks:

Assume $h_s$ supporting a DFS. $\mathcal{I}(h_{sf})$ is preserved by $T_*$, i.e., the orthogonal projection onto $h_{sf}$ is subarmonic. When such a projection is a fixed point, the following results hold:

1. $\mathcal{B}(h_{sf})$ is invariant for $T$,

2. $T_t(x_s \otimes x_f) = U^*_t x_s U_t \otimes T^f_t(x_f) \quad \forall x_s \in \mathcal{B}(h_s), \ x_f \in \mathcal{B}(h_f),$

3. $\mathcal{B}(h_s) \otimes 1_f \subseteq \mathcal{N}(T)$. 
Assume $h_s$ supporting a DFS. $\mathcal{E}(h_{sf})$ is preserved by $T_*$, i.e, the orthogonal projection onto $h_{sf}$ is subharmonic. When such a projection is a fixed point, the following results hold:

1. $\mathcal{B}(h_{sf})$ is invariant for $T$,
2. $T_t(x_s \otimes x_f) = U_t^* x_s U_t \otimes T_t^f(x_f)$ \quad $\forall x_s \in \mathcal{B}(h_s), \ x_f \in \mathcal{B}(h_f)$,
3. $\mathcal{B}(h_s) \otimes 1_f \subseteq \mathcal{N}(T)$.

If there exists a faithful invariant state, then the orthogonal projection onto $h_{sf}$ is harmonic.
Assume $h_s$ supporting a DFS. $\mathcal{I}(h_{sf})$ is preserved by $T_*$, i.e, the orthogonal projection onto $h_{sf}$ is subarmonic. When such a projection is a fixed point, the following results hold:

1. $B(h_{sf})$ is invariant for $T$,
2. $T_t(x_s \otimes x_f) = U_t^* x_s U_t \otimes T_t^f(x_f) \quad \forall x_s \in B(h_s), \ x_f \in B(h_f)$,
3. $B(h_s) \otimes 1_f \subseteq \mathcal{N}(T)$.

If there exists a faithful invariant state, then the orthogonal projection onto $h_{sf}$ is harmonic.

If $\mathcal{N}(T)$ is atomic, $h = (k_i \otimes m_i) \oplus \left( \bigoplus_{j \in I \setminus \{i\}} k_j \otimes m_j \right)$, $k_i$ supports a DFS.
Problem

\( \mathcal{N}(T) \) atomic implies \( h = \bigoplus_{i \in I} k_i \otimes m_i \). 

\( h_s \) supports a DFS for \( T \) then \( h = (h_s \otimes h_f) \oplus h_r \).

IS THERE SOME RELATIONSHIP BETWEEN THESE DECOMPOSITIONS OF \( h \)?
\[ \mathcal{N}(T) \text{ atomic implies } h = \bigoplus_{i \in I} k_i \otimes m_i \]

\[ h_s \text{ supports a DFS for } T \text{ then } h = (h_s \otimes h_f) \oplus h_r \]

**Remark:**

- We need the existence of faithful invariant state!
Essential and maximal DFSs

**Definition:**
Let \( h = (h_s \otimes h_f) \oplus h_r \) be such that \( h_s \) support a DFS for \( \mathcal{T} \).

\( h_s \) supports a **maximal** DFS with respect to \( h_f \) if \( \mathcal{T}^f \) has no non trivial DFSs in \( h_f \).

**Assume \( \mathcal{T} \) has a faithful normal invariant state:**
- If \( h_s \) is not maximal with respect to \( h_f \),

\[
h_{sf} = \bigoplus_{j \in J} (k_{s_j} \otimes k_{f_j})
\]

for some sequences \((k_{s_j})_j \) and \((k_{f_j})_j \) such that each \( k_{s_j} \) supports a maximal DFS wrt \( k_{f_j} \).
Essential and maximal DFSs

Definition:
Let \( h = (h_s \otimes h_f) \oplus h_r \) be such that \( h_s \) support a DFS for \( \mathcal{T} \). \( h_s \) supports an \textit{essential} DFS with respect to \( h_f \) if \( \mathcal{N}(\mathcal{T}^f) = \mathbb{C}1_f \).

Assume \( \mathcal{T} \) has a faithful normal invariant state:

- If \( h_s \) is essential, then it is maximal.
- If \( h_s \) is maximal but not essential wrt \( h_f \), we can split the tensor factor \( h_f \) into a direct sum of spaces \( h_{f_j} \) with \( h_s \) essential wrt any \( h_{f_j} \)

\[
h_{sf} = \bigoplus_{j \in J} (h_s \otimes h_{f_j})
\]
Main results:

**Question**

We have a decomposition of $h$ induced by DFS and another one induced by the atomicity of $\mathcal{N}(T)$: are they the same? are they connected in some way??

**Theorem [S., Umanità]**

Assume there exists a faithful normal invariant state, $\mathcal{N}(T)$ atomic. If $h_s$ supports a **essential DFS** wrt $h_f$, then there exists a unique $j \in I$ such that $h_{sf} = k'_j \otimes m_j$ for some $k'_j$ non zero subspace of $k_j$ preserved by $K_j$. 
Sketch of the proof:

1. \( I(h_{inv}) \) is invariant with respect to \( T_* \) if and only if \( h_{inv} \) is of the form \( k'_i \otimes m_i \) where \( k'_i \) is a non zero subspace of \( k_i \) invariant under the Hamiltonian \( K_i \).

2. \( h_{sf} = \bigoplus_{i \in I} p_i h_{sf} = \bigoplus_{i \in I'} (k'_i \otimes m_i) \) with \( p_i \) projection on \( k_i \otimes m_i \).

3. \( \bigoplus_{i \in I'} (B(k'_i) \otimes 1_{m_i}) \subseteq \mathcal{N}(T^{sf}) \).

4. We assume that there are at least two indexes, \( i_1 \) and \( i_2 \) in \( I' \), such that \( k'_{i_1} \) and \( k'_{i_2} \) are different from 0. This clearly implies that the center of \( \mathcal{N}(T^{sf}) \) is not trivial (it contains \( \bigoplus_{h=1,2} (1_{k'_{i_h}} \otimes 1_{m_{i_h}}) \)), in contradiction to the fact that \( \mathcal{N}(T^{sf})(= B(h_{sf}) \otimes \mathcal{N}(T^f)) \) is a factor.
Main results:

**Question**
We have a decomposition of \( h \) induced by DFS and another one induced by the atomicity of \( \mathcal{N}(T) \):
are they the same? are they connected in some way??

**Theorem [S. Umanità]**
Assume \( T \) has a faithful normal invariant state and \( \mathcal{N}(T) \) is atomic. Then there exists a countable set \( L \) such that, up to a unitary operator we have

\[
h = \bigoplus_{l \in L} (h_{s_l} \otimes h_{f_l})
\]

with \( h_{s_l} \) supporting an essential DFS. Moreover, for all \( l \in L \) we have

\[
h_{s_l} \otimes h_{f_l} = k_{i_l}^{l_j} \otimes m_{i_l} \text{ for a unique } i_l \in I, \text{ where } k_{i_l}^{l_j} \text{ is invariant with respect to the Hamiltonian } K_{i_l}.
\]
There is **environment induced decoherence** (EID) on the system described by $\mathcal{T}$, if there exists a $\mathcal{T}_t$-invariant and $*$-invariant weak* closed subspace $\mathcal{M}_2$ of $\mathcal{B}(\mathcal{H})$ such that:

**EID1**  $\mathcal{B}(\mathcal{H}) = \mathcal{N}(\mathcal{T}) \oplus \mathcal{M}_2$ with $\mathcal{M}_2 \neq \{0\}$,

**EID2**  $w^* \lim_{t \to \infty} \mathcal{T}_t(x) = 0$ for all $x \in \mathcal{M}_2$. 

Theorem [Fagnola, S., Umanità] Assume that there exists a faithful normal invariant state $\rho$. Then $\mathcal{N}(\mathcal{T})$ is atomic if and only if EID.
There is **environment induced decoherence** (EID) on the system described by $\mathcal{T}$, if there exists a $\mathcal{T}_t$-invariant and $\ast$-invariant weak* closed subspace $\mathcal{M}_2$ of $\mathcal{B}(\mathcal{H})$ such that:

**EID1** $\mathcal{B}(\mathcal{H}) = \mathcal{N}(\mathcal{T}) \oplus \mathcal{M}_2$ with $\mathcal{M}_2 \neq \{0\}$,

**EID2** $w^* - \lim_{t \to \infty} \mathcal{T}_t(x) = 0$ for all $x \in \mathcal{M}_2$.

**Theorem [Fagnola, S., Umanità]**

Assume that there exists a faithful normal invariant state $\rho$. Then $\mathcal{N}(\mathcal{T})$ is atomic if and only if EID.
References:


